Dirichlet Problem for Quasi-linear Elliptic Equations*

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Introduction

The motive of this paper is the study of the weak solutions of the quasilinear elliptic equation in $\mathbb{R}^d$, $(d \geq 2)$:

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \mathcal{A}_i(x, u(x), \nabla u(x)) + \mathcal{B}(x, u(x), \nabla u(x)) = 0 \qquad (1)$$

where $\mathcal{A}_i : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ and $\mathcal{B} : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ are given Carathéodory functions satisfying the conditions introduced in section 1.

As an example of equation (1) is the perturbed $p - Laplace$ equation

$$-\mathrm{div}(|\nabla u|^{p-2} \nabla u) + \mathcal{B}(., u, \nabla u) = 0, \quad 1 < p < d. \qquad (2)$$

When $p = 2$, equation (2) reduces to the perturbed *Laplace* equation

$$-\Delta u + \mathcal{B}(., u, \nabla u) = 0. \qquad (3)$$

Another example included in this study is the following linear equation

$$\mathcal{L}u = -\sum_j (\sum_i a_{ij} \frac{\partial u}{\partial x_i} + d_j u) + (\sum_j b_j \frac{\partial u}{\partial x_j} + cu) = 0,$$

where $\mathcal{L}$ is assumed to satisfies the conditions of [Sta65] (see also [HeHe68]).

The equation of type (1) was considered and investigated in many interesting papers [Ser64], [Tru67], [HKM93], [MaZ97]...

Several papers have introduced an axiomatic potential theory for nonlinear equations (2) when $\mathcal{B} = 0$ (see [HKM93]) and for equation of type (3) (see[Baa98] and [BBM98]).

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This paper consists of four sections. First, we recall some definitions and properties for the (weak) sub, super and solutions of the equation (1). In section 2 we give some conditions that allows us to have the comparison principle for sub and supersolutions. After this preparation we are able in section 3 to solve the Dirichlet problem related to the equation (1), so at first we prove the existence of solutions to associated variational problem, after what solve the Dirichlet problem for continuous data boundary. Note that the growth of $\mathcal{B}$ in ∇u is allowed to go until $p-1+\frac{p}{d}$. In the last section we define a potential theory related to the equation (1), so we obtain that the sheaf of solutions of (1) satisfies the Bauer axiomatic theory [Bou98a]. We prove also that the set of all hyperharmonic and the set of all hypoharmonic functions are sheaves.

Notation:

Throughout this paper we will use the following notations:

$\mathbb{R}^d$ is the real Euclidean $d-$space, $d \geq 2$. For an open set U of $\mathbb{R}^d$, we note by $SCI(U)$ (resp.$SCS(U)$) the set of lower (resp. upper) semicontinuous functions defined on U. While $C^k(U)$ is the set of functions which kth- derivative is continuous for k positive integer, $C^\infty(U) = \underset{k\geq 1}{\cap} C^k(U)$ and $C_0^\infty(U)$ is the set of all functions in $C^\infty(U)$ with compact support. $L^q(E)$ is the space of all q^{th}-power Lebesgue integrable functions defined on measurable set E. $W^{1,q}(U)$ is the $(1,q)$-Sobolev space on U. $W_0^{1,q}(U)$ is the closure of $C_0^\infty(U)$ in $W^{1,q}(U)$ relatively to its norm. $W^{-1,q'}(U)$ denotes the dual of $W_0^{1,q}(U)$, $q' = \frac{q}{q-1}$. For a Lebesgue measurable set E, $|E|$ denotes the Lebesgue measure of E. $u \vee v$ and $u \wedge v$ design respectively the supremum and the infimum of u and v. $u^+ = u \vee 0$ and $u^- = u \wedge 0$. We write $\rightharpoonup$(resp. $\rightarrow$) to design the weak (resp. strong) convergence.

1 Supersolutions of (1)

Let Ω denote a bounded domain in $\mathbb{R}^d$ $(d \geq 2)$ with smooth boundary $\partial\Omega$ and let $\mathcal{L}$ be a quasi-linear elliptic differential operator in divergence form

$$\mathcal{L}u(x) = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\mathcal{A}_i(x, u(x), \nabla u(x)) + \mathcal{B}(x, u(x), \nabla u(x)) \qquad a.e. x \in \Omega.$$

Where $\mathcal{A}_i : \mathbb{R}^d\mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ and $\mathcal{B} : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ are given Carathéodory functions. Let $\mathcal{A} = (\mathcal{A}_1, ..., \mathcal{A}_d)$ and $1 < p < d$, we suppose that the following conditions are fulfilled: for a.e. $x \in \Omega, \forall\zeta \in \mathbb{R}$ and ξ, $\xi' \in \mathbb{R}^d$

$(\mathcal{P}_1)$:

$$|\mathcal{A}(x, \zeta, \xi)| \leqslant k_0(x) + b_0(x) |\zeta|^{p-1} + a |\xi|^{p-1},$$

$(\mathcal{P}_2)$:

$$(\mathcal{A}(x, \zeta, \xi) - \mathcal{A}(x, \zeta, \xi^{'}))(\xi - \xi^{'}) > 0$$

if $\xi \neq \xi'$.

$(\mathcal{P}_3)$:

$$\mathcal{A}(x,\zeta,\xi)\xi \geq \alpha\,|\xi|^p - d_0(x)\,|\zeta|^p - e(x),$$

$(\mathcal{P}_4)$:

$$|\mathcal{B}(x,\zeta,\xi)| \leqslant k(x) + b(x)\,|\zeta|^{\alpha} + c\,|\xi|^{r},$$

$0 < r < \frac{p}{(p^*)'}$, $\alpha \geq 0$.

Here a, c and α are positive constants, $p' = \frac{p}{p-1}$, $p^* = \frac{dp}{d-p}$, while k_0, b_0, d_0, e, k and b are measurable functions on Ω satisfying

$k_0 \in L^{p'}$, $b_0 \in L^{\frac{d}{p-1}}$, $k \in L^q$, $(p^*)' < q < (\frac{d}{p-\epsilon} \wedge \frac{p}{r})$ and d_0, e, $b \in L^{\frac{d}{p-\epsilon}}$, $(0 < \epsilon < 1)$.

We can easily show that if $u \in \mathcal{W}^{1,p}(\Omega)$, then $A(.,u,\nabla u) \in L^{p'}$ and if $\alpha \leqslant p-1$ then $\mathcal{B}(.,u,\nabla u) \in L^{(p^*)'}$.

Definition 1.1 *We say that a function $u \in \mathcal{W}^{1,p}_{loc}(\Omega)$ is a (weak) solution of the equation* (1)*, provided*

$$\begin{cases} \mathcal{B}(.,u,\nabla u) \in L^{p^{*\prime}}, \\ \int\limits_{\Omega} \mathcal{A}(.,u,\nabla u)\nabla\varphi + \int\limits_{\Omega} \mathcal{B}(.,u,\nabla u)\varphi = 0, \end{cases} \tag{S_1}$$

for all $\varphi \in \mathcal{W}^{1,p}_0(\Omega)$.

We say that $u \in \mathcal{W}^{1,p}_{loc}(\Omega)$ is a supersolution (resp. subsolution) of (1) *if*

$$\begin{cases} \mathcal{B}(.,u,\nabla u) \in L^{(p^*)'} \\ \int\limits_{\Omega} \mathcal{A}(.,u,\nabla u)\nabla\varphi + \int\limits_{\Omega} \mathcal{B}(.,u,\nabla u)\varphi \geq 0 \quad (\textit{resp.} \leqslant 0), \end{cases}$$

for every nonnegative function $\varphi \in \mathcal{W}^{1,p}_0(\Omega)$.

Note that if u is a supersolution of (1) then $-u$ is a subsolution of the equation

$$-div\widehat{\mathcal{A}} + \widehat{\mathcal{B}} = 0$$

where $\widehat{\mathcal{A}}(x,\zeta,\xi) = -\mathcal{A}(x,-\zeta,-\xi)$ and $\widehat{\mathcal{B}}(x,\zeta,\xi) = -\mathcal{B}(x,-\zeta,-\xi)$. Furthermore the structure of $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ is similar to that of $\mathcal{A}$ and $\mathcal{B}$.

We recall that if u is a bounded supersolution (resp. subsolution)), then u is upper (resp. lower) semicontinuous in Ω [MaZ97, Corollary 4.10].

Proposition 1.2 *Let u and v be two subsolutions of* (1) *in Ω such that*

$$(\mathcal{A}(x,v,\nabla u) - \mathcal{A}(x,u,\nabla u)).(v-u) \geqslant 0, \quad a.e.\ x \in \Omega$$

Then $\max(u,v)$ *is also a subsolution. A similar statement holds for the minimum of two supersolutions.*

Proof. Let u and v be two subsolutions of *(1)* and $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\varphi \geq 0$. Let $\Omega_1 = \{x \in \Omega : u > v\}, \Omega_2 = \{x \in \Omega : u \leqslant v\}$ and put $I = \int\limits_\Omega \mathcal{A}(., u \vee v, \nabla(u \vee v))\nabla\varphi = I_1 + I_2$ where

$$I_1 = \int\limits_{\Omega_1} \mathcal{A}(., u, \nabla u)\nabla\varphi \quad \text{and} \quad I_2 = \int\limits_{\Omega_2} \mathcal{A}(., v, \nabla v)\nabla\varphi.$$

Let $\rho_n : \mathbb{R} \to \mathbb{R}$ such that $\rho_n \in \mathcal{C}^1(\mathbb{R})$,

$$\rho_n(t) = \begin{cases} 1 & \text{if} \quad t \quad \geq \quad \frac{1}{n} \\ 0 & \text{if} \quad t \quad \leqslant \quad 0 \end{cases},$$

such that $\rho_n' > 0$ on $\left]0, \frac{1}{n}\right[$ and put $q_n(x) = \rho_n((u - v)(x))$. We see that $q_n \in \mathcal{W}_{loc}^{1,p}(\Omega)$, $q_n \to 1_{\Omega_1}$ and $\|q_n\|_\infty \leqslant 1$, it follows by Lebesgue theorem of dominated convergence that $I_1 = \lim_{n\to\infty} \int\limits_{\Omega_1} q_n\mathcal{A}(., u, \nabla u).\nabla\varphi$ and $I_2 = \lim_{n\to\infty} \int\limits_{\Omega_2} (1 - q_n)\mathcal{A}(., v, \nabla v).\nabla\varphi$. hence

$$\begin{aligned} \int\limits_\Omega q_n\mathcal{A}(., u, \nabla u).\nabla\varphi &= \int\limits_\Omega \mathcal{A}(., u, \nabla u)\nabla.(q_n\varphi) - \int\limits_\Omega \mathcal{A}(., u, \nabla u)\varphi.\nabla(q_n) \\ &\leqslant -\int\limits_\Omega \mathcal{B}(., u, \nabla u)(q_n\varphi) - \int\limits_{\Omega_n} \mathcal{A}(., u, \nabla u)\varphi.\nabla(q_n) \end{aligned}$$

where $\Omega_n = \left\{x \in \Omega : v < u < v + \frac{1}{n}\right\}$.

Put $I_n = \int\limits_\Omega q_n\mathcal{A}(., u, \nabla u).\nabla\varphi$ and $J_n = \int\limits_\Omega (1 - q_n)\mathcal{A}(., v, \nabla v).\nabla\varphi$, then similarly, we have

$$\int\limits_\Omega (1 - q_n)\mathcal{A}(., v, \nabla v).\nabla\varphi \leqslant -\int\limits_\Omega (1 - q_n)\mathcal{B}(., v, \nabla v)\varphi + \int\limits_{\Omega_n} \mathcal{A}(., v, \nabla v)\varphi.\nabla(q_n)$$

hence we get

$$\begin{aligned} I_n + J_n \quad \leqslant \quad & -\int\limits_\Omega \mathcal{B}(., u, \nabla u)(q_n\varphi) - \int\limits_\Omega (1 - q_n)\mathcal{B}(., v, \nabla v)\varphi + \\ & + \int\limits_{\Omega_n} (\mathcal{A}(., v, \nabla v) - \mathcal{A}(., u, \nabla u))\varphi.\nabla(q_n). \end{aligned}$$

knowing that $\nabla(q_n) = \rho_n'(u-v)\nabla(u-v)$ we get

$$\begin{aligned} I_n + J_n &\leqslant -\int\limits_\Omega \mathcal{B}(.,u,\nabla u)(q_n\varphi) - \int\limits_\Omega (1-q_n)\mathcal{B}(.,v,\nabla v)\varphi + \\ & \quad -\int\limits_{\Omega_n} \rho_n'(u-v)(\mathcal{A}(.,v,\nabla v) - \mathcal{A}(.,u,\nabla u))\varphi.\nabla(v-u) \\ &\leqslant -\int\limits_\Omega \mathcal{B}(.,u,\nabla u)(q_n\varphi) - \int\limits_\Omega (1-q_n)\mathcal{B}(.,v,\nabla v)\varphi \end{aligned}$$

Finally we get

$$\int\limits_\Omega \mathcal{A}(.,u\vee v,\nabla(u\vee v).\nabla\varphi + \int\limits_\Omega \mathcal{B}(.,u\vee v,\nabla(u\vee v)\varphi \leqslant 0$$

□

We say that $\mathcal{L}$ *satisfies the property* $(\pm)$, if for every $k>0$ and every supersolution (resp. subsolution) u of (1) then $u+k$ (resp. $u-k$) is also a supersolution (resp. subsolution) of (1).

Remark 1.3 1- *Suppose that for all* $u \in \mathcal{W}^{1,p}_{loc}(\Omega)$ *and each* $k \in \mathbb{R}$,

$$k\int \left[\mathcal{A}(.,u+k,\nabla u) - \mathcal{A}(.,u,\nabla u)\right].\nabla\varphi + k\int \left[\mathcal{B}(.,u+k,\nabla u) - \mathcal{B}(.,u,\nabla u)\right]\varphi \geq 0 \tag{4}$$

for every nonnegative function $\varphi \in \mathcal{W}^{1,p}_0(\Omega)$. *Then* $\mathcal{L}$ *satisfies the property* $(\pm)$.

2- *Note that if* $\mathcal{L}(u) = -\sum\limits_j \frac{\partial}{\partial x_j}(\sum\limits_i a_{ij}\frac{\partial u}{\partial x_i} + d_j u) + (\sum\limits_i b_i \frac{\partial u}{\partial x_i} + cu)$ *is a linear elliptic operator of second order satisfying the conditions of* [HeHe68], *then* 4 *is equivalent to* $(c - \sum\limits_j \frac{\partial d_j}{\partial x_j}) \geq 0$ *in the distributional sense.*

3- *Suppose that* $\mathcal{A}(x,\zeta,\xi) = \mathcal{A}(x,\xi)$ *and for a.e.* $x \in \Omega$ *and* $\xi \in \mathbb{R}^d$ *the map:* $\zeta \to \mathcal{B}(x,\zeta,\xi)$ *is increasing. Then the property* $(\pm)$ *holds.*

2 Comparison principle

In this section, we will give some conditions which give us the comparison principle. This principle makes it possible to solve the Dirichlet problem and to develop a potential theory in our case.

We say that the *comparison principle* holds for $\mathcal{L}$, if for every supersolution u and every subsolution v of (1) on Ω, such that

$$\limsup_{x\to y} v(x) \leqslant \liminf_{x\to y} u(x)$$

for all $y \in \partial\Omega$ and both sides of the inequality are not simultaneously $+\infty$ or $-\infty$, we have $v \leqslant u$ a.e. in Ω.

Theorem 2.1 *Suppose that the operator $\mathcal{L}$ satisfies the property* $(\pm)$ *and the following strict monotony condition* (see [Nec83])*:*

$$(\mathcal{A}(x,\zeta,\xi) - \mathcal{A}(x,\zeta',\xi')).(\xi - \xi') + (\mathcal{B}(x,\zeta,\xi) - \mathcal{B}(x,\zeta',\xi'))(\zeta - \zeta') > 0$$

for $(\zeta,\xi) \neq (\zeta',\xi')$. Let u be a supersolution and v be a subsolution of (1)*, on Ω, such that*

$$\limsup_{x\to y} v(x) \leqslant \liminf_{x\to y} u(x)$$

for all $y \in \partial\Omega$ and both sides of the inequality are not simultaneously $+\infty$ or $-\infty$, then $v \leqslant u$ a.e. in Ω.

Proof. Let $\varepsilon > 0$ and K a compact subset of Ω such that $v - u \leqslant \varepsilon$ on $\Omega\backslash K$, then the function $\varphi = (v - u - \varepsilon)^+ \in \mathcal{W}_0^{1,p}(\Omega)$. Testing by φ, we obtain that

$$\begin{aligned} 0 &\leqslant \int_{\{v>u+\varepsilon\}} (\mathcal{A}(.,u+\varepsilon,\nabla u) - \mathcal{A}(.,v,\nabla v))\nabla(v-u-\varepsilon) + \\ &+ \int_{\{v>u+\varepsilon\}} (\mathcal{B}(.,u+\varepsilon,\nabla u) - (\mathcal{B}(.,v,\nabla v))(v-u-\varepsilon) \\ &\leqslant 0 \end{aligned}$$

Hence $\nabla(v-u-\varepsilon)^+ = 0$ and $(v-u-\varepsilon)^+ = 0$ a.e. in Ω. It follows that $v \leqslant u+\varepsilon$ a.e. in Ω and therefore $v \leqslant u$ a.e. in Ω □

Corollary 2.2 *we suppose that $\mathcal{A}(x,\zeta,\xi) = \mathcal{A}(x,\xi)$ and $\mathcal{B}(x,\zeta,\xi) = \mathcal{B}(\zeta)$ such that the map $\zeta \to \mathcal{B}(x,\zeta)$ is increasing for a.e. x in Ω. Then the comparison principle holds.*

The idea of the proof of the following theorem is owing to Professor J. Malý.

Theorem 2.3 *Suppose that*

i) $[\mathcal{A}(x,\zeta,\xi) - \mathcal{A}(x,\zeta',\xi')].(\xi-\xi') \geq \gamma\,|\xi - \xi'|^p$ *for all ζ, $\zeta' \in \mathbb{R}$, all $\xi,\xi' \in \mathbb{R}^d$ and a.e. x in Ω. Where $0 < \gamma \in \mathbb{R}$.*

ii) *For a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^d$, the map $\zeta \to \mathcal{B}(x,\zeta,\xi)$ is increasing,*

iii) $|\mathcal{B}(x,\zeta,\xi) - \mathcal{B}(x,\zeta,\xi')| \leqslant b(x,\zeta)\,|\xi - \xi'|^{p-1}$ *for a.e. $x \in \Omega$, all $\zeta \in \mathbb{R}$ and all $\xi,\xi' \in \mathbb{R}^d$. Where $\sup_{|\zeta|\leqslant M} b(.,\zeta) \in L^s_{loc}(\Omega)$, $s > d$, for all $M > 0$.*

Then the comparison principle holds.

Proof. Let $\rho > 0$, $M = \sup(v-u)$ and put $w = v-u-\rho$. Take w^+ as test function . Then we get

$$\int_\Omega \left[\mathcal{A}(.,u,\nabla u) - \mathcal{A}(.,v,\nabla v)\right].\nabla w^+ + \int_\Omega \left[\mathcal{B}(.,u,\nabla u) - \mathcal{B}(.,v,\nabla v)\right] w^+ \geq 0$$

and by consequence

$$\begin{aligned} \gamma \int_\Omega \left|\nabla w^+\right|^p &\leqslant \int_\Omega b(x,v)\left|\nabla w^+\right|^{p-1} w^+ \\ &\leqslant C\left[\int_\Omega \left|\nabla w^+\right|^p\right]^{\frac{p-1}{p}} \left[\int_\Omega \left(w^+\right)^{p^*}\right]^{\frac{1}{p^*}} \left|A_\rho\right|^{\frac{s-d}{sd}} \\ &\leqslant C\left\|\nabla w^+\right\|_p^p \left|A_\rho\right|^{\frac{s-d}{sd}}. \end{aligned}$$

where $A_\rho = \{\rho < v-u < M\}$. Hence we get $|A_\rho| \to 0$ when $\rho \to M$, which is impossible if $M > 0$. Thus $v \leqslant u$ on Ω □

3 Dirichlet Problem

3.1 Existence of solution in the case $0 \leqslant \alpha \leqslant p-1$ and $0 \leqslant r \leqslant p-1$

Definition 3.1 *Let $g \in \mathcal{W}^{1,p}(\Omega)$, we say that u is a solution of the problem* (P) *if*

$$\begin{cases} u - g \in \mathcal{W}_0^{1,p}(\Omega), \\ \int_\Omega \mathcal{A}(.,u,\nabla u).\nabla\varphi + \int_\Omega \mathcal{B}(.,u,\nabla u)\varphi = 0 \end{cases}$$

$\forall \varphi \in \mathcal{W}_0^{1,p}(\Omega)$.

Remark 3.2 *If we put $v = u - g$, then u is a solution of* (P) *iff v is a solution of*

$$(P)' \qquad \begin{cases} u \in \mathcal{W}_0^{1,p}(\Omega) \\ \int_\Omega \mathcal{A}_g(.,u,\nabla u).\nabla\varphi + \int_\Omega \mathcal{B}_g(.,u,\nabla u)\varphi = 0, \quad \forall\varphi \in \mathcal{W}_0^{1,p}(\Omega). \end{cases}$$

where $\mathcal{A}_g(.,u,\nabla u) = \mathcal{A}(.,u+g,\nabla(u+g))$ and $\mathcal{B}_g(.,u,\nabla u) = \mathcal{B}(.,u+g,\nabla(u+g))$.

Let T the operator defined by

$$\begin{array}{llll} T: & \mathcal{W}_0^{1,p}(\Omega) & \to & \mathcal{W}_0^{-1,p'}(\Omega) \\ & u & \to & T(u) \end{array}$$

where $\langle T(u), v\rangle = \int \mathcal{A}_g(., u, \nabla u).\nabla v + \int \mathcal{B}_g(., u, \nabla u)v$ for all $v \in \mathcal{W}_0^{1,p}(\Omega)$.

In the next we will establish the existence of solution of $(P)'$ when $0 \leqslant \alpha \leqslant p-1$ and $0 \leqslant r \leqslant p-1$.

Proposition 3.3 *Suppose that* $0 \leqslant \alpha \leqslant p-1$ *and* $0 \leqslant r \leqslant p-1$. *If* Ω *is small, then the operator* T *is coercive.*

Proof. we have

$$\begin{aligned} \langle T(u), u\rangle &= \int \mathcal{A}(u+g, \nabla(u+g))\nabla u + \int \mathcal{B}(u+g, \nabla(u+g))u \\ &\geq (\alpha - C\|d_0\|_{\frac{n}{p}} - C\|b\|_{\frac{n}{p}})\|\nabla u\|_p^p - H_1(\|u\|, \|\nabla u\|, \|g\|, \|\nabla g\|) \end{aligned}$$

where $C = C(n,p)$ and the growth of H_1 in $\|u\|$ and $\|\nabla u\|$ is less then $p-1$. So let Ω be small enough such that $\alpha > C(\|d_0\|_{\frac{n}{p}} + \|b\|_{\frac{n}{p}})$. Hence $\frac{\langle T(u), u\rangle}{\|\nabla u\|_p} \to +\infty$ as $\|\nabla u\|_p \to +\infty$ and therefore the operator T is coercive. □

In the next, we recall the following proposition (cf. [MaZ97])

Proposition 3.4 *Suppose that* $0 \leqslant \alpha \leqslant p-1$ *and* $0 \leqslant r \leqslant p-1$. *Then the operator* T *is pseudomonotone and satisfies the property* (S_+).

Theorem 3.5 *Suppose that* T *satisfies the* coercive condition *on* Ω. *Then the problem* (P') *has at least a weak solution in* $\mathcal{W}_0^{1,p}(\Omega)$.

Proof. The operator T is pseudomonotone, bounded continuous and coercive. Hence, by [Nec83] it's surjective □

Now we will show the existence of solutions for variational problem in the general case when $\alpha \geq 0$ and $p-1 < r < \frac{p}{(p^*)'}$.

3.2 Existence of solution in the case $p-1 < r < \frac{p}{(p^*)'}$

Definition 3.6 *Let* $g \in \mathcal{W}^{1-\frac{1}{p}}(\partial\Omega)$ *be given.*

- We say that a function u is a solution of the problem *(5)* with boundary value g if

$$\begin{cases} u \in \mathcal{W}^{1,p}(\Omega),\ \mathcal{B}(., u, \nabla u) \in L_{loc}^{p^{*'}}(\Omega), \\ u = g \text{ in } \mathcal{W}^{1-\frac{1}{p}}(\partial\Omega), \\ \int\limits_\Omega \mathcal{A}(., u, \nabla u)\nabla\varphi + \int\limits_\Omega \mathcal{B}(., u, \nabla u)\varphi = 0 \end{cases} \tag{5}$$

  $\forall \varphi \in \mathcal{W}_0^{1,p}(\Omega)$.

(For definition and property of the space $\mathcal{W}^{1-\frac{1}{p}}(\partial\Omega)$ see e.g. [Lio69]).

- We say that a function u is an upper supersolution of the problem *(5)* with boundary value g provided

$$\begin{cases} u \in \mathcal{W}^{1,p}(\Omega),\ \mathcal{B}(.,u,\nabla u) \in L^{p^{*'}}_{loc}(\Omega), \\ u \geq g \text{ in } \mathcal{W}^{1-\frac{1}{p}}(\partial\Omega), \\ \int\limits_{\Omega} \mathcal{A}(.,u,\nabla u)\nabla\varphi + \int\limits_{\Omega} \mathcal{B}(.,u,\nabla u)\varphi \geq 0 \end{cases}$$

  $\forall \varphi \in \mathcal{W}^{1,p}_0(\Omega), \varphi \geq 0.$

Similarly, *a lower subsolution* is characterized by the reverse inequality signs in the above definition.

Remark 3.7 *Let $\widetilde{g} \in \mathcal{W}^{1,p}(\Omega)$ denotes an extension of the function g to Ω. We may choose $\widetilde{g}$ such that $u \leqslant \widetilde{g} \leqslant v$. Thus a solution of* (5) *is such that*

$$\begin{cases} u - \widetilde{g} \in \mathcal{W}^{1,p}_0(\Omega),\ \mathcal{B}(.,u,\nabla u) \in L^{p^{*'}}_{loc}(\Omega), \\ \int\limits_{\Omega} \mathcal{A}(.,u,\nabla u)\nabla\varphi + \int\limits_{\Omega} \mathcal{B}(.,u,\nabla u)\varphi = 0 \end{cases}$$

$\forall \varphi \in \mathcal{W}^{1,p}_0(\Omega).$
If we put $w = u - \widetilde{g}$ then w is a solution of

$$\begin{cases} w \in \mathcal{W}^{1,p}_0(\Omega),\ \mathcal{B}_g(.,w,\nabla w) \in L^{p^{*'}}_{loc}(\Omega), \\ \int\limits_{\Omega} \mathcal{A}_g(.,w,\nabla w)\nabla\varphi + \int\limits_{\Omega} \mathcal{B}_g(.,w,\nabla w)\varphi = 0 \end{cases} \tag{6}$$

$\forall \varphi \in \mathcal{W}^{1,p}_0(\Omega)$. *Where $\mathcal{A}_g(.,w,\nabla w) = \mathcal{A}(.,w+g,\nabla(w+g))$ and $\mathcal{B}_g(.,w,\nabla w) = \mathcal{B}(.,w+g,\nabla(w+g))$.*

We recall the following result given in [Leo97, Theorem 2.2].

Theorem 3.8 *Suppose that there exists an ordered pair $\varphi \leqslant \psi$ of subsolution and supersolution of* (5) *and $|\mathcal{B}(x,\zeta,\xi)| \leqslant k(x) + c\,|\xi|^r$ a.e. $x \in \Omega$ $\forall \zeta : \varphi(x) \leqslant \zeta \leqslant \psi(x)$ and $\forall \xi \in \mathbb{R}^d$, where $k \in L^q(\Omega)$, $q > p^{*'}$. Then* (5) *has at least one solution $u \in \mathcal{W}^{1,p}_0(\Omega)$ such that $\varphi \leqslant u \leqslant \psi$.*

Proposition 3.9 *Suppose that* (5) *admits a pair of bounded lower subsolution u and upper supersolution v, then there exists a solution w of* (5) *such that $u \leqslant w \leqslant v$.*

Proof. Let M a positive real such that $\|u\|_\infty, \|v\|_\infty, \|g\|_\infty \leqslant M$, then for each ζ such that $u(x) - g(x) \leqslant \zeta \leqslant v(x) - g(x)$ we have $|\mathcal{B}_g(x,\zeta,\xi)| \leqslant k(x) + b(x)M^\alpha + 2^r c\,|\nabla g|^r + c\,|\xi|^r$ for a.e. $x \in \Omega$. In addition, $u - g$ (resp.$v - g$) is a lower subsolution (resp. upper supersolution) of (6). Hence by the last theorem, there exists a solution w_1 of (6) such that $u - g \leqslant w_1 \leqslant v - g$. It's clear that $w = w_1 + g$ is a solution of (5) $\square$

Corollary 3.10 *Suppose that all positive constants are supersolutions and all negative constants are subsolutions, then for each* $g \in \mathcal{W}^{1,p}(\overline{\Omega}) \cap L^{\infty}(\Omega)$ *there exists a bounded solution* w *of* (5) *such that* $\|w\|_{\infty} \leqslant \|g\|_{\infty}$.

Proof. We see that for $v = \|g\|_{\infty}$ is an upper supersolution and $u = -\|g\|_{\infty}$ is a lower subsolution and hence by the proposition above we get a solution $u \leqslant w \leqslant v$ □

3.3 Dirichlet Problem

In the next, we assume that $\mathcal{A}(.,0,0) = 0$ and $\mathcal{B}(.,0,0) = 0$ a.e. in Ω, that the property $(\pm)$ is satisfied and that the comparison principle holds.

We suppose that the open set Ω is regular ($p-$regular) see [MaZ97] or [HKM93]. It's known that if u is a solution of (1) in Ω satisfying $u - f \in \mathcal{W}_0^{1,p}(\Omega)$ with $f \in \mathcal{W}^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$, then

$$\lim_{x \to z} u(x) = f(z), \ \ \forall z \in \partial\Omega.$$

Definition 3.11 *Let* f *be a continuous function on* $\partial\Omega$*. We say that* $u \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{W}_{loc}^{1,p}(\Omega)$ *solves the Dirichlet problem with boundary value* f *if* u *is a solution of* (1) *such that* $\lim\limits_{x \to z} u(x) = f(z)$ *, for all* $z \in \partial\Omega$.

We have the following theorem:

Theorem 3.12 *For each* $f \in \mathcal{C}(\partial\Omega)$*, there exists* $u \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{W}_{loc}^{1,p}(\Omega)$ *solving the Dirichlet problem with boundary value* f.

Proof. By the Tietze's extension theorem, we can assume that $f \in \mathcal{C}_c^{\infty}(\mathbb{R}^d)$. Let (f_n) be a sequence of mollifer of f such that $\|f_n - f\| \leqslant \frac{1}{2^n}$ on $\overline{\Omega}$. Let u_n designs the continuous solution of

$$\begin{cases} u_n - f_n \in \mathcal{W}_0^{1,p}(\Omega) \\ \int\limits_{\Omega} \mathcal{A}(., u_n, \nabla u_n)\nabla\varphi - \int\limits_{\Omega} (\mathcal{B}(., u_n, \nabla u_n)\varphi = 0 \ \forall \varphi \in \mathcal{W}_0^{1,p}(\Omega). \end{cases} \tag{7}$$

so we get by the comparison principle that $|u_n - u_m| \leqslant \frac{1}{2^n} + \frac{1}{2^m}$. Hence, the sequence (u_n) is a Cauchy and therefore converges uniformly on $\overline{\Omega}$ to a continuous function u. Let M be a positive real such that $|f_n| + |f| \leqslant M$ and $|u_n| + |u| \leqslant M$ on Ω for all n .

Let $G \subset \overline{G} \subset \Omega$, take φ as a test function in (7) such that $\varphi = \eta^p u_n, \eta \in C_c^{\infty}(\Omega), 0 \leqslant \eta \leqslant 1$ and $\eta = 1$ on G. Then we get

$$\int\limits_{\Omega} \mathcal{A}(., u_n, \nabla u_n)\eta^p \nabla u_n = -p \int\limits_{\Omega} \mathcal{A}(., u_n, \nabla u_n) u_n \eta^{p-1} \nabla\eta - \int\limits_{\Omega} \mathcal{B}(., u_n, \nabla u_n) u_n \eta^p$$

Using the assumptions on $\mathcal{A}$ and $\mathcal{B}$, we get

$$\begin{aligned}
\alpha\int\limits_{\Omega}\eta^p\left|\nabla u_n\right|^p &\leqslant pM\int\limits_{\Omega}k_0\left|\nabla\eta\right| + pM^p\int\limits_{\Omega}b_0\left|\nabla\eta\right| + pM\int\limits_{\Omega}a\left|\nabla u_n\right|^{p-1}\eta^{p-1}\left|\nabla\eta\right| \\
&\quad +cM\int\limits_{\Omega}\left|\nabla u_n\right|^r\eta^p + \int\limits_{\Omega}(M^p d_0 + Mk + M^{\alpha+1}b + e) \\
&\leqslant a(p-1)^{-1}M\varepsilon^{\frac{p}{p-1}}(\int\limits_{\Omega}\left|\nabla u_n\right|^p\eta^p) + crp^{-1}M\varepsilon^{\frac{p}{r}}(\int\limits_{\Omega}\left|\nabla u_n\right|^p\eta^p) + \\
&\quad +C(M,\Omega,\eta,\nabla\eta)
\end{aligned}$$

Thus for ε small enough, We obtain that

$$\int\limits_{G}\left|\nabla(u_n)\right|^p \leqslant C(M,\Omega,\eta,\nabla\eta,\varepsilon).$$

So $(\nabla u_n)_n$ is bounded in $L^p(G)$ and therefore ∇u_n converges weakly to ∇u in $(L^p(G))^d$.

Fix an open D subset of G and let $\eta \in C_0^\infty(G)$ such that $0 \leqslant \eta \leqslant 1$ and $\eta = 1$ on D. Take $\psi = \eta(u - u_n)$ as test function, then

$$\begin{aligned}
-\int\limits_{\Omega}\eta\mathcal{A}(.,u_n,\nabla u_n).\nabla(u_n-u) &= \int\limits_{\Omega}(u_n-u)\mathcal{A}(.,u_n,\nabla u_n).\nabla\eta + \\
&\quad \int\limits_{\Omega}\mathcal{B}(.,u_n,\nabla u_n)(u_n-u)\eta.
\end{aligned}$$

Since $\mathcal{A}(.,u_n,\nabla u_n)\eta$ is bounded in $L^{p'}(G)$ and $\mathcal{B}(.,u_n,\nabla u_n)$ is bounded in $L^q(G)$ then

$$\lim_{n\to\infty}\int\limits_{G}(u_n-u)\mathcal{A}(.,u_n,\nabla u_n).\nabla\eta = 0$$

and

$$\lim_{n\to\infty}\int\limits_{G}\mathcal{B}(.,u_n,\nabla u_n)(u_n-u)\eta = 0.$$

By consequence we get $\lim\limits_{n\to\infty}\int\limits_{D}\mathcal{A}(.,u_n,\nabla u_n).\nabla(u_n-u) = 0$.

To finish the proof, we need to prove that ∇u_n converge to ∇u a.e. in Ω. That's the aim of the following lemma:

Lemma 3.13 *Let $G \subset \Omega$ and suppose that the sequence (∇u_n) is bounded in $L^p(G)$ and*

$$\lim_{n\to\infty} \int_G [\mathcal{A}(.,u_n,\nabla u_n) - \mathcal{A}(.,u,\nabla u)] . \nabla(u_n - u) = 0$$

then

$$\mathcal{A}(.,u_n,\nabla u_n) \to \mathcal{A}(.,u,\nabla u) \quad \textit{weakly in} \;\; L^{p'}(G).$$

Proof. Put $v_n = [\mathcal{A}(.,u_n,\nabla u_n) - \mathcal{A}(.,u_n,\nabla u)] . \nabla(u_n - u)$. Since

$$\begin{aligned} \int_G v_n &= \int_G [\mathcal{A}(.,u_n,\nabla u_n) - \mathcal{A}(.,u,\nabla u)] . \nabla(u_n - u) \\ & \quad - \int_G [\mathcal{A}(.,u_n,\nabla u) - \mathcal{A}(.,u,\nabla u)] . \nabla(u_n - u) \end{aligned}$$

then for a subsequence, we get

$$\lim_{n\to\infty} [\mathcal{A}(.,u_n,\nabla u_n) - \mathcal{A}(.,u_n,\nabla u)] . \nabla(u_n - u) = 0$$

a.e. $x \in G \setminus N$ with $|N| = 0$. Let $x \in G \setminus N$, by the assumptions on $\mathcal{A}$ we have

$$v_n(x) \geq \alpha \left|\nabla u_n(x)\right|^p - F(\left|\nabla u_n(x)\right|^{p-1}, \left|\nabla u(x)\right|^{p-1})$$

Consequently $(\nabla u_n(x))$ is bounded and converges to $\xi \in \mathbb{R}^d$. It yields that $[\mathcal{A}(.,u,\xi) - \mathcal{A}(.,u,\nabla u)] . (\xi - \nabla u) = 0$ and hence $\xi = \nabla u$. finally we conclude that $\mathcal{A}(.,u_n,\nabla u_n) \to \mathcal{A}(.,u,\nabla u)$ a.e. in G and $\mathcal{A}(.,u_n,\nabla u_n)$ converge weakly to $\mathcal{A}(.,u,\nabla u)$ in $L^{p'}(G)$ □

. (*Proof of* Theorem 3.12 *continued*) Now using the Lemma 3.13 , we conclude that $\nabla u_n \to \nabla u$ a.e. in Ω and $\mathcal{A}(.,u_n,\nabla u_n) \rightharpoonup \mathcal{A}(.,u,\nabla u)$ in $L^{p'}(D)$, so we obtain

$$\int_D \mathcal{A}(.,u,\nabla u) . \nabla\varphi + \int_D \mathcal{B}(.,u,\nabla u)\varphi = 0$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$.

We conclude that u is a solution for (1). Moreover, using the fact that

$$-\frac{1}{2^n} - \frac{1}{2^m} \leqslant u_n - u_m \leqslant \frac{1}{2^n} + \frac{1}{2^m} \text{ for all } n,\, m$$

we get

$$-\frac{1}{2^n} - u_n \leqslant u \leqslant \frac{1}{2^n} + u_n. \text{ for all } n$$

so we deduce that for all n

$$-\frac{1}{2^n} + f_n(z) \leqslant \liminf_{\Omega \ni x \to z} u(z) \leqslant \limsup_{\Omega \ni x \to z} u(z) \leqslant \frac{1}{2^n} + f_n(z)$$

for all $z \in \partial\Omega$, which gives us that $\lim_{x \to z} u(x) = f(z)$. □

Remark 3.14 *Using the same techniques as in the proof of the* Theorem 3.12 *we can show that for every increasing and locally bounded sequence* $(u_n)_n$ *of supersolutions of* (1) *in* Ω *is locally bounded in* $\mathcal{W}^{1,p}(\Omega)$ *and that* $u = \lim_n u_n$ *is a supersolution of* (1) *in* Ω.

4 Sheaf property for Superharmonic functions

4.0.1 The Obstacle Problem

Definition 4.1 *Let* $f,\ h \in \mathcal{W}^{1,p}(\Omega)$ *and let*

$$K_{f,h} = \left\{ u \in \mathcal{W}^{1,p}(\Omega) : h \leqslant u \quad a.e.\ in\ \Omega,\ u - f \in \mathcal{W}_0^{1,p}(\Omega) \right\}.$$

If $f = h$, *we note* $K_{f,h} = K_f$.

We say $u \in K_{f,h}$ *is a solution to the obstacle problem in* $K_{f,h}$ *if*

$$\int_\Omega \mathcal{A}(.,u,\nabla u).\nabla(v-u) + \int_\Omega \mathcal{B}(.,u,\nabla u)(v-u) \geq 0$$

whenever $v \in K_{f,h}$.

u *is called solution of the problem with obstacle* h *and boundary value* f .

Remark 4.2 *Since* $u + \varphi \in K_{f,h}$ *for all nonnegative* $\varphi \in \mathcal{W}_0^{1,p}(\Omega)$, *the solution* u *to the obstacle problem is always a supersolution of* (1) *in* Ω. *Conversely, a supersolution of* (1) *is always a solution to the obstacle problem in* $K_u(D)$ *for all open* $D \subset \overline{D} \subset \Omega$.

Theorem 4.3 *Let* h *and* f *in* $\mathcal{W}^{1,p}(\Omega) \cap L^\infty(\Omega)$. *If* v *is an upper bounded supersolution of* (5) *with boundary value* f *such that* $v \geq h$, *then there exists a solution* u *to the obstacle problem in* $K_{f,h}$ *with* $u \leqslant v$.

Proof. Without loss of generality we my suppose that $h \leqslant f \leqslant v$ and by the change of variable $u \longmapsto u - f$, we reduce the problem to the case $f = 0$. We thus assume in future that $f = 0$ and $h \leqslant 0 \leqslant v$ a.e. in Ω, and search for solution in $\mathcal{W}_0^{1,p}(\Omega)$.

We introduce as in [Leo97] the function g defined by

$$g(x,\zeta,\xi) = \begin{cases} \mathcal{B}(x,\zeta,\xi) & \text{if} \quad \zeta \leqslant v(x) \\ \mathcal{B}(x,v,\nabla v) & \text{if} \quad \zeta > v(x) \end{cases}$$

and as in [Hes78] the function $\mathbf{a}$ defined by

$$\mathbf{a}(x,\zeta,\xi) = \begin{cases} \mathcal{A}(x,\zeta,\xi) & \text{if} \quad \zeta \leqslant v(x) \\ \mathcal{A}(x,v,\xi) & \text{if} \quad \zeta > v(x). \end{cases}$$

The functions $\mathbf{a}$ still satisfy the conditions $(\mathcal{P}_1)$, $(\mathcal{P}_2)$ and $(\mathcal{P}_3)$. A Lemma in [Deu74, p.52] proves that the map $u \to g(x,u,\nabla u)$ from $\mathcal{W}^{1,p}(\Omega)$ to $L^{p'}(\Omega)$ is bounded and continuous. Without loss of generality we can suppose that $r \geq p-1$. Let $l = \max\left\{q', \frac{p}{p-r}\right\} - 1$, and define the following penalty term

$$\gamma(x,s) = \left[(s - v(x))^+\right]^l$$

for all $x \in \Omega$, $s \in \mathbb{R}$.

Let $M > 0$ and consider the map $T : K_{0,h} \to \mathcal{W}^{-1,p'}(\Omega)$ defined by

$$\langle T(u), w\rangle = \int_\Omega \mathbf{a}(.,u,\nabla u).\nabla w + \int_\Omega g(.,u,\nabla u)w + M\int_\Omega \gamma(.,u)w.$$

Further, for any $u, w \in K_{0,h}$, we have

$$\left|\int_\Omega g(x,u,\nabla u)w\right| \leqslant c_1 \|w\|_{l+1} + c_2 \|\nabla u\|_p^r \|w\|_{l+1},$$

$$\left|\int_\Omega \gamma(x,u)w\right| \leqslant c_3 \|w\|_{l+1} + c_4 \|u\|_{l+1}^l \|w\|_{l+1},$$

and for each $u \in K_{f,h} - f$, we have

$$\int_\Omega \gamma(.,u)u \geq c_5 \|u\|_{l+1}^{l+1} - c_6.$$

An easy computation shows that for $\varepsilon > 0$

$$\begin{aligned}\langle T(u), u\rangle \;\; &\geq \;\; (\alpha - c_2\varepsilon)\|\nabla u\|_p^p - (c\|u\|_p^p + c_1\|u\|_{l+1}^{l+1} + c_2 c(\varepsilon)\|u\|_{l+1}^{l+1}) \\ &\quad + Mc_5\|u\|_{l+1}^{l+1} - Mc_6 - c_1c_7.\end{aligned}$$

where $c(\varepsilon)$ is a constant which depends on ε and $c > 0$. Now, we choose M large to get the operator T coercive. Since it's bounded , pseudomonotone and continuous, then by a Theorem found in [Nec83], there exists $w \in K_{0,h}$ such that $\langle T(w), u - w\rangle \geq 0$ for all $u \in K_{0,h}$.

Our claim is to show that $w \leqslant v$. Since $w - ((w-v)\vee 0) \in K_{0,h}$ and since v is a supersolution of (5), it yields that

$$\begin{aligned}&\int_{\{w>v\}} \left[\mathcal{A}(.,v,\nabla w) - \mathcal{A}(.,v,\nabla v)\right].\nabla(w-v) \\ \leqslant \;\; & M\int_{\{w>v\}} \gamma(.,w)(v-w)\end{aligned}$$

and by $(\mathcal{P}_2)$ we get that $(w-v)_+ = 0$ a.e. in Ω, and hence $w \leqslant v$ on Ω. Finally if we take $w_1 = w + f$, we obtain a solution of the obstacle problem in $K_{f,h}$ □

4.1 Nonlinear Harmonic Space

Definition 4.4 *Let V be a regular set. For every $f \in C(\partial V)$, we denote by $H_V f$ the solution of the Dirichlet problem with the boundary data f.*

Proposition 4.5 *Let f and $g \in C(\partial V)$ such that $f \leqslant g$. Then*

i) $H_V f \leqslant H_V g$.

ii) *For every $k \geq 0$, we have $H_V(k+f) \leqslant H_V(f)+k$ and $H_V(f)-k \leqslant H_V(f-k)$.*

Definition 4.6 *Let U be an open . We note by $\mathcal{U}(U)$ the set of all open, regular subset of U which are relatively compact in U.*

We say that a function u is harmonic on U, if $u \in C(U)$ and u is solution of (1). We note by $\mathcal{H}(U)$ the set of harmonic functions on U.Then

$$\mathcal{H}(U) = \{u \in C(U) : H_V u = u \text{ for every } V \in \mathcal{U}(U)\}.$$

A function $u \in SCI(U)$ is said to be hyperharmonic on U, if

- $-\infty < u$,
- *$u \neq \infty$ in each component of U*
- *For each regular set $V \subset \overline{V} \subset \Omega$ and for every $f \in \mathcal{H}(V) \cap C(\overline{V})$, the inequality $f \leqslant u$ on ∂V implies $f \leqslant u$ in V .*

  We note by ${}^*\mathcal{H}(U)$ the set of hyperharmonic functions on U.

A function $u \in SCS(U)$ is said to be hypoharmonic on U, if

- $u < +\infty$,
- *$u \neq \infty$ in each component of U*
- *For each regular set $V \subset \overline{V} \subset \Omega$ and each $f \in \mathcal{H}(\overline{V}) \cap C(\overline{V})$, the inequality $f \geq u$ on ∂V implies $f \geq u$ in V .*

We note by $\mathcal{H}_*(U)$ the set of hypoharmonic functions on U.

Proposition 4.7 *Let $u \in^* \mathcal{H}(U)$ and $v \in \mathcal{H}_*(U)$, then for each $k \geq 0$ we have $u + k \in^* \mathcal{H}(U)$ and $v - k \in \mathcal{H}_*(U)$.*

Proposition 4.8 *Let u be a superharmonic function and v be a subharmonic function on U such that*

$$\limsup_{x \to z} v(x) \leqslant \liminf_{x \to z} u(x)$$

for all $z \in \partial U$, and both sides of the previous inequality are not simultaneously $+\infty$ or $-\infty$, then $v \leqslant u$ in U.

Proof. Let $x \in U$ and $\varepsilon > 0$. Choose a regular open set $V \subset \overline{V} \subset U$ such that $x \in V$ and $v < u + \varepsilon$ on ∂V. Let $(\varphi_i) \in C^\infty(\Omega)$ be a decreasing sequence converging to v in $\overline{V}$. Then $\varphi_i \leqslant u + \varepsilon$ on ∂V for i large. Let $h = H_V(\varphi_i)$, then $v \leqslant h \leqslant u + \varepsilon$ on V. By letting $\varepsilon \to 0$, we get $v(x) \leqslant u(x)$. □

Theorem 4.9 *The space* $(\mathbb{R}^d, \mathcal{H})$ *satisfies the Bauer convergence property.*

Proof. Let $(u_n)_n$ be an increasing sequence in $\mathcal{H}(U)$ locally bounded. For every $V \subset \overline{V} \subset U$, by [MaZ97, Theorem 4.11] the set $\{u_n(x), x \in \overline{V}, n \in \mathbb{N}\}$ is equicontinuous . Then the sequence converges locally and uniformly in U to a continuous function u. Take $\varepsilon > 0$, since $u - \varepsilon \leqslant u_n \leqslant u + \varepsilon$, we get $H_V(u) - \varepsilon \leqslant u_n \leqslant H_V(u) + \varepsilon$, and therefore $H_V(u) = u$ □

Theorem 4.10 *Suppose that the conditions in* subsection 3.3 *are satisfied,* $k_0 = e = k = 0$ *and* $\alpha \geqslant p - 1$. *Then* $(\mathbb{R}^d, \mathcal{H})$ *is a nonlinear Bauer harmonic space.*

Proof. It is clear that $\mathcal{H}$ is a sheaf of continuous functions and by Theorem 3.12 there exists a basis of regular sets stable by intersection. The Bauer convergence property is fulfilled by Theorem 4.9. Since $k_0 = e = k = 0$ and $\alpha \geqslant p - 1$ we have the following form of the Harnack inequality (e.g. [MaZ97],[Tru67] or [Ser64]): for every non empty open set U in $\mathbb{R}^d$, for every constant $M > 0$ and every compact K in U, there exists a constant $C = C(K, M)$ such hat

$$\sup_K u \leq C \inf_K u$$

for every $u \in \mathcal{H}^+(U)$ with $u \leq M$. It follows that the sheaf $\mathcal{H}$ is non degenerate. □

Theorem 4.11 *Suppose that the condition of strict monotony holds. Let* $u \in \mathcal{H}^*(\Omega) \cap L^\infty(\Omega)$. *Then* u *is a supersolution on* U.

Proof. Let $V \subset \overline{V} \subset \Omega$. Let $(\varphi_i)_i$ be an increasing sequence in $\mathcal{C}_c^\infty(\Omega)$ such that $u = \sup_i \varphi_i$ on $\overline{V}$. Let

$$K_{\varphi_i} = \left\{ w \in \mathcal{W}^{1,p}_{loc}(\Omega) : \varphi_i \leqslant w,\ \ w - \varphi_i \in \mathcal{W}^{1,p}_0(V) \right\}.$$

We know by Theorem 4.3 that there exists a solution u_i to the obstacle problem K_{φ_i} such that $\|u_i\|_\infty \leqslant \|\varphi_i\|_\infty$. We claim that $(u_i)_i$ is increasing. In fact $u_i \wedge u_{i+1} \in K_{\varphi_i}$, then

$$\int\limits_{\{u_i > u_{i+1}\}} (\mathcal{A}(., u_i, \nabla u_i) - \mathcal{A}(., u_{i+1}, \nabla u_{i+1})).\nabla(u_{i+1} - u_i) + \int\limits_{\{u_i > u_{i+1}\}} \mathcal{B}(., u_i, \nabla u_i) - \mathcal{B}(., u_{i+1}, \nabla u_{i+1})(u_{i+1} - u_i) \qquad \geq 0.$$

Hence $\nabla(u_{i+1} - u_i)^+ = 0$ a.e. which yields that $u_i \leqslant u_{i+1}$a.e. in V .

In the other hand, for each i the function u_i is a solution of (1) in $D_i := \{\varphi_i < u_i\}$. Indeed, let $\psi \in C_c^\infty(\omega)$,$\omega \subset \overline{\omega} \subset D_i$, and $\varepsilon > 0$ such that $\varepsilon \|\psi\| \leqslant \inf_{\overline{\omega}}(u_i - \varphi_i)$ then we get $u_i + \varepsilon\psi \in K_{\varphi_i}$ and

$$\int_\omega \mathcal{A}(., u_i, \nabla u_i).\nabla\psi + \int_\omega \mathcal{B}(., u_i, \nabla u_i)\psi = 0.$$

Since

$$\liminf_{x\to y} u(x) \geq u(y) \geq \varphi_i(y) = \lim_{x\to y} u_i(x)$$

for all $y \in \partial D_i$, it yields, by the comparison principle, that $u \geq u_i$ in D_i. Hence $u \geq u_i$ in D. Thus $u = \lim_{i\to\infty} \varphi_i \leqslant \lim_{i\to\infty} u_i \leqslant u$. Finally we conclude by the Remark 3.14 □

Theorem 4.12 *Suppose that the condition of strict monotony holds. Then $\mathcal{H}^*$ is a sheaf.*

Proof. The proof is the same as in [BB99, Theorem 4.2]. □

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