A TWO PARAMETERS MODEL OF RELATIVISTIC POINT INTERACTIONS IN ONE DIMENSION

J.Shabani[1]
UNESCO Regional Office for Education in Africa
B.P 3311 Dakar, Senegal
E-mail: j.shabani@unesco.org

and

A.Vyabandi
Universität Bielefeld. Fakultät fur Physik.BIBOS
Universitätsstrasse 25.D-33615 Bielefeld. Germany
and
Institut de Mathématiques et des Sciences Physiques
Université Nationale du Bénin
B.P 613 Porto-Novo, Bénin
E-mail: vyabandi@hotmail.com

Abstract

We introduce and study a new 2-parameters model of relativistic point interactions in one dimension formally given by

$$D_{\underline{\underline{\alpha}},y} = D + \underline{\underline{\alpha}}\delta(x-y); x \in \mathbb{R}, \quad y > 0$$

where D is the free Dirac Hamiltonian and $\underline{\underline{\alpha}}$ is a 2×2 matrix given by

$$\underline{\underline{\alpha}} = \begin{pmatrix} \alpha & 0 \\ 0 & \tilde{\alpha} \end{pmatrix}, \quad \alpha, \ \tilde{\alpha} \in \mathbb{R}$$

$D_{\underline{\underline{\alpha}},y}$ provides a generalisation of two models of relativistic point interactions considered in [lett. Math. Phys$\underline{13}$, 345-358 (1987)].

We define $D_{\underline{\underline{\alpha}},y}$ using the theory of self adjoint extensions of symmetric closed operators in Hilbert spaces, derive its resolvent equation, analyze its spectral properties and discuss scattering theory for the pair $(D_{\underline{\underline{\alpha}},y}, D)$. We also study the nonrelativistic limit corresponding to $D_{\underline{\underline{\alpha}},y}$ which provides a new 2-parameters model of nonrelativistic point interactions in one dimension.

1 Introduction

Relativistic point interactions in one dimension have been discussed for a long time in various areas of physics, in particular in connection with the Kronig-Penney type models and Saxon-Hutner conjecture(see e.g[1-9] and references therein).

The first rigorous mathematical formulation of these interactions was given in [9] using the theory of self adjoint extensions of symmetric closed operators in Hilbert spaces.

Indeed [9] defines two models $D_{\alpha,y}$ and $T_{\beta,y}$ of relativistic point interactions which provide natural generalisation of nonrelativistic one dimensional δ-interactions of the first and the second type [10].

This paper consider a 2- parameters model $D_{\underline{\underline{\alpha}},y}$of relativistic point interactions in one dimension formally given by:

$$D_{\underline{\underline{\alpha}},y} = D + \underline{\underline{\alpha}}\delta(x-y); x \in \mathbb{R}, y > 0$$

[1]On leave of absence from University of Burundi, Faculty of Science, BP 2700 Bujumbura, Burundi

where D is the free Dirac Hamiltonian and $\underline{\underline{\alpha}}$ is a 2×2 matrix of the form

$$\underline{\underline{\alpha}} = \begin{pmatrix} \alpha & 0 \\ 0 & \tilde{\alpha} \end{pmatrix}, \quad \alpha,\ \tilde{\alpha} \in \mathbb{R}.$$

To the best of our knowledge, this model is new. It provides a straightforward generalisation of the models $D_{\alpha,y}$ and $T_{\beta,y}$ discussed in [9] which correspond to the special cases $\alpha \neq 0$, $\tilde{\alpha} = 0$ and $\alpha = 0$, $\tilde{\alpha} = -c^2\beta \neq 0$ respectively.

The paper is organized as follows. In Section 2, we define the quantum Hamiltonian $D_{\underline{\underline{\alpha}},y}$ following the strategy used in [11, 12] in the case of relativistic δ-sphere interactions.We also derive the resolvent equation of $D_{\underline{\underline{\alpha}},y}$ analyse its spectral properties and carry out a systematic study of the scattering theory for the pair $(D_{\underline{\underline{\alpha}},y}, D)$.

The nonrelativistic limit corresponding to $D_{\underline{\underline{\alpha}},y}$ defines a new 2-parameters model $\Delta_{\alpha,\beta,y}$ of nonrelativistic point interactions in one dimension.

Section 3 is devoted to the study of $\Delta_{\alpha,\beta,y}$.

In forthcoming paper[13] we generalize the results of sections 2 and 3 to finitely and infinitely many relativistic point interactions as well as random interactions.

2 THE RELATIVISTIC POINT INTERACTION

A. Definition of the Hamiltonian

The quantum Hamiltonian describing a relativistic point interaction is formally given by

$$H = D + \underline{\underline{\alpha}}\delta(x-y); x \in \mathbb{R}, \quad y > 0 \tag{1}$$

where $\underline{\underline{\alpha}}$ is a 2×2 matrix of the form

$$\underline{\underline{\alpha}} = \begin{pmatrix} \alpha & 0 \\ 0 & \tilde{\alpha} \end{pmatrix}, \quad \alpha,\ \tilde{\alpha} \in \mathbb{R} \tag{2}$$

and the one-dimension free Dirac operator in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}) \bigotimes \mathbb{C}^2$ is defined by[9]

$$\begin{aligned} D &= -\mathrm{i}c\frac{d}{dx} \bigotimes \sigma_1 + \left(\frac{c^2}{2}\right) \bigotimes \sigma_3 \\ &= \begin{pmatrix} \frac{c^2}{2} & -\mathrm{i}c\frac{d}{dx} \\ -\mathrm{i}c\frac{d}{dx} & -\frac{c^2}{2} \end{pmatrix} \\ \mathcal{D}(D) &= H^{2,1}(\mathbb{R}) \bigotimes \mathbb{C}^2 \end{aligned} \tag{3}$$

where

(i) $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are Pauli matrices in $\mathbb{C}^2$

(ii) c is the velocity of light

(iii) $H^{m,n}(\Omega)$ is the Sobolev space of indices(m, n).

We consider the symmetric closed operator $\dot{D}_y$ defined by

$$\begin{aligned} \dot{D}_y &= D, \\ \mathcal{D}(\dot{D}_y) &= \left\{ g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in H^{2,1}(\mathbb{R}) \bigotimes \mathbb{C}^2 | g(y\pm) = 0 \right\}. \end{aligned} \tag{4}$$

The adjoint $\dot{D}_y^*$ of $\dot{D}_y$ reads

$$\dot{D}_y^* = D,$$

$$\mathcal{D}(\dot{D}_y^*) \quad = \quad \left\{ g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in H^{2,1}(\mathbb{R}) \bigotimes \mathbb{C}^2 | g \in \mathrm{AC}_{loc}(\mathbb{R} - \{y\}) \right\}. \tag{5}$$

$\mathrm{AC}_{loc}(\Omega)$ denotes the set of locally absolutely continuous functions on Ω.

A straightforward computation shows that the equation

$$\dot{D}_y^* g(z) = z g(z), \;\; g \in \mathcal{D}(\dot{D}_y^*), \;\; z \in \mathbb{C} - \left\{ (-\infty, -\frac{c^2}{2}] \bigcup [\frac{c^2}{2}, \infty) \right\} \tag{6}$$

has the solutions

$$g^{(1)}(z,x) = \left\{ \begin{array}{ll} \binom{\mathrm{e}^{ik'(x-y)}}{\mathrm{e}^{ik'(x-y)}} & x > y \\ \binom{0}{0} & x < y \end{array} \right., \;\; g^{(2)}(z,x) = \left\{ \begin{array}{ll} \binom{0}{0} & x < y \\ \binom{\mathrm{e}^{ik'(y-x)}}{\mathrm{e}^{ik'(y-x)}} & x < y \end{array} \right., \mathrm{Im}k' > 0 \tag{7}$$

where

$$k' = \frac{1}{c}\sqrt{z^2 - \frac{c^4}{4}} \equiv k'(z). \tag{8}$$

Thus $\dot{D}_y$ has deficiency indices (2,2) and hence it has a four-parameter family of self-adjoint extensions. Let us now construct the self-adjoint extension corresponding to the free Dirac operator with the potentiel

$$V(x) = \underline{\underline{\alpha}}\delta(x-y), \;\; \underline{\underline{\alpha}} = \begin{pmatrix} \alpha & 0 \\ 0 & \tilde{\alpha} \end{pmatrix}, \;\; \alpha, \tilde{\alpha} \in \mathbb{R}. \tag{9}$$

assume that g satisfies the equation

$$\begin{array}{l} [D + \underline{\underline{\alpha}}\delta(x-y)]g = zg, \\ D = \begin{pmatrix} \frac{c^2}{2} & -\mathrm{i}c\frac{d}{dx} \\ -\mathrm{i}c\frac{d}{dx} & -\frac{c^2}{2} \end{pmatrix}, \;\; \underline{\underline{\alpha}} = \begin{pmatrix} \alpha & 0 \\ 0 & \tilde{\alpha} \end{pmatrix}, \;\; \alpha, \tilde{\alpha} \in \mathbb{R} \end{array} \tag{10}$$

and the limits $g(y\pm)$ exist. Integrating Eq (10) over $(y-\epsilon, y+\epsilon)$ and taking the limit $\epsilon \to 0$ we get

$$\left(1 + i\frac{\tilde{\tau}_0\underline{\underline{\alpha}}}{2c}\right) g(y+) - \left(1 - i\frac{\tilde{\tau}_0\underline{\underline{\alpha}}}{2c}\right) g(y-) = 0 \tag{11}$$

where

$$\tilde{\tau}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{12}$$

As indicated in [11], the boundary conditions in (11) defines a self-adjoint operator of $\dot{D}_y$ iff $\underline{\underline{\alpha}} = \underline{\underline{\alpha}}^+$.

Consider in $L^2(\mathbb{R}) \bigotimes \mathbb{C}^2$ the operator $D_{\underline{\underline{\alpha}},y}$ defined by

$$\begin{array}{rcl} D_{\underline{\underline{\alpha}},y} & = & \begin{pmatrix} \frac{c^2}{2} & -ic\frac{d}{dx} \\ -ic\frac{d}{dx} & -\frac{c^2}{2} \end{pmatrix} \\ \mathcal{D}(D_{\underline{\underline{\alpha}},y}) & = & \left\{ g \in \mathcal{D}(\dot{D}_y^*) \middle| \left(1 + i\frac{\tilde{\tau}_0\underline{\underline{\alpha}}}{2c}\right) g(y+) - \left(1 - i\frac{\tilde{\tau}_0\underline{\underline{\alpha}}}{2c}\right) g(y-) = 0 \right\}. \end{array} \tag{13}$$

According to [11], the operator $D_{\underline{\underline{\alpha}},y}$ provides the mathematical definition of the formal expression (1).

The case $\underline{\underline{\alpha}} = 0$ ($i.e$ $\alpha = \tilde{\alpha} = 0$) in Eq (13) yields the free Dirac Hamiltonian $D_{0,y} \equiv D$.

The case $\alpha \neq 0$, $\tilde{\alpha} = 0$ in Eq (13) yields the Hamiltonian $D_{\alpha,y}$ which describes the relativistic δ-point interaction of the first type centered at $y \in \mathbb{R}$ defined by[9].

$$D_{\alpha,y} \quad = \quad D,$$

$$\begin{aligned}\mathcal{D}(D_{\alpha,y}) &= \{g \in H^{2,1}(\mathbb{R}-\{y\})\bigotimes\mathbb{C}^2 | g_2 \in \mathrm{AC}_{loc}(\mathbb{R}), g_1 \in \mathrm{AC}_{loc}(\mathbb{R}-\{y\}); \\ & g_2(y+) - g_2(y-) = -(i\alpha/c)g_1(y)\}, \quad -\infty < \alpha \le \infty. \end{aligned} \tag{14}$$

The case $\alpha = 0$, $\tilde{\alpha} = -c^2\beta \neq 0$ in Eq (13) yields the Hamiltonian $T_{\beta,y}$ which describes the relativistic δ-point interaction of the second type centered at $y \in \mathbb{R}$ defined by[9].

$$\begin{aligned} T_{\beta,y} &= D, \\ \mathcal{D}(T_{\beta,y}) &= \{g \in H^{2,1}(\mathbb{R}-\{y\})\bigotimes\mathbb{C}^2 | g_2 \in \mathrm{AC}_{loc}(\mathbb{R}-\{y\}), g_1 \in \mathrm{AC}_{loc}(\mathbb{R}); \\ & g_1(y+) - g_1(y-) = i\beta c g_2(y)\}, \quad -\infty < \beta \le \infty. \end{aligned} \tag{15}$$

Following [11], we note that all the results corresponding to $D_{\underline{\underline{\alpha}},y}$ could be generalised to the model $D_{\hat{\alpha},y}$ formally given by

$$H = D + \hat{\alpha}\delta(x-y); x \in \mathbb{R}, y > 0 \tag{16}$$

where $\hat{\alpha}$ is a non diagonal 2×2 matrix with $\hat{\alpha} = \hat{\alpha}^+$.

B. The resolvent equation

From the Krein resolvent formula [14] and after a straightforward computation(see, e.g., [10]), we obtain

$$\begin{aligned}(D_{\underline{\underline{\alpha}},y} - z)^{-1} &= (D-z)^{-1} - \frac{1}{(2c+i\alpha\zeta)(2c+i\tilde{\alpha}\zeta^{-1})}\{\alpha(\overline{\tilde{f}_{k'}(.-y)},.)f_{k'}(.-y) + \\ &+ \tilde{\alpha}(\overline{\tilde{g}_{k'}(.-y)},.)g_{k'}(.-y) + i\frac{\alpha\tilde{\alpha}}{2c}(\overline{\tilde{f}_{k'}(.-y)},.)\hat{f}_{k'}(.-y) + \\ &+ i\frac{\alpha\tilde{\alpha}}{2c}(\overline{\tilde{g}_{k'}(.-y)},.)\hat{g}_{k'}(.-y)\},\ z \in \rho(D_{\underline{\underline{\alpha}},y}),\ \mathrm{Im}k' > 0 \end{aligned} \tag{17}$$

where $R_{k'} = (D-z)^{-1}$, $z \in \mathbb{C} - \left\{(-\infty, -\frac{c^2}{2}]\bigcup[\frac{c^2}{2},\infty)\right\}$ is the free Dirac resolvent with integral kernel[9]

$$\begin{aligned} R_{k'}(x-x') &= \frac{i}{2c}\begin{pmatrix} \zeta & \mathrm{sgn}(x-x') \\ \mathrm{sgn}(x-x') & \zeta^{-1}\end{pmatrix} \mathrm{e}^{ik'|x-x'|}, \\ \zeta &= [z + \frac{c^2}{2}]/k'(z), \mathrm{Im}k'(z) \ge 0, z \in \mathbb{C} \end{aligned} \tag{18}$$

and

$$\begin{aligned} f_{k'}(x) &= \begin{pmatrix} \zeta \\ \mathrm{sgn}(x)\end{pmatrix}\mathrm{e}^{ik'|x|},\ \tilde{f}_{k'}(x) = \begin{pmatrix} -\zeta \\ \mathrm{sgn}(x)\end{pmatrix}\mathrm{e}^{ik'|x|}, \\ g_{k'}(x) &= \begin{pmatrix} \mathrm{sgn}(x) \\ \zeta^{-1}\end{pmatrix}\mathrm{e}^{ik'|x|},\ \tilde{g}_{k'}(x) = \begin{pmatrix} \mathrm{sgn}(x) \\ -\zeta^{-1}\end{pmatrix}\mathrm{e}^{ik'|x|}, \\ \hat{f}_{k'}(x) &= \begin{pmatrix} 1 \\ \mathrm{sgn(x)}\zeta^{-1}\end{pmatrix}\mathrm{e}^{ik'|x|},\ \hat{g}_{k'}(x) = \begin{pmatrix} \mathrm{sgn}(x)\zeta \\ 1\end{pmatrix}\mathrm{e}^{ik'|x|}, \\ & z \in \mathbb{C} - \left\{(-\infty, -\frac{c^2}{2}] \cup [\frac{c^2}{2},\infty)\right\},\ \mathrm{Im}k' > 0. \end{aligned} \tag{19}$$

Remark 1 *:* From Eq (17), a straightforward computation shows
(i) As $\tilde{\alpha} \to 0$, the Hamiltonian $D_{\underline{\underline{\alpha}},y}$ converge in the norm resolvent sense to $D_{\alpha,y}$

$$n.\lim_{\tilde{\alpha}\to 0}(D_{\underline{\underline{\alpha}},y} - z)^{-1} = (D_{\alpha,y} - z)^{-1},\ \ z \in \rho(D_{\underline{\underline{\alpha}},y}) \cap \rho(D_{\alpha,y}) \tag{20}$$

where[9]

$$(D_{\alpha,y} - z)^{-1} = (D-z)^{-1} - \frac{\alpha}{2c(2c+i\alpha\zeta)}(\overline{\tilde{f}_{k'}(.-y)},.)f_{k'}(.-y),$$

$$z \in \rho(D_{\alpha,y}), \ \ \mathrm{Im}k' > 0 \tag{21}$$

(ii) Let $\tilde{\alpha} = -\beta c^2,\ \beta \in \mathbb{R}$. Then as $\alpha \to 0$, the Hamiltonian $D_{\underline{\underline{\alpha}},y}$ converge in the norm resolvent sense to $T_{\beta,y}$

$$n.\lim_{\alpha\to 0}(D_{\underline{\underline{\alpha}},y} - z)^{-1} = (T_{\beta,y} - z)^{-1}, \ \ z \in \rho(D_{\underline{\underline{\alpha}},y}) \cap \rho(T_{\beta,y}) \tag{22}$$

where[9]

$$\begin{aligned}(T_{\beta,y} - z)^{-1} \ &= \ (D-z)^{-1} + \frac{\beta}{2(2 - i\beta c\zeta^{-1})}(\overline{\tilde{g}_{k'}(.-y)},.)g_{k'}(.-y), \\ & \qquad z \in \rho(T_{\beta,y}), \ \ \mathrm{Im}k' > 0.\end{aligned} \tag{23}$$

The following theorem gives the additional information on the domain of $D_{\underline{\underline{\alpha}},y}$.

Theorem 2.1 *:* The domain $\mathcal{D}(D_{\underline{\underline{\alpha}},y}),\ -\infty < \alpha, \tilde{\alpha} \le \infty,\ \ y \in \mathbb{R}$, consists of all elements $\psi_{\underline{\underline{\alpha}}}$ of the type

$$\begin{aligned}\psi_{\underline{\underline{\alpha}}}(x) \ &= \ \phi_{k'}(x) - \frac{2ic}{(2c+i\alpha\zeta)\,(2c+i\tilde{\alpha}\zeta^{-1})}\{\alpha\phi_{k',1}(y)f_{k'}(x-y) + \tilde{\alpha}\phi_{k',2}(y)g_{k'}(x-y) + \\ & \quad + i\frac{\alpha\tilde{\alpha}}{2c}\phi_{k',1}(y)\hat{f}_{k'}(x-y) + i\frac{\alpha\tilde{\alpha}}{2c}\phi_{k',2}(y)\hat{g}_{k'}(x-y)\}, \ \ x \ne y.\end{aligned} \tag{24}$$

where $\phi_{k'} = \binom{\phi_{k',1}}{\phi_{k',2}} \in \mathcal{D}(D) = H^{2,1}(\mathbb{R}) \bigotimes \mathbb{C}^2$ and Im $k' > 0$. The decomposition (24) is unique and with $\psi_{\underline{\underline{\alpha}}}$ of this form we obtain

$$(D_{\underline{\underline{\alpha}},y} - z)\psi_{\underline{\underline{\alpha}}} = (D-z)\phi_{k'}. \tag{25}$$

Let $\psi_{\underline{\underline{\alpha}}} \in \mathcal{D}(D_{\underline{\underline{\alpha}},y})$ and assume that $\psi_{\underline{\underline{\alpha}}} = 0$ in an open set $\vartheta \in \mathbb{R}$. Then $D_{\underline{\underline{\alpha}},y}\psi_{\underline{\underline{\alpha}}} = 0$ in ϑ, i.e., $D_{\underline{\underline{\alpha}},y}$ describes a local interaction.
Proof. The following relation

$$\begin{aligned}\mathcal{D}(D_{\underline{\underline{\alpha}},y}) \ &= \ (D_{\underline{\underline{\alpha}},y} - z)^{-1}(D-z)\mathcal{D}(D) \\ &= \ \{R_{k'} - \frac{1}{(2c+i\alpha\zeta)(2c+i\tilde{\alpha}\zeta^{-1})}[\alpha(\tilde{f}_{k'}(.-y),.)f_{k'}(.-y) + \\ & \quad + \tilde{\alpha}(\tilde{g}_{k'}(.-y),.)g_{k'}(.-y) + i\frac{\alpha\tilde{\alpha}}{2c}(\tilde{f}_{k'}(.-y),.)\hat{f}_{k'}(.-y) + \\ & \quad + i\frac{\alpha\tilde{\alpha}}{2c}(\tilde{g}_{k'}(.-y),.)\hat{g}_{k'}(.-y)]\}(D-z)\mathcal{D}(D), \ \ z \in \rho(D_{\underline{\underline{\alpha}},y}), \ \ \mathrm{Im}k' > 0\end{aligned} \tag{26}$$

proves (24).
Next let $\psi_{\underline{\underline{\alpha}}} = 0$, then

$$\begin{aligned}\phi_{k'}(x) \ &= \ \frac{2ic}{(2c+i\alpha)(2c+i\tilde{\alpha}\zeta^{-1})}\{\alpha\phi_{k',1}(y)f_{k'}(x-y) + \tilde{\alpha}\phi_{k',2}(y)g_{k'}(x-y) + \\ & \quad + i\frac{\alpha\tilde{\alpha}}{2c}\phi_{k',1}(y)\hat{f}_{k'}(x-y) + i\frac{\alpha\tilde{\alpha}}{2c}\phi_{k',2}(y)\hat{g}_{k'}(x-y)\}\end{aligned} \tag{27}$$

and $\phi_{k'} \in C^0(\mathbb{R})$, implies $\phi_{k'} = 0$ which prove the uniqueness of (24). Relation (25) follows from

$$\begin{aligned}&(D_{\underline{\underline{\alpha}},y} - z)^{-1}(D-z)\phi_{k'} = \phi_{k'} - \frac{1}{(2c+i\alpha\zeta)(2c+i\tilde{\alpha}\zeta^{-1})}\{\alpha(\tilde{f}_{k'}(.-y),(D-z)\phi_{k'})f_{k'}(.-y) + \\ &+ \tilde{\alpha}(\tilde{g}_{k'}(.-y),(D-z)\phi_{k'})g_{k'}(.-y) + i\frac{\alpha\tilde{\alpha}}{2c}(\tilde{f}_{k'}(.-y),(D-z)\phi_{k'})\hat{f}_{k'}(.-y) + \\ &+ i\frac{\alpha\tilde{\alpha}}{2c}(\tilde{g}_{k'}(.-y),(D-z)\phi_{k'})\hat{g}_{k'}(.-y)\} \\ &= \psi_{\underline{\underline{\alpha}}}, z \in \rho(D_{\underline{\underline{\alpha}},y}), \ \ \mathrm{Im}k' > 0.\end{aligned} \tag{28}$$

Let now prove locality. We assume fisrt $y \notin \vartheta$. then

$$
\begin{aligned}
((D-z)(\alpha\phi_{k',1}(y)f_{k'}(.-y) &+ \tilde{\alpha}\phi_{k',2}(y)g_{k'}(.-y) + i\frac{\alpha\tilde{\alpha}}{2c}\phi_{k',1}(y)\hat{f}_{k'}(.-y)+ \\
&+ i\frac{\alpha\tilde{\alpha}}{2c}\phi_{k',1}(y)\hat{g}_{k'}(.-y)))(x) = 0
\end{aligned} \tag{29}
$$

implies that

$$
\begin{aligned}
(D_{\underline{\underline{\alpha}},y}\psi_{\underline{\underline{\alpha}}})(x) &= z\psi_{\underline{\underline{\alpha}}}(x) + ((D-z)\phi_{k'})(x) \\
&= \frac{2ic}{(2c+i\alpha\zeta)(2c+i\tilde{\alpha}\zeta^{-1})}((D-z)(\alpha\phi_{k',1}(y)f_{k'}(.-y) + \tilde{\alpha}\phi_{k',2}(y)g_{k'}(.-y)+ \\
&+ i\frac{\alpha\tilde{\alpha}}{2c}\phi_{k',1}(y)\hat{f}_{k'}(.-y) + i\frac{\alpha\tilde{\alpha}}{2c}\phi_{k',1}(y)\hat{g}_{k'}(.-y)))(x) = 0, \;\; x \in \vartheta
\end{aligned} \tag{30}
$$

Second, if $y \in \vartheta$ then $\psi_{\underline{\underline{\alpha}}}(y) = 0$ and $\phi_{k'} \in C^0(\mathbb{R})$ implies $\phi_{k'} = 0$ and hence

$$
(D_{\underline{\underline{\alpha}},y}\psi_{\underline{\underline{\alpha}}})(x) = z\psi_{\underline{\underline{\alpha}}}(x) = 0, \;\; x \in \vartheta. \tag{31}
$$

C. Spectral properties

The spectral properties of $D_{\underline{\underline{\alpha}},y}$ follow from (17). For $\alpha, \tilde{\alpha} \in \mathbb{R}$ the essential spectrum is purely absolutely continuous and coincide with $(-\infty, -\frac{c^2}{2}] \bigcup [\frac{c^2}{2}, \infty)$. The point spectrum of $D_{\underline{\underline{\alpha}},y}$ in $[-\frac{c^2}{2}, \frac{c^2}{2}]$ contains the poles of the resolvent equation (17). Then $D_{\underline{\underline{\alpha}},y}$ has two eigenvalues in $[-\frac{c^2}{2}, \frac{c^2}{2}]$ iff $\alpha, \tilde{\alpha} < 0$

$$
\sigma_p(D_{\underline{\underline{\alpha}},y}) = \begin{cases} \left\{ \frac{c^2(4c^2-\alpha^2)}{2(4c^2+\alpha^2)}, \frac{c^2(\tilde{\alpha}^2-4c^2)}{2(4c^2+\tilde{\alpha}^2)} \right\}, & \alpha, \tilde{\alpha} < 0 \\ \emptyset, & \alpha, \tilde{\alpha} \geq 0, \; \alpha = \tilde{\alpha} = \infty \end{cases} \tag{32}
$$

and two resonances iff $\alpha, \tilde{\alpha} > 0$.

Following the strategy of [9], one proves that the operator $(D_{\underline{\underline{\alpha}},y} - \frac{c^2}{2})$ converges in the norm resolvent sense to the Schrödinger operator $\Delta_{\alpha,\beta,y}$

$$
n - \lim_{c\to\infty}(D_{\underline{\underline{\alpha}},y} - \frac{c^2}{2} - z)^{-1} = (\Delta_{\alpha,\beta,y} - z)^{-1} \bigotimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \;\; \alpha, \beta \in \mathbb{R} \tag{33}
$$

where

$$
\begin{aligned}
\Delta_{\alpha,\beta,y} &= -\frac{d^2}{dx^2}, \\
\mathcal{D}(\Delta_{\alpha,\beta,y}) &= \left\{ g \in H^{2,2}(\mathbb{R} - \{y\}) \,\middle|\, \begin{array}{l} g'(y+) - g'(y-) = \frac{\alpha}{2}[g(y+) + g(y-)], \\ g(y+) - g(y-) = \frac{\beta}{2}[g'(y+) + g'(y-)] \end{array} \right\}, \;\; -\infty < \alpha, \beta \leq \infty.
\end{aligned} \tag{34}
$$

The Hamiltonian $\Delta_{\alpha,\beta,y}$ defines a new exactly solvable model of nonrelativistic point interaction. In section 2.A we will discuss the properties of the above Hamiltonian.

In particular, as $c \to \infty$, the two eigenvalues of $D_{\underline{\underline{\alpha}},y}$ (rest energy subtracted) $(E^{(1)}_{\underline{\underline{\alpha}},y} - \frac{c^2}{2})$, $(E^{(2)}_{\underline{\underline{\alpha}},y} - \frac{c^2}{2})$ give their respective nonrelativistic limits

$$
\begin{aligned}
\lim_{c\to\infty}(E^{(1)}_{\underline{\underline{\alpha}},y} - \frac{c^2}{2}) &= \lim_{c\to\infty}\left(\frac{c^2(4c^2-\alpha^2)}{2(4c^2+\alpha^2)} - \frac{c^2}{2}\right) \\
&= -\frac{\alpha^2}{4}\lim_{c\to\infty}\left[1 + \frac{\alpha^2}{4c^2}\right]^{-1} \\
&= -\frac{\alpha^2}{4},
\end{aligned} \tag{35}
$$

$$\begin{aligned}
\lim_{c\to\infty}(E^{(2)}_{\underline{\underline{\alpha}},y}-\frac{c^2}{2}) &= \lim_{c\to\infty}\left(\frac{c^2(\tilde{\alpha}^2-4c^2)}{2(4c^2+\tilde{\alpha}^2)}-\frac{c^2}{2}\right)\\
&= -\frac{4}{\beta^2}\lim_{c\to\infty}\left[1+\frac{4}{\beta^2c^2}\right]^{-1}, \quad \beta=-\frac{\tilde{\alpha}}{c^2}\\
&= -\frac{4}{\beta^2}. \qquad (36)
\end{aligned}$$

In section 2.C we will show that $-\frac{\alpha^2}{4}$ and $-\frac{4}{\beta^2}$ are the two eigenvalues of $\Delta_{\alpha,\beta,y}$ [see Eq(69)].

D. Scattering theory of the pair $(D_{\underline{\underline{\alpha}},y}, D)$

From (24), the scattering wave functions of $D_{\underline{\underline{\alpha}},y}$ are defined by

$$\begin{aligned}
\psi_{\underline{\underline{\alpha}},y}(k,\sigma,x) &= \begin{pmatrix} e^{ik'\sigma x}\\ \sigma\zeta^{-1}e^{ik'\sigma x}\end{pmatrix} - \frac{2ice^{ik'\sigma y}}{(2c+i\alpha\zeta)(2c+i\tilde{\alpha}\zeta^{-1})}\left\{\alpha\left\{\begin{array}{ll}\binom{\zeta}{1}e^{ik'(x-y)}, & x>y\\ \binom{\zeta}{-1}e^{ik'(y-x)}, & x<y\end{array}\right\}+\right.\\
&\quad +\tilde{\alpha}\sigma\zeta^{-1}\left\{\begin{array}{ll}\binom{1}{\zeta^{-1}}e^{ik'(x-y)}, & x>y\\ \binom{-1}{\zeta^{-1}}e^{ik'(y-x)}, & x<y\end{array}\right\}+i\frac{\alpha\tilde{\alpha}}{2c}\left\{\begin{array}{ll}\binom{1}{\zeta^{-1}}e^{ik'(x-y)}, & x>y\\ \binom{1}{-\zeta^{-1}}e^{ik'(y-x)}, & x<y\end{array}\right\}+\\
&\quad \left.+i\frac{\sigma\zeta^{-1}\alpha\tilde{\alpha}}{2c}\left\{\begin{array}{ll}\binom{\zeta}{1}e^{ik'(x-y)}, & x>y\\ \binom{-\zeta}{1}e^{ik'(y-x)}, & x<y\end{array}\right\}\right\},\\
&\quad x,y\in\mathbb{R},\ k'\geq 0,\ \sigma=\pm 1, -\infty<\alpha,\tilde{\alpha}\leq\infty. \qquad (37)
\end{aligned}$$

A straightforward computations shows that $\psi_{\underline{\underline{\alpha}},y}(k,\sigma)$ are eigenfunctions associated with $D_{\underline{\underline{\alpha}},y}$ corresponding to left ($\sigma=+1$) and right ($\sigma=-1$) incidence[10].

The asymptotic forms of $\psi_{\underline{\underline{\alpha}},y}$ are defined by[10, 15]

$$\begin{aligned}
\psi_{\underline{\underline{\alpha}},y}(z,+1,x) &= \left\{\begin{array}{lll}\mathcal{T}^l_{\underline{\underline{\alpha}},y}(z)\psi(z,+1,x) & \text{as} & x\to\infty\\ \psi(z,+1,x)+\mathcal{R}^l_{\underline{\underline{\alpha}},y}(z)\psi(z,-1,x) & \text{as} & x\to-\infty,\end{array}\right.\\
\psi_{\underline{\underline{\alpha}},y}(z,-1,x) &= \left\{\begin{array}{lll}\psi(z,-1,x)+\mathcal{R}^r_{\underline{\underline{\alpha}},y}(z)\psi(z,+1,x) & \text{as} & x\to\infty,\\ \mathcal{T}^r_{\underline{\underline{\alpha}},y}(z)\psi(z,-1,x) & \text{as} & x\to-\infty\end{array}\right. \qquad (38)
\end{aligned}$$

where $\psi(z,\sigma,x)$ is the solution of $D\psi=z\psi$ given by

$$\psi(z,\sigma,x)=\begin{pmatrix} e^{i\sigma k'x}\\ \sigma\zeta^{-1}e^{i\sigma k'x}\end{pmatrix}, \quad \sigma=\pm 1 \qquad (39)$$

with k' and ζ are defined by (8) and (18) respectively. Then the reflection and transmission coefficients from the left ($\sigma=+1$) and the right ($\sigma=-1$) are defined by

$$\begin{aligned}
\mathcal{R}^l_{\underline{\underline{\alpha}},y}(z) &= \lim_{x\to-\infty}\frac{1}{2}\left(e^{ik'x},-\zeta e^{ik'x}\right)\left[\psi_{\underline{\underline{\alpha}},y}(z,+1,x)-\begin{pmatrix}e^{ik'x}\\ \zeta^{-1}e^{ik'x}\end{pmatrix}\right]\\
\mathcal{R}^r_{\underline{\underline{\alpha}},y}(z) &= \lim_{x\to+\infty}\frac{1}{2}\left(e^{-ik'x},\zeta e^{-ik'x}\right)\left[\psi_{\underline{\underline{\alpha}},y}(z,-1,x)-\begin{pmatrix}e^{-ik'x}\\ -\zeta^{-1}e^{-ik'x}\end{pmatrix}\right]\\
\mathcal{T}^l_{\underline{\underline{\alpha}},y}(z) &= \lim_{x\to+\infty}\frac{1}{2}\left(e^{-ik'x},\zeta e^{-ik'x}\right)\psi_{\underline{\underline{\alpha}},y}(z,+1,x)\\
\mathcal{T}^r_{\underline{\underline{\alpha}},y}(z) &= \lim_{x\to-\infty}\frac{1}{2}\left(e^{ik'x},-\zeta e^{ik'x}\right)\psi_{\underline{\underline{\alpha}},y}(z,-1,x),\ k'\geq 0,\ -\infty<\alpha,\tilde{\alpha}\leq\infty,\ \ y\in\mathbb{R}. \qquad (40)
\end{aligned}$$

After a straightforward computation, one obtains

Theorem 2.2 *:* Let $\alpha,\tilde{\alpha}\in\mathbb{R}-\{0\}$, $y\in\mathbb{R}$. Then the unitary on-shell scattering matrix $\mathcal{S}_{\underline{\underline{\alpha}},y}(z)$ in $\mathbb{C}^2$ associated with the paire $(D_{\underline{\underline{\alpha}},y}, D)$ reads

$$\mathcal{S}_{\underline{\underline{\alpha}},y}(z) = \begin{bmatrix}\mathcal{T}^l_{\underline{\underline{\alpha}},y}(z) & \mathcal{R}^r_{\underline{\underline{\alpha}},y}(z)\\ \mathcal{R}^l_{\underline{\underline{\alpha}},y}(z) & \mathcal{T}^r_{\underline{\underline{\alpha}},y}(z)\end{bmatrix}, \quad k'\geq 0,\ -\infty<\alpha,\tilde{\alpha}\leq\infty,\ y\in\mathbb{R} \qquad (41)$$

with

$$\mathcal{T}^l_{\underline{\underline{\alpha}},y}(z) = 1-\frac{2ic}{\left(2c+i\alpha\zeta\right)\left(2c+i\tilde{\alpha}\zeta^{-1}\right)}\left(\alpha\zeta+\tilde{\alpha}\zeta^{-1}+i\frac{\alpha\tilde{\alpha}}{c}\right) = \mathcal{T}^r_{\underline{\underline{\alpha}},y}(z), \tag{42}$$

$$\mathcal{R}^l_{\underline{\underline{\alpha}},y}(z) = -\frac{2ic}{\left(2c+i\alpha\zeta\right)\left(2c+i\tilde{\alpha}\zeta^{-1}\right)}\left(\alpha\zeta-\tilde{\alpha}\zeta^{-1}\right)\mathrm{e}^{2ik'y} \tag{43}$$

$$\mathcal{R}^r_{\underline{\underline{\alpha}},y}(z) = -\frac{2ic}{\left(2c+i\alpha\zeta\right)\left(2c+i\tilde{\alpha}\zeta^{-1}\right)}\left(\alpha\zeta-\tilde{\alpha}\zeta^{-1}\right)\mathrm{e}^{-2ik'y} \tag{44}$$

In particular, as $c\to\infty$, the unitary on-shell scattering matrix $\mathcal{S}_{\underline{\underline{\alpha}},y}(z)$ gives its nonrelativistic limit $\mathcal{S}_{\alpha,\beta,y}(k)$ [see Eq(74)].
Indeed

$$\lim_{c\to\infty}\mathcal{T}^l_{\underline{\underline{\alpha}},y}(z) = \lim_{c\to\infty}\left\{1-\frac{2ic}{\left(2c+i\alpha\zeta\right)\left(2c+i\tilde{\alpha}\zeta^{-1}\right)}\left(\alpha\zeta+\tilde{\alpha}\zeta^{-1}+i\frac{\alpha\tilde{\alpha}}{c}\right)\right\} \tag{45}$$

Let $k^2=z-\frac{c^2}{2},\ k>0$ and $\tilde{\alpha}=-\beta c^2, \beta\in\mathbb{R}$, then after a straightforward computation (45) reads

$$\begin{aligned}
\lim_{c\to\infty}\mathcal{T}^l_{\underline{\underline{\alpha}},y}(z) &= \lim_{c\to\infty}\left\{1-\frac{2i\alpha\left(\frac{k^2}{c^2}+1\right)\left(\frac{k^2}{c^2}+1\right)}{\left[2\left(\frac{k^4}{c^2}+k^2\right)^{\frac{1}{2}}+i\alpha\left(\frac{k^2}{c^2}+1\right)\right]\left[2\left(\frac{k^2}{c^2}+1\right)-i\beta\left(\frac{k^4}{c^2}+k^2\right)^{\frac{1}{2}}\right]}+\right.\\
&\quad+\frac{2i\beta\left(\frac{k^4}{c^2}+k^2\right)\left(\frac{k^2}{c^2}+1\right)}{\left[2\left(\frac{k^4}{c^2}+k^2\right)^{\frac{1}{2}}+i\alpha\left(\frac{k^2}{c^2}+1\right)\right]\left[2\left(\frac{k^2}{c^2}+1\right)-i\beta\left(\frac{k^4}{c^2}+k^2\right)^{\frac{1}{2}}\right]\left(\frac{k^2}{c^2}+1\right)}-\\
&\quad\left.-\frac{-2\alpha\beta\left(\frac{k^4}{c^2}+k^2\right)^{\frac{1}{2}}\left(\frac{k^2}{c^2}+1\right)}{\left[2\left(\frac{k^4}{c^2}+k^2\right)^{\frac{1}{2}}+i\alpha\left(\frac{k^2}{c^2}+1\right)\right]\left[2\left(\frac{k^2}{c^2}+1\right)-i\beta\left(\frac{k^4}{c^2}+k^2\right)^{\frac{1}{2}}\right]}\right\}\\
&= i\frac{\left(\frac{\alpha\beta}{4}-1\right)}{4k^2\left(\frac{\alpha}{4k}-\frac{i}{2}\right)\left(\frac{1}{2k^2}-i\frac{\beta}{4k}\right)}\\
&= \lim_{c\to\infty}\mathcal{T}^r_{\underline{\underline{\alpha}},y}(z),
\end{aligned} \tag{46}$$

$$\begin{aligned}
\lim_{c\to\infty}\mathcal{R}^l_{\underline{\underline{\alpha}},y}(z) &= \lim_{c\to\infty}\left\{-\frac{2ic}{\left(2c+i\alpha\zeta\right)\left(2c+i\tilde{\alpha}\zeta^{-1}\right)}\left(\alpha\zeta-\tilde{\alpha}\zeta^{-1}\right)\mathrm{e}^{2ik'y}\right\}\\
&= \lim_{c\to\infty}\left\{-\frac{2i\alpha\left(\frac{k^2}{c^2}+1\right)\left(\frac{k^2}{c^2}+1\right)\mathrm{e}^{2i\left(\frac{k^4}{c^2}+k^2\right)^{\frac{1}{2}}y}}{\left[2\left(\frac{k^4}{c^2}+k^2\right)^{\frac{1}{2}}+i\alpha\left(\frac{k^2}{c^2}+1\right)\right]\left[2\left(\frac{k^2}{c^2}+1\right)-i\beta\left(\frac{k^4}{c^2}+k^2\right)^{\frac{1}{2}}\right]}-\right.\\
&\quad\left.-\frac{2i\beta\left(\frac{k^4}{c^2}+k^2\right)\left(\frac{k^2}{c^2}+1\right)\mathrm{e}^{2i\left(\frac{k^4}{c^2}+k^2\right)^{\frac{1}{2}}y}}{\left[2\left(\frac{k^4}{c^2}+k^2\right)^{\frac{1}{2}}+i\alpha\left(\frac{k^2}{c^2}+1\right)\right]\left[2\left(\frac{k^2}{c^2}+1\right)-i\beta\left(\frac{k^4}{c^2}+k^2\right)^{\frac{1}{2}}\right]\left(\frac{k^2}{c^2}+1\right)}\right\}\\
&= -\frac{\left(\frac{\alpha}{k}+\beta k\right)}{8k^2\left(\frac{\alpha}{4k}-\frac{i}{2}\right)\left(\frac{1}{2k^2}-i\frac{\beta}{4k}\right)}\mathrm{e}^{2iky}
\end{aligned} \tag{47}$$

and

$$\lim_{c\to\infty}\mathcal{R}^r_{\underline{\underline{\alpha}},y}(z) = \lim_{c\to\infty}\left\{-\frac{2ic}{\left(2c+i\alpha\zeta\right)\left(2c+i\tilde{\alpha}\zeta^{-1}\right)}\left(\alpha\zeta-\tilde{\alpha}\zeta^{-1}\right)\mathrm{e}^{-2ik'y}\right\}$$
$$= -\frac{\left(\frac{\alpha}{k}+\beta k\right)}{8k^2\left(\frac{\alpha}{4k}-\frac{i}{2}\right)\left(\frac{1}{2k^2}-i\frac{\beta}{4k}\right)}\mathrm{e}^{-2iky}. \quad (48)$$

3 THE NONRELATIVISTIC POINT INTERACTION

A. Basic properties

Consider in the Hilbert space $L^2(\mathbb{R})$ the closed and nonnegative operator $\tilde{H}_y$ defined by

$$\tilde{H}_y = -\frac{d^2}{dx^2}$$
$$\mathcal{D}(\tilde{H}_y) = \{g\in H^{2,2}(\mathbb{R})|g(y)=g'(y)=0\}. \quad (49)$$

The adjoint $\tilde{H}_y^*$ of $\tilde{H}_y$ is defined by

$$\tilde{H}_y^* = -\frac{d^2}{dx^2}$$
$$\mathcal{D}(\tilde{H}_y^*) = H^{2,2}(\mathbb{R}-\{y\}),\;\; y\in\mathbb{R}. \quad (50)$$

Hence the equation

$$\tilde{H}_y^*f(k)=k^2f(k),\;\; f(k)\in\mathcal{D}(\tilde{H}_y^*),\;\; k^2\in\mathbb{C}-\mathbb{R},\;\; \mathrm{Im}k>0, \quad (51)$$

has two linearly independent solutions

$$f_1(k,x)=\begin{cases}\mathrm{e}^{ik(x-y)}, & x>y,\\ 0, & x<y,\end{cases}\;,\;\; f_2(k,x)=\begin{cases}0, & x>y,\\ \mathrm{e}^{ik(y-x)}, & x<y,\end{cases}\quad \mathrm{Im}k>0. \quad (52)$$

Therefore $\tilde{H}_y$ has deficiency indices (2,2) and hence it has a four-parameter family of self-adjoint extensions. We consider in $L^2(\mathbb{R})$ the operator $\Delta_{\alpha,\beta,y}$ defined by[Eq (34)]

$$\Delta_{\alpha,\beta,y} = -\frac{d^2}{dx^2},$$
$$\mathcal{D}(\Delta_{\alpha,\beta,y}) = \left\{g\in H^{2,2}(\mathbb{R}-\{y\})\;\middle|\;\begin{array}{l} g'(y+)-g'(y-)=\frac{\alpha}{2}[g(y+)+g(y-)],\\ g(y+)-g(y-)=\frac{\beta}{2}[g'(y+)+g'(y-)]\end{array}\right\},\;\; -\infty<\alpha,\beta\le\infty. \quad (53)$$

Let $\alpha\beta-4=0,\;\alpha,\beta\in\mathbb{R}$, then the integration by parts shows $\Delta_{\alpha,\beta,y}$ is symmetric and since $\tilde{H}_y$ has deficiency indices (2,2) and the 2 boundary conditions in (53) are symmetric and linearly independent, it follows that $\Delta_{\alpha,\beta,y}$ is self -adjoint([16], Theorem XII.4.30). We will accept those α,β which satisfy the condition $\alpha\beta-4=0,\;\alpha,\beta\in\mathbb{R}$.
The case $\alpha=0,\beta=0$ in equation (53) yields the kinetic energy Hamiltonian Δ_0 in $L^2(\mathbb{R})$

$$\Delta_0=-\frac{d^2}{dx^2},\;\; \mathcal{D}(\Delta_0)=H^{2,2}(\mathbb{R}). \quad (54)$$

The case $\alpha\neq 0,\beta=0$ in equation (53) gives the δ-point interaction of the first type, whereas $\alpha=0,\beta\neq 0$ leads to a δ-point interaction of the second type[10].

B. Resolvent equation

The resolvent of $\Delta_{\alpha,\beta,y}$ is given by the following theorem

Theorem 3.1 *:* The resolvent of $\Delta_{\alpha,\beta,y}$ is given by

$$
\begin{aligned}
(\Delta_{\alpha,\beta,y}-k^2)^{-1} &= G_k + \frac{1}{2\left(\frac{\alpha}{4k}-i\frac{1}{2}\right)\left(\frac{1}{2k^2}-i\frac{\beta}{4k}\right)}\Big\{ i\frac{\alpha}{2k^2}(\overline{G_k(.-y)},.)G_k(.-y)+ \\
&+ i\frac{\beta}{2}(\overline{\tilde{\tilde{G}}_k(.-y)},.)\tilde{\tilde{G}}_k(.-y)+\frac{\alpha\beta}{4k}(\overline{G_k(.-y)},.)G_k(.-y) \\
&- \frac{\alpha\beta}{4k}(\overline{\tilde{\tilde{G}}_k(.-y)},.)\tilde{\tilde{G}}_k(.-y)\Big\}, \\
& k^2\in\rho(\Delta_{\alpha,\beta}),\ \ \mathrm{Im}k>0,\ \ -\infty<\alpha,\beta\le\infty,\ \ y\in\mathbb{R}.
\end{aligned}
\tag{55}
$$

with integral kernel

$$
\begin{aligned}
&(\Delta_{\alpha,\beta,y}-k^2)^{-1}(x,x') = i\frac{1}{2k}\mathrm{e}^{ik|x-x'|}-\frac{1}{8k^2\left(\frac{\alpha}{4k}-i\frac{1}{2}\right)\left(\frac{1}{k^2}-i\frac{\beta}{4k}\right)}\times \\
&\times\left\{i\frac{\alpha}{2k^2}\left\{\begin{array}{ll}\mathrm{e}^{ik(x-y)}, & x>y\\ \mathrm{e}^{ik(y-x)}, & x<y\end{array}\right\}.\left\{\begin{array}{ll}\mathrm{e}^{ik(x'-y)}, & x'>y\\ \mathrm{e}^{ik(y-x')}, & x'<y\end{array}\right\}+i\frac{\beta}{2}\left\{\begin{array}{ll}\mathrm{e}^{ik(x-y)}, & x>y\\ -\mathrm{e}^{ik(y-x)}, & x<y\end{array}\right\}\times\right. \\
&\times\left\{\begin{array}{ll}\mathrm{e}^{ik(x'-y)}, & x'>y\\ -\mathrm{e}^{ik(y-x')}, & x'<y\end{array}\right\}+\frac{\alpha\beta}{4k}\left\{\begin{array}{ll}\mathrm{e}^{ik(x-y)}, & x>y\\ \mathrm{e}^{ik(y-x)}, & x<y\end{array}\right\}.\left\{\begin{array}{ll}\mathrm{e}^{ik(x'-y)}, & x'>y\\ \mathrm{e}^{ik(y-x')}, & x'<y\end{array}\right\}- \\
&\left.-\frac{\alpha\beta}{4k}\left\{\begin{array}{ll}\mathrm{e}^{ik(x-y)}, & x>y\\ \mathrm{e}^{ik(y-x)}, & x<y\end{array}\right\}.\left\{\begin{array}{ll}\mathrm{e}^{ik(x'-y)}, & x'>y\\ \mathrm{e}^{ik(y-x')}, & x'<y\end{array}\right\}\right\},\ k^2\in\rho(\Delta_{\alpha,\beta,y}),\mathrm{Im}k>0,\ x,x'\in\mathbb{R},
\end{aligned}
\tag{56}
$$

where

$$
G_k(x-y) = \frac{i}{2k}\left\{\begin{array}{ll}\mathrm{e}^{ik(x-y)}, & x>y\\ \mathrm{e}^{ik(y-x)}, & x<y,\ \ \mathrm{Im}k>0\end{array}\right. \tag{57}
$$

$$
\tilde{\tilde{G}}_k(x-y) = \frac{i}{2k}\left\{\begin{array}{ll}\mathrm{e}^{ik(x-y)}, & x>y\\ -\mathrm{e}^{ik(y-x)}, & x<y,\ \ \mathrm{Im}k>0.\end{array}\right. \tag{58}
$$

Proof
We use the resolvent formula

$$
(\Delta_{\alpha,\beta,y}-k^2)^{-1} = G_k-\frac{1}{4k^2}\sum_{i,j=1}^{2}\lambda_{ij}(k)(f_j(-\bar{k}),.)f_i(k) \tag{59}
$$

where $f_j,\ j=1,2$ are defined by (52).
Next consider $h\in L^2(\mathbb{R})$ and define the function $g\in\mathcal{D}(\Delta_{\alpha,\beta,y})$ by

$$
g(k,x) = ((\Delta_{\alpha,\beta,y}-k^2)^{-1}h)(x). \tag{60}
$$

After imposing the boundary conditions in (53), one obtain

$$
\lambda(k) = \left[\begin{array}{cc}\lambda_{11} & \lambda_{12}\\ \lambda_{21} & \lambda_{22}\end{array}\right] \tag{61}
$$

with

$$
\begin{aligned}
\lambda_{11} &= \frac{1}{2\left[\frac{\alpha}{4k}-i\frac{1}{2}\right]\left[\frac{\alpha}{4k}-i\frac{1}{2}\right]}\left(i\frac{\alpha}{2k^2}+i\frac{\beta}{2}\right)\\
\lambda_{12} &= \frac{1}{2\left[\frac{\alpha}{4k}-i\frac{1}{2}\right]\left[\frac{\alpha}{4k}-i\frac{1}{2}\right]}\left(i\frac{\alpha}{2k^2}+\frac{\alpha\beta}{2k}-i\frac{\beta}{2}\right)
\end{aligned}
$$

$$\lambda_{21} = \frac{1}{2\left[\frac{\alpha}{4k} - i\frac{1}{2}\right]\left[\frac{\alpha}{4k} - i\frac{1}{2}\right]}\left(i\frac{\alpha}{2k^2} + \frac{\alpha\beta}{2k} - i\frac{\beta}{2}\right)$$

$$\lambda_{22} = \frac{1}{2\left[\frac{\alpha}{4k} - i\frac{1}{2}\right]\left[\frac{\alpha}{4k} - i\frac{1}{2}\right]}\left(i\frac{\alpha}{2k^2} + i\frac{\beta}{2}\right). \tag{62}$$

Inserting (62) in (59) one obtain (55).

Remark 2 *:* From (55) one obtain the following results

(i) As $\beta \to 0$, the Hamiltonian $\Delta_{\alpha,\beta,y}$ converge in the norm resolvent sense to $-\Delta_{\alpha,y}$

$$n.\lim_{\beta\to 0}(\Delta_{\alpha,\beta,y} - z)^{-1} = (-\Delta_{\alpha,y} - z) \;\; z \in \rho(\Delta_{\alpha,\beta,y}) \cap \rho(-\Delta_{\alpha,y}) \tag{63}$$

where[10]

$$(-\Delta_{\alpha,y} - k^2)^{-1} = G_k - \frac{2\alpha k}{i\alpha + 2k}(\overline{G_k(.-y)},.)G_k(.-y),$$
$$k^2 \in \rho(-\Delta_{\alpha,y}), \;\; \mathrm{Im}k > 0, \;\; -\infty < \alpha \leq \infty, \;\; y \in \mathbb{R} \tag{64}$$

with G_k defined by (57).

(ii) As $\alpha \to 0$, the Hamiltonian $\Delta_{\alpha,\beta,y}$ converge in the norm resolvent sense to $-\Delta_{\beta,y}$

$$n.\lim_{\alpha\to 0}(\Delta_{\alpha,\beta,y} - z)^{-1} = (-\Delta_{\beta,y} - z) \;\; z \in \rho(\Delta_{\alpha,\beta,y}) \cap \rho(-\Delta_{\beta,y}) \tag{65}$$

where[10]

$$(-\Delta_{\beta,y}) - k^2)^{-1} = G_k - \frac{2\beta k^2}{2 - i\beta k}(\overline{\tilde{\tilde{G}}_k(.-y)},.)\tilde{\tilde{G}}_k(.-y),$$
$$k^2 \in \rho(-\Delta_{\alpha,y}), \;\; \mathrm{Im}k > 0, \;\; -\infty < \beta \leq \infty, \;\; y \in \mathbb{R} \tag{66}$$

with $\tilde{\tilde{G}}_k$ defined by (58). The additional information on the domain of $\Delta\alpha, \beta, y$ is given by the following theorem

Theorem 3.2 *:* The domain $\mathcal{D}(\Delta_{\alpha,\beta,y})$, $-\infty < \alpha, \beta \leq \infty$, $y \in \mathbb{R}^3$, consists of all elements $\psi_{\alpha,\beta}$ of the type

$$\psi_{\alpha,\beta}(x) = \varphi_k(x) + \frac{1}{2\left(\frac{\alpha}{4k} - i\frac{1}{2}\right)\left(\frac{1}{2k^2} - i\frac{\beta}{4k}\right)}\left\{i\frac{\alpha}{2k^2}\varphi_k(y)G_k(x-y) - \frac{\beta}{2k}\varphi'(y)\tilde{\tilde{G}}_k(x-y) + \right.$$
$$\left. + \frac{\alpha\beta}{4k}\varphi_k(y)G_k(x-y) - i\frac{\alpha\beta}{4k^2}\varphi_k'(y)\tilde{\tilde{G}}_k(x-y)\right\}, \;\; x \neq y. \tag{67}$$

where $\varphi_k \in \mathcal{D}(\Delta_0) = H^{2,2}(\mathbb{R})$ and Im $k > 0$. The decomposition (67) is unique and with $\psi_{\alpha,\beta}$ of this form we obtain

$$(\Delta_{\alpha,\beta,y} - z)\psi_{\alpha,\beta} = (\Delta_0 - z)\varphi_k. \tag{68}$$

Let $\psi_{\alpha,\beta} \in \mathcal{D}(\Delta_{\alpha,\beta,y})$ and assume that $\psi_{\alpha,\beta} = 0$ in an open set $\tilde{\vartheta} \in \mathbb{R}^3$. Then $\Delta_{\alpha,\beta,y}\psi_{\alpha,\beta} = 0$ in $\tilde{\vartheta}$, i.e., $\Delta_{\alpha,\beta,y}$ describes a local interaction.

Proof.

Similar to the proof of theorem 2.1.

C. Spectral properties

For $\alpha, \beta \in \mathbb{R}$, the essential spectrum of $\Delta_{\alpha,\beta,y}$ is purely absolutely continuous and coincide with $[0, \infty)$ and the singular spectrum is empty. The point spectrum of $\Delta_{\alpha,\beta,y}$ are given as the pole of the resolvent equation (55), one obtain

$$\sigma_p = \begin{cases} \left\{-\frac{\alpha^2}{4}, -\frac{4}{\beta^2}\right\} & \alpha, \beta < 0 \\ 0 & \alpha, \beta \geq 0 \end{cases} \tag{69}$$

For $\alpha, \beta > 0$, $\Delta_{\alpha,\beta,y}$ has two resonances at $k_1 = -\frac{2i}{\beta}$ and $k_2 = -\frac{i\alpha}{2}$ with resonance functions respectively given by

$$\psi_{k_1}(x) = \begin{cases} e^{\frac{\alpha}{2}(x-y)}, & x > y, \\ e^{\frac{\alpha}{2}(y-x)}, & x < y, \end{cases} \quad \alpha > 0 \tag{70}$$

$$\psi_{k_2}(x) = \begin{cases} e^{\frac{2}{\beta}(x-y)}, & x > y, \\ -e^{\frac{2}{\beta}(y-x)}, & x < y, \end{cases} \quad \beta > 0. \tag{71}$$

D. Scattering theory of the paire $(\Delta_{\alpha,\beta,y}, \Delta_0)$

From (67) one can define the generalized function associated with $\Delta_{\alpha,\beta,y}$ by

$$\begin{aligned} \psi_{\alpha,\beta,y} \; = \; & e^{ik\sigma x} + \frac{1}{4k\left(\frac{\alpha}{4k} - i\frac{1}{2}\right)\left(\frac{1}{2k^2} - i\frac{\beta}{4k}\right)} \left\{ -\frac{\alpha}{2k^2} e^{ik\sigma y} \left\{ \begin{array}{ll} e^{ik(x-y)}, & x > y \\ e^{ik(y-x)}, & x < y \end{array} \right\} + \right. \\ & + \frac{\sigma\beta}{2} e^{ik\sigma y} \left\{ \begin{array}{ll} e^{ik(x-y)}, & x > y \\ -e^{ik(y-x)}, & x < y \end{array} \right\} + i\frac{\alpha\beta}{4k} e^{ik\sigma y} \left\{ \begin{array}{ll} e^{ik(x-y)}, & x > y \\ e^{ik(y-x)}, & x < y \end{array} \right\} + \\ & \left. + i\frac{\sigma\alpha\beta}{4k} e^{ik\sigma y} \left\{ \begin{array}{ll} e^{ik(x-y)}, & x > y \\ -e^{ik(y-x)}, & x < y \end{array} \right\} \right\}, \\ & x, y \in \mathbb{R},\; k > 0, \sigma = \pm 1, -\infty < \alpha, \beta \leq \infty. \end{aligned} \tag{72}$$

The corresponding reflection and transmission cofficients from the left ($\sigma = +1$) and the right ($\sigma = -1$) are defined by[10]

$$\begin{aligned} \mathcal{R}^l_{\alpha,\beta,y}(k) &= \lim_{x\to-\infty} e^{ikx}\left[\psi_{\alpha,\beta,y}(z, +1, x) - e^{ikx}\right] \\ \mathcal{R}^r_{\alpha,\beta,y}(k) &= \lim_{x\to+\infty} e^{-ikx}\left[\psi_{\alpha,\beta,y}(k, -1, x) - e^{-ikx}\right] \\ \mathcal{T}^l_{\alpha,\beta,y}(k) &= \lim_{x\to+\infty} e^{-ikx}\psi_{\alpha,\beta,y}(k, +1, x) \\ \mathcal{T}^r_{\alpha,\beta,y}(k) &= \lim_{x\to-\infty} e^{ikx}\psi_{\alpha,\beta,y}(k, -1, x), \quad k \geq 0, \; -\infty < \alpha, \beta \leq \infty, \; y \in \mathbb{R}. \end{aligned} \tag{73}$$

After a straightforward computation, one obtain

Theorem 3.3 *: Let $\alpha, \beta \in \mathbb{R} - \{0\}$, $y \in \mathbb{R}$. Then the unitary on-shell scattering matrix $\mathcal{S}_{\alpha,\beta,y}(k)$ in $\mathbb{C}^2$ associated with the paire $(\Delta_{\alpha,\beta,y}, \Delta_0)$ reads*

$$\mathcal{S}_{\alpha,\beta,y}(k) \; = \; \begin{bmatrix} \mathcal{T}^l_{\alpha,\beta,y}(k) & \mathcal{R}^r_{\alpha,\beta,y}(k) \\ \mathcal{R}^l_{\alpha,\beta,y}(k) & \mathcal{T}^r_{\alpha,\beta,y}(k) \end{bmatrix}, \quad k \geq 0, \; -\infty < \alpha, \beta \leq \infty, \; y \in \mathbb{R} \tag{74}$$

with

$$\mathcal{T}^l_{\alpha,\beta,y}(k) \; = \; i\frac{\left(\frac{\alpha\beta}{4} - 1\right)}{4k^2\left(\frac{\alpha}{4k} - \frac{i}{2}\right)\left(\frac{1}{2k^2} - i\frac{\beta}{4k}\right)} = \mathcal{T}^r_{\alpha,\beta,y}(k), \tag{75}$$

$$\mathcal{R}^{l}_{\alpha,\beta,y}(k) \;=\; -\frac{\left(\frac{\alpha}{k}+\beta k\right)}{8k^2\left(\frac{\alpha}{4k}-\frac{i}{2}\right)\left(\frac{1}{2k^2}-i\frac{\beta}{4k}\right)}\mathrm{e}^{2iky} \tag{76}$$

$$\mathcal{R}^{r}_{\alpha,\beta,y}(k) \;=\; -\frac{\left(\frac{\alpha}{k}+\beta k\right)}{8k^2\left(\frac{\alpha}{4k}-\frac{i}{2}\right)\left(\frac{1}{2k^2}-i\frac{\beta}{4k}\right)}\mathrm{e}^{-2iky}. \tag{77}$$

We note that the limit $\beta \to 0$ (respectively $\alpha \to 0$) in Eq (74) give the unitary on-shell scattering matrix $\mathcal{S}_{\alpha,y}(k)$ ($\mathcal{S}_{\beta,y}(k)$) associated with the paire $(-\Delta_{\alpha,y}, -\Delta)$ and $(-\Delta_{\beta,y}, -\Delta)$ repectively[10]. One obtain

$$\mathcal{S}_{\alpha,\beta,y}(k) \xrightarrow[\beta\to 0]{} (2k+i\alpha)^{-1}\begin{bmatrix} 2k & -i\alpha\mathrm{e}^{-2iky} \\ -i\alpha\mathrm{e}^{2iky} & 2k \end{bmatrix} = \mathcal{S}_{\alpha,y}(k), k \geq 0,\ -\infty < \alpha \leq \infty,\ y \in \mathbb{R}, \tag{78}$$

$$\mathcal{S}_{\alpha,\beta,y}(k) \xrightarrow[\alpha\to 0]{} (2-i\beta k)^{-1}\begin{bmatrix} 2 & -i\beta k\mathrm{e}^{-2iky} \\ -i\beta k\mathrm{e}^{2iky} & 2 \end{bmatrix} = \mathcal{S}_{\beta,y}(k), k \geq 0,\ -\infty < \beta \leq \infty,\ y \in \mathbb{R}. \tag{79}$$

Acknowledgments

V.A would like to thank Prof Blanchard, Universität Bielefeld, for his invitation to the Research Institute BIBOS and for interesting discussions.

References

[1] S.G. DAVISON and M.STESLICKA, J.Phys C2, 1802(1969)

[2] S.G. DAVISON and M.STESLICKA, Inter J Quantum Chem **4**, 445(1971)

[3] M.STESLICKA and S.G. DAVISON, Phys Rev **B1**, 1858(1970)

[4] R.SUBRAMANIAN and K.V. BHAGWAT, J. Phys **C5**, 798(1972)

[5] R.SUBRAMANIAN and K.V. BHAGWAT, Phys Stat. Sol **B 48**, 399(1971)

[6] W.M.FAIRBAIRN,M.L. GLASSER and M. STESLICKA, M. Surf. Sci **36**, 462(1973)

[7] G.GUMBS, Phys Rev **32 A**, 1208 (1985)

[8] I.R. LAPIDUS, Am J.Phys **51**, 1036 (1983)

[9] F.GESZTESY and P.SEBA: Lett Math Phys**13**, 345-358(1987)

[10] S.ALBEVERIO, F.GESZTESY, R.HOEGH-KROHN and H.HOLDEN: *Solvable models in Quantum Mechanics. Texts and Monographs in Physics* (Springer-verlag,Berlin 1988)

[11] J.DITTRICH,P.EXNER and P.SEBA, J.Math.Phys 30, 2275-2282(1989)

[12] J. SHABANI and A. VYABANDI:*Exactly Solvable Models of Relativistic δ-sphere Interactions in Quantum Mechanics* preprint $N^0$00-10-27, University of Bielefeld, Germany October 2000

[13] J.SHABANI and A.VYABANDI: In preparation

[14] N.I. AKHIEZER and I.M. GLAZMAN: *Theory of Linear operators in Hilbert Space.Vol.2* (Pitman Publishing Inc. Boston, London, Melbourne, 1981).

[15] R.G.NEWTON, J.Maths.Phys **21**, 493-505(1980)

[16] N.DUNFORD and J.T. SCHWARTZ: *Linear operators. Part II: Spectral Theory.* Interscience, New York- London, 1963.