# Asymptotic behavior of kernels associated to singular perturbations in two dimensions

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**Abstract**

Explicit large time asymptotics for the heat kernel and the unitary propagator kernel associated to point interaction in two dimensions are derived. For the sphere interaction in two dimensions, an estimate of the heat kernel is given.

## 1 Introduction

In [1] the authors derived an explicit formulae for the kernels of the unitary propagators and the heat operators associated to the operator $H_{\delta,\alpha} = -\triangle + \alpha\delta$ on $\mathbb{R}^d$ for $d = 1,\ 2,\ 3$ where $-\infty < \alpha \leq +\infty$ is related to the inverse scattering length for $d = 2, 3$, is the coupling constant for $d = 1$ and $\delta$ is the Dirac measure with support 0. They also gave asymptotic expansions for both kernels for small time. They remark that for $d = 2$ the situation is somewhat different due to the existence of Logarithmic terms in the asymptotic.

In this paper we are interested to investigate large time behavior of the kernels of the operators $\exp(-tH_{\delta,\alpha})$ and $\exp(-itH_{\delta,\alpha})$ in two dimensions. We will treat first the Hamiltonians describing point interactions placed at the origin where we give large time asymptotic for the associated kernels. We then treat Hamiltonians describing sphere interactions. Let's recall from [1] the formulae of the kernels associated to $\exp(-tH_{\delta,\alpha})$ and $\exp(-itH_{\delta,\alpha})$, which are denoted following Albeverio et al.[1] respectively $P^\alpha(t;x,y)$ and $P^\alpha(it;x,y)$.

$$\begin{aligned} P^\alpha(t;x,y) &= P(t;x,y) + \frac{1}{2\pi}\int_0^{+\infty} t^{u-1}\frac{\exp(-\alpha u)}{\Gamma(u)} \\ &\int_1^{+\infty}(z-1)^{u-1}z^{-u}\exp(-z\frac{|x|^2+|y|^2}{4t})K_0(\frac{|x||y|}{2t}z)dzdu \end{aligned} \quad (1)$$

where $P(t;x,y) = \frac{1}{4\pi t}\exp(-\frac{|x-y|^2}{4t})$ is the classical heat kernel associated to the free Hamiltonian in $I\!R^2$. We will denote the second term in(1) $\widetilde{P^\alpha}(t;x,y)$. The kernel associated to $\exp(-itH_{\delta,\alpha})$ is obtained by which notation replacing $t$ by $it$ in formula (1). We will be first interested to investigate the behavior of both operators as $t$ tends to infinity. It is known that for the free Hamiltonian both $P(t;x,y)$ and $P(;it,x,y)$ decay at infinity like $t^{-1}$. For a large class of potentials $V$ it is known [15](proposition B.6.7) that the heat kernel of $H_V = -\Delta + V$, $e^{-tH_V}(x,y)$ satisfies: For every $\epsilon > 0$ there is a constant $C_\epsilon$ such that

$$|e^{-tH_V}(x,y)| \leq \; C_\epsilon t^{-1}e^{At}\exp(-\frac{|x-y|^2}{2(1+\epsilon)t}) \quad (2)$$

If $d=2,\ V<0$ then one can claim that $A>0$. Indeed: suppose that $A\leq 0$. It is easily seen that

$$\sup_{x\in I\!R^2}\int_{I\!R^2}|e^{-tH_V}(x,y)|dy \leq C(t) := CC_\epsilon t^{-1}e^{At} \quad (3)$$

thus $\|e^{-tH_V}\| \leq C(t)$. On the other hand it is known that [6] $e^{-tH}$ has at least one negative eigenvalue $-\lambda<0$, thereby we have

$$e^{\lambda t} \leq \; \|e^{-tH_V}\| \leq C(t) \quad (4)$$

Letting $t$ goes to infinity we get a contradiction.

# 2 Asymptotic behavior for large time

Let us recall how Hamiltonians describing point interaction can be interpreted via Dirichlet forms [2].
We omit the case $\alpha = +\infty$, which corresponds to the free Hamiltonian. Let $\varphi_\alpha$ be the following function

$$\varphi_\alpha(x) = H_0^{(1)}(2ie^{-2\pi\alpha+\Psi(1)}|x|),\ x \in \mathbb{R}^2 \setminus \{0\} \tag{5}$$

where $H_0^{(1)}$ is a Hankel function. Denote by $H_{\varphi_\alpha}$ the operator associated to the local positive Dirichlet form

$$\mathcal{E}_{\varphi_\alpha} : \mathcal{D}(\mathcal{E}_{\varphi_\alpha}) \subset\ L^2(\varphi_\alpha^2 dx),\ \mathcal{E}_{\varphi_\alpha}(f,g) = \int_{\mathbb{R}^2} \nabla f \bar{\nabla} g \varphi_\alpha^2 dx. \tag{6}$$

Then the Hamiltonian $H_{\delta,\alpha}$ is related to this Dirichlet form by [2]

$$H_{\delta,\alpha} = \varphi_\alpha [H_{\varphi_\alpha} - \beta I] {\varphi_\alpha}^{-1} \tag{7}$$

where $\beta = 4e^{2(-2\pi\alpha+\Psi(1))}$. Clearly $\exp(-tH_{\delta,\alpha}) = e^{\beta t}\varphi_\alpha \exp(-tH_{\varphi_\alpha}){\varphi_\alpha}^{-1}$. Now if we denote by $q^\alpha(t;x,y)$ the heat kernel of the operator $\exp(-tH_{\varphi_\alpha})$, then $P^\alpha(t;x,y) = e^{\beta t}\varphi_\alpha(x){\varphi_\alpha}^{-1}(y)q^\alpha(t;x,y)$. So to give an estimate of the heat kernel $q^\alpha$ one can think about the method proposed by [18]. Unfortunately this method does not work. For we need the measure $\mu(x) = \varphi_\alpha^2 dx$ to satisfy the global doubling property on $\mathbb{R}^2$ i.e.: There is a constant $N$ such that for every $x \in \mathbb{R}^2$ we have

$$\int_{B_{2r}(x)} \varphi_\alpha^2 dx \leq\ 2^N \int_{B_r(x)} \varphi_\alpha^2 dx \tag{8}$$

But if this property is satisfied then [18]p.286 $\inf \sigma(H_{\varphi_\alpha}) = 0$, which is absurd since [2] $\inf \sigma(H_{\varphi_\alpha}) = \beta \neq 0$, for $\alpha \neq +\infty$. So we are lead to use a direct method to do it.

**Proposition 2.1** *For large $t$ we have*

$$P^\alpha(t;x,y)\ \leq C \tag{9}$$

**Proof**: Since the Dirichlet form $\mathcal{E}_{\varphi_\alpha} - \beta I$ is local and positive, then it has a Markovian kernel $\tilde{q}^\alpha(t;x,y)$ [11] and then $0 < \tilde{q}^\alpha(t;x,y) \leq C$ for large $t$. Now we have on the diagonal the following estimate

$$P^\alpha(t;x,x) \ \leq \ C \tag{10}$$

Using the Chapman-Kolmogorov equation:

$$P^\alpha(t+s;x,y) = \int_{\mathbb{R}^2} P^\alpha(s;x,z)P^\alpha(t;z,y)dz \tag{11}$$

we get $P^\alpha(t;x,y) \leq \ C$ which completes the proof. □

# 3 Asymptotics

We are going to give asymptotic development of both kernels $P^\alpha(t;x,y)$ and $P^\alpha(it;x,y)$. Let's mention that in [1] asymptotics are given for small time. The technique used here is quite similar to that used in [1].

## 3.1 Asymptotic for $P^\alpha(t;x,y)$

Let's recall from ([1] Eq. (3.16)) that $P^\alpha(t;x,y)$ can also be written

$$\begin{aligned} P^\alpha(t;x,y) \ &= \ P(t;x,y) + \frac{\exp(-\frac{A}{t})}{(4\pi t|x||y|)^{\frac{1}{2}}} \int_0^{+\infty} \frac{t^u \exp(-\alpha u)}{\Gamma(u)} \\ & \int_0^{+\infty} \frac{r^{u-1}}{(r+1)^{u+\frac{1}{2}}} \exp(-\frac{A}{t}r)\widetilde{K_0}(\frac{|x||y|}{2t}(r+1))drdu \end{aligned}$$

where $A = \frac{(|x|+|y|)^2}{4}$ and $\widetilde{K_0}(z) = \sqrt{\frac{2z}{\pi}}\exp(z)K_0(z)$. Let's recall some properties of the function $\widetilde{K_0}$ established in [1].

$$\sup_{r\geq 0} |r^k \widetilde{K_0}^{(k)}(r)| \ = \ M_{k,1} < +\infty \tag{12}$$

and

$$\sup_{r\geq 0} |r^k \widetilde{K_0}^{(k)}(ir)| \ = \ M_{k,2} < +\infty \tag{13}$$

Denote

$$I(t) = \int_0^{+\infty} \frac{t^u \exp(-\alpha u)}{\Gamma(u)} \int_0^{+\infty} \frac{r^{u-1}}{(r+1)^{u+\frac{1}{2}}} \exp(-\frac{A}{t}r)\widetilde{K_0}(\frac{|x||y|}{2t}(r+1))drdu$$

And $F$ the function defined by $F(r) = \widetilde{K_0}(\frac{|x||y|}{2t}(r+1))$ for every $r \geq 0$, then $F^{(k)}(r) = (\frac{|x||y|}{2t})^k \widetilde{K_0}^{(k)}(\frac{|x||y|}{2t}(r+1))$. It follows from (12) that $\sup_{r\geq 0} |F^{(k)}(r)| = M_k < +\infty$.
Let $N \in I\!N$, the Taylor series of $F$ near zero at the order $N$ is

$$F(r) = \sum_{0\leq k\leq N} \frac{r^k}{k!} F^{(k)}(0) + \frac{r^{(N+1)}}{(N+1)!} \int_0^1 F^{(N+1)}(s)ds \tag{14}$$

$$= \sum_{0\leq k\leq N} \frac{r^k}{k!} (\frac{|x||y|}{2t})^k \widetilde{K_0}^{(k)}(\frac{|x||y|}{2t}) + \frac{r^{(N+1)}}{(N+1)!} \int_0^1 F^{(N+1)}(s)ds \tag{15}$$

Which gives

$$|F(r) - \sum_{0\leq k\leq N} \frac{r^k}{k!} (\frac{|x||y|}{2t})^k \widetilde{K_0}^{(k)}(\frac{|x||y|}{2t})| \leq \frac{r^{(n+1)}}{(N+1)!} M_{N+1} \tag{16}$$

Denote

$$\begin{aligned} I_k(t) &= \frac{1}{k!}(\frac{|x||y|}{2t})^k \widetilde{K_0}^{(k)}(\frac{|x||y|}{2t}) \int_0^{+\infty} \frac{t^u \exp(-\alpha u)}{\Gamma(u)} \\ & \quad \int_0^{+\infty} \frac{r^{u+k-1}}{(r+1)^{u+\frac{1}{2}}} \exp(-\frac{A}{t}r)drdu \end{aligned} \tag{17}$$

and

$$J_k(t) = \int_0^{+\infty} \frac{t^u \exp(-\alpha u)}{\Gamma(u)} \int_0^{+\infty} \frac{r^{u+k-1}}{(r+1)^{u+\frac{1}{2}}} \exp(-\frac{A}{t}r)drdu \tag{18}$$

We first prove that $J_k(t) < +\infty$ for every $k \in I\!N,\ t > 0$. For $k = 0$ with the change of variable $s = \frac{r}{r+1}$ one get

$$\int_0^{+\infty} \frac{r^{u-1}}{(r+1)^{u+\frac{1}{2}}} \exp(-\frac{A}{t}r)dr \leq \frac{\Gamma(u)\Gamma(\frac{3}{2})}{\Gamma(u+\frac{3}{2})}$$

Hence $J_0(t) \;\leq\; \frac{\Gamma(\frac{3}{2})}{t^{\frac{1}{2}}}\exp(\frac{\alpha}{2})\nu(t\exp(-\frac{\alpha}{2}))$ where the function $\nu$ is as defined in[10].
Now for $k \in I\!N^*$ we have:

$$\begin{aligned}\int_0^{+\infty} \frac{r^{u+k-1}}{(r+1)^{u+\frac{1}{2}}}\exp(-\frac{A}{t}r)dr \;&=\; \int_0^{+\infty}(\frac{r}{r+1})^{u+\frac{1}{2}}r^{k-\frac{3}{2}}\exp(-\frac{A}{t}r)dr \\ &\leq \int_0^{+\infty} r^{k-\frac{3}{2}}\exp(-\frac{A}{t}r)dr \leq (\frac{t}{A})^{k-\frac{1}{2}}\Gamma(k-\frac{1}{2})\end{aligned}$$

We then get

$$J_k(t) \;\leq\; (\frac{t}{A})^{k-\frac{1}{2}}\Gamma(k-\frac{1}{2})\int_0^{+\infty}\frac{t^u\exp(-\alpha u)}{\Gamma(u)}du < +\infty$$

Let's come back to the devellopment

$$\begin{aligned}|I(t) - \sum_{0\leq k\leq N} I_k(t)| \;&\leq\; \frac{M_{N+1}}{(N+1)!}\int_0^{+\infty}\frac{t^u\exp(-\alpha u)}{\Gamma(u)}\int_0^{+\infty}\frac{r^{u+N}}{(r+1)^{u+\frac{1}{2}}}\exp(-\frac{A}{t}r)drdu \\ &= \frac{M_{N+1}}{(N+1)!}\int_0^{+\infty}\frac{t^u\exp(-\alpha u)}{\Gamma(u)}J_{N+1}(t)du\end{aligned}$$

We now investigate the behaviour of $J_{N+1}(t)$ as $N$ tends to infinity. Clearly $J_{N+1}(t)$ is the Laplace transform of the function $\frac{r^{u+N}}{(r+1)^{u+\frac{1}{2}}}$ at the point $\frac{A}{t}$, hence (Cf([13]) p.24)

$$J_{N+1}(t) = (\frac{t}{A})^{\frac{N}{2}+\frac{3}{4}}\exp(\frac{A}{2t})\Gamma(u+N+1)W_{u-\frac{\mu}{2},\frac{\mu}{2}}(\frac{A}{t}) \qquad (19)$$

where $\mu = N + \frac{1}{2}$ and $W_{\chi,\mu}$ is the Whittaker function. We then achieve the following inequality

$$\begin{aligned}|I(t) - \sum_{0\leq k\leq N} I_k(t)| \;\;\leq\;\; & \frac{M_{N+1}}{(N+1)!}(\frac{t}{A})^{\frac{N}{2}+\frac{3}{2}}\exp(\frac{A}{2t}) \\ & \int_0^{+\infty}\frac{t^u\exp(-\alpha u)}{\Gamma(u)}\Gamma(u+N+1)W_{u-\frac{\mu}{2},\frac{\mu}{2}}(\frac{A}{t})du\end{aligned}$$

For $N \to +\infty$ we have (Cf [8] p.95)

$$W_{u-\frac{\mu}{2},\frac{\mu}{2}}(\frac{A}{t}) \;\sim\; (\frac{A}{t})^{\frac{N}{2}+\frac{3}{4}}\frac{\exp(-\frac{A}{2t})}{\Gamma(N+\frac{3}{2})}[\frac{1}{\Gamma(\frac{1}{2}-u)} - \frac{1}{\Gamma(N+1-u)}]$$

Put

$$R_{N+1}(t) = \frac{M_{N+1}}{(N+1)!}(\frac{t}{A})^{\frac{N}{2}+\frac{3}{2}} \exp(\frac{A}{2t}) \int_0^{+\infty} \frac{t^u \exp(-\alpha u)}{\Gamma(u)} \Gamma(u+N+1) W_{u-\frac{\mu}{2},\frac{\mu}{2}}(\frac{A}{t}) du \qquad (20)$$

Since the integrals $J_k(t)$ are absolutely convergent we get

$$R_{N+1}(t) \sim \frac{M_{N+1}}{(N+1)!} \frac{1}{\Gamma(N+\frac{3}{2})} \int_0^{+\infty} \frac{t^u \exp(-\alpha u)}{\Gamma(u)} \frac{\Gamma(u+N+1)}{\Gamma(\frac{1}{2}-u)} du - \frac{M_{N+1}}{(N+1)!} \frac{1}{\Gamma(N+\frac{3}{2})} \int_0^{+\infty} \frac{t^u \exp(-\alpha u)}{\Gamma(u)} \frac{\Gamma(u+N+1)}{\Gamma(N+1+u)} du$$

for $N \to +\infty$. We now need an explicit computation for $M_{N+1}$. From formula (Cf [12] Eq5.11.9. p.123) we get

$$M_{N+1} = \frac{((2N+1)!)^2}{2^{4N+3}(N+1)(N!)^3} \qquad (21)$$

On the other hand $N! \sim N^N \exp(-N)\sqrt{2\pi N}$ for large $N$. This yields

$$M_{N+1} \le C \exp(-N-2) \frac{(2N+1)^{N+2}}{2^{4N+3}(N+1)} \qquad (22)$$

and

$$\frac{M_{N+1}}{(N+1)!} \le \frac{C \exp(-2)(2N+1)^{N+2}}{2^{4N+3}(\sqrt{2\pi(N+1)})(N+1)^{N+2}} \sim \frac{C \exp(-2)(2N+1)}{\sqrt{(2\pi)} 2^{4N+3}(N+1)^{N+\frac{3}{2}}}$$

Let's inspect now the term

$$\frac{\Gamma(u+N+1)}{\Gamma(N+\frac{3}{2})}[\frac{1}{\Gamma(\frac{1}{2}-u)} - \frac{1}{\Gamma(N+1-u)}]$$

for $N$ large. Using that fact that $\Gamma(z+1) \sim z^z\sqrt{2\pi z}\exp(-z)$, $|z| \to +\infty$ we get $\Gamma(u+N+1) \sim (u+N)^{u+N}\sqrt{2\pi(u+N)}\exp(-u-N)$ and

$\Gamma(-u+N+1) \sim (-u+N)^{-u+N}\sqrt{2\pi(-u+N)}\exp(-u+N)$ as $N \to +\infty$. Which gives

$$\frac{\Gamma(u+N+1)}{\Gamma(N+1-u)} \sim \frac{(u+N)^{u+N}}{(-u+N)^{-u+N}}\exp(-2u)\sqrt{2\pi\frac{N+u}{N-u}} \tag{23}$$

$$\sim \sqrt{2\pi}N^{2u}\exp(-2u) \tag{24}$$

as $N \to +\infty$ uniformly in $u > 0$. We then get

$$\lim_{N\to+\infty} \frac{\Gamma(u+N+1)}{\Gamma(N+\frac{3}{2})\Gamma(N+1-u)} = 0$$

uniformly in $u$. Let's now see what happens with the term

$$\frac{\Gamma(u+N+1)}{\Gamma(N+\frac{3}{2})\Gamma(\frac{1}{2}-u)}$$

As before one has $\frac{\Gamma(u+N+1)}{\Gamma(N+\frac{3}{2})} \sim \frac{N^u}{\sqrt{(N+\frac{1}{2})}}\exp(-u)$. Using the well known identity $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ we get

$$\lim_{N\to+\infty} \frac{1}{2^N}\frac{\Gamma(u+N+1)}{\Gamma(\frac{1}{2}-u)\Gamma(N+\frac{3}{2})} = 0 \tag{25}$$

Putting all to together we finally get

$$\lim_{N\to+\infty} R_{N+1}(t) = 0$$

uniformly in $t$. We have thus proved

**Proposition 3.1** *:*

$$I(t) \sim \sum_{k\geq 0}\frac{1}{k!}(\frac{|x||y|}{2t})^k\widetilde{K_0}^{(k)}(\frac{|x||y|}{2t})\int_0^{+\infty} t^u\frac{\exp(-\alpha u)}{\Gamma(u)}W_{u-\frac{\mu_k}{2},\frac{\mu_k}{2}}(\frac{A}{t})du$$

*for large $t$, where $\mu_k = k+\frac{1}{2}$.*

This gives the following asymptotic expansion

**Theorem 3.1**

$$P^\alpha(t;x,y) \sim \frac{C}{t} + \frac{C'}{(t|x||y|)^{\frac{1}{2}}}\sum_{k\geq 0}\frac{1}{k!}(\frac{|x||y|}{2t})^k\widetilde{K_0}^{(k)}(\frac{|x||y|}{2t}) \int_0^{+\infty} t^u\frac{\exp(-\alpha u)}{\Gamma(u)}W_{u-\frac{\mu_k}{2},\frac{\mu_k}{2}}(\frac{A}{t})du$$

*for large $t$, where $C$ and $C'$ are two positive constants.*

## 3.2 Asymptotic for $P^\alpha(it;x,y)$

The method used before is no more applicable to derive asymptotic expansion for $P^\alpha(it,x,y)$. This is because the corresponding integrals $J_k(it)$ are no longer absolutely convergent. So we follow the method used in [1] with a slight modification. It is known that

$$\begin{aligned} P^\alpha(it;x,y) &= P(it;x,y) + \frac{\exp(\frac{A}{4t})}{(4\pi it|x||y|)^{\frac{1}{2}}} \int_0^{+\infty} \frac{(it)^u \exp(-\alpha u)}{\Gamma(u)} \\ & \int_0^{+\infty} \frac{r^{u-1}}{(r+1)^{u+\frac{1}{2}}} \exp(-\frac{iA}{t} r) \widetilde{K_0}(\frac{|x||y|}{2it}(r+1)) dr du \quad (26) \end{aligned}$$

We denote $\widetilde{P^\alpha}(it;x,y)$ the second term in the right hand side of Eq(26). Let $(h_n)$ be a sequence of functions such that for every $n \in I\!N$, $h_n \in C_0^\infty((0,+\infty))$; $0 \le h_n \le 1$ and

$$h_n(x) = \begin{cases} 1 & ; \quad 0 \le x \le n \\ 0 & ; \quad x \ge n+1 \end{cases}$$

Let's write $\widetilde{P^\alpha}(it;x,y) = \widetilde{P^\alpha}_{1,n}(it;x,y) + \widetilde{P^\alpha}_{2,n}(it;x,y)$ where

$$\begin{aligned} \widetilde{P^\alpha}_{1,n}(it;x,y) &= \int_0^{+\infty} \frac{(it)^u \exp(-\alpha u)}{\Gamma(u)} \\ & \int_0^{+\infty} \frac{r^{u-1}}{(r+1)^{u+\frac{1}{2}}} \exp(-\frac{iA}{t} r) h_n(r) \widetilde{K_0}(\frac{|x||y|}{2it}(r+1)) dr du \end{aligned}$$

and $\widetilde{P^\alpha}_{2,n}(it;x,y)$ is the remainder. The term $\widetilde{P^\alpha}_{1,n}(it;x,y)$ is absolutely convergent and can be treated as before. If we put

$$F_n(r) = h_n(r) \widetilde{K_0}(\frac{|x||y|}{2it}(r+1)) \quad (27)$$

Observing that $h_n^{(k)}(0) = 0$ for every $k \ge 1$ we get for every $N \in I\!N$

$$F_n(r) = \sum_{0 \le k \le N} \frac{r^k}{k!} (\frac{|x||y|}{2it})^k \widetilde{K_0}^{(k)}(\frac{|x||y|}{2it}) + \frac{r^{(N+1)}}{(N+1)!} \int_0^1 F^{(N+1)}(s) ds$$

and therefore

$$|F_n(r) - \sum_{0 \le k \le N} \frac{r^k}{k!} (\frac{|x||y|}{2it})^k \widetilde{K_0}^{(k)}(\frac{|x||y|}{2it})| \le \frac{r^{(N+1)}}{(N+1)!} M_{N+1}$$

Which yields by the same arguments as before

**Proposition 3.2** *For every $n \in I\!N$ we have*

$$\widetilde{P^{\alpha}}_{1,n}(it;x,y) \sim \frac{\exp(\frac{iA}{4t})}{(4\pi it|x||y|)^{\frac{1}{2}}}\sum_{k\geq 0}\frac{1}{k!}(\frac{|x||y|}{2it})^k\widetilde{K_0}^{(k)}(\frac{|x||y|}{2it}) \int_0^{+\infty}(it)^u\frac{\exp(-\alpha u)}{\Gamma(u)}\int_0^{+\infty}\frac{r^{u+k-1}}{(r+1)^{u+\frac{1}{2}}}\exp(i\frac{A}{t}r)drdu$$

*for large $t$.*

What we get is that the asymptotic expansion (28) is independent of $n$. Now we get for every $n \in I\!N$

$$\widetilde{P^{\alpha}}(it;x,y) \sim \frac{\exp(\frac{iA}{4t})}{(4\pi it|x||y|)^{\frac{1}{2}}}\sum_{k\geq 0}\frac{1}{k!}(\frac{|x||y|}{2it})^k\widetilde{K_0}^{(k)}(\frac{|x||y|}{2it})\int_0^{+\infty}(it)^u\frac{\exp(-\alpha u)}{\Gamma(u)} \int_0^{+\infty}\frac{r^{u+k-1}}{(r+1)^{u+\frac{1}{2}}}\exp(i\frac{A}{t}r)drdu + \widetilde{P^{\alpha}}_{2,n}(it;x,y) \tag{28}$$

Letting $n \to +\infty$ one get $\widetilde{P^{\alpha}}_{2,n}(it;x,y) \to 0$ uniformly in $t$. Finally we get

**Theorem 3.2** *For large $t$ we have*

$$\widetilde{P^{\alpha}}(it;x,y) \sim \frac{\exp(\frac{iA}{4t})}{(4\pi it|x||y|)^{\frac{1}{2}}}\sum_{k\geq 0}\frac{1}{k!}(\frac{|x||y|}{2it})^k\widetilde{K_0}^{(k)}(\frac{|x||y|}{2it}) \int_0^{+\infty}(it)^u\frac{\exp(-\alpha u)}{\Gamma(u)}\int_0^{+\infty}\frac{r^{u+k-1}}{(r+1)^{u+\frac{1}{2}}}\exp(i\frac{A}{t}r)drdu$$

# 4 The sphere interaction in two dimensions

Let $\bar{H}$ be the closure of the operator $-\triangle$ with domain $C_0^{\infty}(I\!R^2 \setminus S^1)$. Then according to the classical decomposition into angular momenta

$$\bar{H} = \oplus_{l\in I\!N}U^{-1}\bar{h}_l U \otimes 1 \tag{29}$$

where $U : L^2((0,+\infty),rdr) \to L^2((0,+\infty),dr)$, $(Uf)(r) = rf(r)$ and

$$D(\bar{h}_l) = \{f \in L^2((0,+\infty)),\ f,f' \in AC_{loc}((0,+\infty)) \quad ; \quad f(0^+)=0,$$
$$if,\ l=0,\ f(R^{+,-})=0,\ -f''+\frac{l^2-4^{-1}}{r^2}f \in L^2((0,+\infty))\}$$

For each $l$ the adjoint operator of $\bar{h_l}$, $\bar{h_l}^*$ is defined by $\bar{h_l}^* = -\frac{d^2}{dr^2} + \frac{l^2-4^{-1}}{r^2}$ with domain

$$D(\bar{h_l}^*) = \{f \in\ L^2((0,+\infty)),\ f, f' \in\ AC_{loc}((0,+\infty)\setminus R) \quad ; \quad f(0^+) = 0$$
$$if,\ l = 0;\ f(R^+) = f(R^-),\ -f'' + \frac{l^2-4^{-1}}{r^2} f \in L^2((0,+\infty))\}$$

The equation

$$\bar{h}_l^* \varphi_l(k) = k^2 \varphi_l(k) \tag{30}$$

with $k^2 \in \mathbf{C} \setminus I\!R,\ \mathrm{Im(k)} > 0,\ \varphi_l(k) \in \mathrm{D}(\bar{h_l}^*)$ has a unique solution (up to a constant) which is given by

$$\varphi_l(k) = \begin{cases} i\frac{\pi}{2} R^{\frac{1}{2}} H_l^{(1)}(kR) r^{\frac{1}{2}} J_l(kr) & ; \quad r \le R \\ i\frac{\pi}{2} R^{\frac{1}{2}} J_l(kR) r^{\frac{1}{2}} H_l^{(1)}(kr) & ; \quad r \ge R \end{cases}$$

Hence the deficiency indices of $\bar{h_l}$ are $(1,1)$. All self-adjoint extensions of $\bar{h_l}$ can be described as follows (Cf [3]).

$$h_{l,\alpha_l} = -\frac{d^2}{dr^2} + \frac{l^2-4^{-1}}{r^2} \tag{31}$$

$D(h_{l,\alpha_l}) = \{f \in L^2((0,+\infty)); f, f' \in AC_{loc}((0,+\infty)\setminus R), \lim_{r\to 0^+} \frac{f(r)}{[r^{\frac{1}{2}}\ln(r)]^{-1}} = 0,\ if,\ l = 0;\ f(R^+) = f(R^-) = f(R),\ f'(R^+) - f'(R^-) = \alpha_l f(R),\ -f'' + \frac{l^2-4^{-1}}{r^2} f \in L^2((0,+\infty))\}$.
The case $\alpha_l = 0$ leads us the free Hamiltonian $h_{l,0}$ for fixed angular momentum $l$.
Let $\alpha = \{\alpha_l\}_{l\in I\!N}$ and introduce in $L^2(I\!R^2)$ the operator

$$H_{\delta,\alpha} \ = \ \oplus_{l\in I\!N} U^{-1} h_{l,\alpha_l} U \otimes 1 \tag{32}$$

The operator $H_{\delta,\alpha}$ provides the formal expression $-\triangle + \alpha\delta(|x| = R)$.
We now proceed to the computation of the resolvent of $h_{l,\alpha_l}$. The resolvent of $H_{\delta,\alpha}$ is then obtained using the composition(32).

**Proposition 4.1** *For every* $l \in I\!N,\ k^2 \in \mathbf{C}\setminus I\!R,\ \mathrm{Im(k)} > 0$ *we have*

$$(h_{l,\alpha_l} - k^2)^{-1} = (h_{l,0} - k^2)^{-1} - p_l(k)(.,\varphi_l(\bar{k}))\varphi_l(k) \tag{33}$$

*where* $p_l(k) = \alpha(1+\alpha g_{l,k}(R,R))^{-1}$ *and* $g_{l,k}$ *is the kernel of the free Hamiltonian given by*

$$g_{l,k}(r,r') = \begin{cases} i\frac{\pi}{2}(r')^{\frac{1}{2}}H_l^{(1)}(kr')r^{\frac{1}{2}}J_l(kr) & ; \quad r \le r' \\ i\frac{\pi}{2}(r)^{\frac{1}{2}}J_l(kr)r'^{\frac{1}{2}}H_l^{(1)}(kr') & ; \quad r \ge r' \end{cases}$$

**Proof**: As in [2], [3] one get this formula by applying Krein's Formula and taking into account the boundary conditions. □

As a consequence one get a full description of the spectrum of $H_{\delta,\alpha}$.

**Proposition 4.2** *we have* $\sigma_{\text{ess}}(H_{\delta,\alpha}) = \sigma_{\text{ac}}(H_{\delta,\alpha}) = [0,+\infty[$, $\sigma_{\text{sc}} = \emptyset$. *While* $\sigma_p(H_{\delta,\alpha})$ *is empty if* $\alpha \ge 0$ *and consists of exactly one negative eigenvalue if* $\alpha < 0$ *which we denote by* $E_\alpha$.

**Proof**: The first part of the proposition can be proved as in [2] or [3]. Clearly $\sigma_p(H_{\delta,\alpha}) \cap [0;+\infty[\ = \emptyset$ while the negative eigenvalues are poles of $p_l$. Put $k = i\sqrt{E}$ where $E > 0$, now the equation $p_l(k) = 0$ yields

$$1 + \alpha K_l(R\sqrt{E})I_l(R\sqrt{E}) = 0 \tag{34}$$

This equation has no solution if $\alpha \ge -2l$ and it has solution $-E$ if $\alpha < -2l$ This completes the proof. □

For Hamiltonians describing point interactions, it is known [2] that these operators can be related to (local) Dirichlet forms on some $L^2$-space. For the sphere interactions in two and three dimensions, recent works [7], [17], [16], [5] about perturbations of the Laplace operator by measures make it possible to describe this Hamiltonians by means of local Dirichlet forms. Indeed the measure $\mu = \delta(|x| = R)$ is in the Kato-class and construction of $-\triangle + \alpha\mu$ is straightforward. It has been recently [4] proved that such operators can be correctly constructed on any $L^2(\nu)$-space where $\nu$ is a positive measure whose fine support is $\mathbb{R}^d$. In [7] equivalence between construction via Dirichlet forms and the construction in [3] and [14] using boundary conditions on the sphere $S^2$ is proved. In [5] and [17] the existence of the heat semigroups associated to these operators was proved and was given using additive functionals associated to measures and a point-wise estimate for their kernels was given. It is proved there that for measure in the Kato class the heat kernel associated to $-\triangle + \mu$ is exponentially bounded. For instance in [5](theorem 5.2) the following point-wise estimate was proved

$$0 \le \ P^\mu(t,x,y) \ \le \ C\exp(\beta t)P(2t,x,y) \tag{35}$$

where $C, \beta$ are positive constants, $P^\mu$ is the kernel of the semi-group of the perturbed operator and $P$ the kernel of the semi-group associated to $-\triangle$ but no informations about $C$ and $\beta$ are given. On the other hand the following estimates proved in [17] are somewhat more informative

$$P^\mu(t,x,y) \ \leq \ Ct^{-\frac{d}{2}} \tag{36}$$

for every $0 < t \leq 1$. For $t \geq 1$ one has

$$P^\mu(t,x,y) \ \leq \ C\exp(-\lambda^\mu(t-1)) \tag{37}$$

where $\lambda^\mu$ is the spectral bound of the operator $H_\mu = -\triangle + \mu$ this is $\lambda^\mu = \inf(\sigma(H_\mu))$. Applying these results to our situation one get

**Proposition 4.3** *For large $t$ the heat kernel associated to $H_{\delta,\alpha}$ in two dimensions satisfies*

$$P^\alpha(t,x,y) \ \leq \ C\exp(-E_\alpha t) \tag{38}$$

*if $\alpha < 0$ and*

$$P^\alpha(t,x,y) \ \leq \ C \tag{39}$$

*if $\alpha \geq 0$.*

**Proof**: By the spectral properties of $H_{\delta,\alpha}$ one get $\lambda^\mu = E_\alpha$ and estimate(37) gives the result. □

Let us observe that the heat kernels associated to $-\Delta + \alpha\delta(|x| = R)$ satisfy the same type estimate as those associated to $H_V = -\Delta + V$ [15] where $V$ is such that $V_+$ is in the local Kato class and $V_-$ in the Kato class.

Let us finally mention that estimate of the type (38) is proved in [9] for higher order differential operators perturbed by potentials in the corresponding Kato class.

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