n-covariation and symmetric SDE's driven by finite cubic variation processes

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Abstract

The $n-$variation of a continuous process and the $n-$covariation of a vector of continuous processes, are defined through a regularization procedure. We calculate explicitly the n-variation process, when it exists, of a martingale convolution. For processes having finite cubic variation, a basic stochastic calculus is developed. We prove an Itô formula and we study existence and uniqueness of the solution of a stochastic differential equation, in a symmetric-Stratonovich sense, with respect to those processes.

Key words and phrases : n-covariation, martingale convolutions, symmetric integral, stochastic differential equation, finite cubic variation process, Hu-Meyer formula.

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1 Introduction

In the last twenty years, many authors have tried to develop a stochastic calculus beyond semimartingales. The strategy of this paper consists in 'mimicking' a "pathwise theory" for the same purpose. Pathwise type integrals are defined very often using discretization, as limit of Riemann sums: an interesting survey on the subject is a book of R.M. Dudley and R. Norvaisa ([12]). They emphasize a big historical literature in the deterministic case. The first contribution in the stochastic framework has been provided by H. Föllmer in 1981; through this significant and simply written contribution, the author wished to discuss integration with respect to a Dirichlet process X, that is to say a local martingale plus a zero quadratic variation (or sometimes zero energy) process.

In the sequel this approach has been continued and performed by J. Bertoin ([5]); an important break through has been realized by T. Lyons and coauthors, i.e. [21], [22]. They are able to integrate pathwise with respect to processes having some p-variation, for p greater than 2.

Since [1993], F. Russo and P. Vallois have developed a regularization procedure, whose philosophy is similar to the discretization. More recently, several papers have followed those techniques, see for instance [24], [25], [26], [32], [14]. All those articles aimed at developing an efficient calculus with respect to a finite quadratic variation process. This paper is a first attempt to discuss (in the "pathwise framework") integration with respect to cubic finite variation processes.

For applications, an important subclass of such processes can be given by Gaussian processes, in particular fractional Brownian motions B^H, where H is the Hurst index. Those processes are in fact very rarely semimartingales. Integration with respect to Gaussian processes has been attacked using Malliavin calculus techniques (Skorohod integrals), see for instance [9], [7] [1] and [2]. Those techniques are quite powerful and they allow to treat integration with respect to processes, whose variation is larger than 2. However, they cannot be easily related to Riemann sums limits. The regularization or discretization technique for those processes has been recently performed by [14], [15], [20], [27], [30] and [31] in the case of finite quadratic variation.

This article develops in particular a calculus with respect to processes having a cubic variation. A processes $X = \{X_t,\ t \in [0,1]\}$ will be said to have a finite cubic variation (or 3-variation), denoted by $[X;3]$, equals to Y if

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^{\cdot} \left(X_{s+\epsilon} - X_s\right)^3 ds = Y \qquad ucp,$$

$$\sup_{0<\epsilon\leq 1} \frac{1}{\epsilon} \int_0^1 \left|X_{s+\epsilon} - X_s\right|^3 ds < +\infty \quad a.s. \tag{1.1}$$

If $H = \frac{1}{3}$, Proposition 3.4 of [27] says that,

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \left|B^H_{s+\epsilon} - B^H_s\right|^3 ds = \rho_H\, t$$

ucp, where $\rho_H = E|G|^3$, G being a $N(0,1)$ random variable. This shows (1.1). On the other hand, the same techniques as previous proposition allow to show that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^{\cdot} \left(B^H_{s+\epsilon} - B^H_s\right)^3 ds = 0.$$

So that the process $B^{\frac{1}{3}}$ is of finite cubic variation process and $[B^{\frac{1}{3}};3] = 0$.

If a process X is of finite cubic variation, we will prove that for every $f \in C^3$,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) d^\circ X_s - \frac{1}{12} \int_0^t f^{(3)}(X_s) d[X,X,X]_s, \tag{1.2}$$

where, $\int_0^t f'(X_s)d^\circ X$ is the symmetric integral (an extension of Stratonovich integral) defined as the *ucp* limit of

$$\frac{1}{2\epsilon} \int_0^{\cdot} f'(X_s)\left(X_{s+\epsilon} - X_{s-\epsilon}\right) ds.$$

We briefly discuss now the content of the paper. In the second section we define the concept of $n-$variation of a continuous process and $n-$covariation of n continuous processes. The $n-$variation of local martingale convolutions and related processes is explicitly given. If the martingale is a Brownian motion, our Proposition 2.13 constitutes a generalization of Hu-Meyer formula which appears for instance in [4], [19], [3] and [28]. Even in the case $n = 2$

we discuss finite quadratic variation processes which are not Dirichlet. For those processes we obtain however an unique decomposition. In the section 3 we prove an Itô formula of the type (1.2) with related calculus and, in section 4, we study a SDE of symmetric type driven by a bounded variation process and a finite cubic variation process.

2 The n-covariation and n-variation processes

Throughout this paper all the processes, are assumed to be continuous, indexed by the *time* variable t in $[0,1]$ and defined on the same complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathcal{F} = \{\mathcal{F}_t,\ t \in [0,1]\}$ satisfying the usual assumptions. Two continuous processes X and Y indistinguishable will be considered equal. Where, if $X(t) = Y(t)$ $a.s.$ for all $t \in [0,1]$, X and Y are then indistinguishable. We recall, also, that a sequence of continuous processes $\{H_n(t),\ t \in [0,1]\}$ converges in the sense of the uniform convergence in probability (*ucp*) if there exists a process H such that the sequence of random variable $\sup_{0\le t\le 1} |H_n(t) - H(t)|$ converges to 0 in probability. Martingales will stand for local continuous martingales.

We denote by BV the space of continuous functions which have bounded variation on $[0,1]$. We equip BV with the metrizable topology that is associated with the following convergence. A sequence (v_n) in BV converges to a function v if and only if $v_n(0) \to v(0)$ and $dv_n \to dv$ holds with respect to the weak $*$-topology.

Along the paper, for every process X, we freely interchange $X(t)$ and X_t.

Remark 2.1 *Let (v_n) be a sequence in BV such that*

$$\sup_n \int_0^1 d|v_n| < \infty. \tag{2.1}$$

Let v_n^+, v_n^- be the increasing functions such that

$$v_n = v_n^+ - v_n^- \quad \textit{and} \quad |v_n| = v_n^+ + v_n^-.$$

Then there is a subsequence (n_k) such that $(v_{n_k}^+)$ and $(v_{n_k}^-)$ converge in BV. In fact, (2.1) implies

$$\sup_n \int_0^1 dv_n^\pm < \infty.$$

By the Helly extraction argument, there is a subsequence (n_k) such that $(v^+_{n_k})$ and $(v^-_{n_k})$ converge respectively to some v^1 and v^2. In particular, the subsequence $(|v_{n_k}|)$ of the total variations converges in BV to $v^1 + v^2$.

2.1 Definitions, notations and basic calculus

Let $n \geq 2$, and $(X^1, X^2, \cdots, X^n)$ be a vector of continuous processes. For any $\epsilon > 0$ and $t \in [0,1]$, we set

$$[X^1, X^2, \cdots, X^n]_\epsilon(t) = \frac{1}{\epsilon} \int_0^t \prod_{k=1}^n \left(X^k_{s+\epsilon} - X^k_s\right) ds,$$

and

$$\big\|[X^1, X^2, \cdots, X^n]_\epsilon\big\| = \frac{1}{\epsilon} \int_0^1 \prod_{k=1}^n \left|X^k_{s+\epsilon} - X^k_s\right| ds.$$

If $[X^1, X^2, \cdots, X^n]_\epsilon$ converges *ucp*, when $\epsilon \to 0$, then the limiting process is called the $n-$**covariation** (process) of the vector $(X^1, X^2, \cdots, X^n)$, and denoted $[X^1, X^2, \cdots, X^n]$. If furthermore

$$\sup_{0<\epsilon\leq 1} \big\|[X^1, X^2, \cdots, X^n]_\epsilon\big\| := \big\|[X^1, X^2, \cdots, X^n]\big\| < +\infty \tag{2.2}$$

we will say that it exists in the **strong sense**. Otherwise we will only say that it exists.

If the processes $\{X^k,\ k = 1, 2, \cdots, n\}$ are all equal to a real valued process X, then we will simple denote $[X; n]$ the $n-$covariation process of the considered vector. This will be called the n-**variation** (process) of X. If $n = 2$ it is the quadratic variation and denoted simply by $[X]$ or $[X, X]$. Cubic variation will often indicate 3-variation. If X has a quadratic (resp. strong cubic) variation, such process will be stand for**finite quadratic** (resp. **strong cubic) variation** process.

Remark 2.2

1) *By definition, the $n-$covariation is a continuous process.*

2) *The map $(X^1, X^2, \cdots, X^n) \to [X^1, X^2, \cdots, X^n]$, when it is well defined, is a multi-linear symmetric application with values in the space of real valued continuous processes.*

3) *If n is even then the existence in the strong sense of the $n-$variation is equivalent to the existence.*

Definition 2.3 *A vector $(X^1, X^2, \cdots, X^n)$ of continuous processes is said to have all its* **mutual** *(resp.* **strong)** $m-$**covariations** *if $[X^{i_1}, X^{i_2}, \cdots, X^{i_m}]$ exists (resp. exists in the strong sense) for any choice (even with repetition) of indices $i_1, i_2, \cdots, i_m \in \{1, 2, \cdots, n\}$.*

Remark 2.4 *If $n = 2$ and $(X^1, \cdots, X^m)$ has all its mutual brackets (or 2-covariations) then, using* Remark 2.2 3) *and polarization, $[X^i, X^j]$, $i, j = 1, 2, \cdots, m$, exist in the strong sense. In particular this happens when $X^1, X^2, \cdots, X^m$ are $\mathcal{F}-$semimartingales.*

Proposition 2.5 *If (2.2) holds then $[X^1, X^2, \cdots, X^n]$ has bounded variation whenever it exists.*

Proof. According to assumption (2.2), ω *a.s.*, the total variations of the measures $[X^1, X^2, \cdots, X^n]_\epsilon$ are bounded. Then since $[X^1, X^2, \cdots, X^n]$ exists, using Remark 2.1, it must be, ω *a.s.*, of bounded variation. ■

Remark 2.6

1) *If n is even then $[X; n]$ exists strongly and it is an increasing continuous process.*

2) *If for every $k = 1, 2, \cdots, n$, $\|[X^k; n]\|$ is finite then $[X^1, X^2, \cdots, X^n]$ exists in the strong sense whenever it exists. Moreover by Hölder inequality, we have*

$$\big|[X^1, X^2, \cdots, X^n]\big|^n \leq \|[X^1; n]\| \; \|[X^2; n]\| \cdots \|[X^n; n]\|.$$

3) *If the $n-$variation $[X; n]$ exists in the strong sense for some n, then $[X; m] = 0$ for all $m > n$. In particular, for any semimartingale S, $[S; n] = 0$ for all $n \geq 3$.*

4) *Suppose that $[X; n]$ exists in the strong sense, then for every continuous process Y and every $m > n$ such that $[Y; m]$ exists in the strong sense, we have*

$$[X, \underbrace{Y, Y, \cdots, Y}_{(m-1) times}] = 0.$$

In fact,

$$\begin{aligned}\Big|[X,\underbrace{Y,Y,\cdots,Y}_{(m-1)times}]_\epsilon(t)\Big| &\leq \|[X,Y,\cdots,Y]_\epsilon\| \\ &= \frac{1}{\epsilon}\int_0^1 \big|X_{s+\epsilon}-X_s\big|\big|Y_{s+\epsilon}-Y_s\big|^{m-1}\,ds \\ &\leq \Big(\frac{1}{\epsilon}\int_0^1 \big|X_{s+\epsilon}-X_s\big|^n\,ds\Big)^{\frac{1}{n}}\Big(\frac{1}{\epsilon}\int_0^1 \big|Y_{s+\epsilon}-Y_s\big|^{\frac{n(m-1)}{n-1}}\,ds\Big)^{\frac{n-1}{n}} \\ &\leq \Big(\sup_{s\in[0,1]}\big|Y_{s+\epsilon}-Y_s\big|^{\frac{m-n}{n-1}}\Big)^{\frac{n-1}{n}}\|[X;n]_\epsilon\|^{\frac{1}{n}}\,\|[Y;m]_\epsilon\|^{\frac{n-1}{n}},\end{aligned}$$

whose limit, using the uniform continuity of the process Y in $[0,1]$, is equal to zero.

5) *If $(X^1,\cdots,X^n)$ has a strong $n-$covariation , then for every vector $(Y^1,Y^2,\cdots,Y^m)$ of continuous processes, $(X^1,\cdots,X^n,Y^1,Y^2,\cdots,Y^m)$ has its strong $(n+m)-$covariation equal to zero.*

6) *Let $(X^1,\cdots,X^n)$ be a vector having a strong $n-$covariation and Y a continuous process. Then*

$$\frac{1}{\epsilon}\int_0^\cdot Y_s\prod_{k=1}^n\big(X^k_{s+\epsilon}-X^k_s\big)\,ds=\int_0^\cdot Y_s\,d[X^1,X^2,\cdots,X^n]_\epsilon(s),$$

converges ucp to

$$\int_0^\cdot Y\,d[X^1,X^2,\cdots,X^n],$$

because, ω a.s., $\big([X^1,X^2,\cdots,X^n]_\epsilon\big)$ converges in BV, up to a subsequence, to $[X^1,\cdots,X^n]$.

We have a stability result through C^1 transformation.

Proposition 2.7 *Let F^1, F^2, $\cdots$, F^m be m functions in $C^1(\mathbb{R}^n)$. Let $X=(X^1,X^2,\cdots,X^n)$ be a vector of continuous processes having all its strong mutual n-covariations. Then the vector $(F^1(X),F^2(X),\cdots,F^m(X))$ have the same property and*

$$[F^1(X),\cdots,F^m(X)](t)= \sum_{1\leq i_1,\cdots,i_n\leq n}\int_0^t \partial_{i_1}F^1(X)\cdots\partial_{i_n}F^m(X)d[X^{i_1},\cdots,X^{i_n}].$$

Proof. Let $\epsilon > 0$. For every $t \in [0,1]$ we express

$$\big[F^1(X), \cdots, F^m(X)\big]_\epsilon(t) = \sum_{1 \le i_1 \cdots, i_n \le n} \int_0^t \partial_{i_1} F^1(X_s) \cdots \partial_{i_n} F^m(X_s)\, d[X^{i_1}, \cdots, X^{i_n}]_\epsilon(s) + R_\epsilon(t),$$

where R_ϵ is a rest which converges *ucp* to zero because of the uniform continuity of the derivatives of $F^1, \cdots, F^m$ on compacts. On the other hand, Remark 2.6 6) shows that

$$\int_0^\cdot \partial_{i_1} F^1(X_s) \cdots \partial_{i_n} F^m(X_s)\, d[X^{i_1}, \cdots, X^{i_n}]_\epsilon(s), \tag{2.3}$$

converges *ucp*, for every $1 \le i_1 \cdots, i_n \le n$, to

$$\int_0^\cdot \partial_{i_1} F^1(X) \cdots \partial_{i_n} F^m(X)\, d[X^{i_1}, \cdots, X^{i_n}]. \tag{2.4}$$

Remark 2.1 implies the strong existence. ■

In the following subsections we aim at calculating the $n-$variation of processes of the type

$$X = \Big\{X(t) = \int_0^t G(t,s)\, dM(s); \quad t \in [0,1]\Big\}, \tag{2.5}$$

where $M = \{M(t),\ t \ge 0\}$ is a local $(\mathcal{F}_t)$-continuous martingale and $G : \{0 \le s \le t \le 1\} \longrightarrow \mathbb{R}$, is a $\mathcal{F}_0$-measurable random field, which we prolongate to $\mathbb{R}^2$ by setting,

$$G(t,s) = G(s,s) \quad \text{if} \ \ s \ge t.$$

The convolution case, i.e $X(t) = \int_0^t \mathcal{G}(t-s)\, dM(s)$, will be a particular case setting $G(t,s) = \mathcal{G}(t-s)$. We remark that the process X is not in general a semimartingale unless that $\mathcal{G}$ is enough regular. It is, for instance, the case when $\mathcal{G}$ has paths in $W^{1,2}_{loc}$. When M is a Brownian motion and $\mathcal{G}$ deterministic, Goldys and Musiela [17] have shown that this is a necessary and sufficient condition. We remark that [6] gives necessary and sufficient conditions on $\mathcal{G}$ so that X is continuous.

From now on, for the sequel of this section 2, we decompose the process X as

$$X = N + Z, \tag{2.6}$$

where N will stand for the local $(\mathcal{F}_t)$-continuous martingale given by

$$N_t = \int_0^t G(s,s)\, dM_s, \quad t \in [0,1],$$

and Z for the $(\mathcal{F}_t)$-adapted process,

$$Z_t = \int_0^t H(t,s)\, dM_s = \int_0^1 H(t,s)\, dM_s,$$

where $H(t,s) = G(t,s) - G(s,s), \quad (t,s) \in [0,1]^2$. For $(t,s) \notin [0,1]^2$, H will be extended putting zero.

2.2 Quadratic variation of martingale convolutions

The case $n = 2$ was partially considered by the same authors in [14]. Here we give some recalls and complements.

We formulate the following assumption :

$$\text{(H2)} \quad \big[G(\cdot,u), G(\cdot,v)\big]_\epsilon(t),$$

converges *ucp* for (u,v,t) belonging to $[0,1]^3$. In particular $\big[G(\cdot,u), G(\cdot,v)\big]$ exists (in the strong sense) for all $(u,v) \in [0,1]^2$.

Proposition 2.8 *Under assumption* (H2) *we set*

$$A(t) = A_1(t) + A_2(t), \quad t \in [0,1],$$

where,

$$A_1(t) = \int_0^t [G(\cdot,s); G(\cdot,s)] d[M]_s$$
$$A_2(t) = 2\int_0^t \int_0^{s_2} [G(\cdot,s_1); G(\cdot,s_2)] dM_{s_1} dM_{s_2}.$$

If the process A is continuous, then,

$$[Z,Z] = A \quad \text{and} \quad [Z,N] = 0. \tag{2.7}$$

In particular $[X,X] = [N] + A$.

Remark 2.9 *Since* $[Z, Z]$ *and* A_2 *are increasing processes,* A_1 *is forced to be of bounded variation.*

When $G(t, s)$, for all $t \in [0, 1]$, is only $(\mathcal{F}_s)$-adapted, even if we do not evaluate the quadratic variation, with the same decomposition (2.6) we have the following result.

Lemma 2.10 *Suppose that* $G(\cdot, s)$ *is* $(\mathcal{F}_s)$*-adapted (without necessarily assumption* (H2)*). Then for every local* $(\mathcal{F}_t)$*-martingale* Y*,* $[Z, Y] = 0$*. In particular*

$$[X, Y] = [N, Y]. \tag{2.8}$$

Proof. Let Y be a local $(\mathcal{F}_t)$-continuous martingale, $\epsilon > 0$ and $t \in [0, 1]$. We have

$$\begin{aligned}
[Z, Y]_\epsilon(t) =& \frac{1}{\epsilon} \int_0^t \big(Y_{s+\epsilon} - Y_s\big)\big(Z_{s+\epsilon} - Z_s\big)\, ds \\
=& \frac{1}{\epsilon} \int_0^t \big(Y_{s+\epsilon} - Y_s\big)\bigg(\int_0^s (H(s+\epsilon, u) - H(s, u))\, dM_u \\
& \qquad\qquad + \int_s^{s+\epsilon} H(s+\epsilon, u)\, dM_u \bigg)\, ds \\
:=& I_1(\epsilon, t) + I_2(\epsilon, t),
\end{aligned}$$

where,

$$\begin{aligned}
I_1(\epsilon, t) &= \frac{1}{\epsilon} \int_0^t \big(Y_{s+\epsilon} - Y_s\big) \int_0^s (H(s+\epsilon, u) - H(s, u))\, dM_u\, ds \\
&= \frac{1}{\epsilon} \int_0^t \int_s^{s+\epsilon} \int_0^s \big(H(s+\epsilon, u) - H(s, u)\big)\, dM_u\, dY_v\, ds \\
= \frac{1}{\epsilon} \int_0^t \int_0^{v\wedge t} & \int_{0\vee u \vee v-\epsilon}^{v\wedge t} 1_{\{v \le t+\epsilon\}} \big(H(s+\epsilon, u) - H(s, u)\big)\, ds\, dM_u\, dY_v,
\end{aligned}$$

using stochastic Fubini's theorem on the last equation. Using localization arguments, we will reduce to the case where Y and M are square integrable martingales. Then Doob's inequality, continuity of H and the fact that

$H(s,s)=0$, for every $s\in[0,1]$, show that $I_1(\epsilon,t)$ converges *ucp* to zero. It remains to calculate the limit of $I_2(\epsilon,\cdot)$. Recall that

$$I_2(\epsilon,t)=\frac{1}{\epsilon}\int_0^t \big(Y_{s+\epsilon}-Y_s\big)\int_s^{s+\epsilon} H(s+\epsilon,u)dM_u\,ds.$$

Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
E\Big(\sup_{t\in[0,1]}\big|I_2(\epsilon,t)\big|\Big) &\le E\Big(\frac{1}{\epsilon}\int_0^1 \big|Y_{s+\epsilon}-Y_s\big|\Big|\int_s^{s+\epsilon} H(s+\epsilon,u)dM_u\Big|ds\Big)\\
&\le E\Big(\frac{1}{\epsilon}\Big[\int_0^1 \big(Y_{s+\epsilon}-Y_s\big)^2 ds\Big]^{\frac{1}{2}}\Big[\int_0^1\Big(\int_s^{s+\epsilon} H(s+\epsilon,u)dM_u\Big)^2 ds\Big]^{\frac{1}{2}}\Big)\\
&\le \Big[E\frac{1}{\epsilon}\int_0^1 \big(Y_{s+\epsilon}-Y_s\big)^2 ds\Big]^{\frac{1}{2}}\Big[E\int_0^1\frac{1}{\epsilon}\Big(\int_s^{s+\epsilon} H(s+\epsilon,u)dM_u\Big)^2 ds\Big]^{\frac{1}{2}}\\
&= \Big[E\Big(\frac{1}{\epsilon}\int_0^1 \big(Y_{s+\epsilon}-Y_s\big)^2 ds\Big)\Big]^{\frac{1}{2}}\Big[E\int_0^1\frac{1}{\epsilon}\int_u^{u+\epsilon} H(s,u)^2 ds d[M]_u\Big]^{\frac{1}{2}},
\end{aligned}$$

where in the last equality we use stochastic Fubini's theorem. It is clear that $E\frac{1}{\epsilon}\int_0^1\big(Y_{s+\epsilon}-Y_s\big)^2ds$ converges to $E\big([Y]_1\big)$. The continuity of H and the fact that $H(s,s)=0$, for every $s\in[0,1]$, together with dominated convergence theorem show that $I_2(\epsilon,\cdot)$ converges *ucp* to zero. ■

Corollary 2.11 *The process X of type (2.5) can be uniquely decomposed in $N+Z$, where N is a local $(\mathcal{F}_t)$-martingale such that $N(0)=0$ and Z is a finite quadratic variation process, if we require moreover that $[Z,Y]=0$, for any $(\mathcal{F}_t)$-local martingale Y. Moreover the process Z, and therefore X, is not a Dirichlet process unless $[Z]=0$.*

Proof. (2.6) provides one such decomposition also because of Lemma 2.10. As far as uniqueness is concerned, suppose that, $N+Z=0$. Using, again Lemma 2.10, we get that $[N+Z]=[N]+[Z]=0$, which gives that $N=0$ and therefore $Z=0$.

Suppose now that the process Z is Dirichlet, i.e $Z=Y+A$ where Y is a martingale and $[A]=0$. Then Lemma 2.10 tells that $0=[Z,Y]=[Z]$, using also the fact that $[Z,A]=0$ because of $[A]=0$ and Remark 2.6 2). ■

We remark that under (H2), $[Z,Z]$ can be written as

$$[Z,Z](t)=2\int_0^t\int_0^{s_2}[G(\cdot,s_1);G(\cdot,s_2)]d^\circ M_{s_1}d^\circ M_{s_2}, \tag{2.9}$$

where d° means that the integral is in Stratonovich sense.

In general let $f(t,s);\ (t,s) \in [0,1]^2$, be a $\mathcal{F}_0$-measurable continuous random-field. We set

$$I_2^\circ(f)(t) = \int_0^t \int_0^{s_2} f(s_1,s_2) d^\circ M_{s_1} d^\circ M_{s_2}.$$

Then we can prove that,

$$I_2^\circ(f)(t) = \int_0^t \int_0^{s_2} f(s_1,s_2) dM_{s_1} dM_{s_2} + \frac{1}{2}\int_0^t f(s,s)\, d[M]_s. \tag{2.10}$$

In fact, it is clear that,

$$\int_0^{s_2} f(s_1,s_2) d^\circ M_{s_1} = \int_0^{s_2} f(s_1,s_2) dM_{s_1}.$$

So classical Itô-Stratonovich calculus (Remark 3.3 2)) implies

$$\begin{aligned} I_2^\circ(f)(t) &= \int_0^t \Big(\int_0^{s_2} f(s_1,s_2) dM_{s_1}\Big) d^\circ M_{s_2} \\ &= \int_0^t \int_0^{s_2} f(s_1,s_2) dM_{s_1} dM_{s_2} + \frac{1}{2}\Big[\int_0^{\cdot} f(s_1,\cdot) dM_{s_1}\ ,\ M\Big](t). \end{aligned}$$

Using (2.8) we get,

$$I_2^\circ(f)(t) = \int_0^t \int_0^{s_2} f(s_1,s_2) dM_{s_1} dM_{s_2} + \frac{1}{2}\int_0^t f(s,s) d[M]_s,$$

which shows in particular (2.9).

2.2.1 The Gaussian case

We suppose that the martingale M is a Brownian motion $W = \{W_t,\ t \in [0,1]\}$, so that

$$X_t = \int_0^t G(t,s)\, dW_s \tag{2.11}$$

Under assumption (H2), Proposition 2.8 gives the following expression for the 2-variation (quadratic variation):

$$[X]_t = \int_0^t G(s,s)^2\, ds + A_t,$$

where $A = A_1 + A_2$ with,

$$A_1(t) = \int_0^t \big[G(\cdot, s); G(\cdot, s)\big]\, ds$$

$$A_2(t) = 2\int_0^t \int_0^{s_2} \big[G(\cdot, s_1); G(\cdot, s_2)\big]\, dW_{s_1} dW_{s_2}, \quad t \in [0,1].$$

We will now make the link with the study of the quadratic variation of a Gaussian process given in [27].

[27] considered a (mean zero) Gaussian process with covariance function

$$K(u,v) = E\big(X_u\, X_v\big), \quad u, v \in [0,1].$$

If X is of the form (2.11) then obviously,

$$K(u,v) = \int_0^{u\wedge v} G(u,s)G(v,s)\, ds.$$

[27] defined the concept of 2-planar variation for K which was given by

$$\lim_{\epsilon,\delta\to 0} \frac{1}{\epsilon\delta} \int_{[0,1]^2} \Big(\triangle_{\epsilon,\delta} K(u,v)\Big)^2\, dudv, \tag{2.12}$$

with

$$\triangle_{\epsilon,\delta} K(u,v) = K(u+\epsilon, v+\delta) + K(u,v) - K(u, v+\delta) - K(u+\epsilon, v),$$

provided that the limit in (2.12) exists for any $t \in [0,1]$. In [27] the concept of energy process $En(X)$ was defined as,

$$En(X)(t) = \lim_{\epsilon\to 0} E\Big(\frac{1}{\epsilon}\int_0^t \big(X_{s+\epsilon} - X_s\big)^2\, ds\Big).$$

It was easily shown that

$$En(X)(t) = \lim_{\epsilon\to 0} \int_0^t \triangle_{\epsilon,\epsilon} K(s,s)\, ds. \tag{2.13}$$

Remark 2.12

1) *A careful analysis on (2.12) and (2.13) shows the following properties.*

a) The 2-planar variation of K equals

$$4\int_0^t \int_0^{s_2} \big[G(\cdot, s_1); G(\cdot, s_2)\big]^2\, ds_1 ds_2 = E\big(A_2^2(t)\big).$$

b)

$$En(X)(t) = \int_0^t G(s,s)^2\,ds + \int_0^t \big[G(\cdot,s);G(\cdot,s)\big]\,ds$$
$$= [\int_0^{\cdot} G(s,s)dW_s](t) + A_1(t).$$

For illustration consider the case $G(t,s) = B_{t-s}$, where B is a Brownian motion independent of W, which can be considered $\mathcal{F}_0-$measurable. Then the process $\{X_t = \int_0^t B_{t-s}\,dW_s,\ t\in[0,1]\}$ is well defined and Proposition 2.8 gives that,

$$[X,X](t) = \frac{t^2}{2}.$$

In fact in this case $A_2(t) = 0$ since $[B, B_{\cdot-s}] = 0$ for all $s\in[0,1]$.

2.3 $n-$variation of martingale convolutions

To extend this calculus in order to evaluate the $n-$variation process of X, we will need an explicit expression of $(N)^n$ for any $n \geq 3$ and N continuous $(\mathcal{F}_t)-$martingale. As we will see, this expression will be a particular case of a generalization of (2.10).

Notation. Let $n\geq 3$, $k\in\{1,2,\cdots,[\frac{n}{2}]\}$. We denote by $\sigma = (\sigma^1,\cdots,\sigma^{n-k})$ a permutation of $\{1,2,\cdots,n-k\}$ such that the first k elements of σ^{-1} are chosen arbitrarily among $\{1,2,\cdots,n-k\}$ and the $n-2k$ remaining are taken at the sequel. We denote by Σ_k^n that family of permutations σ. We remark that its cardinal is given by $C_{n-k}^k = \frac{(n-k)!}{k!(n-2k)!}$.

Let Y be a finite quadratic variation process, and $k\in\{1,2,\cdots,[\frac{n}{2}]\}$. With $\sigma\in\Sigma_k^n$ we associate

$$\sigma_Y = \sigma\big(\underbrace{[Y],\cdots,[Y]}_{k\ times},\underbrace{Y,\cdots,Y}_{n-2k\ times}\big) = (\sigma_Y^1,\cdots,\sigma_Y^{n-k});$$

we remark that, for all $l\in\{1,2,\cdots,n-k\}$,

$$\sigma_Y^l = \begin{cases} [Y] & \text{if} \quad \sigma(l)\in\{1,2,\cdots,k\} \\ Y & \text{if} \quad \sigma(l)\in\{k+1,k+2,\cdots,n-k\}. \end{cases} \tag{2.14}$$

We denote by $P_k^n(Y)$ the set of σ_Y where $\sigma \in \Sigma_k^n$.

Now we give a generalization of Hu-Meyer formula. For the proof see the Appendix.

Proposition 2.13 *Let $n \geq 3$ and $\{f(s_1, \cdots, s_n);\ (s_1, \cdots, s_n) \in [0,1]^n\}$ be a continuous, symmetric and $\mathcal{F}_0-$measurable random-fields. We set,*

$$I_n^\circ(f)(t) := \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} f(s_1, \cdots, s_n) d^\circ M_{s_1} \cdots d^\circ M_{s_n}.$$

Then

$$I_n^\circ(f)(t) = \sum_{k=0}^{[\frac{n}{2}]} \frac{1}{2^k} \sum_{\sigma \in \Sigma_k^n} \int_0^t \int_0^{s_{n-k}} \cdots \int_0^{s_2} f\big(s_{\sigma^{-1}(1)}, s_{\sigma^{-1}(1)}, \cdots, s_{\sigma^{-1}(k)}, s_{\sigma^{-1}(k)}, s_{\sigma^{-1}(k+1)}, s_{\sigma^{-1}(k+2)}, \cdots, s_{\sigma^{-1}(n-k)}\big) d\sigma_M^1(s_1) \cdots d\sigma_M^{n-k}(s_{n-k}). \quad (2.15)$$

Corollary 2.14 *Let $M = \{M_t, t \in [0,1]\}$ be any $(\mathcal{F}_t)$-continuous martingale, then for all $n \geq 3$,*

$$\big(M_t\big)^n = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{2^k} \sum_{\sigma \in \Sigma_k^n} \int_0^t \int_0^{s_{n-k}} \cdots \int_0^{s_2} d\sigma_M^1(s_1) \cdots d\sigma_M^{n-k}(s_{n-k}). \quad (2.16)$$

Proof. Itô-Stratonovich formula shows that,

$$\big(M_t\big)^n = n!\, I_n^\circ(1)(t),$$

so that we can apply Proposition 2.13. ■

As a consequence of Proposition 2.13 we also obtain the following.

Proposition 2.15 *If martingale M is a Brownian motion W then (2.15) and (2.16) become respectively,*

$$I_n^\circ(f)(t) = \sum_{k=0}^{[\frac{n}{2}]} \frac{1}{2^k} \int_0^t \int_0^{s_{n-k}} \cdots \int_0^{s_{k+2}} \int_0^{\mathbf{t}} \int_0^{s_k} \cdots \int_0^{s_2} f\big(s_1, s_1, \cdots, s_k, s_k, s_{k+1}, s_{k+2}, \cdots, s_{n-k}\big) ds_1 \cdots ds_k dW_{s_{k+1}} \cdots dW_{s_{n-k}}.$$

and

$$(W_t)^n = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!\,t^k}{2^k k!} \int_0^t \int_0^{s_{n-2k}} \cdots \int_0^{s_2} dW_{s_1} \cdots dW_{s_{n-2k}}, \quad t \in [0,1].$$

Proof. $(W_t)^n$ follows immediately from the first expression. The evaluation of $I_n^\circ(f)$ is given in the Appendix. ■

Using Proposition 2.13 and classical convergence properties of Itô integrals we get the following.

Corollary 2.16 *Let* $\Big(F_\epsilon(s_1, \cdots, s_n); (s_1, \cdots, s_n) \in [0,1]^n\Big)_{\epsilon>0}$ *be a sequence of* $\mathcal{F}_0$*-measurable continuous and symmetric random fields which converges ucp to a continuous random field* F *when* ϵ *goes to 0. Then*

$$\int_0^t \int_0^{s_n} \cdots \int_0^{s_2} F_\epsilon(s_1, \cdots, s_n) d^\circ M_{s_1} \cdots d^\circ M_{s_n}$$
$$\longrightarrow_{\epsilon\to 0} \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} F(s_1, \cdots, s_n)\, d^\circ M_{s_1} \cdots d^\circ M_{s_n}$$

where the convergence holds ucp.

We are ready now for evaluating the $n-$variation of process of the type (2.5). Assumption (H2) will be generalized in :

(Hn) $\big[G(\cdot, s_1), \cdots, G(\cdot, s_n)\big]_\epsilon(t)$

converges *ucp* for $(s_1, s_2, \cdots, s_n, t)$ belonging to $[0,1]^{n+1}$. In particular $\big[G(\cdot, s_1), \cdots, G(\cdot, s_n)\big]$ exists for all $(s_1, \cdots, s_n) \in [0,1]^n$.

Theorem 2.17 *Under assumption* (Hn) *we set*

$$A(t) = n! \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} \big[G(\cdot, s_1), \cdots, G(\cdot, s_n)\big](t)\, d^\circ M_{s_1} \cdots d^\circ M_{s_n},$$

$t \in [0,1]$. *If the process* A *is continuous, then*

$$[Z; n] = A,$$

where, we recall that $Z_t = \int_0^t \big(G(t,s) - G(s,s)\big)\, dM_s$.

Proposition 2.18 *Let us make the assumptions and conventions of Theorem 2.17. If* $[N, \underbrace{Z, \cdots, Z}_{(n-1)\ times}]$ *exists, then*

$$[X;n] = [Z;n] + n\,[N, \underbrace{Z, \cdots, Z}_{(n-1)\ times}]. \tag{2.17}$$

Proof (of the Proposition).
Using the multi-linearity of the $n-$covariation map we have that

$$[X;n] = \sum_{p=0}^{n} C_n^p\,[\underbrace{N, \cdots, N}_{p\ times}, \underbrace{Z, \cdots, Z}_{(n-p)\ times}].$$

On the other hand, existence of $[N;2]$ in the strong sense and Remark 2.6 5), imply

$$[\underbrace{N, \cdots, N}_{p\ times}, \underbrace{Z, \cdots, Z}_{(n-p)\ times}] = 0, \quad \text{for all} \quad p \geq 2.$$

So it remains $[Z;n]$ and $[N, \underbrace{Z, \cdots, Z}_{(n-1)\ times}]$. ■

Remark 2.19
1) If $[Z;n]$ *exists strongly then* $[N, \underbrace{Z, \cdots, Z}_{(n-1)\ times}] = 0$ *using Remark 2.6 4). This happens for instance when* $[Z;n]$ *exists and* n *is even.*

2) The equality $[X;n] = [Z;n]$ *can be insured reinforcing Assumption (Hn). Suppose that*

$$\sup_{s_1, \cdots, s_n \in [0,1]} \|[G(\cdot, s_1); \cdots; G(\cdot, s_n)]\| < \infty.$$

In particular $[G(\cdot, s_1); \cdots; G(\cdot, s_n)]$ *exists strongly for each* $s_1, \cdots, s_n \in [0,1]$*. Similarly as for Remark 2.6 3), Hölder inequality implies that Assumption (H*$2(n-1)$*) holds and that the corresponding limit is zero. Theorem 2.17 implies that* $[Z; 2(n-1)]$ *exists (strongly) and it equals zero. Since* N *has finite 2-variation and because of Remark 2.6 4) we can conclude that* $[N, \underbrace{Z, \cdots, Z}_{(n-1)\ times}] = 0.$

Proof (of Theorem 2.17).

We recall the decomposition of the process X,

$$X_t = N_t + Z_t, \quad t \in [0,1],$$

where $Z = \{\int_0^1 H(t,s)\, dM_s \,, t \in [0,1]\}$, with $H(t,s) = G(t,s) - G(s,s)$, is a continuous $(\mathcal{F}_t)$-adapted process. We recall that

$$H(t,s) = 0 \quad \text{for all } \ 0 \le t \le s \le 1.$$

Now we take $\epsilon > 0$ and $t \in [0,1]$,

$$\begin{aligned}[Z;n]_\epsilon(t) &= \frac{1}{\epsilon}\int_0^t \left(Z_{s+\epsilon} - Z_s\right)^n ds \\ &= \frac{1}{\epsilon}\int_0^t \left(\int_0^1 \Big(H(s+\epsilon,u) - H(s,u)\Big)\, dM_u\right)^n ds.\end{aligned}$$

We set

$$F_{s,\epsilon}(u) = H(s+\epsilon,u) - H(s,u); \quad \text{and} \quad N_{s,\epsilon}(v) := \int_0^v F_{s,\epsilon}(u) dM_u.$$

for all $s < t$, and $u, v \in [0,1]$. This gives

$$[Z;n]_\epsilon(t) = \frac{1}{\epsilon}\int_0^t \Big(N_{s,\epsilon}(1)\Big)^n ds.$$

Let $v \in [0,1]$. Itô-Stratonovich formula gives

$$\Big(N_{s,\epsilon}(v)\Big)^n = n! \int_0^v \int_0^{s_n} \cdots \int_0^{s_2} \prod_{k=1}^n \Big(H(s+\epsilon, s_k) - H(s,s_k)\Big)^2 d^\circ M_{s_1} \cdots d^\circ M_{s_n}.$$

Using the stochastic Fubini's theorem we get

$$[Z;n]_\epsilon(t) = n! \int_0^1 \int_0^{s_n} \cdots \int_0^{s_2} \Big[H(\cdot,s_1), \cdots, H(\cdot,s_n)\Big]_\epsilon(t) \; d^\circ M_{s_1} \cdots d^\circ M_{s_n}.$$

Since $[G(\cdot,s_1), \cdots, G(\cdot,s_n)]$ exists and it is equal to $[H(\cdot,s_1), \cdots, H(\cdot,s_n)]$, Corollary 2.16 and the fact that,

$$[H(\cdot,s_1), H(\cdot,s_2), \cdots, H(\cdot,s_n)](t) = 0, \quad s_n > t,$$

allow to show that $[Z;n] = A$. ■

Under suitable assumptions formulated in Remark 2.19 2), applying Theorem 2.17 and Proposition 2.13 we obtain the following illustration.

Example 2.20 *1) The cubic variation of the process X is given by*

$$
\begin{aligned}
[X;3](t) =& 3!\int_0^t\int_0^{s_3}\int_0^{s_2}[G(.,s_1);G(.,s_2);G(.,s_3)](t)d^\circ M_{s_1}d^\circ M_{s_2}d^\circ M_{s_3}\\
=& 3!\int_0^t\int_0^{s_3}\int_0^{s_2}\Big[G(.,s_1);G(.,s_2);G(.,s_3)\Big](t)dM_{s_1}dM_{s_2}dM_{s_3}\\
&+\frac{3!}{2}\int_0^t\int_0^{s_2}\Big[G(.,s_1);G(.,s_1);G(.,s_2)\Big](t)d[M]_{s_1}dM_{s_2}\\
&+\frac{3!}{2}\int_0^t\int_0^{s_2}\Big[G(.,s_1);G(.,s_2);G(.,s_2)\Big](t)dM_{s_1}d[M]_{s_2}
\end{aligned}
$$

2) Suppose that the martingale M is a Brownian Motion W. Then the cubic variation and 4-variation of the process X are given by

$$
\begin{aligned}
[X;3](t) =& 3!\int_0^t\int_0^{s_3}\int_0^{s_2}\Big[G(.,s_1);G(.,s_2);G(.,s_3)\Big](t)d^\circ W_{s_1}d^\circ W_{s_2}d^\circ W_{s_3}\\
=& 3!\int_0^t\int_0^{s_3}\int_0^{s_2}\Big[G(.,s_1);G(.,s_2);G(.,s_3)\Big](t)dW_{s_1}dW_{s_2}dW_{s_3}\\
&+\frac{3!}{2}\int_0^t\int_0^t\Big[G(.,s_1);G(.,s_1);G(.,s_2)\Big](t)ds_1dW_{s_2}
\end{aligned}
$$

$$
\begin{aligned}
[X;4](t) =& 4!\int_0^t\int_0^{s_4}\int_0^{s_3}\int_0^{s_2}\Big[G(.,s_1);G(.,s_2);G(.,s_3);G(.,s_4)\Big](t)\\
& \qquad d^\circ W_{s_1}d^\circ W_{s_2}d^\circ W_{s_3}d^\circ W_{s_4}\\
=& 4!\int_0^t\int_0^{s_4}\int_0^{s_3}\int_0^{s_2}\Big[G(.,s_1);G(.,s_2);G(.,s_3);G(.,s_4)\Big](t)\\
& \qquad dW_{s_1}dW_{s_2}dW_{s_3}dW_{s_4}\\
&+\frac{4!}{2}\int_0^t\int_0^{s_3}\int_0^t\Big[G(.,s_1);G(.,s_1);G(.,s_2);G(.,s_3)\Big](t)ds_1dW_{s_2}dW_{s_3}\\
&+\frac{4!}{2^2}\int_0^t\int_0^{s_2}\Big[G(.,s_1);G(.,s_1);G(.,s_2);G(.,s_2)\Big](t)ds_1ds_2.
\end{aligned}
$$

We observe that those variations can also be obtained using Hu-Meyer formula.

2.4 The true convolution Case

Suppose that $G(t,s) = \mathcal{G}(t-s)$ for all $\{0 \le s,t \le 1\}$. The process X is then given by

$$X(t) = \int_0^t \mathcal{G}(t-s)\, dM_s.$$

Assumption (Hn) becomes here:

$$\text{(Hn')} \quad [\theta_{s_1}\mathcal{G}, \theta_{s_2}\mathcal{G}, \cdots, \theta_{s_n}\mathcal{G}]_\epsilon(t)$$

where $\theta_s\mathcal{G}(t) = \mathcal{G}(t-s)$, converges uniformly for $(s_1, s_2, \cdots, s_n, t)$ belonging to $[0,1]^{n+1}$. In particular $\left[\theta_{s_1}\mathcal{G}, \cdots, \theta_{s_n}\mathcal{G}\right]$ exists for all $(s_1, \cdots, s_n) \in [0,1]^n$.

Corollary 2.21 *Under assumption (Hn') if the following process,*

$$A(t) = \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} \left[\theta_{s_1}\mathcal{G}, \cdots \theta_{s_n}\mathcal{G}\right](t) d^\circ M_{s_1} \cdots d^\circ M_{s_n},$$

is continuous, then the process X has the following decomposition,

$$X = \mathcal{G}(0)M + Z,$$

where $Z(t) = \int_0^t \big(\mathcal{G}(t-s) - \mathcal{G}(0)\big)\, dM(s)$ and $[Z;n] = A$.

3 Stochastic calculus with respect to finite cubic variation continuous processes

We are interested in a stochastic calculus with respect to finite strong cubic variation (or 3-variation) continuous processes. A simple example of continuous process, is the fractional Brownian motion B^H of Hurst index H. We recall that B^H is a Gaussian process with covariance,

$$Cov(B^H_u, B^H_v) = \frac{1}{2}\big(u^{2H} + v^{2H} - |u-v|^{2H}\big) \quad \text{for} \quad u,v \ge 0.$$

Proposition 3.1 *B^H has a strong cubic variation if $H \ge \frac{1}{3}$. Moreover $[B^H;3] = 0$.*

Proof. Proposition 3.4 of [27] tells us that, if $H = \frac{1}{3}$ then

$$\frac{1}{\epsilon}\int_0^{\cdot} |B^H_{s+\epsilon} - B^H_s|^3\, ds \tag{3.1}$$

converges *ucp*, when $\epsilon \to 0$, to $\rho_H\, t$, where $\rho_H = E[|G|^3]$, G being a centered Gaussian random variable with unit variance. If $H > \frac{1}{3}$ it is then immediate, by Hölder inequality, to verify that

$$\frac{1}{\epsilon}\int_0^{\cdot} \left(B^H_{s+\epsilon} - B^H_s\right)^3 ds \tag{3.2}$$

converges *ucp* to 0. In that case (3.1) implies that B^H has a strong cubic variation which equals zero. We will see that even for $H = \frac{1}{3}$,

$$\frac{1}{\epsilon}\int_0^{\cdot} (B^H_{s+\epsilon} - B^H_s)^3\, ds$$

converges *ucp* to 0. From now on, in this proof, H will stand for $\frac{1}{3}$.

We proceed as in the proof of Proposition 3.14 of [27]. We denote by $K(u,v) = Cov(B^H_u, B^H_v)$, $(u,v) \in [0,1]^2$ the covariance of B^H, and we set

$$\triangle_{\epsilon,\epsilon} K(u,v) = K(u+\epsilon, v+\epsilon) + K(u,v) - K(u+\epsilon, v) - K(u, v+\epsilon).$$

and

$$C^H_\epsilon(t) = \frac{1}{\epsilon}\int_0^t \left(B^H_{s+\epsilon} - B^H_s\right)^3 ds, \quad t \in [0,1].$$

A straightforward calculation shows that

$$\triangle_{\epsilon,\epsilon} K(u,v) = \frac{1}{2}\Big(\big|u+\epsilon-v\big|^{2H} + \big|v+\epsilon-u\big|^{2H} - 2\big|u-v\big|^{2H}\Big).$$

Using Lemma 3.1 of [27], C^H_ϵ converges *ucp* to zero if for every fixed $t \geq 0$, $C^H_\epsilon(t)$ converges in probability to zero. We have, in fact,

$$E\Big(C^H_\epsilon(t)^2\Big) = \frac{2}{\epsilon^2}\int_{[0,1]^2} \mu_\epsilon\, 1_{\{u<v\}} du dv \tag{3.3}$$

where

$$\mu_\epsilon = E\Big(\left(B^H_{u+\epsilon} - B^H_u\right)^3 \left(B^H_{v+\epsilon} - B^H_v\right)^3\Big).$$

Using linear regression, we obtain

$$\mu_\epsilon = E\Big[(\epsilon^H N_1)^3 \Big(\frac{\theta_\epsilon}{\epsilon^H} N_1 + N_2 \sqrt{\epsilon^{2H} - \frac{\theta_\epsilon^2}{\epsilon^{2H}}}\ \Big)^3\Big]$$

where $\theta_\epsilon = \triangle_{\epsilon,\epsilon} K(u,v)$, N_1 and N_2 being two independent Gaussian random variables with distribution $N(0,1)$. Then,

$$\frac{\mu_\epsilon}{\epsilon^2} = E\Big[N_1^3 \Big(\frac{\theta_\epsilon}{\epsilon^{2H}} N_1 + N_2 \sqrt{1 - (\frac{\theta_\epsilon}{\epsilon^{2H}})^2}\ \Big)^3\Big].$$

We recall that in the proof of Proposition 3.14 of [27] it was proved that,

$$\begin{aligned} &\Big|\frac{\theta_\epsilon}{\epsilon^{2H}}\Big| \leq C, \qquad u \leq v; \\ &\lim_{\epsilon \to 0} \frac{\theta_\epsilon}{\epsilon^{2H}} = 0, \quad u < v. \end{aligned} \tag{3.4}$$

Therefore,

$$\frac{\mu_\epsilon}{\epsilon^2} = E\Big[N_1^3 N_2^3 \Big(1 - (\frac{\theta_\epsilon}{\epsilon^{2H}})^{\frac{3}{2}} + R(\epsilon, u, v)\Big)\Big], \tag{3.5}$$

where,

$$\big|R(\epsilon, u, v)\big| \leq \frac{\theta_\epsilon}{\epsilon^{2H}} Z, \tag{3.6}$$

and Z is an integrable random variable not depending on ϵ, u, v.

Using (3.4), (3.5), (3.6), the Lebesgue convergence theorem, (3.3) and the fact that $E(N_1^3 N_2^3) = 0$, (3.2) is finally obtained. ∎

In this stochastic calculus, the symmetric integral will play a similar role of the forward integral in the case of the stochastic calculus with respect to finite quadratic variation continuous processes, see [27]. We start by recalling, from [25], the definition and some properties.

3.1 Symmetric integral

Definition 3.2 *Let X, Y be two continuous processes. For any $\epsilon > 0$ and $t \in [0,1]$ we set,*

$$I_\epsilon^\circ(Y, dX)(t) = \frac{1}{2\epsilon} \int_0^t Y_s \big(X_{s+\epsilon} - X_{s-\epsilon}\big)\, ds.$$

If the process $I_\epsilon^\circ(Y, dX)$ converge ucp, when ϵ goes to zero, then the limit will be denoted by $\int_0^t Y d^\circ X$, and called the **symmetric** *integral.*

Remark 3.3

1) *It is easy to prove that the symmetric integral* $\int_0^t Y d^\circ X$*, if it exists, is the ucp limit of*

$$J_\epsilon^\circ(Y, dX)(t) = \frac{1}{2\epsilon}\int_0^t \big(Y_{s+\epsilon} + Y_s\big)\big(X_{s+\epsilon} - X_s\big)\, ds.$$

2) *The symmetric integral* $\int_0^t Y d^\circ X$ *coincides with the Stratonovich one when* X *and* Y *are two semimartingales. More precisely,*

$$\int_0^t Y\, d^\circ X = \int_0^t Y\, dX + \frac{1}{2}[Y, X]. \tag{3.7}$$

3) *If the process* X *is of bounded variation then* $\int_0^t Y d^\circ X$ *is well defined, it is equal to the integral* $\int_0^t Y dX$ *in Stieltjes sense, and has bounded variation, see [25].*

4) *By definition, the symmetric integral is a continuous process. If both processes* X *and* Y *are* $\{\mathcal{F}_t;\ t \in [0,1]\}$*-adapted then, since the filtration satisfies the usual assumptions, the integral process* $\int_0^\cdot Y\, d^\circ X$*, if it exists, is an adapted process.*

5) *We have an integration by parts formula,*

$$\int_0^t Y d^\circ X = YX(t) - YX(0) - \int_0^t X d^\circ Y,$$

provided that one of the two integrals exists.

3.2 Itô formulae

We recall that, e.g. [27], in the case where X is a continuous process with finite quadratic variation and $f \in C^2$, we have

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) d^\circ X.$$

In the case of finite strong cubic variation continuous processes we have the following.

Proposition 3.4 *Let* X *be a real valued process with finite strong cubic variation, and* $f \in C^3$*. Then*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) d^\circ X_s - \frac{1}{12}\int_0^t f^{(3)}(X_s) d[X, X, X]_s. \tag{3.8}$$

Remark 3.5

1) *In particular the symmetric integral above exists.*

2) *Using Proposition 2.7, equation (3.8) is equivalent to*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) d^\circ X_s - \frac{1}{12}[f''(X), X, X](t).$$

Proof. Recall a Taylor-type formula,

$$\begin{aligned} f(b) = f(a) + f'(a)(b-a) + \frac{1}{2} f''(a)(b-a)^2 \\ + \frac{1}{6} f^{(3)}(a)(b-a)^3 + R(a,b)(b-a)^3, \end{aligned} \tag{3.9}$$

for every $a, b \in \mathbb{R}$, where,

$$R(a,b) = \int_0^1 \frac{\alpha^2}{2} \Big(f^{(3)}(\alpha a + (1-\alpha)b) - f^{(3)}(a) \Big) \, d\alpha.$$

Let $\epsilon > 0$ and $s \in [0,1]$. Applying (3.9) we get that

$$\begin{aligned} f(X_{s+\epsilon}) = f(X_s) + f'(X_s)(X_{s+\epsilon} - X_s) + \frac{1}{2} f''(X_s)(X_{s+\epsilon} - X_s)^2 \\ + \frac{1}{6} f^{(3)}(X_s)(X_{s+\epsilon} - X_s)^3 + R(X_s, X_{s+\epsilon})(X_{s+\epsilon} - X_s)^3 \end{aligned} \tag{3.10}$$

and,

$$\begin{aligned} f(X_s) = f(X_{s+\epsilon}) - f'(X_{s+\epsilon})(X_{s+\epsilon} - X_s) + \frac{1}{2} f''(X_{s+\epsilon})(X_{s+\epsilon} - X_s)^2 \\ - \frac{1}{6} f^{(3)}(X_{s+\epsilon})(X_{s+\epsilon} - X_s)^3 - R(X_{s+\epsilon}, X_s)(X_{s+\epsilon} - X_s)^3. \end{aligned} \tag{3.11}$$

Calculating the difference between (3.10) and (3.11), dividing by 2ϵ and integrating over $[0,t]$ we get,

$$\begin{aligned} \frac{1}{\epsilon} \int_0^t (f(X_{s+\epsilon}) - f(X_s)) ds = \frac{1}{2\epsilon} \int_0^t (f'(X_{s+\epsilon}) + f'(X_s))(X_{s+\epsilon} - X_s) ds \\ - \frac{1}{4\epsilon} \int_0^t (f''(X_{s+\epsilon}) - f''(X_s))(X_{s+\epsilon} - X_s)^2 \, ds \\ + \frac{1}{12\epsilon} \int_0^t (f^{(3)}(X_{s+\epsilon}) + f^{(3)}(X_s))(X_{s+\epsilon} - X_s)^3 \, ds \\ + \frac{1}{2\epsilon} \int_0^t \big(R(X_s, X_{s+\epsilon}) + R(X_{s+\epsilon}, X_s) \big)(X_{s+\epsilon} - X_s)^3 \, ds. \end{aligned}$$

First, since $f(X)$ is a continuous process, we observe that

$$\frac{1}{\epsilon}\int_0^t (f(X_{s+\epsilon}) - f(X_s))\, ds$$

converges *ucp* to $f(X_t) - f(X_0)$. Remark 3.3 1) tells that,

$$\frac{1}{2\epsilon}\int_0^t (f'(X_{s+\epsilon}) + f'(X_s))(X_{s+\epsilon} - X_s)\, ds$$

converges *ucp* to $\int_0^t f'(X) d^\circ X$. By definition of cubic variation,

$$\frac{1}{4\epsilon}\int_0^t (f''(X_{s+\epsilon}) - f''(X_s))(X_{s+\epsilon} - X_s)^2\, ds,$$

converges *ucp* to $\frac{1}{4}[f''(X), X, X](t)$. We have,

$$\frac{1}{12\epsilon}\int_0^t (f^{(3)}(X_{s+\epsilon}) + f^{(3)}(X_s))(X_{s+\epsilon} - X_s)^3\, ds = \frac{1}{6\epsilon}\int_0^t f^{(3)}(X_s)(X_{s+\epsilon} - X_s)^3\, ds + \frac{1}{12}\left[f^{(3)}(X), X, X, X\right]_\epsilon(t).$$

The first term converges *ucp* to $\frac{1}{6}\int_0^t f^{(3)}(X)\ d[X, X, X]$ by Remark 2.6 6). The second one converges to the bracket $[f^{(3)}(X), X, X, X]$ that vanishes following Remark 2.6 5). Finally, using the uniform continuity of both f'' and X on compacts, and the fact that $[X; 3]$ exists strongly, we have that

$$\frac{1}{\epsilon}\int_0^t \big(R(X_s, X_{s+\epsilon}) + R(X_{s+\epsilon}, X_s)\big)(X_{s+\epsilon} - X_s)^3\, ds$$

converges *ucp* to 0. So the result follows. ■

We give now multi-dimensional extension of Proposition 3.4. For this aim we introduce some notations.

Notations and Definitions. Let $X = (X^1, X^2, \cdots, X^n)$, and $Y = (Y^1, Y^2, \cdots, Y^n)$ be two vectors of continuous processes. We set

$$I_\epsilon^\circ(Y, \cdot dX)(t) = \frac{1}{2\epsilon}\sum_{k=1}^n \int_0^t Y_s^k \left(X_{s+\epsilon}^k - X_{s-\epsilon}^k\right) ds.$$

If the process $I_\epsilon^\circ(Y, \cdot dX)$ converges *ucp*, when ϵ goes to zero, then the limiting process will be denoted by $\int_0^\cdot Y \cdot d^\circ X$. This integral is in the spirit of [8] for semimartingales.

If for every $k = 1, 2, \cdots, n$, $\int_0^{\cdot} Y^k d^\circ X^k$ is well defined, then

$$\int_0^{\cdot} Y \cdot d^\circ X = \sum_{k=1}^{n} \int_0^{\cdot} Y^k \, d^\circ X^k.$$

Let $Z = (Z^{i,j})_{1 \le i,j \le n}$ be a $\mathbb{R}^{n \times n}$ matrix of continuous processes. We set,

$$[X, Z, Y^T]_\epsilon(t) = \frac{1}{\epsilon} \sum_{i,j=1}^{n} \int_0^t (X^i_{s+\epsilon} - X^i_s)(Z^{i,j}_{s+\epsilon} - Z^{i,j}_s)(Y^j_{s+\epsilon} - X^j_s) \, ds,$$

where Y^T is the transposition of vector Y. If the process $[X, Z, Y^T]_\epsilon$ converges *ucp*, when ϵ goes to zero, then the limiting process, denoted by $[X, Z, Y^T]$, will define the **3-covariation** of (X, Z, Y). If furthermore,

$$\big\| [X, Z, Y^T] \big\| := \sup_{0 < \epsilon \le 1} \frac{1}{\epsilon} \sum_{i,j=1}^{n} \int_0^1 |X^i_{s+\epsilon} - X^i_s| \, |Z^{i,j}_{s+\epsilon} - Z^{i,j}_s| |Y^j_{s+\epsilon} - X^j_s| \, ds \; < \infty,$$

then we will say that $[X, Z, Y^T]$ exists in the strong sense.

If, for every $i, j = 1, 2, \cdots, n$, the 3-covariation process (or strong 3- covariation) $[X^i, Z^{i,j}, Y^j]$ exists, then $[X, Z, Y^T]$ is equal (in the strong sense) to $\sum_{i,j} [X^i, Z^{i,j}, Y^j]$.

If F is a function of class C^1, we set $\nabla F(X) = \big(\partial_1 F(X), \cdots, \partial_n F(X)\big)$, and $\triangle F(X) = (\partial_{i,j} F(X))_{\substack{1 \le i \le n \\ 1 \le j \le n}}$ in the case when F is of class C^2.

Proposition 3.6 *Let $F \in C^3(\mathbb{R}^n)$ and $X = (X^1, X^2, \cdots, X^n)$ be a vector of continuous processes having all its mutual strong 3-covariations. Then*

$$F(X_t) = F(X_0) + \int_0^t \nabla F(X) \cdot d^\circ X - \frac{1}{12} [X, \triangle F(X), X^T](t).$$

In particular the symmetric integral above exists.

Remark 3.7 *If $\int_0^{\cdot} \partial_k F(X) \, d^\circ X^k$ exists for all $k = 1, 2, \cdots, n$, then $F(X)$ is given explicitly by,*

$$F(X_t) = F(X_0) + \sum_{i=1}^{n} \int_0^t \partial_i F(X) \, d^\circ X^i - \frac{1}{12} \sum_{1 \le i,j,k \le n} \int_0^t \partial_{ijk} F(X) \, d[X^i, X^j, X^k].$$

Proposition 3.8 *Let $X = (X^1, X^2, \cdots, X^m)$ be a vector of continuous processes having its mutual strong 3-covariations and $S = (S^1, \cdots, S^p)$ a vector of continuous processes having its mutual 2-covariations. We set $Y = (X, S)$. Let $F \in C^{3,2}(\mathbb{R}^m \times \mathbb{R}^p)$. Then*

$$\begin{aligned} F(Y_t) =& F(Y_0) + \int_0^t \nabla F(Y) \cdot d^\circ Y \\ & - \frac{1}{12} \sum_{1 \le i,j,k \le m} \int_0^t \partial_{ijk} F(Y) d[X^i, X^j, X^k]. \end{aligned} \tag{3.12}$$

Remark 3.9 *If we replace S with V, a vector of bounded variation processes then (3.12) holds even when F belongs to $C^{3,1}(\mathbb{R}^m \times \mathbb{R}^p)$ only. Moreover, if $\int_0^t \partial_i F(Y)\, d^\circ X^i$ exists for all $i = 1, \cdots, m$, then*

$$\begin{aligned} F(Y_t) =& F(Y_0) + \sum_{i=1}^{m+p} \int_0^t \partial_i F(Y) d^\circ Y^i \\ & - \frac{1}{12} \sum_{1 \le i,j,k \le m} \int_0^t \partial_{ijk} F(Y) d[X^i, X^j, X^k]. \end{aligned}$$

3.3 On chain-rule formulae

Let $X = (X^1, X^2, \cdots, X^n)$ be a vector of continuous processes, and Z a real process. Suppose that (Z, X) has all its mutual strong 3-covariations. The aim here, is to evaluate integrals of the type $\int_0^t Z\, d^\circ \varphi(X)$, where $\varphi \in C^3(\mathbb{R}^n)$.

Proposition 3.10 *If $\int_0^\cdot Z \nabla \varphi(X) \cdot d^\circ X$ exists then $\int_0^\cdot Z d^\circ \varphi(X)$ is well defined and it is given by,*

$$\begin{aligned} & \int_0^t Z d^\circ \varphi(X) = \int_0^t Z \nabla \varphi(X) \cdot d^\circ X \\ & - \frac{1}{4}[X, Z\nabla\varphi(X), X^T](t) + \frac{1}{6} \int_0^t Z d[X, \triangle \varphi(X), X^T], \quad t \in [0, 1]. \end{aligned} \tag{3.13}$$

Remark 3.11

1) *Using Proposition 2.7, (3.13) is explicitly given by*

$$\int_0^t Z d^\circ \varphi(X) = \int_0^t Z \nabla \varphi(X) \cdot d^\circ X - \frac{1}{4} \sum_{1 \le i,j \le n} \int_0^t \partial_{ij} \varphi(X) d[Z, X^i, X^j]$$
$$- \frac{1}{12} \sum_{1 \le i,j,k \le n} \int_0^t Z \, \partial_{ijk} \, \varphi(X) \, d[X^i, X^j, X^k]. \quad (3.14)$$

Recall that, when all the integrals $\int_0^t Z \partial_i \varphi(X) \, d^\circ X^i$, $i = 1 \cdots, n$, *are well defined then,* $\int_0^t Z \nabla \varphi(X) \cdot d^\circ X = \sum_{i=1}^n \int_0^t Z \partial_i \varphi(X) \, d^\circ X^i$.

2) *If* X *is a real valued process with finite strong cubic variation, and* φ *a* C^3 *function such that* $\int_0^t Z \varphi'(X) \, d^\circ X$ *exists, then (3.14) becomes*

$$\begin{aligned} \int_0^t Z d^\circ \varphi(X) = \int_0^t Z \varphi'(X) \, d^\circ X & - \frac{1}{4} \int_0^t \varphi''(X) \, d[Z, X, X] \\ & - \frac{1}{12} \int_0^t Z \, \varphi^{(3)}(X) \, d[X, X, X]. \end{aligned} \quad (3.15)$$

3) *As an application, we get an integration by parts formula. Let* (X, Y, Z) *be a vector of continuous processes having all its mutual strong 3-covariations. Suppose that,* $\int_0^t ZX \, d^\circ Y$ *and* $\int_0^t ZY \, d^\circ X$ *exist. Setting* $\varphi(x, y) = xy$, *Proposition 3.10 gives*

$$[X, Y, Z](t) = 2 \Big(\int_0^t ZX \, d^\circ Y + \int_0^t ZY \, d^\circ X - \int_0^t Z \, d^\circ X Y \Big).$$

Proof. We suppose that $n = 1$. The proof of general case is similar.

Let $\epsilon > 0$, and $s \in [0, 1]$. We multiply respectively equations (3.10) and (3.11) (identifying f with φ) by Z_s and $Z_{s+\epsilon}$. Then calculating the difference, dividing by 2ϵ and integrating over $[0, t]$ we get

$$\begin{aligned}
&\frac{1}{2\epsilon}\int_0^t (Z_{s+\epsilon}+Z_s)(\varphi(X_{s+\epsilon})-\varphi(X_s))\,ds = \\
&\qquad \frac{1}{2\epsilon}\int_0^t (Z_{s+\epsilon}\varphi'(X_{s+\epsilon})+Z_s\varphi'(X_s))(X_{s+\epsilon}-X_s)\,ds \\
&\qquad -\frac{1}{4\epsilon}\int_0^t (Z_{s+\epsilon}\varphi''(X_{s+\epsilon})-Z_s\varphi''(X_s))(X_{s+\epsilon}-X_s)^2\,ds \\
&\qquad +\frac{1}{12\epsilon}\int_0^t (Z_{s+\epsilon}\varphi^{(3)}(X_{s+\epsilon})+Z_s\varphi^{(3)}(X_s))(X_{s+\epsilon}-X_s)^3\,ds \\
&\qquad +\frac{1}{2\epsilon}\int_0^t \big(Z_s R(X_s,X_{s+\epsilon})+Z_{s+\epsilon}R(X_{s+\epsilon},X_s)\big)(X_{s+\epsilon}-X_s)^3\,ds.
\end{aligned}$$

The term on the left hand side of the equality converges *ucp* to $\int_0^t Z\,d^\circ\varphi(X)$. Using similar arguments as those in the proof of the Proposition 3.4 we see that, the first term on the right-hand side converges to $\int_0^t Z\,d^\circ\varphi'(X)$; the second one converges to $-\frac{1}{4}\,[Z\,\varphi''(X),\,X,\,X]$ and Proposition 2.7 tells that,

$$[Z\,\varphi''(X),X,X](t)=\int_0^t \varphi''(X)d[Z,X,X]+\int_0^t Z\,\varphi^{(3)}(X)d[X,X,X].$$

Using the fact that the third term converges to,

$$\frac{1}{6}\int_0^t Z\,\varphi^{(3)}(X)\,d[X,X,X]$$

and the last term to zero we get (3.15). ■

We give a small generalization of Proposition 3.10.

Remark 3.12 *Let $X=(X^1,X^2,\cdots,X^n)$ be a vector of continuous processes, and Z a continuous process such that (Z,X) has all its mutual strong 3-covariations. Let $S=(S^1,\cdots,S^m)$ be a vector of continuous processes with finite mutual 2-covariations (resp. with bounded variation). We set $Y=(X,S)$. If for every function $\varphi\in C^{3,2}(\mathbb{R}^{n+m})$ (resp. $\in C^{3,1}(\mathbb{R}^{n+m})$)*

$$\int_0^t Z\partial_i\varphi(Y)\,d^\circ Y^i \quad \textit{exist for all} \quad i=1,\cdots,n+m;$$

(resp. $\int_0^t Z\partial_i\varphi(Y)\,d^\circ X^i$ *exist for all* $i = 1,\cdots,n$*) then* $\int_0^\cdot Zd^\circ\varphi(Y)$ *exists and given by,*

$$\int_0^t Zd^\circ\varphi(Y) = \sum_{i=1}^{n+m}\int_0^t Z\partial_i\varphi(Y)d^\circ Y^i - \frac{1}{4}\sum_{1\le i,j\le n}\int_0^t \partial_{ij}\varphi(Y)d[Z,X^i,X^j]$$
$$- \frac{1}{12}\sum_{1\le i,j,k\le n}\int_0^t Z\,\partial_{ijk}\,\varphi(Y)\,d[X^i,X^j,X^k].$$

3.4 Generalized symmetric vector Itô processes

Let X be a real valued continuous process with finite strong cubic variation. Using Proposition 3.4, we easily see that the integral $\int_0^\cdot f(X)\,d^\circ X$ is well defined for every $f \in C^2$. However, if $X = (X^1, X^2, \cdots, X^n)$ is a vector of continuous processes, the existence of its mutual strong 3-covariations, is not a sufficient condition for guaranteeing the existence of $\int_0^t f(X)\,d^\circ X^k$; $k = 1,2,\cdots,n$. For this reason, we need a concept of 3-symmetric vector Itô process.

Definition 3.13 *A vector of continuous processes* $X = (X^1, X^2, \ldots, X^n)$ *is a 3-symmetric vector Itô process if the following assumptions are fulfilled.*

(i) $(X^1, X^2, \cdots, X^n)$ *has all its mutual strong 3-covariations,*

(ii) $\displaystyle\int_0^\cdot f(X)\,d^\circ X^i$ *exists for every* $f \in C^2(\mathbb{R}^n)$*,* $i = 1,2,\cdots,n$*,*

(iii)

$$[\int_0^\cdot f_1(X)\,d^\circ X^i, \int_0^\cdot f_2(X)\,d^\circ X^j, \int_0^\cdot f_3(X)\,d^\circ X^k] =$$
$$\int_0^\cdot f_1(X)f_2(X)f_3(X)\,d[X^i,X^j,X^k],$$

for every $f_1, f_2, f_3 \in C^2(\mathbb{R}^n)$ *and* $1 \le i,j,k \le n$*.*

Now we state some results which we will need in the next section.

Lemma 3.14 *Let* $X = (X^1, X^2, \cdots, X^m)$ *be a 3-symmetric vector Itô process, and* $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ *a function of class* C^3*. We set*

$$Y = \big(\varphi_1(X), \varphi_2(X), \cdots, \varphi_n(X)\big) = \varphi(X).$$

Then Y is again a 3-symmetric vector Itô process.

Proof. We will prove (i), (ii) and (iii) of Definition 3.13.
(i) Y has all its mutual strong 3-covariations because of Proposition 2.7.
(ii) For every $f \in C^2(\mathbb{R}^n)$ and $i = 1, 2, \cdots, n$, $\int_0^\cdot f(Y)\, d^\circ Y^i$ exists using the fact that X is a 3-symmetric vector Itô process and Remark 3.11 1).
(iii) Let $f_k \in C^2(\mathbb{R}^n)$, $k = 1, 2, 3$. We apply successively Proposition 3.10 with $Z = f_1(\varphi(X))$, $f_2(\varphi(X))$ and $f_3(\varphi(X))$. We obtain

$$\int_0^\cdot f_k(Y)\, d^\circ Y^i = \sum_{j=1}^m \int_0^\cdot f_k(\varphi(X))\partial_j \varphi_i(X)\, d^\circ X^j + V_i(\cdot)$$

for every $i = 1, \cdots, n$ and $k = 1, 2, 3$, where V_i is a bounded variation continuous process. Consequently (iii) follows from the fact that X is a 3-symmetric vector Itô process, Remark 2.6 3) and 4). ■

Proposition 3.15 *Let X be a real valued continuous process with finite strong cubic variation and $V = (V^1, V^2, \cdots, V^n)$ a vector of bounded variation processes. Then (X, V) is a 3-symmetric vector Itô process.*

Proof. Using Remark 2.6 3) 4), we observe that (X, V) has all its mutual strong 3-covariations. Concerning (ii), let $f \in C^2(\mathbb{R}^{1+n})$. Remark 3.3 3) says that $\int_0^\cdot f(X, V)\, d^\circ V^i$, $i = 1, \cdots, n$ coincide with the classical Stieltjes integrals. On the other hand, we set $F(x, v) = \int_0^x f(y, v)\, dy$, where $v = (v_1, \cdots, v_n)$; F belongs to $C^{3,2}(\mathbb{R}^{1+n})$. Remark 3.9 (Itô formula) tells that $\int_0^t f(X, V)\, d^\circ X$ is well defined by

$$\int_0^t f(X, V)\, d^\circ X = F(X_t, V_t) + A(t),$$

where

$$A(t) = -F(X_0, V_0) - \sum_{i=2}^{n+1} \int_0^t \partial_i F(X_s, V_s)\, d^\circ V_s^i - \frac{1}{12} \int_0^t \frac{\partial^2 f}{\partial x^2}(X_s, V_s) d[X, X, X], \quad t \in [0, 1].$$

Now we prove (iii). Let $g, h \in C^2(\mathbb{R}^{1+n})$. Since

$$\int_0^\cdot f(X_s, V_s)\, d^\circ V^i, \int_0^\cdot g(X_s, V_s)\, d^\circ V^i \text{ and } \int_0^\cdot h(X_s, V_s)\, d^\circ V^i,$$

$i = 1, \cdots, n$, are bounded variation processes, Remark 2.6 4) tells that it remains to prove

$$[\int_0^\cdot f(X,V)\, d^\circ X, \int_0^\cdot g(X,V)\, d^\circ X, \int_0^\cdot h(X,V)\, d^\circ X] = \int_0^\cdot f(X,V) g(X,V) h(X,V)\, d[X,X,X]. \tag{3.16}$$

As at the beginning of the proof, we can write

$$\int_0^t g(X,V)\, d^\circ X = G(X_t, V_t) + B(t)$$
$$\int_0^t h(X,V)\, d^\circ X = H(X_t, V_t) + C(t), \quad t \in [0,1],$$

where $G(x,v) = \int_0^x g(y,v)\, dy$ and $H(x,v) = \int_0^x h(y,v)\, dy$. Processes B and C have bounded variation. Using Remark 2.6 4) and Proposition 2.7 we obtain (3.16). ■

Corollary 3.16 *A finite strong cubic variation process is a 3-symmetric vector Itô process.*

As a consequence of Itô formula (Remark 3.9), we have the following result.

Remark 3.17 *Let ξ be a finite strong cubic variation process. We denote by $\mathcal{V}(\xi)$ the class of processes*

$$X_t = X_0 + \int_0^t \varphi(\xi, V^1, \cdots, V^n)\, d^\circ \xi + V_t^0,$$

where $n \in \mathbb{N}^*$, $V^1, \cdots, V^n, V^0$, are bounded variation processes and $\varphi \in C^{2,1}(\mathbb{R}^{1+n})$. $\mathcal{V}(\xi)$ coincides with the set of processes $\{\psi(\xi_t, V_t^1, \cdots, V_t^m),\ t \in [0,1]\}$, where $m \in \mathbb{N}^*$, $V^1, \cdots, V^m$ are bounded variation processes and $\psi \in C^{3,1}(\mathbb{R}^{1+m})$.

We conclude this section with a useful lemma which provides a chain rule formula for differentiating integrator processes.

Lemma 3.18 *Let ξ, Z be two continuous processes, $V = (V^1, \cdots, V^n)$ a vector of bounded variation processes. We suppose that (ξ, Z) has all its* *mutual strong 3-covariations. Let $\varphi = (\varphi(r,v)) \in C^{2,1}(\mathbb{R}^{1+n})$, where $v = (v_1, \cdots, v_n)$. We set*

$$X_t = \int_0^t \varphi(\xi, V)\, d^\circ \xi, \quad t \in [0,1].$$

Then the integral process $\int_0^\cdot Z\, d^\circ X$ exists and it is given by

$$\int_0^\cdot Z\, d^\circ X = \int_0^\cdot Z\varphi(\xi, V)\, d^\circ \xi - \frac{1}{4}\int_0^\cdot \frac{\partial \varphi}{\partial r}(\xi, V)\, d[Z, \xi, \xi]. \tag{3.17}$$

Proof. We set $\phi(r,v) = \int_0^r \varphi(u,v)\, du$. ϕ is obviously of class $C^{3,1}$. Since (ξ, V) is a 3-symmetric vector Itô process, applying Proposition 3.6 we get,

$$\begin{aligned}\phi(\xi_t, V_t) =& \phi(\xi_0, V_0) + \int_0^t \varphi(\xi, V)\, d^\circ \xi + \sum_{i=1}^n \int_0^t \frac{\partial \phi}{\partial v_i}(\xi, V) dV^i \\ & - \frac{1}{12}\int_0^t \frac{\partial^2 \varphi}{\partial r^2}(\xi, V) d[\xi, \xi, \xi].\end{aligned}$$

So

$$\begin{aligned}\int_0^t Z\, d^\circ X = & \int_0^t Z\, d^\circ \phi(\xi, V) - \sum_{i=1}^n \int_0^t Z\, \frac{\partial \phi}{\partial v_i}(\xi, V) dV^i \\ & + \frac{1}{12}\int_0^t \frac{\partial^2 \varphi}{\partial r^2}(\xi, V) d[\xi, \xi, \xi].\end{aligned} \tag{3.18}$$

On the other hand, Remark 3.12 tells that,

$$\begin{aligned}\int_0^t Z\, d^\circ \phi(\xi, V) = & \int_0^t Z\, \varphi(\xi, V)\, d^\circ \xi + \sum_{i=1}^n \int_0^t Z \frac{\partial \phi}{\partial v_i}(\xi, V) dV^i \\ & -\frac{1}{4}\int_0^t \frac{\partial \varphi}{\partial r}(\xi, V)\, d[Z, \xi, \xi] - \frac{1}{12}\int_0^t \frac{\partial^2 \varphi}{\partial r^2}(\xi, V) d[\xi, \xi, \xi].\end{aligned} \tag{3.19}$$

(3.18) and (3.19) show (3.17). ■

Remark 3.19 *Let $(X^2, \cdots, X^n)$ be a vector of bounded variation processes and X^1 a finite strong cubic variation process. Then the conclusion of Lemma 3.14 holds even when $\varphi \in C^{3,1}(\mathbb{R} \times \mathbb{R}^{n-1})$.*

4 On a SDE which is driven by finite cubic variation continuous processes

We aim here to study stochastic differential equations driven by finite strong cubic variation continuous process. We will operate with Doss-Sussmann ([11],[29]) transformation.

Let $\xi = \{\xi_t,\ t \in [0,1]\}$ (resp. $V = \{V_t,\ t \in [0,1]\}$) be a real process with finite strong cubic variation (resp. bounded variation).
We are interested in one equation of the type:

$$\begin{cases} dX_t = \sigma(X_t)\ \circ d\xi_t + b(t, X_t)\, dV_t \\ X_0 = \alpha \end{cases} \tag{4.1}$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ (resp. $b : [0,1] \times \mathbb{R} \to \mathbb{R}$) is of class $C^3(\mathbb{R})$ (resp. continuous), such that, σ' σ'' are bounded. We suppose moreover that b is locally Lipschitz with linear growth (uniformly in t, with respect to the second variable x), α any random variable $\mathcal{F}_1-$measurable.

Definition 4.1 *A real process X will be called* **solution of equation** (4.1) *if the following assumptions are fulfilled*

(i) $X_0 = \alpha$

(i) (X, ξ) is a 3-symmetric vector Itô process.

(iii) For every $Z = \psi(X, \xi)$, where $\psi \in C^\infty(\mathbb{R}^2)$, we have,

$$\begin{aligned} \int_0^t Z\, d^\circ X = \int_0^t Z\, \sigma(X)\, d^\circ \xi - \frac{1}{4} \int_0^t \sigma\sigma'(X)\, d[Z, \xi, \xi] \\ + \int_0^t Z_s\, b(s, X_s)\, dV_s. \end{aligned} \tag{4.2}$$

Remark 4.2

1) *If X is a solution of* (4.1)*, taking $Z = 1$, we observe in particular that X solves the integral equation*

$$X_t = \alpha + \int_0^t \sigma(X)\, d^\circ \xi + \int_0^t b(s, X_s)\, dV_s.$$

2) *If X is a solution of* (4.1) *then* (4.2) *remains true for every $Z = \psi(X, \xi)$, with $\psi \in C^2(\mathbb{R}^2)$.*

In fact, using Banach Steinhaus theorem (ch.II.1 *of* [13])*, and Proposition 2.7 we can prove that*

$$\begin{aligned} \psi \longmapsto \int_0^t \psi(X, \xi) d^\circ X = \int_0^t \psi(X, \xi)\, \sigma(X) d^\circ \xi \\ - \frac{1}{4} \int_0^t \sigma\sigma'(X)\, d[\psi(X, \xi), \xi, \xi] + \int_0^t \psi(X_s, \xi_s) b(s, X_s) dV_s, \end{aligned}$$

is linear and continuous operator with values in the space of continuous processes, equipped with the uniform convergence in probability. So by regularizing and passing to limit we have (iii) for every $\psi \in C^2$.

Let $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ be the flow generated by σ, defined as the solution of the following equation :

$$\begin{cases} \frac{\partial F}{\partial r}(r,x) &= \sigma\big(F(r,x)\big) \\ F(0,x) &= x. \end{cases} \tag{4.3}$$

Since σ is of class C^3, for any $r \in \mathbb{R}$, $F(r,.)$ is a C^3-diffeomorphism on $\mathbb{R}$. We set

$$H(r,x) = F^{-1}(r,x), \tag{4.4}$$

where the inverse is taken with respect to the variable x. H is again of class C^3.

We state, first, the uniqueness result.

Proposition 4.3 *There is at most one solution to* (4.1). *Moreover if X is a solution of* (4.1)*, then it is equal to $F(\xi, Y)$ where Y is the unique solution to the following equation :*

$$\begin{aligned} Y_t = H(\xi_0, \alpha) &+ \int_0^t \frac{\partial H}{\partial x}(\xi_s, F(\xi_s, Y_s))\, b(s, F(\xi_s, Y_s)) dV_s \\ &+ \frac{1}{12} \int_0^t \big(\sigma\sigma'^2(F(\xi_s, Y_s)) + \sigma^2\sigma''(F(\xi_s, Y_s))\big) \frac{\partial H}{\partial x}(\xi_s, F(\xi_s, Y_s))\, d[\xi, \xi, \xi]_s. \end{aligned} \tag{4.5}$$

Proof. We recall some relations involving F and H established in [27].

$$\frac{\partial H}{\partial r}(r,x) = -\sigma(x) \frac{\partial H}{\partial x}(r,x). \tag{4.6}$$

$$\frac{\partial^2 H}{\partial r \partial x}(r,x) = -\sigma'(x) \frac{\partial H}{\partial x}(r,x) - \sigma(x) \frac{\partial^2 H}{\partial x^2}(r,x). \tag{4.7}$$

Deriving those relations, we can prove the following:

$$\frac{\partial^2 H}{\partial r^2}(r,x) = \sigma\sigma'(x)\frac{\partial H}{\partial x}(r,x) + \sigma^2(x)\frac{\partial^2 H}{\partial x^2}(r,x). \tag{4.8}$$

$$\frac{\partial^3 H}{\partial r\partial x^2}(r,x) = -\sigma''(x)\frac{\partial H}{\partial x}(r,x) - 2\sigma'(x)\frac{\partial^2 H}{\partial x^2}(r,x) - \sigma(x)\frac{\partial^3 H}{\partial x^3}(r,x). \tag{4.9}$$

$$\begin{aligned}\frac{\partial^3 H}{\partial r^2\partial x}(r,x) = \big(\sigma'^2(x)+\sigma\sigma''(x)\big)\frac{\partial H}{\partial x}(r,x)\\ +3\sigma\sigma'(x)\frac{\partial^2 H}{\partial x^2}(r,x) + \sigma^2(x)\frac{\partial^3 H}{\partial x^3}(r,x).\end{aligned} \tag{4.10}$$

$$\begin{aligned}\frac{\partial^3 H}{\partial r^3}(r,x) = -\big(\sigma\sigma'^2(x) + \sigma^2\sigma''(x)\big)\frac{\partial H}{\partial x}(r,x)\\ - 3\sigma^2\sigma'(x)\frac{\partial H^2}{\partial x^2}(r,x) - \sigma^3(x)\frac{\partial^3 H}{\partial x^3}(r,x).\end{aligned} \tag{4.11}$$

Now, let X be a solution of (4.1) and set $Y = H(\xi, X)$. Obviously $X = F(\xi, Y)$. Since (ξ, X) is a 3-symmetric vector Itô process, in particular all its mutual strong 3-covariations exist; H is of class C^3 so Proposition 3.6 tells then that

$$\begin{aligned}Y_t = Y_0 + \int_0^t \frac{\partial H}{\partial r}(\xi, X)\, d^\circ \xi\\ + \int_0^t \frac{\partial H}{\partial x}(\xi, X)\, d^\circ X - \frac{1}{12}[(\xi, X), \Delta H(\xi, X), (\xi, X)^T](t),\end{aligned}$$

where $Y_0 = H(\xi_0, \alpha)$, and

$$\begin{aligned}&[(\xi, X), \Delta H(\xi, X), (\xi, X)^T](t) = \\ &\int_0^t \frac{\partial^3 H}{\partial r^3}(\xi_s, X_s)\, d[\xi,\xi,\xi]_s + 3\int_0^t \frac{\partial^3 H}{\partial r^2\partial x}(\xi_s, X_s)\, d[\xi,\xi,X]_s\\ &+3\int_0^t \frac{\partial^3 H}{\partial r\partial x^2}(\xi_s, X_s)\, d[\xi,X,X]_s + \int_0^t \frac{\partial^3 H}{\partial x^3}(\xi_s, X_s)\, d[X,X,X]_s.\end{aligned}$$

X being a solution of (4.1), we choose $Z = \frac{\partial H}{\partial x}$ in (4.2) and Remark 4.2 2) we obtain

$$\int_0^t \frac{\partial H}{\partial x}(\xi, X)\, d^\circ X = \int_0^t \frac{\partial H}{\partial x}(\xi, X)\sigma(X)\, d^\circ \xi$$
$$- \frac{1}{4}\int_0^t \sigma\sigma'(X)\, d[\frac{\partial H}{\partial x}(\xi, X), \xi, \xi] + \int_0^t \frac{\partial H}{\partial x}(\xi_s, X_s) b(s, X_s)\, dV_s.$$
$$= \int_0^t \frac{\partial H}{\partial x}(\xi, X)\sigma(X)\, d^\circ \xi - \frac{1}{4}\int_0^t \sigma\sigma'(X)\frac{\partial^2 H}{\partial x^2}(\xi, X)\, d[X, \xi, \xi]$$
$$- \frac{1}{4}\int_0^t \sigma\sigma'(X)\frac{\partial^2 H}{\partial r \partial x}(\xi, X)\, d[\xi, \xi, \xi] + \int_0^t \frac{\partial H}{\partial x}(\xi_s, X_s) b(s, X_s)\, dV_s,$$

where in the second equality we use Proposition 2.8.

Using Remark 4.2 1) and the fact that (ξ, X) a 3-symmetric vector Itô process we get

$$d[\xi, \xi, X]_s = \sigma(X_s) d[\xi, \xi, \xi]_s$$
$$d[\xi, X, X]_s = \sigma^2(X_s) d[\xi, \xi, \xi]_s$$
$$d[X, X, X]_s = \sigma^3(X_s) d[\xi, \xi, \xi]_s.$$

So, identities (4.6), (4.9), (4.10) and (4.11) show that

$$[(\xi, X), \Delta H(\xi, X), (\xi, X)^T](t) =$$
$$\int_0^t \big(2\sigma\sigma'^2(X_s) - \sigma^2\sigma''(X_s)\big)\frac{\partial H}{\partial x}(\xi_s, X_s) d[\xi, \xi, \xi]_s.$$

and

$$\int_0^t \frac{\partial H}{\partial x}(\xi, X)\, d^\circ X = \int_0^t \frac{\partial H}{\partial x}(\xi_s, X_s) b(s, X_s)\, dV_s$$
$$- \int_0^t \frac{\partial H}{\partial r}(\xi, X) d^\circ \xi + \frac{1}{4}\int_0^t \sigma\sigma'^2(X)\frac{\partial H}{\partial x}(\xi, X)\, d[\xi, \xi, \xi].$$

This gives that

$$Y_t = Y_0 + \int_0^t \frac{\partial H}{\partial x}(\xi_s, X_s)\, b(s, X_s) dV_s$$
$$+ \frac{1}{12}\int_0^t \big(\sigma\sigma'^2(X_s) + \sigma^2\sigma''(X_s)\big)\frac{\partial H}{\partial x}(\xi_s, X_s) d[\xi, \xi, \xi]_s.$$

X being equal to $F(\xi, Y)$, Y is then a solution of (4.5).

The proof will be concluded by the following remark. ■

Remark 4.4 *Equation* (4.5) *is in fact a random differential equation which is driven by bounded variation processes and it has a unique solution.*

In fact, in the proof of Proposition 5.3 of [27] there are elements to prove that,

$$(t,x) \longmapsto \Big(\sigma\sigma'^2(F(\xi_t,x)) + \sigma^2\sigma''(F(\xi_t,x))\Big)\frac{\partial H}{\partial x}(\xi_t, F(\xi_t,x))$$
$$(t,x) \longmapsto b\big(t, F(\xi_t,x)\big)\frac{\partial H}{\partial x}(\xi_t, F(\xi_t,x))$$

belong, ω a.s., to the LL class constituted by locally Lipschitz and of linear growth functions. Therefore equation (4.5) has exactly one solution because of classical propositions of [23].

We can now state the most important result of this section.

Theorem 4.5 *Let ξ (resp. V) be a finite strong cubic (resp. bounded) variation real process. Let $\sigma : \mathbb{R} \to \mathbb{R}$ (resp. $b : [0,1] \times \mathbb{R} \to \mathbb{R}$) is of class $C^3(\mathbb{R})$ (resp. continuous), such that, σ' σ'' are bounded. We suppose moreover that b is locally Lipschitz with linear growth (uniformly in t, with respect to the second variable x), α any random variable. Let Y be the unique solution to* (4.5) *and F the flow generated by σ. Then $X = F(\xi, Y)$ is the unique solution of* (4.1).

Proof. Recall that Y is a bounded variation continuous process. We have to prove that $X = F(\xi, Y)$ solves (4.1), i.e. (i), (ii) and (iii) of Definition 4.1.

(i) $X_0 = F(\xi_0, H(\xi_0, \alpha)) = \alpha$ because of (4.4).

(ii) Using Proposition 3.15 and Lemma 3.14, we get that $F(\xi, Y)$ is a 3-symmetric vector Itô process.

(iii) Let $\psi \in C^\infty(\mathbb{R}^2)$ and $Z = \psi(X, \xi)$. We apply Proposition 3.10 and Remark 3.11 1) to get

$$\int_0^t Z d^\circ X = \int_0^t Z\frac{\partial F}{\partial r}(\xi, Y)d^\circ\xi + \int_0^t Z\,\frac{\partial F}{\partial x}(\xi, Y)dY$$
$$- \frac{1}{4}\int_0^t \frac{\partial^2 F}{\partial r^2}(\xi, Y)d[Z,\xi,\xi] - \frac{1}{12}\int_0^t Z\frac{\partial^3 F}{\partial r^3}(\xi, Y)d[\xi,\xi,\xi].$$

We remark that $\int_0^t Z \frac{\partial F}{\partial r}(\xi, Y) d^\circ \xi$ exists since (Z, ξ, Y) is a 3-symmetric vector Itô process by Lemma 3.14.

We need a few relations involving F and H. Since $F(r, H(r, x)) = x$, taking the derivative with respect to x we obtain

$$\frac{\partial F}{\partial x}(r, H(r, x)) \frac{\partial H}{\partial x}(r, x) = 1. \tag{4.12}$$

We apply the operator $\frac{\partial}{\partial r}$ to the first identity of (4.3):

$$\frac{\partial^2 F}{\partial r^2} = \sigma\sigma'(F(r, x)). \tag{4.13}$$

$$\frac{\partial^3 F}{\partial r^3} = \sigma\sigma'^2(F(r, x)) + \sigma^2\sigma''(F(r, x)). \tag{4.14}$$

Using (4.3), (4.12), (4.13), (4.14), and the fact that Y is the solution of (4.5), we have

$$\begin{aligned} \int_0^t Z_s d^\circ X_s = \int_0^t Z_s\, \sigma(X_s)\, d^\circ \xi_s + \int_0^t Z_s\, b(s, X_s)\, dV_s \\ -\frac{1}{4} \int_0^t \sigma\sigma'(X)\, d[Z, \xi, \xi]. \end{aligned}$$

This implies that X solves (4.1). ∎

4.1 On the integral equation

The definition we gave, of a solution to the differential problem (4.1), may appear unusual. One may ask if the following integral problem is well stated:

$$X(t) = \alpha + \int_0^t \sigma(X)\, d^\circ \xi + \int_0^t b(s, X_s)\, dV_s. \tag{4.15}$$

For this integral equation, it is hard to imagine that uniqueness will hold in the class of all continuous processes, even adapted if α is $\mathcal{F}_0-$measurable. However uniqueness (and existence) will be shown in the class $\mathcal{V}(\xi)$, defined in Remark 3.17, of processes

$$X_t = X_0 + \int_0^t \varphi(\xi_s, V_s^1, \cdots, V_s^n)\, d^\circ \xi_s + V_t^0,$$

where $\varphi = \varphi(r, v) \in C^{2,1}(\mathbb{R}^{1+n})$ for some positive integer n and bounded variation processes $V^1, \cdots, V^n$, V^0.

Proposition 4.6 *Integral equation* (4.15) *has a unique solution in the class* $\mathcal{V}(\xi)$*; this one coincides with the solution of differential problem* (4.1).

Proof. Existence is provided by Theorem 4.5 setting $Z = 1$. In order to prove uniqueness we will show that a solution to (4.15) in $\mathcal{V}(\xi)$ will solve problem (4.1).

Let X be such a solution. Clearly we have $X_0 = \alpha$. Using Proposition 3.15, Remark 3.19, Lemma 3.14 and Remark 3.17, it follows that (X, ξ) is 3-symmetric vector Itô process. It remains to prove (iii) of Definition 4.1.

Let $Z = \psi(X, \xi)$, where ψ a C^∞ function.

$$\begin{aligned}\int_0^t Z\, d^\circ X &= \int_0^t Z_s\, d^\circ\Big(\alpha + \int_0^s \sigma(X)\, d^\circ\xi + \int_0^s b(u, X_u)\, dV_u\Big)\\ &= \int_0^t Z_s\, d^\circ\Big(\int_0^s \sigma(X)\, d^\circ\xi\Big) + \int_0^t Z_s\, b(s, X_s)\, dV_s.\end{aligned}$$

We observe in fact that (ξ, Z, X) is a 3-symmetric vector Itô process; Remark 3.17 says that X is of the form $\psi(\xi, V^1, \cdots, V^m)$; therefore Lemma 3.18 tells that,

$$\begin{aligned}&\int_0^t Z_s\, d^\circ\Big(\int_0^s \sigma(X)\, d^\circ\xi\Big)\\ &\qquad= \int_0^t Z\sigma(X)\, d^\circ\xi - \frac{1}{4}\int_0^t \frac{\partial\psi}{\partial r}(\xi, V^1, \cdots, V^m)\sigma'(X)\, d[Z, \xi, \xi]\\ &\qquad= \int_0^t Z\sigma(X)\, d^\circ\xi - \frac{1}{4}\int_0^t \sigma'(X)\, d[Z, X, \xi],\end{aligned}$$

where in the second equality we use Proposition 2.7. Using the 3-symmetric vector Itô properties and (4.15), we get that

$$d[Z, X, \xi] = \sigma(X) d[Z, \xi, \xi].$$

So,

$$\begin{aligned}\int_0^t Z\, d^\circ X = \int_0^t Z\, \sigma(X)\, d^\circ\xi &- \frac{1}{4}\int_0^t \sigma\sigma'(X)\, d[Z, \xi, \xi]\\ &+ \int_0^t Z_s b(s, X_s)\, dV_s.\end{aligned}$$

(iii) is then established. ∎

4.2 Example : Linear SDE

Let ξ be a process with finite strong cubic variation, and α any random variable. We consider the following equation:

$$\begin{cases} dX_t = X_t \circ d\xi_t \\ X_0 = \alpha. \end{cases} \tag{4.16}$$

In this case the flow is given by $F(r,x) = x\, exp(r)$, and $H(r,x) = x\, exp(-r)$. The ordinary differential equation (4.5) becomes,

$$Y_t = \alpha\, exp(-\xi_0) + \frac{1}{12}\int_0^t Y_s\, d[\xi,\xi,\xi]_s,$$

whose unique solution is the following process

$$Y_t = \alpha\, exp(-\xi_0)\, exp\big(\frac{1}{12}[\xi,\xi,\xi]_t\big), \quad t \in [0,1].$$

This gives, using the result of Theorem 4.5, that the unique solution of (4.16) is given by

$$X_t = \alpha\, exp(-\xi_0)\, exp\big(\frac{1}{12}[\xi,\xi,\xi](t) + \xi(t)\big), \quad t \in [0,1].$$

5 Appendix

Proof (of Proposition 2.13).
We will proceed by recurrence. We will start with $n = 3$, however the case $n = 2$ may be considered done by (2.10). Using Itô-Stratonovich calculus we get

$$\begin{aligned} I_3^\circ(f)(t) &= \int_0^t \int_0^{s_3} \int_0^{s_2} f(s_1,s_2,s_3)\, d^\circ M_{s_1}\, d^\circ M_{s_2}\, d^\circ M_{s_3} \\ &= \int_0^t \int_0^{s_3} \int_0^{s_2} f(s_1,s_2,s_3)\, dM_{s_1}\, d^\circ M_{s_2}\, d^\circ M_{s_3} \\ &:= I_1 + I_2, \end{aligned}$$

where,

$$\begin{aligned} I_1 &= \int_0^t \Big(\int_0^{s_3} \int_0^{s_2} f(s_1,s_2,s_3)\, dM_{s_1}\, dM_{s_2} \Big)\, d^\circ M_{s_3} \\ I_2 &= \frac{1}{2}\int_0^t \Big(\Big[\int_0^{\cdot} f(s_1,\cdot,s_3)\, dM_{s_1}\, ,\, M\Big]_{s_3}\Big) d^\circ M_{s_3}. \end{aligned}$$

Now

$$I_1 = \int_0^t \int_0^{s_3} \int_0^{s_2} f(s_1, s_2, s_3)\, dM_{s_1}\, dM_{s_2}\, dM_{s_3} + \frac{1}{2}\Big[\int_0^{\cdot} \int_0^{s_2} f(s_1, s_2, \cdot)\, dM_{s_1}\, dM_{s_2}\, ,\, M\Big](t)$$

Using Lemma 2.10, with $G(t, s_2) = \int_0^{s_2} f(s_1, s_2, t) dM_{s_1}$, we get

$$I_1 = \int_0^t \int_0^{s_3} \int_0^{s_2} f(s_1, s_2, s_3)\, dM_{s_1}\, dM_{s_2}\, dM_{s_3} + \frac{1}{2} \int_0^t \int_0^{s_2} f(s_1, s_2, s_2)\, dM_{s_1}\, d[M]_{s_2};$$

moreover, using again the same Lemma with $G(t, s_1) = f(s_1, t, s_3)$, we get

$$\Big[\int_0^{\cdot} f(s_1, \cdot, s_3)\, dM_{s_1}\, ,\, M\Big]_t = \int_0^t f(s_1, s_1, s_3)\, d[M]_{s_1}.$$

This implies that

$$\begin{aligned} I_2 =& \frac{1}{2} \int_0^t \int_0^{s_3} f(s_1, s_1, s_3)\, d[M]_{s_1}\, d^{\circ} M_{s_3} \\ =& \frac{1}{2} \int_0^t \int_0^{s_2} f(s_1, s_1, s_2)\, d[M]_{s_1}\, dM_{s_2}. \end{aligned}$$

This gives

$$\begin{aligned} I_3^{\circ}(f)(t) = \int_0^t \int_0^{s_3} \int_0^{s_2} & f(s_1, s_2, s_3) dM_{s_1} dM_{s_2} dM_{s_3} \\ & + \frac{1}{2} \int_0^t \int_0^{s_2} f(s_1, s_2, s_2) dM_{s_1} d[M]_{s_2} \\ & + \frac{1}{2} \int_0^t \int_0^{s_2} f(s_1, s_1, s_2) d[M]_{s_1} dM_{s_2}. \end{aligned}$$

Now let $n \geq 3$, and suppose that the statement of Proposition 2.13 holds for all $m \leq n$. We have

$$\begin{aligned} I_{n+1}^{\circ}(f)(t) =& \int_0^t \int_0^{s_{n+1}} \cdots \int_0^{s_2} f(s_1, \cdots, s_{n+1}) d^{\circ} M_{s_1} \cdots d^{\circ} M_{s_{n+1}} \\ =& \int_0^t \Big(\int_0^{s_{n+1}} \cdots \int_0^{s_2} f(s_1, \cdots, s_{n+1}) d^{\circ} M_{s_1} \cdots d^{\circ} M_{s_n} \Big) dM_{s_{n+1}} \\ & + \frac{1}{2} \Big[\int_0^{\cdot} \int_0^{s_n} \cdots \int_0^{s_2} f(s_1, \cdots, s_n, \cdot) d^{\circ} M_{s_1} \cdots d^{\circ} M_{s_{n-1}} dM_{s_n}, M\Big]_t \\ :=& I_1 + I_2, \end{aligned}$$

where s_n plays the role of t.

Using induction hypothesis for $m = n$, and renaming the variable s_{n+1} with s_{n+1-k}, we have

$$I_1 = \sum_{k=0}^{[\frac{n}{2}]} \frac{1}{2^k} \sum_{\sigma \in \Sigma_k^n} \int_0^t \int_0^{s_{n+1-k}} \int_0^{s_{n-k}} \cdots \int_0^{s_2}$$

$$f\big(s_{\sigma^{-1}(1)}, s_{\sigma^{-1}(1)}, \cdots, s_{\sigma^{-1}(k)}, s_{\sigma^{-1}(k)},$$
$$s_{\sigma^{-1}(k+1)}, s_{\sigma^{-1}(k+2)}, \cdots, s_{\sigma^{-1}(n-k)}, s_{n+1-k}\big)$$
$$d\sigma_M^1(s_1) \cdots d\sigma_M^{n-k}(s_{n-k}) dM_{s_{n+1-k}};$$

$\sigma \in \Sigma_k^n$ is extended trivially to $\sigma \in \Sigma_k^{n+1}$ setting $\sigma^{n+1-k} = n+1-k$.

Using Lemma 2.10 with

$$G(t, s_n) = \int_0^{s_n} \cdots \int_0^{s_2} f(s_1, \cdots, s_n, t) d^\circ M_{s_1} \cdots d^\circ M_{s_{n-1}}$$

we get

$$I_2 = \frac{1}{2} \int_0^t \left(\int_0^{s_n} \cdots \int_0^{s_2} f(s_1, \cdots, s_n, s_n) d^\circ M_{s_1} \cdots d^\circ M_{s_{n-1}} \right) d[M]_{s_n}.$$

Using this time the induction hypothesis for $m = n-1$ and renaming the variable s_n with s_{n-l}, we obtain

$$I_2 = \sum_{l=0}^{[\frac{n-1}{2}]} \frac{1}{2^{l+1}} \sum_{\sigma \in \Sigma_l^{n-1}} \int_0^t \int_0^{s_{n-l}} \int_0^{s_{(n-1)-l}} \cdots \int_0^{s_2}$$

$$f\big(s_{\sigma^{-1}(1)}, s_{\sigma^{-1}(1)}, \cdots, s_{\sigma^{-1}(l)}, s_{\sigma^{-1}(l)},$$
$$s_{\sigma^{-1}(l+1)}, s_{\sigma^{-1}(l+2)}, \cdots, s_{\sigma^{-1}(n-1-l)}, s_{n-l}, s_{n-l}\big)$$
$$d\sigma_M^1(s_1) \cdots d\sigma_M^{n-1-l}(s_{n-1-l}) d[M]_{s_{n-l}}.$$

We change the variable l with $k = l+1$, so $1 \le k \le [\frac{n-1}{2}] + 1 = [\frac{n+1}{2}]$ which gives that

$$I_2 = \sum_{k=1}^{[\frac{n+1}{2}]} \frac{1}{2^k} \sum_{\sigma\in\Sigma_{k-1}^{n-1}} \int_0^t \int_0^{s_{(n+1)-k}} \cdots \int_0^{s_2}$$

$$f\big(s_{\sigma^{-1}(1)}, s_{\sigma^{-1}(1)}, \cdots, s_{\sigma^{-1}(k-1)}, s_{\sigma^{-1}(k-1)},$$
$$s_{\sigma^{-1}(k)}, s_{\sigma^{-1}(k+1)}, \cdots, s_{\sigma^{-1}(n-k)}, s_{n+1-k}, s_{n+1-k}\big)$$
$$d\sigma_M^1(s_1)\cdots d\sigma_M^{(n-1)-(k-1)}(s_{n-k}) d[M]_{s_{(n+1)-k}}.$$

For given $\sigma \in \Sigma_{k-1}^{n-1}$, we define $\tilde{\sigma} \in \Sigma_k^{n+1}$ by

$$\tilde{\sigma}^i = \begin{cases} \sigma^i & \text{for} \quad i = 1, 2, \cdots n-k \\ k & \text{for} \quad i = n+1-k. \end{cases}$$

Recall that we have to prove that,

$$I_1 + I_2 = \sum_{k=0}^{[\frac{n+1}{2}]} \frac{1}{2^k} \sum_{\sigma\in\Sigma_k^{n+1}} \int_0^t \int_0^{s_{(n+1)-k}} \cdots \int_0^{s_2}$$

$$f\big(s_{\sigma^{-1}(1)}, s_{\sigma^{-1}(1)}, \cdots, s_{\sigma^{-1}(k)}, s_{\sigma^{-1}(k)},$$
$$s_{\sigma^{-1}(k+1)}, s_{\sigma^{-1}(k+2)}, \cdots, s_{\sigma^{-1}((n+1)-k)}\big)$$
$$d\sigma_M^1(s_1)\cdots d\sigma_M^{n-k}(s_{n-k}) d\sigma_M^{(n+1)-k}(s_{n+1-k}).$$

To obtain it we have just to see that,

$$P_0^{n+1}(M) = \Big\{\Big(\underbrace{M, M, \cdots, M}_{(n+1)\ times}\Big)\Big\} = \Big\{\sigma\Big(\underbrace{M, M, \cdots, M}_{(n+1)\ times}\Big)\Big\},$$

where $\sigma^i = i$, $i = 1, \cdots, n+1$. For all $1 \le k \le [\frac{n+1}{2}]$, we have

$$P_k^{n+1}(M) = \Big\{\Big(\sigma(\underbrace{[M], \cdots, [M]}_{k\ times}, \underbrace{M, \cdots, M}_{n-2k\ times}), M\Big);\quad \sigma \in \Sigma_k^n\Big\}$$
$$\bigcup \Big\{\Big(\sigma(\underbrace{[M], \cdots, [M]}_{(k-1)\ times}, \underbrace{M, \cdots, M}_{(n-1)-2(k-1)\ times}), [M]\Big);\ \sigma \in \Sigma_{k-1}^{n-1}\Big\}$$
$$= \Big\{\big(\sigma_M, M\big); \sigma \in \Sigma_k^n\Big\} \bigcup \Big\{\big(\sigma_M, [M]\big);\ \sigma \in \Sigma_{k-1}^{n-1}\Big\}$$

where previous union is disjoint. ■

Remark *We recall that the cardinality of $P_k^n(M)$ (resp. $P_{k-1}^{n-1}(M)$) equals C_{n-k}^k (resp. C_{n-k}^{k-1}). Therefore the cardinality of $P_k^{n+1}(M)$ is C_{n+1-k}^k as expected by combinatorial calculus.*

Proof (of Proposition 2.15).

It remains to prove the following property:

for all $n \geq 3$, and every $k = 1, \cdots, [\frac{n}{2}]$,

$$\begin{aligned}
L_k^n(t) &:= \sum_{\sigma \in \Sigma_k^n} \int_0^t \int_0^{s_{n-k}} \cdots \int_0^{s_2} \\
& f\big(s_{\sigma^{-1}(1)}, s_{\sigma^{-1}(1)}, \cdots, s_{\sigma^{-1}(k)}, s_{\sigma^{-1}(k)}, s_{\sigma^{-1}(k+1)}, \cdots, s_{\sigma^{-1}(n-k)}\big) \\
& \qquad d\sigma_W^1(s_1) \cdots d\sigma_W^{n-k}(s_{n-k}) \\
&= \int_0^t \int_0^{s_{n-k}} \cdots \int_0^{s_{k+2}} \int_0^{\mathbf{t}} \int_0^{s_k} \cdots \int_0^{s_2} \\
& f\big(s_1, s_1, \cdots, s_k, s_k, s_{k+1}, s_{k+2}, \cdots, s_{n-k}\big) \\
& \qquad ds_1 \cdots ds_k dW_{s_{k+1}} \cdots dW_{s_{n-k}}.
\end{aligned}$$

We will prove by induction. We start proving the case $n = 3$. We have,

$$\begin{aligned}
P_0^3(W) &= \big\{ \big(W, W, W\big) \big\} \\
P_1^3(W) &= \big\{ \big([W], W\big), \big(W, [W]\big) \big\}.
\end{aligned}$$

This gives

$$L_0^3(t) = \int_0^t \int_0^{s_3} \int_0^{s_2} f(s_1, s_2, s_3)\, dW_{s_1} dW_{s_2} dW_{s_3},$$

and

$$\begin{aligned}
L_1^3(t) &= \int_0^t \int_0^{s_2} f(s_1, s_1, s_2)\, ds_1 dW_{s_2} + \int_0^t \int_0^{s_2} f(s_1, s_2, s_2)\, dW_{s_1} ds_2 \\
&= \int_0^t \int_0^{s_2} f(s_1, s_1, s_2)\, ds_1 dW_{s_2} + \int_0^t \int_{s_1}^t f(s_1, s_2, s_2)\, ds_2 dW_{s_1} \\
&= \int_0^t \int_0^{s_2} f(s_1, s_1, s_2)\, ds_1 dW_{s_2} + \int_0^t \int_{s_2}^t f(s_1, s_1, s_2)\, ds_1 dW_{s_2} \\
&= \int_0^t \int_0^t f(s_1, s_1, s_2)\, ds_1 dW_{s_2}.
\end{aligned}$$

This means that the Proposition is true for $n = 3$.

Now let $n \geq 3$, and suppose that the statement of the Proposition holds for all $m \leq n$. Recall that, for all $1 \leq k \leq [\frac{n+1}{2}]$, we have

$$P_k^{n+1}(W) = \Big\{ \big(\sigma_W, W\big) \,; \sigma \in \Sigma_k^n \Big\} \bigcup \Big\{ \big(\sigma_W, [W]\big) \,;\; \sigma \in \Sigma_{k-1}^{n-1} \Big\},$$

where the union is disjoint. Therefore

$$L_k^{n+1}(t) = I_1 + I_2,$$

where

$$I_1 = \sum_{\sigma \in \Sigma_k^n} \int_0^t \Bigg(\int_0^s \int_0^{s_{n-k}} \cdots \int_0^{s_2}$$
$$f\big(s_{\sigma^{-1}(1)}, s_{\sigma^{-1}(1)}, \cdots, s_{\sigma^{-1}(k)}, s_{\sigma^{-1}(k)}, s_{\sigma^{-1}(k+1)}, \cdots, s_{\sigma^{-1}(n-k)}, s\big)$$
$$d\sigma_W^1(s_1) \cdots d\sigma_W^{n-k}(s_{n-k}) \Bigg) dW_s.$$

and

$$I_2 = \sum_{\sigma \in \Sigma_{k-1}^{n-1}} \int_0^t \Bigg(\int_0^s \int_0^{s_{n-k}} \cdots \int_0^{s_2}$$
$$f\big(s_{\sigma^{-1}(1)}, \cdots, s_{\sigma^{-1}(k-1)}, s_{\sigma^{-1}(k-1)}, s_{\sigma^{-1}(k)}, \cdots, s_{\sigma^{-1}(n-k)}, s, s\big)$$
$$d\sigma_W^1(s_1) \cdots d\sigma_W^{n-k}(s_{n-k}) \Bigg) ds.$$

Using the hypothesis for $m = n$, and renaming s with s_{n+1-k}, we have

$$I_1 = \int_0^t \int_0^{s_{n+1-k}} \int_0^{s_{n-k}} \cdots \int_0^{s_{k+2}} \int_0^{\mathbf{s_{n+1-k}}} \int_0^{s_k} \cdots \int_0^{s_2}$$
$$f\big(s_1, s_1, \cdots, s_k, s_k, s_{k+1}, s_{k+2}, \cdots, s_{n+1-k}\big)$$
$$ds_1 \cdots ds_k dW_{s_{k+1}} \cdots dW_{s_{n-k}} dW_{s_{n+1-k}}.$$

Using now the induction hypothesis for $m = n - 1$, we have,

$$I_2 = \int_0^t \int_0^s \int_0^{s_{n-k}} \cdots \int_0^{s_{k+1}} \int_0^{\mathbf{s}} \int_0^{s_{k-1}} \cdots \int_0^{s_2} f\big(s_1, s_1, \cdots, s_{k-1}, s_{k-1}, s_k, s_{k+1}, \cdots, s_{n-k}, s, s\big) \, ds_1 \cdots ds_{k-1} dW_{s_k} \cdots dW_{s_{n-k}} ds$$

$$= \int_0^t \int_{s_{n-k}}^t \int_0^{s_{n-k}} \cdots \int_0^{s_{k+1}} \int_0^{\mathbf{s}} \int_0^{s_{k-1}} \cdots \int_0^{s_2} f\big(s_1, s_1, \cdots, s_{k-1}, s_{k-1}, s_k, s_{k+1}, \cdots, s_{n-k}, s, s\big) \, ds_1 \cdots ds_{k-1} dW_{s_k} \cdots dW_{s_{n-k-1}} ds dW_{s_{n-k}}$$

$$= \int_0^t \int_0^{s_{n-k}} \cdots \int_0^{s_{k+1}} \int_{s_{n-k}}^t \int_0^{\mathbf{s}} \int_0^{s_{k-1}} \cdots \int_0^{s_2} f\big(s_1, s_1, \cdots, s_{k-1}, s_{k-1}, s_k, s_{k+1}, \cdots, s_{n-k}, s, s\big) \, ds_1 \cdots ds_{k-1} \, ds \, dW_{s_k} \cdots dW_{s_{n-k-1}} dW_{s_{n-k}}.$$

Renaming s with s_k, s_k with s_{k+1}, $\cdots$, s_{n-k} with s_{n+1-k}, we obtain

$$I_1 = \int_0^t \int_0^{s_{n+1-k}} \cdots \int_0^{s_{k+2}} \int_{\mathbf{s_{n+1-k}}}^t \int_0^{s_k} \int_0^{s_{k-1}} \cdots \int_0^{s_2} f\big(s_1, s_1, \cdots, s_k, s_k, s_{k+1}, s_{k+2}, \cdots, s_{n+1-k}\big) \, ds_1 \cdots \, ds_k \, dW_{s_{k+1}} \cdots dW_{s_{n+1-k}}.$$

This gives that,

$$I_1 + I_2 = \int_0^t \int_0^{s_{n+1-k}} \cdots \int_0^{s_{k+2}} \int_0^{\mathbf{t}} \int_0^{s_k} \int_0^{s_{k-1}} \cdots \int_0^{s_2} f\big(s_1, s_1, \cdots, s_k, s_k, s_{k+1}, s_{k+2}, \cdots, s_{n+1-k}\big) \, ds_1 \cdots \, ds_k \, dW_{s_{k+1}} \cdots dW_{s_{n+1-k}}.$$

$\blacksquare$

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