EUCLIDEAN GIBBS STATES OF QUANTUM LATTICE SYSTEMS

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ABSTRACT. An approach to the description of the Gibbs states of lattice models of interacting quantum anharmonic oscillators, based on integration in infinite dimensional spaces, is described in a systematic way. Its main feature is the representation of the local Gibbs states by means of certain probability measures (local Euclidean Gibbs measures). This makes possible to employ the machinery of conditional probability distributions, known in classical statistical physics, and to define the Gibbs state of the whole system as a solution of the equilibrium (Dobrushin-Lanford-Ruelle) equation. With the help of this representation the Gibbs states are extended to a certain class of unbounded multiplication operators, which includes the order parameter and the fluctuation operators describing the long range ordering and the critical point respectively. It is shown that the local Gibbs states converge, when the mass of the particle tends to infinity, to the states of the corresponding classical model. A lattice approximation technique, which allows one to prove for the local Gibbs states analogs of known correlation inequalities, is developed. As a result, certain new inequalities are derived. By means of them, a number of results describing physical properties of the model are obtained. Among them are: the existence of the long-range order for low temperatures and large values of the particle's mass; the suppression of the critical point behaviour for small values of the mass and for all temperatures; the uniqueness of the Euclidean Gibbs states for all temperatures and for the values of the mass less than a certain threshold value, dependent on the temperature.

CONTENTS

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1. Introduction

Gibbs states of quantum systems living on a lattice $\mathbb{L}$ are constructed as positive normalized functionals on von Neumann algebras whose elements (observables) represent physical quantities characterizing a given system (see [26], [51]). If the algebra of observables of each subsystem in a finite $\Lambda \subset \mathbb{L}$ may be regarded as a C^*–algebra of bounded operators on a Hilbert space, the construction of the Gibbs states is performed within an algebraic approach, which now is quite well elaborated [26]. But if one needs to include into consideration also unbounded operators, the situation becomes much more complicated and the construction of Gibbs states even for simple models turns into a very hard task (for more details on this see the discussion in [49], Chapter IV, pp. 169, 170 and [50]).

In 1975, in [1], an approach to the construction of Gibbs states of lattice systems of interacting quantum particles performing D-dimensional oscillations around their equilibrium positions has been initiated. This approach employs the integration theory in path spaces (see also [2], [16], [17], [20], [43], [44], [51], [60], [76]). It is based on the fact, discovered by R. Høegh-Krohn [48], that the C^*-algebra of observables of every subsystem in a finite subset $\Lambda \subset \mathbb{L}$ is spanned by the operators of a certain type constructed with bounded multiplication operators. The essence of the approach is that the states at a given temperature $T = \beta^{-1}$ taken on such operators (Green functions) are written as expectations with respect to a probability measure $\mu_{\beta,\Lambda}$ on a certain infinite-dimensional space, obtained as a perturbation of a Gaussian measure. Then, for 'nice' perturbating functions, it was proved that, in the thermodynamic limit when $\Lambda \nearrow \mathbb{L}$, the weak limit $\mu_{\beta,\Lambda} \Rightarrow \mu_\beta$ exists. The Gibbs state of the whole system as a functional was reconstructed by means of μ_β, analogously to the case of the Euclidean quantum field theory (see [1], Section 4, in particular Theorem 4.1 for the reconstruction of the Gibbs state). That is the reason why μ_β is often called *a Euclidean Gibbs state* of a quantum system. This approach was further developed in [16] – [20], [43], [44], [52]. As a result, it has become possible to develop substantially the theory of Gibbs states in the models of quantum anharmonic crystals employing unbounded operators. In particular, for a model of this type, the convergence at the critical point of the states taken on fluctuation operators - the only result of this kind obtained for quantum models - was proven [4]. In this article, we intend to describe the most important aspects of this approach in a systematic way. Though being mainly a review article based on our works [5] – [8], [18], [19], [52] – [56], this paper contains some new results (see the last paragraph of this introduction).

Since the Euclidean Gibbs state μ_β is a measure, in order to establish the set of all possible such states, one can apply the machinery of conditional probability distributions, known in classical statistical physics (see [33], [34], [42]). This was done in [8], [11] – [14], [63]. Certain information regarding the properties of the systems with large values of D may be obtained by means of perturbation arguments with respect to $1/D$, as it has been done in [15]. Starting from [1], as a main tool in studying such states, various cluster expansion techniques were employed [10], [59], [62], [64], [68]. As a result, the existence and certain properties of the Euclidean Gibbs states (ergodicity, decay of correlations) were obtained for high temperatures [64], or for all temperatures in the case of the one-dimensional lattice [62], [64]. In [59], for small values of the particle's mass, the convergence of corresponding cluster expansions was proved for all values of the temperature including zero. This made possible to prove the existence of temperature and ground states and to describe a number of properties of these states. The convergence of cluster expansions implies analyticity in the coupling parameter which, for systems of particles moving on compact manifolds (considered in [10]), or for the case of 'gentle' anharmonicity (studied in [1]), corresponds to the uniqueness of the Euclidean Gibbs states. However, for systems with unbounded oscillations (and hence described by unbounded operators), as in the case considered in this work, it is impossible to recover the uniqueness of the states from the convergence of a cluster expansion. An alternative approach consists in establishing correlation inequalities, as it has been used in solving various problems of classical statistical physics [25], [27], [35] – [41], [47], [61], [74], [83]. To apply such inequalities to the Euclidean Gibbs states one should approximate them by classical (i.e. non-quantum) Gibbs measures. In the Euclidean quantum field theory this is known as the lattice approximation technique [75], [76]. As it has been mentioned above, an essential role in the theory of equilibrium properties of the models considered is played by unbounded operators. Starting from the early seventies great efforts to generalize the traditional algebraic schemes of the construction of states on C^*-algebras to the algebras of unbounded operators have been done [67], [71], [72]. The *status quo* in this domain, as well as an extensive bibliography, may be found in [49], [50]. It should be stressed here that within such an algebraic approach only the states for *finite* families of particles of the type considered in this work have been constructed. Thus the Euclidean approach remains so far the only method which allows one to construct the Gibbs states for the infinite systems of quantum particles described by unbounded operators.

We consider the following quantum lattice system. To each point of the lattice $\mathbb{L} = \mathbb{Z}^d$, $d \in \mathbb{N}$, there is attached a quantum particle (oscillator) with the reduced mass $\mathfrak{m} = \mathfrak{m}_{\rm ph}/\hbar^2$ ($\mathfrak{m}_{\rm ph}$ being the physical mass), which has an unstable equilibrium position at this point. Such particles perform D-dimensional oscillations around their equilibrium positions and interact among themselves via an attractive potential. Similar objects have been studied for many years as quite realistic models of crystalline substance undergoing structural phase transitions – one of the most spectacular phenomena of contemporary statistical physics (see [29], [30], [70], [80]). They also are used as parts of the models which describe strong electron-electron correlations caused by the interaction of electrons with oscillating ions [40], [81], [82]. In the case considered, the phase transition is connected with the appearance of macroscopic displacements of particles (a long–range order), which break the $O(D)$-symmetry possessed by the model, when the dimensions d, D, the mass $\mathfrak{m}$, the temperature β^{-1}, and the parameters of the potential energy satisfy certain conditions. These phenomena were studied mathematically in various papers, see e.g. [18], [52], [66], [85], [86]. The essential problem in this context is to understand how does a quantum model become more and more classical, i.e. how (and whether) do the quantum Gibbs states converge to the corresponding classical Gibbs states. On the other hand, of the same importance is to understand the role of quantum effects in phase transitions in such models. As was justified on the physical level [70] and observed experimentally (see [84] and Chapter 2.5.4.3 of the book [29]), quantum effects may suppress the long-range ordering. For the one-dimensional oscillations (i.e. for $D = 1$), this was proved in [86]. Later on it was shown in [5], [6] ($D = 1$), and [53], [54], [55] ($D \in \mathbb{N}$) that not only the long-range order but also any critical anomaly of the displacements of particles from the equilibrium positions are suppressed at all temperatures if the model is 'strongly quantum', which may occur in particular if the mass $\mathfrak{m}$ is small enough.

Another important problem of the mathematical theory of models which exhibit such phenomena is the uniqueness of their Gibbs states. Such uniqueness would imply the absence of all critical anomalies and all the more of the long– range ordering. Therefore, one may expect the uniqueness of Gibbs states at all values of the temperature for 'strongly quantum' models. First the uniqueness of the Euclidean Gibbs states for the model considered in this work (for $D = 1$) was proved to occur under conditions which were irrelevant to the 'quantumness' of the model (e.g. for high temperatures). This was done in [11] – [14] by means of logarithmic Sobolev inequalities. Then in [8] the mentioned uniqueness was proved to hold for $D = 1$ and for every fixed inverse temperature β if the mass $\mathfrak{m}$ is less than some threshold value $\mathfrak{m}_*$ (depending on β).

The present paper is organized as follows. In Section 2 we describe the models which will be considered throughout the article. Necessary facts from the theory of local Gibbs states of such models are also presented there. Thereafter, we introduce a Gaussian measure on an infinite-dimensional Hilbert space. This measure plays a key role in our approach. Then its properties, which we use in the sequel, are described in details. By means of this measure we define local Euclidean Gibbs measures corresponding to different boundary conditions. The Green functions constructed by bounded multiplication operators for the periodic and zero boundary conditions are written as moments of the Euclidean Gibbs measures. Moreover, by means of such measures, we introduce the Green functions corresponding to nonzero boundary conditions. Then we give the definition of the Euclidean Gibbs state for the whole system as a solution of the Dobrushin-Lanford-Ruelle equation. In Section 3, the results of which were announced in [7], we show that such states converge, when $\mathfrak{m} \to +\infty$, to states isomorphic to the Gibbs states of the corresponding classical models. Section 4 is based on [53] – [56]. It is dedicated to the extension of the Green functions (and hence of the local Gibbs states) to a certain class of unbounded operators, which includes the order parameter and fluctuation operators describing the long-range ordering and the critical points of the models considered. In Section 5 we prove that the local Euclidean Gibbs measures may be approximated by finite-dimensional measures corresponding to general ferromagnets. This allows us to prove analogs of known correlation inequalities for the moments of the local Euclidean Gibbs states (Section 6). In Section 7 we use these inequalities to prove a number of new inequalities, such as scalar domination, zero boundary domination, refined Gaussian upper bound. In Section 8, which is based on [5], [6], [8], [18], [52] – [56], we apply these results to the description of certain physical properties of the models considered. Thus, we prove the existence of the long-range order (Theorem 8.1). By means of the scalar domination inequality we show that the fluctuations of the displacement of particles remain normal, at all temperatures and for all dimensions of the oscillations, if the energy of zero-point oscillations of a given particle exceeds a certain value proportional to the energy of its interaction with the rest of the particles. In particular, this occurs when the smallest distance between the energy levels of the corresponding one-dimensional isolated oscillator is large enough or its mass is small enough (Theorem 8.3). Under a similar condition we prove that the Euclidean Gibbs state of the whole system is unique (Theorem 8.4). To this end we use the zero boundary domination inequality. General infinite dimensional methods we use in this article may be found in [22], [58].

Now let us mention which new results are contained in the present article. In Section 2 we give a complete description of the properties of the basic Gaussian measure (Lemmas 2.2 – 2.4). In Section 3 we give a complete proof of Theorems 3.2, 3.3 - in [7] these theorems were only announced. In Section 4 we prove that the Green functions, constructed in the Euclidean region by certain unbounded operators, may be analytically continued to the same domain as the functions corresponding to bounded operators, although the former functions cannot be bounded uniformly in this domain (Theorem 4.1). Here we also prove that the Green functions corresponding to nonempty boundary conditions, and constructed by certain unbounded multiplication operators, are continuous in the Euclidean domain (Theorem 4.2). The lattice approximation technique was known in the context of quantum fields at least since the seventies [75]. Section 5 gives a version of this technique with a complete proof adapted to the models we consider. The proof of Theorem 7.4 is also new. A similar statement was proved in [6] but by means of a much more complicated technique. Theorem 8.2, proved in Section 8, is a generalization of a similar statement proved in [53]. Finally, the uniqueness of Euclidean Gibbs states (Theorem 8.4) here is proved for more general models than it was done in [8].

2. Euclidean Formalism for Quantum Gibbs States

2.1. **Local Gibbs States.** As it was mentioned above, we consider a countable system of interacting quantum particles with the reduced mass $\mathfrak{m}$, performing D-dimensional oscillations around their equilibrium positions which form a lattice $\mathbb{L} = \mathbb{Z}^d$. The oscillations of the particle having its equilibrium position at $l \in \mathbb{L}$ are described by the momentum and displacement operators $\{p_l, q_l\}$ obeying the canonical commutation relations and densely defined on the complex Hilbert space $\mathcal{H}_l = L^2(\mathbb{R}^D_l)$. The whole system is described by the formal Hamiltonian

$$H = \frac{1}{2}\sum_{l,l' \in \mathbb{L}} d_{ll'}(q_l, q_{l'}) + \sum_{l \in \mathbb{L}} H_l, \tag{2.1}$$

$$H_l = \frac{1}{2\mathfrak{m}}(p_l, p_l) + \frac{1}{2}(q_l, q_l) + V(q_l), \tag{2.2}$$

where $(\, . \, , \, . \,)$ stands for the scalar product in $\mathbb{R}^D$ and $d_{ll'}$ form a dynamical matrix. The one-particle potential V is supposed to be $O(D)$-invariant, i.e.,

$$V(x) = v((x,x)). \tag{2.3}$$

Generally, regarding the function v we will assume that it is continuous on $\mathbb{R}_+ \stackrel{\rm def}{=} [0, +\infty)$ and obeys the following condition

$$v(\xi) \geq a\xi + b, \qquad \forall \xi \in \mathbb{R}_+, \tag{2.4}$$

with certain positive a and $b \in \mathbb{R}$. Sometimes we will impose more restrictive conditions:

- **(V1)** v is a polynomial of order $r \geq 2$, convex on $\mathbb{R}_+$;
- **(V2)** v has the form

$$v(\xi) = \frac{1}{2}a\xi + \sum_{s=2}^{r} b_s \xi^s, \quad r \geq 2, \ a \in \mathbb{R}, \ b_s \geq 0, \ b_r > 0. \tag{2.5}$$

Clearly, a function v which obeys **(V2)**, also obeys **(V1)**. For $p \in \mathbb{Z}$, let

$$\mathcal{S}_p = \left\{ x = (x_l)_{l \in \mathbb{L}} \ \Big| \ \sum_{l \in \mathbb{L}} (1 + |l|)^{2p} x_l^2 < \infty \right\},$$

where $|l|$ is the Euclidean distance on $\mathbb{L} = \mathbb{Z}^d \subset \mathbb{R}^d$. Let also

$$\mathcal{S} \stackrel{\rm def}{=} \bigcap_{p\in\mathbb{N}} \mathcal{S}_p, \qquad \mathcal{S}' \stackrel{\rm def}{=} \bigcup_{p\in\mathbb{N}} \mathcal{S}_{-p}. \tag{2.6}$$

These sets, equipped with the projective-limit ($\mathcal{S}$) and inductive-limit ($\mathcal{S}'$) topologies respectively, constitute a mutually dual, with respect to the Hilbert space $\mathcal{S}_0 = l^2(\mathbb{L})$, pair of Schwartz spaces.

The dynamical matrix $(d_{ll'})_{l,l'\in\mathbb{L}}$ is supposed to possess the following properties:

- **(D1)** $d_{ll'}$ is invariant under translations on $\mathbb{L}$;
- **(D2)** $d_{ll'} \leq 0$ (ferroelectricity), $d_{ll} = 0$;
- **(D3)** for every $l \in \mathbb{L}$, $(d_{ll'})_{l'\in\mathbb{L}}$ belongs to $\mathcal{S}$.

The formal Hamiltonian cannot be defined directly and is "represented" by local Hamiltonians H_Λ – indexed by finite subsets $\Lambda \subset \mathbb{L}$ essentially self-adjoint and lower bounded (due to (2.4)) operators acting in the complex Hilbert space $\mathcal{H}_\Lambda = L^2(\mathbb{R}^{D|\Lambda|})$, ($|\cdot|$ stands for cardinality). In standard situations, also in this article, it is enough to consider Hamiltonians indexed by the boxes

$$\Lambda = \{l = (l_1, \ \ldots, \ l_d) \mid l_j^0 \leq l_j \leq l_j^1, \ j = 1, \ \ldots, d; \ l_j^0 < l_j^1, \ l_j^0, l_j^1 \in \mathbb{Z}\}.$$

For a box Λ, let $\mathcal{P}(\Lambda)$ denote the partition of $\mathbb{L}$ by the boxes which are obtained as translations of Λ. Let also $\mathfrak{T}$ be the group of all translations of $\mathbb{L}$, and $\mathfrak{T}(\Lambda) \subset \mathfrak{T}$ be its subgroup consisting of the translations which generate $\mathcal{P}(\Lambda)$, i.e. $\mathcal{P}(\Lambda) = \{t(\Lambda) \mid t \in \mathfrak{T}(\Lambda)\}$, where $t(\Lambda) = \{t(l) \mid l \in \Lambda\}$. Then the dynamical matrix $(d^\Lambda_{ll'})_{l,l'\in\Lambda}$, obeying periodic conditions on the boundaries of Λ, and the corresponding local Hamiltonian H_Λ are introduced as follows

$$d^\Lambda_{ll'} = \min\{d_{lt(l')} \ : \ t \in \mathfrak{T}(\Lambda)\}, \tag{2.7}$$

$$H_\Lambda = \frac{1}{2}\sum_{l,l'\in\Lambda} d^\Lambda_{ll'}(q_l, q_{l'}) + \sum_{l\in\Lambda} H_l. \tag{2.8}$$

The dynamical matrix $(d^\Lambda_{ll'})_{l,l'\in\Lambda}$ is invariant with respect to the translations on the torus which one obtains by identifying the boundaries of the box Λ. These translations constitute a factor-group $\mathfrak{T}/\mathfrak{T}(\Lambda)$. The local Hamiltonian which corresponds to the zero boundary conditions is

$$H^{(0)}_\Lambda = \frac{1}{2}\sum_{l,l'\in\Lambda} d_{ll'}(q_l, q_{l'}) + \sum_{l\in\Lambda} H_l. \tag{2.9}$$

For a box Λ, a local periodic Gibbs state $\gamma_{\beta,\Lambda}$ at a given value of the temperature $T = \beta^{-1}$ is defined on $\mathfrak{A}_\Lambda$ – the C^*-algebra of all bounded operators on $\mathcal{H}_\Lambda$, as the following positive normalized functional

$$\gamma_{\beta,\Lambda}(A) = \frac{\mathrm{trace}\,(A\exp(-\beta H_\Lambda))}{\mathrm{trace}\exp(-\beta H_\Lambda)}. \tag{2.10}$$

The state $\gamma^{(0)}_{\beta,\Lambda}$ corresponding to the zero boundary conditions is defined in the same way but with the Hamiltonian $H^{(0)}_\Lambda$ (2.9) instead of H_Λ.

Given a box Λ and $t \in \mathbb{R}$, we introduce the following automorphisms of $\mathfrak{A}_\Lambda$

$$\begin{aligned} \mathfrak{a}^\Lambda_t(A) &= \exp(itH_\Lambda)\,A\exp(-itH_\Lambda), \\ \mathfrak{a}^{0,\Lambda}_t(A) &= \exp\left(itH^{(0)}_\Lambda\right)A\exp\left(-itH^{(0)}_\Lambda\right). \end{aligned} \tag{2.11}$$

A significant role in the construction of the Gibbs states on the algebras $\mathfrak{A}_\Lambda$ is played by multiplication operators. Recall that, for a function $A : \mathbb{R}^{D|\Lambda|} \to \mathbb{C}$, the multiplication operator $A \in \mathfrak{A}_\Lambda$ acts on $\Psi \in \mathcal{H}_\Lambda$ as follows

$$(A\Psi)(x) = A(x)\Psi(x).$$

The components $q^{(\alpha)}_l$, $\alpha = 1, 2, \ldots, D$, $l \in \Lambda$ of the displacement operator are multiplication operators, but they do not belong to $\mathfrak{A}_\Lambda$ since they are unbounded. R. Høegh-Krohn in [48] proved the following assertion (for more details see also [1] and [44]).

Proposition 2.1. *Let $t_1, \ldots, t_n \in \mathbb{R}$ and $A_1, \ldots A_n$ be bounded continuous functions $A_j : \mathbb{R}^{D|\Lambda|} \to \mathbb{C}$. Then $\mathfrak{A}_\Lambda$ is the smallest strongly closed linear space containing all operators of the form*

$$\mathfrak{a}^\Lambda_{t_1}(A_1)\mathfrak{a}^\Lambda_{t_2}(A_2)\ldots\mathfrak{a}^\Lambda_{t_n}(A_n).$$

The same remains true if one replaces $\mathfrak{a}^\Lambda_t$ with $\mathfrak{a}^{0,\Lambda}_t$.

For $A_1, \ldots, A_n \in \mathfrak{A}_\Lambda$ and $t_1, \ldots t_n \in \mathbb{R}$, the temporal Green functions corresponding to the periodic and zero boundary conditions are

$$G^{\beta,\Lambda}_{A_1,\ldots,A_n}(t_1, \ldots, t_n) = \gamma_{\beta,\Lambda}\left(\mathfrak{a}^\Lambda_{t_1}(A_1)\ldots\mathfrak{a}^\Lambda_{t_n}(A_n)\right), \tag{2.12}$$

$$G^{0,\beta,\Lambda}_{A_1,\ldots,A_n}(t_1, \ldots, t_n) = \gamma^{(0)}_{\beta,\Lambda}\left(\mathfrak{a}^{0,\Lambda}_{t_1}(A_1)\ldots\mathfrak{a}^{0,\Lambda}_{t_n}(A_n)\right). \tag{2.13}$$

For a domain $\mathcal{O} \subset \mathbb{C}^n$, let $Hol(\mathcal{O})$ stand for the set of all holomorphic in $\mathcal{O}$ complex valued functions. Let also

$$\begin{aligned} \mathcal{D}^\beta_n \stackrel{\mathrm{def}}{=} \{&(t_1, \ldots, t_n) \in \mathbb{C}^n \mid \\ &0 < \Im(t_1) < \Im(t_2) \cdots < \Im(t_n) < \beta\}. \end{aligned} \tag{2.14}$$

By virtue of [1], Sect.3 and [48], Sect.2, we prove the following statement.

Lemma 2.1. *For every* $A_1, \ldots, A_n \in \mathfrak{A}_\Lambda$,

(a) $G^{\beta,\Lambda}_{A_1,\ldots,A_n}$ *may be extended to a holomorphic function on* $\mathcal{D}^\beta_n$;

(b) *this extension (which will also be written as* $G^{\beta,\Lambda}_{A_1,\ldots,A_n}$*) is continuous on the closure* $\overline{\mathcal{D}}^\beta_n$ *of* $\mathcal{D}^\beta_n$*, moreover, for all* $(t_1, \ldots, t_n) \in \overline{\mathcal{D}}^\beta_n$,

$$\left| G^{\beta,\Lambda}_{A_1,\ldots,A_n}(t_1, \ldots, t_n) \right| \leq \|A_1\| \cdot \cdots \cdot \|A_n\|, \tag{2.15}$$

where $\|\cdot\|$ *stands for operator norm;*

(c) *for every* $\xi_1, \ldots, \xi_n \in \mathbb{R}$*, the set*

$$\mathcal{D}^\beta_n(\xi_1, \ldots, \xi_n) \tag{2.16}$$
$$\stackrel{\text{def}}{=} \{(t_1, \ldots, t_n) \in \mathcal{D}^\beta_n \mid \Re(t_j) = \xi_j, \quad j = 1, \ldots, n\},$$

is such that for arbitrary $F, G \in Hol(\mathcal{D}^\beta_n)$*, the equality* $F = G$ *on* $\mathcal{D}^\beta_n(\xi_1, \ldots, \xi_n)$ *implies that these functions are equal on the whole* $\mathcal{D}^\beta_n$.

The Green function $G^{0,\beta,\Lambda}_{A_1,\ldots,A_n}$ *has the same properties.*

Proof. It is known (see [23], p. 57) that the Hamiltonian H_Λ (2.8) has a discrete spectrum consisting of positive eigenvalues E_s, $s \in \mathbb{N}$. The corresponding eigenfunctions Ψ_s constitute an orthonormal base of the space $L^2(\mathbb{R}^{D|\Lambda|})$. We set

$$H_\Lambda \Psi_s = E_s \Psi_s, \quad A_{s,s'} = (A\Psi_s, \Psi_{s'})_{L^2(\mathbb{R}^{D|\Lambda|})}. \tag{2.17}$$

Then

$$\begin{aligned} & G^{\beta,\Lambda}_{A_1,\ldots,A_n}(t_1, \ldots, t_n) \\ & \quad = \frac{1}{Z_{\beta,\Lambda}} \sum_{s_1,\ldots,s_n \in \mathbb{N}} (A_1)_{s_1,s_2} \exp\left[i(t_2 - t_1)E_{s_2}\right] \times \\ & \times \cdots \times (A_{n-1})_{s_{n-1},s_n} \exp\left[i(t_n - t_{n-1})E_{s_n}\right] \times \\ & \times (A_n)_{s_n,s_1} \exp\left[i(t_1 - t_n + i\beta)E_{s_1}\right], \end{aligned} \tag{2.18}$$

where

$$Z_{\beta,\Lambda} = \text{trace}\left\{\exp\left(-\beta H_\Lambda\right)\right\}. \tag{2.19}$$

Each element of the Dirichlet series (2.18) is an entire function of $(t_1, \ldots, t_n)$. Hence its module achieves the maximal on $\overline{\mathcal{D}}^\beta_n$ value on the boundaries of this set, that is at the points $\Im(t_1) = \Im(t_2) = \cdots = \Im(t_k) = 0$ and $\Im(t_{k+1}) = \cdots = \Im(t_n) = \beta$ with k running from 1 to n. For such $(t_1, \ldots, t_n)$, one has

$$\left| \left(\mathfrak{a}^{\Lambda}_{t_1}(A_1) \ldots \mathfrak{a}^{\Lambda}_{t_n}(A_n) \exp\left[-\beta H_\Lambda\right] \Psi_s, \Psi_s \right)_{L^2(\mathbb{R}^{N|\Lambda|})} \right| \tag{2.20}$$
$$\leq \left| (K_{k+1} \ldots K_n K_1 \ldots K_k \Psi_s, \Psi_s)_{L^2(\mathbb{R}^{D|\Lambda|})} \right| \exp\left[-\beta E_s\right],$$

where

$$K_j = \mathfrak{a}^{\Lambda}_{\theta_j}(A_j), \quad \theta_j = \Re(t_j), \quad j = 1, \ldots, n. \tag{2.21}$$

The number k depends on s. Obviously,

$$\left| (K_{k+1} \ldots K_n K_1 \ldots K_k \Psi_s, \Psi_s)_{L^2(\mathbb{R}^{D|\Lambda|})} \right| \leq \|K_{k+1} \ldots K_n K_1 \ldots K_k\|$$
$$\leq \|K_1\| \ldots \|K_n\|,$$

yielding

$$\mathrm{trace}\left\{ \mathfrak{a}^{\Lambda}_{t_1}(A_1) \ldots \mathfrak{a}^{\Lambda}_{t_n}(A_n) \exp\left[-\beta H_\Lambda\right] \right\} \leq \|K_1\| \ldots \|K_n\| Z_{\beta,\Lambda}.$$

Moreover,

$$\|K_j\| = \|A_j\|,$$

since $\mathfrak{a}^{\Lambda}_{\theta}$ is a norm preserving automorphism of $\mathfrak{A}_\Lambda$. Thus, the mentioned Dirichlet series converges uniformly on $\overline{\mathcal{D}}^{\beta}_n$, which proves claims (a) and (b). To prove (c) one observes that $\mathcal{D}^{\beta}_n(\xi_1, \ldots, \xi_n)$ is a generating manifold (see e.g. [73], p. 444), hence it is an inner uniqueness set for the functions from $Hol(\mathcal{D}^{\beta}_n)$. The latter means that every $F \in Hol(\mathcal{D}^{\beta}_n)$, which is zero on this set is identically zero on the whole $\mathcal{D}^{\beta}_n$. □

The restrictions of the functions $G^{\beta,\Lambda}$, $G^{0,\beta,\Lambda}$ to $\mathcal{D}^{\beta}_n(0, \ldots, 0)$, i.e.

$$\Gamma^{\beta,\Lambda}_{A_1,\ldots,A_n}(\tau_1, \ldots \tau_n) = G^{\beta,\Lambda}_{A_1,\ldots,A_n}(i\tau_1, \ldots i\tau_n), \tag{2.22}$$
$$\Gamma^{0,\beta,\Lambda}_{A_1,\ldots,A_n}(\tau_1, \ldots \tau_n) = G^{0,\beta,\Lambda}_{A_1,\ldots,A_n}(i\tau_1, \ldots i\tau_n), \tag{2.23}$$

are called temperature (Matsubara) Green functions. Writing them in the form of the series (2.18) one immediately concludes that they have the following property

$$\Gamma^{\beta,\Lambda}_{A_1,\ldots,A_n}(\tau_1 + \theta, \ldots \tau_n + \theta) = \Gamma^{\beta,\Lambda}_{A_1,\ldots,A_n}(\tau_1, \ldots \tau_n), \tag{2.24}$$
$$\Gamma^{0,\beta,\Lambda}_{A_1,\ldots,A_n}(\tau_1 + \theta, \ldots \tau_n + \theta) = \Gamma^{0,\beta,\Lambda}_{A_1,\ldots,A_n}(\tau_1, \ldots \tau_n),$$

for every $\theta \in \mathcal{I}_\beta \stackrel{\mathrm{def}}{=} [0, \beta]$, where addition is modulo β.

In view of Proposition 2.1, the Green functions defined by (2.12), (2.13) with bounded multiplication operators fully determine the states $\gamma_{\beta,\Lambda}$, $\gamma^{(0)}_{\beta,\Lambda}$. Claim (c) of the latter assertion yields in turn that these states are determined by the Matsubara functions (2.22), (2.23) constructed with such operators.

2.2. **Basic Gaussian Measure.** The essence of the Euclidean approach is that the Matsubara functions may be written as moments of probability measures. We begin the construction of such measures with the introduction of a Gaussian measure, which plays a key role in the sequel. Given β, let $\mathcal{X}_\beta$ stand for the real Hilbert space $L^2(\mathcal{I}_\beta \to \mathbb{R}^D)$ equipped with the scalar product and norm respectively

$$\langle \omega, \omega' \rangle_\beta = \int_{\mathcal{I}_\beta} (\omega(\tau), \omega'(\tau)) d\tau, \qquad \|\omega\|_\beta = \sqrt{\langle \omega, \omega \rangle_\beta}. \tag{2.25}$$

On this space we define the following operator

$$S_\beta = (-\mathfrak{m}\Delta_\beta + 1)^{-1} \mathbf{1}, \tag{2.26}$$

where Δ_β is the Laplacian, $\mathfrak{m}$ is the reduced mass, and $\mathbf{1}$ is the identity operator in $\mathbb{R}^D$. This operator is strictly positive and trace class. Thus it determines on $\mathcal{X}_\beta$ an isotropic (i.e. $O(D)$–invariant) Gaussian measure χ_β having the Laplace transform

$$\int_{\mathcal{X}_\beta} \exp\left\{\langle \varphi, \omega \rangle_\beta\right\} \chi_\beta(d\omega) = \exp\left\{ \frac{1}{2} \langle S_\beta \varphi, \varphi \rangle_\beta \right\}. \tag{2.27}$$

This measure describes a D-dimensional quantum harmonic oscillator with the mass $\mathfrak{m}$. Sometimes to indicate its dependence on the mass we shall write $\chi_\beta^{\mathfrak{m}}$. The integral kernel of the operator (2.26) may be written as follows

$$S_\beta^{\alpha\alpha'}(\tau, \tau') = \tag{2.28}$$
$$= \frac{\delta_{\alpha\alpha'}}{2\sqrt{\mathfrak{m}}} \cdot \frac{\exp\left((\beta - |\tau - \tau'|)/\sqrt{\mathfrak{m}}\right) + \exp(|\tau - \tau'|/\sqrt{\mathfrak{m}})}{\exp(\beta/\sqrt{\mathfrak{m}}) - 1},$$

where $\delta_{\alpha\alpha'}$, $\alpha, \alpha' = 1, \ldots, D$ stands for the Kronecker delta. Employing this kernel one can show that

$$< (\omega(\tau) - \omega(\tau'), \omega(\tau) - \omega(\tau')) >_{\chi_\beta} \leq \frac{D}{\mathfrak{m}} \cdot |\tau - \tau'|_\beta, \tag{2.29}$$

where $|\tau - \tau'|_\beta \stackrel{\mathrm{def}}{=} \min\{|\tau - \tau'|, \beta - |\tau - \tau'|\}$. Here and further on we write

$$< f >_\mu = \int f d\mu. \tag{2.30}$$

Given $\tau, \tau' \in \mathcal{I}_\beta$, we set

$$\xi_1 = \omega(\tau) - \omega(\tau'), \qquad \xi_2 = \omega(\tau), \qquad |\xi_j|^2 = (\xi_j, \xi_j).$$

For the random variables ξ_j, $j = 1, 2$, one can show that

$$< |\xi_j|^{2p} >_{\chi_\beta} = \left[C_p < |\xi_j|^2 >_{\chi_\beta}\right]^p, \quad p \in \mathbb{N}, \tag{2.31}$$

where C_p is a constant depending only on p and D. Thus, one has from (2.29)

$$< |\omega(\tau) - \omega(\tau')|^{2p} >_{\chi_\beta} \leq (C_p D/\mathfrak{m})^p \, |\tau - \tau'|^p_\beta. \tag{2.32}$$

Further, by means of (2.29), (2.31) one gets that

$$\int_{\mathcal{X}_\beta} \exp\left[a(\omega(\tau), \omega(\tau))\right] \chi_\beta(d\omega) < \infty, \quad \forall a < a_*, \tag{2.33}$$

where

$$a_* = \frac{2\sqrt{\mathfrak{m}}}{D} \cdot \frac{e^{\beta/\sqrt{\mathfrak{m}}} - 1}{e^{\beta/\sqrt{\mathfrak{m}}} + 1}. \tag{2.34}$$

We set

$$\mathcal{C}_\beta = \{\omega \in C(\mathcal{I}_\beta) \mid \omega(0) = \omega(\beta)\}, \tag{2.35}$$

and

$$\mathcal{C}^\sigma_\beta = \{\omega \in \mathcal{C}_\beta \mid (\forall \sigma \in (0, 1/2)) \; (\exists K_\sigma(\omega) > 0) \tag{2.36}$$
$$(\forall \tau, \tau' \in \mathcal{I}_\beta) \; |\omega(\tau) - \omega(\tau')| \leq K_\sigma(\omega)|\tau - \tau'|^\sigma_\beta\}.$$

Clearly, $\mathcal{C}_\beta$ is a subspace of the Banach space $C(\mathcal{I}_\beta)$, thus in the topology induced from this space it is also a Banach space. The periodicity of the functions from $\mathcal{C}_\beta$ is related to the property (2.24).

Lemma 2.2. *The measure χ_β is concentrated on $\mathcal{C}^\sigma_\beta$. There exists $a > 0$ such that*

$$\int_{\mathcal{X}_\beta} \exp\{a\|\omega\|^2_{\mathcal{C}_\beta}\} \chi_\beta(d\omega) < \infty. \tag{2.37}$$

Proof. The proof of the first statement follows from the estimate (2.32) and Theorem 5.1 from [76], p. 43. Since the measure χ is concentrated on $\mathcal{C}_\beta \supset \mathcal{C}^\sigma_\beta$, one can apply Fernique's theorem (see e.g. [32], p. 16), which gives (2.37). □

The result just proven allows us to consider χ_β as a measure on the Banach space $\mathcal{C}_\beta$. Recall that a family of probability measures $\mathcal{M}$ on a topological space X is called *tight* in this space if, for any $\varepsilon > 0$, there exists a compact subset $A_\varepsilon \subset X$ such that $\mu(X \setminus A) \leq \varepsilon$ for all $\mu \in \mathcal{M}$. A measure μ is called tight if the family $\{\mu\}$ is so.

Lemma 2.3. *For every $\mathfrak{m}_0 > 0$, the family of measures $\{\chi^{\mathfrak{m}}_\beta \mid \mathfrak{m} \geq \mathfrak{m}_0\}$ is tight in $\mathcal{C}_\beta$.*

The proof of this lemma will be based on a tightness criterium, for which we take Theorem 8.2 from Billingsley's book [24], p. 55 [1]. The modulus of continuity of a $\omega \in \mathcal{C}_\beta$ is set as follows

$$\phi(\omega, \delta) = \sup\{|\omega(\tau) - \omega(\tau')| \mid |\tau - \tau'|_\beta < \delta\}, \quad 0 < \delta \leq \beta/2. \quad (2.38)$$

Proposition 2.2. *The family of measures* $\{\mu_\theta \mid \theta \in \Theta\}$ *is tight in* $\mathcal{C}_\beta$ *if and only if these two conditions hold:*

(i) *For each positive* η*, there exists an* a *such that*

$$\mu_\theta(\{\omega \mid |\omega(0)| > a\}) \leq \eta, \qquad \forall\ \theta \in \Theta. \quad (2.39)$$

(ii) *For each positive* ε *and* η*, there exists a* $\delta \in (0, \beta/2)$ *such that*

$$\mu_\theta(\{\omega \mid \phi(\omega, \delta) \geq \varepsilon\}) \leq \eta, \qquad \forall\ \theta \in \Theta. \quad (2.40)$$

If $\{\mu_\theta \mid \theta \in \Theta\}$ *is a sequence* $\{\mu_M \mid M \in \mathbb{N}\}$*, the above condition is to be satisfied only for* $M > M_0$*, with* M_0 *depending on* ε *and* η *only.*

To employ this criterium we shall use the Chebyshev inequality (see e.g. [24], p. 223)

$$\mu_\theta(\{\omega \mid F(\omega) \geq a\}) \leq \frac{1}{a} \cdot < F >_{\mu_\theta}, \quad (2.41)$$

which holds for any nonnegative and integrable function.

Proof of Lemma 2.3. First we prove that the condition (i) holds. By (2.28) and (2.41) one has

$$\begin{aligned} \chi_\beta^{\mathfrak{m}}(\{\omega \mid |\omega(0)| > a\}) = & \quad (2.42) \\ = \chi_\beta^{\mathfrak{m}}(\{\omega \mid |\omega(0)|^2 > a^2\}) \leq \frac{1}{a^2} \cdot < (\omega(0), \omega(0)) >_{\chi_\beta^{\mathfrak{m}}} & \\ = \sum_{\alpha=1}^{D} S_\beta^{\alpha\alpha}(0,0) = \frac{D}{2a^2\sqrt{\mathfrak{m}}} \cdot \frac{\exp(\beta/\sqrt{\mathfrak{m}}) + 1}{\exp(\beta/\sqrt{\mathfrak{m}}) - 1} & \\ \leq \frac{D}{2a^2\sqrt{\mathfrak{m}_\circ}} \cdot \frac{\exp(\beta/\sqrt{\mathfrak{m}_0}) + 1}{\exp(\beta/\sqrt{\mathfrak{m}_0}) - 1}, \qquad \forall \mathfrak{m} \geq \mathfrak{m}_0. & \end{aligned}$$

To prove (ii) we shall use the estimates obtained in [21] by means of the Garsia-Rodemich-Rumsey lemma. For $\omega \in \mathcal{C}_\beta^\sigma$, one has (see (2.36))

$$\phi(\omega, \delta) \leq K_\sigma(\omega)\delta^\sigma, \qquad \forall \sigma \in (0, 1/2). \quad (2.43)$$

[1] This theorem gives a criterium for sequences, but just after the proof the author remarks how it can be generalized to an arbitrary family of measures.

Given $\sigma \in (0, 1/2)$, let us take $p \in \mathbb{N}$ such that $p > (1-2\sigma)^{-1}$. For this σ and p, one has by 2.41) and (2.43)

$$\chi_\beta^{\mathfrak{m}}(\{\omega \mid \phi(\omega,\delta) \geq \varepsilon\}) = \tag{2.44}$$
$$= \chi_\beta^{\mathfrak{m}}(\{\omega \mid [\phi(\omega,\delta)]^{2p} \geq \varepsilon^{2p}\}) \leq \frac{\delta^{2p\sigma}}{\varepsilon^{2p}} \cdot < [K_\sigma(\omega)]^{2p} >_{\chi_\beta^{\mathfrak{m}}}$$

Taking into account (2.32) and applying the estimate (3b) from [21], p. 203, we get

$$< [K_\sigma(\omega)]^{2p} >_{\chi_\beta^{\mathfrak{m}}} \leq \frac{1}{\mathfrak{m}^p} \cdot \frac{C_{p,\sigma} D^p}{p(1-2\sigma)-1} \beta^{p(1-2\sigma)},$$

with a constant $C_{p,\sigma}$ depending only on D, p, and σ. Employing this estimate in (2.44) one gets (2.40). □

As a strictly positive trace class operator, S_β possesses eigenvectors, the set of which, $\mathcal{E}_\beta$, spans the space $\mathcal{X}_\beta$. This set may be written as follows

$$\mathcal{E}_\beta = \{\epsilon_k \mid k \in \mathcal{K}\}, \qquad \mathcal{K} \stackrel{\text{def}}{=} \{k = \frac{2\pi}{\beta}\kappa \mid \kappa \in \mathbb{Z}\}, \tag{2.45}$$
$$\epsilon_k = (\epsilon_k^\alpha)_{\alpha=1,\dots,D}\,, \qquad \epsilon_k^\alpha(\tau) = e_k(\tau)\iota^\alpha,$$
$$e_k(\tau) = \sqrt{\frac{2}{\beta}} \cos k\tau \quad (k > 0), \quad e_k(\tau) = -\sqrt{\frac{2}{\beta}} \sin k\tau \quad (k < 0),$$
$$e_0(\tau) = 1/\sqrt{\beta},$$

where ι^α, $\alpha = 1, \dots, D$ form the canonical base of $\mathbb{R}^D$. Let P_k^α, $k \in \mathcal{K}$, $\alpha = 1, \dots, D$ stand for the projector from $\mathcal{X}_\beta$ onto the subspace spanned by ϵ_k^α. Then the operator S_β may be written in the canonical form

$$S_\beta = \sum_{\alpha=1}^{D} \sum_{k \in \mathcal{K}} (\mathfrak{m}k^2 + 1)^{-1} P_k^\alpha. \tag{2.46}$$

Below we consider the sequences $\{\chi_{\lambda,M} \mid M \in \mathbb{Z}_+\}$, $\mathbb{Z}_+ \stackrel{\text{def}}{=} \mathbb{N} \cup \{0\}$ of Gaussian measures on $\mathcal{X}_\beta$ having zero means and the covariance operators

$$S_{\lambda,M} = \sum_{\alpha=1}^{D} \sum_{k \in \mathcal{K}} \lambda_k^{(M)} P_k^\alpha, \qquad \lambda_k^{(M)} \geq 0. \tag{2.47}$$

We shall assume that each a sequence $\{\lambda^{(M)} = (\lambda_k^{(M)})_{k \in \mathcal{K}} \mid M \in \mathbb{Z}_+\}$ converges in l^1 to $\lambda = ([\mathfrak{m}k^2 + 1]^{-1})_{k \in \mathcal{K}}$, when $M \to \infty$. Therefore, the sequence of operators $\{S_{\lambda,M} \mid M \in \mathbb{Z}_+\}$ converges to S_β in the trace norm. Given a measure $\chi_{\lambda,M}$ (resp. χ_β), *a finite-dimensional* *approximation* $\chi^{(N)}_{\lambda,M}$ (resp. $\chi^{(N)}_{\beta}$), $N \in 2\mathbb{Z}_+$, i.e., $N = 2L$, $L \in \mathbb{Z}_+$, is the measure which has the covariance operator $S^{(N)}_{\lambda,M}$ (resp. $S^{(N)}_{\beta}$), given as follows

$$S^{(N)}_{\lambda,M} = \sum_{\alpha=1}^{D} \sum_{k\in\mathcal{K}_N} \lambda_k^{(M)} P_k^{\alpha}, \tag{2.48}$$

$$S^{(N)}_{\beta} = \sum_{\alpha=1}^{D} \sum_{k\in\mathcal{K}_N} (\mathfrak{m}k^2+1)^{-1} P_k^{\alpha}.$$

Here

$$\mathcal{K}_N \stackrel{\mathrm{def}}{=} \{k = \frac{2\pi}{\beta}\kappa \mid \kappa = -(L-1), \ldots, L\}. \tag{2.49}$$

Throughout this paper we deal with the weak convergence of measures on metric spaces (see e.g. [24], [65]). For a measure space $(X, \mathfrak{B}(X))$, where X is a real separable metric space and $\mathfrak{B}(X)$ is the Borel σ-algebra of its subsets, let $\mathcal{M}(X)$ be the space of all probability measures defined on X. Let $C_{\mathrm{b}}(X)$ stand for the space of all bounded real-valued continuous functions on X. The *weak topology* on the space $\mathcal{M}(X)$ is defined in such a way that a net of measures $\{\mu_\theta\}$ converges to $\mu \in \mathcal{M}(X)$ (then we write $\mu_\theta \Rightarrow \mu$) in this topology if

$$\int f d\mu_\theta \to \int f d\mu, \quad \forall f \in C_{\mathrm{b}}(X).$$

Regarding the measures on separable Hilbert spaces, Lemma 5.1 of [65], p.182 implies the following

Proposition 2.3. *Let a net of Gaussian measures $\{\chi_\theta\}$ on a separable Hilbert space $\mathcal{H}$ be given. Let also each χ_θ have zero mean and covariance S_θ, which is a positive trace class operator on $\mathcal{H}$. Suppose that the net $\{S_\theta\}$ converges in the trace norm to an operator S. Then there exists a Gaussian symmetric measure on $\mathcal{H}$, such that its covariance operator is S and $\chi_\theta \Rightarrow \chi$ in $\mathcal{H}$.*

Employing this fact we prove the following lemma.

Lemma 2.4. *Let the sequence $\{S_{\lambda,M} \mid M \in \mathbb{Z}_+\}$ converges to S_β in the trace norm. Then the sequence of measures $\{\chi_{\lambda,M} \mid M \in \mathbb{Z}_+\}$ converges weakly in the Banach space $\mathcal{C}_\beta$ to the measure χ_β, i.e., for every $F \in C_{\mathrm{b}}(\mathcal{C}_\beta)$, one has*

$$\int_{\mathcal{C}_\beta} F(\omega)\chi_{\lambda,M}(d\omega) \longrightarrow \int_{\mathcal{C}_\beta} F(\omega)\chi_\beta(d\omega), \qquad M \to \infty.$$

Proof. By Proposition 2.3 the assumed convergence of the sequence $\{S_{\lambda,M}\}$ yields the weak convergence in $\mathcal{X}_\beta$ of the sequences of finite-dimensional approximations $\chi^{(N)}_{\lambda,M}$ to $\chi^{(N)}_\beta$ for every $N \in 2\mathbb{Z}_+$. Since all these measures are concentrated on finite-dimensional subspaces of $\mathcal{C}_\beta \subset \mathcal{X}_\beta$, each a sequence $\{\chi^{(N)}_{\lambda,M} \mid N \in 2\mathbb{Z}_+\}$ converges weakly to $\chi^{(N)}_\beta$ also in $\mathcal{C}_\beta$. If we show that the sequence $\{\chi_{\lambda,M} \mid M \in \mathbb{Z}_+\}$ is tight in $\mathcal{C}_\beta$, the stated convergence will follow from Theorem 8.1 of Billingsley's book [24], p. 54. One observes that

$$< (\omega(0), \omega(0)) >_{\chi_{\lambda,M}} = \sum_{\alpha=1}^{D} S^{\alpha\alpha}_{\lambda,M}(0,0) = \mathrm{trace} S_{\lambda,M}.$$

Since the sequence $\{\mathrm{trace} S_{\lambda,M} \mid M \in \mathbb{Z}_+\}$ is bounded, the condition (i) of Proposition 2.2 is satisfied. Similarly,

$$\begin{aligned} &< (\omega(\tau) - \omega(\tau'), \omega(\tau) - \omega(\tau') >_{\chi_{\lambda,M}} = \\ &= 2 \sum_{\alpha\alpha}^{D} [S^{\alpha\alpha}_{\lambda,M}(0,0) - S^{\alpha\alpha}_{\lambda,M}(\tau,\tau')]. \end{aligned} \tag{2.50}$$

But

$$\begin{aligned} S^{\alpha\alpha}_{\lambda,M}(0,0) - S^{\alpha\alpha}_{\lambda,M}(\tau,\tau') = \\ S^{\alpha\alpha}_{\beta}(0,0) - S^{\alpha\alpha}_{\beta}(\tau,\tau') + \\ +[S^{\alpha\alpha}_{\lambda,M}(0,0) - S^{\alpha\alpha}_{\beta}(0,0)] - \\ -[S^{\alpha\alpha}_{\lambda,M}(\tau,\tau') - S^{\alpha\alpha}_{\beta}(\tau,\tau')] \\ \stackrel{\mathrm{def}}{=} S^{\alpha\alpha}_{\beta}(0,0) - S^{\alpha\alpha}_{\beta}(\tau,\tau') + \\ +I_M(0,0) - I_M(\tau,\tau'). \end{aligned} \tag{2.51}$$

Further,

$$I_M(0,0) = \mathrm{trace}\,[S_{\lambda,M} - S_\beta] \to 0, \quad M \to +\infty, \tag{2.52}$$

$$\begin{aligned} |I_M(\tau,\tau')| &= |\sum_{k\in\mathcal{K}} ([S_{\lambda,M} - S_\beta]\,\epsilon_k, \epsilon_k)\,| \\ &\leq \frac{2D}{\beta} \sum_{k\in\mathcal{K}} |\lambda^{(M)}_k - (\mathfrak{m}k^2 + 1)^{-1}| \to 0, \quad M \to +\infty. \end{aligned} \tag{2.53}$$

Taking into account (2.52), (2.53) in (2.51) and (2.32), one concludes that there exists M_0 such that the estimate

$$< |\omega(\tau) - \omega(\tau')|^{2p} >_{\chi_{\lambda,M}} \leq (C_p D/\mathfrak{m})^p\, |\tau - \tau'|^p_\beta,$$

holds for all $M > M_0$. Now we may proceed as in proving Lemma 2.3, where the estimate (2.32) and the Garsia-Rodemich-Rumsey lemma implied (ii) of Proposition 2.2. Thus the sequence $\{\chi_{\lambda,M} \mid M \in \mathbb{Z}_+\}$ is tight. □

2.3. **Euclidean Gibbs States.** Given β and a box Λ, we write

$$\Omega_{\beta,\Lambda} = \{\omega_\Lambda = (\omega_l)_{l\in\Lambda} \mid \omega_l \in \mathcal{C}_\beta\}, \tag{2.54}$$

and

$$\mathcal{X}_{\beta,\Lambda} = \{\omega_\Lambda = (\omega_l)_{l\in\Lambda} \mid \omega_l \in \mathcal{X}_\beta\}. \tag{2.55}$$

Since Λ is finite, one may equip $\Omega_{\beta,\Lambda}$ and $\mathcal{X}_{\beta,\Lambda}$ with the usual Banach space and Hilbert space structures respectively. Then the space $\Omega_{\beta,\Lambda}$ may be densely embedded into $\mathcal{X}_{\beta,\Lambda}$. Let $\mathfrak{B}(\Omega_{\beta,\Lambda})$ stand for the Borel σ–algebra of the subsets of $\Omega_{\beta,\Lambda}$. Further, set

$$\chi_{\beta,\Lambda}(d\omega_\Lambda) = \bigotimes_{l\in\Lambda} \chi_\beta(d\omega_l). \tag{2.56}$$

The latter measure is concentrated on

$$\Omega^\sigma_{\beta,\Lambda} = \{\omega_\Lambda = (\omega_l)_{l\in\Lambda} \mid \omega_l \in \mathcal{C}^\sigma_\beta\}. \tag{2.57}$$

Set

$$E^V_{\beta,\Lambda}(\omega_\Lambda) = \frac{1}{2}\sum_{l,l'\in\Lambda} d^\Lambda_{ll'}\langle\omega_l,\omega_{l'}\rangle_\beta + \sum_{l\in\Lambda}\int_{\mathcal{I}_\beta} V(\omega_l(\tau))d\tau, \tag{2.58}$$

and

$$E^V_{\beta,\Lambda}(\omega_\Lambda|0) = \frac{1}{2}\sum_{l,l'\in\Lambda} d_{ll'}\langle\omega_l,\omega_{l'}\rangle_\beta + \sum_{l\in\Lambda}\int_{\mathcal{I}_\beta} V(\omega_l(\tau))d\tau. \tag{2.59}$$

Under the assumptions made regarding V and $(d_{ll'})_{l,l'\in\mathbb{L}}$, both $E^V_{\beta,\Lambda}$, $E^V_{\beta,\Lambda}(\cdot|0)$ are continuous functions from $\Omega_{\beta,\Lambda}$ to $\mathbb{R}$.

Thereafter, we may introduce the local Euclidean Gibbs measures corresponding to the periodic and zero boundary conditions. These are respectively the following probability measures on the Hilbert space $\mathcal{X}_{\beta,\Lambda}$, supported on $\Omega_{\beta,\Lambda}$,

$$\mu_{\beta,\Lambda}(d\omega_\Lambda) = \frac{1}{Z_{\beta,\Lambda}}\exp\left\{-E^V_{\beta,\Lambda}(\omega_\Lambda)\right\}\chi_{\beta,\Lambda}(d\omega_\Lambda), \tag{2.60}$$

$$\mu_{\beta,\Lambda}(d\omega_\Lambda|0) = \frac{1}{Z_{\beta,\Lambda}(0)}\exp\left\{-E^V_{\beta,\Lambda}(\omega_\Lambda|0)\right\}\chi_{\beta,\Lambda}(d\omega_\Lambda), \tag{2.61}$$

where $Z_{\beta,\Lambda}$, $Z_{\beta,\Lambda}(0)$ are the normalizing constants.

By means of these measures one can write the Green functions (2.22), (2.23), constructed with the multiplication operators $A_1, \dots A_n \in \mathfrak{A}_\Lambda$, as follows [1], [44]

$$\Gamma^{\beta,\Lambda}_{A_1,\ldots,A_n}(\tau_1,\ldots,\tau_n) = \int_{\mathcal{X}_{\beta,\Lambda}} A_1(\omega_\Lambda(\tau_1))\ldots A_n(\omega_\Lambda(\tau_n))\mu_{\beta,\Lambda}(d\omega_\Lambda), \tag{2.62}$$

$$\Gamma^{0,\beta,\Lambda}_{A_1,\ldots,A_n}(\tau_1,\ldots,\tau_n) = \int_{\mathcal{X}_{\beta,\Lambda}} A_1(\omega_\Lambda(\tau_1))\ldots A_n(\omega_\Lambda(\tau_n))\mu_{\beta,\Lambda}(d\omega_\Lambda|0). \tag{2.63}$$

The Gibbs states of the whole lattice system which correspond to the periodic and zero boundary conditions are obtained as limits of the above states $\gamma_{\beta,\Lambda}$, $\gamma^{(0)}_{\beta,\Lambda}$ when $\Lambda \nearrow I\!\!L$. More precisely, let $\mathcal{L}$ be a sequence of boxes ordered by inclusion and such that $\cup_{\Lambda\in\mathcal{L}}\Lambda = I\!\!L$. For $\Lambda_1 \subset \Lambda_2$, one may introduce a natural norm-preserving embedding $\mathfrak{A}_{\Lambda_1} \subset \mathfrak{A}_{\Lambda_2}$, which defines an increasing sequence of algebras $\{\mathfrak{A}_\Lambda \mid \Lambda \in \mathcal{L}\}$. In a standard way [26], this sequence defines a quasi-local algebra of observables $\mathfrak{A}$. Two sequences $\mathcal{L}$, $\mathcal{L}'$ are set to be equivalent if the corresponding quasi-local algebras coincide. A standard sequence $\mathcal{L}$ is the sequence of boxes $\{\Lambda_L \mid L \in \mathbb{N}\}$, where $\Lambda_L = (-L, L]^d \cap \mathbb{Z}^d$. In the sequel, all (thermodynamic) limits $\Lambda \nearrow I\!\!L$ are taken over a sequence $\mathcal{L}$, which is equivalent to the standard one. The mentioned Gibbs states of the whole lattice system are defined as the thermodynamic limits of the local Gibbs states $\gamma_{\beta,\Lambda}$, $\gamma^{(0)}_{\beta,\Lambda}$. The existence of periodic Gibbs states for similar models was shown in [20] (see also [62] – [64]).

As it was mentioned above, the great advantage of the Euclidean approach lies in the fact that due to the above relationship between the Green functions and local Gibbs states one may apply to the quantum case the machinery of conditional probability distributions, which form the base of modern classical equilibrium statistical physics (see e.g. [33], [34], [42] and the references therein). To this end, along with the Gibbs measures (2.60), (2.61), which correspond to the periodic and zero boundary conditions respectively, we introduce conditional local Gibbs measures. They will describe the Gibbs states of the particles contained in the box Λ and interacting between themselves and with fixed configurations of particles outside Λ. Such configurations determine conditions for the measures we are going to introduce.

Since the complements of boxes Λ, in which we shall fix configurations, are infinite subsets of the lattice $I\!\!L$, we employ the spaces $\Omega_{\beta,\Lambda}$, introduced (2.54), (2.55) also for infinite subsets Λ, in particular, we shall use Ω_β standing for $\Omega_{\beta,I\!\!L}$. We equip such spaces with the product topology and with the σ-algebra $\mathfrak{B}(\Omega_{\beta,\Lambda})$ generated by the cylinder subsets

$$\{\omega_\Lambda = (\omega_l)_{l\in\Lambda} \mid (\omega_l)_{l\in\Delta} \in B_\Delta\}, \quad B_\Delta = \times_{l\in\Delta} B_l,$$

with finite $\Delta \subset \mathbb{L}$ and Borel subsets $\{B_l \subset \mathcal{C}_\beta \mid l \in \Delta\}$. For $\Delta \subset \Lambda \subset \mathbb{L}$, we write $\omega_\Delta \times \zeta_{\Lambda\setminus\Delta}$ for the configuration $(\xi_l)_{l\in\Lambda}$ such that $\xi_l = \omega_l$ for $l \in \Delta$, and $\xi_l = \zeta_l$ for $l \in \Lambda \setminus \Delta$. Given a sequence of boxes $\mathcal{L}$, in order to have the collections of all the spaces $\{\Omega_{\beta,\Lambda}, \Lambda \in \mathcal{L}\}$ ordered by inclusion, we introduce the following mappings. For $\Delta \subset \Lambda$, we set $\omega_\Delta \mapsto \omega_\Delta \times 0_{\Lambda\setminus\Delta} \in \Omega_{\beta,\Lambda}$, where 0_Λ is the zero configuration in $\Omega_{\beta,\Lambda}$. Hence one may consider every configuration ω_Δ as an element of all $\Omega_{\beta,\Lambda}$ with $\Delta \subset \Lambda$. Besides, we define

$$\Omega_{\beta,\Lambda} \ni \omega_\Lambda \mapsto (\omega_\Lambda)_{\Lambda'} \in \Omega_{\beta,\Lambda'},$$

as a configuration such that $\omega_l = 0$ for $l \in \Lambda' \setminus \Lambda$. Obviously, $(\omega_\Lambda)_{\Lambda'} = 0_{\Lambda'}$ if $\Lambda \cap \Lambda' = \emptyset$. Let

$$\Omega^t_\beta \stackrel{\text{def}}{=} \{\zeta \in \Omega_\beta \mid (\|\zeta_l\|_\beta)_{l\in\mathbb{L}} \in \mathcal{S}'\}. \tag{2.64}$$

Given $\zeta \in \Omega_\beta$ and a box Λ, we put

$$\mu_{\beta,\Lambda}(B|\zeta) = 0, \quad \zeta \in \Omega_\beta \setminus \Omega^t_\beta, \quad B \in \mathfrak{B}(\Omega_{\beta,\Lambda}), \tag{2.65}$$

and for $\zeta \in \Omega^t_\beta$,

$$\mu_{\beta,\Lambda}(d\omega_\Lambda|\zeta) = \frac{1}{Z_{\beta,\Lambda}(\zeta)} \exp\left\{-E^V_{\beta,\Lambda}(\omega_\Lambda|\zeta)\right\} \chi_{\beta,\Lambda}(d\omega_\Lambda), \tag{2.66}$$

where

$$Z_{\beta,\Lambda}(\zeta) \stackrel{\text{def}}{=} \int_{\Omega_{\beta,\Lambda}} \exp\left\{-E^V_{\beta,\Lambda}(\omega_\Lambda|\zeta)\right\} \chi_{\beta,\Lambda}(d\omega_\Lambda),$$

is the local partition function subject to the external boundary condition ζ_{Λ^c}, $\Lambda^c = \mathbb{L} \setminus \Lambda$, and

$$E^V_{\beta,\Lambda}(\omega_\Lambda|\zeta) = E_{\beta,\Lambda}(\omega_\Lambda|\zeta) + \sum_{l\in\Lambda} \int_{\mathcal{I}_\beta} V(\omega_l(\tau))d\tau, \tag{2.67}$$

$$E_{\beta,\Lambda}(\omega_\Lambda|\zeta) = \frac{1}{2}\sum_{l,l'\in\Lambda} d_{ll'}\langle\omega_l, \omega_{l'}\rangle_\beta + \sum_{l\in\Lambda, l'\in\Lambda^c} d_{ll'}\langle\omega_l, \zeta_{l'}\rangle_\beta. \tag{2.68}$$

Here V is the same as in (2.2). Under the assumptions made regarding V and $d_{ll'}$, both $E_{\beta,\Lambda}(\cdot|\zeta)$, $E^V_{\beta,\Lambda}(\cdot|\zeta)$ are continuous functions from $\Omega_{\beta,\Lambda}$ to $\mathbb{R}$ for all $\zeta \in \Omega^t_\beta$. The function $E_{\beta,\Lambda}(\cdot|\zeta)$ describes the interaction between the particles in Λ and with the fixed configuration ζ_{Λ^c}. Clearly, for $\zeta \in \Omega^t_\beta$, $\mu_{\beta,\Lambda}(\cdot|\zeta)$ is a probability measure. For $\zeta = 0$, it coincides with the measure (2.61).

Thus, along with the Green functions (2.62), (2.63) we introduce the temperature Green function which corresponds to the external boundary condition ζ_{Λ^c}

$$\Gamma^{\zeta,\beta,\Lambda}_{A_1,\ldots,A_n}(\tau_1,\ldots,\tau_n) = \int_{\mathcal{X}_{\beta,\Lambda}} A_1(\omega_\Lambda(\tau_1))\ldots A_n(\omega_\Lambda(\tau_n))\mu_{\beta,\Lambda}(d\omega_\Lambda|\zeta). \quad (2.69)$$

Here $A_1,\ldots,A_n$ are multiplication operators such that for every $\tau_1,\ldots,\tau_n \in \mathcal{I}_\beta$, the function

$$\Omega_{\beta,\Lambda} \ni \omega_\Lambda \mapsto A_1(\omega_\Lambda(\tau_1))\ldots A_n(\omega_\Lambda(\tau_n)),$$

is $\mu_{\beta,\Lambda}(\cdot|\zeta)$ integrable for every $\zeta \in \Omega_\beta$, which obviously holds for $A_1,\ldots,A_n \in \mathfrak{A}_\Lambda$. Note that the above temperature Green function is defined only for multiplication operators, there is no *a priori* information regarding its analytic and continuity properties (except for $\zeta = 0$), even in the case of bounded operators.

For $B \in \mathfrak{B}(\Omega_\beta)$ and $\omega \in \Omega_\beta$, let $\delta_B(\omega)$ take values 1, resp. 0, if ω belongs, resp. does not belong, to B. Then for a finite $\Lambda \subset \mathbb{L}$, $\zeta \in \Omega_\beta$, $B \in \mathfrak{B}(\Omega_\beta)$, we set

$$\pi_{\beta,\Lambda}(B|\zeta) \stackrel{\text{def}}{=} \int_{\Omega_{\beta,\Lambda}} \delta_B(\omega_\Lambda \times \zeta_{\Lambda^c})\mu_{\beta,\Lambda}(d\omega_\Lambda|\zeta). \quad (2.70)$$

These probability kernels satisfy the consistency conditions

$$\int_{\Omega_\beta} \pi_{\beta,\Lambda}(d\omega|\zeta)\pi_{\beta,\Delta}(B|\omega) = \pi_{\beta,\Lambda}(B|\zeta), \quad (2.71)$$

which holds for arbitrary pairs of finite subsets $\Delta \subset \Lambda \subset \mathbb{L}$, and any $B \in \mathcal{B}(\Omega_\beta)$, $\zeta \in \Omega_\beta$ (for the meaning of such consistency conditions see e.g. [42]).

Definition 2.1. *A probability measure μ on the measure space $(\Omega_\beta, \mathcal{B}(\Omega_\beta))$ is said to be a Euclidean Gibbs state of the model considered at the inverse temperature β if it satisfies the Dobrushin-Lanford-Ruelle (DLR) equilibrium equation*

$$\int_{\Omega_\beta} \mu(d\omega)\pi_{\beta,\Lambda}(B|\omega) = \mu(B), \quad (2.72)$$

for all finite $\Lambda \subset \mathbb{L}$ and $B \in \mathfrak{B}(\Omega_\beta)$.

In order to exclude the states with no physical relevance we impose some *a priori* conditions restricting the growth of the sequences of moments (see [11], [46]).

Definition 2.2. *The class $\mathcal{G}_\beta$ of tempered Gibbs measures consists of the Gibbs states μ defined above, the moments of which obey the condition*

$$(< \|\omega_l\|_\beta >_\mu)_{l \in \mathbb{L}} \in \mathcal{S}'.$$

3. Classical Limits

In this section $\mathcal{L}$, $\mathcal{L}_{\rm fin}$ will stand for the set of all, respectively of all finite, subsets of $\mathbb{L}$. Given $\Lambda \in \mathcal{L}$, let us consider the subset of $\Omega_{\beta,\Lambda}$ consisting of constant trajectories, that is

$$\Omega^{\rm qc}_{\beta,\Lambda} \stackrel{\rm def}{=} \{\omega_\Lambda \in \Omega_{\beta,\Lambda} \mid (\forall l \in \Lambda)\ (\exists x_l \in \mathbb{R}^D) \\ (\forall \tau \in \mathcal{I}_\beta)\ \omega_l(\tau) = x_l\} \simeq \left(\mathbb{R}^D\right)^\Lambda. \tag{3.1}$$

We also set

$$\Omega_\beta \supset \Omega^{\rm qc}_{\beta,\mathbb{L}} \stackrel{\rm def}{=} \Omega^{\rm qc}_\beta \simeq \left(\mathbb{R}^D\right)^{\mathbb{L}}. \tag{3.2}$$

For $\Lambda \in \mathcal{L}$, let $\mathfrak{B}^{\rm qc}_{\beta,\Lambda}$ be the σ-algebra generated by the cylinder subsets of $\Omega^{\rm qc}_{\beta,\Lambda}$, which is isomorphic to the corresponding σ-algebra $\mathfrak{B}\left(\left(\mathbb{R}^D\right)^\Lambda\right)$ generated by the cylinder subsets of $\left(\mathbb{R}^D\right)^\Lambda$ but, on the other hand, is a subalgebra of $\mathfrak{B}_{\beta,\Lambda} \stackrel{\rm def}{=} \mathfrak{B}\left(\Omega_{\beta,\Lambda}\right)$. For every $B \in \mathfrak{B}_{\beta,\Lambda}$, let

$$\mathrm{C}(B) \stackrel{\rm def}{=} B \cap \Omega^{\rm qc}_{\beta,\Lambda}. \tag{3.3}$$

We write

$$\mathfrak{B}^{\rm qc}_{\beta,\Lambda} \ni C \simeq A \in \mathfrak{B}\left(\left(\mathbb{R}^D\right)^\Lambda\right), \tag{3.4}$$

for the pair of subsets $C \in \mathfrak{B}^{\rm qc}_{\beta,\Lambda}$, $A \in \mathfrak{B}\left(\left(\mathbb{R}^D\right)^\Lambda\right)$ which are isomorphic in the above sense. This means that they consist of exactly those ω_Λ and x_Λ, for which $\omega_l(\tau) = x_l$ for all $\tau \in \mathcal{I}_\beta$ and $l \in \Lambda$.

Consider the following Gaussian measure

$$\varpi_{\beta,\Lambda}(dx_\Lambda) \stackrel{\rm def}{=} \prod_{l \in \Lambda} \varpi_\beta(dx_l), \quad x_\Lambda \in \left(\mathbb{R}^D\right)^\Lambda, \quad \Lambda \in \mathcal{L}_{\rm fin}, \tag{3.5}$$

$$\varpi_\beta(dx_l) \stackrel{\rm def}{=} \left(\frac{\beta}{2\pi}\right)^{D/2} \exp\left\{-\frac{\beta}{2}(x_l, x_l)\right\} dx_l, \quad x_l \in \mathbb{R}^D. \tag{3.6}$$

For $\Lambda \in \mathcal{L}_{\rm fin}$, let $\chi^{\rm qc}_{\beta,\Lambda}$ be the Gaussian measure on $\Omega_{\beta,\Lambda}$ such that for every $B \in \mathfrak{B}_{\beta,\Lambda}$ one has

$$\chi^{\rm qc}_{\beta,\Lambda}(B) = \varpi_{\beta,\Lambda}(A), \tag{3.7}$$

where $A \simeq \mathrm{C}(B)$, which is defined by (3.3), (3.4). This means that

$$\chi^{\rm qc}_{\beta,\Lambda}(B) = \chi^{\rm qc}_{\beta,\Lambda}(\mathrm{C}(B)), \tag{3.8}$$

i.e. $\chi^{\rm qc}_{\beta,\Lambda}$ is supported on $\mathfrak{B}^{\rm qc}_{\beta,\Lambda}$. Making use of these measures we construct the periodic and conditional Gibbs measures following the scheme (2.60), (2.58) and (2.66) – (2.67). Thus we set

$$\mu^{\rm qc}_{\beta,\Lambda}(d\omega_\Lambda) \stackrel{\rm def}{=} \frac{1}{Z^{\rm qc}_{\beta,\Lambda}} \exp\left\{-E^V_{\beta,\Lambda}(\omega_\Lambda)\right\} \chi^{\rm qc}_{\beta,\Lambda}(d\omega_\Lambda), \tag{3.9}$$

and for $\zeta \in \Omega^t_\beta$,

$$\mu^{\rm qc}_{\beta,\Lambda}(d\omega_\Lambda|\zeta) \stackrel{\rm def}{=} \frac{1}{Z^{\rm qc}_{\beta,\Lambda}(\zeta)} \exp\left\{-E^V_{\beta,\Lambda}(\omega_\Lambda|\zeta)\right\} \chi^{\rm qc}_{\beta,\Lambda}(d\omega_\Lambda), \tag{3.10}$$

where $E^V_{\beta,\Lambda}(\cdot)$ and $E^V_{\beta,\Lambda}(\cdot|\zeta)$ are given by (2.58) and (2.67) respectively. Here, as above,

$$Z^{\rm qc}_{\beta,\Lambda} \stackrel{\rm def}{=} \int_{\Omega_{\beta,\Lambda}} \exp\left\{-E^V_{\beta,\Lambda}(\omega_\Lambda)\right\} \chi^{\rm qc}_{\beta,\Lambda}(d\omega_\Lambda), \tag{3.11}$$

and

$$Z^{\rm qc}_{\beta,\Lambda}(\zeta) \stackrel{\rm def}{=} \int_{\Omega_{\beta,\Lambda}} \exp\left\{-E^V_{\beta,\Lambda}(\omega_\Lambda|\zeta)\right\} \chi^{\rm qc}_{\beta,\Lambda}(d\omega_\Lambda), \tag{3.12}$$

are the normalizing constants. We remark that the measures (3.9), (3.10) are defined on the same space as $\mu_{\beta,\Lambda}(\cdot)$ and $\mu_{\beta,\Lambda}(\cdot|\zeta)$ given by (2.60) and (2.66) respectively. Further, (3.8) implies that

$$\mu^{\rm qc}_{\beta,\Lambda}(B) = \mu^{\rm qc}_{\beta,\Lambda}(\mathrm{C}(B));\ \mu^{\rm qc}_{\beta,\Lambda}(B|\zeta) = \mu^{\rm qc}_{\beta,\Lambda}(\mathrm{C}(B)|\zeta),\ \forall \zeta \in \Omega_\beta. \tag{3.13}$$

By means of the conditional Gibbs measures (3.10) we define the family of probability kernels $\{\pi^{\rm qc}_{\beta,\Lambda}(\cdot|\zeta) \mid \Lambda \in \mathcal{L}_{\rm fin}\}$ (setting as above $\pi^{\rm qc}_{\beta,\Lambda}(\cdot|\zeta) = 0$ for $\zeta \in \Omega_\beta \setminus \Omega^t_\beta$), and hence the corresponding Euclidean Gibbs states. The family of such Euclidean tempered Gibbs states will be denoted $\mathcal{G}^{\rm qc}_\beta$. The members of this family will be called *quasiclassical* Gibbs states.

Now let us construct the Gibbs states for the classical model described by the Hamiltonian

$$H^{\rm cl} = \sum_{l \in \mathbb{L}} [(x_l, x_l)/2 + V(x_l)] + \frac{1}{2} \sum_{l,l' \in \mathbb{L}} d_{ll'}(x_l, x_{l'}), \tag{3.14}$$

where V is the same as in (2.2), which means that in this case only the potential energy of the oscillators described by (2.1) – (2.3) is taken into account. Heuristically, this potential energy may be obtained from (2.1) by passing to the limit $\mathfrak{m} \to +\infty$. For $\Lambda \in \mathcal{L}_{\rm fin}$, we set

$$I_\Lambda(x_\Lambda) = \sum_{l\in\Lambda} V(x_l) + \frac{1}{2}\sum_{l,l'\in\Lambda} d^\Lambda_{ll'}(x_l, x_{l'}), \tag{3.15}$$

and

$$I_\Lambda(x_\Lambda|y) = \sum_{l\in\Lambda} V(x_l) + \frac{1}{2}\sum_{l,l'\in\Lambda} d_{ll'}(x_l, x_{l'}) + \sum_{l\in\Lambda, l'\in\Lambda^c} d_{ll'}(x_l, y_{l'}), \tag{3.16}$$

where $y = (y_l)_{l\in\mathbb{L}} \in \mathcal{S}'$ determines the boundary conditions outside Λ and plays here the same role as ζ does in the case of quantum Euclidean Gibbs states. It is not difficult to show that $I_\Lambda(\cdot)$ and $I_\Lambda(\cdot|y)$ are continuous functions on $\left(\mathbb{R}^D\right)^\Lambda$, $\Lambda \in \mathcal{L}_{\rm fin}$. The periodic and conditional Gibbs measures for the classical model are introduced respectively as

$$\rho_{\beta,\Lambda}(dx_\Lambda) = \frac{1}{Y_{\beta,\Lambda}} \exp\left\{-\beta I_\Lambda(x_\Lambda)\right\} \varpi_{\beta,\Lambda}(dx_\Lambda), \tag{3.17}$$

$$\rho_{\beta,\Lambda}(dx_\Lambda|y) = \frac{1}{Y_{\beta,\Lambda}(y)} \exp\left\{-\beta I_\Lambda(x_\Lambda|y)\right\} \varpi_{\beta,\Lambda}(dx_\Lambda), \tag{3.18}$$

where $Y_{\beta,\Lambda}$, $Y_{\beta,\Lambda}(y)$ are the corresponding normalizing constants. As above, $\{\rho_{\beta,\Lambda}(\cdot|y) \mid \Lambda \in \mathcal{L}_{\rm fin}\}$ defines the family of probability kernels, and, thereby, the family of classical Gibbs states. We will denote this family by $\mathcal{G}^{\rm cl}_\beta$.

For $\zeta, \tilde{\zeta} \in \Omega_\beta$, we write $\zeta \sim \tilde{\zeta}$ if for every $l \in \mathbb{L}$,

$$\int_{\mathcal{I}_\beta} \zeta_l(\tau) d\tau = \int_{\mathcal{I}_\beta} \tilde{\zeta}_l(\tau) d\tau. \tag{3.19}$$

For $y \in \left(\mathbb{R}^D\right)^{\mathbb{L}}$, let $\Upsilon_\beta(y)$ be the equivalence class consisting of such ζ that

$$\beta^{-1}\int_{\mathcal{I}_\beta} \zeta_l(\tau) d\tau = y_l, \quad \forall l \in \mathbb{L}. \tag{3.20}$$

We write $y \in \Upsilon_\beta(y)$ assuming that the former y stands for the constant loop $\omega_l(\tau) = y_l$, $l \in \mathbb{L}$ and $\tau \in \mathcal{I}_\beta$.

Since all the quasiclassical kernels $\pi^{\rm qc}_{\beta,\lambda}(\cdot|\zeta)$, as measures on Ω_β, are concentrated on $\Omega^{\rm qc}_\beta$ (see (3.8), (3.10)), every solution of the DLR equation constructed by means of such kernels has the same property .

Proposition 3.1. *For every $\mu \in \mathcal{G}^{\rm qc}_\beta$ and all $B \in \mathfrak{B}_\beta$,*

$$\mu(B) = \mu({\rm C}(B)), \tag{3.21}$$

i.e. every quasiclassical Euclidean Gibbs state is supported on the configurations consisting of constant loops.

Our first theorem in this section establishes the relationship between the families $\mathcal{G}_\beta^{\rm qc}$ and $\mathcal{G}_\beta^{\rm cl}$.

Theorem 3.1. *For every $\mu \in \mathcal{G}_\beta^{\rm qc}$, there exists $\rho \in \mathcal{G}_\beta^{\rm cl}$, such that*

$$\rho(A) = \mu(B) = \mu(\mathrm{C}(B)), \tag{3.22}$$

for all $A \in \mathfrak{B}\left(\left(\mathbb{R}^D\right)^{\mathbb{L}}\right)$ and $B \in \mathfrak{B}_\beta$, where $\mathrm{C}(B) \simeq A$ in the sense (3.2). The mapping $\mu \mapsto \rho$ (3.22) is a bijection.

Proof. By construction, the measure spaces $\left(\Omega_\beta^{\rm qc}, \mathfrak{B}\left(\Omega_\beta^{\rm qc}\right)\right)$ and $\left(\left(\mathbb{R}^D\right)^{\mathbb{L}}, \mathfrak{B}\left(\left(\mathbb{R}^D\right)^{\mathbb{L}}\right)\right)$ are isomorphic. On the other hand, since every equivalence class Υ_β contains exactly one element of $\Omega_\beta^{\rm qc}$, the latter space and the corresponding factor space are isomorphic as well. Also by construction (3.7), (3.10), (3.18), every solution μ of the DLR equation constructed with the help of the quasiclassical kernels defines by (3.22) a measure ρ on $\left(\left(\mathbb{R}^D\right)^{\mathbb{L}}, \mathfrak{B}\left(\left(\mathbb{R}^D\right)^{\mathbb{L}}\right)\right)$, which solves the corresponding DLR equation in this space, and *vice versa*. $\square$

In this section $\chi_\beta^{\mathfrak{m}}$, $\chi_{\beta,\Lambda}^{\mathfrak{m}}$, $\mu_{\beta,\Lambda}^{\mathfrak{m}}(\cdot|\zeta)$ will stand for the measures (2.27), (2.56), (2.66) respectively. In such a way we indicate their dependence on the mass $\mathfrak{m}$. We shall speak about a net of measures $\{\mu_{\beta,\Lambda}^{\mathfrak{m}}\}$ assuming the net $\{\mu_{\beta,\Lambda}^{\mathfrak{m}}(\cdot|\zeta) \mid \mathfrak{m} \geq \mathfrak{m}_0\}$ with a certain positive $\mathfrak{m}_0$.

Theorem 3.2. *Let $\beta > 0$, $\Lambda \in \mathcal{L}_{\rm fin}$, and $y \in \left(\mathbb{R}^D\right)^{\mathbb{L}}$ be chosen. Then for every $\zeta \in \Upsilon_\beta(y)$, the net of measures $\{\mu_{\beta,\Lambda}^{\mathfrak{m}}(\cdot|\zeta)\}$ converges weakly in $\Omega_{\beta,\Lambda}$, when $\mathfrak{m} \to +\infty$, to the measure $\mu_{\beta,\Lambda}^{\rm qc}(\cdot|\zeta) = \mu_{\beta,\Lambda}^{\rm qc}(\cdot|y)$*

Theorem 3.3. *For every $\beta > 0$, $\Lambda \in \mathcal{L}_{\rm fin}$, and $\xi \in \Omega_\beta^{\rm qc}$, the conditional Gibbs measure $\mu_{\beta,\Lambda}^{\rm qc}(\cdot|\xi)$, given by (3.10), is a weak limit in $\Omega_{\beta,\Lambda}$, when $\mathfrak{m} \to +\infty$, of the net of measures $\{\mu_{\beta,\Lambda}^{\mathfrak{m}}(\cdot|\zeta)\}$ with arbitrary $\zeta \in \Upsilon_\beta(\xi)$.*

Remark 3.1. *Similar statements may be proven also for the measures (2.60) and corresponding quasiclassical periodic measures.*

The proof of the two just stated theorems is based upon the following lemmas.

Lemma 3.1. *For every box Λ and any $\beta > 0$, the net of measures $\chi_{\beta,\Lambda}^{\mathfrak{m}}$ converges weakly in the Hilbert space $\mathcal{X}_{\beta,\Lambda}$ to the measure $\chi_{\beta,\Lambda}^{\rm qc}$ given by (3.7).*

Proof. Since, for a box Λ, $\chi^{\mathfrak{m}}_{\beta,\Lambda}$ is a product measure (see (2.56)), it is enough to prove this lemma for a one-point box. By (3.6), (3.7), one has

$$\int_{\mathcal{X}_\beta} \exp\{\langle \varphi, \omega\rangle_\beta\}\chi^{\rm qc}_\beta(d\omega) \tag{3.23}$$
$$= (\beta/2\pi)^{D/2}\int_{\mathbb{R}^D} \exp\left\{\left(x, \int_{\mathcal{I}_\beta}\varphi(\tau)d\tau\right)\right\}\times$$
$$\times \exp\{-\beta(x,x)/2\}dx =$$
$$= \exp\left\{-\frac{1}{2\beta}\int_{\mathcal{X}_\beta}\int_{\mathcal{X}_\beta}(\varphi(\tau), \varphi(\tau'))d\tau d\tau'\right\} =$$
$$= \exp\{-\langle \epsilon_0, \varphi\rangle^2_\beta/2\},$$

where ϵ_k belongs to the base $\mathcal{E}_\beta$ given by (2.45). This implies that the covariance operator $S^{\rm qc}_\beta$ of this measure may be written as follows

$$S^{\rm qc}_\beta = \sum_{\alpha=1}^{D} P^\alpha_0. \tag{3.24}$$

Then by (2.46) one obtains

$$\text{trace}\left(S_\beta - S^{\rm qc}_\beta\right) = \sum_{k\in\mathcal{K}\setminus\{0\}}\frac{D}{\mathfrak{m}k^2+1} \leq \sum_{k\in\mathcal{K}\setminus\{0\}}\frac{D}{\mathfrak{m}k^2} \tag{3.25}$$
$$= \frac{\beta^2 D}{2\pi^2\mathfrak{m}}\sum_{n\in\mathbb{N}} n^{-2} \longrightarrow 0, \qquad \text{when} \quad \mathfrak{m}\to+\infty.$$

Now one may use Proposition 2.3 which yields the convergence to be proven. □

Lemma 3.2. *For every box Λ and any $\beta > 0$, the net of measures $\{\chi^{\mathfrak{m}}_{\beta,\Lambda}\}$ converges weakly in the Banach space $\Omega_{\beta,\Lambda}$ to the measure $\chi^{\rm qc}_{\beta,\Lambda}$. Hence, for arbitrary $F\in C_{\rm b}(\Omega_{\beta,\Lambda})$, one has*

$$\int_{\Omega_{\beta,\Lambda}} F(\omega_\Lambda)\chi^{\mathfrak{m}}_{\beta,\Lambda}(d\omega_\Lambda) \longrightarrow \int_{\Omega_{\beta,\Lambda}} F(\omega_\Lambda)\chi^{\rm qc}_{\beta,\Lambda}(d\omega_\Lambda), \qquad \mathfrak{m}\to+\infty.$$

Proof. Again, it is enough to prove this lemma for a one-element box Λ. By Lemma 2.3 the net $\{\chi^{\mathfrak{m}}_\beta\}$ is tight in the Banach space $\mathcal{C}_\beta$. On the other hand, by the above lemma, each a net of finite-dimensional approximations $\{\chi^{\mathfrak{m},M}_\beta\}$ converges to $\chi^{{\rm qc},M}_\beta$ in $\mathcal{C}_\beta$ since it converges in the Hilbert space $\mathcal{X}_\beta$ (see also the proof of Lemma 2.4). Thus, by mentioned Theorem 8.1 of Billingsley's book [24], p. 54, the same convergence holds for the net $\{\chi^{\mathfrak{m}}_\beta\}$. □

Proof of Theorems 3.2 and 3.3. Let $y \in \left(\mathbb{R}^D\right)^{\mathbb{L}} \setminus \mathcal{S}'$, then every ζ belongs to $\Omega_\beta \setminus \Omega_\beta^t$ since by (3.20)

$$\|\zeta_l\|_\beta \leq \|\zeta_l\|_{C(\mathcal{I}_\beta)} \leq |y_l|, \qquad \forall l \in \mathbb{L}.$$

Thus, every member of the net $\{\mu^{\mathfrak{m}}_{\beta,\Lambda}(\cdot|\zeta)\}$, as well as its limit, are zero measures. For $y \in \mathcal{S}'$, one has $\zeta \in \Omega_\beta^t$, and the members of the net given by (2.66) - (2.67) now may be written as follows

$$\mu^{\mathfrak{m}}_{\beta,\Lambda}(d\omega_\Lambda|\zeta) = F_{\beta,\Lambda}(\omega_\Lambda|\zeta)\chi^{\mathfrak{m}}_{\beta,\Lambda}(d\omega_\Lambda), \tag{3.26}$$

where

$$F_{\beta,\Lambda}(\omega_\Lambda|\zeta) = \frac{1}{Z_{\beta,\Lambda}(\zeta)} \exp\left\{-\sum_{l\in\Lambda, l'\in\Lambda^c} d_{ll'}\langle\omega_l, \zeta_{l'}\rangle_\beta\right\} \times \\ \times \Psi_{\beta,\Lambda}(\omega_\Lambda), \tag{3.27}$$

where

$$\Psi_{\beta,\Lambda}(\omega_\Lambda) = \exp\left\{-\frac{1}{2}\sum_{l,l'\in\Lambda} d_{ll'}\langle\omega_l, \omega_{l'}\rangle_\beta - \right. \\ \left. - \sum_{l\in\Lambda}\int_{\mathcal{I}_\beta} V(\omega_l(\tau))d\tau\right\}. \tag{3.28}$$

Since $\zeta \in \Omega_\beta^t$ and the dynamical matrix satisfies **(D3)**, both $F_{\beta,\Lambda}(\cdot|\zeta)$, $\Psi_{\beta,\Lambda}$ belong to $C_{\rm b}(\Omega_{\beta,\Lambda})$. Moreover, $GF(\cdot|\zeta) \in C_{\rm b}(\Omega_{\beta,\Lambda})$, for all $\zeta \in \Omega_\beta^t$ and any $G \in C_{\rm b}(\Omega_{\beta,\Lambda})$. Thus by Lemma 3.2, one has

$$\int_{\Omega_{\beta,\Lambda}} G(\omega_\Lambda)\mu^{\mathfrak{m}}_{\beta,\Lambda}(d\omega_\Lambda|\zeta) = \\ = \int_{\Omega_{\beta,\Lambda}} G(\omega_\Lambda)F_{\beta,\Lambda}(\omega_\Lambda|\zeta)\chi^{\mathfrak{m}}_{\beta,\Lambda}(d\omega_\Lambda) \\ \longrightarrow \int_{\Omega_{\beta,\Lambda}} G(\omega_\Lambda)F_{\beta,\Lambda}(\omega_\Lambda|\zeta)\chi^{\rm qc}_{\beta,\Lambda}(d\omega_\Lambda), \tag{3.29}$$

when $\mathfrak{m} \to \infty$. But

$$\begin{aligned}
&\int_{\Omega_{\beta,\Lambda}} G(\omega_\Lambda) F_{\beta,\Lambda}(\omega_\Lambda|\zeta)\chi^{\mathrm{qc}}_{\beta,\Lambda}(d\omega_\Lambda) = \\
&\quad = \frac{1}{Z_{\beta,\Lambda}(\zeta)} \int_{\Omega_{\beta,\Lambda}} G(\omega_\Lambda)\Psi_{\beta,\Lambda}(\omega_\Lambda) \times \\
&\times \exp\left\{ - \sum_{l\in\Lambda, l'\in\Lambda^c} d_{ll'} \langle \omega_l, \zeta_{l'}\rangle_\beta \right\} \chi^{\mathrm{qc}}_{\beta,\Lambda}(d\omega_\Lambda) \\
&= \frac{1}{Z_{\beta,\Lambda}(\zeta)} \int_{\mathbb{R}^{D|\Lambda|}} \hat{G}(x_\Lambda)\hat{\Psi}_{\beta,\Lambda}(x_\Lambda) \times \\
&\times \exp\left\{ - \sum_{l\in\Lambda, l'\in\Lambda^c} d_{ll'} \left(x_l, \int_{\mathcal{I}_\beta} \zeta_{l'}(\tau) d\tau \right) \right\} \varpi_{\beta,\Lambda}(dx_\Lambda) \\
&= \frac{1}{Z_{\beta,\Lambda}(\zeta)} \int_{\Omega_{\beta,\Lambda}} G(\omega_\Lambda)\Psi_{\beta,\Lambda}(\omega_\Lambda) \times \\
&\times \exp\left\{ - \sum_{l\in\Lambda, l'\in\Lambda^c} d_{ll'} \langle \omega_l, y_{l'}\rangle_\beta \right\} \chi^{\mathrm{qc}}_{\beta,\Lambda}(d\omega_\Lambda) \\
&= \int_{\Omega_{\beta,\Lambda}} G(\omega_\Lambda)\mu^{\mathrm{qc}}_{\beta,\Lambda}(d\omega_\Lambda|y),
\end{aligned}$$

where

$$\hat{G}(x_\Lambda) = G(\omega_\Lambda), \quad \omega_\Lambda(\tau) = x_\Lambda,$$

and similarly $\hat{\Psi}_{\beta,\Lambda}$. The proof of Theorem 3.3 is straightforward. □

4. Green Functions for Unbounded Operators

The most spectacular phenomenon described by the model considered in this work is the spontaneous $O(D)$-symmetry breaking, which occurs when the fluctuations of displacements of particles become large. Since the displacement operators q_l, $l \in \mathbb{L}$ are unbounded, to study this phenomenon we should extend the local Gibbs states, as well as the corresponding Green functions, to certain classes of unbounded multiplication operators. To this end we will use representations like (2.62), (2.63), which makes possible to replace bounded functions by suitable integrable unbounded functions.

Theorem 4.1. *Let the functions* $A_1, \ldots, A_n : \mathbb{R}^{D|\Lambda|} \to \mathbb{C}$ *be such that for every* $\beta > 0$ *and every* $\tau \in \mathcal{I}_\beta$*, the functions* $\Omega_{\beta,\Lambda} \ni \omega_\Lambda \mapsto A_j(\omega_\Lambda(\tau))$, $j = 1, \ldots n$*, are* $\mu_{\beta,\Lambda}$ *(resp.* $\mu^{(0)}_{\beta,\Lambda}$*) integrable. Then, for the corresponding multiplication operators* $A_1, \ldots, A_n$*, the Green function*

(2.62) (resp. (2.63)) can be analytically continued to the domain $\mathcal{D}_n^\beta$ *defined by (2.14).*

Proof. In view of the statement (c) of Lemma 2.1, it is enough to show that there exists a function $F \in Hol(\mathcal{D}_n^\beta)$, such that its restriction to the set $\mathcal{D}_n^\beta(0,\ldots,0)$ coincides with the function (2.62) (resp. (2.63)). Let us show this in the case of periodic boundary conditions. By (2.62), for any $\delta > 0$, the operators $\hat{A}_j \stackrel{\rm def}{=} A_j \exp(-\delta H_\lambda)$, $j = 1,\ldots,n$ are bounded since

$$\mathrm{trace}\,\{A_j \exp(-\delta H_\lambda)\} = Z_{\delta,\Lambda} \int_{\mathcal{X}_\delta} A_j(\omega_\Lambda(0))\mu_{\delta,\Lambda}(d\omega_\Lambda) < \infty. \quad (4.1)$$

Given $\delta \in (0,\beta)$, we take positive $\delta_1,\ldots,\delta_n$, such that $\delta_1 + \cdots + \delta_n = \delta$. Then, for $0 \le \tau_1 \le \cdots \le \tau_n \le \beta$, one has

$$\Gamma^{\beta,\Lambda}_{A_1,\ldots,A_n}(\tau_1,\ldots,\tau_n) = \frac{Z_{\beta-\delta,\Lambda}}{Z_{\beta,\Lambda}} \Gamma^{\beta-\delta,\Lambda}_{\hat{A}_1,\ldots,\hat{A}_n}(\hat{\tau}_1,\ldots,\hat{\tau}_n), \quad (4.2)$$

where the arguments of the Green function on the right-hand side are

$$\hat{\tau}_1 = \tau_1, \ \hat{\tau}_k = \tau_k - (\delta_1 + \cdots + \delta_{k-1}), \ k = 2,\ldots,n, \quad (4.3)$$

and satisfy the condition

$$0 \le \hat{\tau}_1 \le \cdots \le \hat{\tau}_n \le \beta - \delta.$$

By Lemma 2.1, the function on the right-hand side of (4.2) can be continued to a function holomorphic in $(\hat{t}_1,\ldots,\hat{t}_n) \in \mathcal{D}_n^{\beta-\delta}$. Let $\widehat{\mathcal{D}}^{\beta-\delta}_{\delta_1,\ldots,\delta_n}$ stand for the set of values of $(t_1,\ldots,t_n) \in \mathcal{D}_n^\beta$, such that $t_1 = \hat{t}_1$, $t_k = \hat{t}_k + i(\delta_1 + \cdots + \delta_{k-1})$, $k = 2,\ldots,n$ with $(\hat{t}_1,\ldots,\hat{t}_n) \in \mathcal{D}_n^{\beta-\delta}$. Then the left-hand side of (4.2) can be continued to a function of $(t_1,\ldots,t_n)$ holomorphic in $\widehat{\mathcal{D}}^{\beta-\delta}_{\delta_1,\ldots,\delta_n}$, which is an open subset of $\mathcal{D}_n^\beta$. But

$$\mathcal{D}_n^\beta = \bigcup \widehat{\mathcal{D}}^{\beta-(\delta_1+\cdots+\delta_n)}_{\delta_1,\ldots,\delta_n},$$

where summation is taken over all $\delta_1,\ldots,\delta_n$ running through the interval $(0,\beta)$ and obeying the condition $\delta_1 + \cdots + \delta_n < \beta$. Thus $\Gamma^{\beta,\Lambda}_{A_1,\ldots,A_n}$ can be continued to the whole $\mathcal{D}_n^\beta$. ☐

In contrast to the case of bounded operators (c.f. statement (b) of Lemma 2.1), one cannot expect that, for unbounded operators, the extended Green functions $G^{\beta,\Lambda}_{A_1,\ldots,A_n}$ are uniformly bounded on $\overline{\mathcal{D}}_n^\beta$ and continuous on its boundaries. To get such a continuity we impose additional restrictions on the functions $A_1,\ldots,A_n$.

Definition 4.1. *A continuous function* $A : \mathbb{R}^{D|\Lambda|} \to \mathbb{C}$ *belongs to the family* $\mathfrak{P}_\Lambda^{(D)}$ *if for arbitrary* $a > 0$*, the function*

$$\mathbb{R}^{D|\Lambda|} \ni x_\Lambda \mapsto |A(x_\Lambda)| \exp\left\{ -a \sum_{l\in\Lambda} |x_l|^2 \right\}, \tag{4.4}$$

is bounded on $\mathbb{R}^{D|\Lambda|}$.

Here $|x_l|$ stands for the Euclidean norm of $x_l \in \mathbb{R}^D$. In the case of one-point boxes, i.e. for $|\Lambda| = 1$, we will simply write $\mathfrak{P}^{(D)}$. It is worth noting that under point-wise multiplication $\mathfrak{P}_\Lambda^{(D)}$ is an algebra.

Corollary 4.1. *For arbitrary* $A_1, \ldots, A_n \in \mathfrak{P}_\Lambda^{(D)}$*, the temperature Green functions (2.62), (2.63) may be continued analytically in accordance with Theorem 4.1.*

Indeed, by (2.33), functions from $\mathfrak{P}_\Lambda^{(D)}$ are integrable.

Theorem 4.2. *Given a box* Λ*, let* $A_1, \ldots, A_n$ *belong to* $\mathfrak{P}_\Lambda^{(D)}$*. Then for all* $\zeta \in \Omega_\beta$*, the Green functions (2.62), (2.63), (2.69) are continuous functions of* $(\tau_1, \ldots, \tau_n) \in \mathcal{I}_\beta^n$*.*

Proof. In view of (2.66) one may rewrite (2.69) as follows

$$\begin{aligned} \Gamma^{\zeta,\beta,\Lambda}_{A_1,\ldots,A_n}(\tau_1, \ldots, \tau_n) = \int_{\Omega_{\beta,\Lambda}} & A_1(\omega_\Lambda(\tau_1)) \ldots A_n(\omega_\Lambda(\tau_n)) \\ & \times \Psi_{\beta,\Lambda}(\omega_\Lambda|\zeta)\chi_{\beta,\Lambda}(d\omega_\Lambda), \end{aligned} \tag{4.5}$$

where

$$\Psi_{\beta,\Lambda}(\omega_\Lambda|\zeta) \stackrel{\text{def}}{=} \frac{1}{Z_{\beta,\Lambda}(\zeta)} \exp\left\{ -E^V_{\beta,\Lambda}(\omega_\Lambda|\zeta) \right\}. \tag{4.6}$$

All A_j are continuous, thus all the functions $\Omega_{\beta,\Lambda} \ni \omega_\Lambda \mapsto A_j(\omega_\Lambda(\tau))$ are continuous as well. Set

$$R(\omega_\Lambda) \stackrel{\text{def}}{=} \max_{j=1,\ldots,n} \sup_{\tau_j\in\mathcal{I}_\beta} |A_j(\omega_\Lambda(\tau_j))|. \tag{4.7}$$

By Lemma 2.2 the latter function is $\chi_{\beta,\Lambda}$-integrable since all A_j belong to $\mathfrak{P}_\Lambda^{(D)}$. Hence

$$\phi(d\omega_\Lambda) \stackrel{\text{def}}{=} [R(\omega_\Lambda)]^n \, \Psi_{\beta,\Lambda}(\omega_\Lambda|\zeta)\chi_{\beta,\Lambda}(d\omega_\Lambda),$$

is a measure on $\Omega_{\beta,\Lambda}$. It is tight because, for finite Λ, $\Omega_{\beta,\Lambda}$ is a Polish space. Therefore, for every $\varepsilon > 0$, there exists a compact subset $\Omega^\varepsilon_{\beta,\Lambda} \subset \Omega_{\beta,\Lambda}$ such that

$$\phi(\Omega_{\beta,\Lambda} \setminus \Omega^\varepsilon_{\beta,\Lambda}) < \frac{\varepsilon}{4}. \tag{4.8}$$

For $\delta > 0$, let

$$\Upsilon_\delta \stackrel{\mathrm{def}}{=} \sup \left| \Gamma^{\zeta,\beta,\Lambda}_{A_1,\dots,A_n}(\tau_1,\dots,\tau_n) - \Gamma^{\zeta,\beta,\Lambda}_{A_1,\dots,A_n}({\tau_1}',\dots,{\tau_n}') \right|, \qquad (4.9)$$

where the supremum is taken over the subset of $\mathcal{I}_\beta^{2n}$ defined by the condition

$$\max_{j=1,\dots,n} |\tau_j - {\tau_j}'| < \delta.$$

For such δ and $\omega_\Lambda \in \Omega_{\beta,\Lambda}$, we set

$$W_\delta(\omega_\Lambda) \stackrel{\mathrm{def}}{=} \max_{j=1,\dots,n} \sup_{|\tau_j - {\tau_j}'| < \delta} |A_j(\omega_\Lambda(\tau_j)) - A_j(\omega_\Lambda({\tau_j}'))| . \qquad (4.10)$$

Since all A_j are continuous functions from $\mathbb{R}^{D|\Lambda|}$ to $\mathbb{C}$, in order that $\Omega^\varepsilon_{\beta,\Lambda}$ be compact it is necessary and sufficient that the following conditions be satisfied simultaneously (see [65] p. 213):

$$\text{(i)} \ \lim_{\delta \searrow 0} \sup_{\omega_\Lambda \in \Omega^\varepsilon_{\beta,\Lambda}} W_\delta(\omega_\Lambda) = 0, \qquad (4.11)$$

$$\text{(ii)} \ \sup_{\omega_\Lambda \in \Omega^\varepsilon_{\beta,\Lambda}} R(\omega_\Lambda) < \infty, \qquad (4.12)$$

where R was defined by (4.7). Now let us estimate Υ_δ. From (4.9) and (4.7), (4.10) one obtains

$$\Upsilon_\delta \leq n \int_{\Omega^\varepsilon_{\beta,\Lambda}} W_\delta(\omega_\Lambda) \left[R(\omega_\Lambda)\right]^{n-1} \Psi_{\beta,\Lambda}(\omega_\Lambda|\zeta) \chi_{\beta,\Lambda}(d\omega_\Lambda) + 2\phi(\Omega_{\beta,\Lambda} \setminus \Omega^\varepsilon_{\beta,\Lambda}).$$

In view of (4.11), (4.12) one can choose δ small enough making the first term in the right-hand side of the latter formula less than $\varepsilon/2$. The second one has already been estimated by (4.8). The stated continuity of the Green functions (2.62), (2.63) may be proven just in the same way. □

5. Lattice Approximation

In the following two sections our aim is to prove, for the Euclidean measures (2.60), (2.61), (2.66), correlation inequalities analogous to the inequalities known in the Euclidean quantum field theory (see e.g. [75], [76]). In the subsequent sections we use these basic inequalities to get a number of new correlation inequalities, which in turn are used in studying physical properties of our models. The basic inequalities we are going to prove concern the one-dimensional oscillations, that does not preclude from their application to the vector case which will be given below. Thus, we put in this section and in the next one $D = 1$. Since we will use the mentioned inequalities for the moments not only of the above introduced local Gibbs measures, we prove them in a more general setting.

Given a box Λ and $\xi \in \Omega_{\beta,\Lambda}$, we define the following measure

$$\begin{aligned} \varrho_{\beta,\Lambda}(d\omega_\Lambda|\xi) &= \Phi_{\beta,\Lambda}(\omega_\Lambda|\xi)\chi_{\beta,\Lambda}(d\omega_\Lambda) \\ &\stackrel{\text{def}}{=} \frac{1}{Y_{\beta,\Lambda}(\xi)} \exp\left\{ -\frac{1}{2}\sum_{l,l'\in\Lambda} J_{ll'}\langle\omega_l,\omega_{l'}\rangle_\beta - \sum_{l\in\Lambda}\langle\omega_l,\xi_l\rangle_\beta \right. \\ &\qquad \left. - \sum_{l\in\Lambda}\int_{\mathcal{I}_\beta} W(\omega_l(\tau))d\tau \right\} \chi_{\beta,\Lambda}(d\omega_\Lambda), \end{aligned} \tag{5.1}$$

with certain nonpositive $J_{ll'} = J_{l'l}$, $\xi \in \mathcal{X}_\beta$ and $W(x) = w((x,x))$, w being a polynomial satisfying (2.4). Clearly, every measure (2.60), (2.61), (2.66) may be written in this form.

The Gaussian measure χ_β is determined by its covariance operator S_β given by (2.26). Since $D = 1$, the base $\mathcal{E}_\beta$ (2.45) consists of the eigenfunctions $\{e_k \mid k \in \mathcal{K}\}$. In this case the canonical representation (2.46) may be rewritten as follows

$$S_\beta = \sum_{k\in\mathcal{K}} \frac{1}{\mathfrak{m}k^2+1} P_k. \tag{5.2}$$

Now we choose $N = 2L$, $L \in \mathbb{N}$ and set

$$\lambda_k^{(N)} \stackrel{\text{def}}{=} \frac{1}{\mathfrak{m}\left(\frac{2N}{\beta}\right)^2 \left[\sin\left(\frac{\beta}{2N}\right)k\right]^2 + 1}, \tag{5.3}$$

and

$$S_\beta^{(N)} \stackrel{\text{def}}{=} \sum_{k\in\mathcal{K}_N} \lambda_k^{(N)} P_k, \tag{5.4}$$

It is a technical exercise to prove the following statement.

Proposition 5.1. *The sequence of finite-rank operators $\{S_\beta^{(N)}\}$ converges in the trace norm, when $N \to \infty$, to the operator S_β.*

Let $\chi_\beta^{(N)}$ be the Gaussian measure on $\mathcal{X}_\beta$ having $S_\beta^{(N)}$ as a covariance operator. This measure may also be written in a coordinate representation. To this end we introduce Gaussian measures on $\mathbb{R}$, $\sigma_k^{(N)}$, $k \in \mathcal{K}$, such that

$$\int_{\mathbb{R}} \exp(ixy)\,\sigma_k^{(N)}(dy) = \exp\left\{-\frac{1}{2}\lambda_k^{(N)}x^2\right\}, \tag{5.5}$$

with $\lambda_k^{(N)}$ given by (5.3). Then

$$\chi_\beta^{(N)}(d\omega) = \bigotimes_{k\in\mathcal{K}_N} \sigma_k^{(N)}(d\hat{\omega}_k) \bigotimes_{k\in\mathcal{K}\setminus\mathcal{K}_N} \delta(\hat{\omega}_k)d\hat{\omega}_k, \tag{5.6}$$

where

$$\omega(\tau) = \sum_{k\in\mathcal{K}} \hat{\omega}(k)e_k(\tau), \quad \hat{\omega}(k) = \int_{\mathcal{I}_\beta} \omega(\tau)e_k(\tau)d\tau, \tag{5.7}$$

and δ is the Dirac δ-function on $\mathbb{R}$. Proposition 5.1 and Lemma 2.4 immediately yield

Lemma 5.1. *The sequence of Gaussian measures* $\{\chi_\beta^{(N)}\}$ *converges weakly in the Banach space* $\mathcal{C}_\beta$ *to the measure* χ_β.

Employing the sequence $\{\chi_\beta^{(N)} \mid N \in \mathbb{N}\}$ we will construct by means of (2.60), (2.66) approximations of the measure $\varrho_{\beta,\Lambda}(\cdot|\xi)$ (5.1), and hence of its moments such as the Green functions (5.14). This means that, for integrable functions $F : \Omega_{\beta,\Lambda} \to \mathbb{C}$, the integrals

$$\begin{aligned} < F >_{\varrho_{\beta,\Lambda}(\cdot|\xi)} &\stackrel{\text{def}}{=} \int_{\Omega_{\beta,\Lambda}} F(\omega_\Lambda)\varrho_{\beta,\Lambda}(d\omega_\Lambda|\xi) \\ &= \int_{\Omega_{\beta,\Lambda}} F(\omega_\Lambda)\Phi_{\beta,\Lambda}(\omega_\Lambda|\xi)\chi_{\beta,\Lambda}(d\omega_\Lambda), \end{aligned} \tag{5.8}$$

will be approximated by

$$\begin{aligned} &\int_{\Omega_{\beta,\Lambda}} F(\omega_\Lambda)\Phi_{\beta,\Lambda}(\omega_\Lambda|\xi)\chi_{\beta,\Lambda}^{(N)}(d\omega_\Lambda) \\ &\qquad = \int_{\Omega_{\beta,\Lambda}} F^{(N)}(\omega_\Lambda)\Phi_{\beta,\Lambda}^{(N)}(\omega_\Lambda|\xi)\chi_{\beta,\Lambda}^{(N)}(d\omega_\Lambda), \end{aligned} \tag{5.9}$$

where

$$\chi_{\beta,\Lambda}^{(N)}(d\omega_\Lambda) \stackrel{\text{def}}{=} \bigotimes_{l\in\Lambda} \chi_\beta^{(N)}(d\omega_l), \tag{5.10}$$

and

$$F^{(N)}(\omega_\Lambda) \stackrel{\text{def}}{=} F(\omega_\Lambda^{(N)}), \quad \Phi_{\beta,\Lambda}^{(N)}(\omega_\Lambda|\xi) \stackrel{\text{def}}{=} \Phi_{\beta,\Lambda}(\omega_\Lambda^{(N)}|\xi), \tag{5.11}$$

$$\omega_\Lambda^{(N)} = \left(\omega_l^{(N)}\right)_{l\in\Lambda}, \quad \omega_l^{(N)} \stackrel{\text{def}}{=} \sum_{k\in\mathcal{K}_N} P_k\omega_l. \tag{5.12}$$

The reason to use such approximations is that the integrals on the right-hand side of (5.9) may be rewritten as integrals over finite-dimensional spaces. To the latter integrals one may apply the classical ferromagnetic interpretation, which would yield the correlation inequalities we are going to get. To do this we should make more precise definition of the class of functions F, for which such an interpretation may make sense. First of all we will need the following functions

$$F(\omega_\Lambda) = A_1(\omega_\Lambda(\tau_1)) \dots A_n(\omega_\Lambda(\tau_n)), \tag{5.13}$$

with $A_1, \dots, A_n \in \mathfrak{P}^{(1)}_\Lambda$, which determine the Green functions

$$\begin{aligned} \Gamma^{(\xi)}_{A_1,\dots,A_n}(\tau_1, \dots, \tau_n) &= \int_{\Omega_{\beta,\Lambda}} A_1(\omega_\Lambda(\tau_1)) \dots A_n(\omega_\Lambda(\tau_n)) \varrho_{\beta,\Lambda}(d\omega_\Lambda|\xi) \\ &\stackrel{\text{def}}{=} < A_1(\omega_\Lambda(\tau_1)) \dots A_n(\omega_\Lambda(\tau_n)) >_{\varrho_{\beta,\Lambda}(\cdot|\xi)} . \end{aligned} \tag{5.14}$$

The latter functions are continuous on $\mathcal{I}^n_\beta$ in view of Theorem 4.2. Since the $\omega^{(N)}_\Lambda$ belong to a finite-dimensional subspace of $\mathcal{X}_{\beta,\Lambda}$, they can be written as linear combinations of $\omega_\Lambda((\nu/N)\beta)$, $\nu = 0, \dots, N-1$, which may be chosen as variables for the mentioned finite-dimensional integrals. Therefore, for F given by (5.13), it would be much more convenient to construct such approximations if the arguments $\tau_1, \dots, \tau_n$ belonged to $\mathcal{Q}_\beta \subset \mathcal{I}_\beta$, where $\mathcal{Q}_\beta$ consists of the values of τ, for which τ/β is rational. Then, for given $\tau_1, \dots, \tau_n \in \mathcal{Q}_\beta$, one can find $\nu_1, \dots, \nu_n, N \in \mathbb{N}$, such that $\tau_j = (\nu_j/N)\beta$, $j = 1, \dots, n$. In this case, we obtain ferromagnetic approximations of the Green functions (5.14) only for the arguments belonging to $\mathcal{Q}_\beta$. But in view of the continuity of these functions, this will be enough since $\mathcal{Q}_\beta$ is dense in $\mathcal{I}_\beta$.

Following this way we will deal with such basic types of functions $\Omega_{\beta,\Lambda} \to \mathbb{R}$:

$$\begin{aligned} &\text{(i)} \quad \omega_\Lambda \mapsto \omega_\Lambda(\tau), \quad \tau \in \mathcal{Q}_\beta; \\ &\text{(ii)} \quad \omega_\Lambda \mapsto \langle \omega_l, \omega_{l'} \rangle_\beta, \quad \omega_\Lambda \mapsto \langle \omega_l, \xi_l \rangle_\beta, \quad l, l' \in \Lambda, \quad \xi \in \Omega_{\beta,\Lambda}; \\ &\text{(iii)} \quad \omega_\Lambda \mapsto \int_{\mathcal{I}_\beta} W(\omega_l(\tau)) d\tau. \end{aligned} \tag{5.15}$$

Definition 5.1. *Given $\tau_1, \dots, \tau_n$, $n \in \mathbb{Z}_+$, the family $\mathfrak{F}_{\beta,\Lambda}(\tau_1, \dots, \tau_n)$ consists of the continuous functions $F : \Omega_{\beta,\Lambda} \to \mathbb{C}$ which are compositions of the functions*

$$\omega_\Lambda \mapsto (\omega_\Lambda(\tau_1), \dots, \omega_\Lambda(\tau_n)),$$

with functions $\mathbb{R}^{n|\Lambda|} \to \mathbb{C}$, such that for all $a_1, \dots, a_n > 0$,

$$|F(\omega_\Lambda)| \exp\left\{ -\sum_{l\in\Lambda} \sum_{j=1}^n a_j \left[\omega_l(\tau_j)\right]^2 \right\} < \infty. \tag{5.16}$$

Clearly, the functions F having the form (5.13) with $A_1, \dots, A_n \in \mathfrak{P}^{(1)}_\Lambda$ belong to the family just introduced.

Thereafter, we choose $\tau_1, \ldots, \tau_n \in \mathcal{Q}_\beta$, $n \in \mathbb{Z}_+$, $\xi \in \Omega_{\beta,\Lambda}$ and keep them fixed. Then for $n \geq 1$, there exist tending to infinity sequences $\{N^{(m)} \mid m \in \mathbb{N}\}$, $\{\nu_j^{(m)} \mid m \in \mathbb{N}\}$, $j = 1, \ldots, n$, such that for all $m \in \mathbb{N}$,

$$\tau_j = \frac{\nu_j^{(m)}}{N^{(m)}} \beta, \quad j = 1, \ldots, n. \tag{5.17}$$

Below we drop the superscript (m) assuming that N and ν_j tend to infinity in such a way that (5.17) holds. We also suppose that all N are even. The set of values of such N satisfying (5.17) depends on the choice of $\{\tau_j, j = 1, \ldots, n\}$, we denote it by $\mathcal{N}(\tau_1, \ldots, \tau_n)$.

The basic element of our construction is the following statement.

Theorem 5.1. *For every $F \in \mathfrak{F}_{\beta,\Lambda}(\tau_1, \ldots, \tau_n)$ and all $\xi \in \Omega_{\beta,\Lambda}$, the following convergence, when $\mathcal{N}(\tau_1, \ldots, \tau_n) \ni N \to \infty$,*

$$\begin{aligned} &\int_{\Omega_{\beta,\Lambda}} F^{(N)}(\omega_\Lambda) \Phi^{(N)}(\omega_\Lambda|\xi) \chi_{\beta,\Lambda}^{(N)}(d\omega_\Lambda) \\ &\qquad \longrightarrow \int_{\Omega_{\beta,\Lambda}} F(\omega_\Lambda) \Phi(\omega_\Lambda|\xi) \chi_{\beta,\Lambda}(d\omega_\Lambda), \end{aligned} \tag{5.18}$$

holds.

Proof. By (5.6) and (5.10) one has

$$\begin{aligned} &\int_{\Omega_{\beta,\Lambda}} F^{(N)}(\omega_\Lambda) \Phi^{(N)}(\omega_\Lambda|\xi) \chi_{\beta,\Lambda}^{(N)}(d\omega_\Lambda) = \\ &\qquad = \int_{\Omega_{\beta,\Lambda}} F(\omega_\Lambda) \Phi(\omega_\Lambda|\xi) \chi_{\beta,\Lambda}^{(N)}(d\omega_\Lambda). \end{aligned}$$

By Lemma 5.1, for any box Λ, the sequence of product measures $\{\chi_{\beta,\Lambda}^{(N)} \mid N \in \mathcal{N}(\tau_1, \ldots, \tau_n)\}$ converges weakly in $\Omega_{\beta,\Lambda}$ to the measure $\chi_{\beta,\Lambda}$. On the other hand, for every $\xi \in \Omega_{\beta,\Lambda}$, the function $F\Phi(\cdot|\xi)$ is bounded and continuous on $\Omega_{\beta,\Lambda}$, which yields (5.18). □

Now let $G : \mathbb{R}^{n|\Lambda|} \to \mathbb{C}$ be such that $F(\omega_\Lambda) = G(\omega_\Lambda(\tau_1), \ldots, \omega_\Lambda(\tau_n))$. Then $F^{(N)}(\omega_\Lambda) = G(\omega_\Lambda^{(N)}(\tau_1), \ldots, \omega_\Lambda^{(N)}(\tau_n))$. Our next statement gives the ferromagnetic representation of the above approximating integrals.

Theorem 5.2. *For every $\tau_1, \ldots, \tau_n \in \mathcal{Q}_\beta$, all $F \in \mathfrak{F}_{\beta,\Lambda}(\tau_1, \ldots, \tau_n)$, any $\xi \in \Omega_{\beta,\Lambda}$, and all $N \in \mathcal{N}(\tau_1, \ldots, \tau_n)$ the following representation* *holds*

$$\begin{aligned}&\int_{\Omega_{\beta,\Lambda}} F^{(N)}(\omega_\Lambda)\Phi^{(N)}_{\beta,\Lambda}(\omega_\Lambda|\xi)\chi^{(N)}_{\beta,\Lambda}(d\omega_\Lambda) \\ &= K^{(N)}_{\beta,\Lambda}(\xi)\int_{\mathbb{R}^{N|\Lambda|}} G\left(S_\Lambda(\nu_1),\ldots,S_\Lambda(\nu_n)\right)\rho_{\beta,\Lambda}(dS_\Lambda|X) \\ &= K^{(N)}_{\beta,\Lambda}(\xi) < G\left(S_\Lambda(\nu_1),\ldots,S_\Lambda(\nu_n)\right) >_{\rho_{\beta,\Lambda}(\cdot|X)} .\end{aligned} \tag{5.19}$$

Here $K^{(N)}_{\beta,\Lambda}(\xi)$ *is a positive constant,* $\nu_j = (\tau_j/\beta)N$, $j = 1,\ldots,n$, *and the probability measure* $\rho_{\beta,\Lambda}(\cdot|X)$ *has the form*

$$\begin{aligned}\rho_{\beta,\Lambda}(dS_\Lambda|X) &\stackrel{\text{def}}{=} \frac{1}{C_{\beta,\Lambda}(X)}\times \\ &\times\exp\left\{-\frac{1}{2}\sum_{l,l'\in\Lambda} J_{ll'}\sum_{\nu=0}^{N-1} S_l(\nu)S_{l'}(\nu) - \sum_{l\in\Lambda}\sum_{\nu=0}^{N-1} S_l(\nu)X_l(\nu)\right. \\ &\left. -\frac{\mathfrak{m}N^2}{2\beta^2}\sum_{l\in\Lambda}\sum_{\nu=0}^{N-1}\left[S(\nu+1)-S(\nu)\right]^2\right\}\bigotimes_{l\in\Lambda}\bigotimes_{\nu=0}^{N-1}\sigma^{(N)}(dS_l(\nu)),\end{aligned} \tag{5.20}$$

where the coefficients $J_{ll'}$ *are the same as in (5.1),* $\{X_l(\nu) \mid l\in\Lambda, \nu = 0,\ldots,N-1\}$ *is a certain, dependent on* ξ, *real vector (*$X = 0$ *for* $\xi = 0$*), and*

$$\begin{aligned}&\sigma^{(N)}(dS_l(\nu)) \\ &= \exp\left\{-\frac{\beta}{N}W\left(\sqrt{\frac{N}{\beta}}S_l(\nu)\right) - \frac{1}{2}\left[S_l(\nu)\right]^2\right\}dS_l(\nu).\end{aligned} \tag{5.21}$$

It should be noted here that by [75] the measure $\rho_{\beta,\Lambda}(\cdot|X)$ corresponds to a general type ferromagnet, whereas $\rho_{\beta,\Lambda}(\cdot|0)$ corresponds to an even ferromagnet.

Proof of Theorem 5.2 . First we change the variables in the integral on the right-hand side of (5.9) by means of the following Fourier transformations (c.f. (5.7))

$$\omega_l(\tau) = \sum_{k\in\mathcal{K}}\hat{\omega}_l(k)e_k(\tau), \quad \hat{\omega}_l(k) = \int_{\mathcal{I}_\beta}\omega_l(\tau)e_k(\tau)d\tau, \tag{5.22}$$

where the functions e_k, $k\in\mathcal{K}$ were defined by (2.45). Then, for $\mathcal{Q}_\beta \ni \tau = (\nu/N)\beta$, one has

$$\omega^{(N)}_l(\tau) = \sum_{k\in\mathcal{K}_N}\hat{\omega}_l(k)e_k\left(\frac{\nu}{N}\beta\right) = \sqrt{\frac{N}{\beta}}\sum_{p\in\mathcal{P}_N}\hat{\omega}_l\left(\frac{N}{\beta}p\right)\varepsilon_p(\nu), \tag{5.23}$$

where

$$\mathcal{P}_N \stackrel{\rm def}{=} \{p = \frac{2\pi}{N}\kappa \mid \kappa = -(L-1), \dots, L\}, \tag{5.24}$$

and for $\nu = 0, 1, \dots, N-1$, (c.f. (2.45))

$$\varepsilon_p(\nu) = \sqrt{\frac{2}{N}} \cos p\nu \ (p > 0), \qquad \varepsilon_p(\nu) = -\sqrt{\frac{2}{N}} \sin p\nu \ (p < 0),$$
$$\varepsilon_0(\nu) = \frac{1}{\sqrt{N}}. \tag{5.25}$$

For the functions of the type (ii) taken at $\omega_\Lambda^{(N)}$, one has

$$\langle \omega_l^{(N)}, \omega_{l'}^{(N)} \rangle_\beta = \sum_{k \in \mathcal{K}_N} \hat{\omega}_l(k) \hat{\omega}_{l'}(k) = \sum_{p \in \mathcal{P}_N} \hat{\omega}_l \left(\frac{N}{\beta} p \right) \hat{\omega}_{l'} \left(\frac{N}{\beta} p \right), \tag{5.26}$$

and

$$\langle \omega_l^{(N)}, \xi_l \rangle_\beta = \sum_{k \in \mathcal{K}_N} \hat{\omega}_l(k) \hat{\xi}_l(k), \quad \hat{\xi}_l(k) = \int_{\mathcal{I}_\beta} \xi_l(\tau) e_k(\tau) d\tau. \tag{5.27}$$

As for functions of the type (iii), instead of (5.22) it is more convenient to use the following transformation

$$\omega_l(\tau) = \frac{1}{\sqrt{\beta}} \sum_{k \in \mathcal{K}} \tilde{\omega}_l(k) \exp(ik\tau), \tag{5.28}$$
$$\tilde{\omega}_l(k) = \frac{1}{\sqrt{\beta}} \int_{\mathcal{I}_\beta} \omega_l(\tau) \exp(-ik\tau)\, d\tau.$$

Then one has

$$\int_{\mathcal{I}_\beta} W\left(\omega_l^{(N)}(\tau)\right) d\tau = \sum_{s=1}^{r} w_s \int_{\mathcal{I}_\beta} \left[\omega_l^{(N)}(\tau)\right]^{2s} d\tau. \tag{5.29}$$

Further

$$\begin{aligned}\int_{\mathcal{I}_\beta} \left[\omega_l^{(N)}(\tau)\right]^{2s} d\tau = \beta^{-s} & \sum_{k_1, \dots, k_{2s} \in \mathcal{K}_N} \tilde{\omega}_l(k_1) \dots \tilde{\omega}_l(k_{2s}) \\ & \times \int_{\mathcal{I}_\beta} \exp\left[i(k_1 + \dots + k_{2s})\tau\right] d\tau \\ & = \beta^{-s+1} \sum_{k_1, \dots, k_{2s} \in \mathcal{K}_N} \tilde{\omega}_l(k_1) \dots \tilde{\omega}_l(k_{2s}) \\ & \times \delta\left(k_1 + \dots + k_{2s}\right). \end{aligned} \tag{5.30}$$

Here $\delta(0) = 1$, $\delta(k) = 0$ if $k \neq 0$. Now we introduce the variables (quasi-spins) $S_l(\nu)$:

$$S_l(\nu) = \sqrt{\frac{\beta}{N}}\omega_l\left(\frac{\nu}{N}\beta\right), \quad l \in \Lambda, \quad \nu = 0, 1, \ldots, N-1; \tag{5.31}$$
$$\hat{S}_l(p) = \hat{\omega}_l\left(\frac{N}{\beta}p\right), \quad \tilde{S}_l(p) = \tilde{\omega}_l\left(\frac{N}{\beta}p\right), \quad p \in \mathcal{P}_N;$$

for which one has

$$S_l(\nu) = \sum_{p\in\mathcal{P}_N} \hat{S}_l(p)\varepsilon_p(\nu) = \frac{1}{\sqrt{N}}\sum_{p\in\mathcal{P}_N} \tilde{S}_l(p)\exp(ip\nu); \tag{5.32}$$
$$\hat{S}_l(p) = \sum_{\nu=0}^{N-1} S_l(\nu)\varepsilon_p(\nu), \quad \tilde{S}_l(p) = \frac{1}{\sqrt{N}}\sum_{\nu=0}^{N-1} S_l(\nu)\exp(-ip\nu).$$

Then (5.30) may be rewritten in the following way

$$\begin{aligned}
&\int_{\mathcal{I}_\beta}\left[\omega_l^{(N)}(\tau)\right]^{2s} d\tau = \\
&\beta^{-s+1}\sum_{k_1,\ldots,k_{2s}\in\mathcal{K}_N} \tilde{S}_l\left(\frac{\beta}{N}k_1\right)\ldots\tilde{S}_l\left(\frac{\beta}{N}k_{2s}\right)\times \\
&\times\delta\left(k_1+\cdots+k_{2s}\right) \\
&= \frac{1}{N^s\beta^{s-1}}\sum_{\nu_1,\ldots,\nu_{2s}=0}^{N-1} S_l(\nu_1)\ldots S_l(\nu_{2s})\times \\
&\times\sum_{k_1,\ldots k_{2s}\in\mathcal{K}_N}\delta(k_1+\cdots+k_{2s})\times \\
&\times\exp\left\{-\frac{i\beta}{N}(k_1\nu_1+\cdots+k_{2s}\nu_{2s})\right\} \\
&= \frac{1}{N^s\beta^{s-1}}\sum_{\nu_1,\ldots,\nu_{2s}=0}^{N-1} S_l(\nu_1)\ldots S_l(\nu_{2s})\times \\
&\times\sum_{\kappa_1,\ldots,\kappa_{2s-1}=-L+1}^{L}\exp\left\{-\frac{2\pi i}{N}\kappa_1(\nu_1-\nu_{2s})\right\}\times\ldots \\
&\times\exp\left\{-\frac{2\pi i}{N}\kappa_{2s-1}(\nu_{2s-1}-\nu_{2s})\right\} = \\
&= \frac{N^{2s-1}}{N^s\beta^{s-1}}\sum_{\nu=0}^{N-1}\left[S_l(\nu)\right]^{2s} = \frac{\beta}{N}\sum_{\nu=0}^{N-1}\left[\sqrt{\frac{N}{\beta}}S_l(\nu)\right]^{2s}.
\end{aligned}$$

Returning to (5.29) one obtains

$$\int_{\mathcal{I}_\beta} W\left(\omega_l^{(N)}(\tau)\right) d\tau = \frac{\beta}{N}\sum_{\nu=0}^{N-1} W\left(\sqrt{\frac{N}{\beta}}S_l(\nu)\right). \tag{5.33}$$

Accordingly,

$$\langle \omega_l^{(N)}, \omega_{l'}^{(N)}\rangle_\beta = \sum_{p\in\mathcal{P}_N} \hat{S}_l(p)\hat{S}_{l'}(p) = \sum_{\nu=1}^{N-1} S_l(\nu)S_{l'}(\nu), \tag{5.34}$$

and

$$\langle \omega_l^{(N)}, \xi_l\rangle_\beta = \sum_{k\in\mathcal{K}_N} \hat{\omega}_l(k)\xi_l(k) = \sum_{\nu=0}^{N-1} S_l(\nu)X_l(\nu), \tag{5.35}$$

$$X_l(\nu) \stackrel{\mathrm{def}}{=} \sum_{p\in\mathcal{P}_N} \xi_l\left(\frac{N}{\beta}p\right)\varepsilon_p(\nu).$$

At last, (5.23) takes the form

$$\omega_l(\tau_j) = \omega_l\left(\frac{\nu_j}{N}\beta\right) = \sqrt{\frac{N}{\beta}}S_l(\nu_j), \quad j = 1,\ldots,n. \tag{5.36}$$

The next step is to introduce the measure on finite-dimensional space which would have the above mentioned ferromagnetic properties and such that the integral on the right-hand side of (5.9) could be substituted by the integral over this finite-dimensional space. Here we use the representation (5.6) and construct a finite-dimensional analogue of $\chi_\beta^{(N)}$. To this end we introduce the following Gaussian measure on $\mathbb{R}^N$:

$$\phi_\beta^{(N)}(d\hat{S}) = \bigotimes_{p\in\mathcal{P}_N} \phi_p^{(N)}(d\hat{S}(p)), \tag{5.37}$$

where the measure $\phi_p^{(N)}$ satisfies (5.5) with (c.f.(5.3))

$$\theta_p^{(N)} \stackrel{\mathrm{def}}{=} \frac{1}{\mathfrak{m}\left(\frac{2N}{\beta}\right)^2(\sin p/2)^2 + 1} = \lambda_{Np/\beta}^{(N)}. \tag{5.38}$$

It is clear that $\varpi_k^{(N)} = \phi_{\beta k/N}^{(N)}$, where the former measure defines by (5.6) the measure $\chi_\beta^{(N)}$. On the other hand, this new Gaussian measure may also be written in the coordinates $\{S(\nu), \nu = 0, \ldots N-1\}$, related to

$\{\hat{S}(p), p \in \mathcal{P}_N\}$ by the transformation (5.32), as follows

$$\phi_\beta^{(N)}(dS) = \frac{1}{C_{\beta,N}} \exp\left\{ -\frac{\mathfrak{m}N^2}{2\beta^2} \sum_{\nu=0}^{N-1} [S(\nu+1) - S(\nu)]^2 - \frac{1}{2}\sum_{\nu=0}^{N-1} [S(\nu)]^2 \right\} \bigotimes_{\nu=0}^{N-1} dS(\nu), \tag{5.39}$$

with the convention $S(N) = S(0)$ and the normalizing constant $C_\beta^{(N)}$. Therefore, the measure $\phi_\beta^{(N)}$ may be regarded as the Gibbs measure of a chain of unbounded (Gaussian) spins. Due to the choice of the numbers (5.38), the interaction is ferromagnetic and of the nearest-neighbor type.

Now we define the measure which will correspond to $\chi_{\beta,\Lambda}^{(N)}$ given by (5.10). It is

$$\phi_{\beta,\Lambda}^{(N)}(dS_\Lambda) \stackrel{\text{def}}{=} \bigotimes_{l\in\Lambda} \phi_\beta^{(N)}(dS_l) \tag{5.40}$$

$$= \frac{1}{[C_{\beta,N}]^{|\Lambda|}} \exp\left\{ -\frac{\mathfrak{m}N^2}{2\beta^2} \sum_{l\in\Lambda}\sum_{\nu=0}^{N-1} [S_l(\nu+1) - S_l(\nu)]^2 - \frac{1}{2}\sum_{l\in\Lambda}\sum_{\nu=0}^{N-1} [S_l(\nu)]^2 \right\} \bigotimes_{l\in\Lambda}\bigotimes_{\nu=0}^{N-1} dS_l(\nu).$$

By construction we have that

$$\int_{\Omega_{\beta,\Lambda}} F^{(N)}(\omega_\Lambda)\Phi_{\beta,\Lambda}^{(N)}(\omega_\Lambda|\xi)\chi_{\beta,\Lambda}^{(N)}(d\omega_\Lambda) \tag{5.41}$$

$$= \frac{C_{\beta,\Lambda}(X)}{Y_{\beta,\Lambda}(\xi)} \left(\frac{N}{\beta}\right)^{\frac{n}{2}} \int_{\mathbb{R}^{N|\Lambda|}} G\left(S_\Lambda(\nu_1), \ldots, S_\Lambda(\nu_n)\right) \rho_{\beta,\Lambda}(dS_\Lambda|X),$$

where the measure $\rho_{\beta,\Lambda}(\cdot|X)$ is given by (5.20), (5.21). □

6. Basic Inequalities

In this section we use the lattice approximation to prove a number of basic inequalities for the moments, like (5.8), of the measure (5.1) with the function w satisfying **(V1)**.

Theorem 6.1. [FKG Inequality] *Given Λ, β, and $\tau_1, \ldots, \tau_n \in \mathcal{I}_\beta$, let the functions $F, G \in \mathfrak{F}_{\beta,\Lambda}$ increase when every chosen $\omega_l(\tau_j)$, $l \in \Lambda$, $j = 1, \ldots, n$ increases. Then the following inequality*

$$< FG >_{\varrho_{\beta,\Lambda}(\cdot|\xi)} \geq < F >_{\varrho_{\beta,\Lambda}(\cdot|\xi)} < G >_{\varrho_{\beta,\Lambda}(\cdot|\xi)}, \tag{6.1}$$

holds for all $\xi \in \Omega_{\beta,\Lambda}$.

The proof follows from Theorem 5.1, 5.2 and the fact that the measure (5.20) corresponds to a general type ferromagnet, for which the FKG inequality holds (see Theorem VIII.16 of [75]).

Below $\varrho_{\beta,\Lambda}$ will stand for the measure (5.1) with $\xi = 0$.

Theorem 6.2. [**GKS Inequalities**] *Given* Λ *and* β, *let the real valued functions* $A_1, \dots, A_{n+m} \in \mathfrak{P}_\Lambda^{(1)}$, $n, m \in \mathbb{N}$ *have the properties:*

(a) *every* A_j *depends only on the values of* x_{l_j} *with certain* $l_j \in \Lambda$;

(b) *every* A_j *is either an odd monotone growing function of* x_{l_j} *or an even positive function, monotone growing on* $[0, +\infty)$.

Then the following inequalities hold

$$< A_1(\omega_\Lambda(\tau_1)) \dots A_n(\omega_\Lambda(\tau_n)) >_{\varrho_{\beta,\Lambda}} \geq 0, \tag{6.2}$$

$$\begin{aligned} < A_1(\omega_\Lambda(\tau_1)) \dots A_n(\omega_\Lambda(\tau_n)) \times & \\ \times\ A_{n+1}(\omega_\Lambda(\tau_{n+1})) \dots A_{n+m}(\omega_\Lambda(\tau_{n+m})) >_{\varrho_{\beta,\Lambda}} & \\ \geq\ < A_1(\omega_\Lambda(\tau_1)) \dots A_n(\omega_\Lambda(\tau_n)) >_{\varrho_{\beta,\Lambda}} \times & \\ \times\ < A_{n+1}(\omega_\Lambda(\tau_{n+1})) \dots A_{n+m}(\omega_\Lambda(\tau_{n+m})) >_{\varrho_{\beta,\Lambda}} . & \end{aligned} \tag{6.3}$$

The proof follows from Theorem 5.1, 5.2 and the fact that the measure (5.20) with $X = 0$ corresponds to an even ferromagnet, for which the GKS inequalities hold (see Theorem VIII.14 of [75]).

Corollary 6.1. *For all* $\tau_1, \dots, \tau_{n+m} \in \mathcal{I}_\beta$, *the Green functions (2.62), (2.63) obey the following inequalities:*

$$\Gamma^{\beta,\Lambda}_{A_1,\dots,A_n}(\tau_1, \dots, \tau_n) \geq 0, \qquad \Gamma^{0,\beta,\Lambda}_{A_1,\dots,A_n}(\tau_1, \dots, \tau_n) \geq 0; \tag{6.4}$$

$$\begin{aligned} \Gamma^{\beta,\Lambda}_{A_1,\dots,A_{n+m}}(\tau_1, \dots, \tau_{n+m}) \geq & \\ & \Gamma^{\beta,\Lambda}_{A_1,\dots,A_n}(\tau_1, \dots, \tau_n) \times \\ & \times \Gamma^{\beta,\Lambda}_{A_{n+1},\dots,A_{n+m}}(\tau_{n+1}, \dots, \tau_{n+m}) \\ \Gamma^{0,\beta,\Lambda}_{A_1,\dots,A_{n+m}}(\tau_1, \dots, \tau_{n+m}) \geq & \\ & \Gamma^{0,\beta,\Lambda}_{A_1,\dots,A_n}(\tau_1, \dots, \tau_n) \times \\ & \times \Gamma^{0,\beta,\Lambda}_{A_{n+1},\dots,A_{n+m}}(\tau_{n+1}, \dots, \tau_{n+m}). \end{aligned} \tag{6.5}$$

For $\varphi_\Lambda = (\varphi_l)_{l\in\Lambda} \in \mathcal{X}_{\beta,\Lambda}$, we set

$$F(\varphi_\Lambda) = \int_{\mathcal{X}_{\beta,\Lambda}} \exp\left\{ \sum_{l\in\Lambda} \langle \varphi_l, \omega_l \rangle_\beta \right\} \varrho_{\beta,\Lambda}(d\omega_\Lambda). \tag{6.6}$$

$F(\varphi_\Lambda)$ is an entire real analytic function, which means that the expansion

$$F(\varphi_\Lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(\varphi_\Lambda, \ldots, \varphi_\Lambda), \tag{6.7}$$

where every $F^{(n)}(\cdot, \ldots, \cdot)$ is a n–linear bounded functional on $\mathcal{X}_{\beta,\Lambda}$, converges absolutely on the whole Hilbert space $\mathcal{X}_{\beta,\Lambda}$. These functionals may be written in the integral form

$$F^{(n)}(\varphi_\Lambda, \ldots, \varphi_\Lambda) = \sum_{l_1,\ldots,l_n \in \Lambda} \int_{\mathcal{I}_\beta^n} F_{l_1,\ldots,l_n}(\tau_1, \ldots, \tau_n) \varphi_{l_1}(\tau_1) \ldots \varphi_{l_n}(\tau_n) d\tau_1 \ldots d\tau_n, \tag{6.8}$$

with the kernels being the moments of the measure $\varrho_{\beta,\Lambda}$, i.e.

$$F_{l_1,\ldots,l_n}(\tau_1, \ldots, \tau_n) =< \omega_{l_1}(\tau_1) \ldots \omega_{l_n}(\tau_n) >_{\varrho_{\beta,\Lambda}},$$

which means in turn that they are the Green functions (5.14) with $A_j(x_\Lambda) = x_{l_j}$. These kernels are continuous as functions of $\tau_1, \ldots, \tau_n$. Since $F(0) = 1$, the function $\log F(\varphi_\Lambda)$ is a real analytic function in a neighborhood of the point $\varphi_\Lambda = 0$, where it can be expanded similarly to (6.7)

$$U(\varphi_\Lambda) \stackrel{\text{def}}{=} \log F(\varphi_\Lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} U^{(n)}(\varphi_\Lambda, \ldots, \varphi_\Lambda), \tag{6.9}$$

and

$$U^{(n)}(\varphi_\Lambda, \ldots, \varphi_\Lambda) = \sum_{l_1,\ldots,l_n \in \Lambda} \int_{\mathcal{I}_\beta^n} U_{l_1,\ldots,l_n}(\tau_1, \ldots, \tau_n) \varphi_{l_1}(\tau_1) \ldots \varphi_{l_n}(\tau_1) d\tau_n \ldots d\tau_n. \tag{6.10}$$

Theorem 6.3. [**Lebowitz Inequalities**] *Given Λ and β, the following inequality*

$$U_{l_1,\ldots,l_4}(\tau_1, \ldots, \tau_4) \leq 0, \tag{6.11}$$

holds for all $\tau_1, \ldots, \tau_4 \in \mathcal{I}_\beta$ and $l_1, \ldots, l_4 \in \Lambda$.

The proof follows from Theorem 5.1, 5.2 and the fact that the Lebowitz inequality holds for the measure (5.20) with the function w satisfying **(V1)** and with $X = 0$ (see Theorem 2.4 and Corollary 2.5 of [83]).

For Gaussian random variables $X_1, \ldots, X_{2n}$, $n \in \mathbb{N}$ with zero mean, one has

$$< X_1 \ldots X_{2n} >= \sum_{\sigma \in S_n} \prod_{k=1}^{n} < X_{\sigma(2k-1)} X_{\sigma(2k)} >,$$

where the sum is taken over all partitions of the set $\{1, \ldots, 2n\}$ into the pairs. If in the latter expression one has "$\leq$" instead of "$=$", the variables $X_1, \ldots, X_{2n}$ are said to obey the Gaussian upper bound principle.

Theorem 6.4. [Gaussian Upper Bound] *Given Λ and β, the following inequality*

$$
\begin{aligned}
& < \omega_{l_1}(\tau_1) \ldots \omega_{l_{2n}}(\tau_{2n}) >_{\varrho_{\beta,\Lambda}} \\
\leq & \sum_{\sigma \in S_n} \prod_{k=1}^{n} < \omega_{l_{\sigma(2k-1)}}(\tau_{\sigma(2k-1)}) \omega_{l_{\sigma(2k)}}(\tau_{\sigma(2k)}) >_{\varrho_{\beta,\Lambda}}, \qquad (6.12)
\end{aligned}
$$

holds for all values of $l_1, \ldots, l_{2n} \in \Lambda$ and $\tau_1, \ldots, \tau_{2n} \in \mathcal{I}_\beta$.

The proof follows from Theorem 5.1, 5.2 and the fact that the Gaussian upper bound principle holds for the measure (5.20) with $X = 0$ (see Section 12 of [39])

Theorem 6.5. *Under the conditions of Theorem 6.3 let the potential W, which defines the measures (5.1), (5.20), have the form*

$$
W(x) = \frac{1}{2} a x^2 + b x^4, \quad a \in \mathbb{R}, \quad b > 0. \qquad (6.13)
$$

Then the following inequalities

$$
(-1)^{n-1} U_{l_1, \ldots, l_{2n}}(\tau_1, \ldots, \tau_{2n}) \geq 0, \qquad (6.14)
$$

hold for all $\in \mathbb{N}$, all $\tau_1, \ldots, \tau_{2n} \in \mathcal{I}_\beta$ and $l_1, \ldots, l_{2n} \in \Lambda$

The proof follows from Theorem 5.1, 5.2 and the fact that the above sign rule holds for the measure (5.20) with $X = 0$ and W given by (6.13), which can be deduced from Shlosman's results [74] for the Ising model by means of the classical Ising approximation (for more details see Ch.IX of book [75]).

7. More Inequalities

7.1. **Scalar Domination.** Here we assume that the measures (2.60), (2.61) and the Green functions (2.62), (2.63) describe the vector model (2.1) – (2.9) with $D > 1$ and with the potential V (2.3) obeying the condition **(V2)**. Since we will compare the Green functions for this model with similar functions for the corresponding scalar model, we need a special notation for the latter ones. Let the Green functions $\widetilde{\Gamma}^{\beta,\Lambda}$ and $\widetilde{\Gamma}^{0,\beta,\Lambda}$ be defined also by (2.62) and (2.63) respectively but for the model (2.1) - (2.9) with $D = 1$.

Theorem 7.1. [**Scalar Domination**] *For a box Λ, let the local Gibbs measures be defined by (2.60), (2.61) with the potential V obeying the condition* **(V2)** *and with arbitrary $D > 1$. Let the functions $A_1, \ldots, A_n \in \mathfrak{P}_\Lambda^{(D)}$, $n \in \mathbb{N}$ have the following property: there exists $\alpha \in \{1, 2, \ldots, D\}$ and the functions $\widetilde{A}_1, \ldots, \widetilde{A}_n \in \mathfrak{P}_\Lambda^{(1)}$ satisfying the conditions of Theorem 6.2 and such that $A_j(x_\Lambda) = \widetilde{A}_j(x_\Lambda^\alpha)$, $j = 1, \ldots, n$. Then for arbitrary $\tau_1, \ldots, \tau_n \in \mathcal{I}_\beta$,*

$$0 \leq \Gamma^{\beta,\Lambda}_{A_1,\ldots,A_n}(\tau_1, \ldots, \tau_n) \leq \widetilde{\Gamma}^{\beta,\Lambda}_{\widetilde{A}_1,\ldots,\widetilde{A}_n}(\tau_1, \ldots, \tau_n). \tag{7.1}$$

Remark 7.1. *It is important that all A_j depend on their x_Λ^α with one and the same α. The first above inequality is a D-dimensional version of the first GKS inequality (6.4). The second inequality in (7.1) describes scalar domination. The same inequalities hold for zero boundary conditions as well.*

Proof of Theorem 7.1. For the α mentioned in the hypothesis, let us decompose

$$\mathcal{X}_{\beta,\Lambda} = \bar{\mathcal{X}}_{\beta,\Lambda} \times \mathcal{X}_{\beta,\Lambda}^\alpha, \quad \mathcal{X}_\beta = \bar{\mathcal{X}}_\beta \times \mathcal{X}_\beta^\alpha,$$

which means that every $\omega_\Lambda \in \mathcal{X}_{\beta,\Lambda}$ is regarded as $\omega_\Lambda = (\bar{\omega}_\Lambda, \omega_\Lambda^\alpha)$, where a $(D-1)$-dimensional vector $\bar{\omega}_\Lambda$ belongs to $\bar{\mathcal{X}}_{\beta,\Lambda}$ (or to $\bar{\mathcal{X}}_\beta$ for one-element Λ), whereas the scalar ω_Λ^α is supposed to belong to the Hilbert space $\mathcal{X}_{\beta,\Lambda}^\alpha$ (respectively to $\mathcal{X}_\beta^\alpha$ for one-element Λ). Then the Gaussian measure χ_β can also be decomposed

$$\chi_\beta(d\omega) = (\bar{\chi}_\beta \otimes \chi_\beta^\alpha)(d\bar{\omega}, d\omega^\alpha), \tag{7.2}$$

where the Gaussian measures $\bar{\chi}_\beta$, χ_β^α are defined on the Hilbert spaces $\bar{\mathcal{X}}_\beta$ and $\mathcal{X}_\beta^\alpha$ respectively. The potential V (2.3), may be written

$$V(x) = v((\bar{x}, \bar{x})) + v((x^\alpha)^2) + \sum_{s=2}^{r-1} (x^\alpha)^{2s} B_s(\bar{x}), \quad B_s(\bar{x}) \geq 0. \tag{7.3}$$

The nonnegativity of $B_s(\bar{x})$ follows from the condition **(V2)**. Set

$$Q(\bar{\omega}, \omega^\alpha) \stackrel{\text{def}}{=} \int_{\mathcal{I}_\beta} \sum_{s=2}^{r-1} (\omega^\alpha(\tau))^{2s} B_s(\bar{\omega}(\tau))\, d\tau. \tag{7.4}$$

Then by means of the decomposition (7.2) one may write the measure (2.60) as follows

$$\mu_{\beta,\Lambda}(d\omega_\Lambda) = \tag{7.5}$$
$$= C_{\beta,\Lambda} \exp\left\{ -\sum_{l\in\Lambda} Q(\bar{\omega}_l, \omega_l^\alpha) \right\} (\bar{\mu}_{\beta,\Lambda} \otimes \mu_{\beta,\Lambda}^\alpha)(d\bar{\omega}_\Lambda, d\omega_\Lambda^\alpha),$$

where $C_{\beta,\Lambda}$ is the normalization constant and the Gibbs measures $\bar{\mu}_{\beta,\Lambda}$ and $\mu^{\alpha}_{\beta,\Lambda}$ describe systems of $(D-1)-$ and one-dimensional interacting anharmonic oscillators respectively. This allows us to rewrite (2.62) in the following way

$$\Gamma^{\beta,\Lambda}_{A_1,\dots,A_n}(\tau_1,\dots,\tau_n) = \\ = C_{\beta,\Lambda}\int_{\bar{\mathcal{X}}_{\beta,\Lambda}} \Xi(1|\bar{\omega}_\Lambda,\tau_1,\dots,\tau_n)\Theta(1|\bar{\omega}_\Lambda)\bar{\mu}_{\beta,\Lambda}(d\bar{\omega}_\Lambda), \tag{7.6}$$

where, for $\vartheta\in[0,1]$, we have set

$$\Xi(\vartheta|\bar{\omega}_\Lambda,\tau_1,\dots,\tau_n) = \frac{1}{\Theta(\vartheta|\bar{\omega}_\Lambda)}\int_{\mathcal{X}^{\alpha}_{\beta,\Lambda}} A_1(\omega_\Lambda(\tau_1))\dots A_n(\omega_\Lambda(\tau_n))\times \\ \times\exp\left\{-\vartheta\sum_{l\in\Lambda}Q(\bar{\omega}_l,\omega^{\alpha}_l)\right\}\mu^{\alpha}_{\beta,\Lambda}(d\omega^{\alpha}_{\beta,\Lambda}), \tag{7.7}$$

and

$$\Theta(\vartheta|\bar{\omega}_{\beta,\Lambda}) = \int_{\mathcal{X}^{\alpha}_{\beta,\Lambda}}\exp\left\{-\vartheta\sum_{l\in\Lambda}Q(\bar{\omega}_l,\omega^{\alpha}_l)\right\}\mu^{\alpha}_{\beta,\Lambda}(d\omega^{\alpha}_{\Lambda}). \tag{7.8}$$

Now let the functions $\widetilde{A}_1,\dots,\widetilde{A}_n$ be such that $A_j(\omega_\Lambda(\tau_j)) = \widetilde{A}_j(\omega^{\alpha}_\Lambda(\tau_j))$, as it is supposed in Theorem 7.1. Then

$$\Xi(0|\bar{\omega}_\Lambda,\tau_1,\dots,\tau_n) = \int_{\mathcal{X}^{\alpha}_{\beta,\Lambda}}\widetilde{A}_1(\omega^{\alpha}_\Lambda(\tau_1))\dots\widetilde{A}_n(\omega^{\alpha}_\Lambda(\tau_n))\mu^{\alpha}_{\beta,\Lambda}(d\omega^{\alpha}_\Lambda) \\ = \widetilde{\Gamma}^{\beta,\Lambda}_{\widetilde{A}_1,\dots,\widetilde{A}_n}(\tau_1,\dots,\tau_n). \tag{7.9}$$

As a function of ϑ, Ξ is continuous on $[0,1]$ and differentiable on $(0,1)$, where

$$\frac{\partial}{\partial\vartheta}\Xi(\vartheta|\bar{\omega}_\Lambda,\tau_1,\dots,\tau_n) = -\sum_{l\in\Lambda}\sum_{s=1}^{r-1}\int_{\mathcal{I}_\beta}B_s(\bar{\omega}_l(t))\times \\ \times\Big\{<\widetilde{A}_1(\omega^{\alpha}_\Lambda(\tau_1))\dots\widetilde{A}_n(\omega^{\alpha}_\Lambda(\tau_n))\,(\omega^{\alpha}_l(t))^{2s}>_\phi \\ - <\widetilde{A}_1(\omega^{\alpha}_\Lambda(\tau_1))\dots\widetilde{A}_n(\omega^{\alpha}_\Lambda(\tau_n))>_\phi\times \\ \times<(\omega^{\alpha}_l(t))^{2s}>_\phi\Big\}\,dt. \tag{7.10}$$

Here (see (2.30)), for a fixed $\bar{\omega}_\Lambda\in\bar{\mathcal{X}}_{\beta,\Lambda}$, the measure ϕ is defined on $\mathcal{X}^{\alpha}_{\beta,\Lambda}$ as follows

$$\phi(d\omega^{\alpha}_\Lambda) = \frac{1}{\Theta(\vartheta|\bar{\omega}_\Lambda)}\exp\left\{-\vartheta\sum_{l\in\Lambda}Q(\bar{\omega}_l,\omega^{\alpha}_l)\right\}\mu^{\alpha}_{\beta,\Lambda}(d\omega^{\alpha}_\Lambda).$$

Since the measure $\mu^{\alpha}_{\beta,\Lambda}$ and the functions $\tilde{A}_1, \ldots, \tilde{A}_n$, $\omega^{\alpha}_{\Lambda}(t) \mapsto (\omega^{\alpha}_l(t))^{2s}$, satisfy the conditions of Theorem 6.2, the estimate (6.5) yields in (7.10)

$$\frac{\partial}{\partial \vartheta} \Xi(\vartheta | \bar{\omega}_{\Lambda}, \tau_1, \ldots, \tau_n) \leq 0,$$

for all $\vartheta \in (0,1)$, $\bar{\omega}_{\Lambda} \in \bar{\mathcal{X}}_{\beta,\Lambda}$, and $\tau_1, \ldots, \tau_n \in \mathcal{I}_{\beta}$. The latter fact and the estimate (6.4) yield in turn

$$\begin{aligned} 0 \leq \Xi(1 | \bar{\omega}_{\Lambda}, \tau_1, \ldots, \tau_n) &\leq \Xi(0 | \bar{\omega}_{\Lambda}, \tau_1, \ldots, \tau_n) \\ &= \widetilde{\Gamma}^{\beta,\Lambda}_{\widetilde{A}_1, \ldots, \widetilde{A}_n}(\tau_1, \ldots, \tau_n). \end{aligned} \tag{7.11}$$

Using this double inequality in (7.6) we obtain

$$\begin{aligned} & 0 \leq \Gamma^{\beta,\Lambda}_{A_1,\ldots,A_n}(\tau_1, \ldots, \tau_n) \leq \tilde{\Gamma}^{\beta,\Lambda}_{\tilde{A}_1,\ldots,\tilde{A}_n}(\tau_1, \ldots, \tau_n) \times \\ & \times C_{\beta,\Lambda} \int_{\mathcal{X}^{\alpha}_{\beta,\Lambda}} \int_{\bar{\mathcal{X}}_{\beta,\Lambda}} \exp\left\{ - \sum_{l \in \Lambda} Q(\bar{\omega}_l, \omega^{\alpha}_l) \right\} (\bar{\mu}_{\beta,\Lambda} \otimes \mu^{\alpha}_{\beta,\Lambda})(d\bar{\omega}_{\beta,\Lambda}, d\omega^{\alpha}_{\Lambda}) \\ & = \tilde{\Gamma}^{\beta,\Lambda}_{\tilde{A}_1,\ldots,\tilde{A}_n}(\tau_1, \ldots, \tau_n) \int_{\mathcal{X}_{\beta,\Lambda}} \mu_{\beta,\Lambda}(d\omega_{\Lambda}) = \tilde{\Gamma}^{\beta,\Lambda}_{\tilde{A}_1,\ldots,\tilde{A}_n}(\tau_1, \ldots, \tau_n). \end{aligned}$$

□

The above theorem admits a generalization. One observes that (7.1) may be rewritten

$$\begin{aligned} 0 \ &\leq \ < A_1(\omega_{\Lambda}(\tau_1)) \ldots A_n(\omega_{\Lambda}(\tau_n)) >_{\mu_{\beta,\Lambda}} \leq \\ &\leq \ < \widetilde{A}_1(\omega_{\Lambda}(\tau_1)) \ldots \widetilde{A}_n(\omega_{\Lambda}(\tau_n)) >_{\mu^{\alpha}_{\beta,\Lambda}} . \end{aligned}$$

Theorem 7.2. *Let the conditions of Theorem 7.1 be satisfied. Then for every $\mu_{\beta,\Lambda}$-integrable function $F : \Omega_{\beta,\Lambda} \to \mathbb{R}_+$, which does not depend on x^{α}_{Λ} mentioned in this theorem, the following inequalities*

$$\begin{aligned} 0 \ &\leq \ < A_1(\omega_{\Lambda}(\tau_1)) \ldots A_n(\omega_{\Lambda}(\tau_n)) F(\omega_{\Lambda}) >_{\mu_{\beta,\Lambda}} \leq \\ &\leq \ < \widetilde{A}_1(\omega_{\Lambda}(\tau_1)) \ldots \widetilde{A}_n(\omega_{\Lambda}(\tau_n)) >_{\mu^{\alpha}_{\beta,\Lambda}} \cdot < F(\omega_{\Lambda}) >_{\mu_{\beta,\Lambda}} \end{aligned} \tag{7.12}$$

hold for all $\tau_1, \ldots, \tau_n \in \mathcal{I}_{\beta}$.

To prove this theorem one writes (c.f. (7.6))

$$\begin{aligned} & < A_1(\omega_{\Lambda}(\tau_1)) \ldots A_n(\omega_{\Lambda}(\tau_n)) F(\omega_{\Lambda}) >_{\mu_{\beta,\Lambda}} = \\ & \quad = C_{\beta,\Lambda} \int_{\bar{\mathcal{X}}_{\beta,\Lambda}} \Xi(1 | \bar{\omega}_{\Lambda}, \tau_1, \ldots, \tau_n) F(\omega_{\Lambda}) \Theta(1 | \bar{\omega}_{\Lambda}) \bar{\mu}_{\beta,\Lambda}(d\bar{\omega}_{\Lambda}). \end{aligned}$$

Then employing (7.11) one gets (7.12).

7.2. **Zero Boundary Domination.** Here we consider the scalar case, thus the measures (2.60), (2.61), and (2.66) describe the model (2.1) - (2.9) with $D = 1$. The potential V is supposed to obey **(V2)**. This model will be compared with the model described by the Hamiltonian (2.1), (2.2) but with the following one-particle potential

$$\hat{V}(x) \stackrel{\rm def}{=} 2V\left(\frac{x}{\sqrt{2}}\right) = \frac{1}{2}ax^2 + \sum_{s=2}^{r} 2^{1-s} b_s x^{2s}, \quad x \in \mathbb{R}, \tag{7.13}$$

instead of V given by (2.3). Here the parameters a, and all b_s, $s = 2, \ldots, r$ are the same as in (2.3). The polynomials V, $\hat{V}$ obey the relation

$$V\left(\frac{x+y}{\sqrt{2}}\right) + V\left(\frac{x-y}{\sqrt{2}}\right) = \hat{V}(x) + \hat{V}(y) + W(x|y), \tag{7.14}$$

where

$$W(x|y) = W(y|x) = \sum_{s=1}^{r-1} b_s(x) y^{2s}, \tag{7.15}$$
$$b_s(x) = \sum_{p=s+1}^{r} \binom{2p}{2s} 2^{1-p} b_p x^{2(p-s)}.$$

Then the measures constructed with $\hat{V}$ by (2.60) and (2.61) will be written as $\hat{\mu}_{\beta,\Lambda}$, and $\hat{\mu}^{(0)}_{\beta,\Lambda}$, respectively. Let also

$$K^{\zeta}_{ll'}(\tau,\tau') \stackrel{\rm def}{=} <\omega_l(\tau)\omega_{l'}(\tau')> - <\omega_l(\tau)><\omega_{l'}(\tau')>, \tag{7.16}$$
$$\zeta \in \Omega_\beta, \quad l, l' \in \Lambda, \quad \tau, \tau' \in \mathcal{I}_\beta,$$

where the expectations are taken with respect to the measure $\mu_{\beta,\Lambda}(.|\zeta)$ (2.66). Further, for the same l, l' and τ, τ', let

$$\hat{K}^{0}_{ll'}(\tau,\tau') \stackrel{\rm def}{=} \int_{\mathcal{X}_{\beta,\Lambda}} \omega_l(\tau)\omega_{l'}(\tau')\hat{\mu}^{(0)}_{\beta,\Lambda}(d\omega_\Lambda). \tag{7.17}$$

Theorem 7.3. *For arbitrary $\zeta \in \Omega_\beta$, all $\tau, \tau' \in \mathcal{I}_\beta$ and $l, l' \in \Lambda$, the following estimates hold*

$$0 \le K^{\zeta}_{ll'}(\tau,\tau') \le \hat{K}^{0}_{ll'}(\tau,\tau'). \tag{7.18}$$

Proof. The case $\zeta \in \Omega_\beta \setminus \Omega_\beta^t$ is trivial. For $\zeta \in \Omega_\beta^t$, we rewrite (7.16) as follows

$$
\begin{aligned}
K_{ll'}^{\zeta}(\tau, \tau') = & \frac{1}{[Z_{\beta,\Lambda}(\zeta)]^2} \int \int_{\mathcal{X}_{\beta,\Lambda} \times \mathcal{X}_{\beta,\Lambda}} \frac{\omega_l(\tau) - \tilde{\omega}_l(\tau)}{\sqrt{2}} \times \\
& \times \frac{\omega_{l'}(\tau') - \tilde{\omega}_{l'}(\tau')}{\sqrt{2}} \exp\Bigg\{ - \sum_{\lambda \in \Lambda, \lambda' \in \Lambda^c} d_{\lambda\lambda'} \langle \omega_\lambda + \tilde{\omega}_\lambda, \zeta_{\lambda'} \rangle_\beta - \\
& - \frac{1}{2} \sum_{\lambda, \lambda' \in \Lambda} d_{\lambda\lambda'} \left[\langle \omega_\lambda, \omega_{\lambda'} \rangle_\beta + \langle \tilde{\omega}_\lambda, \tilde{\omega}_{\lambda'} \rangle_\beta \right] - \\
& - \sum_{\lambda \in \Lambda} \int_{\mathcal{I}_\beta} \left[V(\omega_\lambda(t)) + V(\tilde{\omega}_\lambda(t)) \right] dt \Bigg\} \times \\
& \times \bigotimes_{\lambda \in \Lambda} (\chi_\beta \otimes \chi_\beta)(d\omega_\lambda, d\tilde{\omega}_\lambda).
\end{aligned}
$$

Now we apply the following orthogonal transformation in the space $\mathcal{X}_{\beta,\Lambda} \times \mathcal{X}_{\beta,\Lambda}$:

$$
\begin{aligned}
\xi_l(\tau) &= (\omega_l(\tau) - \tilde{\omega}(\tau)) / \sqrt{2}, \quad \omega_l(\tau) = (\xi_l(\tau) + \eta_l(\tau)) / \sqrt{2}, \\
\eta_l(\tau) &= (\omega_l(\tau) + \tilde{\omega}(\tau)) / \sqrt{2}; \quad \tilde{\omega}_l(\tau) = (-\xi_l(\tau) + \eta_l(\tau)) / \sqrt{2};
\end{aligned} \tag{7.19}
$$

which yields

$$
\begin{aligned}
K_{ll'}^{\zeta}(\tau, \tau') = & [Z_{\beta,\Lambda}(\zeta)]^{-2} \int \int_{\mathcal{X}_{\beta,\Lambda} \times \mathcal{X}_{\beta,\Lambda}} \xi_l(\tau) \xi_{l'}(\tau') \times \\
& \times \exp\Bigg\{ -\sqrt{2} \sum_{\lambda \in \Lambda, \lambda' \in \Lambda^c} d_{\lambda\lambda'} \langle \eta_\lambda, \zeta_{\lambda'} \rangle_\beta - \\
& - \frac{1}{2} \sum_{\lambda, \lambda' \in \Lambda} d_{\lambda\lambda'} \left[\langle \xi_\lambda, \xi_{\lambda'} \rangle_\beta + \langle \eta_\lambda, \eta_{\lambda'} \rangle_\beta \right] - \\
& - \sum_{\lambda \in \Lambda} Q(\xi_\lambda, \eta_\lambda) - \\
& - \sum_{\lambda \in \Lambda} \int_{\mathcal{I}_\beta} \left[\hat{V}(\xi_\lambda(t)) + \hat{V}(\eta_\lambda(t)) \right] dt \Bigg\} \times \\
& \times \bigotimes_{\lambda \in \Lambda} (\chi_\beta \otimes \chi_\beta)(d\xi_\lambda, d\eta_\lambda),
\end{aligned} \tag{7.20}
$$

where (see (7.14), (7.15))

$$Q(\xi_\lambda, \eta_\lambda) = \int_{\mathcal{I}_\beta} W(\xi_\lambda(t)|\eta_\lambda(t))dt$$
$$= \sum_{p=1}^{r-1} \int_{\mathcal{I}_\beta} b_p(\eta_\lambda(t))[\xi_\lambda(t)]^{2p} dt. \tag{7.21}$$

Since, for V obeying the condition **(V2)**, all b_p are nonnegative, all the coefficients $b_p(\eta_\lambda(t))$ are nonnegative for all $\eta_\lambda(t)$. For $\vartheta \in [0,1]$, we set

$$\Xi_{ll'}(\vartheta|\eta_\Lambda, \tau, \tau') \stackrel{\text{def}}{=} < \xi_l(\tau)\xi_{l'}(\tau') >_{\phi_\vartheta(\cdot|\eta_\Lambda)}, \tag{7.22}$$

where the expectation is taken with respect to the measure

$$\phi_\vartheta(d\xi_\Lambda|\eta_\Lambda) = \frac{1}{\Theta(\vartheta|\eta_\Lambda)} \exp\Bigg\{ -\vartheta \sum_{\lambda\in\Lambda} Q(\xi_\lambda, \eta_\lambda) - $$
$$- \frac{1}{2} \sum_{\lambda,\lambda'\in\Lambda} d_{\lambda\lambda'} \langle \xi_\lambda, \xi_{\lambda'}\rangle_\beta -$$
$$- \sum_{\lambda\in\Lambda} \int_{\mathcal{I}_\beta} \hat{V}(\xi_\lambda(t))dt \Bigg\} \bigotimes_{\lambda\in\Lambda} \chi_\beta(d\xi_\lambda), \tag{7.23}$$

where

$$\Theta(\vartheta|\eta_\Lambda) \stackrel{\text{def}}{=} \int_{\mathcal{X}_{\beta,\Lambda}} \exp\Bigg\{ -\vartheta \sum_{\lambda\in\Lambda} Q(\xi_\lambda, \eta_\lambda) \tag{7.24}$$
$$-\frac{1}{2} \sum_{\lambda,\lambda'\in\Lambda} d_{\lambda\lambda'} \langle \xi_\lambda, \xi_{\lambda'}\rangle_\beta - \sum_{\lambda\in\Lambda} \int_{\mathcal{I}_\beta} \hat{V}(\xi_\lambda(t))dt \Bigg\} \bigotimes_{\lambda\in\Lambda} \chi_\beta(d\xi_\lambda).$$

One observes that $\Xi_{ll'}$ is a continuous function of $\vartheta \in [0,1]$. It is differentiable on $(0,1)$, where its derivative is

$$\frac{\partial}{\partial\vartheta} \Xi_{ll'}(\vartheta|\eta_\Lambda, \tau, \tau')$$
$$= -\frac{1}{\Theta(\vartheta|\eta_\Lambda)} \sum_{p=1}^{r-1} \sum_{\lambda\in\Lambda} \int_{\mathcal{I}_\beta} b_p(\eta_\lambda(t)) \left\{ < [\xi_\lambda(t)]^{2p} \, \xi_l(\tau)\xi_{l'}(\tau') >_{\phi_\theta(\cdot|\eta_\Lambda)} \right.$$
$$\left. - < [\xi_\lambda(t)]^{2p} >_{\phi_\theta(\cdot|\eta_\Lambda)} < \xi_l(\tau)\xi_{l'}(\tau') >_{\phi_\theta(\cdot|\eta_\Lambda)} \right\} dt.$$

For every $\eta_\Lambda \in \Omega_{\beta,\Lambda}$, the measure (7.23) has the form (5.1) with $\xi = 0$, thus the GKS inequalities (6.2), (6.3) hold for its moments. This yields

$$\frac{\partial}{\partial\vartheta} \Xi_{ll'}(\vartheta|\eta_\Lambda, \tau, \tau') \leq 0,$$

hence

$$0 \leq \Xi_{ll'}(0|\eta_\Lambda, \tau, \tau') \leq \Xi_{ll'}(1|\eta_\Lambda, \tau, \tau') = \hat{K}^0_{ll'}(\tau, \tau'), \tag{7.25}$$

for all η_Λ, $l, l' \in \Lambda$, $\tau, \tau' \in \mathcal{I}_\beta$. On the other hand, (7.20) may be rewritten

$$\begin{aligned} K^\zeta_{ll'}(\tau, \tau') = [Z_{\beta,\Lambda}(\zeta)]^{-2} \int_{\mathcal{X}_{\beta,\Lambda}} \Xi_{ll'}(1|\eta_\Lambda, \tau, \tau')\Theta(1|\eta_\Lambda) \times \\ \times \exp\left\{ -\sqrt{2} \sum_{\lambda \in \Lambda, \lambda' \in \Lambda^c} d_{\lambda\lambda'} (\eta_\lambda, \zeta_{\lambda'}\rangle_\beta - \frac{1}{2} \sum_{<\lambda,\lambda'>\in\Lambda} d_{\lambda\lambda'}\langle \eta_\lambda, \eta_{\lambda'}\rangle_\beta - \right. \\ \left. - \sum_{\lambda\in\Lambda} \int_{\mathcal{I}_\beta} \hat{V}(\eta_\lambda(t))dt \right\} \bigotimes_{\lambda\in\Lambda} \chi_\beta(d\eta_\lambda). \end{aligned} \tag{7.26}$$

Applying here (7.25) one arrives at (7.18). □

Now let us return to the measures (2.60), (2.61), for which one may write

$$\mu_{\beta,\Lambda}(d\omega_\Lambda) = \frac{Z^{(0)}_{\beta,\Lambda}}{Z_{\beta,\Lambda}} \exp\left\{ -\frac{1}{2} \sum_{l,l'\in\Lambda} \left[d^\Lambda_{ll'} - d_{ll'} \right] \langle \omega_l, \omega_{l'}\rangle_\beta \right\} \mu^{(0)}_{\beta,\Lambda}(d\omega_\Lambda).$$

Taking into account **(D2)** and the GKS inequalities, one easily proves the following statement.

Proposition 7.1. *For every pair $l, l' \in \Lambda$ and all $\tau, \tau' \in \mathcal{I}_\beta$, the following estimate holds*

$$\hat{K}^0_{ll'}(\tau, \tau') \leq \int_{\mathcal{X}_{\beta,\Lambda}} \omega_l(\tau)\omega_{l'}(\tau')\hat{\mu}_{\beta,\Lambda}(d\omega_\Lambda) \stackrel{\text{def}}{=} \hat{K}_{ll'}(\tau, \tau'). \tag{7.27}$$

Combining this estimate with (7.18) one obtains that

$$K^\zeta_{ll'}(\tau, \tau') \leq \hat{K}_{ll'}(\tau, \tau'), \tag{7.28}$$

which holds for arbitrary $\zeta \in \Omega_\beta$, all $\tau, \tau' \in \mathcal{I}_\beta$ and $l, l' \in \Lambda$.

7.3. **Refined Gaussian Upper Bound.** For the periodic local Gibbs measure (2.60), we write

$$K_{ll'}(\tau, \tau') = \int_{\mathcal{X}_{\beta,\Lambda}} \omega_l(\tau)\omega_{l'}(\tau')\mu_{\beta,\Lambda}(d\omega_\Lambda), \tag{7.29}$$

and

$$\begin{aligned} K_\Lambda &= \frac{1}{\beta|\Lambda|} \sum_{l,l'\in\Lambda} \int\int_{\mathcal{I}^2_\beta} K_{ll'}(\tau, \tau')d\tau d\tau' \\ &= \sum_{l'\in\Lambda} \int_{\mathcal{I}_\beta} K_{ll'}(\tau, \tau')d\tau'. \end{aligned} \tag{7.30}$$

For the one-point box $\Lambda = \{l\}$, we write simply K. This parameter depends on β, $\mathfrak{m}$, and the parameters of the potential V. We show that varying these quantities, K can be made arbitrarily small. Set

$$\mathfrak{d} = -\sum_{l' \in \mathbb{L}} d_{ll'}. \tag{7.31}$$

Recall that the limit $\Lambda \nearrow \mathbb{L}$ is taken over a sequence of boxes $\mathcal{L}$.

Theorem 7.4. *For the model described by the Hamiltonian (2.1), (2.2) with $D = 1$ and with the one-particle potential V satisfying* **(V1)***, let K, defined by (7.30) with $\Lambda = \{l\}$, obey the condition*

$$K < 1/\mathfrak{d}. \tag{7.32}$$

Then there exist $\Lambda_0 \in \mathcal{L}$ such that for all $\Lambda > \Lambda_0$, the following estimate

$$K_\Lambda \leq \frac{K}{1 - \mathfrak{d}K}, \tag{7.33}$$

holds.

Proof. For the potential V satisfying **(V1)**, we set

$$\sigma_\beta(d\omega_l) \ \stackrel{\rm def}{=} \ \exp\left\{ - \int_{\mathcal{I}_\beta} V(\omega_l(t))dt \right\} \chi_\beta(d\omega_l), \tag{7.34}$$

where χ_β is the Gaussian measure defined by (2.27). Clearly, σ_β is a finite measure on $\mathcal{X}_\beta$, which belongs to the BFS class (see [39]). For $\vartheta \in [0, 1]$, we set

$$\mu(d\omega_\Lambda) \stackrel{\rm def}{=} \frac{1}{Z_{\beta,\Lambda}(\vartheta)} \exp\left\{ -\frac{\vartheta}{2} \sum_{l,l' \in \Lambda} d^\Lambda_{ll'} \langle \omega_l, \omega_{l'} \rangle_\beta \right\} \bigotimes_{l \in \Lambda} \sigma_\beta(d\omega_l), \tag{7.35}$$

$$Z_{\beta,\Lambda}(\vartheta) \stackrel{\rm def}{=} \int_{\mathcal{X}_{\beta,\Lambda}} \exp\left\{ -\frac{\vartheta}{2} \sum_{l,l' \in \Lambda} d^\Lambda_{ll'} \langle \omega_l, \omega_{l'} \rangle_\beta \right\} \bigotimes_{l \in \Lambda} \sigma_\beta(d\omega_l).$$

This measure has the form (5.1) with $\xi = 0$, thus its moments obey the GKS and the FKG inequalities. Let us set

$$K_{ll'}(\vartheta|\tau, \tau') = < \omega_l(\tau)\omega_{l'}(\tau') >_\mu . \tag{7.36}$$

For every $\vartheta \in [0, 1]$, this is a nonnegative (e.g., by (6.1)) and continuous (by Theorem 4.2) function of τ, τ'. One can easily show that it is also continuous on $[0, 1]$ and differentiable on $(0, 1)$, as a function of ϑ.

Further, set (see (6.9), (6.10))

$$\begin{aligned} U_{l_1,\ldots,l_4}(\vartheta|t_1,\ldots,t_4) =& < \omega_{l_1}(t_1)\ldots\omega_{l_4}(t_4) >_\mu \\ &- < \omega_{l_1}(t_1)\omega_{l_2}(t_2) >_\mu < \omega_{l_3}(t_3)\omega_{l_4}(t_4) >_\mu \\ &- < \omega_{l_1}(t_1)\omega_{l_3}(t_3) >_\mu < \omega_{l_2}(t_2)\omega_{l_4}(t_4) >_\mu \\ &- < \omega_{l_1}(t_1)\omega_{l_4}(t_4) >_\mu < \omega_{l_2}(t_2)\omega_{l_3}(t_3) >_\mu . \end{aligned} \tag{7.37}$$

For V satisfying **(V1)** and for a ferroelectric interaction $\vartheta d_{ll'}^{\Lambda}$, the semi-invariant $U_{l_1,\ldots,l_4}$ satisfies the Lebowitz inequality (6.11). It is also continuous as a function of $t_1,\ldots,t_4$ and ϑ. From (7.35), (7.36) one has

$$\begin{aligned} \frac{\partial}{\partial\vartheta}K_{ll'}(\vartheta|\tau,\tau') = -\frac{1}{2}\sum_{\lambda,\lambda'\in\Lambda} d_{\lambda\lambda'}^{\Lambda}\int_{\mathcal{I}_\beta}\{U_{\lambda,\lambda',l,l'}(\vartheta|t,t,\tau,\tau') \\ + 2K_{\lambda l}(\vartheta|t,\tau)K_{\lambda' l'}(\vartheta|t,\tau')\}\, dt. \end{aligned} \tag{7.38}$$

Setting

$$\begin{aligned} K_\Lambda(\vartheta) &= \frac{1}{\beta|\Lambda|}\sum_{l,l'\in\Lambda}\int\int_{\mathcal{I}_\beta^2} K_{ll'}(\vartheta|\tau,\tau')d\tau d\tau' \\ &= \sum_{l'\in\Lambda}\int_{\mathcal{I}_\beta} K_{ll'}(\vartheta|\tau,\tau')d\tau', \end{aligned} \tag{7.39}$$

we get from (7.38)

$$\frac{d}{d\vartheta}K_\Lambda(\vartheta) = -\Psi(\vartheta) + \mathfrak{d}_\Lambda\left[K_\Lambda(\vartheta)\right]^2. \tag{7.40}$$

Here

$$\mathfrak{d}_\Lambda \stackrel{\mathrm{def}}{=} -\sum_{l'\in\Lambda} d_{ll'}^{\Lambda} \nearrow \mathfrak{d}, \quad \Lambda \nearrow \mathbb{L}, \tag{7.41}$$

and

$$\Psi(\vartheta) \stackrel{\mathrm{def}}{=} \frac{1}{2|\Lambda|\beta}\sum_{l_1,\ldots,l_4\in\Lambda} d_{l_1 l_2}^{\Lambda} U_{l_1,\ldots,l_4}(\vartheta|\tau,\tau',t,t)d\tau d\tau' dt \geq 0, \tag{7.42}$$

for all $\vartheta\in[0,1]$. Where we have taken into account (2.7), (6.11), and **(D2)**. Set

$$R_\Lambda(\vartheta) = \frac{K_\Lambda(0)}{1-\mathfrak{d}_\Lambda K_\Lambda(0)\vartheta}. \tag{7.43}$$

By (7.35), (7.39) $K_\Lambda(0) = K$. In view of (7.32) and (7.41) one has

$$K\mathfrak{d}_\Lambda < 1, \quad \forall\Lambda\in\mathcal{L},$$

which means that, for any Λ, $R_\Lambda(\vartheta)$ is differentiable on $(0,1)$, where

$$\frac{d}{d\vartheta}R_\Lambda(\vartheta) = \mathfrak{d}_\Lambda\left[R_\Lambda(\vartheta)\right]^2. \tag{7.44}$$

Set

$$P_\Lambda(\vartheta) = K_\Lambda(\vartheta) + R_\Lambda(\vartheta) \geq 0, \qquad Q_\Lambda(\vartheta) = K_\Lambda(\vartheta) - R_\Lambda(\vartheta). \quad (7.45)$$

Employing (7.40), (7.44) one has

$$\frac{d}{d\vartheta} Q_\Lambda(\vartheta) = -\Psi(\vartheta) + \mathfrak{d}_\Lambda P_\Lambda(\vartheta) Q_\Lambda(\vartheta), \qquad Q_\Lambda(0) = 0. \quad (7.46)$$

In view of (7.42) one has

$$\frac{d}{d\vartheta} Q_\Lambda(0) \leq 0,$$

which implies

$$Q_\Lambda(\vartheta) \leq 0, \qquad \forall \vartheta \in [0,1]. \quad (7.47)$$

In fact, $Q_\Lambda(\vartheta) \leq 0$ in a right neighborhood of zero. Since the function Q_Λ is continuous, to become positive it should vanish at a point, where its derivative should be positive. But this is impossible in view of (7.46) and (7.42). Then $Q_\Lambda(1) \leq 0$, which yields (7.33). □

Remark 7.2. *The measure (7.35) would be Gaussian if one took V in (7.34) to be identically zero. In this case one would get an equality in (7.35). This is the reason why the latter estimate is called Gaussian upper bound. It is in fact a refined upper bound because K is computed for the non-Gaussian measure (7.34).*

8. Applications

8.1. **Existence of the Long Range Order.** The appearance of the long range order is an effect of the phase transition, which occurs when the fluctuations of the displacements of particles become large. In this subsection, we show this for the model (2.1) – (2.3) with $D = 1$ and

$$d_{ll'} = -J\delta_{|l-l'|,1}; \quad J > 0. \quad (8.1)$$

To describe the appearance of the long range order one introduces *an order parameter*. Here we will use the following one (more on this theme one may find in [37])

$$\Pi(\beta) = \lim_{\Lambda \nearrow \mathbb{L}} \gamma_{\beta,\Lambda} \left\{ \left(\frac{1}{|\Lambda|} \sum_{l \in \Lambda} q_l \right)^2 \right\}, \quad (8.2)$$

where $\gamma_{\beta,\Lambda}$ is the periodic local Gibbs state introduced in (2.10). The value β_* of the inverse temperature β, such that

$$\Pi(\beta) = 0, \quad \text{for} \quad \beta \leq \beta_*, \quad \text{and} \quad \Pi(\beta) > 0, \quad \text{for} \quad \beta > \beta_*,$$

will be called *a critical inverse temperature.*

Theorem 8.1. *For the system of anharmonic oscillators described by the Hamiltonian (2.1) – (2.3) with $D = 1$, $d \geq 3$, and with the polynomial v which is strictly convex on $\mathbb{R}_+$ and such that the polynomial $\xi/2 + v(\xi)$ has a minimum at some $\xi = \xi_0$, there exists $\bar{\mathfrak{m}}$ such that, for $\mathfrak{m} > \bar{\mathfrak{m}}$ there exists a critical inverse temperature.*

Proof. Having in mind the periodic conditions on the boundaries of the box

$$\Lambda = (-L, L]^d \bigcap \mathbb{L}, \quad L \in \mathbb{N},$$

we will use the Fourier transformation of the following form

$$\hat{q}_p = \frac{1}{\sqrt{|\Lambda|}} \sum_{l \in \Lambda} q_l \exp(ipl), \quad p = (p_1, \ldots, p_d) \in \Lambda^*, \tag{8.3}$$

$$\Lambda^* \stackrel{\text{def}}{=} \{p \mid p_j = -\pi + \frac{\pi}{L}\nu_j, \ \nu_j = 1, 2, \ldots, 2L, \ j = 1, \ldots, d\}.$$

Denote

$$D_\Lambda(p) = \int_{\mathcal{I}_\beta} \Gamma^{\beta,\Lambda}_{\hat{q}_{-p},\hat{q}_p}(0, \tau) d\tau. \tag{8.4}$$

Suppose that there exist positive B_p and C_p, independent of Λ and such that

$$D_\Lambda(p) \leq B_p, \quad \gamma_{\beta,\Lambda} \{[\hat{q}_p, [H_\Lambda, \hat{q}_{-p}]]\} \leq C_p, \tag{8.5}$$

where $[.,.]$ stands for the commutator. By means of the estimate obtained in [37], p.363 one gets the following bound

$$\gamma_{\beta,\Lambda} \{\hat{q}_p \hat{q}_{-p}\} \leq \frac{1}{2}\sqrt{B_p C_p} \coth\left(\frac{\beta}{2}\sqrt{\frac{C_p}{B_p}}\right). \tag{8.6}$$

In our case

$$[\hat{q}_p, [H_\Lambda, \hat{q}_{-p}]] = \frac{1}{\mathfrak{m}}.$$

On the other hand, the infrared estimates [37], [41] yield

$$D_\Lambda(p) \leq \frac{1}{J E(p)}, \tag{8.7}$$

where J is taken from (8.1) and

$$E(p) = \sum_{j=1}^{d} [1 - \cos(p_j)].$$

Employing these estimates in (8.6) one obtains

$$\gamma_{\beta,\Lambda}\left\{\hat{q}_p\hat{q}_{-p}\right\} \leq \frac{1}{2}\cdot\frac{1}{\sqrt{2\mathfrak{m}JE(p)}}\coth\left(\beta\sqrt{\frac{JE(p)}{2\mathfrak{m}}}\right). \tag{8.8}$$

By (8.3)

$$\left(\frac{1}{|\Lambda|}\sum_{l\in\Lambda}q_l\right)^2 = \frac{1}{|\Lambda|}\hat{q}_0^2. \tag{8.9}$$

On the other hand,

$$\sum_{l\in\Lambda}q_l^2 = \hat{q}_0^2 + \sum_{p\in\Lambda^*\setminus\{0\}}\hat{q}_p\hat{q}_{-p}, \tag{8.10}$$

which yields

$$\gamma_{\beta,\Lambda}\{\hat{q}_0^2\} = |\Lambda|\gamma_{\beta,\Lambda}\{q_l^2\} - \sum_{p\in\Lambda^*\setminus\{0\}}\gamma_{\beta,\Lambda}\{\hat{q}_p\hat{q}_{-p}\}. \tag{8.11}$$

Here we have taken into account that the periodic Gibbs state $\gamma_{\beta,\Lambda}$ is invariant under translations from $\mathfrak{T}/\mathfrak{T}(\Lambda)$. Then by (8.9)

$$\begin{aligned}\gamma_{\beta,\Lambda}&\left\{\left(\frac{1}{|\Lambda|}\sum_{l\in\Lambda}q_l\right)^2\right\} \qquad (8.12)\\ &= \gamma_{\beta,\Lambda}\{q_l^2\} - \frac{1}{|\Lambda|}\sum_{p\in\Lambda^*\setminus\{0\}}\gamma_{\beta,\Lambda}\{\hat{q}_p\hat{q}_{-p}\}.\end{aligned}$$

Making use of (8.8) and passing to the limit $\Lambda \nearrow I\!\!L$ one obtains the following estimate for the order parameter (8.2)

$$\begin{aligned}\Pi(\beta) \geq \gamma_{\beta,\Lambda}&\{q_l^2\} - \qquad (8.13)\\ &- \frac{1}{2}\cdot\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\frac{1}{\sqrt{2\mathfrak{m}JE(p)}}\coth\left(\beta\sqrt{\frac{JE(p)}{2\mathfrak{m}}}\right)dp.\end{aligned}$$

The latter integral is convergent for $d \geq 3$. Getting back to the representations (2.11), (2.12), (2.22), (2.62), (7.35), and (7.36) one obtains

$$\gamma_{\beta,\Lambda}\{q_l^2\} = K_{ll}(1|0,0),$$

with $d_{ll'}$ given by (8.1). By means of (7.37) one may rewrite (7.38) as follows

$$\begin{aligned}\frac{\partial}{\partial\vartheta}K_{ll'}(\vartheta|\tau,\tau') = -\frac{1}{2}\sum_{\lambda,\lambda'\in\Lambda}d^{\Lambda}_{\lambda\lambda'}\int_{\mathcal{I}_\beta}&\{<\omega_l(\tau)\omega_{l'}(\tau')\omega_\lambda(t)\omega_{\lambda'}(t)>_\mu\\ &- <\omega_l(\tau)\omega_{l'}(\tau')>_\mu<\omega_\lambda(t)\omega_{\lambda'}(t)>_\mu\} \geq 0.\end{aligned}$$

Here, to obtain the latter estimate, we have used the GKS inequality (6.3), which obviously holds for the moments of the measure (7.35). This estimate yields

$$\gamma_{\beta,\Lambda}\{q_l^2\} = K_{ll}(1|0,0) \geq K_{ll}(0|0,0) =< \omega_l(0)^2 >_{\sigma_\beta} \\ = \frac{\text{trace}\,\{q_l^2 \exp(-\beta H_l)\}}{\text{trace}\,\{\exp(-\beta H_l)\}}, \tag{8.14}$$

where the Hamiltonian H_l and the measure σ_β are given by (2.2) and (7.34) respectively. Now, as above, we shall use the spectral properties of the Hamiltonian H_l. Its spectrum consists of nondegenerate eigenvalues ϵ_s, $s \in \mathbb{N}$, $\epsilon_s < \epsilon_{s+1}$, which correspond to the eigenfunctions ψ_s constituting an orthonormal base of the space $L^2(\mathbb{R})$. Setting $q_s^2 = (q_l^2 \psi_s, \psi_s)_{L^2(\mathbb{R})}$, we have

$$K_{ll}(0|0,0) = \left(\sum_{s\in\mathbb{N}} q_s^2 e^{-\beta\epsilon_s}\right) / \left(\sum_{s\in\mathbb{N}} e^{-\beta\epsilon_s}\right).$$

Multiplying numerator and denominator by $e^{\beta\epsilon_1}$ and passing to the limit $\beta \to +\infty$ we get

$$\lim_{\beta\to+\infty} K_{ll}(0|0,0) = \int_{\mathbb{R}} x^2 \psi_1^2(x) dx. \tag{8.15}$$

In [31, 77, 78] there was proven the following semi-classical result. For a double-well potential $V(x) + x^2/2$ possessing nondegenerate minima at the points $\pm x_0$, and for any $\varepsilon > 0$, one has

$$\lim \inf_{\mathfrak{m}\to+\infty} \int_{B_\varepsilon^\pm} \psi_1(x)^2 dx = \frac{1}{2}, \tag{8.16}$$

where $B_\varepsilon^\pm = [\pm x_0 - \varepsilon, \pm x_0 + \varepsilon]$. Therefore, given $\varepsilon > 0$ and any $\delta > 0$, one may find $\mathfrak{m}_{\varepsilon,\delta} > 0$, such that for all $\mathfrak{m} \geq \mathfrak{m}_{\varepsilon,\delta}$,

$$\int_{B_\varepsilon}^{\pm} \psi_1^2(x) dx \geq \frac{1}{2}(1-\delta).$$

Then, for such $\mathfrak{m}$,

$$\int_{\mathbb{R}} x^2 \psi_1^2(x) dx \geq 2 \int_{B_\varepsilon^+} x^2 \psi_1^2(x) dx \\ \geq (x_0 - \varepsilon)^2 \int_{B_\varepsilon^+} \psi_1^2(x) dx \geq (x_0 - \varepsilon)^2 (1-\delta).$$

Suppose that the parameters of the model are such that the following inequality

$$x_0^2 > \frac{1}{\sqrt{8\mathfrak{m}J}} \cdot \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{dp}{\sqrt{E(p)}} \stackrel{\mathrm{def}}{=} \frac{I_d}{\sqrt{8\mathfrak{m}J}}, \tag{8.17}$$

holds. The latter integral converges for $d \geq 3$. Then one may choose positive ε and δ such that

$$(x_0 - \varepsilon)^2(1-\delta) > \frac{I_d}{\sqrt{8\mathfrak{m}J}}. \tag{8.18}$$

Since $\coth x$ is a monotone decreasing function on $\mathbb{R}_+$ and $\coth x > 1$ for all $x > 0$, one may find, taking into account also (8.15), $\beta_0(\varepsilon, \delta) > 0$ such that,

$$K_{ll}(0|0,0) > \\ > \frac{1}{2} \cdot \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{1}{\sqrt{2\mathfrak{m}JE(p)}} \coth\left(\beta\sqrt{\frac{JE(p)}{2\mathfrak{m}}}\right) dp,$$

for all $\beta > \beta_0(\varepsilon, \delta)$. Then by (8.14) and (8.13) one gets

$$\Pi(\beta) > 0,$$

for $\mathfrak{m} \geq \mathfrak{m}(\varepsilon, \delta)$ and $\beta > \beta_0(\varepsilon, \delta)$. Now we fix ε and δ in such a way that $\beta_0(\varepsilon, \delta)$ has its smallest possible value β_*. Then we put $\bar{\mathfrak{m}}$ being the value of $\mathfrak{m}(\varepsilon, \delta)$ at such ε and δ. $\square$

8.2. **Normality of Fluctuations and Suppression of Critical Points.** In this subsection, we consider the model described by (2.1) - (2.3) with arbitrary $D \in \mathbb{N}$ and with the potential V satisfying **(V2)**. At the critical point of the model the strong dependence between the oscillations of particles appears. At this point the fluctuations of the displacement of particles become large (abnormal). More about abnormal fluctuations in such and similar systems one may find in [3], [28], [45], [85]. To describe the fluctuations we introduce *a fluctuation operator*

$$Q_\Lambda \stackrel{\mathrm{def}}{=} \frac{1}{\sqrt{|\Lambda|}} \sum_{l \in \Lambda} q_l, \tag{8.19}$$

where Λ is again a box. If the Green functions (2.62), (2.63), constructed with the help of Q_Λ, remain bounded when $\Lambda \nearrow \mathbb{L}$, the fluctuations may be regarded as normal (under certain additional conditions this implies normality in the usual sense [61]). At the critical point the fluctuations become so large that in order to preserve the mentioned boundedness one should use *an abnormal normalization*, i.e., to describe them one should employ the following operator

$$Q_{\lambda,\Lambda} = \lambda(\Lambda) Q_\Lambda = \frac{\lambda(\Lambda)}{\sqrt{|\Lambda|}} \sum_{l \in \Lambda} q_l, \tag{8.20}$$

where $\{\lambda(\Lambda) \in \mathbb{R} \mid \Lambda \in \mathcal{L}\}$ is a converging to zero sequence and $\mathcal{L}$ is a sequence of boxes exhausting the lattice $\mathbb{L}$. Typically, $\lambda(\Lambda) \sim |\Lambda|^{-\sigma}$, where $\sigma < 1/2$ is *a critical exponent*. For $\beta > \beta_*$, the fluctuations destroy the $O(D)$-symmetry and $\lambda(\Lambda)$ is to be set $|\Lambda|^{-1/2}$ (c.f. (8.2)). In what follows, the above mentioned normality of fluctuations corresponds to the suppression of the critical point behaviour of the model considered.

Definition 8.1. *Given $\beta > 0$, let the sequence of the Green functions*

$$\{\Gamma^{\beta,\Lambda}_{Q^{(\alpha_1)},\dots,Q^{(\alpha_{2n})}}(\tau_1, \dots, \tau_{2n}) \mid \Lambda \in \mathcal{L}\},$$

be bounded uniformly on $\mathcal{I}_\beta$ for all $n \in \mathbb{N}$, any $\alpha_1, \dots \alpha_{2n} = 1, 2, \dots, D$, and any sequence of boxes $\mathcal{L}$. Then the fluctuations of the displacements of particles are said to be normal at this temperature.

Let H_l be the Hamiltonian (2.2) describing a one-dimensional (i.e. $D = 1$) oscillator. Its spectrum consists of the nondegenerate eigenvalues ϵ_s, $s \in \mathbb{N}$. Set

$$\Delta = \min\{\epsilon_{s+1} - \epsilon_s \ : \ s \in \mathbb{N}\}. \tag{8.21}$$

Theorem 8.2. *Let the particle mass $\mathfrak{m}$, the interaction parameter $\mathfrak{d}$ given by (7.31), and the spectral parameter Δ obey the condition*

$$\mathfrak{m}\Delta^2 > \mathfrak{d}. \tag{8.22}$$

Then, for any $D \in \mathbb{N}$, the fluctuations of the displacements of particles remain normal at all temperatures.

The proof of this theorem will be given below. Our next statement shows that the condition (8.22) may be satisfied for small values of the mass $\mathfrak{m}$.

Theorem 8.3. *There exists $\kappa > 0$ such that*

$$\lim_{\mathfrak{m} \searrow 0} \mathfrak{m}^{(r-1)/(r+1)} \left(\mathfrak{m}\Delta^2\right) = \kappa, \tag{8.23}$$

where r is the same as in (2.5).

Proof. Recall that the Hamiltonian H_l acts in the Hilbert space $\mathcal{H}_l = L^2(\mathbb{R})$. Given $\delta > 0$, consider the following unitary operator on $\mathcal{H}_l$

$$(U_\delta \psi)(x) = \delta^{1/2}\psi(\delta x).$$

Then

$$U_\delta \left(\frac{d}{dx}\right) U_\delta^{-1} = \delta^{-1}\left(\frac{d}{dx}\right), \qquad U_\delta q U_\delta^{-1} = \delta q. \tag{8.24}$$

Set $\delta = \mathfrak{m}^{-1/(2r+2)}$. Then the operator

$$H_{\mathfrak{m}} = \mathfrak{m}^{-r/(r+1)} R, \qquad R \stackrel{\text{def}}{=} R_0 + \mathfrak{m}^{1/(r+1)} R_1, \tag{8.25}$$

is unitary equivalent to H_l given by (2.2). Here

$$R_0 = -\frac{1}{2}\left(\frac{d}{dx}\right)^2 + b_r q^{2r},$$

and

$$R_1 = \frac{1}{2}(1+a)\mathfrak{m}^{\frac{r-2}{r+1}} q^2 + \sum_{s=2}^{r-1} \mathfrak{m}^{\frac{r-s-1}{r+1}} b_s q^{2s}.$$

Let Δ_R and Δ_0 be defined by (8.21) but with the eigenvalues of the operators R and R_0 respectively. Then

$$\Delta = \mathfrak{m}^{-\frac{r}{r+1}} \Delta_R. \tag{8.26}$$

One observes that the operator R is a perturbation of R_0, which is analytic with respect to the variable $\lambda = \mathfrak{m}^{1/(r+1)}$ at the point $\lambda = 0$. Thus

$$\lim_{\mathfrak{m} \searrow 0} \Delta_R = \Delta_0.$$

Taking into account (8.26) one gets (8.23). $\square$

Due to the $O(D)$-symmetry of the model the following function (c.f. (7.29))

$$K_{ll'}(\tau, \tau') = \int_{\mathcal{X}_{\beta,\Lambda}} \omega_l^{(\alpha)}(\tau)\omega_{l'}^{(\alpha)}(\tau')\mu_{\beta,\Lambda}(d\omega_\Lambda),$$

does not depend on $\alpha = 1, \ldots, D$. Let K_Λ be defined by (7.30) with the above function $K_{ll'}$. Further, set

$$K_\Lambda(\nu) = \int_{\mathcal{I}_\beta} \Gamma^{\beta,\Lambda}_{Q_\Lambda^{(\alpha)}, Q_\Lambda^{(\alpha)}}(0,\tau)\cos\left(2\pi\nu\tau/\beta\right) d\tau, \quad \nu \in \mathbb{Z}. \tag{8.27}$$

Then $K_\Lambda = K_\Lambda(0)$ and

$$\Gamma^{\beta,\Lambda}_{Q^{(\alpha)}_\Lambda, Q^{(\alpha)}_\Lambda}(\tau,\tau') = \frac{1}{\beta}\sum_{\nu\in\mathbb{Z}} K_\Lambda(\nu)\cos\left[\frac{2\pi\nu}{\beta}(\tau-\tau')\right]. \tag{8.28}$$

Lemma 8.1. *The following estimate*

$$0 \leq K_\Lambda(\nu) \leq \frac{\beta^2}{4\mathfrak{m}\pi^2\nu^2}, \tag{8.29}$$

holds for all $\nu \in \mathbb{Z}\setminus\{0\}$.

Proof. By (2.12), (2.22)

$$\Gamma^{\beta,\Lambda}_{Q^{(\alpha)}_\Lambda, Q^{(\alpha)}_\Lambda}(0,\tau) = \frac{1}{Z_{\beta,\Lambda}}\mathrm{trace}\left\{Q^{(\alpha)}_\Lambda \exp\left[-\tau H_\Lambda\right] Q^{(\alpha)}_\Lambda \exp\left[-(\beta-\tau)H_\Lambda\right]\right\}.$$

The Hamiltonian H_Λ has a discrete spectrum consisting of positive eigenvalues E_s, $s\in\mathbb{N}$ (see (2.17)). We set

$$Q_{ss'} = (Q^{(\alpha)}\Psi_s, \Psi_{s'})_{L^2(\mathbb{R}^{D|\Lambda|})}.$$

This yields in (8.27)

$$\begin{aligned} K_\Lambda(\nu) = \frac{1}{Z_{\beta,\Lambda}}\sum_{s,s'\in\mathbb{N}} Q^2_{ss'}\frac{E_s - E_{s'}}{(E_s-E_{s'})^2 + (2\pi\nu/\beta)^2}\times \\ \times\left[\exp(-\beta E_{s'}) - \exp(-\beta E_s)\right]. \end{aligned} \tag{8.30}$$

Thus $K_\Lambda(\nu)\geq 0$. Further, for $\nu\neq 0$,

$$\begin{aligned} K_\Lambda(\nu) &\leq \frac{\beta^2}{(2\pi\nu)^2 Z_{\beta,\Lambda}}\sum_{s,s'\in\mathbb{N}} Q^2_{ss'}[E_s - E_{s'}]^2\times \\ &\times\left[\exp(-\beta E_{s'}) - \exp(-\beta E_s)\right] \\ &= \frac{\beta^2}{(2\pi\nu)^2}\gamma_{\beta,\Lambda}\left\{\left[Q^{(\alpha)}_\Lambda, \left[H_\Lambda, Q^{(\alpha)}_\Lambda\right]\right]\right\} = \frac{\beta^2}{4\mathfrak{m}\pi^2\nu^2}, \end{aligned} \tag{8.31}$$

where $[\cdot,\cdot]$ stands for commutator. □

As a corollary of (8.29) one gets from (8.28)

$$\Gamma^{\beta,\Lambda}_{Q^{(\alpha)}_\Lambda, Q^{(\alpha)}_\Lambda}(\tau,\tau') \leq \Gamma^{\beta,\Lambda}_{Q^{(\alpha)}_\Lambda, Q^{(\alpha)}_\Lambda}(0,0), \quad \forall \tau,\tau'\in\mathcal{I}_\beta. \tag{8.32}$$

Below we will use the scalar domination estimate (7.1). To this end we compare the D-dimensional model we consider with the corresponding scalar model. Let us set

$$\widetilde{\Gamma}^{\beta,\Lambda}_{2n}(\tau_1,\ldots,\tau_{2n}) = \widetilde{\Gamma}^{\beta,\Lambda}_{\widetilde{Q}_\Lambda,\ldots,\widetilde{Q}_\Lambda}(\tau_1,\ldots,\tau_{2n}), \tag{8.33}$$

where $\widetilde{Q}_\Lambda$ is defined by (8.19) but for the one-dimensional model. For this model, the Gaussian domination inequality (6.12) and the estimate (8.32) imply that the following estimate

$$0 \leq \widetilde{\Gamma}_{2n}^{\beta,\Lambda}(\tau_1, \ldots, \tau_{2n}) \leq \frac{(2n)!}{2^n n!} \left[\widetilde{\Gamma}_2^{\beta,\Lambda}(0,0)\right]^n \tag{8.34}$$

holds for all $n \in \mathbb{N}$. Let $\widetilde{K}_\Lambda$ be defined by (8.27) with $\nu = 0$ and with $\widetilde{\Gamma}$ instead of Γ (i.e., it is K_Λ for the one-dimensional model). As above, $\widetilde{K}$ will stand for a one-point box Λ. Since the estimates (8.29) are valid for all D, they hold also for $\widetilde{K}_\Lambda$. Moreover, the scalar domination inequality (7.1) yields

$$K_\Lambda \leq \widetilde{K}_\Lambda. \tag{8.35}$$

Lemma 8.2. *Let Δ be defined by (8.21). Then*

$$\widetilde{K} \leq \frac{1}{\mathfrak{m}\Delta^2}. \tag{8.36}$$

Proof. By (8.30)

$$\begin{aligned} \widetilde{K} &= \frac{1}{\widetilde{Z}_\beta} \sum_{s,s' \in \mathbb{N}} q_{ss'}^2 \frac{(\epsilon_s - \epsilon_{s'})[e^{-\beta\epsilon_{s'}} - e^{-\beta\epsilon_s}]}{(\epsilon_s - \epsilon_{s'})^2} \\ &\leq \frac{1}{\Delta^2} \cdot \frac{1}{\widetilde{Z}_\beta} \sum_{s,s' \in \mathbb{N}} q_{ss'}^2 (\epsilon_s - \epsilon_{s'})[e^{-\beta\epsilon_{s'}} - e^{-\beta\epsilon_s}] = \frac{1}{\mathfrak{m}\Delta^2}, \end{aligned}$$

where

$$\widetilde{Z}_\beta = \mathrm{trace} \exp[-\beta \widetilde{H}_l],$$

and $\widetilde{H}_l$ is the one-particle Hamiltonian (2.2) for a one-dimensional oscillator. □

Corollary 8.1. *Let (8.22) hold. Then the following estimate*

$$K_\Lambda \leq \widetilde{K}_\Lambda \leq \frac{1}{\mathfrak{m}\Delta^2 - \mathfrak{d}}, \tag{8.37}$$

holds for all β, Λ, and D.

Lemma 8.3. *Let (8.22) hold. Then for every $\beta > 0$, the sequence $\{\widetilde{\Gamma}_2^{\beta,\Lambda}(0,0) \mid \Lambda \in \mathcal{L}\}$ is bounded.*

Proof. By (8.28), (8.33),

$$\widetilde{\Gamma}_2^{\beta,\Lambda}(0,0) = \frac{1}{\beta} \sum_{\nu \in \mathbb{Z}} \widetilde{K}_\Lambda(\nu) = \frac{1}{\beta} \left[\widetilde{K}_\Lambda + 2 \sum_{\nu=1}^{+\infty} \widetilde{K}_\Lambda(\nu) \right],$$

hence by (8.29), which hold also for $D = 1$, and by (8.37)

$$\widetilde{\Gamma}_2^{\beta,\Lambda}(0,0) \leq \frac{1}{\beta}\widetilde{K}_\Lambda + \frac{\beta}{12\mathfrak{m}} \leq \frac{\beta^{-1}}{\mathfrak{m}\Delta^2 - \mathfrak{d}} + \frac{\beta}{12\mathfrak{m}} \stackrel{\text{def}}{=} \Gamma_\beta. \tag{8.38}$$

Thus, the stated property follows from the boundedness of the sequence $\{\widetilde{K}_\Lambda \mid \Lambda \in \mathcal{L}\}$, which in turn follows from (8.37). □

Proof of Theorem 8.2. To estimate the Green functions

$$\Gamma^{\beta,\Lambda}_{Q^{(\alpha_1)},\ldots,Q^{(\alpha_{2n})}}(\tau_1,\ldots,\tau_{2n}), \quad ,\alpha_1,\ldots,\alpha_{2n} = 1,\ldots,D, \tag{8.39}$$

we use the scalar domination inequality (7.1) and the Gaussian upper bound (8.34). We recall that one may apply (7.1) only to the functions with *coinciding* α_j. Let us gather the indices α_j in (8.39) into the groups g_k, $k = 1,2,\ldots,\delta \leq D$ numbered in such a way that $|g_k| \geq |g_{k+1}|$. Set $|g_k| = s_k$, then $s_1 + \cdots + s_\delta = 2n$. Hence

$$\Gamma^{\beta,\Lambda}_{Q^{(\alpha_1)},\ldots,Q^{(\alpha_{2n})}}(\tau_1,\ldots,\tau_{2n}) = < X_1 \ldots X_\delta >_{\mu_{\beta,\Lambda}}, \tag{8.40}$$

where

$$X_k \stackrel{\text{def}}{=} \prod_{j:\alpha_j \in g_k} \left(\frac{1}{\sqrt{|\Lambda|}} \sum_{l\in\Lambda} \omega_l^{(\alpha_j)}(\tau_j) \right). \tag{8.41}$$

Now we apply repeatedly the Schwarz inequality and obtain

$$| < X_1 \ldots X_\delta >_{\mu_{\beta,\Lambda}} | \leq \prod_{k=1}^{\delta-1} \left[< X_k^{2^k} >_{\mu_{\beta,\Lambda}} \right]^{2^{-k}} \cdot \left[< X_\delta^{2^{\delta-1}} >_{\mu_{\beta,\Lambda}} \right]^{2^{-(\delta-1)}} \tag{8.42}$$

The Green function $< X_k^{2^k} >_{\mu_{\beta,\Lambda}}$ contains $Q_\Lambda^{(\alpha)}$ with the same α, thus we may employ the scalar domination inequality (7.1) and the Gaussian upper bound (8.34). This yields

$$< X_k^{2^k} >_{\mu_{\beta,\Lambda}} \leq \Theta_k(s_k) \left[\widetilde{\Gamma}_2^{\beta,\Lambda}(0,0) \right]^{2^{k-1}s_k}, \tag{8.43}$$

$$\Theta_k(s) \stackrel{\text{def}}{=} 1 \cdot 3 \cdot 5 \ldots (2^k s - 1) = (2^k s - 1)!!.$$

In Appendix we show that, for all $n \in \mathbb{Z}_+$, all $D \in \mathbb{N}$, and for all possible combinations of $\alpha_1,\ldots,\alpha_{2n}$, the following estimate holds

$$\frac{1}{(2n)!} \prod_{k=1}^{\delta-1} [\Theta_k(s_k)]^{2^{-k}} [\Theta_\delta(s_\delta)]^{2^{-(\delta-1)}} \leq c_D^n, \tag{8.44}$$

where

$$c_1 = 1, \quad c_D = 2\left[(2^{D-2})!\right]^{2^{2-D}}, \quad D \geq 2. \tag{8.45}$$

Thus

$$\prod_{k=1}^{\delta-1}\left[< X_k^{2^k} >_{\mu_{\beta,\Lambda}}\right]^{2^{-k}} \cdot \left[< X_\delta^{2^{\delta-1}} >_{\mu_{\beta,\Lambda}}\right]^{2^{-(\delta-1)}} \leq (2n)!\left[c_D\Gamma_\beta\right]^n, \quad (8.46)$$

where we have taken into account (8.38). Applying this estimate in (8.42) one gets the boundedness to be proven. □

8.3. **Uniqueness of Gibbs States.** In this subsection we again consider the scalar version of the model (2.1) - (2.5) with the potential V obeying **(V2)**. As it has been proved in [14] (see also [11] and the references therein), the class of tempered Gibbs measures $\mathcal{G}_\beta$ (see Definition 2.2), for this model, is actually nonempty. Moreover, by Theorem 8.1, the model has a critical point, which implies that, for one and the same value of the model parameters, $\mathcal{G}_\beta$ contains more than one element. The suppression of the critical points, proved above, implies in turn that one may have uniqueness of tempered Gibbs measures for small values of $\mathfrak{m}$. In fact, we prove this in the current subsection.

Theorem 8.4. *For the model with the Hamiltonian described by (2.1) - (2.5) with the potential obeying* **(V2)***, for every β, there exists a positive $\mathfrak{m}_* = \mathfrak{m}_*(\beta)$ such that for all values of the mass $\mathfrak{m} \in (0, \mathfrak{m}_*)$, the class of tempered Gibbs measures $\mathcal{G}_\beta$ consists of exactly one element, that is $|\mathcal{G}_\beta|$.*

To prove this theorem we need to create corresponding tools, which is done just below. Let $(\mathcal{X}, \rho)$ be a complete separable metric space and $\mathcal{B}(\mathcal{X})$ be the Borel algebra of its subsets. Let also $\mathcal{M}$ be the set of all probability measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, and

$$\mathcal{M}_1 \stackrel{\rm def}{=} \left\{\mu \in \mathcal{M} \mid \int_{\mathcal{X}} \rho(y, y_0)\mu(dy) < \infty\right\}, \quad (8.47)$$

for some $y_0 \in \mathcal{X}$. Further, $\mathrm{Lip}(\mathcal{X})$ will stand for the set of Lipschitz functions $f : \mathcal{X} \to \mathbb{R}$, for which we write

$$[f]_{\rm Lip} = \sup\left\{\frac{|f(x) - f(y)|}{\rho(x, y)} \ : \ x, y \in \mathcal{X}, \quad x \neq y\right\}, \quad (8.48)$$

$$\mathrm{Lip}_1(\mathcal{X}) = \{f \in \mathrm{Lip}(\mathcal{X}) \mid [f]_{\rm Lip} \leq 1\}. \quad (8.49)$$

Given $\mu_1, \mu_2 \in \mathcal{M}_1$, we set

$$R(\mu_1, \mu_2) \quad (8.50)$$
$$\stackrel{\rm def}{=} \sup\left\{\left|\int_{\mathcal{X}} f(x)\mu_1(dx) - \int_{\mathcal{X}} f(x)\mu_2(dx)\right| \ : \ f \in \mathrm{Lip}_1(\mathcal{X}\right\}.$$

A key role in the proof of Theorem 8.4 will be played by Dobrushin's matrix. It is defined by the conditional Gibbs measures $\mu_{\beta,\Lambda}(\cdot|\zeta)$, given by (2.65) - (2.68), with $\zeta \in \Omega^t_\beta$ and a one-point box $\Lambda = \{l\}$. To simplify notations we set

$$\zeta_{\{l\}^c} = \zeta_l^c, \qquad \mu_{\beta,\{l\}}(\cdot|\zeta) = \mu_l(\cdot|\zeta). \tag{8.51}$$

Then the elements of Dobrushin's matrix $(C_{ll'})_{l,l'\in\mathbb{L}}$ are

$$C_{ll'} = \sup\left\{ \frac{R\left(\mu_l(\cdot|\xi), \mu_l(\cdot|\eta)\right)}{\|\xi_{l'} - \eta_{l'}\|_\beta} \ : \ \xi^c_{l'} = \eta^c_{l'} \right\}. \tag{8.52}$$

They will be used to check Dobrushin's condition [33], [34], [42], [57].

Proposition 8.1. [Dobrushin's Uniqueness Condition] *Let*

$$\sup\left\{ \sum_{l'\in\mathbb{L}\setminus\{l\}} C_{ll'} \ : \ l \in \mathbb{L} \right\} < 1. \tag{8.53}$$

Then there exists exactly one tempered Gibbs measure.

Taking into account **(D2)** one has from (2.65) - (2.68)

$$\mu_l(d\omega|\xi) = \frac{1}{Z_l(\xi)} \exp\left\{ \langle \omega, \varphi_l(\xi)\rangle_\beta - \int_{\mathcal{I}_\beta} V(\omega(t))dt \right\} \chi_\beta(d\omega), \tag{8.54}$$

where

$$\varphi_l(\xi) = -\sum_{\lambda\in\{l\}^c} d_{l\lambda}\xi_\lambda, \tag{8.55}$$

and $Z_l(\xi)$ is the normalization constant. Given $x \in \mathcal{X}_\beta$, set

$$\mu^x(d\omega) = \frac{1}{Z_x} \exp\left\{ \langle \omega, x\rangle_\beta - \int_{\mathcal{I}_\beta} V(\omega(t))dt \right\} \chi_\beta(d\omega), \tag{8.56}$$

and

$$C \stackrel{\rm def}{=} \sup\left\{ \frac{R(\mu^x, \mu^y)}{\|x - y\|_\beta} \ : \ x, y \in \mathcal{X}_\beta, \ x \neq y \right\}. \tag{8.57}$$

For $\xi \neq \eta$, such that $\xi^c_{l'} = \eta^c_{l'}$, one has

$$\|\varphi_l(\xi) - \varphi_l(\eta)\|_\beta = -d_{ll'}\|\xi_{l'} - \eta_{l'}\|_\beta.$$

Then by (8.52), (8.54) - (8.57)

$$\sup\left\{ \sum_{l'\in\mathbb{L}\setminus\{l\}} C_{ll'} \ : \ l \in \mathbb{L} \right\} \leq \mathfrak{d}C, \tag{8.58}$$

where $\mathfrak{d}$ was defined by (7.36). Then the condition (8.53) would be satisfied if

$$C < \frac{1}{\mathfrak{d}}. \tag{8.59}$$

Having this in mind let us estimate $R(\mu^x, \mu^y)$. To this end we will estimate the variance of the following function

$$\mathcal{X}_\beta \ni x \mapsto < f >_{\mu^x} = \int_{\mathcal{X}_\beta} f(\omega)\mu^x(d\omega) \in \mathbb{R}, \tag{8.60}$$

with a fixed $f \in \mathrm{Lip}_1(\mathcal{X}_\beta)$. This function is Fréchet differentiable [11], its derivative on a certain $\psi \in \mathcal{X}_\beta$ has the following form

$$\begin{aligned} \langle \nabla_x < f >_{\mu^x}, \psi\rangle_\beta \ &= < fg >_{\mu^x} \ - \ < f >_{\mu^x} < g >_{\mu^x} \\ = \mathrm{Cov}_{\mu^x}(f, g), \qquad & g(\omega) \ \stackrel{\mathrm{def}}{=} \ \langle \omega, \psi\rangle_\beta. \end{aligned} \tag{8.61}$$

By the Schwarz inequality one has

$$|\langle \nabla_x < f >_{\mu^x}, \psi\rangle_\beta| \leq \sqrt{\mathrm{Var}_{\mu^x} f} \cdot \sqrt{\mathrm{Var}_{\mu^x} g}, \tag{8.62}$$

where

$$\mathrm{Var}_{\mu^x} f = \frac{1}{2}\int_{\mathcal{X}_\beta}\int_{\mathcal{X}_\beta} [f(\omega) - f(\omega')]^2 \mu^x(d\omega)\mu^x(d\omega'), \tag{8.63}$$

$$\mathrm{Var}_{\mu^x} g = \frac{1}{2}\int_{\mathcal{X}_\beta}\int_{\mathcal{X}_\beta} \langle \omega - \omega', \psi\rangle_\beta^2 \mu^x(d\omega)\mu^x(d\omega'). \tag{8.64}$$

The idea how to prove Theorem 8.4 may be outlined as follows. Suppose that we have estimated, uniformly for all $x \in \mathcal{X}_\beta$, the first variance by a positive continuous function of β, of the parameters of the potential V (2.3), (2.5), and of the mass $\mathfrak{m}$. Let also the second variance be bounded by a positive function of the same parameters multiplied by $\|\psi\|_\beta$. Then the mean-value theorem together with (8.57) would imply that the condition (8.59) be satisfied provided the product of the mentioned bounds is sufficiently small. Below we shall implement this idea.

One observes that (8.64) defines a quadratic form on $\mathcal{X}_\beta$

$$\mathrm{Var}_{\mu^x} g = \langle T^x \psi, \psi\rangle_\beta,$$

with the operator T^x given as follows

$$(T^x\psi)(\tau) = \int_{\mathcal{I}_\beta} T^x(\tau, \tau')\psi(\tau')d\tau', \quad \tau \in \mathcal{I}_\beta. \tag{8.65}$$

The kernel of this integral operator is

$$T^x(\tau,\tau') = \frac{1}{2}\int_{\mathcal{X}_\beta}\int_{\mathcal{X}_\beta}[\omega(\tau)-\omega'(\tau)]\cdot[\omega(\tau')-\omega'(\tau')]\mu^x(d\omega)\mu^x(d\omega'). \tag{8.66}$$

Comparing this kernel with the function given by (7.16) with $\zeta \in \Omega^t_\beta$, and taking into account (8.56) and (2.66) - (2.68), one concludes that

$$T^x(\tau,\tau') = K^\zeta_{ll}(\tau,\tau'), \qquad x = -\sum_{l'\in\mathbb{L}} d_{ll'}\zeta_{l'}. \tag{8.67}$$

This yields, in particular, that $T^x(\tau,\tau')$ is a continuous nonnegative function of $\tau,\tau' \in \mathcal{I}_\beta$ (see Theorem 4.2). Clearly, for every $x \in \mathcal{X}_\beta$, the operator T^x is symmetric and positive. Moreover,

$$\mathrm{trace}(T^x) = \frac{1}{2}\int_{\mathcal{X}_\beta}\int_{\mathcal{X}_\beta}\|\omega-\omega'\|^2_\beta\mu^x(d\omega)\mu^x(d\omega') < \infty, \tag{8.68}$$

which follows from (2.37). Let $\hat{K} : \mathcal{X}_\beta \to \mathcal{X}_\beta$ stand for the integral operator with the kernel $\hat{K}(\tau,\tau')$ defined by (7.17) with $l' = l$. Then this operator is also positive and trace class, its trace may be computed as above with the help of the measure $\hat{\mu}^{(0)}_{\beta,\{l\}}$.

For a bounded linear operator $A : \mathcal{X}_\beta \to \mathcal{X}_\beta$, let $\sigma(A)$ be its pure point spectrum and $\|A\|$ stand for its operator norm. For a positive compact operator, one has

$$\|A\| = \max\sigma(A). \tag{8.69}$$

On the other hand, for such an operator (see e.g. [69], p.216)

$$\|A\| = \sup\left\{\frac{\langle A\psi,\psi\rangle_\beta}{\|\psi\|^2_\beta} \ : \ \psi \in \mathcal{X}_\beta \setminus \{0\}\right\}. \tag{8.70}$$

The construction of the above mentioned bounds is based upon the following lemmas, which will be proved at the end of this subsection.

Lemma 8.4. *For every $x,\psi \in \mathcal{X}_\beta$, one has*

$$\langle T^x\psi,\psi\rangle_\beta \leq \|\hat{K}\|\|\psi\|_\beta. \tag{8.71}$$

Lemma 8.5. *For every $x \in \mathcal{X}_\beta$, one has*

$$\mathrm{trace}(T^x) \leq \mathrm{trace}(\hat{K}). \tag{8.72}$$

Lemma 8.6. *The following estimate holds*

$$\max\sigma(\hat{K}) \leq \frac{1}{\mathfrak{m}\hat{\Delta}^2}, \tag{8.73}$$

where $\hat{\Delta}$ is defined by (8.21) but with the eigenvalues of the one-particle Hamiltonian (2.2) with the potential $\hat{V}$ (7.13) instead of V (2.3), (2.5).

Lemma 8.7. *Let r in (2.5) be set $r = 2$. Then, for all $x \in \mathcal{X}_\beta$ and any $f \in \mathrm{Lip}_1(\mathcal{X}_\beta)$, one has*

$$\mathrm{Var}_{\mu^x} f \ \leq \ h e^{\beta\delta_0}, \qquad \delta_0 = \frac{25}{288}\left(\frac{a}{\sqrt{b_2}}\right)^2, \tag{8.74}$$

where the constant h depends only on the interaction parameter $\mathfrak{d}$.

The proof of the above statement may be done by means of the logarithmic Sobolev inequality, just as it was done in [11]. Another estimate of the variance of f is linear in β. We will use it for $r > 2$.

Lemma 8.8. *There exists a parameter h_0, independent of $\mathfrak{m}$ and β, such that the estimate*

$$\mathrm{Var}_{\mu^x} f \ \leq \ \beta h_0 \mathfrak{m}^{-1/(r+1)}, \tag{8.75}$$

holds for all $x \in \mathcal{X}_\beta$ and any $f \in \mathrm{Lip}_1(\mathcal{X}_\beta)$.

Proof of Theorem 8.4. First we estimate $\mathrm{Var}_{\mu^x} g$ given by (8.64). By Lemma 8.4, (8.69), and Lemma 8.6 one has

$$\mathrm{Var}_{\mu^x} g = \langle T^x \psi, \psi\rangle_\beta \leq \|\hat{K}\| \|\psi\|^2_\beta = \max \sigma(\hat{K}) \|\psi\|^2_\beta \leq \frac{\|\psi\|^2_\beta}{\mathfrak{m}\hat{\Delta}^2}.$$

By Theorem 8.3, one may find $\mathfrak{m}_0$ and κ_0 such that, for $\mathfrak{m} \in (0, \mathfrak{m}_0)$, the following estimate holds

$$\frac{1}{\mathfrak{m}\hat{\Delta}^2} \leq \kappa_0 \mathfrak{m}^{(r-1)/(r+1)}. \tag{8.76}$$

Then one has

$$\mathrm{Var}_{\mu^x} g \ \leq \ \kappa_0 \mathfrak{m}^{(r-1)/(r+1)} \|\psi\|^2_\beta, \tag{8.77}$$

that holds for $\mathfrak{m} \in (0, \mathfrak{m}_0)$. For $r > 2$, one may use (8.75), which yields the following estimate of the distance (8.50)

$$R(\mu^x, \mu^y) \leq \|x - y\|_\beta \sqrt{\beta h_0 \kappa_0} \cdot \mathfrak{m}^{\frac{r-2}{2(r+1)}},$$

holding for all $\mathfrak{m} \in (0, \mathfrak{m}_0)$. Employing this estimate in (8.57) one obtains that the uniqueness condition (8.59) holds true if

$$\mathfrak{m} < \mathfrak{m}_*(\beta) = \min\left\{\mathfrak{m}_0, \left[\beta h_0 \kappa_0 \mathfrak{d}^2\right]^{-\frac{r+1}{r-2}}\right\}. \tag{8.78}$$

For $r = 2$, we use (8.74) and obtain

$$R(\mu^x, \mu^y) \leq \|x - y\|_\beta \sqrt{\beta h \kappa_0} e^{\beta\delta_0/2} \mathfrak{m}^{1/6},$$

which yields that the uniqueness condition holds in this case if

$$\mathfrak{m} < \mathfrak{m}_*(\beta) = \min\left\{\mathfrak{m}_0, \frac{e^{-3\beta\delta_0}}{[h\kappa_0\mathfrak{d}^2]^3}\right\}. \tag{8.79}$$

□

Proof of Lemma 8.4. Here we use (8.67) and the zero boundary domination estimate (7.18). Then taking into account that the kernel $T^x(\tau,\tau')$ is nonnegative one obtains

$$\langle T^x\psi,\psi\rangle_\beta = |\langle T^x\psi,\psi\rangle_\beta| \leq \int_{\mathcal{I}_\beta}\int_{\mathcal{I}_\beta} T^x(\tau,\tau')|\psi(\tau)||\psi(\tau')|d\tau d\tau'$$

$$\leq \int_{\mathcal{I}_\beta}\int_{\mathcal{I}_\beta} \hat{K}(\tau,\tau')|\psi(\tau)||\psi(\tau')|d\tau d\tau' \leq \|\hat{K}\||\psi||_\beta^2 = \|\hat{K}\|\|\psi\|_\beta^2.$$

□

The proof of Lemma 8.5 immediately follows from the estimate (7.18).

Proof of Lemma 8.4. By (8.27), (8.30), and (8.69) one has

$$\max\sigma(\hat{K}) = \|\hat{K}\| = \int_{\mathcal{I}_\beta}\hat{K}(0,\tau)d\tau$$

$$= \frac{1}{\hat{Z}_\beta}\sum_{s,s'\in\mathbb{N}} q_{ss'}^2(\hat{\epsilon}_s - \hat{\epsilon}_{s'})\frac{e^{-\beta\hat{\epsilon}_{s'}} - e^{-\beta\hat{\epsilon}_s}}{(\hat{\epsilon}_s - \hat{\epsilon}_{s'})^2}$$

$$\leq \frac{1}{\hat{\Delta}^2}\cdot\frac{1}{\hat{Z}_\beta}\sum_{s,s'\in\mathbb{N}} q_{ss'}^2(\hat{\epsilon}_s - \hat{\epsilon}_{s'})\left\{e^{-\beta\hat{\epsilon}_{s'}} - e^{-\beta\hat{\epsilon}_s}\right\} = \frac{1}{\mathfrak{m}\hat{\Delta}^2},$$

□

Proof of Lemma 8.8. For a Lipschitz function f, one obtains by means of (8.48) , (8.49), (8.68), and Lemma 8.5

$$\mathrm{Var}_{\mu^x} f \leq \int_{\mathcal{X}_\beta}\int_{\mathcal{X}_\beta} \|\omega - \omega'\|_\beta \mu^x(d\omega)\mu^x(d\omega') = \mathrm{trace}(T^x)$$

$$\leq \mathrm{trace}(\hat{K}) = \int_{\mathcal{I}_\beta}\hat{K}(\tau,\tau) = \beta\hat{K}(0,0). \tag{8.80}$$

Further, as in (8.31) one has

$$\hat{K}(0,0) = \frac{1}{\hat{Z}_\beta}\mathrm{trace}\left[q^2\exp\{-\beta\hat{H}_l\}\right] \stackrel{\mathrm{def}}{=} < q^2 > .$$

It turns out that $\max\sigma(\hat{K})$ may be expressed in terms of the Duhamel two-point functions [37] and hence estimated from below as follows

$$\beta < q^2 > f\left(\frac{\beta}{4\mathfrak{m} < q^2 >}\right) \leq \max\sigma(\hat{K}), \tag{8.81}$$

where the function $f : (0, +\infty) \to (0, +\infty)$ was introduced and estimated in [37]. It has the following bound

$$\frac{1}{t}(1 - e^{-t}) \leq f(t). \tag{8.82}$$

Then by (8.73) and (8.76) one gets in (8.81)

$$< q^2 > \leq \frac{1}{2}\sqrt{\kappa_0}\mathfrak{m}^{-1/(r+1)},$$

that holds for $\mathfrak{m} \in (0, \mathfrak{m}_0)$. Applying this estimate in (8.80) one obtains (8.75). □

9. Appendix

Here we prove the estimate (8.44). For $s \in \mathbb{N}$ and $\mathbb{N} \ni k \geq 2$, we write

$$\Theta_k(s) = (2s-1)!! \prod_{k=2}^{2^{k-1}} \Upsilon_{2k}(s) \leq \tag{9.1}$$
$$\leq 2^s s! 2^{s(2^{k-1}-1)} \left[\left(2^{k-1}\right)!\right]^s s^{s(2^{k-1}-1)},$$

where

$$\Upsilon_{2k}(s) \stackrel{\text{def}}{=} \frac{(2ks-1)!!}{(2(k-1)s-1)!!} \leq (2ks)^s.$$

Let M_n stand for the left-hand side of (8.44). One may write

$$M_n = K_{1,0} \cdot K_{2,0} \dots K_{\delta-1,0} \cdot L_0, \tag{9.2}$$

where

$$K_{k,0} = [\Theta_k(s_k)/(2n)!]^{2^{-k}}, \quad L_0 = [\Theta_\delta(s_\delta)/(2n)!]^{2^{-(\delta-1)}}. \tag{9.3}$$

From now on we fix n. For $\mathbb{N} \ni s < 2n$, we write

$$[s] \stackrel{\text{def}}{=} (s+1)\dots(2n) = \frac{(2n)!}{s!}. \tag{9.4}$$

Then one has

$$K_{1,0} = [\Theta_1(s_1)/(2n)!]^{1/2} = \left[\frac{(2s_1-1)!!}{2^{s_1}s_1!} \cdot \frac{2^{s_1}}{[s_1]}\right]^{1/2} \leq 2^{s_1/2}[s_1]^{-1/2}.$$

Applying this estimate in (9.2) we obtain

$$M_n \leq 2^{s_1/2} K_{2,1} \cdot K_{3,1} \dots K_{\delta-1,1} L_1, \tag{9.5}$$

where

$$K_{k,1} \stackrel{\text{def}}{=} K_{k,0} \cdot [s_1]^{-2^{-k}}, \quad L_1 \stackrel{\text{def}}{=} L_0 \cdot [s_1]^{-2^{-(\delta-1)}}. \tag{9.6}$$

Let us estimate $K_{2,1}$ as follows

$$K_{2,1} = \left[\frac{(2S_2-1)!!}{2^{s_2}(s_2)!}\cdot 2^{s_2}\cdot \frac{\Upsilon_4(s_2)}{[s_1][s_2]}\right]^{1/4} \le 2^{s_2/2}(2!)^{s_2/4}\left[\frac{(s_2)^{s_2}}{[s_1][s_2]}\right]^{1/4}. \tag{9.7}$$

All multipliers in the products $[s_1]$, $[s_2]$ are greater than s_2 (we recall that $s_k \ge s_{k+1}$ for all $k = 1,2,\ldots,\delta$, and $s_1 + s_2 + \cdots + s_\delta = 2n$). Therefore, one may find the numbers $\sigma_2 > s_1$, $\sigma_3 > s_2$, both less than $2n$, such that $\sigma_2 \ge \sigma_3$ and $\sigma_2 + \sigma_3 = s_1 + 2s_2$. Then one gets

$$\frac{(s_2)^{s_2}}{[s_1][s_2]} = \frac{(s_2)^{s_2}}{(s_1+1)\ldots\sigma_2\cdot(s_2+1)\ldots\sigma_3}\cdot\frac{1}{[\sigma_2][\sigma_3]} \le \frac{1}{[\sigma_2][\sigma_3]}.$$

Here we have taken onto account that the number of multipliers in the product $(s_1+1)\ldots\sigma_2\cdot(s_2+1)\ldots\sigma_3$ is $\sigma_2 - s_1 + \sigma_3 - s_2 = s_2$ and that every such multiplier is greater than s_2. This yields in (9.7)

$$K_{2,1} \le 2^{s_2/2}(2!)^{s_2/4}\{[\sigma_2]\cdot[\sigma_3]\}^{-1/4}.$$

Applying this estimate in (9.5) we get

$$M_n \le 2^{(s_1+s_2)/2}(2!)^{s_2/4}K_{3,2}K_{4,2}\ldots K_{\delta-1,2}L_2, \tag{9.8}$$

where we have set $\sigma_1 = s_1$ and

$$K_{k,2} = K_{k,0}\{[\sigma_1]\cdot[\sigma_2]\cdot[\sigma_3]\}^{-2^{-k}}, \quad L_2 = L_0\{[\sigma_1]\cdot[\sigma_2]\cdot[\sigma_3]\}^{-2^{-(\delta-1)}}.$$

Proceeding in this way one obtains

$$M_n \le 2^{(s_1+\cdots+s_k)/2}(2!)^{s_2/4}(4!)^{s_3/8}\ldots[(2^{k-1})!]^{s_k/2^k}\times \times K_{k+1,k}\ldots K_{\delta-1,k}L_k, \tag{9.9}$$

where $k = 2,3,\ldots,\delta-1$ and for $j = 2,3,\ldots,k+1$,

$$K_{j,k} = K_{j,0}\{[\sigma_1][\sigma_2]\ldots[\sigma_{2^k-1}]\}^{-2^{-j}},$$
$$L_k = L_0\{[\sigma_1][\sigma_2]\ldots[\sigma_{2^k-1}]\}^{-2^{-(\delta-1)}},$$

$$\sigma_{2^{l-1}} + \sigma_{2^{l-1}+1} + \cdots + \sigma_{2^l-1} = 2^{l-1}s_l + 2^{l-2}(s_{l-1}+\cdots+s_1),$$
$$\sigma_1 + \sigma_2 + \cdots + \sigma_{2^l-1} = 2^{l-1}(s_l + s_{l-1} + \cdots + s_1)$$
$$\sigma_{l+1} \le \sigma_l < 2n.$$

Finally, we arrive at

$$M_n \le 2^{(s_1+\ldots s_\delta)/2}(2!)^{s_2/4}(4!)^{s_3/8}\ldots[(2^{\delta-1})!]^{(s_{\delta-1}+s_\delta)/2^{\delta-1}}$$
$$2^n(2!)^{s_2/4}(4!)^{s_3/8}\ldots[(2^{\delta-1})!]^{(s_{\delta-1}+s_\delta)/2^{\delta-1}}.$$

Taking into account that $[(2^j)!]^{s_j/2^j} \leq [(2^{j+1})!]^{s_{j+1}/2^{j+1}}$, $j \in \mathbb{N}$, and that $\delta \leq D$, one obtains (8.44).

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References

[1] S. Albeverio, R. Høegh–Krohn, Homogeneous Random Fields and Quantum Statistical Mechanics,*J. Funct. Anal.,* **19** 242–279 (1975).

[2] S. Albeverio, R. Høegh–Krohn, *Mathematical Theory of Feynman Path Integrals,* Lecture Notes in Math. 523, Springer, Berlin, 1976.

[3] S. Albeverio, A. Daletskii, Yu. Kondratiev, M. Röckner, Fluctuations and Their Glauber Dynamics in Lattice Systems, *J. Func. Anal.* **166**, 148–167 (1999)

[4] S. Albeverio, Yu. Kondratiev, Yu. Kozitsky, Quantum Hierarchical Model, *Methods of Funct. Anal. Topology,* **2**, 1–35 (1996).

[5] S. Albeverio, Yu. Kondratiev, Yu. Kozitsky, Absence of Critical Points for a Class of Quantum Hierarchical Models, *Commun. Math. Phys.,* **187**, 1–18 (1997).

[6] S. Albeverio, Yu. Kondratiev, Yu. Kozitsky, Suppression of Critical Fluctuations by Strong Quantum Effects in Quantum Lattice Systems, *Commun. Math. Phys.,* **194**, 493-512 (1998).

[7] S. Albeverio, Yu. Kondratiev, Yu. Kozitsky, Classical Limits of Euclidean Gibbs States for Quantum Lattice Models, *Letters in Math. Phys.,* **48**, 221-233 (1999).

[8] S. Albeverio, Yu. Kondratiev, Yu. Kozitsky, M. Röckner, Uniqueness for Gibbs Measures of Quantum Lattices in Small Mass Regime, *Ann. Inst. H. Poincaré,* **37**, 43-69 (2001)

[9] S. Albeverio, Yu. G. Kondratiev, R.A. Minlos, A.L. Rebenko, Small Mass Behaviour of Quantum Gibbs States for Lattice Models with Unbounded Spins, *J. Stat. Phys.,* **92**, 1153-1172 (1998)

[10] S. Albeverio, Yu. G. Kondratiev, R.A. Minlos, G.V. Shchepan'uk, Uniqueness Problem for Quantum Lattice Systems with Compact Spins, *Letters in Math. Phys.,* **52**, 185-195 (2000)

[11] S. Albeverio, Yu. Kondratiev, M.Röckner, T.V. Tsikalenko, Uniqueness of Gibbs States for Quantum Lattice Systems, *Probab. Theory Relat. Fields,* **108**, 193-218 (1997).

[12] S. Albeverio, Yu. Kondratiev, M. Röckner, T.V. Tsikalenko, Uniqueness of Gibbs States on Loop Lattices, *C.R. Acad. Sci. Paris, Probabilités/Probability Theory,* **342**, Série 1, 1401–1406 (1997)

[13] S. Albeverio, Yu. Kondratiev, M. Röckner, T.V. Tsikalenko, Dobrushin's Uniqueness for Quantum Lattice Systems with Nonlocal Interaction, *Commun. Math. Phys.,* **189**, 621-630 (1997)

[14] S. Albeverio, Yu. Kondratiev, M.Röckner, T.V. Tsikalenko, Glauber Dynamics for Quantum Lattice Systems, *Rev. Math. Phys.,* (to appear)

[15] N. Angelescu, A. Verbeure, V.A. Zagrebnov, Quantum n-vector Anharmonic Crystal. I. $1/n$-expansion, *Comm. Math. Phys.,* **205**, 81–95 (1999).

[16] V.S. Barbulyak, Yu.G. Kondratiev, Functional Integrals and Quantum Lattice Systems: I. Existence of Gibbs States, *Rep. Nat. Acad. Sci of Ukraine,* No 9, 38-40 (1991).

[17] V.S. Barbulyak, Yu.G. Kondratiev, Functional Integrals and Quantum Lattice Systems: II. Periodic Gibbs States, *Rep. Nat. Acad. Sci of Ukraine,* No 8, 31-34 (1991).

[18] V.S. Barbulyak, Yu.G. Kondratiev, Functional Integrals and Quantum Lattice Systems: III. Phase Transitions, *Rep. Nat. Acad.Sci of Ukraine,* No 10, 19–21 (1991).

[19] V.S. Barbulyak, Yu.G. Kondratiev, The Quasiclassical Limit for the Schrödinger Operator and Phase Transitions in Quantum Statistical Physics, *Func. Anal. Appl.,* **26(2)**, 61–64 (1992).

[20] V.S. Barbulyak, Yu.G. Kondratiev, A Criterion for the Existence of Periodic Gibbs States of Quantum Lattice Systems, *Selecta Math. (N.S.),* **12**, 25–35 (1993).

[21] M.T. Barlow, M. Yor, Semi-martingale Inequalities via the Garsia-Rodemich-Rumsey Lemma, and Applications to Local Times, *J. Func. Anal.,* **49**, 198–229 (1982).

[22] Yu.M. Berezansky, Yu.G. Kondratiev, *Spectral Methods in Infinite Dimensional Analysis,* Kluwer Academic Publishers, Dordrecht Boston London, 1994.

[23] F.A. Berezin, M.A. Shubin, *The Schrödinger Equation,* Kluwer Academic Publishers, Dordrecht Boston London, 1991.

[24] P. Billingsley, *Convergence of Probability Measures,* John Wiley & Sons, Inc., New York, 1968

[25] T. Bodineau, B. Helffer, The *log*-Sobolev Inequality for Unbounded Spin Systems, *J. Func. Anal.* **166**, 168–178 (1999)

[26] O. Bratteli, D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics,* I, II, Springer, New York, 1981.

[27] J. Bricmont, The Gaussian Inequality for Multicomponent Rotators, *J. Stat. Phys.* **17**, 289–300 (1997)

[28] M. Broidoi, B. Momont, A. Verbeure, Lie Algebra of Anomalously Scaled Fluctuations, *J. Math. Phys.* **36**, 6746–6757 (1995)

[29] A.D. Bruce, R.A. Cowley, *Structural Phase Transitions,* Taylor and Francis Ltd., 1981.

[30] Ph. Choquard, *The Anharmonic Crystal,* W.A. Benjamin, New York, 1967

[31] J.M. Combes, P. Duclos, R. Seiler, Krein's Formula and One-Dimensional Multiple Well, *J. Funct. Anal.,* **52**, 257–301 (1983)

[32] J.-D. Deuschel, D.W. Strook, *Large Deviations,* Academic Press, Inc., London, 1989.

[33] R.L. Dobrushin, Prescribing a System of Random Variables by Conditional Distributions, *Theory Prob. Appl.* **15**, 458–486 (1970).

[34] R.L. Dobrushin, S.B. Shlosman, "Constructive Criterion for the Uniqueness of Gibbs field", pp. 347–370 in *Statistical Physics and Dynamical Systems. Rigorous Results,* Birkhaeuser, Basel, 1985.

[35] W. Driessler, L. Landau, J.F.Perez, Estimates of Critical Lengths and Critical Temperatures for Classical and Quantum Lattice Systems, *J. Stat. Phys.* **20**, 123-162 (1979).

[36] F. Dunlop, Correlation Inequalities for Multicomponent Rotators, *Comm. Math. Phys.* **49**, 247–256 (1976)

[37] F.J. Dyson, E.H. Lieb, B. Simon, Phase transitions in quantum spin systems with isotropic and nonisotropic interactions, *J. Stat. Phys.* **18**, 335–383, (1978).

[38] M. Fannes, A. Verbeure, Correlation Inequalities and Equilibrium States, *Comm. Math. Phys.* **55**, 125–131 (1977)

[39] R. Fernandez, J. Fröhlich, A. Sokal, *Random Walks, Critical Phenomena and Triviality in Quantum Field Theory,* Springer, Berlin - Heidelberg - New York - Paris - Tokyo - Hong Kong, 1992.

[40] J.K. Freericks, Mark Jarrell, G.D. Mahan, The Anharmonic Electron-Phonon Problem, *Phys. Rev. Lett.,* **77**, 4588–4591 (1996).

[41] J. Fröhlich, B. Simon, T. Spencer, Infrared Bounds, Phase Transitions and Continuous Symmetry Breaking, *Commun. Math. Phys.* **50**, 79-85 (1976).

[42] H.O. Georgii, *Gibbs Measures and Phase Transitions.* Vol 9, Walter de Gruyter, Springer, Berlin New York, 1988.

[43] S.A. Globa, A Class of Quantum Lattice Models and their Gibbs States, *Ukrainian Math.J.* **40**, 787–792 (1988).

[44] S.A. Globa, Yu.G. Kondratiev, The Construction of Gibbs States of Quantum Lattice Systems, *Selecta Math. Sov.,* **9**, 297–307 (1990).

[45] D. Goderis, A. Verbeure, P. Vets, Dynamics of Fluctuations for Quantum Lattice Systems, *Comm. Math. Phys.* **128**, 533–549 (1990)

[46] F. Guerra, L. Rosen, B.Simon, Boundary Conditions for the $P(\varphi)_2$ Euclidean Field Theory, *Ann. Inst. H.Poincaré* **15**, 231-234 (1976).

[47] B. Helffer, Splitting in Large Dimensions and Infrared Estimates. II. Moment Inequalities, *J. Math. Phys.* **39**, 760–776 (1998)

[48] R. Høegh–Krohn, Relativistic Quanum Statistical Mechanics in Two-dimensional Space-time, *Comm. Math. Phys.* **38**, 195–224 (1974)

[49] A. Inoue, *Tomita-Takesaki Theory in Algebras of Unbounded Operators.* Lecture Notes in Math. 1699, Springer-Verlag, Berlin Heidelberg New York, 1998.

[50] A. Inoue, A Survey of Tomita-Takesaki Theory in Algebras of Unbounded Operators. II. Physical Applications, *Fukuoka Univ. Sci. Rep.* **30**, 49–66 (2000).

[51] A. Klein, L. Landau, Stochastic Processes Associated with KMS States, *J. Funct. Anal.* **42**, 368–428 (1981).

[52] Ju.G. Kondratiev, "Phase Transitions in Quantum Models of Ferroelectrics", pp. 465–475 in *Stochastic Processes, Physics, and Geometry* II, World Scientific, Singapore New Jersey, 1994.

[53] Yu. Kozitsky, Quantum Effects in a Lattice Model of Anharmonic Vector Oscillators, *Lett. Math. Phys.* **51**, 71–81 (2000).

[54] Yu. Kozitsky, Scalar Domination and Normal Fluctuation in N-Vector Quantum Anharmonic Crystals, *Lett. Math. Phys.* **53**, 289–303 (2000).

[55] Yu. Kozitsky, Quantum Effects in Lattice Models of Vector Anharmonic Oscillators, in: *Stochastic Processes, Physics and Geometry: New Interplays, II (Leipzig, 1999)*, 403–411, CMS Conf. Proc., 29, *Amer. Math. Soc., Providence, RI, 2000.*

[56] Yu. Kozitsky, Gibbs States of a Lattice Systems of Quantum Anharmonic Oscillators, in: *Noncommutative Structures in Mathematics and Physics,* NATO ASI Series (eds. S. Duplij and J. Wess) (to appear).

[57] H. Künsch, Decay of Correlations under Dobrushin's Uniqueness Condition and its Applications, *Commun. Math. Phys.* **84**, 207-222 (1982)

[58] V.A. Malyshev, R.A. Minlos, *Linear Infinite Particle Operators,* AMS, 1995.

[59] R.A. Minlos, A. Verbeure, V.A. Zagrebnov, A Quantum Crystal Model in the Light-Mass Limit: Gibbs States, *Reviews in Math. Phys.* **12**, 981–1032 (2000)

[60] E. Nelson, Feynman Integrals and the Schrödinger Equation, *J. Math. Phys.* **5**, 332–343 (1964)

[61] C.M. Newman, Normal Fluctuations and the FKG inequalities, *Comm. Math. Phys.* **74**, 119–128 (1980).

[62] E. Olivieri, P.Picco, Yu.M. Suhov, On the Gibbs States for One-Dimensional Lattice Boson Systems with a Long-Range Interaction, *J. Stat. Phys.* **70**, 985-1028 (1993)

[63] Y.M. Park, H.H Yoo, A Characterization of Gibbs States of Lattice Boson Systems, *J. Stat. Phys.* **75**, 215–239 (1994).

[64] Y.M. Park, H.H Yoo, Uniqueness and Clustering Properties of Gibbs States for Classical and Quantum Unbounded Spin Systems, *J. Stat. Phys.* **80**, 223–271 (1995)

[65] K.R. Parthasarathy, *Probability Measures on Metric Spaces,* Academic Press, New York, 1967.

[66] L.A. Pastur. B.A. Khoruzhenko, Phase Transitions in Quantum Models of Rotators and Ferroelectrics, *Theoret. Math.Phys.* **73**, 111– 124 (1987).

[67] R.T. Powers, Self-Adjoint Algebras of Unbounded Operators, *Comm. Math. Phys.* **21**, 85–124 (1971)

[68] A. Procacci, B. Scoppola, On decay of Correlations for Unbounded Spin Systems with Arbitrary Boundary Conditions, Preprint IAMP 00-462 (2000) (http://www.ma.utexas.edu/mp_arc/)

[69] M. Reed, B. Simon, *Methods of Modern Mathematical Physics. I. Functional Analysis,* Academic Press: New York, London, 1972

[70] T. Schneider, H. Beck, E. Stoll, Quantum Effects in an n-component Vector Model for Structural Phase Transitions, *Phys. Rev.* **B13**, 1123–1130 (1976).

[71] G.L. Sewell, Unbounded Local Observables in Quantum Statistical Mechanics, *J. Math. Phys.* **11**, 1868–1884 (1970)

[72] G.L. Sewell, *Quantum Theory of Collective Phenomena,* Clarendon Press, Oxford, 1986

[73] B.V. Shabat, *Introduction to Complex Analysis. II: Functions of Several Variables,* Nauka, Moscow, 1985 (in russian)

[74] S. Shlosman, Signs of the Ising Model Ursell Functions, *Commun. Math. Phys.* **102**, 679-686, (1986).

[75] B.Simon, *The $P(\varphi)_2$ Euclidean (Quantum) Field Theory,* Princeton Univ. Press, Princeton, 1974.

[76] B. Simon, *Functional Integration and Quantum Physics,* Academic Press, New York San Francisco London, 1979.

[77] B.Simon, Instantons, Double Wells and Large Deviations, *Bulletin of the AMS (New Series)* **8**, 323–326 (1983).

[78] B.Simon, Semiclassical Analysis of Low Lying Eigenvalues, II. Tunneling, *Ann. Math.* **120**, 89–118 (1984).

[79] B.Simon, Schrödinger Operators in the Twentieth Century, Preprint IAMP 00-77 (2000) (http://www.ma.utexas.edu/mp_arc/)

[80] S. Stamenković, Unified Model Description of Order-Disorder and Displacive Structural Phase Transitions, *Condensed Matter Physics (Lviv),* **1(14)**, 257-309 (1998).

[81] I.V. Stasyuk, Local Anharmonic Effects in High-T_c Superconductors. Pseudospin-Electron Model, *Condensed Matter Physics (Lviv),* **2(19)**, 435-446 (1999)

[82] I.V. Stasyuk, Approximate Analitical Dynamical Mean-Field Approach to Strongly Correlated Electron Systems, *Condensed Matter Physics (Lviv),* **3(22)**, 437-456 (2000)

[83] G.Sylvester, Inequalities for Continuous-Spin Ising Ferrromagnets, *J. Stat. Phys.* **15**, 327-341 (1976)

[84] J.E. Tibballs, R.J. Nelmes, G.J. McIntyre, The Crystal Structure of Tetragonal KH_2 PO_4 and KD_2 PO_4 as a Function of Temperature and Pressure, *J. Phys. C: Solid State Phys.,* **15**, 37–58 (1982).

[85] A. Verbeure, V.A. Zagrebnov, Phase Transitions and Algebra of Fluctuation Operators in Exactly Soluble Model of a Quantum Anharmonic Crystal, *J. Stat. Phys.* **69**, 37-55 (1992).

[86] A. Verbeure, V.A. Zagrebnov, No–Go Theorem for Quantum Structural Phase Transition, *J.Phys.A: Math.Gen.* **28**, 5415–5421 (1995).

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