Convolution Calculus and Applications to Stochastic Differential Equations

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Abstract
The present work is mostly based on a functional analytic point of view. In this paper we develop a convolution calculus over a family of spaces of generalized functions. We use this calculus to discuss new solutions of some stochastic differential equations.

1 Introduction
Let $X$ be a real nuclear Frechet space. Assume that its topology is defined by an increasing family of Hilbertian norms $\{\|\cdot\|_p, p \in \mathbb{N}\}$. Then $X$ is represented as

$$X = \bigcap_{p \in \mathbb{N}} X_p,$$

where for $p \in \mathbb{N}$ the space $X_p$ is the completion of $X$ with respect to the norm $\|\cdot\|_p$. Denote by $X_{-p}$ the dual space of $X_p$, then the dual space $X'$ of $X$ is represented as

$$X' = \bigcup_{p \in \mathbb{N}} X_{-p},$$

and it is equipped with the inductive limit topology. Let $N$ (resp. $N_p$) be the complexification of $X$ (resp. $X_p$), i.e. $N = X + iX$ and $N_p = X_p + iX_p$. 
\( p \in \mathbb{Z} \). For any \( n \in \mathbb{N} \) we denote by \( N^{\otimes n} \) the \( n \)-th symmetric tensor product of \( N \) equipped with the \( \pi \)-topology and by \( N^{\otimes n}_p \) the symmetric Hilbertian tensor product of \( N_p \). We will preserve the notation \( |.|_p \) and \( |.|_{-p} \) for the norms on \( N^{\otimes n}_p \) and \( N^{-\otimes n}_p \) respectively. Let \( \theta \) be a Young function on \( IR_+ \), \( i.e. \theta \) is continuous, convex, increasing function and satisfies \( \lim_{x \to +\infty} \frac{\theta(x)}{x} = +\infty \), see [9]. We define the conjugate function \( \theta^* \) of \( \theta \) by

\[
\forall x \geq 0, \quad \theta^*(x) := \sup_{t \geq 0} (tx - \theta(t)).
\]

For a such Young function \( \theta \) we denote by \( G_{\theta}(N) \) the space of holomorphic functions on \( N \) with exponential growth of order \( \theta \) and of arbitrary type, and by \( F_{\theta}(N) \) the space of holomorphic functions on \( N \) with exponential growth of order \( \theta \) and of minimal type. For every \( p \in \mathbb{Z} \) and \( m > 0 \), we denote by \( Exp(N_p, \theta, m) \) the space of entire functions \( f \) on the complex Hilbert space \( N_p \) such that

\[
||f||_{\theta, p, m} := \sup_{x \in N_p} |f(x)|e^{-\theta(m|x|_p)} < +\infty.
\]

Then the spaces \( F_{\theta}(N') \) and \( G_{\theta}(N) \) are represented as

\[
F_{\theta}(N') = \bigcap_{p \in \mathbb{N}} \bigcup_{m > 0} Exp(N_{-p}, \theta, m)
\]

\[
G_{\theta}(N) = \bigcup_{p \in \mathbb{N}} \bigcap_{m > 0} Exp(N_p, \theta, m),
\]

and equipped with the projective limit topology and the inductive limit topology respectively. The spaces \( F_{\theta}(N') \) and its dual \( F'_{\theta}(N') \) equipped with the strong topology are called the test functions space and the distributions space respectively.

Let \( p \in \mathbb{N} \) and \( m > 0 \), we define the Hilbert spaces

\[
F_{\theta, m}(N_p) = \{ f = (f_n)_{n=0}^{\infty}, f_n \in N^{\otimes n}_p; \sum_{n \geq 0} \theta_n^{-2} m^{-n} |f_n|_p^2 < +\infty \}
\]

\[
G_{\theta, m}(N_{-p}) = \{ \phi = (\phi_n)_{n=0}^{\infty}, \phi_n \in N^{-\otimes n}_p; \sum_{n \geq 0} (n! \theta_n)^{2} m^{n} |\phi_n|_{-p}^2 < +\infty \},
\]

where \( \theta_n = \inf_{r > 0} \frac{\theta^{(r)}(r)}{r^n} \), \( n \in \mathbb{N} \), and put

\[
F_{\theta}(N) = \bigcap_{p \in \mathbb{N}} F_{\theta, m}(N_p),
\]
The space $F_\theta(N)$ equipped with the projective limit topology is a nuclear Frechet space [3], and $G_\theta(N')$ carries the dual topology of $F_\theta(N)$ with respect to the $\mathbb{C}$-bilinear form $\langle \ldots \rangle$:

$$
\langle \tilde{\phi}, \tilde{f} \rangle = \sum_{n \geq 0} n! \langle \phi_n, f_n \rangle, \quad \tilde{\phi} = (\phi_n) \in G_\theta(N'), \quad \tilde{f} = (f_n) \in F_\theta(N).
$$

It was proved in [3] that the Taylor series map, denoted by $S.T$, yields a topological isomorphism between $\mathcal{F}_\theta(N')$ (resp. $G_{\theta^*}(N)$) and $F_\theta(N)$ (resp. $G_\theta(N')$). Then the action of a distribution $\phi \in \mathcal{F}_\theta(N')$ on a test function $f \in \mathcal{F}_\theta(N')$ is given by

$$
\langle \phi, f \rangle = \langle \tilde{\phi}, \tilde{f} \rangle,
$$

where $\tilde{\phi} = [(S.T)^*]^{-1}(\phi)$ and $\tilde{f} = (S.T)(f)$. It is easy to see that for every $\xi \in N$, the exponential function $\epsilon_\xi : z \mapsto e^{\langle z, \xi \rangle}$, $z \in N'$ belongs to the test space $\mathcal{F}_\theta(N')$ for any Young function $\theta$. Then we define the Laplace transform of a distribution $\phi \in \mathcal{F}_\theta(N')$ by

$$
\hat{\phi}(\xi) := \langle \phi, \epsilon_\xi \rangle, \quad \xi \in N.
$$

In [3], the authors prove the important duality theorem: the Laplace transform realizes a topological isomorphism of $\mathcal{F}_\theta(N')$ on $G_{\theta^*}(N)$.

In this paper we develop a new convolution calculus over the generalized functionals spaces $\mathcal{F}_\theta(N')$. Unlike the Wick calculus studied by many authors [7][10][12], the convolution calculus is developed independently of the gaussian analysis. In fact, we define the convolution product $\phi_1 \ast \phi_2$ of two distributions $\phi_1, \phi_2$ in $\mathcal{F}_\theta(N')$ by a naturaly way using convolution operators. Then we give a sens to the expression $f^*(\phi) = \sum f_n \phi^m$, for any entire function $f(z) = \sum_{n \geq 0} f_n z^n$, $z \in \mathbb{C}$ with exponential growth, and for any distribution $\phi \in \mathcal{F}_\theta(N')$. In particular, the important convolution exponential functional $\exp^* \phi = \sum_{n \geq 0} \frac{\phi^m}{m!}$, which cannot be in general an element of the usual white noise distributions spaces introduced in [8], is well defined in the $\mathcal{F}_\theta(N')$-spaces. This permits to solve some stochastic differential equations in the distributions spaces of type $\mathcal{F}_\theta(N')$. Moreover, this solutions as elements of the $\mathcal{F}_\theta(N')$-spaces have more regularity and properties than those of the bigger distributions space $(N)^{-1}$ of Kondratiev-Streit type, systematically used for example by Oksendal in [13].
2 Convolution of distributions

In infinite dimension complex analysis [2], a convolution operator on the test space $\mathcal{F}_\theta(N')$ is a continuous linear operator from $\mathcal{F}_\theta(N')$ into itself which commutes with translation operators.

Let $x \in N'$, we define the translation operator $\tau_x$ on $\mathcal{F}_\theta(N')$ by

$$\tau_x \varphi(y) = \varphi(x + y), \quad y \in N', \quad \varphi \in \mathcal{F}_\theta(N').$$

It is easy to see that $\tau_x$ is a continuous linear operator from $\mathcal{F}_\theta(N')$ into itself. Now, we define the convolution product of a distribution $\phi \in \mathcal{F}_\theta(N')$ with a test function $\varphi \in \mathcal{F}_\theta(N')$ as follows

$$\phi \ast \varphi(x) = \langle \phi, \tau_x \varphi \rangle, \quad x \in N'.$$

If $\phi$ is represented by $\tilde{\phi} = (\phi_n)_{n \geq 0} \in \mathcal{G}_\theta(N')$, then

$$\phi \ast \varphi(x) = \sum_{n \geq 0} \langle x^{\otimes n}, \psi^{(n)} \rangle,$$

where for every integer $n \in \mathbb{N}$

$$\psi^{(n)} = \sum_{k \geq 0} k! C^n_{n+k} \langle \phi_k, \varphi^{[n+k]} \rangle.$$

A direct calculation shows that the sequence $(\psi^{(n)})_{n \geq 0}$ is an element of $F_\theta(N)$ and consequently $\phi \ast \varphi \in \mathcal{F}_\theta(N')$. It was proved in [4] that $T$ is a convolution operator on $\mathcal{F}_\theta(N')$ if and only if there exists $\phi \in \mathcal{F}_\theta(N')$ such that

$$T(\varphi) = \phi \ast \varphi, \quad \forall \varphi \in \mathcal{F}_\theta(N').$$

We denote the convolution operator $T$ by $T_\phi$. Moreover for every $\varphi \in \mathcal{F}_\theta(N')$ we have

$$T_\phi(\varphi)(0) = \langle \phi, \varphi \rangle.$$

Let $\phi_1, \phi_2 \in \mathcal{F}_\theta(N')$ and $T_{\phi_1}, T_{\phi_2}$ be the associated convolution operators respectively. It is clear that the composition $T_{\phi_1} \circ T_{\phi_2}$ is also a convolution operator. Consequently there exists a unique element of $\mathcal{F}_\theta(N')$ denoted by $\phi_1 \ast \phi_2$ such that

$$T_{\phi_1} \circ T_{\phi_2} = T_{\phi_1 \ast \phi_2}.$$

The distribution $\phi_1 \ast \phi_2$, defined by (3) is called the convolution product of $\phi_1$ and $\phi_2$. 

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Proposition 1  For every \( \varphi \in \mathcal{F}_\theta(N) \) we have

\[
\langle \varphi_1 \ast \varphi_2, \varphi \rangle := [(\varphi_1 \ast \varphi_2) \ast \varphi](0) = [\varphi_1 \ast (\varphi_2 \ast \varphi)](0).
\]

Moreover, \( \forall \varphi_1, \varphi_2 \in \mathcal{F}_\theta(N') \) it holds that

\[
\hat{\varphi_1} \ast \hat{\varphi_2} = \hat{\varphi_1 \ast \varphi_2}.
\]

Proof

Let \( \varphi \in \mathcal{F}_\theta(N') \), in view of \((\text{(2)})\) and \((\text{(3)})\) we obtain

\[
[(\varphi_1 \ast \varphi_2) \ast \varphi](x) = [\varphi_1 \ast (\varphi_2 \ast \varphi)](x), \quad \forall x \in N'.
\]

In particular if we put \( x = 0 \) then we get

\[
\langle \varphi_1 \ast \varphi_2, \varphi \rangle = [\varphi_1 \ast (\varphi_2 \ast \varphi)](0),
\]

from which follows \((\text{(4)})\) by taking \( \varphi(x) = e^{(x, \xi)}, \xi \in N. \)

Let \( \mathcal{L}_\varphi^c \) be the space of convolution operators on \( \mathcal{F}_\theta(N') \). Taking \((\text{(3)})\) into consideration, we immediately obtain

Lemma 1

\[
(\mathcal{F}_\theta(N'), \ast) \longrightarrow (\mathcal{L}_\varphi^c, \circ)
\]

\[
\phi \longmapsto T_\phi
\]

is an isomorphism of algebra.

It follows from \((\text{(4)})\) that \((\mathcal{F}_\theta(N'), \ast)\) is a commutative algebra. Hence we deduce from lemma 1 that so is \((\mathcal{L}_\varphi^c, \circ)\).

Theorem 1  Let \( \gamma \) be a Young function on \( IR_+ \) which does not necessarily satisfy \( \lim_{x \to +\infty} \frac{\gamma(x)}{x} = +\infty \) and \( f \in \text{Exp}(C, \gamma, m) \) for some \( m > 0 \). Then for every distribution \( \phi \in \mathcal{F}_\theta(N') \), the functional \( f^*(\phi) \) defined by

\[
f^*(\phi) = f(\hat{\phi})
\]

belongs to \( \mathcal{F}_\lambda(N') \), where \( \lambda = (\gamma \circ e^\theta)^* \).
Proof

By the duality theorem, it is sufficient to prove that \( f(\hat{\phi}) \in \mathcal{G}_\lambda^r(N) \). In fact let \( \phi \in \mathcal{F}_0^r(N') \), then there exist \( p \in \mathbb{N}, m' > 0 \) and \( c' > 0 \) such that

\[
|\hat{\phi}(\xi)| \leq c'e \theta^r(m'|d\nu|), \quad \xi \in N.
\]

On the other hand there exists \( c > 0 \) such that

\[
|f(z)| \leq c e^{\gamma(m|z|)}, \quad z \in \mathbb{C}.
\]

Then combining the last inequality we get

\[
|f(\hat{\phi}(\xi))| \leq c e^{\gamma(m'e \theta^r(m'|d\nu|))}, \quad \xi \in N
\]

\[
\leq \begin{cases} 
  c e^{\gamma(e^\theta^r(m'|d\nu|))} & \text{if } mc' \leq 1 \\
  c e^{\gamma(e^\theta^r(m'+m'|d\nu|))} & \text{if } mc' > 1.
\end{cases}
\]

This inequality with the holomorphy of \( f(\hat{\phi}) \) on \( N \) show that \( f(\hat{\phi}) \in \mathcal{G}_\lambda^r(N) \).

If we take \( \gamma(x) = x^k \), \( x \in \mathbb{R}_+ \) and \( f(z) = e^{x^k} \), \( z \in \mathbb{C} \) in theorem 1, we get the following result

**Corollary 1** Let \( \phi \in \mathcal{F}_0^r(N') \), then the convolution exponential function of \( \phi \), denoted by \( e^{\phi} \), is an element of \( \mathcal{F}_0^r(e^{\theta^r},(N')). \) If in addition \( \hat{\phi}(\xi) \) is a polynomial in \( \xi \) of degree \( k \in \mathbb{N} \), \( k \geq 2 \) then \( e^{\phi} \in \mathcal{F}_\lambda^r(N) \), where \( \lambda(x) = x^{k-1}, x \geq 0 \).

A similar result of corollary 1, in the particular case where \( \hat{\phi} \) is a polynomial, was proved in [12] with Wick product.

### 3 Applications to stochastic differential equations

A one parameter generalized stochastic process with values in \( \mathcal{F}_0^r(N') \) is a family of distributions \( \{\phi_t, t \in I\} \subset \mathcal{F}_0^r(N') \), where \( I \) is an interval, without loss generality we can assume that \( 0 \in I \). The process \( \phi_t \) is said to be continuous if the map \( t \mapsto \phi_t \) is continuous. In order to introduce generalized stochastic integrals, we need the following result proved in [17].

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Proposition 2 [17] Let \((\phi_n)_{n \geq 0}\) be a sequence in \(\mathcal{F}_\theta(N')\). Then \((\phi_n)\) converges in \(\mathcal{F}'_\theta(N')\) if and only if the following conditions hold:

(D1) There exist \(p \geq 0, m > 0\) and \(c > 0\) such that for every integer \(n\)

\[
|\widehat{\phi}_n(\xi)| \leq c e^{\alpha n|\xi|_p}, \quad \forall \xi \in N.
\]

(D2) The sequence \(\widehat{\phi}_n(\xi)\) converges in \(C\) for each \(\xi \in N\).

Let \(\{\phi_t\}_{t \in I}\) be a continuous \(\mathcal{F}'_\theta(N')\)-process and put

\[
\phi_n = \frac{t}{n} \sum_{k=0}^{n-1} \phi_{nk} \quad n \in IN^*, \quad t \in I.
\]

It is easy to prove that the sequence \((\phi_n)\) is bounded in \(\mathcal{G}_\theta(N')\) and for every \(\xi \in N\), \((\phi_n(\xi))_n\) converges to \(\int_0^t \widehat{\phi}_s(\xi)ds\). Thus we conclude by proposition 2 that \((\phi_n)\) converges in \(\mathcal{F}_\theta(N')\). We denote its limit by

\[
\int_0^t \phi_s ds := \lim_{n \to +\infty} \phi_n \text{ in } \mathcal{F}'_\theta(N').
\]

Proposition 3 \(E_t = \int_0^t \phi_s ds\), \(t \in I\) is a continuous \(\mathcal{F}'_\theta(N')\)-process which satisfies

\[
\int_0^t \phi_s ds = \int_0^t \widehat{\phi}_s ds.
\]

Moreover, the process \(E_t\) is differentiable in \(\mathcal{F}_\theta(N')\) i.e. \(\frac{dE_t}{dt} = \phi_t, \ t \in I\).

Proof

Since the map \(s \mapsto \widehat{\phi}_s \in \mathcal{G}_\theta(N)\) is continuous, \(\{\widehat{\phi}_s, s \in [0, t]\}\) becomes a compact set, in particular it is bounded in \(\mathcal{G}_\theta(N)\) i.e. there exist \(p \in IN\), \(m > 0\) and \(C_t > 0\) such that for every \(\xi \in N_p\) we have

\[
|\widehat{\phi}_s(\xi)| \leq C_t e^{\alpha m|\xi|_p}, \quad \forall s \in [0, t]. \quad (6)
\]

Then inequality (6) show that the function \(\xi \mapsto \int_0^t \widehat{\phi}_s(\xi)ds\) belongs to \(\mathcal{G}_\theta(N)\). Consequently the pointwise convergence of the sequence of functions \((\widehat{\phi}_n)\) to \(\int_0^t \phi_s ds\) becomes a convergence in \(\mathcal{G}_\theta(N)\) and we get

\[
\int_0^t \phi_s ds = \int_0^t \widehat{\phi}_s ds.
\]
Let $t_0 \in I$ and let $\varepsilon > 0$ such that $[t_0 - \varepsilon, t_0 + \varepsilon] \subset I$. It then follows from (6) that
\[
\|\hat{E}_t - \hat{E}_{t_0}\|_{\sigma, p, m} \leq \int_{t_0}^{t} \|\hat{\phi}_s\|_{\sigma, p, m} ds \\
\leq |t - t_0| C_{t_0 + \varepsilon}.
\]
This proves the continuity of the map $t \in I \mapsto \hat{E}_t \in G_{\theta^*}(N)$ which is equivalent to the continuity of the process $E_t$. By the same argument we prove the differentiability of $E_t$.

3.1 Stochastic Volterra equation

Let $J : [0, T] \rightarrow \mathcal{F}_0(N'), K : [0, T] \times [0, T] \rightarrow \mathcal{F}_0(N')$ be two continuous generalized processes. We consider the stochastic Volterra equation
\[
E(t) = J(t) + \int_0^t K(t, s) \ast E(s) ds, \quad 0 \leq t \leq T. \tag{7}
\]

**Theorem 2** Suppose that there exist $p \in \mathbb{N}$, $m > 0$ and $M > 0$ such that
\[
\|\hat{K}(t, s)\|_{\sigma, p, m} \leq M, \quad \forall \ 0 \leq s \leq t \leq T,
\]
then there exists a unique continuous $\mathcal{F}_{(\sigma^*)}(N')$-process that solves (7). The solution $E(t)$ is given by
\[
E(t) = J(t) + \int_0^t H(t, s) \ast J(s) ds \tag{8}
\]
where $H(t, s) = \sum_{n \geq 1} K_n(t, s)$ with $K_n$ given inductively by
\[
K_{n+1}(t, s) = \int_s^t K_n(t, u) \ast K(u, s) du, \quad n \geq 1
\]
and $K_1(t, s) = K(t, s)$.

**Proof**
The solution is given by Picard iteration. In fact, put $E_0(t) = J(t)$ and consider
\[
E_{n+1}(t) = J(t) + \int_0^t K(t, s) \ast E_n(s) ds, \quad n \geq 0 \tag{9}
\]
By iteration we get

\[ E_n(t) = J(t) + \int_0^t H_n(t, s) * J(s)ds, \quad n \geq 1 \]

where \( H_n(t, s) = \sum_{l=1}^n K_l(t, s) \). Now, we use proposition 2 to prove that for every \( t, s \in [0, T] \) the sequence \( H_n(t, s) \) converges in \( \mathcal{F}(e^{\theta^*}, (N')) \). By assumption we have

\[ |\hat{K}(t, s)(\xi)| \leq M e^{\theta^*(m|d|\nu)}, \quad \xi \in N_p. \]

Thus by induction we get

\[ |\hat{H}_n(t, s)(\xi)| \leq \sum_{l=1}^n \frac{M^l(t-s)^{l-1}}{(l-1)!} (e^{\theta^*(m|d|\nu)})^l. \tag{10} \]

Then, summing up both sides of (10) we come to

\[ |\hat{H}_n(t, s)(\xi)| \leq \sum_{l=1}^n \frac{M^l(t-s)^{l-1}}{(l-1)!} (e^{\theta^*(m|d|\nu)})^l \]

\[ \leq M e^{\theta^*(m|d|\nu)} \exp[M(t-s)e^{\theta^*(m|d|\nu)}] \]

\[ \leq M e^{\theta^*(m|d|\nu)} \exp[M^2(t-s)^2 + e^{\theta^*(3m|d|\nu)}] \]

\[ \leq M e^{A_1(t-s)^2} \exp(e^{\theta^*(3m|d|\nu)}). \]

Hence we get the first condition (D1) of proposition 2. For the second condition (D2) we just note that for every \( 0 \leq s \leq t \leq T \) and \( \xi \in N \), \( \hat{H}_n(t, s)(\xi)_{n \geq 0} \) is a Cauchy sequence in \( C \). We have thus proved that the infinite series \( H(t, s) = \sum_{l=1} K_l(t, s) \) converges in \( \mathcal{F}(e^{\theta^*}, (N')). \) Consequently, the sequence \( E_n(t)_{n \geq 0} \) converges also in \( \mathcal{F}(e^{\theta^*}, (N')) \) to \( E(t) = J(t) + \int_0^t H(t, s) * J(s)ds \). By equation (9), \( E(t) \) is a solution of the stochastic Volterra equation. Finally, we use the Gronwall inequality to prove the uniqueness.  

\[ \square \]

### 3.2 Differential equations associated with convolution operators

Let \( \theta_1 \) and \( \theta_2 \) be two fixed Young functions, and let \( \{\phi_t\}_{t \in I} \) be a continuous \( \mathcal{F}_{\theta_1}(N') \)-process. Consider the Cauchy problem
Theorem 3 If there exists constant $C > 0$ such that $e^{\theta(r)} \leq C \theta_2'(r)$ for $r$ large enough, then the Cauchy problem (11) has a unique solution given by

$$U(t, x) = (e^{\int_0^t \phi_s ds} * f)(x), \quad x \in N', \quad t \in I.$$  \hspace{1cm} (12)

Moreover, $U(t) \in F_{\theta_2}(N') \forall t \in I$. If in addition $\hat{\phi}_t(\xi)$ is a polynomial in $\xi$ of degree $k \geq 2$, $\forall t \in I$, then $U(t)$ given by (12) is also the unique solution of equation (11) with values in $F_{\theta_2}(N')$ whenever $\lim_{r \rightarrow +\infty} \frac{r^k}{\theta_2'(r)}$.

Proof

The solution $U(t)$ is obtained by Picard iteration as in the proof of theorem 2. \hspace{1cm} \Box

As an application of theorem 3 we give the heat equation associated with Gross Laplacian. In fact, let $\varphi(x) = \sum_{n \geq 0} \langle x^{\otimes n}, \varphi^{(n)} \rangle \in F_{\theta}(N)$. The Gross Laplacian [3], [10] of $\varphi$ at $x \in N'$ is given by

$$\Delta_G \varphi(x) = \sum_{n \geq 0} (n + 2)(n + 1) \langle x^{\otimes n}, \langle \tau, \varphi^{(n+2)} \rangle \rangle$$

where $\tau$ is the trace operator defined by

$$\langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle, \quad \xi, \eta \in N.$$

Let $\gamma$ be the standard gaussian measure on $X'$ defined by its characteristic function $f_{X', e^{iy \cdot \xi}} d\gamma(y) = e^{-\frac{|y|^2}{2}}$, see [6],[7],[11] [14].

Corollary 2 Let $\theta$ be a Young function satisfying $\lim_{r \rightarrow +\infty} \frac{\theta(r)}{r^2} < +\infty$ and $f \in F_{\theta}(N')$. Then the heat equation associated with the Gross Laplacian

$$\frac{\partial U}{\partial t} = \frac{1}{2} \Delta_G U, \quad t \geq 0, \quad U(0) = f,$$  \hspace{1cm} (13)

has a unique solution in $F_{\theta}(N')$ given by

$$U(t, x) = \int_{X', f(x + \sqrt{t} y)} d\gamma(y).$$
Pro of
In fact/, the Gross Laplacian $\Delta G$ is a convolution operator. The distribution associated to $\Delta G$ is $\hat{\phi}_r = (0, 0, \tau, 0, \cdots)$, then it follows from equality (2) that

$$\Delta G(\varphi) = \phi_r * \varphi, \quad \forall \varphi \in \mathcal{F}_0(N').$$

Thus the heat equation (13) is equivalent to

$$\frac{\partial U}{\partial t} = \phi_2 * U, \quad t \geq 0, \quad U(0) = f.$$

Since $\hat{\phi}_2(\xi) = \frac{(\xi, \xi)}{2}$, $\xi \in N$ is a polynomial of degree 2, then it follows from theorem 3 that the equation (13) has a unique solution in $\mathcal{F}_0(N')$ given by

$$U(t, x) = (e^{st\phi_2} * f)(x), \quad t \geq 0.$$

On the other hand, since $e^{st\phi_2}(\xi) = e^{it(\xi, \xi)} = \hat{\gamma}_{\sqrt{t}}(\xi), \quad \xi \in N, \quad t \geq 0$ where $\gamma_{\sqrt{t}}$ is a gaussian measure on $X' [10]$, then the solution $U(t)$ can be expressed as

$$U(t, x) = (\gamma_{\sqrt{t}} * f)(x) = \int_{X'} f(x + \sqrt{t}y)d\gamma(y), \quad t \geq 0, \quad x \in N'.$$

Let $\{\phi_t\}$ and $\{M_t\}$ be two continuous $\mathcal{F}_0(N')$-processes. Consider the initial value problem

$$\frac{dX_t}{dt} = \phi_t * X_t + M_t, \quad X(0) = X_0 \in \mathcal{F}_0(N').$$

Then using the Laplace transform we prove the following theorem

**Theorem 4** The stochastic differential equation (14) has a unique solution in $\mathcal{F}_0(N')$, given by

$$X_t = X_0 * e^{\int_0^t \phi_u \, du} + \int_0^t e^{\int_s^t \phi_u \, du} * M_s \, ds.$$

The next example is an application of theorem 4:

In fact, let $\phi(t), \quad t \geq 0$ and $F(x), \quad x \in \mathbb{R}^d$ be two continuous $\mathcal{F}_0(N')$-processes. Suppose that there exist $p \in \mathbb{N}, \quad m > 0$ and a positive function
\( \beta \in L^1(\mathbb{R}, d\lambda) \) such that \(|\tilde{F}(x, \xi)| \leq \beta(x) e^{\alpha |m| d\lambda} \). Then the heat equation with stochastic potential

\[
\begin{cases}
\frac{\partial u(t,x,\omega)}{\partial t} = \frac{\sigma^2}{2} \Delta_x u(t,x,\omega) + \phi(t,\omega) \ast U(t,x,\omega), & t > 0, x \in \mathbb{R}^d \\
u(0,x,\omega) = F(x,\omega), & x \in \mathbb{R}^d
\end{cases}
\]

has a unique solution given by

\[
U(t,x) = \exp(\int_0^t \phi(s) ds) \ast \int_{\mathbb{R}^d} F(y) \frac{e^{-\frac{|x-y|^2}{2\sigma^2 t}}}{\sqrt{2\pi t \sigma}} dy.
\]

Moreover, \( U(t,x) \) is a continuous \( \mathcal{F}_{\phi}\sigma^{*}\mathcal{F}_{N'} \)-process. In particular if \( \phi(t) = W(t) \) the white noise, then \( U(t,x) \) becomes a continuous \( \mathcal{F}_{\phi}\sigma(N') \)-process. See [15] in the case \( \theta(x) = x^k \).

Now, we give an example of non-linear stochastic differential equation:

Let \( \{ \phi_t \} \) be a continuous \( \mathcal{F}_{\phi}\sigma(N') \)-process and consider the Verhulst equation

\[
\begin{cases}
\frac{dX_t}{dt} = X_t \ast (X_t - 1) \ast \phi_t, & t \geq 0 \\
X(0) = x_0 \in [0,1]
\end{cases}
\tag{15}
\]

In an obvious manner we show that

\[
\tilde{X}_t = \frac{1}{1 + (\frac{1}{x_0} - 1)e^{-\int_0^t \phi(s) ds}}, & t \geq 0
\tag{16}
\]

**Lemma 2** [4] Let \( f \in \mathcal{G}_{\phi}(N) \) such that \( f(z) \neq 0, \forall z \in N \), then \( \frac{1}{f} \in \mathcal{G}_{\phi}(N) \).

Since the function \( \xi \mapsto \exp(\int_0^t \hat{\phi}_s(\xi) ds) \) is an element of \( \mathcal{G}_{\phi}\sigma^*(N) \), the above lemma shows that \( \tilde{X}_t \in \mathcal{G}_{\phi}\sigma^*(N) \). Then using the duality theorem, \( X_t \) given by (16) is the unique continuous \( \mathcal{F}_{(\phi\sigma^*)}(N') \)-process that solves equation (15).

In particular if \( \hat{\phi}_t(\xi) \) is a polynomial in \( \xi \) of degree \( k \geq 2 \) then the solution \( X_t \) becomes a continuous \( \mathcal{F}_{\phi^*\sigma}\mathcal{F}_{N'} \)-process.

**Remark**

If the Young function \( \theta \) satisfies \( \lim_{x \to +\infty} \frac{\theta|x|}{x^2} < +\infty \), we get [3]

\[
\mathcal{F}_\theta(N') \hookrightarrow L^2(X',\gamma) \hookrightarrow \mathcal{F}_\theta(N'),
\tag{17}
\]

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where \( \gamma \) is the standard gaussian measure on \( X' \). In this case the test space \( \mathcal{F}_\theta(N') \) coincides with the space \( (X)_\theta \) introduced in [1]. In addition, the function \( \xi \mapsto e^{i\frac{\xi \cdot x}{2}} \), \( \xi \in N \) becomes an element of \( \mathcal{G}_\theta(N) \) and the usual S-transform, denoted by \( S \), is obtained by

\[
S(\phi)(\xi) = \hat{\phi}(\xi) e^{-\frac{\xi \cdot x}{2}}, \quad \xi \in N, \phi \in \mathcal{F}_\theta'(N').
\]

Unlike to the Laplace transform, we see here that the chaotic transform \( S \) can not be defined on all spaces of generalized functions \( \mathcal{F}_\theta(N') \), it is defined only on the space \( \mathcal{F}_\theta'(N') \) with \( \lim_{x \to +\infty} \frac{\theta(x)}{x^2} < +\infty \). Recall that in the gaussian analysis, the Wick product of two generalized functions \( \phi \) and \( \psi \) in \( \mathcal{F}_\theta'(N') \), denoted by \( \phi \circ \psi \), is the unique distribution in \( \mathcal{F}_\theta'(N') \) such that

\[
S(\phi \circ \psi) = S \phi S \psi,
\]

see [7] [10]. Then using (18) we can derive the following relationships between convolution and Wick product

\[
\phi \circ \psi = \phi \ast \psi \ast \nu \quad \text{and} \quad \phi \ast \psi = \phi \circ \psi \circ \gamma \sqrt{\pi},
\]

where \( \nu \) and \( \gamma \sqrt{\pi} \) are two distributions in \( \mathcal{F}_{\mathbb{R}^2}'(N') \) given by there Laplace transforms \( \tilde{\nu}(\xi) = e^{-\frac{1}{2}|\xi|^2} \) and \( \tilde{\gamma} \sqrt{\pi}(\xi) = e^{i\xi \cdot \xi}, \xi \in N \).

A similar convolution calculus can be developed if we replace the space \( \mathcal{F}_\theta(N) \) by a space of test functions with several variables introduced in [16]

**References**


