On mesoscopic spectral universality in the Wishart ensemble*

Anne BOUTET de MONVEL[1,2] and Alexei KHORUNZHY[1,2]

[1]Université Paris-7, Mathématiques, case 7012, Paris, France

[2]Institute for Low Temperature Physics, Kharkov, Ukraine

Abstract

We describe a method to study the eigenvalue distribution of random hermitian (or real symmetric) $N \times N$ matrices in the limit $N \to \infty$. Our approach is related to a regularized version of the empirical eigenvalue distribution function $\rho(\Delta)$ that counts the number of eigenvalues belonging to an interval Δ of the real axis. In this paper we consider the Wishart random matrices widely used in applications. Basing on the calculus developed, we study the eigenvalue density $\rho_N = \rho(\Delta_N)$ with $\Delta_N = (\lambda - \delta_N, \lambda + \delta_N)$ in two different limiting transitions.

In the first case when $\delta_N = \mathrm{O}(N^{-1})$, we show that in the limit $N \to \infty$, the moments of the random variable $\rho(\Delta_N)$ are bounded by factorial-type expressions.

The second case is determined by the power decay of the interval length $\delta_N = \mathrm{O}(N^{-\alpha})$, $0 < \alpha < 1$ as $N \to \infty$. In this asymptotic regime called mesoscopic, we show that the average value $\mathsf{E}\,\rho_N$ converges to the limiting integral eigenvalue distribution of the ensemble. We prove also that the random variables $\gamma_N = (\rho_N - \mathsf{E}\,\rho_N)N^{1-\alpha}$ converge to a gaussian random variable γ.

We find an explicit expression for the corresponding correlation function. It coincides with that obtained for other random matrix ensembles. This proves the universality of the spectral fluctuations of large random matrices in the mesoscopic regime previously observed by physicists.

1 Introduction

In the spectral theory of random matrices, one of the most intensively discussed problems is the universality conjecture put forward by F.J. Dyson [7]. It can be formulated as follows. Let us consider a subset $\mathcal{A}_N \subset \mathbb{C}^{N^2}$ of $N \times N$ hermitian matrices equipped with a probability measure μ_N. Assume that $\mu = \mu_N$ induces a joint distribution of eigenvalues $\lambda_1^{(N)} \leq \cdots \leq \lambda_N^{(N)}$

*November 21, 2000

with density $P^{(N)}(\lambda_1, \dots, \lambda_N)$. The conjecture is that the corresponding marginal distributions $P_k^{(N)}(\lambda_1, \dots, \lambda_k)$ converge in the local scale to certain distributions π_k

$$P_k^{(N)}\left(\lambda + \frac{s_1}{N}, \dots, \lambda + \frac{s_k}{N}\right) \to \pi_k^{(\lambda)}(s_1, \dots, s_k).$$

These functions $\{\pi_k^{(\lambda)}\}$ determine a limiting random field $\{\pi^{(\lambda)}\}$ that does not depend on the details of the initial measure μ. One expects also a very discretized relationship between π and λ. The resulting field depends on whether λ belongs to the edge of the limiting spectrum or is situated in the bulk of it.

Such a universal behavior of the eigenvalue distribution is interesting by itself. It is also important because there exists a series of applications of large random matrices where this universality takes place (see e.g. [3, 6, 11]). However, this Dyson's universality conjecture is not proved in general case and even is not formulated in a full generality. Properties of the universal random field $\{\pi\}$ are studied for several families of random matrix ensembles, where the explicit form of $P^{(N)}$ is known a priori (see e.g. [16, 17] and references therein).

In this paper we describe a new method to study the local properties of the eigenvalue distribution of large random matrices. It does not require the a priori knowledge of the joint probability density $P^{(N)}$ but is aimed at the distribution

$$\sigma_N(\Delta) = \frac{1}{N} \#\{j \mid \lambda_j \in \Delta\}$$

known as the empirical eigenvalue distribution function.

The starting point of our approach is the elementary observation that the imaginary part of the Stieltjes transform $f_N(z)$ of σ_N represents certain regularized measure $\tilde{\sigma}_N$. It is well-known that the value of $\varepsilon = \operatorname{Im} z$ determines a scale on the spectral axis and broadly speaking determines those eigenvalues that provide the principal contribution to $\tilde{\sigma}_N$.

The Stieltjes transform $f_N(z)$ is closely related to the resolvent of the corresponding random matrix. In the spectral theory of random matrices, the resolvent technique is fairly well developed in the case when $\operatorname{Im} z > 0$ (see e.g. [15, 13] and references therein). The main advantage of this approach is that one has no need in an explicit form of $P^{(N)}$ and therefore large classes of random matrices can be studied.

It is natural to try to improve the resolvent approach up to the level that f_N can be studied for $\operatorname{Im} z \to 0$ and to detect universalities there [12]. The first steps of this program were carried out in the case of random real symmetric matrices whose entries are independent identically distributed random variables [4].

Here we consider a more complicated case of random matrices whose entries are statistically dependent random variables. These matrices are widely knowm in mathematical statistics as Wishart matrices (see e.g. [1]) and have been intensively used in statistical mechanics and neural network theory [10].

We derive explicit expressions for the limiting smoothed eigenvalue density and determine the correlation function of its fluctuations. This correlation function coincides with that determined in [4] for random matrices with independent entries. This allows us to claim that the fluctuations of the smoothed eigenvalue density are universal.

The results of this paper are partially announced in [5].

2 Wishart ensemble, main results, discussion

2.1 Wishart random matrices

The Wishart ensemble of $N \times N$ random matrices is given by

$$H^{(N,m)}(x,y) = \frac{1}{N}\sum_{j=1}^{m} \xi_j(x)\xi_j(y), \qquad x,y = 1,\ldots,N \tag{2.1a}$$

where $\{\xi_j(x),\ x,j \in \mathbb{N}\} = \Xi$ is a family of real Gaussian jointly independent identically distributed random variables determined on the same probability space [1]. We assume that $\xi_j(x)$ have zero mathematical expectation and variance v^2;

$$\begin{aligned} \mathsf{E}\,\xi_j(x) &= 0, \\ \mathsf{E}\,\xi_j(x)\xi_k(y) &= \delta_{jk}\delta_{xy}v^2, \end{aligned} \tag{2.1b}$$

where $\mathsf{E}\{\,\cdot\,\}$ denotes the mathematical expectation with respect to the measure generated by Ξ and δ_{xy} is the Kronecker δ-symbol.

The eigenvalue distribution of a $N \times N$ real symmetric (or hermitian) matrix is given by the normalized eigenvalue counting function $\sigma(\lambda; H^{(N,m)})$ (NCF)

$$\sigma(\lambda; H^{(N,m)}) = \frac{1}{N}\,\#\left\{\lambda_j^{(N,m)} \le \lambda\right\} N^{-1}, \tag{2.2}$$

where $\lambda_1^{(N,m)} \le \cdots \le \lambda_N^{(N,m)}$ are the eigenvalues of $H^{(N,m)}$, and its density

$$\rho(\lambda; H^{(N,m)}) = \sigma'(\lambda; H^{(N,m)}) = \frac{1}{N}\sum_{j=1}^{N} \delta\left(\lambda - \lambda_j^{(N,m)}\right),$$

where $\delta(\lambda)$ is the Dirac delta-function.

Our main subject is the asymptotic behaviour of the regularized density $\hat\rho$ determined by the formula

$$\hat\rho^{(\epsilon)}(\lambda; H^{(N,m)}) \equiv \hat\rho^{(\epsilon)}_{N,m}(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\epsilon}{(\lambda-\mu)^2+\epsilon^2}\,\rho(\mu; H^{(N,m)})\,\mathrm{d}\mu. \tag{2.3}$$

We assume that

$$\epsilon = \frac{1}{\tau N^\alpha}$$

with $\tau, \alpha > 0$ and denote by $\hat{\rho}_{N,m}^{(\alpha,\tau)}$ the corresponding function.

Regarding (2.3), we study the properties of the random field

$$\mathcal{R}_\lambda^{(N)}(\vartheta) \equiv \hat{\rho}_{N,m}^{(\alpha,\tau)}\left(\lambda + \frac{\vartheta}{\tau N^\alpha}\right), \quad \vartheta \in \mathbb{R} \tag{2.4}$$

in the limit

$$\begin{aligned} &m, N \to \infty \\ &\frac{m}{N} \to c > 0. \end{aligned} \tag{2.5}$$

2.2 Main results

We will consider (2.4) in two different limiting transitions.

The first is given by the condition $\alpha = 1$ and is referred to as the *local asymptotic regime.*

Theorem 2.1 (existence of a local random field). *Assume that $\alpha = 1$ and $\tau < 1$. Then the moments of $\mathcal{R}_\lambda^{(N)}(\vartheta)$ are bounded:*

$$\limsup_{N\to\infty} \mathsf{E}\left[\mathcal{R}_\lambda^{(N)}(\vartheta)\right]^k \leq C^k k! \tag{2.6}$$

where C is some constant independent from N.

The second limit of (2.4) corresponds to values $0 < \alpha < 1$ and can be called the *mesoscopic asymptotic regime.*

Theorem 2.2 (mesoscopic density of eigenvalues). *Assume that $\tau = 1$ and $0 < \alpha < 1$. Then for any fixed λ such that*

$$\lambda \in \left(v^2(1-\sqrt{c})^2, v^2(1+\sqrt{c})^2\right) \tag{2.7}$$

and any fixed $k \in \mathbb{N}$, we have

$$\mathsf{E}[\hat{\rho}_{N,m}^{(\alpha,\tau)}(\lambda)]^k \to [\rho_c(\lambda)]^k, \tag{2.8}$$

in the limit (2.5), with

$$\rho_c(\lambda) = (2\pi\lambda v^2)^{-1}\sqrt{4cv^4 - [\lambda - (1+c)v^2]^2}.$$

Remarks. 1. (2.7) implies the two inequalities $\lambda > 0$ and $\rho_c(\lambda) > 0$. It should be also noted that the function

$$\tilde{\rho}_c(\lambda) = \begin{cases} \rho_c(\lambda), & \text{if } c > 1 \\ \rho_c(\lambda) + (1-c)\delta(\lambda), & \text{if } 0 < c < 1 \end{cases}$$

is proved to be the limiting eigenvalue density of the ensemble (2.1) [15]. More precisely, it was proved in [15] that the NCF (2.2) of (2.1) weakly converges in probability, as $N \to \infty$, to a nonrandom distribution. This means that for any fixed $\epsilon > 0$ the convergence

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\epsilon}{(\lambda-\mu)^2+\epsilon^2}\, \mathrm{d}\sigma(\mu; H^{(N,m)}) \to \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\epsilon}{(\lambda-\mu)^2+\epsilon^2} \tilde{\rho}_c(\mu)\, \mathrm{d}\mu$$

holds in probability. In our terms this convergence corresponds to the limiting transition $\alpha = 0, N \to \infty$ known as the *global spectral regime* for large random matrices.

3. Let us also mention the paper [2], where convergence in probability (2.7) of random variables (2.4) was established for $\alpha \le 1/4$ and $\tau \to 0$. The random matrix ensemble considered in [2] is defined as (2.1a) where $\xi_j(x)$ are independent arbitrary distributed random variables.

The next statement is related with the centered random variables

$$\gamma_\lambda^{(N)}(\vartheta) = N^{1-\alpha}\left[\mathcal{R}_\lambda^{(N)}(\vartheta) - \mathsf{E}\,\mathcal{R}_\lambda^{(N)}(\vartheta)\right], \qquad (*)$$

where $\tau = 1$ and $\mathsf{E}\{\,\cdot\,\}$ denotes the mathematical expectation with respect to the measure generated by the family Ξ.

Theorem 2.3. *Under conditions of Theorem* 2.2 *and notation* $(*)$ *the vector* $\left\{\gamma_\lambda^{(N)}(\vartheta_1), \ldots, \gamma_\lambda^{(N)}(\vartheta_k)\right\}$ *converges in distribution in the limit* (2.5) *to a Gaussian random k-dimensional vector* $\vec{\gamma}_\lambda = \{\gamma(\vartheta_1), \ldots, \gamma(\vartheta_k)\}$ *with zero mathematical expectation and covariance matrix determined by*

$$\mathsf{E}\,\gamma(\vartheta_1)\,\gamma(\vartheta_2) = -\frac{4-(\vartheta_1-\vartheta_2)^2}{\pi^2\left[4+(\vartheta_1-\vartheta_2)^2\right]^2}. \qquad (2.9)$$

Remark. Theorem 2.3 implies that the convergence (cf. (2.8))

$$\hat{\rho}_{N,m}^{(\alpha,\tau)}(\lambda) \to \rho_c(\lambda) \qquad (2.10)$$

holds with probability 1.

2.3 Discussion

To discuss our results, let us note first that similar statements were proved in [4-I] for the ensemble of random real symmetric matrices

$$A_{ij}^{(N)} = \frac{1}{\sqrt{N}} a_{ij}, \quad i,j = 1,\ldots,N, \qquad (2.11)$$

whose entries $\{a_{ij}, i \le j\}$ are gaussian independent random variables with zero average and variance u^2.

This ensemble is known as the Gaussian orthogonal ensemble of random matrices (GOE) [14]. Together with its unitary and symplectic analogues, it plays a central role in the spectral theory of random matrices.

Regarding this gaussian ensemble (2.11), it was proved in [4-I] an analogue of (2.10), where $\rho_c(\lambda)$ is replaced by the semi-circle distribution

$$\rho_w = \frac{1}{2\pi u^2} \begin{cases} \sqrt{4u^2 - \lambda^2}, & \text{if } |\lambda| \le 2u; \\ 0, & \text{if } |\lambda| > 2u \end{cases} \tag{2.12}$$

known also as the Wigner law [18]. The difference between (2.1) and (2.11) is that the matrix elements of $H^{(N,m)}$ are statistically dependent random variables. This implies the difference between the densities of eigenvalue distributions $\tilde{\rho}_c$ and ρ_w.

However, the gaussian fluctuations of the densities coincide in these two different ensembles. This means that the vectors $\{\vec{\gamma}_\lambda\}$ determine the same random process that is a very good candidate for the universal mesoscopic spectral field.

It should be also pointed out that statements analogues to Theorem 2.3 are proved in [4-II] for the Wigner ensembles of random matrices (2.11), where the set $\{a_{ij}, i \le j\}$ is given by independent identically distributed random variables that are not necessarily Gaussian.

Finally, let us stress that the universality of the mesoscopic eigenvalue density has been predicted and respective mechanisms have been discussed in a series of physical papers (see [8] for the review of related results). Let us just note in the asymptotics of large distance $1 \ll |\vartheta_1 - \vartheta_2| \ll N^{-\alpha}$, the eigenvalue density correlation function (2.10) leads to the expression

$$-\frac{1}{\pi^2(\vartheta_1 - \vartheta_2)^2}(1 + o(1)).$$

This expression coincides with the large-distance limit of the correlation functions derived by F. Dyson [7].

Our proof is based on the derivation and asymptotic analysis of the system of relations for the moments (2.4) of the smoothed eigenvalue density $\hat{\rho}^{(\alpha,\tau)}_{N,m}$. We follow the route suggested in [12] and further developed in [4]. The main difference between the present work and the results of [4-I] is that here we start by considering the local eigenvalue density determined by $\hat{\rho}$ with $\alpha = 1$ that was not studied in [4-I] at all. The study of this local asymptotic regime results in more detailed information than that obtained in [4-I].

Basing on the estimates obtained in the local asymptotic regime, the analysis of the mesoscopic density $\hat{\rho}$ with $\alpha \in (0, 1)$ is much more easy than that carried out in [4-I]. In the study of the correlation function of $\hat{\rho}^{(\alpha,\tau)}_{N,m}$, we follow the lines of [4-I] and [4-II] with necessary changes.

To complete this section, let us describe the structure of this paper. In Section 3 we derive an infinite system of relations for the correlation function of the eigenvalue density. We give preliminary estimates that shows that this system consists of the main part and that one that vanishes in the limit either $N \to \infty$ and $\tau \to 0$. In Section 4 we consider this system in the case of local asymptotic regime and prove Theorem 2.1. Theorem 2.2 concerning the limiting eigenvalue density is proved in Section 5. In Section 6 we establish the selfaveraging property of the mesoscopic density and prove Theorem 2.3. In Section 7 we summarize our results and formulate final conclusions.

3 Relations for the moments

3.1 Basic calculus

In this paper we follow the scheme proposed in [12] and developed in [4] to study the smoothed eigenvalue density of random matrices with independent entries. Let us adapt the corresponding formulas to the case of Wishart matrices. As in [12, 4], we introduce the matrices

$$\begin{aligned} A(x,y) &= \tau N^{\alpha} H^{(N,m)} \\ A_\lambda(x,y) &= \tau N^{\alpha}\left(H^{(N,m)} - \lambda I\right)(x,y) \end{aligned} \tag{3.1}$$

and denote

$$\begin{aligned} R &= \left(I + A_\lambda^2\right)^{-1} \\ Q &= A_\lambda R = R A_\lambda. \end{aligned} \tag{3.2}$$

It is clear that for the resolvent

$$G^{(N,m)}(z) = (H^{(N,m)} - z)^{-1}$$

with $z = \lambda + \mathrm{i}\varepsilon$, we have

$$G \equiv G^{(N,m)}(z) = Q + \mathrm{i}R \tag{3.3}$$

and

$$\frac{1}{N}\operatorname{Tr} G\left(\lambda + \frac{\mathrm{i}}{\tau N^{\alpha}}\right) = \frac{\tau}{N^{1-\alpha}} \sum_{x=1}^{N} \{Q(x,x) + \mathrm{i}R(x,x)\}. \tag{3.4}$$

Here and below we omit superscripts N and m when this does not lead to confusions.

Let us denote by the bar sign the normalized traces:

$$\overline{Q} = \frac{\tau}{N^{1-\alpha}} \sum_{x=1}^{N} Q(x,x),$$

$$\overline{R} = \frac{\tau}{N^{1-\alpha}} \sum_{x=1}^{N} R(x,x).$$

Now it is not hard to observe that definition (2.2) and equality (3.3) imply

$$\hat{\rho}^{(\epsilon)}(\lambda; H^{(N,m)}) = \frac{1}{\pi N} \operatorname{Im} \operatorname{Tr} G^{(N,m)}(\lambda + \mathrm{i}\varepsilon) = \frac{1}{\pi} \overline{R}.$$

Our aim is to derive a system of relations for the moments of $\operatorname{Tr} G(z)$. To do this, we consider the moments of the random variables $r = \overline{R}$ and $q = \overline{Q}$. An important observation proved below is that the family

$$\begin{aligned} L_k &= \mathsf{E}\, r^k \\ M_k &= \mathsf{E}\, q r^k, \\ P_k &= \mathsf{E}\, q^2 r^k \end{aligned}$$

is sufficient to our purposes.

We derive our relations with the help of the elementary formula

$$\mathsf{E}\, \gamma F(\gamma) = \mathsf{E}\, \gamma^2\, \mathsf{E}\, F'(\gamma). \tag{3.5}$$

for a Gaussian random variable γ with zero average. Here $F(x)$, $x \in \mathbb{R}$ is a nonrandom function such that all integrals in (3.5) exist. It is easy to prove (3.5) by integration by parts.

We will need also the following expressions that follow from definitions (3.1) and (3.2):

$$\begin{aligned} \frac{\partial R(x,y)}{\partial \xi_j(s)} &= -\frac{\tau}{N^{1-\alpha}} \sum_{t=1}^{N} [Q(x,s)R(t,y) + Q(x,t)R(s,y)]\xi_j(t) \\ &\quad - \frac{\tau}{N^{1-\alpha}} \sum_{t=1}^{N} [R(x,s)Q(t,y) + R(x,t)Q(s,y)]\, \xi_j(t) \end{aligned} \tag{3.6a}$$

and

$$\begin{aligned} \frac{\partial Q(x,y)}{\partial \xi_j(s)} &= -\frac{\tau}{N^{1-\alpha}} \sum_{t=1}^{N} [Q(x,s)Q(t,y) + Q(x,t)Q(s,y)]\, \xi_j(t) \\ &\quad + \frac{\tau}{N^{1-\alpha}} \sum_{t=1}^{N} [R(x,s)R(t,y) + R(x,t)R(s,y)]\, \xi_j(t). \end{aligned} \tag{3.6b}$$

Relations (3.5) and (3.6) form a basis for the calculus we apply to the moments L_k, M_k, and P_k.

3.2 Derivation of the main relations

Let us start with L_k. We write $L_k = \mathsf{E}\, r^{k-1} r$ and apply to the last factor identity

$$R = I - RA_\lambda^2 = I + \lambda\tau N^\alpha Q - QA. \tag{3.7}$$

We obtain equality

$$L_k = \tau N^\alpha L_{k-1} + \lambda\tau N^\alpha M_{k-1} - \frac{\tau^2}{N^{2-2\alpha}} \sum_{x,s=1}^{N} \sum_{j=1}^{m} \mathsf{E}\, r^{k-1} Q(x,s)\xi_j(s)\xi_j(x). \tag{3.8}$$

To compute the last mathematical expectation, we use (3.5) with $\gamma = \xi_j(s)$ and $F = r^{k-1}Q(x,s)\xi_j(x)$. Taking into account (2.2), we can write that

$$\mathsf{E}\, r^{k-1} Q(x,s)\xi_j(s)\xi_j(x) = \delta_{xs}\, \mathsf{E}\, r^{k-1} Q(x,x) + v^2\, \mathsf{E}\, \frac{\partial r^{k-1} Q(x,s)}{\partial \xi_j(s)} \xi_j(x).$$

We compute the partial derivatives with the help of (3.6) and derive from (3.8) the relation

$$\begin{aligned} L_k &= \tau N^\alpha L_{k-1} - \lambda v^2 \tau N^\alpha L_{k+1} + \tau N^\alpha \left[\lambda + v^2(1-c_N)\right] M_{k-1} \\ &\quad + \lambda v^2 \tau N^\alpha P_{k-1} + \lambda v^2 \tau N^\alpha \left\{ \Phi_k^{(1)}(N) + \Psi_k^{(1)}(N) \right\}, \end{aligned} \tag{3.9}$$

where $c_N = m/N$,

$$\begin{aligned} \Phi_k^{(1)}(N) = &-\frac{2}{\lambda\tau N^\alpha} M_k - \frac{1}{\lambda N}\, \mathsf{E}\, r^{k-1} \left[2\overline{RQ} - \overline{Q}\right] \\ &+ \frac{4(k-1)\tau}{\lambda N^{2-\alpha}}\, \mathsf{E}\, r^{k-2} \left[\overline{RQ} - \overline{R^2 Q}\right], \end{aligned}$$

and

$$\Psi_k^{(1)}(N) = \frac{4(k-1)\tau^2}{N^{2-2\alpha}}\, \mathsf{E}\, r^{k-2} \left[\overline{R^2} - \overline{R^3}\right] + \frac{\tau}{N^{1-\alpha}}\, \mathsf{E}\, r^{k-1} \left[\overline{Q^2} - \overline{R^2}\right].$$

In these computations we used definitions (3.1) and (3.2) together with the elementary identity $Q^2 = R - R^2$.

Let us denote

$$\theta_N = \frac{1}{v^2} + \frac{1 - c_N}{\lambda}$$

and rewrite (3.9) in the form

$$L_{k+1} = \frac{1}{\lambda v^2} L_{k-1} + \theta_N M_{k-1} + P_{k-1} + \Psi_k^{(1)}(N) + \Phi_k^{(1)}(N) + \frac{1}{\tau N^\alpha} \left\{L_k\right\}. \tag{3.10}$$

We have derived our first main equality.

To explain the structure of this relation, let us note that the terms Φ will be shown vanishing in the limit $N \to \infty$ in both asymptotic regimes $\alpha = 1$ and $\alpha \in (0,1)$ while the terms Ψ vanish when $\alpha \in (0,1)$ but survive when $\alpha = 1$.

Using the same set of formulas (3.5), (3.6) and (3.7), we derive after certain amount of simple computations two other main relations:

$$M_{k-1} = -\frac{1}{2}\theta_N L_{k-1} + \Psi^{(2)}_{k-2}(N) + \Phi^{(2)}_{k-2}(N) \tag{3.11}$$

$$P_{k-1} = -\frac{1}{2}\theta_N M_{k-1} + \Psi^{(3)}_{k-2}(N) + \Phi^{(3)}_{k-2}(N). \tag{3.12}$$

In these relations we still separate the terms that provide non-zero contribution $N \to \infty$ for $\alpha = 1$ and those that do not. The first group of terms is given by expressions

$$\begin{aligned}
\Psi^{(2)}_{k-2}(N) &= -\frac{2(k-2)\tau}{N^{1-\alpha}} \mathsf{E}\, r^{k-3}\overline{R^2Q} - \frac{\tau}{N^{1-\alpha}} \mathsf{E}\, r^{k-2}\overline{RQ},\\
\Psi^{(3)}_{k-2}(N) &= -\frac{2(k-2)\tau}{N^{1-\alpha}} \mathsf{E}\, qr^{k-3}\overline{R^2Q} - \frac{\tau}{N^{1-\alpha}} \mathsf{E}\, qr^{k-2}\overline{RQ}\\
&\quad + \frac{\tau^2}{N^{2-2\alpha}} \mathsf{E}\, r^{k-2}\left[2\overline{R^3} - \overline{R^2}\right],
\end{aligned}$$

and the second one is represented by the terms

$$\begin{aligned}
\Phi^{(2)}_{k-2}(N) &= \frac{2(k-2)\tau}{\lambda N^{2-\alpha}} \mathsf{E}\, r^{k-3}\left[\overline{R^2} - \overline{R^3}\right] - \frac{1}{\lambda N} \mathsf{E}\, r^{k-2}\left[r - \overline{R^2}\right]\\
&\quad + \frac{1}{2\lambda\tau N^{\alpha}}\left\{L_k - P_{k-2} - \frac{1}{v^2}M_{k-2}\right\},
\end{aligned}$$

and

$$\begin{aligned}
\Phi^{(3)}_{k-2}(N) &= -\frac{1}{2v^2\lambda\tau N^{\alpha}}P_{k-2} - \frac{2(k-2)\tau}{\lambda N^{2-\alpha}} \mathsf{E}\, qr^{k-3}\left[\overline{R^2} - \overline{R^3}\right]\\
&\quad - \frac{1}{\lambda N} \mathsf{E}\, qr^{k-2}\left[r - \overline{R^2}\right] - \frac{1}{2\lambda\tau N^{\alpha}} \mathsf{E}\, q^3r^{k-2}\\
&\quad + \frac{1}{2\lambda\tau N^{\alpha}}M_k + \frac{\tau}{\lambda v^2 N^{2-\alpha}} \mathsf{E}\, r^{k-2}\left[2\overline{R^2Q} - \overline{RQ}\right],
\end{aligned}$$

respectively.

The full system of relations (3.10)-(3.12) together with the terms Φ and Ψ looks somewhat cumbersome. However, the positive thing is that it provides a complete description of the moments of the smoothed eigenvalue density. By these means all of three asymptotic regimes $\alpha = 0$, $\alpha = 1$ and $\alpha \in (0,1)$ can be studied.

3.3 General estimates

It is clear that the a priori estimate $|G^{(N,m)}(z)| \le |\mathrm{Im}\, z|^{-1}$ implies inequalities

$$\bar{R} \le \frac{1}{\tau N^{\alpha}} \quad \text{and} \quad |\bar{Q}| \le \frac{1}{\tau N^{\alpha}}$$

that diverge when $N \to \infty$. Thus, the absolute estimates of the terms $\Phi(N)$ and $\Psi(N)$ do not show that these terms vanish in the limit $N \to \infty$. In this section we estimate them in terms of $L_k^{(N)}$, $M_k^{(N)}$, and $P_k^{(N)}$.

It is obvious that $\|R\| \leq 1$ and $\|Q\| \leq 1$. These estimates, together with positivity of R, imply following inequalities (see e.g. [9], p. 182)

$$\begin{aligned} \overline{R^{k+1}} &\leq r, \\ \overline{RQ} &\leq r, \\ \overline{R^2Q} &\leq r. \end{aligned} \tag{3.13}$$

Using them, one can easily derive that

$$|\Phi_k^{(1)}(N)| \leq \frac{2}{\lambda\tau N^\alpha}|M_k| + \frac{1}{\lambda N}(2L_k + |M_{k-1}|) + \frac{8(k-1)\tau}{\lambda N^{2-\alpha}} L_{k-1} \tag{3.14a}$$

and

$$|\Psi_k^{(1)}(N)| \leq \frac{8(k-1)\tau^2}{N^{2-2\alpha}} L_{k-1} + \frac{3\tau}{N^{1-\alpha}} L_k. \tag{3.14b}$$

Taking into account that $|q| \leq 1 + q^2$ and

$$L_{k-1} \leq [L_k]^{(k-1)/k} \leq \max\{1, L_k\} \equiv \bar{L}_k,$$

we can deduce from (3.14) inequalities

$$|\Phi_k^{(1)}(N)| \leq \frac{5 + 8(k-1)\tau^2}{\lambda\tau N^\alpha} \bar{L}_k + \frac{2}{\lambda\tau N^\alpha} P_k + \frac{1}{\lambda N} P_{k-1} \tag{3.15a}$$

and

$$|\Psi_k^{(1)}(N)| \leq \frac{8(k-1)\tau^2 + 3\tau}{N^{1-\alpha}} \bar{L}_k. \tag{3.15b}$$

Next, regarding $\Phi_{k-2}^{(2)}(N)$ and $\Psi_{k-2}^{(2)}(N)$, we can write that

$$|\Phi_{k-2}^{(2)}(N)| \leq \frac{C_k(\tau,\lambda)}{\tau N^\alpha} \overline{L}_k + \frac{1 + v^{-2}}{2\lambda\tau N^\alpha} P_{k-2}, \tag{3.16a}$$

where $C_k(\tau,\lambda) = [8(k-2)\tau^2 + 4\tau + 2 + v^{-2}](1 + \lambda^{-1})$ and

$$|\Psi_{k-2}^{(2)}(N)| \leq \frac{(2k-3)\tau}{\lambda N^{1-\alpha}} \bar{L}_{k-1}. \tag{3.16b}$$

To estimate the third group of terms we will need some more computations. The term $\Psi^{(3)}$ is directly bounded:

$$|\Psi_{k-2}^{(3)}(N)| \leq \frac{(2k-3)\tau + 3\tau^2}{N^{1-\alpha}} \bar{L}_{k-1} + \frac{2(k-2)\tau}{N^{1-\alpha}} P_{k-1} + \frac{\tau}{N^{1-\alpha}} P_{k-2}. \tag{3.17a}$$

For $\Phi^{(3)}$ we derive a first inequality that involves the terms M:

$$|\Phi^{(3)}_{k-2}(N)| \le \frac{4(k-2)\tau}{\lambda N^{2-\alpha}}|M_{k-2}| + \frac{1}{\lambda N}|M_{k-1}| + \frac{1}{2\lambda\tau N^\alpha}|M_k| + \frac{1}{2v^2\lambda\tau N^\alpha}P_{k-2} + \frac{3\tau}{\lambda v^2 N^{2-\alpha}}L_{k-1} + \frac{1}{2\lambda\tau N^\alpha}|T_{k-2}|, \quad (3.17b)$$

where $T_{k-2} = \mathsf{E}\, q^3 r^k$.

In this inequality, we estimate $|M_{k-2}|$ and $|M_{k-1}|$ by $L_{k-2} + P_{k-2}$ and $L_{k-1} + P_{k-1}$, respectively. To avoid the appearance of P_k in the final expressions, we estimate M_k more precisely.

We use relation (3.11) and write that

$$|M_k| \le \frac{1}{2}\theta_N L_k + |\Phi^{(2)}_{k-1}(N)| + |\Psi^{(2)}_{k-1}(N)|.$$

Using the inequality $L_{k+1} \le \tau N^\alpha L_k$ which is obvious, we conclude that

$$\frac{1}{2\lambda\tau N^\alpha}|M_k| \le \frac{1}{2\lambda\tau N^\alpha}\left(\frac{\theta_N}{2} + C_k(\tau,\lambda) + \frac{(2k-3)\tau}{\lambda N^{1-\alpha}}\right) L_k + \frac{1+v^{-2}}{(2\lambda\tau N^a)^2}P_{k-1}.$$

Now we substitute this relation into (3.17b) and obtain the estimate

$$|\Phi^{(3)}_{k-2}(N)| \le \frac{4C_k(\tau,\lambda)}{\tau N^\alpha}\left(L_k + P_{k-1} + P_{k-2} + T_{k-2}\right). \quad (3.18)$$

Let us turn to the auxiliary terms T_k. For the purposes of the next section, it is sufficient to show that

$$\frac{1}{\tau N^\alpha}|T_{k-2}| \le |\mathsf{E}|q|r^{k-1}| \le L_{k-1} + P_{k-1}. \quad (3.19)$$

The first inequality of (3.19) follows from elementary computations

$$q^2 = \frac{1}{N^{2-2\alpha}}\sum_{i,j}\frac{d_j}{1+d_j^2}\frac{d_i}{1+d_i^2} \le \frac{1}{2N^{2-2\alpha}}\sum_{i,j}\frac{d_i^2+d_j^2}{(1+d_i^2)(1+d_j^2)} < \tau N^\alpha r, \quad (3.20)$$

where $d_i = N^\alpha(\lambda - \lambda_i)$. Now we are ready to study the local spectral asymptotic regime.

4 Estimates in the local asymptotic regime

Let us rewrite the system (3.10)-(3.11) in the following form:

$$L_{k+1} = \left(\frac{1}{\lambda v^2} - \frac{\theta_N^2}{4}\right)L_{k-1} + \Phi^{(1)}_k + \Psi^{(1)}_k + \Phi^{(3)}_{k-2} + \Psi^{(3)}_{k-2} + \frac{\theta_N}{2}\left(\Phi^{(2)}_{k-2} + \Psi^{(2)}_{k-2}\right), \quad (4.1)$$

$$P_{k-1} = \frac{\theta_N^2}{4}L_{k-1} + \Phi^{(3)}_{k-2} + \Psi^{(3)}_{k-2} - \frac{\theta_N}{2}\left(\Phi^{(2)}_{k-2} + \Psi^{(2)}_{k-2}\right) \quad (4.2)$$

Using estimates (3.15b), (3.16b) and (3.17a) for the terms Ψ and gathering terms Φ in $\delta_{N,k}$, we obtain for the case when $\alpha = 1$ that

$$L_{k+1} \leq [k(8\tau^2+\theta_N\lambda^{-1}+2\tau)+3(1+\tau)^2+\delta_N]\bar{L}_k+(2k-3)\tau P_{k-2}+\tau P_{k-1} \quad (4.3)$$

and

$$\begin{aligned} P_{k-1} \leq & \frac{(2k-3)\tau}{1-\tau N^{\alpha-1}} P_{k-2} \\ & + \frac{1}{1-\tau N^{\alpha-1}} \left[(2k-3)\tau \left(1 + \frac{\theta_N}{2\lambda} \right) + \frac{\theta_N^2}{4} + 3\tau^2 + \delta_N \right] \bar{L}_{k-1}, \end{aligned} \quad (4.4)$$

where $\delta_N = \mathrm{O}(k/N)$ vanishes as $N \to \infty$. Let us note here that according to conditions of Theorem 2.1, $0 < \tau < 1$ and we keep the factors $N^{\alpha-1}$ in the right-hand side of (4.4) to use this relation (4.4) also in the next section.

One can rewrite these relations (4.3) and (4.4) as

$$\begin{aligned} L_{k+1} &\leq k a_1 \bar{L}_k + k b_1 P_{k-2} + \tau P_{k-1} \\ P_{k-1} &\leq k a_2 P_{k-2} + k b_2 \bar{L}_{k-1} \end{aligned}$$

with some a_i and b_i that do not depend on N. This system is equal to

$$\begin{aligned} \bar{L}_{k+1} &\leq k \tilde{a}_1 \bar{L}_k + k \tilde{b}_1 P_{k-2} \\ P_{k-1} &\leq k a_2 P_{k-2} + k b_2 \bar{L}_{k-1} \end{aligned} \quad (4.5)$$

Let us denote $\bar{a} = \max\{\tilde{a}_1, a_2\}$ and $\bar{b} = \max\{\tilde{b}_1, b_2\}$. Then elementary calculations show that if

$$\begin{aligned} L_k &\leq (\bar{a}+\bar{b})^k k! LP_2 \\ P_{k-1} &\leq (\bar{a}+\bar{b})^k k! LP_2, \end{aligned} \quad (4.6)$$

where $LP_2 = \max\{L_2, P_0\}$, then (4.5) holds.

End of proof of Theorem 2.1. To complete the proof of Theorem 2.1, it is sufficient to show that

$$LP_2 \leq C\bar{L}_2$$

and that $L_2^{(N)}$ does not increase when $N \to \infty$. Let us consider relation (3.9) with $k = 1$:

$$\begin{aligned} \mathsf{E}\, r^2 = & \frac{1}{\lambda v^2} + \theta_N \,\mathsf{E}\, q + \mathsf{E}\, q^2 + \frac{2}{\lambda \tau N^\alpha} \mathsf{E}\, rq \\ & + \frac{1}{\lambda N} \mathsf{E} \left(q + 2\overline{RQ} \right) + \frac{\tau}{N^{1-\alpha}} \mathsf{E} \left(r - 2\overline{R^2} \right). \end{aligned} \quad (4.7)$$

Using inequality $|\mathsf{E}\, q| \leq (\mathsf{E}\, q^2)^{1/2}$ together with our usual estimates, we derive from (4.7) that

$$\mathsf{E}\, r^2(1+o(1)) \geq \frac{1}{\lambda v^2} - \theta_N^2 + \left(\sqrt{\mathsf{E}\, q^2} - \theta_N \right)^2 (1 - o(1)).$$

Since, $P_0 = \mathsf{E}\, q^2$, then the last inequality implies the expected estimate for LP_2.

Now let us assume that $L_k^{(N)} = \mathrm{O}(f(N))$ when $N \to \infty$. Then (4.6) implies

$$L_k^{(N)} \le k!(\bar a + \bar b)^k (1 + C) L_2^{(N)} \le k!(\bar a + \bar b)^k (1 + C)[L_k^{(N)}]^{1/k}.$$

It follows that $L_k^{(N)}$ increases not faster than $[f(N)]^{1/k}$ that contradicts our assumption. Theorem 2.1 is proved. □

5 Mesoscopic density of eigenvalues

In this section we apply the computations of Section 3 to study the moments of $\hat\rho^{(\epsilon)}(\lambda; H^{(N,m)})$ with $\alpha \in (0, 1)$ when $N \to \infty$. We will need the following result.

Lemma 5.1. *For sufficiently large N, we have the estimate*

$$|T_{k-2}| \le v^2 C \left(aP_{k-2} + b_k P_{k-3} + 6L_{k-2}\right). \tag{5.1}$$

with some constant C and $a = \lambda\theta_N + \lambda(2 + \tau)^2 + 1$, $b_k = (2k - 3)\lambda\tau^2 + k$.

We postpone the proof of this statement to the end of this section.

Proof of Theorem 2.2. The next observation is that estimates (4.6) derived in the case of $\alpha = 1$ remain valid when $\alpha \in (0, 1)$. Then, denoting $\rho_N = (\lambda v^2)^{-1} - \theta_N^2/4$, we can rewrite (4.1) in the form

$$L_{k+1}^{(N)} = \rho_N L_{k-1}^{(N)} + \tilde\Phi_k(N), \tag{5.2}$$

where

$$\tilde\Phi_k(N) = \mathrm{O}((k + 1)N^{-\chi})$$

with

$$\chi = \min\{\alpha, 1 - \alpha\}.$$

Then obviously (2.8) follows that proves Theorem 2.2. □

Proof of Lemma 5.1. We repeat computations of Section 3 and derive

$$T_{k-2} = -\frac{1}{2}\theta_N P_{k-2} + \frac{1}{2\lambda v^2 \tau N^\alpha} T_{k-3} + \Phi_{k-2}^{(4)}(N) + \Psi_{k-2}^{(4)}(N), \tag{5.3}$$

where

$$\begin{aligned}\Phi_{k-2}^{(4)}(N) = {} & \frac{1}{2\lambda\tau N^\alpha} P_{k-1} - \frac{1}{N} P_{k-2} \\ & - \frac{1}{N}\,\mathsf{E}\, q^2 r^{k-3}\overline{R^2} - \frac{k\tau}{2\lambda N^{2-\alpha}}\,\mathsf{E}\, q^2 r^{k-4}\left(\overline{R^2} - \overline{R^3}\right) \\ & - \frac{2\tau}{\lambda N^{2-\alpha}}\,\mathsf{E}\, q r^{k-3}\left(\overline{R^2 Q} - \overline{R^3 Q}\right) + \frac{1}{2\lambda\tau N^\alpha}\,\mathsf{E}\, q^4 r^{k-3}\end{aligned} \tag{5.4}$$

and

$$\Psi^{(4)}_{k-2}(N) = -\frac{\tau}{N^{1-\alpha}} \mathsf{E}\, q^2 r^{k-3}\overline{RQ}$$
$$- \frac{2\tau^2(k-3)}{N^{2-2\alpha}} \mathsf{E}\, q^2 r^{k-4}\overline{R^2 Q} + \frac{2\tau^2}{N^{2-2\alpha}} \mathsf{E}\, q r^{k-3} \left(2\overline{R^3} - \overline{R^2}\right).$$

To estimate the last term of (5.4), we use inequality (3.20) and write that $N^{-\alpha}\, \mathsf{E}\, q^4 r^{k-3} \leq P_{k-2}$. Taking also into account that $P_k \leq \tau N^{\alpha} P_{k-1}$, we derive from (5.3) that

$$T_{k-2} \leq \frac{1}{2\lambda v^2 \tau N^{\alpha}} T_{k-3} + S_{k-2}(N),$$

where

$$|S_{k-2}(N)| \leq \left(\frac{\theta_N}{2} + \frac{1}{\lambda} + (2+\tau)^2\right) P_{k-2}$$
$$+ \frac{2(k-3) + k/\lambda}{N^{2-2\alpha}} P_{k-3} + \frac{6}{N^{2-2\alpha}} L_{k-2}.$$

Therefore

$$T_{k-2} \leq \sum_{j=0}^{k-3} |S_{k-2-j}| \left(\frac{1}{2\lambda v^2 \tau N^{\alpha}}\right)^j. \tag{5.5}$$

Taking once more into account estimates (4.6), we easily arrive at (5.1). Lemma 5.1 is proved. $\square$

6 Mesoscopic spectral fluctuations

In this section we study the centered random variables

$$\begin{cases} r^{\circ} = r - \mathsf{E}\, r \\ q^{\circ} = q - \mathsf{E}\, q \end{cases}$$

given by (3.4) with $\tau = 1$ and $0 < \alpha < 1$.

Let us consider the average $\mathsf{E}\, r^{\circ} r^{\circ} = \mathsf{E}\, r^{\circ} r$ and apply to the last factor identity (3.7). We obtain the relation

$$\mathsf{E}\, r^{\circ} r = -\frac{1}{N^{1-\alpha}} \sum_{x,s=1}^{N} \sum_{j=1}^{m} \mathsf{E}\, r^{\circ} Q(x,s)\xi_j(s)\xi_j(x) + \lambda N^{\alpha}\, \mathsf{E}\, r^{\circ} q$$

that can be treated with the help of formulas (2.1) and (2.3). We compute the mathematical expectation and derive the relation

$$N^{-\alpha}\, \mathsf{E}\, r^{\circ} r = \left(\lambda + (1 - c_N)\, v^2\right) \mathsf{E}\, r^{\circ} q$$
$$+ \lambda v^2 \left(\mathsf{E}\, r^{\circ} qq - \mathsf{E}\, r^{\circ} rr\right) + \Psi_1(N), \tag{6.1a}$$

where

$$\Psi_1(N) = \frac{\lambda v^2}{N^{1-\alpha}} \,\mathsf{E}\, r^\circ r - \frac{v^2}{N^\alpha} \,\mathsf{E}\, r^\circ q r$$
$$- \frac{2v^2}{N} \,\mathsf{E}\, r^\circ \left(\overline{RQ} + \lambda N^\alpha \overline{R^2} \right)$$
$$+ \frac{4v^2}{N^{2-2\alpha}} \,\mathsf{E} \left\{ \overline{RQ^3} + \lambda N^\alpha \overline{RQ^2} \right\}. \tag{6.1b}$$

A similar procedure leads to the relation

$$N^{-\alpha} \,\mathsf{E}\, q^\circ q = -\left(\lambda + (1-c_N) v^2\right) \mathsf{E}\; q^\circ r - 2\lambda v^2 \,\mathsf{E}\, q^\circ q r + \Psi_2(N), \tag{6.2a}$$

where

$$\Psi_2(N) = \frac{v^2}{N^\alpha} \left(\mathsf{E}\, q^\circ r r - \mathsf{E}\, q^\circ q q\right) + \mathrm{O}\left(\frac{1}{N^{1-\alpha}} \,\mathsf{E}\, |q^\circ| + \frac{1}{N^{2-2\alpha}} \right). \tag{6.2b}$$

These relations play a central role in what follows. Basing on them, let us first estimate the order of the variances of the random variables r and q:

$$\mathsf{V}\, r = \mathsf{E}\, r^\circ r^\circ = \mathsf{E}\, r^\circ r$$
$$\mathsf{V}\, q = \mathsf{E}\, q^\circ q^\circ = \mathsf{E}\, q^\circ q.$$

6.1 Selfaveraging property

Let us take the sum of (6.1a) and (6.2a), write it in the form

$$\lambda v^2 \,\mathsf{E}\, r^\circ r \,\mathsf{E}\, r + \lambda v^2 \,\mathsf{E}\, r^\circ r^\circ r + \lambda v^2 \left[2 \,\mathsf{E}\, q^\circ q r - \mathsf{E}\, r^\circ q^2 \right] =$$
$$= -\frac{v^2}{N^\alpha} \left\{ \mathsf{E}\, q^\circ q q - \mathsf{E}\, q^\circ r r \right\}$$
$$+ \mathrm{O}\left(\frac{1}{N^{1-\alpha}} \,\mathsf{E}\, |q^\circ| + \frac{1}{N^{1-\alpha}} \,\mathsf{E}\, |r^\circ| + \frac{1}{N^{2-2\alpha}} \right) \tag{6.3}$$

and consider the term in square brackets. Using identities

$$2 \,\mathsf{E}\, q^\circ q r \equiv 2 \,\mathsf{E}\, q^\circ r \,\mathsf{E}\, q + 2 \,\mathsf{E}\, q^\circ q^\circ r$$
$$\mathsf{E}\, r^\circ q^2 \equiv 2 \,\mathsf{E}\, r^\circ q \,\mathsf{E}\, q + \mathsf{E}\, r^\circ q^\circ q^\circ,$$

we derive that

$$2 \,\mathsf{E}\, q^\circ q r - \mathsf{E}\, r^\circ q^2 = 2 \,\mathsf{E}\, q^\circ q^\circ r - \mathsf{E}\, r^\circ q^\circ q^\circ = \mathsf{E}\; q^\circ q^\circ \,\mathsf{E}\, r + \mathsf{E}\, q^\circ q^\circ r. \tag{6.4}$$

Taking into account the positivity of r, we derive from (6.3) and (6.4) that

$$\lambda v^2 \left(\mathsf{E}\, r^\circ r^\circ + \mathsf{E}\, q^\circ q\right) \mathsf{E}\, r + \frac{v^2}{N^\alpha} \left(\mathsf{E}\, r^\circ r^\circ + \mathsf{E}\, q^\circ q\right) =$$
$$= \mathrm{O}\left(\frac{1}{N^{1-\alpha}} \,\mathsf{E}\, |q^\circ| + \frac{1}{N^{1-\alpha}} \,\mathsf{E}\, |r^\circ| + \frac{1}{N^{2-2\alpha}} \right). \tag{6.5}$$

Since $\mathsf{E}\, r \equiv L_1^{(N)}$, then we can apply Theorem 2.2 and obtain for sufficiently large N the inequality

$$\mathsf{E}\, r \geq \pi \frac{\rho_c}{2} > 0.$$

Then inequality

$$\mathsf{V}\, r + \mathsf{V}\, q \leq N^{\alpha-1} \left(\sqrt{\mathsf{V}\, r} + \sqrt{\mathsf{V}\, q}\right) + \mathrm{O}\left(N^{2\alpha-2}\right)$$

follows from (6.5). This proves that

$$\begin{cases} \mathsf{V}\, r = \mathrm{O}(N^{2\alpha-2}) \\ \mathsf{V}\, q = \mathrm{O}(N^{2\alpha-2}). \end{cases} \tag{6.6}$$

6.2 Limiting gaussian random field

Let us denote

$$\gamma_j = \gamma_{N,m}^{(\alpha)}(\lambda + \vartheta_j N^{-\alpha}) = r^\circ N^{1-\alpha}$$
$$\eta_j = \eta_{N,m}^{(\alpha)}(\lambda + \vartheta_j N^{-\alpha}) = q^\circ N^{1-\alpha}.$$

The main aim of this section is to show that the characteristic function

$$\phi_N^{(k)} \equiv \phi_N(t_1, \dots, t_k) = \mathsf{E} \exp\left\{\mathrm{i}\left(t_1\gamma_1 + \dots + t_k\gamma_k\right)\right\}$$

satisfies in the limit $N \to \infty$ the relation

$$\frac{\partial}{\partial t_l}\phi_N^{(k)} = -\sum_{j=1}^{k} B_{lj} t_j \phi_N^{(k)} \left(1 + o(1)\right), \tag{6.7}$$

where B_{lj} is determined by the right-hand side of (2.5):

$$B_{lj} = -\frac{1}{\pi^2}\frac{4 - (\vartheta_l - \vartheta_j)^2}{[4 + (\vartheta_l - \vartheta_j)^2]^2}. \tag{6.8}$$

Then the statement of Theorem 2.3 will follow from (6.7), (6.8) and from standard arguments of probability theory.

We also will show that the families of random variables γ and η become jointly independent in the limit $N \to \infty$.

According to the definition of $\phi_N^{(k)}$,

$$\frac{\partial}{\partial t_l}\phi_N^{(k)} = \mathsf{E}\left\{\phi_N^{(k)}\gamma_l\right\} = \mathrm{i}N^{1-\alpha}\,\mathsf{E}\, e_k r_l^\circ = \mathrm{i}N^{1-\alpha}\,\mathsf{E}\, e_k^\circ r_l, \tag{6.9}$$

where we denoted $e_k = \exp\{\mathrm{i}(t_1\gamma_1 + \dots + t_k\gamma_k)\}$ and $r_l = \overline{R_l} \equiv \hat\rho_{N,m}^{(\alpha)}(\lambda_l)$, $\lambda_l = \lambda + \tau_l N^{-\alpha}$. In the last expression of (6.9), we apply to the factor r_l formulas (2.1) and (2.3) and obtain

$$\mathsf{E}\, e_k^\circ r_l = -\frac{1}{N^{1-\alpha}}\sum_{x,s=1}^{N}\sum_{j=1}^{m}\mathsf{E}\, e_k^\circ Q_l(x,s)\xi_j(s)\xi_j(x) + \lambda_l N^\alpha\,\mathsf{E}\, e_k^\circ q_l,$$

where

$$q_l = \overline{Q_l} \equiv \overline{R_l A_{\lambda_l}}\,.$$

Computing mathematical expectation and gathering similar terms, we derive

$$\begin{aligned}
2\lambda_l v^2\, \mathsf{E}\, e_k^\circ r_l\ \mathsf{E}\, r_l + \lambda v^2\, \mathsf{E}\ e_k^\circ r_l^\circ r_l^\circ + \frac{1}{N^\alpha}\, \mathsf{E}\, e_k^\circ r_l = \\
= \theta_N\, \mathsf{E}\, e_k^\circ q_l + 2\lambda v^2\, \mathsf{E}\, e_k^\circ q_l\ \mathsf{E}\, q_l \\
+ \lambda v^2\, \mathsf{E}\, e_k^\circ q_l^\circ q_l^\circ + 4iv^2 V_{lj}(N) + \Upsilon_1(N),
\end{aligned} \tag{6.10}$$

where

$$V_{lj}(N) = \sum_{j=1}^{k} t_j\, \mathsf{E}\, \overline{Q_j Q_l R_l}\ \mathsf{E}\, e_k \tag{6.11}$$

$$\begin{aligned}
\Upsilon_1(N) = 4\mathrm{i}v^2 \sum_{j=1}^{k} t_l\, \mathsf{E}\left\{\overline{Q_j Q_l R_l A}\, e_k^\circ\right\} + \frac{v^2}{N}\left(\mathsf{E}\, e_k^\circ q_l + \mathsf{E}\ e_k^\circ r_l\right) \\
+ \frac{2v^2}{N}\, \mathsf{E}\, e_k^\circ \left(\overline{R_l Q_l} + \lambda N^\alpha \overline{R_l^2}\right) \\
- \frac{2v^2}{N^\alpha}\left(\mathsf{E}\, e_k^\circ q_l\, \mathsf{E}\, r_l + \mathsf{E}\, e_k^\circ r_l\, \mathsf{E}\, q_l + \mathsf{E}\, e_k^\circ q_l^\circ r_l^\circ\right).
\end{aligned} \tag{6.12}$$

The proof of relations (6.7) and (6.8) can be splitted into two parts. The first one is to compute the limit of $V_{lj}(N)$ (6.11) as $N \to \infty$. The second part deals with estimates of the term $\Upsilon_1(N)$. These parts use the following results whose proofs we discuss at the end of this section.

Proposition 6.1. *Under conditions of Theorem 2.2 relations*

$$\begin{aligned}
\mathsf{E}\, \overline{R^2} = \mathsf{E}\, r/2\,(1 + o(1))\,, \\
\mathsf{E}\, \overline{R^3} = 3\, \mathsf{E}\, r/8\,(1 + o(1))
\end{aligned} \tag{6.12}$$

and

$$\begin{aligned}
\mathsf{E}\, \overline{Q_l R_k} = o(1), \\
\mathsf{E}\, e_k^\circ q_k = o(1)
\end{aligned} \tag{6.13}$$

hold in the limit $N \to \infty$.

Remark. In fact, we will prove a little more than (6.12):

$$\mathsf{E}\, \overline{R_k R_l} = \frac{\tilde r_k \tilde r_l}{\tilde r_k + \tilde r_l}\,(1 + o(1))\,, \tag{6.14}$$

where $\tilde r = \mathsf{E}\, r$.

Proposition 6.2. *The following relations*

$$\begin{aligned} \mathsf{V}\,\overline{R_k Q_k} &= o(1), \\ \mathsf{V}\,\overline{R_k R_k} &= o(1) \end{aligned} \tag{6.15}$$

hold in the limit $N \to \infty$.

Now let us study the right-hand side of equality (6.11). Taking into account definitions (3.4), we can write that

$$\begin{aligned} \overline{R_l Q_l Q_j} &= \frac{1}{N^{1-\alpha}} \operatorname{Tr} \left(\frac{G_l - \bar{G}_l}{2\mathrm{i}N^\alpha} \cdot \frac{G_l + \bar{G}_l}{2N^\alpha} \cdot \frac{G_j + \bar{G}_j}{2N^\alpha} \right) \\ &= \frac{1}{8\mathrm{i}N^{1+2\alpha}} \operatorname{Tr} \left[G_l^2 G_j - \bar{G}_l^2 G_j + G_l^2 \bar{G}_j - \bar{G}_l^2 \bar{G}_l \right]. \end{aligned} \tag{6.16}$$

The next step is to use the identity

$$G_l G_j = -\frac{G_l - G_j}{z_l - z_j}$$

and its consequence

$$G_l^2 G_j = \frac{G_l^2}{z_l - z_j} - \frac{G_l - G_j}{(z_l - z_j)^2}.$$

It is not hard to see that

$$\begin{aligned} \frac{1}{N^{1+2\alpha}} \operatorname{Tr} G_l^2 &= \frac{1}{N} \operatorname{Tr} \left[Q_l^2 - R_l^2 + 2\mathrm{i} Q_l R_l \right] \\ &= \frac{1}{N^\alpha} \left[\overline{Q_l^2} - \overline{R_l^2} + 2\mathrm{i} \overline{Q_l R_l} \right]. \end{aligned}$$

Now let us observe that (6.12) and (6.13) imply that the right-hand side of the following relation

$$\frac{1}{8\mathrm{i}N^{1+2\alpha}} \cdot \frac{1}{z_l - z_j} \operatorname{Tr} G_l^2 = \frac{1}{8\mathrm{i}(\vartheta_l - \vartheta_j)} \left[\overline{R_l} - 2\overline{R_l^2} + 2\mathrm{i} \overline{Q_l R_l} \right]$$

vanishes as $N \to \infty$. Now elementary computations show that

$$\begin{aligned} \overline{R_l Q_l Q_j} &= -\frac{\overline{R_l} - \overline{R_j}}{4(\vartheta_l - \vartheta_j)^2} \\ &\quad - \frac{(\overline{R_l} + \overline{R_j}) \left[(\vartheta_l - \vartheta_j + 2\mathrm{i})^2 + (\vartheta_l - \vartheta_j - 2\mathrm{i})^2 \right]}{8(\vartheta_l - \vartheta_j + 2\mathrm{i})^2 (\vartheta_l - \vartheta_j - 2\mathrm{i})^2} + o(1) \\ &= \frac{r_k + r_l}{4} \cdot \frac{4 - (\vartheta_l - \vartheta_j)^2}{\left[(\vartheta_l - \vartheta_j)^2 + 4 \right]^2} + o(1). \end{aligned} \tag{6.17}$$

It follows from (6.10), (6.13) and estimates (6.15) that

$$2\lambda_l v^2 \mathsf{E}\, e_k^\circ r_l = \frac{\mathsf{E}\, r_l + \mathsf{E}\, r_j}{\mathsf{E}\, r_l} \cdot \frac{4 - (\vartheta_l - \vartheta_j)^2}{[(\vartheta_l - \vartheta_j)^2 + 4]^2} + o(1) \tag{6.18}$$

provided

$$\mathsf{E}\, e_k^\circ r_l^\circ r_l^\circ = o(N^{2-2\alpha}) \tag{6.19}$$

as $N \to \infty$. This relation can be easily proved with the help of estimates (6.6).

Taking into account equality (6.9), it is not hard to derive (6.7)-(6.8) from (6.18).

Proof of Proposition 6.1. Let us start with the first relation from (6.13) and denote $t_{kl} = \mathsf{E}\, T_{kl} = \mathsf{E}\, \overline{R_k Q_l}$ and $r_{kl} = \mathsf{E}\, R_{kl} = \mathsf{E}\, \overline{R_k R_l}$. The first steps repeat the procedure of derivation of main relations (3.11). We use definition (3.2), relation (3.1), and formulas (3.5) and (3.6) to compute the mathematical expectations. Then we obtain

$$\begin{aligned} \mathsf{E}\, T_{kl} = & -\lambda_l N^\alpha\, \mathsf{E}\, \overline{R_k R_l} + v^2 c_N N^\alpha\, \mathsf{E}\, \overline{R_k R_l} \\ & - v^2\, \mathsf{E}\left[\overline{R_k R_l Q_l A} + \overline{R_k Q_l R_l A}\right] \\ & - \frac{v^2}{N^{1-\alpha}}\, \mathsf{E}\left[\overline{R_k R_l l A Q_l} + \overline{R_k Q_l A R_l}\right] \\ & - 2v^2\, \mathsf{E}\, r_k \overline{Q_k R_l A} - \frac{2v^2}{N^{1-\alpha}}\, \mathsf{E}\, \overline{R_k A Q_k Q_l}. \end{aligned}$$

Using a procedure which is usual for us and assuming that estimates of Proposition 6.2 are true, we derive

$$\lambda_l v^2 \tilde{r}_l t_{kl} = r_{kl}\left[-\lambda_l + v^2(c-1) - \lambda_l v^2 \tilde{q}_l\right] - 2\lambda_l v^2 \tilde{r}_k t_{lk} + o(1), \tag{6.20a}$$

where $\tilde{r}_k = \mathsf{E}\, r_k$ and $\tilde{q}_k = \mathsf{E}\, q_k$. Repeating the same procedure, we obtain

$$\lambda_k v^2 \tilde{r}_k t_{lk} = r_{kl}\left[-\lambda_k + v^2(c-1) - \lambda_l v^2 \tilde{q}_k\right] - 2\lambda_k v^2 \tilde{r}_l T_{kl} + o(1). \tag{6.20b}$$

Subtracting (6.20b) from (6.20a) and taking into account (2.8)

$$-\lambda_k + v^2(c-1) - \lambda_l v^2 \tilde{q}_k = o(1), \quad N \to \infty,$$

we conclude that

$$-\lambda_l \tilde{r}_l t_{kl} + \lambda_k \tilde{r}_k t_{lk} = o(1).$$

Therefore

$$3\lambda_k v^2 \tilde{r}_k t_{lk} = \left[-\lambda_l + v^2(c-1) - \lambda_l v^2 \tilde{q}_l\right] r_{kl} + o(1)$$

and the boundedness of $\mathsf{E}\, \overline{R_k R_l}$ implies the first relation from (6.13).

Let us prove (6.14). Regarding $\mathsf{E}\,\overline{R_k R_l}$, we apply to the last factor identity (3.1) and compute the mathematical expectation with the help of (3.5) and (3.6). Then we obtain

$$\begin{aligned}
\mathsf{E}\,R_{kl} &= \mathsf{E}\,r_k + \lambda_l N^\alpha\,\mathsf{E}\,T_{kl} + v^2\,\mathsf{E}\left\{r_k(q_k - \overline{Q_k R_l})\right\} \\
&\quad + \lambda_l N^\alpha v^2\,\mathsf{E}\,r_k Q_{kl} - N^\alpha v^2 c_N\,\mathsf{E}\,T_{kl} + v^2\,\mathsf{E}\,q_k(r_k - R_{kl}) \\
&\quad + \lambda_l v^2 N^\alpha\,\mathsf{E}\,q_k T_{kl} + v^2 N^\alpha\,\mathsf{E}\,T_{kl} - v^2\,\mathsf{E}\,r_l T_{kl} \\
&\quad + \lambda_l v^2 N^\alpha\,\mathsf{E}\,q_l T_{kl} - v^2\,\mathsf{E}\,R_{kl} q_l - \lambda_l v^2 N^\alpha\,\mathsf{E}\,r_l R_{kl} \\
&\quad + \frac{v^2}{N^{2-2\alpha}}\,\mathsf{E}\left[2\overline{Q_k^2 Q_l} + \overline{Q_l^2 Q_k} - \overline{Q_k R_l^2}\right] \\
&\quad + \frac{v^2\lambda_k N^\alpha}{N^{1-\alpha}}\,\mathsf{E}\left[3\overline{R_k Q_k Q_l} - \overline{R_k R_l^2}\right],
\end{aligned}$$

where we denote $Q_{kl} = \overline{Q_k Q_l}$. Using the selfaveraging property (6.6) and estimates (3.13), we derive from this equality that

$$\left[v^2(1-c_N)\lambda_l + v^2\lambda_l\tau q_l\right] T_{kl} + \lambda_l v^2\tau r_k Q_{kl} - \lambda_l v^2 \tau r_l R_{kl} = o(1)$$

as $N \to \infty$. Finally, observing that

$$Q_{kl} = r_l - R_{kl} + (\lambda_k - \lambda_l) N^\alpha T_{lk}$$

and taking into account the first relation from (6.13), we arrive at (6.14).

The second relation of (6.13) is a consequence of the first one. This can be proved by deriving an analog of (6.10). Proposition 6.1 is proved. □

Proof of Proposition 6.2. The proof of Proposition 6.2 can be easily obtained by derivation of relations for $\mathsf{E}\,\overline{R^2}$ and using estimates (6.6). □

7 Summary

We propose a method to study the fine structure of large random matrix spectra. We develop a resolvent-based approach to study the empirical eigenvalue counting function σ_N; a regularization is used to make σ_N taking into account the eigenvalues from an interval Δ_N whose length decays when $N \to \infty$.

Such a smoothed eigenvalue density $\rho_N = \sigma'_N(\Delta_N)$ is very well studied in the *global regime* when $|\Delta_N| = \mathrm{O}(1)$ in the limit $N \to \infty$. In this case the interval Δ_N contains $\mathrm{O}(N)$ eigenvalues and one can show that in many cases the limiting eigenvalue density ρ of the ensemble under consideration exists and is nonrandom.

In this paper we study two other asymptotic regimes.

The first one called the *local regime* corresponds to $|\Delta_N| = \mathrm{O}(N^{-1})$. The universality conjecture by F.J. Dyson states that the eigenvalue statistics in this scale do not depend on the details of the probability distribution of the random matrix ensemble. In this limiting transition we show that the moments of ρ_N are bounded by factorial-type expressions.

Another asymptotic regime we consider is determined by condition $|\Delta_N| = \mathrm{O}(N^{-\alpha})$, $0 < \alpha < 1$. Due to its intermediate character between the global and local regimes, it can be called the *mesoscopic regime*.

Our main results is that in this case is that ρ_N converges $N \to \infty$ to a limiting eigenvalue distribution ρ while fluctuations of ρ_N are gaussian and do not depend on the particular form of the probability distribution of the random matrix ensemble.

Therefore our results provide an evidence to the fact that the mesoscopic regime leads to a universal behavior of the eigenvalue distribution.

This kind of "soft" universality can be explained as follows. The intervals satisfying condition $1 \ll |\Delta_N| \ll N$, are sufficiently large to obtain in the limit $N \to \infty$ a bounded non-random density of the eigenvalue distribution. At the same time the group of eigenvalues lying in such an interval can be regarded as a local object on a length scale much greater than $|\Delta_N|$. This leads to Dyson's universal asymptotics of the eigenvalue correlation functions.

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