Singular limiting behavior in nonlinear stochastic wave equations

Michael Oberguggenberger
Institut für Mathematik und Geometrie
Universität Innsbruck
A - 6020 Innsbruck, Austria

Francesco Russo
Institut Galilée, Département de Mathématiques
Université Paris 13
F - 93430 Villetaneuse, France

In this paper we study the semilinear stochastic wave equation

$$\begin{aligned} (\partial_t^2 - \Delta)u = F(u) + \dot{W} \quad \text{on} \quad \mathbb{R}^{n+1}, \\ u \mid \{t < 0\} = 0 \end{aligned} \tag{1}$$

where F is globally Lipschitz, and the stochastic excitation $\dot{W}$ is a white noise on the half space $T = \mathbb{R}^n \times [0, \infty)$. The solution to the linear wave equation

$$\begin{aligned} (\partial_t^2 - \Delta)v = \dot{W} \quad \text{on} \quad \mathbb{R}^{n+1}, \\ v \mid \{t < 0\} = 0 \end{aligned} \tag{2}$$

is known to be a generalized stochastic process in space dimensions $n \geq 2$, that is, its sample paths are distributions on $\mathbb{R}^{n+1}$. It is possible to construct solutions to the nonlinear wave equation (1) as Colombeau-type generalized stochastic processes [1, 18]. For various types of nonlinearities F, white noise calculus is applicable as well [7, 9, 11, 15]. In this paper we are concerned with the limiting behavior of regularized solutions

$$(\partial_t^2 - \Delta)u_\varepsilon = F(u_\varepsilon) + \dot{W}_\varepsilon$$

$$(\partial_t^2 - \Delta)v_\varepsilon = \dot{W}_\varepsilon$$

obtained by smoothing white noise, as $\varepsilon \to 0$. Our goal is to demonstrate that the approximate solutions to the nonlinear equation (1) converge to the solution of the linear equation (2) plus a deterministic term which essentially depends only on the behavior of the Fourier transform of F at zero. For example, assume that F is globally Lipschitz and has a limit L at infinity. Then, in space dimensions $n = 2, 3$,

$$\lim_{\varepsilon\to 0} u_\varepsilon = v + \frac{t^2}{2}L \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^{n+1}) \tag{3}$$

in the L^1-sense, that is,

$$\lim_{\varepsilon\to 0} E\Big(\sup_{\varphi\in B}\Big|\langle u_\varepsilon - v - \frac{t^2}{2}L, \varphi\rangle\Big|\Big) = 0$$

for every bounded subset B in the space of test functions $\mathcal{D}(\mathbb{R}^{n+1})$. We present here two methods of proof. The first one is based on a study of the pathwise behavior of the regularized solutions v_ε to the linear wave equation as well as delta-wave estimates from nonlinear hyperbolic theory as in [5, 13, 14, 17]. This first method gives a good explanation of the observed effect: subsequences of the solutions $v_\varepsilon(x,t)$ tend to infinity almost surely, thus only the values of F near infinity are activated. However, this method gives the limiting behavior (3) only in probability.

The second method uses the Fourier transform of F. We single out a large class of nonlinearities for which the result (3) holds: namely, the Fourier transform of F should have *mass L at zero*. This, together with Gaussian properties of the free solution $v_\varepsilon(x,t)$ and the fact that its variance tends to infinity when $n \geq 2$, allows to demonstrate (3) in the L^1-sense. A similar argument actually gives convergence in the L^p-sense for $1 < p < \infty$. This generalizes the results of [1], where the case of nonlinearities F which are Fourier transforms of integrable measures, massless at 0, in space dimension $n = 2$ has been treated.

The observed triviality effect arises with stochastic initial value problems involving white noise as well. It occurs just the same in semilinear parabolic equations ($n \geq 2$), semilinear Schrödinger equations ($n \geq 1$) and semilinear elliptic equations on domains in $\mathbb{R}^n, n \geq 4$; these results will be published elsewhere. For a study of limiting solutions to semilinear parabolic equations with Wick renormalization we refer to [2]. The wave equation with white noise excitation has found quite some attention from the probabilistic side (e.g. [19]). The results presented here can also be seen from the viewpoint of delta-waves: in this spirit, stochastic analysis provides a range of intertesting, highly singular distributions whose effects as inputs in nonlinear equations can be studied.

We conclude this introduction by saying that in the last years, semilinear stochastic wave equations have been intensively studied in the case of space dimension $n > 1$. Among the most significant references, we quote [3, 4, 10, 12, 16]. The driving noise in those papers is Gaussian and homogeneous but not necessarily with nuclear covariance. The equations there still allow classical function valued solutions; the covariance of the noise is situated at the boarder case not to get triviality effects.

1 White noise and the linear wave equation

Denote by $\mathcal{S}(T) = \mathcal{S}(\mathbb{R}^{n+1}) \,|\, T$ the space of rapidly decreasing smooth functions on $T = \mathbb{R}^n \times [0, \infty)$. Let $\Omega = \mathcal{S}'(T)$ with Σ the Borel σ-algebra generated by the weak topology. By the Bochner-Minlos theorem [6, §3.1], there is a unique probability measure μ on (Ω, Σ) such that

$$\int e^{i\langle\omega,\varphi\rangle}\, d\mu(\omega) = e^{-\frac{1}{2}\|\varphi\|^2_{L^2(T)}} \tag{4}$$

for $\varphi \in \mathcal{S}(T)$. *White noise with support in* T is the process

$$\dot{W} \,:\, \Omega \to \mathcal{D}'(\mathbb{R}^{n+1}) \,:\, \langle \dot{W}(\omega), \varphi\rangle = \langle \omega, \varphi \mid T\rangle\,,$$

a generalized Gaussian process with mean zero and variance

$$E(\dot{W}(\varphi)^2) = \|\varphi \mid T\|^2_{L^2(T)} \tag{5}$$

for $\varphi \in \mathcal{D}(\mathbb{R}^{n+1})$.

We shall make use of the following regularizations. Let $\psi \in \mathcal{D}(\mathbb{R}^{n+1})$ with $\int\int \psi(x,t)dxdt = 1$. We define ψ_ε by

$$\psi_\varepsilon(x,t) = \varepsilon^{-n-1}\,\psi(\frac{x}{\varepsilon},\frac{t}{\varepsilon})\,.$$

For computational convenience, we shall assume in this paper that the support of ψ is contained in the interior of T and, in addition, that ψ is of the tensor product form

$$\psi(x,t) = \chi(x)\chi_0(t)$$

with $\chi \in \mathcal{D}(\mathbb{R}^n)$, $\chi_0 \in \mathcal{D}(0,\infty)$. Regularized white noise is defined as

$$\dot{W}_\varepsilon(\omega) = \dot{W}(\omega) * \psi_\varepsilon\,.$$

It belongs to $\mathcal{C}^\infty(\mathbb{R}^{n+1})$ and has its support in T.

The next wellknown proposition states existence and uniqueness of a stochastic solution to the linear wave equation (2). The white noise probability space Ω is as described above and fixed. The proof is included for completeness.

Proposition 1 *There is a unique generalized stochastic process* $v : \Omega \to \mathcal{D}'(\mathbb{R}^{n+1})$ *such that*

$$\text{supp } v(\omega) \subset T$$

$$(\partial_t^2 - \Delta)v(\omega) = \dot{W}(\omega)$$

in $\mathcal{D}'(\mathbb{R}^{n+1})$ *for all* $\omega \in \Omega$.

Proof: Let S be the fundamental solution of the wave operator with support in the forward light cone. Then the convolution

$$v(\omega) = S * \dot{W}(\omega)$$

exists, has its support in T, and satisfies $(\partial_t^2 - \Delta)v(\omega) = \dot{W}(\omega)$ for all $\omega \in \Omega$. The measurability follows from the defining formula

$$\langle S * \dot{W}(\omega), \varphi\rangle = \langle \dot{W}(\omega), \check{S} * \varphi\rangle = \langle \omega, \check{S} * \varphi \mid T\rangle$$

where the hat denotes inflection. Uniqueness can be seen as follows. Assume $z(\omega)$ has its support in T and satisfies $(\partial_t^2 - \Delta)z(\omega) = 0$. Taking a mollifier ψ_ε as described above, we have

$$(\partial_t^2 - \Delta)(z * \psi_\varepsilon) = 0\,, \ \text{supp}\ (z * \psi_\varepsilon) \subset T\,.$$

By classical $\mathcal{C}^\infty$-theory, it follows that $z*\psi_\varepsilon \equiv 0$. But then $z = \lim_{\varepsilon\to 0} z*\psi_\varepsilon \equiv 0$ as well. □

2 Sample path estimates

In this section we shall obtain estimates on the regularized solution of the linear problem

$$\begin{aligned}(\partial_t^2 - \Delta)v_\varepsilon &= \dot{W} * \psi_\varepsilon \\ v_\varepsilon \mid \{t<0\} &= 0\end{aligned} \tag{6}$$

where ψ_ε is a mollifier as described in the previous section, $\psi(x,t) = \chi(x)\chi_0(t)$ with $\text{supp}\,\chi_0 \subset (0,\infty)$. We first show that the variance of $v_\varepsilon(x,t)$ tends to infinity as $\varepsilon \to 0$, uniformly on strips $\mathbb{R}^n \times [t_0,t_1]$, $0 < t_0 < t_1$.

Proposition 2 *For every $t_0 > 0$ there is $\varepsilon_0 > 0$ and positive constants c_0, c_1, (depending only on the mollifier ψ) such that*

$$\frac{c_1}{\varepsilon}\,t \ \le\ E(v_\varepsilon(x,t)^2)\ \le\ \frac{c_0}{\varepsilon}\,t \qquad (n=3)$$

$$(c_1 \log\frac{1}{\varepsilon})\,t \ \le\ E(v_\varepsilon(x,t)^2)\ \le\ (c_0 \log\frac{1}{\varepsilon})\,t \qquad (n=2)$$

for $0 < \varepsilon \le \varepsilon_0$, $x \in \mathbb{R}^n$, $t \ge t_0$.

Proof: The regularized free solution is given by

$$v_\varepsilon(x,t) = \dot{W} * S * \psi_\varepsilon(x,t) = \langle \dot{W}, \theta\,(S * \psi_\varepsilon(x-.,t-.))\rangle$$

where θ is some cut-off function identically equal to one on T and vanishing as $t \to -\infty$. For fixed (x,t), the argument $\theta\,(S * \psi_\varepsilon(x-.,t-.))$ belongs to $\mathcal{D}(\mathbb{R}^{n+1})$. By the characterizing property (5) of white noise,

$$E(v_\varepsilon(x,t)^2) = \int_0^\infty \int_{\mathbb{R}^n} |S * \psi_\varepsilon(x-y,t-s)|^2\,dyds\,.$$

Recalling that S is a smooth function of $t \in [0,\infty)$ with values in the space of compactly supported distributions $\mathcal{E}'(\mathbb{R}^n)$, replacing $x-y$ by y and $t-s$ by s, and observing the support properties of S and ψ_ε this is seen to equal

$$\begin{aligned}
&\int_0^t \| \int_0^s S(.,s-r) \star \psi_\varepsilon(.,r)dr\|^2_{L^2(\mathbb{R}^n)}\, ds \\
&= \int_0^t \int_{\mathbb{R}^n} | \int_0^s \frac{\sin(s-r)|\xi|}{|\xi|} \hat{\psi}_\varepsilon(\xi,r)dr|^2\, d\xi ds
\end{aligned}$$

by Parseval's equality; the latter hat and star denote Fourier transform, respectively convolution, effected in the x-variable only. Recalling that $\hat{\psi}_\varepsilon(\xi,r) = \hat{\chi}(\varepsilon\xi)\frac{1}{\varepsilon}\chi_0(\frac{r}{\varepsilon})$ finally yields

$$E(v_\varepsilon(x,t)^2) = \varepsilon^{3-n} \int_0^{t/\varepsilon} \int_{\mathbb{R}^n} | \int_0^s \frac{\sin(s-r)|\xi|}{|\xi|} \chi_0(r)dr|^2\, |\hat{\chi}(\xi)|^2\, d\xi ds\,.$$

For the remainder of the proof, we work out the case $n=3$ only, the case $n=2$ being similar.

After introducing polar coordinates on $\mathbb{R}^3$ and denoting by M the spherical average of $|\hat{\chi}|^2$, we have

$$\begin{aligned}
\frac{d}{dt} E(v_\varepsilon(x,t)^2) &= \frac{1}{\varepsilon} \int_0^\infty | \int_0^{t/\varepsilon} \sin(\frac{t}{\varepsilon}\rho - r\rho)\, \chi_0(r)dr|^2\, M(\rho)d\rho \\
&\le \frac{1}{\varepsilon} \int_0^\infty M(\rho)d\rho \left(\int_0^\infty |\chi_0(r)dr|\right)^2 \le \frac{c_0}{\varepsilon}\,.
\end{aligned}$$

Thus we have the upper bound

$$E(v_\varepsilon(x,t)^2) \le \frac{c_0}{\varepsilon}\, t\,.$$

To obtain the lower bound, we first observe that

$$| \int_0^{t/\varepsilon} \sin(\frac{t}{\varepsilon}\rho - r\rho)\, \chi_0(r)dr| = \frac{1}{2}|e^{i\frac{t}{\varepsilon}\rho}\, \hat{\chi}_0(\rho) - e^{-i\frac{t}{\varepsilon}\rho}\, \hat{\chi}_0(-\rho)|$$

for sufficiently small ε (uniformly in $t \ge t_0 > 0$) due to the fact that the support of χ_0 is a compact subset of $[0,\infty)$. This in turn equals

$$| \sin(\frac{t}{\varepsilon}\rho)\, \hat{\chi}_0(\rho) + \frac{1}{2i}\, e^{-i\frac{t}{\varepsilon}\rho}\, \left(\hat{\chi}_0(\rho) - \hat{\chi}_0(-\rho)\right) |\,.$$

Using the fact that $\tilde{\chi}(0) = 1, \hat{\chi}_0(0) = 1$, we can find $\alpha > 0$ so that

$$M(\rho) \geq \frac{1}{2}, |\hat{\chi}_0(\rho) - 1| \leq \frac{1}{10}, |\hat{\chi}_0(\rho) - \hat{\chi}_0(-\rho)| \leq \frac{1}{5}$$

for $|\rho| \leq \alpha$. Then $\frac{d}{dt} E(v_\varepsilon(x,t)^2)$ is estimated from below by

$$\frac{1}{2\varepsilon} \int_0^\alpha |\sin(\frac{t}{\varepsilon}\rho) + \sin(\frac{t}{\varepsilon}\rho)(\hat{\chi}_0(\rho) - 1) + \frac{1}{2i} e^{-i\frac{t}{\varepsilon}\rho} (\hat{\chi}_0(\rho) - \hat{\chi}_0(-\rho))|^2 \, d\rho$$

$$\geq \frac{1}{2\varepsilon} \int_0^\alpha \left(|\sin \frac{t}{\varepsilon}\rho|^2 - \frac{2}{5} \right) d\rho = \frac{1}{4} \int_0^{\alpha/\varepsilon} (1 - \cos 2t\rho) \, d\rho - \frac{\alpha}{5\varepsilon} \geq \frac{\alpha}{20\varepsilon} - \frac{1}{8t_0}$$

for $t \geq t_0$. Thus, finally

$$E(v_\varepsilon(x,t)^2) \geq \left(\frac{\alpha}{20\varepsilon} - \frac{1}{8t_0} \right) t \geq \frac{c_1}{\varepsilon} t$$

for $t \geq t_0$ and sufficiently small ε. □

Remark 1 In the case $n = 1$, the variance of $v_\varepsilon(x,t)$ remains bounded. This reflects the fact that for $n = 1$, the solution v to the linear wave equation with white noise excitation is a regular stochastic process, namely a rotated two-dimensional Wiener process [19].

The divergence of the variance as $\varepsilon \to 0$ implies divergence almost surely for suitable subsequences, as is seen from the following lemma of Borel-Cantelli type.

Lemma 1 *Let $(\Omega, \Sigma, \mu), (\Xi, \Sigma', \lambda)$ be probability spaces, $V(\varepsilon, \xi, \omega)$ be a family of random variables on $\Xi \times \Omega$ such that for each $\xi \in \Xi$, $V(\varepsilon, \xi, .)$ is Gaussian with mean zero and variance $\sigma(\varepsilon, \xi)^2 = \int_\Omega V(\varepsilon, \xi, \omega)^2 d\mu(\omega)$. Assume that $\inf_{\xi \in \Xi} \sigma(\varepsilon, \xi) \to \infty$ as $\varepsilon \to 0$. Then there is a subsequence $\varepsilon_k \to 0$ such that*

$$\lambda \times \mu \left\{ (\xi, \omega) \in \Xi \times \Omega \, : \, \lim_{k \to \infty} |V(\varepsilon_k, \xi, \omega)| = \infty \right\} = 1 \, .$$

In addition, every subsequence has a subsequence with this property.

Proof: Let $U(\varepsilon, \xi, \omega) = V(\varepsilon, \xi, \omega)/\sigma(\varepsilon, \xi)$. For fixed $\xi \in \Xi$ and $\varepsilon > 0$, $U(\varepsilon, \xi, .)$ is a Gaussian random variable on Ω with mean zero and variance one. By

Fubini's theorem, the random variable $U(\varepsilon, ., .)$ on the product space $\Xi \times \Omega$ has the same Gaussian distribution. Choose $\varepsilon_k \to 0$ so that $\sigma(\varepsilon, \xi) \geq 2^{2k}$ for all $k \in \mathbb{N}$ and all $\xi \in \Xi$. Fix $\alpha > 0$ and consider the event

$$A_k = \left\{ (\xi, \omega) \,:\, |U(\varepsilon_k, \xi, \omega)| \geq \alpha 2^k \right\} .$$

Using the Gaussian property of $U(\varepsilon, ., .)$ we have

$$(\lambda \times \mu)(A_k) \geq 1 - \alpha (2\pi)^{-1/2}\, 2^{-k+1}$$

and so

$$\lambda \times \mu \left(\bigcap_{k=1}^{\infty} A_k \right) \geq 1 - 2\alpha (2\pi)^{-1/2} .$$

Thus

$$\begin{aligned} &\lambda \times \mu \left\{ (\xi, \omega) \,:\, \lim_{k \to \infty} |V(\varepsilon_k, \xi, \omega)| = \infty \right\} \\ &= \lambda \times \mu \left\{ (\xi, \omega) \,:\, \lim_{k \to \infty} |U(\varepsilon_k, \xi, \omega) \sigma(\varepsilon, \xi)| = \infty \right\} \\ &\geq \lambda \times \mu \left\{ (\xi, \omega) \,:\, |U(\varepsilon_k, \xi, \omega)| \geq \alpha 2^k \text{ for all } k \in \mathbb{N} \right\} \\ &= \lambda \times \mu \left(\bigcap_{k=1}^{\infty} A_k \right) \geq 1 - 2\alpha (2\pi)^{-1/2} . \end{aligned}$$

The assertion follows by letting $\alpha \to 0$. □

Consider now the regularized solutions v_ε of the linear equation (6). For each fixed (x, t), they are mean zero Gaussian random variables on white noise probability space (Ω, Σ, μ). Taking any compact set $\Xi \subset \mathbb{R}^n \times (0, \infty)$ and letting λ be normalized Lebesgue measure on Ξ, we can apply Lemma 1 together with Proposition 2 to get a subsequence diverging almost surely on $\Xi \times \Omega$. Exhausting $\mathbb{R}^n \times (0, \infty)$ by compact sets, a diagonal sequence argument gives the following result:

Corollary 1 *Let v_ε be the solution to (6) in space dimension $n = 2$ or $n = 3$. Then there is a subsequence $\varepsilon_k \to 0$ such that μ-almost surely*

$$\lim_{k \to \infty} |v_{\varepsilon_k}(x, t)| = \infty$$

for almost all $x \in \mathbb{R}^n, t > 0$. In addition, every subsequence has a subsequence with this property. □

The corollary states, in particular, that μ-almost surely $|v_{\varepsilon_k}| \to \infty$ Lebesgue-almost everywhere on T.

3 Limits in the nonlinear equation

In this section we apply the previous results to the regularized nonlinear equation

$$\begin{aligned}(\partial_t^2 - \Delta)u_\varepsilon &= F(u_\varepsilon) + \dot{W} * \psi_\varepsilon \\ u_\varepsilon \mid \{t < 0\} &= 0\end{aligned} \tag{7}$$

where F is assumed to be globally Lipschitz, and the mollifiers ψ_ε are as in Section 1. In space dimension $n = 1, 2, 3$, equation (7) is easily seen to have a unique solution u_ε, a stochastic process with smooth paths. Indeed, fix $\rho > 0$ and let

$$K_\tau = \{(x,t) \in \mathbb{R}^{n+1} \,:\, 0 \le t \le \tau,\ |x| \le \rho - t\}$$

be the conical region cut off at height τ with base the ball of radius ρ. The solution to the linear equation

$$(\partial_t^2 - \Delta)w = h,\ \ w \mid \{t < 0\} = 0$$

with smooth right hand side h is given by Kirchhoff's formula

$$w(x,t) = \frac{1}{4\pi}\int_0^t \frac{ds}{t-s}\int_{|y-x|=t-s} h(y,s)\,d\sigma(y)$$

for $n = 3$ and similarly by Poisson's and D'Alembert's formulas for $n = 2, 1$. In these dimensions, the estimate

$$\|w\|_{L^p(K_\tau)} \le \tau \int_0^\tau \|h\|_{L^p(K_t)}\,dt \tag{8}$$

holds both for $p = 1$ and $p = \infty$. Rewriting (7) as an integral equation and employing the estimate (8) with $p = \infty$ in a fixed point argument yields the existence of a unique solution $u_\varepsilon(x,t,\omega)$, smooth with respect to $(x,t) \in \mathbb{R}^{n+1}$ and measurable with respect to $\omega \in \Omega$.

The pathwise estimates obtained in Section 2 allow to describe the behavior of the regularized solutions u_ε to the nonlinear equation when F has a limit at infinity, say

$$\lim_{|y|\to\infty} F(y) = L\,.$$

In this case, define the function a by $a(x,t) = \frac{t^2}{2}L, t \ge 0; a(x,t) = 0, t < 0$.

Theorem 1 *Let $n = 2$ or $n = 3$. Assume that F is globally Lipschitz and $\lim_{|y|\to\infty} F(y) = L$. Let u_ε be the smooth stochastic process solving problem (7) and let v_ε be the solution of the free equation (6). Then every subsequence of $\varepsilon \to 0$ has a subsequence $\varepsilon_k \to 0$ such that for all compact sets $K \subset \mathbb{R}^{n+1}$*

$$\lim_{k\to\infty} \|u_{\varepsilon_k} - v_{\varepsilon_k} - a\|_{L^1(K)} = 0$$

μ-almost surely.

Proof: Write

$$(\partial_t^2 - \Delta)(u_\varepsilon - v_\varepsilon - a)$$
$$= \int_0^1 F'(\sigma u_\varepsilon + (1-\sigma)(v_\varepsilon + a))d\sigma \ (u_\varepsilon - v_\varepsilon - a) \ + \ F(v_\varepsilon + a) - L$$

so that on any conical compact region K_τ the estimate (8) gives

$$\|u_\varepsilon - v_\varepsilon - a\|_{L^1(K_\tau)} \tag{9}$$
$$\leq \tau \|F'\|_{L^\infty(I\!R)} \int_0^\tau \ \|u_\varepsilon - v_\varepsilon - a\|_{L^1(K_t)}\, dt \ + \ \tau^2 \|F(v_\varepsilon + a) - L\|_{L^1(K_\tau)} \, .$$

By Corollary 1, for every subsequence there is a subsequence $\varepsilon_k \to 0$ such that $|v_{\varepsilon_k}(x,t,\omega)| \to \infty$ almost surely ($\omega \in \Omega$) almost everywhere ($(x,t) \in T$). For such members $\omega \in \Omega$, the bounded sequence $F(v_{\varepsilon_k} + a) - L$ converges to zero almost everywhere. Hence by Lebesgue's theorem and Gronwall's lemma the assertion follows. □

Corollary 2 *Under the assumptions of Theorem 1, let v be the solution to the free wave equation in Propostion 1. Then u_ε converges to $v + a$ with respect to the strong topology of $\mathcal{D}'(\mathbb{R}^{n+1})$, in probability as $\varepsilon \to 0$.*

Proof: Let q be one of the defining seminorms of the strong topology of $\mathcal{D}'(\mathbb{R}^{n+1})$. By Theorem 1, every subsequence of $\varepsilon \to 0$ has a subsequence $\varepsilon_k \to 0$ such that $q(u_{\varepsilon_k} - v_{\varepsilon_k} - a) \to 0$ almost surely. This is equivalent to convergence in probability. □

In case $\lim_{|y|\to\infty} F(y) = 0$, the function a vanishes and so the solutions u_ε to the nonlinear equation exhibit triviality in their behavior: they converge to the solution of the linear equation.

This pathwise study admits a rather intuitive interpretation of the triviality result: pathwise, at least for subsequences, the free solution tends to infinity, and hence the nonlinearity F is activated only near its limiting values at infinity. Thus the behavior of F on finite values has no influence on the solution. On the other hand, these arguments lead only to convergence in probability. We now present two further arguments of increasing generality that demonstrate first the triviality effect in the stronger sense of convergence in $L^1(\Omega)$ and, second, for a larger class of nonlinearities F.

Taking the expectation in (9) we get

$$E\left(\|u_\varepsilon - v_\varepsilon - a\|_{L^1(K_\tau)}\right)$$

$$\leq \tau\|F'\|_{L^\infty(I\!R)} \int_0^\tau E\left(\|u_\varepsilon - v_\varepsilon - a\|_{L^1(K_t)}\right) dt + \tau^2 E\left(\|F(v_\varepsilon + a) - L\|_{L^1(K_\tau)}\right) .$$

The assertion

$$\lim_{\varepsilon\to 0} E\left(\|u_\varepsilon - v_\varepsilon - a\|_{L^1(K_\tau)}\right) = 0$$

is obtained, provided the last term on the right hand side tends to zero. But

$$E\left(\|F(v_\varepsilon + a) - L\|_{L^1(K_\tau)}\right)$$

$$= \iint_{K_\tau} E\left(|F(v_\varepsilon(x,t) - a(x,t)) - L|\right) dxdt$$

$$= \iint_{K_\tau} \int_{-\infty}^{\infty} |F(y - a(x,t)) - L| \frac{1}{\sqrt{2\pi}\sigma_\varepsilon} \exp\left(-\frac{y^2}{2\sigma_\varepsilon^2}\right) dydxdt$$

where $\sigma_\varepsilon = E(v_\varepsilon(x,t)^2)$ denotes the variance of the free solution. Substituting $\sigma_\varepsilon y$ for y in the integral shows that this goes to zero provided $\lim_{|z|\to\infty} F(z) = L$. However, this Gaussian argument can be modified so that a larger class of nonlinear functions F can be treated, which we now define.

Definition 1 A distribution $H \in \mathcal{S}'(\mathbb{R})$ is said to *have mass L at zero* if

$$\lim_{\varepsilon\to 0} \langle H, \eta(\cdot/\varepsilon)\rangle = L\,, \tag{10}$$

for the function $\eta(y) = \exp(-y^2/2)$. If $L = 0$, H is said to be *massless at zero.*

Remark 2 For our applications, we shall be mainly concerned with functions whose Fourier transform has mass L at zero. For $F \in \mathcal{S}'(\mathbb{R})$ the Fourier transform $H = \mathcal{F}F$ has mass L at zero, i. e. satisfies (10), if and only if

$$\lim_{\varepsilon \to 0} \langle F, \frac{\varepsilon}{\sqrt{2\pi}} \eta(\varepsilon \cdot) \rangle = L \,, \tag{11}$$

noting that η is identical with its Fourier transform up to the multiplicative factor $\sqrt{2\pi}$.

Example 1 Let F be a continuous function such that the limit $\lim_{|y| \to \infty} = L$ exists. It follows easily from formula (11) that the Fourier transform $\mathcal{F}F$ has mass L at zero.

Example 2 Let G be a continuous function such that the limits $\lim_{x \to -\infty} = L_-$ and $\lim_{x \to +\infty} = L_+$ exist. It follows from the symmetry of η that the limit as $\varepsilon \to 0$ in (11) equals $(L_- + L_+)/2$. Thus $\mathcal{F}G$ is massless at zero iff $L_- = -L_+$. In particular, this holds when G vanishes at infinity.

Example 3 Let G be a periodic, sufficiently regular function with period π_0. Expanding G in its Fourier series, we see that the limit in (11) equals $\int_0^{\pi_0} G(y) dy$ so that $\mathcal{F}G$ is massless at zero iff G has mean zero along its period.

Example 4 If $G \in L^p(\mathbb{R})$ for some $p \in [1, 2]$ or if $x^{-q} G(x) \in L^1(\mathbb{R})$ then $\mathcal{F}G$ is massless at zero (direct computation). If G is a tempered distribution such that $\mathcal{F}G = \mu$, an integrable measure with $\mu(\{0\}) = 0$, then $\mathcal{F}G$ is massless at zero.

The significance of this notion is exhibited by the following proposition.

Proposition 3 *Let $(V_1(\varepsilon), V_2(\varepsilon))$ be a mean-zero Gaussian vector, nondegenerate for all small $\varepsilon > 0$, and such that $\sigma_2^2(\varepsilon) = \mathrm{Var}\, V_2(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Let $G : \mathbb{R} \to \mathbb{R}$ be a bounded function such that $\mathcal{F}G$ is massless at zero. Then*

$$E\left(G(V_1((\varepsilon))\, G(V_2(\varepsilon))\right) \to 0 \text{ as } \varepsilon \to 0\,.$$

Proof: The covariance matrix $\Sigma(\varepsilon)$ and its inverse are given by

$$\Sigma(\varepsilon) = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}, \; \Sigma(\varepsilon)^{-1} = \begin{pmatrix} a_1^2 & a_{12} \\ a_{12} & a_2^2 \end{pmatrix},$$

Observing the relation

$$\det \Sigma^{-1} = a_1^2 a_2^2 - a_{12}^2 = \frac{a_1^2}{\sigma_2^2}$$

an easy algebraic computation gives that

$$\begin{aligned} &2\pi E\left(G(V_1)G(V_2)\right) \\ &= \int dx_1\, G(x_1) \int dx_2\, G(x_2)\ \det \Sigma^{-1/2}\ \exp\left(-\tfrac{1}{2}[a_1^2 x_1^2 + 2a_{12}x_1x_2 + a_2^2 x_2^2]\right) \\ &= \int dx_2\, \tfrac{1}{\sigma_2} G(x_2)\ \exp\left(\tfrac{-x_2^2}{2\sigma_2^2}\right)\ a_1 \int dx_1\, G(x_1)\ \exp\left(-\tfrac{1}{2}[a_1 x_1 + \tfrac{a_{12}}{a_1} x_2]^2\right). \end{aligned}$$

The second integral is bounded by the L^∞-norm of G, while the first converges to zero by assumption and (11). □

Remark 3 Let $V_1(\varepsilon) = v_\varepsilon(x_1, t_1), V_2(\varepsilon) = v_\varepsilon(x_2, t_2)$, where v_ε is the solution to the free wave equation (6). Then $(V_1(\varepsilon), V_2(\varepsilon))$ is nondegenerate when $|x_2 - x_1| \neq |t_2 - t_1|$. For example, in dimension $n = 3$, this follows from the fact that the covariance remains bounded, while the variances tend to infinity by Proposition 2. This in turn is seen by computing

$$\begin{aligned} &E\left(v_\varepsilon(x_1, t_1) v_\varepsilon(x_2, t_2)\right) \\ &\quad = \int_0^{t_1 \wedge t_2} \int_{\mathbb{R}^3} S * \psi_\varepsilon(x_1 - z, t_1 - r) S * \psi_\varepsilon(x_2 - z, t_2 - r)\, dz dr \\ &\quad\quad = \int_0^{(t_1 \wedge t_2)/\varepsilon} \int_{\mathbb{R}^3} w(z, r) w\left(\tfrac{x_2 - x_1}{\varepsilon} + z, \tfrac{t_2 - t_1}{\varepsilon} + r\right) dz dr \end{aligned}$$

where $w(z, r)$ is the classical solution of the wave equation with right hand side ψ and zero initial data. Combining the support- and decay properties of the solution w in evaluating the latter integral shows that it remains bounded as $\varepsilon \to 0$ when $|x_2 - x_1| \neq |t_2 - t_1|$.

These considerations allow to prove a stronger version of the result in Theorem 1:

Theorem 2 *Let $n = 2$ or $n = 3$. Assume that F is globally Lipschitz, bounded, and its Fourier transform $\mathcal{F}F$ has mass L at zero. Let u_ε be the smooth stochastic process solving problem (7) and let v_ε be the solution of the free equation (6). Then for all compact sets $K \subset \mathbb{R}^{n+1}$,*

$$\lim_{\varepsilon \to 0} E\left(\|u_\varepsilon - v_\varepsilon - a\|_{L^1(K)}\right) = 0\,.$$

Proof: Without restriction of generality we may assume that $L = 0, a \equiv 0$. On every conical compact region K_τ we have that

$$E\left(\|F(v_\varepsilon)\|_{L^1(K_\tau)}\right) \leq \left(E\left(\|F(v_\varepsilon)\|^2_{L^1(K_\tau)}\right)\right)^{1/2}$$

$$\leq \left(\iint_{K_\tau} \iint_{K_\tau} E\Big(F(v_\varepsilon(x_1,t_1))\, F(v_\varepsilon(x_2,t_2))\Big)\, dx_1 dt_1\, dx_2 dt_2\right)^{1/2}.$$

We apply Proposition 3 to the Gaussian vector $(v_\varepsilon(x_1,t_1), v_\varepsilon(x_2,t_2))$, using Remark 3. Then Proposition 2 shows that the integral above tends to zero as $\varepsilon \to 0$. The discussion before Definition 1 entails the desired result. □

Corollary 3 *Under the assumptions of Theorem 2, let v be the solution to the free wave equation in Propostion 1. Then u_ε converges to $v + a$ with respect to the strong topology of $\mathcal{D}'(\mathbb{R}^{n+1})$, in $L^1(\Omega)$ as $\varepsilon \to 0$.* □

It is clear that convergence in $L^p(\Omega)$ for $1 < p < \infty$ can be proven by the same methods.

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