Generalized Integration and Stochastic ODEs. Part II

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Abstract

The generalized stochastic calculus developed in part I (stochastic forward integrals for processes more general than semimartingales, Itô-Wentzell formula and covariation formulae, stochastic equations driven by finite quadratic variation processes and semimartingales) is applied here to treat in a new way uniqueness and regular dependence on parameters for stochastic equations with non regular drift.

Key words:
Stochastic forward integration, stochastic differential equation with non-regular drift, regular dependence on parameters.

AMS-classification: 60H05, 60H10

1 Introduction

This paper follow Part I, [7], in which we developed a generalized stochastic integration involving Dirichlet processes and more general finite quadratic variation processes. In the recent years, many authors have investigated such an integration. Among the most recent references we find for instance [17], [8], [18], [14].

As announced in [7], we consider the one-dimensional stochastic differential equation with non regular drift b:

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \quad \text{for } t \in [0, T] \\ X_0 = x \end{cases} \tag{1}$$

where $b, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfy the assumptions listed in the sequel, and W_t is a one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}_T, P)$, adapted to the filtration $\mathcal{F} = (\mathcal{F}_t)$. Even if b is only continuous and locally bounded (with linear growth to prevent explosion), it is well known that for every initial condition $x \in L^2(\Omega, \mathcal{F}_0, P)$ there exists a unique strong solution X_t, with $X_\cdot \in L^2(\Omega, \mathcal{F}, P; C([0, T]; \mathbb{R}))$ for every $T \geq 0$.

On this subject, especially on strong existence and uniqueness, there is a large literature, we recall in particular [2], [3], [4], [5], [13], [15], [16], [20], [22], [23] or [25].

The aim of this paper is to analyze the dependence of solutions on various perturbations. In particular, we obtain a *new proof of strong uniqueness* (obtained by direct computations on the difference of two solutions, like in the case of Lipschitz continuous b, in contrast to the indirect proofs known in the literature), a *continuously differentiable stochastic flow*, a rigorous way to interpret (and deal with) the *variational equation*

$$\begin{cases} dV_t = \frac{\partial b}{\partial x}(t, X_t)V_t dt + \frac{\partial \sigma}{\partial x}(t, X_t)V_t dW_t \quad \text{for } t \in [0, T] \\ V_0 = v \end{cases} \tag{2}$$

(notice that the drift is a distribution), the continuous dependence with respect to a change of the drift coefficient.

Every time we try to deal with the difference between two solutions or with the derivative of the solution with respect to some parameter (this happens in all the problems mentioned above), we have to deal with an integral of the form

$$\int_0^t \frac{\partial b}{\partial x}(s, X_s)\, ds$$

or with similar expressions. This integral requires a particular definition, since b is not differentiable in x. Moreover, to deal with the variational equation (2), we need to give a meaning to the more difficult integral

$$\int_0^t \frac{\partial b}{\partial x}(s, X_s) V_s \, ds \tag{3}$$

for suitable processes V_t. If we set

$$A_t^{b,X} := \int_0^t \frac{\partial b}{\partial x}(s, X_s) ds,$$

we interpret (3) as

$$\int_0^t V_s \, dA_s^{b,X}. \tag{4}$$

Therefore the plan of the paper is the following one. We first define the stochastic process $A_t^{b,X}$. This step requires a considerable effort, with a long preliminar on a certain parabolic PDE. The general idea is similar to the definition of the local time by Tanaka formula, and it has been used in other contexts for different purposes (see for instance [24], [1]). It turns out that $A_t^{b,X}$ can be defined as a natural extension of the case when b is regular, but it is not bounded variation or a semimartingale, in general. It is a process with zero quadratic variation, hence a Dirichlet process. Meanwhile we give also the definition of another process A_t^{b,X^0,X^1} useful for the sequel, and in section 6 we give the definition of $A_t^{b,X}$ under a second class of assumptions.

The second step is the definition of the stochastic integral (4). Here we rely on the theory developed in part I [7], where integrals of the form (4) have been defined for a large class of integrands V. According to the theory of part I, these integrals allow us to give a meaning and a unique solution to equations like (2). Then we use all these facts to prove the various stability results for the solutions of (1) with respect to perturbations.

Let us stress (see also the introduction to part I) that one of our main motivations is to show that properties of uniqueness and regular dependence on initial conditions can be treated by methods similar to the case of Lipschitz b, i.e. based on estimates on the difference of solutions. We have in mind open problems for stochastic PDE's where such an approach would be natural, and where more classical probabilistic methods for uniqueness do not work. But until now our results are very restricted to the one-dimensional case.

Let us also put in evidence the result of differentiability with respect to the initial conditions, perhaps unexpected when b is not regular. This remark applies as well to the direct analysis of the variational equation (2).

1.1 Notations

We set $Q_T = [0,T] \times \mathbb{R}$. We say that a function $\alpha(t,x)$, $(t,x) \in Q_T$, satisfies condition (C1) if there exists a constant $C > 0$ such that

$$|\alpha(t,x)| \leq C\,(1+|x|) \quad \forall (t,x) \in Q_T.$$

Similarly, $\alpha(t,x)$ satisfies condition (C2) with Hölder exponent $\varepsilon \in (0,1)$ if there exists a constant $C > 0$ such that

$$|\alpha(t,x) - \alpha(t',x')| \leq C\left(|t-t'|^{\frac{\varepsilon}{2}} + |x-x'|^{\varepsilon} \wedge 1\right)(|x|+|x'|) \quad \forall (t,x) \in Q_T.$$

Similarly, $\alpha(t,x)$ satisfies condition (C3) if there exist constants $C, \omega > 0$ such that

$$|\alpha(t,x)| \leq C e^{\omega |x|} \quad \forall (t,x) \in Q_T$$

and satisfies condition (C4) with Hölder exponent $\varepsilon \in (0,1)$ if there exist constants $C, \omega > 0$ such that

$$|\alpha(t,x) - \alpha(t',x')| \leq C\left(|t-t'|^{\frac{\varepsilon}{2}} + |x-x'|^{\varepsilon} \wedge 1\right) e^{\omega(|x|+|x'|)} \quad \forall (t,x) \in Q_T.$$

In plain words, (C1) is linear growth, (C2) is Hölder continuity with linear growth at infinity, (C3) is exponential growth, (C4) is Hölder continuity with exponential growth at infinity.

Remark 1 *(C2)⇒(C1), (C4)⇒(C3), (C3)⇒(C1), (C4)⇒(C2).*

If k and m are two non-negative integers, we denote by $C^{k,m}$ the space of continuous functions $\alpha(t,x)$ having continuous derivatives $\frac{\partial^k \alpha}{\partial t^k}$ and $\frac{\partial^m \alpha}{\partial x^m}$ (a priori we do not require properties of the mixed derivatives). If $\beta, \gamma \in (0,1)$, we denote by $C^{\beta,\gamma}$ the space of continuous functions $\alpha(t,x)$ that are β-Hölder continuous in t uniformly in x on compact sets, and γ-Hölder continuous in

x on compact sets, uniformly in t. Then we denote by $C^{k+\beta,m+\gamma}$ the space of continuous functions $\alpha(t,x)$ with $\frac{\partial^k\alpha}{\partial t^k}$, $\frac{\partial^m\alpha}{\partial x^m}\in C^{\beta,\gamma}$.

For $\varepsilon\in(0,1)$ we denote by $C_{lin}^{\frac{\varepsilon}{2},\varepsilon}$ the space of $\alpha\in C^{0,0}$ satisfying condition (C2) with Hölder exponent ε, for some constant $C>0$. We denote by $C_{lin}^{\frac{\varepsilon}{2},1+\varepsilon}$ the space of $\alpha\in C_{lin}^{\frac{\varepsilon}{2},\varepsilon}\cap C^{0,1}$ such that $\frac{\partial\alpha}{\partial x}\in C_{lin}^{\frac{\varepsilon}{2},\varepsilon}$. We denote by $C_{lin}^{1+\frac{\varepsilon}{2},1+\varepsilon}$ the space of $\alpha\in C_{lin}^{\frac{\varepsilon}{2},\varepsilon}\cap C^{1,1}$ such that $\frac{\partial\alpha}{\partial t},\frac{\partial\alpha}{\partial x}\in C_{lin}^{\frac{\varepsilon}{2},\varepsilon}$. The spaces $C_{\exp}^{\frac{\varepsilon}{2},\varepsilon}$ and $C_{\exp}^{\frac{\varepsilon}{2},1+\varepsilon}$ are defined similarly, replacing (C2) with (C4).

If $\alpha_n(t,x)$, $\alpha(t,x)\in C^{\frac{\varepsilon}{2},\varepsilon}$ (in particular if they belong to $C_{lin}^{\frac{\varepsilon}{2},\varepsilon}$ or $C_{\exp}^{\frac{\varepsilon}{2},\varepsilon}$), we say that

$$\alpha_n \text{ converges to } \alpha \text{ in the } \varepsilon-\text{Hölder norm uniformly over compact sets}$$

if for every $r>0$ we have

$$\begin{aligned}
&\left|(\alpha_n(t,x)-\alpha(t,x))-(\alpha_n(t',x')-\alpha(t',x'))\right| \\
\leq\ & \delta_n^{(r)}\left(|t-t'|^{\frac{\varepsilon}{2}}+|x-x'|^{\varepsilon}\right)\quad \forall\,|x|,|x'|\leq r,\ \forall t,t'\in[0,T],
\end{aligned}$$

and for some sequence $\delta_n^{(r)}\to 0$ as $n\to\infty$.

1.2 Hypotheses

In the sequel we shall always assume that for some $\varepsilon\in(0,1)$

$$\begin{aligned}
b\ \in\ & C_{lin}^{\frac{\varepsilon}{2},\varepsilon},\quad \sigma\in C_{lin}^{1+\frac{\varepsilon}{2},1+\varepsilon},\quad \sigma(t,x)\geq\sigma_0>0, \\
& \frac{\partial\sigma}{\partial x}\text{ uniformly bounded,}\quad \frac{\partial\sigma}{\partial t}\text{satisfying condition (C1).}
\end{aligned}$$

This assumption will be changed only in section 6.

2 Preliminaries on parabolic PDEs

2.1 A linear parabolic equation with constant coefficients

The following result is essentially known when λ,λ_n are bounded, see [11] and also [21]. However, the boundedness assumption is too restrictive for our application. Perhaps it is possible to use a localization argument and reduce ourselves to the bounded case, but we do not know how to do it. The statement under the assumption made below is perhaps known, being a generalization of the known case of bounded functions, but we do not know a reference, so we give a probabilistic proof that can be interesting in itself. In particular, this probabilistic treatment of the regularity of the second derivatives seems to be new.

Theorem 2 *Let $\lambda_n \in C_{\exp}^{\frac{\varepsilon}{2},1+\varepsilon}$, $\lambda \in C_{\exp}^{\frac{\varepsilon}{2},\varepsilon}$, be given, with the constants in condition (C4) uniform in n for λ_n. Assume that $\lambda_n \to \lambda$ in the $\varepsilon-$Hölder norm uniformly over compact sets. Consider the equation*

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}=\lambda, \quad u(T,x)=0, \tag{5}$$

and the similar equation for u_n, λ_n. Then there exists a solution $u \in C^{1+\frac{\varepsilon}{2},2+\varepsilon}$, with $\frac{\partial u}{\partial x}$ that is $\frac{1+\varepsilon}{2}$-Hölder continuous in t, uniformly on compact sets, given by

$$u(t,x)=-\int_t^T E\left[\lambda\left(s,x+W_{s-t}\right)\right]ds, \tag{6}$$

where (W_t) is an auxiliary Brownian motion on some probability space, with expectation E. Similarly, there exists a solution $u_n \in C^{1,3}$, with $\frac{\partial u_n}{\partial x} \in C^{1+\frac{\varepsilon}{2},2+\varepsilon}$, given by

$$u_n(t,x)=-\int_t^T E\left[\lambda_n\left(s,x+W_{s-t}\right)\right]ds.$$

Moreover, $\frac{\partial u_n}{\partial x}$ and $\frac{\partial^2 u_n}{\partial x^2}$ converge to $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ uniformly on compact sets.

Remark 3 *If a function $u(t,x)$ satisfies on $[0,T]\times B_r$ ($B_r=(-r,r)$) the condition*

$$\frac{\partial u}{\partial t},\frac{\partial^2 u}{\partial x^2}\in C^{\frac{\varepsilon}{2}} \text{ in } t, \text{ uniformly in } x,$$

then for the intermediate derivative $\frac{\partial u}{\partial x}$ we have

$$\frac{\partial u}{\partial x}(t,x)\in C^{\frac{1+\varepsilon}{2}} \text{ in } t, \text{ uniformly in } x.$$

Results similar to this one can be found for instance in [11], Ch. II, lemma 3.1, and [9]. We give a proof, suggested to us by A. Lunardi, which uses the ideas of A. Zygmund to treat the Hölderianity of the derivatives. First (see [12]), for C^2 functions u on B_r, there is a constant $C > 0$ (depending on r) such that

$$\|u_x\|_\infty^2 \le C \|u\|_\infty (\|u\|_\infty + \|u_{xx}\|_\infty)$$

where we have denoted by $\|.\|_\infty$ the uniform norm on B_r (and we have used obvious notations for the space derivatives). Moreover, instead of the usual increments $u(t) - u(s)$, one can use the increments

$$u(t) + u(s) - 2u\left(\frac{t+s}{2}\right)$$

to characterize the Hölderianity, with the advantage that this applies also to the case when the derivative is Hölder continuous. Then for our function $u(t,s)$ we have

$$\left\|u_x(t) + u_x(s) - 2u_x\left(\frac{t+s}{2}\right)\right\|_\infty^2 \le C \left\|u(t) + u(s) - 2u\left(\frac{t+s}{2}\right)\right\|_\infty$$

$$\times \left(\left\|u(t) + u(s) - 2u\left(\frac{t+s}{2}\right)\right\|_\infty + \left\|u_{xx}(t) + u_{xx}(s) - 2u_{xx}\left(\frac{t+s}{2}\right)\right\|_\infty\right)$$

$$\le C\,|t-s|^{1+\frac{\varepsilon}{2}} \left(|t-s|^{1+\frac{\varepsilon}{2}} + |t-s|^{\frac{\varepsilon}{2}}\right) \le C\,|t-s|^{\frac{1}{2}+\frac{\varepsilon}{2}}.$$

PROOF.

STEP 1 (Preliminaries). If we assume a sufficiently high regularity of λ (for instance C^∞ with compact support), it is well known that the unique regular solution of equation (5) is given by (6). Then it can be easily verified, either using the density of W_{s-t} and integration by parts, or by means of Bismuth-Elworthy formula, that

$$\frac{\partial u}{\partial x}(t,x) = -\int_t^T E\left[\frac{\partial \lambda}{\partial x}(s, x + W_{s-t})\right] ds$$

$$= -\int_t^T \frac{1}{s-t} E\left[\lambda\left(s, x+W_{s-t}\right) W_{s-t}\right] ds \tag{7}$$

$$\frac{\partial^2 u}{\partial x^2}(t,x) = -\int_t^T E\left[\frac{\partial^2 \lambda}{\partial x^2}\left(s, x+W_{s-t}\right)\right] ds$$

$$= -\int_t^T \frac{1}{s-t} E\left[\frac{\partial \lambda}{\partial x}\left(s, x+W_{s-t}\right) W_{s-t}\right] ds \tag{8}$$

$$\begin{aligned} = \quad & -\int_t^T \frac{1}{s-t} E\left[\left(\frac{\partial \lambda}{\partial x}\left(s, x+W_{s-t}\right) - \frac{\partial \lambda}{\partial x}\left(t, x+W_{s-t}\right)\right) W_{s-t}\right] ds \\ & -\int_t^T E\left[\frac{\partial^2 \lambda}{\partial x^2}\left(t, x+W_{s-t}\right)\right] ds \end{aligned}$$

$$= \int_t^T \frac{1}{s-t} E\left[\left(\lambda\left(s, x+W_{s-t}\right) - \lambda\left(t, x+W_{s-t}\right)\right)\right] ds$$

$$-\int_t^T \frac{1}{(s-t)^2} E\left[\left(\lambda\left(s, x+W_{s-t}\right) - \lambda\left(t, x+W_{s-t}\right)\right)\left(W_{s-t}\right)^2\right] ds$$

$$-E\left[\lambda\left(t, x+W_t\right)\right] - \lambda\left(t,x\right), \tag{9}$$

and of course

$$\frac{\partial u}{\partial t}(t,x) = \lambda\left(t,x\right) - \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x), \tag{10}$$

$$\frac{\partial^2 u}{\partial t \partial x}(t,x) = \frac{\partial \lambda}{\partial x}\left(t,x\right) - \frac{1}{2}\frac{\partial^3 u}{\partial x^3}(t,x). \tag{11}$$

Notice that (7) is well defined as soon as λ satisfies condition (C3), since

$$\left|E\left[\lambda\left(s, x+W_{s-t}\right) W_{s-t}\right]\right| \leq \sqrt{s-t}\sqrt{E\left[\lambda\left(s, x+W_{s-t}\right)^2\right]}$$

$$\leq C_1 e^{\omega|x|}\sqrt{s-t}\sqrt{E\left[e^{2\omega|W_{s-t}|}\right]} \leq C_2 e^{\omega|x|}\sqrt{s-t}.$$

Also (9) is well defined as soon as λ satisfies condition (C4), since the quantity

$$e_2(s,t,x) := E\left[\left(\lambda(s,x+W_{s-t}) - \lambda(t,x+W_{s-t})\right)(W_{s-t})^2\right]$$

satisfies the inequality

$$|e_2(s,t,x)| \leq C_1(s-t)\sqrt{E\left[\left(\lambda(s,x+W_{s-t}) - \lambda(t,x+W_{s-t})\right)^2\right]}$$

$$\leq C_2 e^{\omega|x|}(s-t)\sqrt{E\left[|s-t|^{\varepsilon} e^{2\omega|W_{s-t}|}\right]}$$

$$\leq C_3 e^{\omega|x|}(s-t)^{1+\frac{\varepsilon}{2}}, \tag{12}$$

and similarly

$$e_1(s,t,x) := E\left[\left(\lambda(s,x+W_{s-t}) - \lambda(t,x+W_{s-t})\right)\right]$$

satisfies

$$|e_1(s,t,x)|\, C_4 e^{\omega|x|}(s-t)^{\frac{\varepsilon}{2}}.$$

If $\lambda \in C_{\exp}^{\frac{\varepsilon}{2},\varepsilon}$, we can approximate it with smoother λ such that the previous result hold. From the uniform estimates proved above one can deduce that the function u defined by (6) is C^2 in x, with $\frac{\partial^2 u}{\partial x^2}$ continuous in (t,x). From (10) it then follows that it is also C^1 in t, with $\frac{\partial u}{\partial t}$ continuous in (t,x), and equation (5) holds true. In fact, some technical steps in the uniform convergence of this regularization procedure are not entirely obvious. We show below in step 3 how to treat the most difficult term when λ is mollified.

STEP 2 (Hölder continuity). In view of remark 3 and (10), if we prove that:

$$\frac{\partial^2 u}{\partial x^2} \text{ is } C^{\frac{\varepsilon}{2},\varepsilon} \text{ on } [0,T]\times B_r$$

then $\frac{\partial u}{\partial x}$ will be $\left(\frac{1+\varepsilon}{2}\right)$-Hölder continuous in t, uniformly in $x \in B_r$, justifying the last claim on u.

Therefore let us prove that $\frac{\partial^2 u}{\partial x^2}$ given by formula (9) is $C^{\frac{\varepsilon}{2},\varepsilon}$ on $[0,T]\times B_r$. The last term, $\lambda(t,x)$, has this property by assumption. For the term $g(t,x) := E[\lambda(t,x+W_t)]$ we have (for $|x|\leq r$)

$$|g(t,x)-g(t',x')| \leq E\left[|\lambda(t,x+W_t)-\lambda(t',x'+W_{t'})|\right]$$

$$\leq CE\left[\left(|t-t'|^{\frac{\varepsilon}{2}}+|x+W_t-x'-W_{t'}|^{\varepsilon}\right)e^{\omega(|x+W_t|+|x'+W_{t'}|)}\right]$$

$$\leq C_r\left(|t-t'|^{\frac{\varepsilon}{2}}+|x-x'|^{\varepsilon}\right)+C_rE\left[|W_t-W_{t'}|^{\varepsilon}e^{\omega(|W_t|+|xW_{t'}|)}\right]$$

$$\leq C_r\left(|t-t'|^{\frac{\varepsilon}{2}}+|x-x'|^{\varepsilon}\right)+C_r|t-t'|^{\frac{\varepsilon}{2}}$$

(we have used Hölder inequality with an exponent p such that $p\varepsilon=2$). Let us come to the first two, more lengthy, terms in (9), that we denote by $I_1(t,x)$ and $I_2(t,x)$. Let us treat only $I_2(t,x)$, since $I_1(t,x)$ is very similar and in fact easier. Let us denote $I_2(t,x)$ by $I(t,x)$, and e_2 above by e. We also write

$$I(t,x) = -\int_t^T \frac{1}{(s-t)^2}e(s,t,x)ds.$$

We have:

$$\begin{aligned}
|I(t,x)-I(t',x')| \quad &\leq \quad \int_t^T\left|\frac{1}{(s-t)^2}-\frac{1}{(s-t')^2}\right||e(s,t,x)|\,ds \\
&\quad +\int_t^T\frac{1}{(s-t')^2}|e(s,t,x)-e(s,t',x')|\,ds \\
+\int_{t\wedge t'}^{t\vee t'}\frac{1}{(s-t')^2}|e(s,t',x')|\,ds \quad &= \quad J_1+J_2+J_3.
\end{aligned}$$

By (12),

$$\begin{aligned}
J_1 \quad &\leq \quad C_r\int_t^T\frac{|s-t|^{1+\frac{\varepsilon}{2}}\left|(s-t')^2-(s-t)^2\right|}{(s-t)^2(s-t')^2}ds \\
&\leq \quad C_r|t-t'|\int_t^T\frac{|s-t|+|s-t'|}{(s-t')^2|s-t|^{1-\frac{\varepsilon}{2}}}ds \\
&= \quad C_r|t-t'|\int_t^T\frac{|s-t|^{\frac{\varepsilon}{2}}}{(s-t')^2}ds+C_r|t-t'|\int_t^T\frac{ds}{|s-t'||s-t|^{1-\frac{\varepsilon}{2}}}.
\end{aligned}$$

Using the inequality $|s-t| \leq |s-t'| + |t-t'|$ in the first integral, followed by explicit integration, and either $|s-t'| \leq |s-t|$ or $|s-t| \leq |s-t'|$ (depending on the one which is correct) in the second integral, followed again by explicit integration, one can see that

$$J_1 \leq C_r \, |t-t'|^{\frac{\varepsilon}{2}} \, .$$

Again by (12), we have

$$J_3 \leq C_r \int_{t \wedge t'}^{t \vee t'} \frac{(s-t')^{1+\frac{\varepsilon}{2}}}{(s-t')^2} ds = C_r \left[(s-t')^{\frac{\varepsilon}{2}} \right]_{t \wedge t'}^{t \vee t'} \leq C_r \, |t-t'|^{\frac{\varepsilon}{2}} \, .$$

Finally, to estimate J_2, we first have (the computations are similar to those yielding (12)):

$$\begin{aligned} & |e(s,t,x) - e(s,t',x')| \\ \leq & \; E\left[|\lambda(s, x+W_{s-t}) - \lambda(t, x+W_{s-t})| \, |W_{s-t} - W_{s-t'}| \left(|W_{s-t}| + |W_{s-t'}| \right) \right] \\ + & \; E\left[\left(|\lambda(s, x+W_{s-t}) - \lambda(s, x'+W_{s-t'})| + |\lambda(t, x+W_{s-t}) - \lambda(t', x'+W_{s-t'})| \right) W_{s-t'}^2 \right] \end{aligned}$$

$$\begin{aligned} \leq & \; C_r \, |t-s|^{\frac{\varepsilon}{2}} \sqrt{|t-t'|} \left(\sqrt{|s-t|} + \sqrt{|s-t'|} \right) \\ & + C_r \, |s-t'| \left(2\,|x-x'|^{\varepsilon} + 2\sqrt{E\left[|W_{s-t} - W_{s-t'}|^{2\varepsilon} \right]} + |t-t'|^{\frac{\varepsilon}{2}} \right) \end{aligned}$$

and $\sqrt{E\left[|W_{s-t} - W_{s-t'}|^{2\varepsilon} \right]}$ can be further estimated by $C\,|t-t'|^{\frac{\varepsilon}{2}}$. If one insert this inequality into J_2 and perform computations as we have done for J_1, one can easily find

$$J_2 \leq C_r \left(|t-t'|^{\frac{\varepsilon}{2}} + |x-x'|^{\varepsilon} \right) .$$

The proof of the Hölder continuity is complete.

STEP 3 (Approximation). Starting from (8) as it were (7), and performing the previous analysis at the level of the second and third space derivative, one can show that $\lambda_n \in C_{\exp}^{\frac{\varepsilon}{2}, 1+\varepsilon}$ implies the required regularity of u_n. Finally, if λ_n converges to λ as assumed in the lemma, we prove using (7) and (9) that u_n converges to u in the way claimed in the statement. Let us give only the proof of this convergence for the second, more difficult, term of (9), denoted above by $I(t,x)$. We denote by $e_n(s,t,x)$ the quantity similar to $e(s,t,x)$, defined now in terms of λ_n, and by $I_n(t,x)$ the corresponding integral (as above). We want to prove that I_n converges to I uniformly in (t,x) over $[0,T] \times B_r$. We have:

$$\begin{aligned} |I_n(t,x) - I(t,x)| &\leq \int_t^T \frac{1}{(s-t)^2} |e_n(s,t,x) - e(s,t,x)|\, ds \\ &\leq \int_t^T \frac{1}{(s-t)} \sqrt{E\left[g_n(s,t,x)^2\right]} ds \end{aligned}$$

where

$$\begin{aligned} g_n(s,t,x) &= (\lambda_n(s, x+W_{s-t}) - \lambda_n(t, x+W_{s-t})) \\ &\quad - (\lambda(s, x+W_{s-t}) - \lambda(t, x+W_{s-t})). \end{aligned}$$

Let $\eta > 0$ be given. We look for n_0 such that for all $n \geq n_0$

$$|I_n(t,x) - I(t,x)| \leq C\eta$$

for some constant $C > 0$ and for all $t \in [0,T]$, $|x| \leq r$. For every $R > 0$ let

$$\Omega_R = \left\{ \omega \in \Omega : \sup_{\sigma \in [0,T]} |W_\sigma| \leq R \right\}.$$

We have $\lim_{R \to \infty} P(\Omega_R) = 1$. Then there exists R_η such that

$$P(\Omega_{R_\eta}^c) \leq \eta^2.$$

Then we have

$$E\left[g_n(s,t,x)^2\right] \leq E\left[1_{\Omega_{R_\eta}} g_n(s,t,x)^2\right] + \eta\sqrt{E\left[g_n(s,t,x)^4\right]}$$

$$\begin{aligned} \leq & \left(\delta_n^{(R_\eta)} |t-s|^{\frac{\varepsilon}{2}}\right)^2 + C\eta\sqrt{E\left[|\lambda_n(s, x+W_{s-t}) - \lambda_n(t, x+W_{s-t})|^4\right]} \\ & + C\eta\sqrt{E\left[|\lambda(s, x+W_{s-t}) - \lambda(t, x+W_{s-t})|^4\right]} \end{aligned}$$

where $\delta_n^{(r)}$ denotes a sequence as in the definition of ε-Hölder convergence (see section 1.1). Estimating the second term as in (12) (we use the uniform convergence in (C4) for λ_n) we finally have

$$E\left[g_n(s,t,x)^2\right] \leq \left(\delta_n^{(R_\eta)} |t-s|^{\frac{\varepsilon}{2}}\right)^2 + C\eta\, |t-s|^\varepsilon$$

which, inserted in the inequality for $|I_n(t,x) - I(t,x)|$ above, yields the desired result.

2.2 Properties of a certain change of variables

Lemma 4 *a) The function*

$$h(t,x)=\int_0^x \frac{1}{\sigma(t,x')}dx'$$

is of class $C^{1,2}$*, it satisfies condition* $(C1)$*,* $\sqrt{\left|\frac{\partial h}{\partial t}\right|}$ *satisfies condition* $(C1)$*,* $\frac{\partial h}{\partial x}$ *and* $\frac{\partial^2 h}{\partial x^2}$ *are uniformly bounded. For every* $t\in[0,T]$*, the function* $x\mapsto h(t,x)$ *is a strictly increasing diffeomorphism. Finally,* $\frac{\partial h}{\partial t}, \frac{\partial^2 h}{\partial x^2}\in C^{\frac{\varepsilon}{2},\varepsilon}$.

b) For every $t\in[0,T]$*, Let* $k(t,x)$ *be the solution of the equation*

$$\frac{\partial k(t,x)}{\partial x}=\sigma\left(t,k(t,x)\right),\quad k(t,0)=0.$$

It is of class $C^{1,2}$*,* k*,* $\frac{\partial k}{\partial x}$*,* $\frac{\partial^2 k}{\partial x^2}$*, and* $\frac{\partial k}{\partial t}$ *satisfy condition (C3), and* k *satisfies condition (C4) with* $\varepsilon=1$*. Moreover,*

$$k(t,x)=h^{-1}(t,x)$$

(the inversion is done with respect to the x *variable).*

c) If $f\in C_{lin}^{\frac{\varepsilon}{2},1+\varepsilon}$*, the function*

$$\lambda(t,x)=\int_0^x \frac{\partial f}{\partial x}\left(t,k(t,x')\right)dx'$$

is of class $C_{\exp}^{\frac{\varepsilon}{2},1+\varepsilon}$*. It also satisfies*

$$\begin{aligned}\lambda(t,x) &= \frac{f\left(t,k(t,x)\right)}{\sigma\left(t,k(t,x)\right)}-\frac{f\left(t,0\right)}{\sigma\left(t,0\right)} \\ &\quad +\int_0^x f\left(t,k(t,x')\right)\frac{\frac{\partial\sigma}{\partial x}\left(t,k(t,x')\right)\frac{\partial k}{\partial x}(t,x')}{\sigma^2\left(t,k(t,x')\right)}dx'.\end{aligned} \tag{13}$$

If f *is only of class* $C_{lin}^{\frac{\varepsilon}{2},\varepsilon}$*, formula (13) still defines a function* $\lambda\in C_{\exp}^{\frac{\varepsilon}{2},\varepsilon}$*. If* f_n *is a sequence in* $C_{lin}^{\frac{\varepsilon}{2},\varepsilon}$*, converging to* $f\in C_{lin}^{\frac{\varepsilon}{2},\varepsilon}$ *in the* ε*-Hölder norm uniformly on compact sets, then the corresponding* λ_n *converge to* λ *in the* ε*-Hölder norm uniformly on compact sets.*

d) When $\sigma(t,x)=1$ *(hence* $k(t,x)=x$*), the result of part c) remains true even if the assumption* $f\in C_{lin}^{\frac{\varepsilon}{2},1+\varepsilon}$ *is replaced by* $f\in C_{\exp}^{\frac{\varepsilon}{2},1+\varepsilon}$*, or similarly the assumption* $f\in C_{lin}^{\frac{\varepsilon}{2},\varepsilon}$ *is replaced by* $f\in C_{\exp}^{\frac{\varepsilon}{2},\varepsilon}$*.*

PROOF. The proof is based only on elementary analysis. We outline the proof of the main steps.

Concerning part a), notice that

$$\frac{\partial h}{\partial t}(t,x) = -\int_0^x \frac{\frac{\partial \sigma}{\partial t}}{\sigma^2} dx', \quad \frac{\partial h}{\partial x} = \frac{1}{\sigma}, \quad \frac{\partial^2 h}{\partial x^2} = -\frac{\frac{\partial \sigma}{\partial x}}{\sigma^2}. \tag{14}$$

Then condition (C1) follows from

$$|h(t,x)| \leq \int_0^{|x|} \frac{1}{|\sigma|} dx' \leq C\,|x|\,,$$

$\sqrt{\left|\frac{\partial h}{\partial t}\right|}$ satisfies condition $(C1)$ because

$$\left|\frac{\partial h}{\partial t}(t,x)\right| \leq \int_0^{|x|} \frac{\left|\frac{\partial \sigma}{\partial t}\right|}{\sigma^2} dx' \leq \int_0^{|x|} C\left(|x'|+1\right) dx' \leq C\,|x|^2 + C,$$

$\frac{\partial h}{\partial x}$ and $\frac{\partial^2 h}{\partial x^2}$ are uniformly bounded since

$$\left|\frac{\partial h}{\partial x}(t,x)\right| = \frac{1}{|\sigma|} \leq C, \quad \left|\frac{\partial^2 h}{\partial x^2}(t,x)\right| = \frac{\left|\frac{\partial \sigma}{\partial x}\right|}{\sigma^2} \leq C.$$

The function $x \mapsto h(t,x)$ is a strictly increasing diffeomorphism because $\frac{\partial h}{\partial x} = \frac{1}{\sigma}$ is strictly positive on compact sets, and $\frac{\partial h}{\partial x}$ is continuous. The property $\frac{\partial h}{\partial t}, \frac{\partial^2 h}{\partial x^2} \in C^{\frac{\varepsilon}{2},\varepsilon}$ follows from the Hölderianity of $\frac{\partial \sigma}{\partial t}$ and $\frac{\partial \sigma}{\partial x}$. Notice that we have used all the assumptions on σ.

As to b), since σ is Lipschitz continuous, the function $k(t,x)$ is uniquely defined and C^1 in x. A fortiori it is C^2 in x from the equation itself since σ and k are C^1 in x. It is C^1 in t since σ is C^1 in t (by a classical theorem of continuous dependence on parameters for ordinary differential equations. To prove condition (C3) for k, notice that

$$\frac{\partial}{\partial x} k^2 = 2k\sigma \leq C\,|k|\,(1+|k|) \leq C + Ck^2$$

hence by Gronwall lemma

$$|k(t,x)| \leq M e^{\omega |x|}.$$

Condition (C3) for $\frac{\partial k}{\partial x}$ and $\frac{\partial^2 k}{\partial x^2}$ comes from

$$\left|\frac{\partial k}{\partial x}(t,x)\right| \leq C\left(1+|k|\right)$$

$$\left|\frac{\partial^2 k}{\partial x^2}(t,x)\right| = \left|\frac{\partial \sigma}{\partial x}(t,k(t,x))\frac{\partial k}{\partial x}(t,x)\right| \leq C\left|\frac{\partial k}{\partial x}(t,x)\right|.$$

For $\frac{\partial k}{\partial t}$ we have

$$\frac{\partial}{\partial x}\left(\frac{\partial k}{\partial t}\right) = \frac{\partial \sigma}{\partial t}(t,k(t,x)) + \frac{\partial \sigma}{\partial x}(t,k(t,x))\left(\frac{\partial k}{\partial t}\right)$$

whence

$$\begin{aligned}\frac{\partial k}{\partial t}(t,x) &= e^{\int_0^x \frac{\partial \sigma}{\partial x}(t,k(t,x'))dx'}\frac{\partial k}{\partial t}(t,0) \\ &\quad + \int_0^x e^{\int_{x''}^x \frac{\partial \sigma}{\partial x}(t,k(t,x'))dx'}\frac{\partial \sigma}{\partial t}(t,k(t,x''))dx''\end{aligned}$$

where $\frac{\partial k}{\partial t}(t,0)=0$, hence

$$\begin{aligned}\left|\frac{\partial k}{\partial t}(t,x)\right| &\leq \int_0^{|x|} e^{C|x-x''|}C\left(1+\left|K\left(t,x''\right)\right|\right)dx'' \\ &\leq e^{C|x|}\int_0^{|x|} e^{C|x''|}C\left(1+Me^{\omega|x''|}\right)dx'' \leq M'e^{\omega'|x''|}.\end{aligned}$$

This proves (C3) for $\frac{\partial k}{\partial t}$. Property (C4) with $\varepsilon=1$ for k follows from the mean value theorem and the fact that $\frac{\partial k}{\partial t}$ and $\frac{\partial k}{\partial x}$ satisfy condition (C3). Finally, to prove that $k=h^{-1}$ we compute

$$\frac{\partial h^{-1}(t,x)}{\partial x} = \left.\frac{1}{\frac{\partial h(t,y)}{\partial y}}\right|_{y=h^{-1}(t,x)} = \sigma\left(t,h^{-1}(t,x)\right) \tag{15}$$

and $h^{-1}(t,0)=0$ because $h(t,0)=0$. Then we apply the uniqueness for the equation satisfied by k.

Let us prove c). From $f\in C_{lin}^{\frac{\varepsilon}{2},1+\varepsilon}$ and $k\in C^{1,2}$ we get $\lambda\in C^{\frac{\varepsilon}{2},1+\varepsilon}$ and (13) follows by integration by parts. If $f\in C_{lin}^{\frac{\varepsilon}{2},\varepsilon}$, from $k\in C^{1,2}$, $\sigma\in C^{\frac{\varepsilon}{2},1+\varepsilon}$

and (13) we get $\lambda \in C^{\frac{\varepsilon}{2},\varepsilon}$. property (C4) for λ, and $\frac{\partial \lambda}{\partial x}$ in the more regular case, requires a little more work. First we have

$$\begin{aligned} |\lambda(t,x)| &\leq C\left(1+|k(t,x)|\right)+C \\ +\int_0^{|x|} C\left(1+|k(t,x')|\right)\left|\frac{\partial k}{\partial x}(t,x')\right| dx' &\leq M e^{\omega|x|} \end{aligned}$$

$$\left|\frac{\partial \lambda}{\partial x}(t,x)\right| = \left|\frac{\partial f}{\partial x}(t,k(t,x))\right| \leq C\left(1+|k(t,x)|\right).$$

So λ and $\frac{\partial \lambda}{\partial x}$ satisfy (C3). Now we need some general considerations. Since k satisfies property (C4) with $\varepsilon = 1$, if a function α satisfies (C2), then $\alpha(t,k(t,x))$ satisfies (C4) (with the value of ε of α):

$$\begin{aligned} & |\alpha(t,k(t,x)) - \alpha(t',k(t',x'))| \\ \leq\ & C\left(|t-t'|^{\frac{\varepsilon}{2}} + |k(t,x)-k(t',x')|^{\varepsilon}\right)\left(|k(t,x)|+|k(t',x')|\right) \\ \leq\ & C\left(|t-t'|^{\frac{\varepsilon}{2}} + C^{\varepsilon}\left(|t-t'|+|x-x'|\right)^{\varepsilon} e^{\varepsilon\omega(|x|+|x'|)}\right)\left(Ce^{\omega|x|}+Ce^{\omega|x'|}\right). \end{aligned}$$

Hence $f(t,k(t,x))$, $\sigma(t,k(t,x))$, and $\frac{\partial \sigma}{\partial x}(t,k(t,x))$ satisfy (C4).

If two functions α and β satisfy (C4), then $\alpha\beta$ and $\frac{\alpha}{\beta}$ satisfy (C4) (by triangle inequality). Hence all terms in (13) including the term under the integral, satisfy (C4). If a function α satisfies (C3) and (C4), then $\int_0^x \alpha(t,x')dx'$ satisfies (C4). Therefore λ satisfies (C4). And also $\frac{\partial \lambda}{\partial x}$, since it is equal to $\frac{\partial f}{\partial x}$, that satisfies (C2).

The convergence properties of part c) and the claims of part d) can be easily checked, completing the proof.

2.3 A parabolic equation with non constant coefficients

Theorem 5 *Let $f \in C_{lin}^{\frac{\varepsilon}{2},\varepsilon}$, $f_n \in C_{lin}^{\frac{\varepsilon}{2},1+\varepsilon}$ be given, with the constant C for f_n in property (C2) independent of n. Assume that f_n converges to f in the ε-Hölder norm uniformly on compact sets. Let $\lambda \in C_{\exp}^{\frac{\varepsilon}{2},\varepsilon}$, $\lambda_n \in C_{\exp}^{\frac{\varepsilon}{2},1+\varepsilon}$, be defined in terms of f and f_n respectively, as in part c) of lemma 4 (by (13)). Let h be given by lemma 4, part a). Finally, let u and u_n be the functions given by lemma 4 with respect to λ and λ_n.*

i) The function F_n defined as

$$F_n(t,x) = \frac{\partial u_n}{\partial x}(t, h(t,x))$$

belongs to $C^{1,2}$ and solves the equation

$$\frac{\partial F_n}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 F_n}{\partial x^2} = \frac{\partial f_n}{\partial x} + R_n, \quad F_n(T,x) = 0, \tag{16}$$

where

$$R_n = \sigma \frac{\partial F_n}{\partial x}\left(\frac{\partial h}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 h}{\partial x^2}\right).$$

ii) Let $F \in C^{0,1}$ be the function defined as

$$F(t,x) = \frac{\partial u}{\partial x}(t, h(t,x)).$$

Moreover, let $R = \sigma \frac{\partial F}{\partial x}\left(\frac{\partial h}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 h}{\partial x^2}\right)$. Then F_n, $\frac{\partial F_n}{\partial x}$ and R_n converge to F, $\frac{\partial F}{\partial x}$ and R uniformly on compact sets.

iii) The function F is $\left(\frac{1+\varepsilon}{2}\right)$-Hölder continuous in t, uniformly on compact sets of x.

Remark 6 *F is a distributional solution of a problem similar to (16).*

Remark 7 *We have defined F from f through several steps:*

$$f \mapsto \lambda \mapsto \left.\begin{matrix} u \\ h \end{matrix}\right\} \mapsto F,$$

and similarly for F_n from f_n.

Remark 8 *Perhaps with a better argument it is possible to solve equation (16) and the similar one for F in the more natural case when $R_n = R = 0$. However, we do not know how to treat such case, and on the other side the presence of the compensators R_n and R does not increase the complexity of subsequent applications.*

PROOF. Part i). Let $v_n(t,x)$ be the function defined as

$$v_n(t,x) = \frac{\partial u_n}{\partial x}(t,x).$$

We have $v_n \in C^{1,2}$ and

$$\frac{\partial v_n}{\partial t} + \frac{1}{2}\frac{\partial^2 v_n}{\partial x^2} = \frac{\partial \lambda_n}{\partial x} = \frac{\partial f_n}{\partial x}(t,k(t,x)) \tag{17}$$

(recall the definition of λ_n in lemma 4). We also have

$$F_n(t,x) = v_n(t,h(t,x)).$$

From the regularity of h and v_n it is clear that $F_n \in C^{1,2}$. Moreover,

$$\begin{aligned}
&\frac{\partial F_n}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 F_n}{\partial x^2} \\
= \; & \frac{\partial v_n}{\partial t} + \frac{\partial v_n}{\partial x}\frac{\partial h}{\partial t} + \frac{\sigma^2}{2}\left(\frac{\partial^2 v_n}{\partial x^2}\left(\frac{\partial h}{\partial x}\right)^2 + \frac{\partial v_n}{\partial x}\frac{\partial^2 h}{\partial x^2}\right) \\
= \; & \frac{\partial v_n}{\partial t} + \frac{1}{2}\frac{\partial^2 v_n}{\partial x^2} + \frac{\partial v_n}{\partial x}\left(\frac{\partial h}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 h}{\partial x^2}\right)
\end{aligned}$$

where we have used the equality $\frac{\partial h}{\partial x} = \frac{1}{\sigma}$. Notice that v_n is always computed at $(t,h(t,x))$. Using (17) and the fact that k and h are inverse of each other (see lemma 4.b), and setting

$$\begin{aligned}
R_n(t,x) &= \frac{\partial v_n}{\partial x}(t,h(t,x))\left(\frac{\partial h}{\partial t}(t,x) + \frac{\sigma^2(t,x)}{2}\frac{\partial^2 h}{\partial x^2}(t,x)\right) \\
&= \sigma(t,x)\frac{\partial F_n}{\partial x}(t,x)\left(\frac{\partial h}{\partial t}(t,x) + \frac{\sigma^2(t,x)}{2}\frac{\partial^2 h}{\partial x^2}(t,x)\right)
\end{aligned}$$

we finally obtain the differential equation in (16). The condition $F_n(T,x) = 0$ is obvious.

Part ii). From the regularity of u and h it is clear that $F \in C^{0,1}$. From the proof of lemma 4 it is clear that the constants C, ω in property (C4) for λ_n are independent of n. Moreover, from lemma 4.c, λ_n converge to λ in the ε-Hölder norm uniformly on compact sets. Therefore, from theorem 2, $\frac{\partial u_n}{\partial x}$ and $\frac{\partial^2 u_n}{\partial x^2}$ converge to $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ uniformly on compact sets. It follows that F_n, $\frac{\partial F_n}{\partial x}$ and R_n converge to F, $\frac{\partial F}{\partial x}$ and R uniformly on compact sets.

Part iii). We have

$$\begin{aligned}|F(t,x)-F(s,x)| &\leq \left|\frac{\partial u}{\partial x}(t,h(t,x))-\frac{\partial u}{\partial x}(s,h(t,x))\right| \\ &\quad +\left|\frac{\partial u}{\partial x}(s,h(t,x))-\frac{\partial u}{\partial x}(s,h(s,x))\right|.\end{aligned}$$

We estimate the first term with $C_K |t-s|^{\frac{1+\varepsilon}{2}}$ over a compact set K using the $\left(\frac{1+\varepsilon}{2}\right)$-Hölder continuity of $\frac{\partial u}{\partial x}$ in t (unifrom in x on compact sets) and the continuity of h. On the other side we estimate the second term with $C_K |h(t,x)-h(s,x)|$ over a compact set K, by the mean value theorem, using the property that $\frac{\partial^2 u}{\partial x^2}$ and h are continuous; then we estimate $|h(t,x)-h(s,x)|$ with $C'_K |t-s|$ over the compact set K by means of the continuity of $\frac{\partial u}{\partial t}$. This completes the proof.

3 Generalized integration

We define the processes

$$A_t^{f,X} := \int_0^t \frac{\partial f}{\partial x}(s,X_s)ds, \qquad A_t^{f,X^0,X^1} := \int_0^1 (\int_0^t \frac{\partial f}{\partial x}(s,X_s^\alpha)ds)d\alpha \qquad (18)$$

for certain distributions $\frac{\partial f}{\partial x}$ and certain solutions X, X^0, X^1 to equation (1), X^α being the convex combination of X^0, X^1. The processes $A_t^{f,X}$ and A_t^{f,X^0,X^1}, shortly denoted by A, turn out to be Dirichlet processes, with zero quadratic variation. By the integration theory of part I [7] we can define stochastic integrals with respect to the integrator A. At some places we treat sligthly more general processes X for later purposes.

3.1 Definition of $A_t^{f,X}$

Let $(\Omega, \mathcal{A}, P)$ be a probability space, $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration, $(W_t)_{t\geq 0}$ a Brownian motion adapted to $\mathcal{F}$. Let X be a process satisfying the equation

$$X_t = x_0 + V_t + \int_0^t \sigma(s,X_s)\,dW_s$$

with V_t a continuous $\mathcal{F}$-adapted bounded variation process, and σ satisfying the assumptions of section 1.2. In particular, X could be a solution to equation (1).

Theorem 9 *Given a process X as above, for every $f \in C_{lin}^{\frac{\varepsilon}{2},\varepsilon}$ there exists a process $A_t^{f,X}$ with the following properties. For every sequence $f_n \in C_{lin}^{\frac{\varepsilon}{2},1+\varepsilon}$ converging to f in the ε-Hölder norm uniformly on compact sets, f_n having uniform constant C in property (C2), we have*

$$\int_0^t \frac{\partial f_n}{\partial x}(s, X_s)\, ds \to A_t^{f,X} \quad ucp$$

In other words, the integrals $\int_0^t \frac{\partial f_n}{\partial x}(s, X_s)\, ds$ have the ***same*** *limit, denoted by $A_t^{f,X}$. We shall denote $A_t^{f,X}$ also by $\int_0^t \frac{\partial f}{\partial x}(s, X_s)\, ds$. The process $A_t^{f,X}$ is also given by the formula*

$$A_t^{f,X} = F(t, X_t) - F(0, X_0) - \int_0^t \frac{\partial F}{\partial x}(s, X_s)\, dX_s - \int_0^t R(s, X_s)\, ds \qquad (19)$$

where F and R are defined in the previous theorem. $A_t^{f,X}$ is a Dirichlet process, and it has zero quadratic variation.

Remark 10 *Define the linear operator*

$$A : C_{lin}^{\frac{\varepsilon}{2},1+\varepsilon} \to L^0\left(\Omega; C\left([0,T]; \mathbb{R}\right)\right)$$

as

$$\left(Af\right)(t) = \int_0^t \frac{\partial f}{\partial x}(s, X_s)\, ds.$$

By the previous theorem, it has a unique (by density) extention to $C_{lin}^{\frac{\varepsilon}{2},\varepsilon}$. In other words, when $f \in C_{lin}^{\frac{\varepsilon}{2},\varepsilon}$, the process $A_t^{f,X}$ defined by the theorem is the unique natural extention of the classical definition for functions $f \in C_{lin}^{\frac{\varepsilon}{2},1+\varepsilon}$.

PROOF (of the theorem). By Itô formula we have (using equation (16))

$$\begin{aligned} \int_0^t \frac{\partial f_n}{\partial x}(s, X_s)\, ds &= F_n(t, X_t) - F_n(0, X_0) \\ &\quad - \int_0^t \frac{\partial F_n}{\partial x}(s, X_s)\, dX_s - \int_0^t R_n(s, X_s)\, ds. \end{aligned}$$

By the previous theorem, F_n, $\frac{\partial F_n}{\partial x}$ and R_n converge to F, $\frac{\partial F}{\partial x}$ and R uniformly on compact sets. Then the right-hand-side converges u.c.p. to the process

$A_t^{f,X}$ given by (19). Hence $\int_0^t \frac{\partial f_n}{\partial x}(s,X_s)\,ds$ converges to $A_t^{f,X}$, independently of the sequence f_n. This proves the first claim.

Let us prove that $A_t^{f,X}$ has quadratic variation equal to zero, so in particular it is a Dirichlet process.

Since the term

$$-F(0,X_0)-\int_0^t R(s,X_s)\,ds$$

has zero quadratic variation, all the brackets involving it vanish. Therefore

$$\begin{aligned}\left[A_t^{f,X},A_t^{f,X}\right] &= [F(t,X_t),F(t,X_t)]+\left[\int_0^t \frac{\partial F}{\partial x}(s,X_s)\,dX_s,\int_0^t \frac{\partial F}{\partial x}(s,X_s)\,dX_s\right]\\ &\quad -2\left[F(t,X_t),\int_0^t \frac{\partial F}{\partial x}(s,X_s)\,dX_s\right].\end{aligned}$$

By classical semimartingale theory,

$$\left[\int_0^t \frac{\partial F}{\partial x}(s,X_s)\,dX_s,\int_0^t \frac{\partial F}{\partial x}(s,X_s)\,dX_s\right]_t=\int_0^t\left(\frac{\partial F}{\partial x}(s,X_s)\right)^2 d\,[X,X]_s\,.$$

For the other terms we apply proposition 7 of part I [7]. First we apply it with $X_t(x)=Y_t(x)=F(t,x)$, $A_t=X_t$. The fields $X_t(x)$ and $Y_t(x)$ are not random and are Hölder continuous in $t\in[0,T]$ with exponent $\alpha>\frac{1}{2}$, uniformly in x over compact sets. Therefore (see also example 5 in [7]) they are strict zero quadratic variation processes, in the sense defined in section 2.2 of [7]. Because of this we can apply proposition 7 of [7] and get

$$[F(t,X_t),F(t,X_t)]_t=\int_0^t\left(\frac{\partial F}{\partial x}(s,X_s)\right)^2 d\,[X,X]_s\,.$$

Finally, we apply again proposition 7 of [7] with

$$X_t(x)=F(t,x),\quad Y_t(x)=\int_0^t \frac{\partial F}{\partial x}(s,X_s)\,dX_s,\quad A_t=X_t,$$

and we get

$$\left[F(t,X_t),\int_0^t \frac{\partial F}{\partial x}(s,X_s)\,dX_s\right]_t=\int_0^t\left(\frac{\partial F}{\partial x}(s,X_s)\right)^2 d\,[X,X]_s\,.$$

Summing up these expressions we get the desired result.

Remark 11 *The effort to deal with Dirichlet processes in section 2.3 of [7], and the effort to prove part iii) of theorem 5 of the present paper, are due to the proof given now that* $A_t^{f,X}$ *has zero quadratic variation, of major importance for the sequel.*

Exactly with the same proof (in fact we rely only on theorem 2 and not on theorem 5), we have the following variant, in which the growth of f is relaxed, but we take $\sigma = 1$.

Theorem 12 *Assume* $\sigma = 1$. *Given a process* X *as above, i.e. now a process of the form*

$$X_t = x_0 + V_t + W_t$$

(V as above), for every $f \in C_{\exp}^{\frac{\varepsilon}{2},\varepsilon}$ *there exists a process* $A_t^{f,X}$ *with the following properties. For every sequence* $f_n \in C_{\exp}^{\frac{\varepsilon}{2},1+\varepsilon}$ *converging to* f *in the* ε*-Hölder norm uniformly on compact sets,* f_n *having uniform constant* C *in property (C4), we have*

$$\int_0^t \frac{\partial f_n}{\partial x}(s, X_s)\, ds \to A_t^{f,X} \quad ucp.$$

Therefore we denote $A_t^{f,X}$ *also by* $\int_0^t \frac{\partial f}{\partial x}(s, X_s)\, ds$. *The process* $A_t^{f,X}$ *is also given by the formula*

$$A_t^{f,X} = U(t, X_t) - U(0, X_0) - \int_0^t \frac{\partial U}{\partial x}(s, X_s)\, dX_s \tag{20}$$

where $U = \frac{\partial u}{\partial x}$ *and* u *is defined in theorem 2 with* $\lambda = f$. $A_t^{f,X}$ *is a Dirichlet process, and it has zero quadratic variation.*

3.2 Definition of A_t^{f,X^0,X^1}

In this section we assume $\sigma = 1$ and $f \in C_{\exp}^{\frac{\varepsilon}{2},\varepsilon}$. Let X_t^0 and X_t^1 be two processes of the form

$$X_t^i = x_0^i + V_t^i + W_t$$

with V_t^i continuous $\mathcal{F}$-adapted bounded variation processes. Let

$$X_t^\alpha = \alpha X_t^1 + (1 - \alpha) X_t^0.$$

We have

$$X_t^\alpha = X_0^\alpha + V_t^\alpha + W_t$$

where

$$V_t^\alpha = \alpha V_t^1 + (1-\alpha) V_t^0$$

is a bounded variation continuous process. Recall from the previous section that the process

$$A_t^{f,X^\alpha} = \int_0^t \frac{\partial f}{\partial x}(s, X_s^\alpha)\, ds$$

is well defined (theorem 8). A short inspection in formula (20) (notice that U does not depend on α) shows that A_t^{f,X^α} is integrable in α over $[0,1]$, and that the new process

$$A_t^{f,X^0,X^1} = \int_0^1 (\int_0^t \frac{\partial f}{\partial x}(s, X_s^\alpha) ds) d\alpha$$

is still a zero quadratic variation Dirichlet process. It is given by the formula

$$A_t^{f,X^0,X^1} = \int_0^1 d\alpha \left\{ U(t, X_t^\alpha) - U(0, X_0^\alpha) - \int_0^t \frac{\partial U}{\partial x}(s, X_s^\alpha)\, dX_s^\alpha \right\} \qquad (21)$$

where $U = \frac{\partial u}{\partial x}$, u defined in theorem 2 with $\lambda = f$.

3.3 Generalized stochastic integrals and stochastic equations

Let us briefly recall some facts from part I [7]. Let A be a finite quadratic variation process. In particular, we can take

$$A = A^{f,X} \quad \text{or } A = A^{f,X^0,X^1}. \qquad (22)$$

We denote by $\mathcal{A}_k$ the set of all processes V of the form

$$V_t = X_t(A_t)$$

where $(X_t(x), t \in [0,T], x \in \mathbb{R})$ is a C^k Itô field driven by some semimartingale $N = (N^1, ..., N^m)$, such that (A, N) has all its mutual brackets. For the definition of C^k Itô field and the other concepts used here, see [7]. Then the forward integral

$$\int_0^t V_s d^- A_s$$

exists for all $V \in \mathcal{A}_1$, and it belongs to $\mathcal{A}_2$. This theory applies to the processes A given by (22) without restrictions on the semimartingales griving the Itô fields, since (A, N) has all its mutual brackets for all semimartingales N. This is a consequence of the fact that A has zero quadratic variation (this implies that $[A, N]$ exists and is zero).

Let us show an example of application, useful for the sequel. The space $\mathcal{A}_1$ is an algebra, and contains the semimartingales (see [7]). Therefore $V \cdot S \in \mathcal{A}_1$ if $V \in \mathcal{A}_1$ and S is a semimartingale. Due to this, the right-hand-side of the following equality exists as a forward integral.

Lemma 13 *For every C^1 Itô field $(Y_t(x), t \in [0,T], x \in \mathbb{R})$ driven by W, we have*

$$\int_0^t Y_s(A_s^{f,X^0,X^1}) \left\{ f(s, X_s^0) - f(s, X_s^1) \right\} ds$$

$$= \int_0^t Y_s(A_s^{f,X^0,X^1}) \left(X_s^0 - X_s^1 \right) d^- A_s^{f,X^0,X^1}.$$

Proof. Let $f_n \in f \in C_{\exp}^{\frac{\varepsilon}{2},1+\varepsilon}$ be a sequence converging to f uniformly on compact sets. By the definition of the stochastic integral with respect to A^{f,X^0,X^1}, the continuity properties of A^{f,X^0,X^1} with respect to f, and the continuity of Y, we see that

$$\int_0^t Y_s(A_s^{f,X^0,X^1}) \left(X_s^0 - X_s^1 \right) dA_s^{f,X^0,X^1}$$

$$= u.c.p. - \lim \int_0^t Y_s(A_s^{f_n,X^0,X^1}) \left(X_s^0 - X_s^1 \right) dA_s^{f_n,X^0,X^1}.$$

But we have

$$\int_0^t Y_s(A_s^{f_n,X^0,X^1}) \left(X_s^0 - X_s^1 \right) dA_s^{f_n,X^0,X^1}$$

$$= \int_0^t Y_s(A_s^{f_n,X^0,X^1}) \left(X_s^0 - X_s^1\right) \int_0^1 \frac{\partial f_n}{\partial x}(s, X_s^\alpha)\, d\alpha\, ds$$

$$= \int_0^t Y_s(A_s^{f_n,X^0,X^1}) \left\{ f_n(s, X_s^0) - f_n(s, X_s^1) \right\}\ ds.$$

This process converges u.c.p. to

$$\int_0^t Y_s(A_s^{f,X^0,X^1}) \left\{ f(s, X_s^0) - f(s, X_s^1) \right\}\ ds.$$

Collecting all the facts, we have proved the Lemma.

Finally, we recall theorem 36 of [7], that we repeadetly use in the sequel.

Theorem 14 *Consider the linear SDE*

$$X_t = x + \int_0^t X_s\, d^- A_s + \int_0^t X_s\, dN_s + \int_0^t g_s\, ds \tag{23}$$

where N is a continuous $\mathcal{F}$-semimartingale and A is a continuous $\mathcal{F}$-adapted process such that (N, A) has all its mutual brackets, and g is an adapted process with integrable paths. Then it has a unique solution in $\mathcal{A}_2$, given by

$$X_t = \mathcal{U}_t \left(x + \int_0^t \mathcal{U}_s^{-1} g_s\, ds \right) \tag{24}$$

where

$$\mathcal{U}_t = \exp\left(A_t + N_t - A_0 - N_0 - \frac{1}{2}\left[A + N\right]_t \right). \tag{25}$$

4 Uniqueness and dependence on parameters

4.1 The case $\sigma = 1$

4.1.1 Uniqueness

The following result is known under even weaker assumptions on b like measurability and boundedness (see for instance [20], p. 371, or [2], and references therein). The point here, as we have emphasized in the introduction, is the attempt to prove uniqueness by a technique which is as close as possible to the usual computations for equations with Lipschitz coefficients.

Theorem 15 *If $\sigma = 1$ and $b \in C_{\exp}^{\frac{\varepsilon}{2},\varepsilon}$, strong uniqueness holds true for equation (1).*

Remark 16 *Since the uniqueness is a local property, by localization we can simply ask $b \in C^{\frac{\varepsilon}{2},\varepsilon}$.*

This result is a strightforward consequence of the following proposition. It also provides a *probabilistic representation formula* for the difference of two solutions, in terms of the generalized integral defined in the previous section.

Proposition 17 *Under the previous assumptions, let X_t^0 and X_t^1 be two solutions of equation (1) with initial conditions x_0^0 and x_0^1. Let*

$$X_t^\alpha = \alpha X_t^1 + (1-\alpha) X_t^0.$$

Then $V_t := X_t^0 - X_t^1$ satisfies the equation

$$V_t = V_0 + \int_0^t V_s \, d^- A_s^{b,X^0,X^1} \tag{26}$$

(the forward integral exists since $V \in \mathcal{A}_1$, because it is a semimartingale and even a bounded variation process, see the previous section or [7]) and is given by

$$X_t^0 - X_t^1 = \left(x_0^0 - x_0^1\right) \exp\left(A_t^{b,X^0,X^1}\right). \tag{27}$$

Recall that A_t^{b,X^0,X^1} is given by the formula (21).

We give two proofs.

First Proof. We have

$$\frac{d}{dt} V_t = b(t, X_t^0) - b(t, X_t^1)$$

hence, from lemma 13,

$$\begin{aligned} V_t &= V_0 + \int_0^t \left(b(s, X_s^0) - b(s, X_s^1)\right) \, ds \\ &= V_0 + \int_0^t V_s \, d^- A_s^{b,X^0,X^1}. \end{aligned}$$

We have proved (26) The last claim follows now from theorem 14 (notice that $V \in \mathcal{A}_2$, since it is a semimartingale, and $\left[A^{b,X^0,X^1}\right] \equiv 0$).

Second Proof. We apply the Itô formula of remark 4 of [7] with

$$F(y_1, y_2) = e^{y_1} y_2, \quad Y_t^1 = A_t^{b,X^0,X^1}, \quad Y_t^2 = X_t^0 - X_t^1$$

where we notice that Y_t^1 is a finite quadratic variation process with $[Y^1, Y^1] = 0$, and Y_t^2 is a bounded variation process. Then

$$\exp\left(-A_t^{b,X^0,X^1}\right)(X_t^0 - X_t^1) = \left(X_0^0 - X_0^1\right)$$

$$+ \int_0^t \exp(-A_s^{b,X^0,X^1}) \left\{b(s, X_s^0) - b(s, X_s^1)\right\} ds$$
$$- \int_0^t \exp(-A_s^{b,X^0,X^1}) \left(X_s^0 - X_s^1\right) dA_s^{b,X^0,X^1}$$

which is zero, proving the statement.

4.1.2 Regularity in the initial conditions

Theorem 18 *Assume $\sigma = 1$, $b \in C_{lin}^{\frac{\varepsilon}{2},\varepsilon}$. Then there exists a filtered probability space $\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, P\right)$, a Brownian motion $(W_t)_{t\geq 0}$, and a random field $\left(\tilde{X}_t^x,\ t \in [0,T],\ x \in \mathbb{R}\right)$, continuous in (t,x), C^1 in x (even with locally Hölder continuous derivative in x) such that for all $x \in \mathbb{R}$ the process $\left(\tilde{X}_t^x,\ t \in [0,T]\right)$ is the pathwise unique solution of equation (1). Moreover,*

$$\frac{\partial \tilde{X}_t^x}{\partial x} = \exp\left(A_t^{b,\tilde{X}^x}\right). \tag{28}$$

STEP 0. Preparation.

We start the proof from the following data: a filtered probability space $\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, P\right)$, a Brownian motion $(W_t)_{t\geq 0}$, and a family of solutions of equation (1) denoted by $\left(X_t^{t_0,x},\ t \in [t_0, T]\right)$, with parameters $t_0 \in \mathbb{Q} \cap [0,T)$ and $x \in \mathbb{Q}$. The Brownian motion and the solutions are adapted continuous processes on $\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, P\right)$. By $X^{t_0,x}$ we denote here a solution of equation

(1) over the time interval $[t_0, T]$, with initial condition x at time t_0. The existence of such a data may be deduced from Yamada-Watanabe theorem, since for any given (t_0, x) existence of weak solutions and strong uniqueness (proved above) are known. To avoid this theorem, we may take advantage of the countability of the set of the previous (t_0, x) and perform an elementary proof as follows. One has to repeat the classical proof of existence of a weak solution for a given (t_0, x), treating simultaneously a countable family of identical stochastic equations parametrized by different initial conditions (t_0, x). The usual compactness argument of laws in $C([0,T];\mathbb{R})$ has to be replaced by a compactness in the product space $\prod C([t_0,T];\mathbb{R})$, where the product is taken over all $t_0 \in \mathbb{Q} \cap [0,T)$ and $x \in \mathbb{Q}$. We leave the details to the reader.

STEP 1. Locally Hölder flow for a small time.

Let $t_0 \in \mathbb{Q} \cap [0,T)$ be given. Let $\left(X_t^{t_0,x},\, t \in [t_0,T], x \in \mathbb{Q}\right)$ be the family of solutions given in step 0. We construct a locally Hölder flow over a time interval $[t_0, (t_0+\tau) \wedge T]$ with τ independent of t_0. For simplicity of notations we take $t_0 = 0$ in the sequel, but it is clear that we use only constants which are uniform over $[0,T]$, thus independent of t_0.

We shall prove below that there exists $\tau \in (0,T]$ such that for all $R > 0$ we have

$$C_R := \sup \left\{ E\left[\sup_{t\in[0,\tau]} \exp\left(2A_t^{b,X^x X^y}\right)\right] ;\ x, y \in \mathbb{Q} \cap B(0,R) \right\} < \infty. \quad (29)$$

¿From this estimate and the identity (27) we deduce

$$E\left[\sup_{t\in[0,\tau]} |X_t^x - X_t^y|^2\right] \le C_R |x-y|^2, \quad \forall x, y \in \mathbb{Q} \cap B(0,R).$$

By the argument of Kolmogorov regularity theorem (in Banach spaces, see [20]) it follows that there exists a bicontinuous random field $\left(\tilde{X}_t^x,\, t \in [0,\tau], x \in \mathbb{R}\right)$, locally $\delta-$Hölder continuous in x for some $\delta > 0$, such that $\tilde{X}_t^x = X_t^x$ P-a.s., for all $x \in \mathbb{Q}$ and $t \in [0,\tau]$. From the equation

$$X_t^x = W_t + x + \int_0^t b(s, X_s^x)\, ds,$$

true P-a.s. for all given $x \in \mathbb{Q}$ and $t \in [0, \tau]$, it is easy to deduce that $\left(\tilde{X}_t^x, t \in [0, \tau]\right)$ is a solution of the same equation, for *all* $x \in \mathbb{R}$. This is the local flow claimed in this step. We have only to prove (29). Let us add a remark: in (29) we could take any large number p in place of 2 at the exponent, in order to have a higher Hölder exponent in hte subsequent argument. However, τ depends on p, and we cannot prove that it is bounded below by some $\tau_0 > 0$ when p goes to infinity. We do not need such efforts to increase p, since from a little Hölder continuity we shall prove in step 2 the Lipschitz property.

Recall that

$$A_t^{b,X^x X^y} = \int_0^1 d\alpha \left\{ U\left(t, X_t^\alpha\right) - U\left(0, X_0^\alpha\right) - \int_0^t \frac{\partial U}{\partial x}\left(s, X_s^\alpha\right) b_s^\alpha ds \right. \\ \left. - \int_0^t \frac{\partial U}{\partial x}\left(s, X_s^\alpha\right) dW_s \right\} \quad (30)$$

where: $U = \frac{\partial u}{\partial x}$, u is given by theorem 2 with $\lambda = b$,

$$X_t^\alpha = \alpha X_t^y + (1-\alpha) X_t^x$$

$$b_t^\alpha = \alpha b\left(t, X_t^y\right) + (1-\alpha) b\left(t, X_t^x\right).$$

From (7) we have (we use condition (C1))

$$\left|U(t,x)\right| \leq \int_t^T \frac{1}{s-t} E\left[C\left(1 + |x| + |W_{s-t}|\right)\right] ds$$

so, by easy estimates as those in the proof of theorem 2, we get

$$\left|U(t,x)\right| \leq C\left(1 + |x|\right). \quad (31)$$

Similarly, from (9) we have (recall condition (C2) here, in addition to (C1))

$$\left|\frac{\partial U}{\partial x}(t,x)\right| \leq \int_t^T \frac{1}{s-t} C\left|s-t\right|^{\frac{\varepsilon}{2}} E\left[|x| + |W_{s-t}|\right] ds \\ + \int_t^T \frac{1}{(s-t)^2} C\left|s-t\right|^{\frac{\varepsilon}{2}} E\left[\left(|x| + |W_{s-t}|\right)\left|W_{s-t}\right|^2\right] ds \\ + E\left[C\left(1 + |x| + |W_{s-t}|\right)\right] + C\left(1 + |x|\right)$$

$$\leq C\left(1+|x|\right) \tag{32}$$

(with obvious abuse in the name of the constants, tacitly continued below; notice only that all the constants do not depend on x, y).

From (30), property (C1) for b, (31), (32) and Jensen inequality, for $x, y \in \mathbb{Q} \cap B(0, R)$ we have (τ has to be determined)

$$E\left[\sup_{t\in[0,\tau]} \exp\left(2A_t^{b,X^x X^y}\right)\right]$$

$$\leq \; E\int_0^1 d\alpha \sup_{t\in[0,\tau]} \exp\left\{C\left(1+|X_t^\alpha|\right)+R+\int_0^t C\left(1+|X_s^\alpha|\right)\left(1+|X_s^x|+|X_s^y|\right)ds\right.$$

$$\left.-\int_0^t \frac{\partial U}{\partial x}\left(s, X_s^\alpha\right)dW_s-\lambda\int_0^t\left|\frac{\partial U}{\partial x}\left(s, X_s^\alpha\right)\right|^2 ds+\lambda\int_0^t C\left(1+|X_s^\alpha|^2\right)ds\right\}$$

$$\leq \; E\int_0^1 C_{R,\lambda}\left(\sup_{t\in[0,\tau]}\exp\left(C\left|X_t^\alpha\right|\right)\right)\exp\left(\int_0^\tau C_\lambda\left|X_s^\alpha\right|\left(|X_s^x|+|X_s^y|+|X_s^\alpha|\right)ds\right)$$

$$\cdot\left(\sup_{t\in[0,\tau]}\exp\left(-\int_0^t \frac{\partial U}{\partial x}\left(s, X_s^\alpha\right)dW_s-\lambda\int_0^t\left|\frac{\partial U}{\partial x}\left(s, X_s^\alpha\right)\right|^2 ds\right)\right)d\alpha$$

$$\leq C_{R,\lambda}\int_0^1 a_\alpha^{\frac{1}{3}} b_{\alpha,\lambda}^{\frac{1}{3}} c_{\alpha,\lambda}^{\frac{1}{3}}\, d\alpha$$

where

$$a_\alpha = E\left[\sup_{t\in[0,\tau]}\exp\left(3C\left|X_t^\alpha\right|\right)\right]$$

$$b_{\alpha,\lambda} = E\left[\exp\left(\int_0^\tau 3C_\lambda\left|X_s^\alpha\right|\left(|X_s^x|+|X_s^y|+|X_s^\alpha|\right)ds\right)\right]$$

$$c_{\alpha,\lambda} = E\left[\sup_{t\in[0,\tau]}\exp\left(-3\int_0^t \frac{\partial U}{\partial x}\left(s, X_s^\alpha\right)dW_s-3\lambda\int_0^t\left|\frac{\partial U}{\partial x}\left(s, X_s^\alpha\right)\right|^2 ds\right)\right].$$

Take $3\lambda = \frac{9}{2}$. We shall prove that

$$E\left[\exp\left(9\int_0^\tau \left|\frac{\partial U}{\partial x}(s, X_s^\alpha)\right|^2 ds\right)\right] < \infty \text{ for sufficiently small } \tau > 0 \quad (33)$$

(independent of α, x, y). This implies that

$$c_{\alpha,\lambda} \leq 1$$

by a Doob theorem, and Novikov criterium in the theory of Girsanov transformation. In order to estimate X_s^α etc. in $a_\alpha, b_{\alpha,\lambda}$ and in (33), let us remark that, for any $x \in \mathbb{R}$, the process $Y_t^x = X_t^x - W_t$ satisfies the equation

$$Y_t^x = x + \int_0^t b(s, Y_s^x + W_s)\, ds.$$

Hence, from condition (C1),

$$\begin{aligned} |Y_t^x| &\leq |x| + \int_0^t C(1 + |Y_s^x| + |W_s|)\, ds \\ &\leq \left(|x| + \int_0^\tau C(1 + |W_s|)\, ds\right) + \int_0^t C\, |Y_s^x|\, ds \end{aligned}$$

so, by Gronwall lemma, and the definition of Y_t^x,

$$|X_t^x| \leq |W_t| + e^{c\tau}\left(|x| + \int_0^\tau C(1 + |W_s|)\, ds\right). \quad (34)$$

Since $E\left[e^{\gamma|W_s|}\right] \leq C_\gamma$ for all $\gamma > 0$ and $s \in [0, T]$, by Hölder and Jensen inequality we have

$$a_\alpha \leq E\left[\sup_{t\in[0,\tau]} e^{2|W_t|}\right]^{\frac{1}{2}} Ce^{C|x|}$$

for all $\alpha \in [0, 1]$ and $x \in \mathbb{R}$ (we have used the definition of X_t^α and (34)). It is not difficult to estimate the previous expectation by martingale arguments, so we get

$$a_\alpha \leq C_R$$

for all $\alpha \in [0,1]$ and $x, y \in B(0,R)$. Let us finally prove that, given $C > 0$, there exists $\tau > 0$ such that

$$E\left[\exp\left(C \int_0^\tau |X_t^x|^2 \, dt\right)\right] \leq C_R \tag{35}$$

for all $x \in B(0,R)$. From this inequality we easily deduce a similar one for $b_{\alpha,\lambda}$, and we deduce also (33), by (31). Altogether this proves (29).

From (34) and Jensen and Hölder inequalities, (35) is a consequence of the property: given $C > 0$, there exists $\tau > 0$ such that

$$\sup_{t \in [0,\tau]} E\left[e^{C|W_t|^2}\right] < \infty.$$

This fact may be proved directly from the density of W_t, completing the proof of step 1.

STEP 2. Locally Lipschitz flow for small time.

The continuity in (t,x) and local Hölder continuity in x of $\tilde{X}_t^x$ implies that the random field $(t,x,y) \mapsto A_t^{b,X^x X^y}$ has a continuous modification. This is obvious for the first three terms in (30), while for the last term it requires a result on the Hölder continuity of stochastic integrals that can be found for instance in [10]. To this end, notice also that $\frac{\partial U}{\partial x}(t,x)$ is continuous in (t,x) and local Hölder continuous in x, so the same property holds true for the random field $\frac{\partial U}{\partial x}\left(t, \tilde{X}_t^x\right)$.

From the previous facts we have

$$C_{\tau,R}(\omega) := \sup\left\{A_t^{b,X^x X^y}(\omega)\,;\, t \in [0,\tau]\,,\, x,y \in B(0,R)\right\} < \infty$$

P-a.s. Hence, from (27),

$$\left|\tilde{X}_t^x - \tilde{X}_t^y\right| \leq e^{C_{\tau,R}(\omega)} |x-y|\,.$$

The random firld $\tilde{X}_t^x$ is therefore locally Lipschitz continuous in x, uniformly in t. To justify more carefully the treatment of P-null sets with respect to the parameters, either see [6], or use the following argument. Identity (27) holds true P-a.s., for every a priori given x, y, where X_t^x and X_t^y are two solutions of (1) with initial conditions x, y; thus it holds true for $\tilde{X}_t^x$ and $\tilde{X}_t^y$. Then it holds true uniformly in $x, y \in \mathbb{Q}$ over a full measure set $\Omega' \subset \Omega$.

Therefore also (27) holds true for all $x, y \in \mathbb{Q}$ and all $\omega \in \Omega'$. Together with the continuity $\tilde{X}_t^x$ in x, this implies the local Lipschitz continuity.

STEP 3. Locally Lipschitz flow on $[0,T]$.

In the previous steps we have proved that there exists $\tau > 0$ such that for all $t_0 \in [0,T]$ there exists, on the filtered probability space $\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, P\right)$ and with the Brownian motion $(W_t)_{t\geq 0}$ given a priori, a random field

$$\left(\tilde{X}_t^{t_0,x},\, t \in [t_0, (t_0+\tau)\wedge T]\,,\, x \in \mathbb{R}\right)$$

continuous in (t,x), locally Lipschitz in x, uniformly in t, such that $t \mapsto \tilde{X}_t^{t_0,x}$ solves equation (1) over $[t_0, (t_0+\tau)\wedge T]$ with initial condition x at time t_0. The last fact means that

$$\tilde{X}_t^{t_0,x}(\omega) = W_t(\omega) - W_{t_0}(\omega) + x + \int_{t_0}^t b\left(s, \tilde{X}_s^{t_0,x}(\omega)\right) ds$$

holds true for all $t \in [t_0, (t_0+\tau)\wedge T]$, all $x \in \mathbb{R}$, and all ω in a full measure set Ω_{t_0}. Dividing $[0,T]$ in a finite number of intervals of length not greater than τ, and gluing together the previous local flows, we get a flow over $[0,T]$ with the same regularity properties and satisfying (1) with $t_0 = 0$, for all $x \in \mathbb{R}$, all $t \in [0,T]$, and all ω in a full measure set Ω'.

STEP 4. Differentiability.

Recall from step 2 that the random field $(t,x,y) \mapsto A_t^{b,X^x X^y}$ has a continuous version, that we denote here by $\tilde{A}_t^{x,y}$. Thus we can state (27) in the form (see also the argument at the end of step 2)

$$\tilde{X}_t^x(\omega) - \tilde{X}_t^y(\omega) = (x-y)\, e^{\tilde{A}_t^{x,y}(\omega)}$$

for all $x, y \in \mathbb{R}$, all $t \in [0,T]$, and all ω in a full measure set $\Omega' \subset \Omega$. Given $x_0 \in \mathbb{R}$, to prove differentiability of $x \mapsto \tilde{X}_t^x(\omega)$ at x_0 (t and ω given), it is sufficient to prove that there exists, finite, the limit

$$\lim_{h\to 0} \tilde{A}_t^{x_0, x_0+h}(\omega)\,.$$

But this is true simply because of the continuity of the random field $\tilde{A}$. Thus we have

$$\frac{\partial \tilde{X}_t^x(\omega)}{\partial x} = e^{\tilde{A}_t^{x,x}(\omega)}$$

for all $x \in \mathbb{R}$, $t \in [0,T]$, $\omega \in \Omega'$. For the same reason, the derivative $\frac{\partial \tilde{X}_t^x(\omega)}{\partial x}$ is continuous in x, and even locally Hölder continuous (recall the regularity of $\tilde{A}$ in step 2). Finally, by (30),

$$\tilde{A}_t^{x,x} = \int_0^1 d\alpha \left\{ U\left(t, X_t^x\right) - U\left(0, X_0^x\right) - \int_0^t \frac{\partial U}{\partial x}\left(s, X_s^x\right) dX_s^x \right\},$$

so $\tilde{A}_t^{x,x} = A_t^{b,X^x}$, see theorem 12. The proof of the theorem is complete.

4.1.3 Stability with respect to the drift

Consider the equation

$$dX_t^* = b^*(t, X_t^*)dt + dW_t, \quad t \geq 0,$$

where the coefficient b^* satisfies the same conditions of b. We want to estimate the distance between the solutions X_t and X_t^* in terms of the distance between the coefficients. Similarly to the proof of uniqueness we have

$$V_t := X_t^* - X_t = V_0 + \int_0^t \left(b^*(s, X_s^*) - b(s, X_s)\right) ds$$

$$= \int_0^t \left(b^*(s, X_s^*) - b^*(s, X_s)\right) ds + \int_0^t \left(b^*(s, X_s) - b(s, X_s)\right) ds$$

$$= \int_0^t V_s \, dA_s^{b^*, X^*, X} \quad \text{(by lemma 12)}$$

$$+ \int_0^t \left(b^*(s, X_s) - b(s, X_s)\right) ds$$

where in this section

$$X_t^\alpha = \alpha X_t^* + (1-\alpha) X_t, \ \alpha \in [0,1].$$

We can apply theorem 14 and have:

Proposition 19

$$X_t^* - X_t = U_t \left(X_0^* - X_0 + \int_0^t \left(b^*(s, X_s) - b(s, X_s) \right) U_s^{-1} ds \right)$$

where

$$U_t = \exp\left(A_t^{b^*, X^*, X} \right).$$

Proposition 20 *Assume $X_0^n \to X_0$ in probability, and $b^n(t,x) \to b(t,x)$ uniformly on compact sets, with the constants in (C1), (C2) independent of n. Then the solution X^n of equation (1) with initial condition X_0^n and drift b^n converges to X ucp.*

PROOF. Let us take $X_0^* = X_0^n$ and $b^* = b^n$ in the previous proposition. Repeating step by step the proof of (29) we get

$$\sup_n E\left[\sup_{t\in[0,\tau]} \exp\left(2A_t^{b^n, X^n X} \right) \right] < \infty \tag{36}$$

for sufficiently small $\tau > 0$. This implies $X^n \to X$ ucp on $[0, \tau]$. Then we consider the same problem on $[\tau, 2\tau]$, where we know for the initial conditions that $X_\tau^n \to X_\tau$ in probability. From the proof of (36) it is clear that τ can be chosen such that a property like (36) holds true on $[\tau, 2\tau]$. In a finite number of steps we get the result.

4.2 The general case

In this section we remove the restriction $\sigma = 1$ and go back to the general conditions stated in section 1.2. The reason to distinguish the two levels of generality is that we cannot define the process A_t^{b, X^0, X^1} for general σ; thus we use the machinery of this process when $\sigma = 1$, and then we transform the general case to the previous one. On the contrary, at the level of the variational equation we can deal directly with the general case.

Let $h(t,x)$ and $k(t,x)$ be the functions, inverse of each other in x, given by lemma 4. Recall that they are of class $C^{1,2}$, so we can apply Itô formula. If X is a solution to equation (1), let us set

$$Y_t = h(t, X_t).$$

Since $\frac{\partial h(t,x)}{\partial x}\sigma(t,x)=1$, by Itô formula we have

$$dY_t=\tilde{b}(t,Y_t)dt+dW_t \tag{37}$$

where

$$\begin{aligned}\tilde{b}(t,x) &= \frac{\partial h}{\partial t}(t,k(t,x))+\frac{\partial h}{\partial x}(t,k(t,x))\cdot b(t,k(t,x)) \\ &+\frac{1}{2}\frac{\partial^2 h}{\partial x^2}(t,k(t,x))\cdot\sigma^2(t,k(t,x)).\end{aligned}$$

Under our assumptions we have $\tilde{b}\in C_{\exp}^{\frac{\varepsilon}{2},\varepsilon}$. Indeed, concerning the regularity, the internal function $k\in C^{1,2}$, and: σ (thus also σ^2) is more than $C^{\frac{\varepsilon}{2},\varepsilon}$; $b\in C^{\frac{\varepsilon}{2},\varepsilon}$; $\frac{\partial h}{\partial t},\frac{\partial^2 h}{\partial x^2}\in C^{\frac{\varepsilon}{2},\varepsilon}$, by lemma 4 (and thus also $\frac{\partial h}{\partial x}\in C^{\frac{\varepsilon}{2},\varepsilon}$). As to the exponential growth of $\tilde{b}$, its proof is exactly the same as the proof of the exponential growth of λ in lemma 4.

Notice of course that if Y is a solution of (37) then

$$X_t=k(t,Y_t)$$

is a solution of (1). On this equivalence along with the $C^{1,2}$ regularity of both transformations we base the following arguments.

For equation (37) we have proved above the pathwise uniqueness. This readily imply:

Theorem 21 *Under the assumptions of section 1.2, strong uniqueness holds true for equation (1).*

Similarly, it is easy to prove the following two results.

Theorem 22 *Under the assumptions of section 1.2, the result of the regularity theorem 18 holds true, except for formula (28) for the derivative. Here we have*

$$\frac{\partial\tilde{X}_t^x}{\partial x}=\sigma\left(t,\tilde{X}_t^x\right)\exp\left(A_t^{\tilde{b},\tilde{Y}^y}\right)$$

where

$$\tilde{Y}_t^y=h(t,\tilde{X}_t^x),\quad y=h(0,x).$$

Theorem 23 *Let $\{b^n\}$ and b satisfy the assumptions of section 1.2, together with a given σ. Given initial conditions $\{X_0^n\}$ and X_0, let (X_t^n) and (X_t) be the corresponding solutions of equation (1). If b^n converges to b uniformly on compact sets, with the constants in (C1) and (C2) independent of n, and X_0^n converges to X_0 in probability, then (X_t^n) converges to (X_t) ucp.*

5 Variational equation

The variational equation around a solution X_t^x, associated to equation (1), is formally

$$V_t = 1 + \int_0^t \frac{\partial b}{\partial x}(s, X_s^x) V_s \, ds + \int_0^t \frac{\partial \sigma}{\partial x}(s, X_s^x) V_s \, dW_s.$$

Formally, the solution V of this equation is $\frac{\partial X_t^x}{\partial x}$. We shall prove this result below. As another interpretation, if the initial condition x of equation (1) is peturbed by a small amount h, the corresponding solution X_t^x is approximatively perturbed by $V_t \cdot h$.

The rigorous interpretation we give to this equation is

$$V_t = 1 + \int_0^t V_s \, dA_s^{b,X^x} + \int_0^t \frac{\partial \sigma}{\partial x}(s, X_s^x) V_s \, dW_s, \quad \text{for } t \in [0,T] \tag{38}$$

which is meaningful as soon as $V \in \mathcal{A}_1$, where $\mathcal{A}_k$ is the space defined in section 3.3 with respect to the process $A = A_t^{b,X^x}$. We consider equation (38) in the more restricted class of A_t^{b,X^x}-Itô processes $\mathcal{A}_2$ (see [7]) and prove the following result.

Notice that the following theorem treats in a unified way the variational equation under the assumptions of section 2.1 (we do not need to restrict ourselves first to the case $\sigma = 1$).

Theorem 24 *The variational equation (38) has a unique solution $V \in \mathcal{A}_2$, given by*

$$V_t = \exp\left(A_t^{b,X^x} + B_t^{\sigma,X^x}\right) \tag{39}$$

where

$$B_t^{\sigma,X^x} = \int_0^t \frac{\partial \sigma}{\partial x}(s, X_s^x) dW_s \; - \frac{1}{2} \int_0^t (\frac{\partial \sigma}{\partial x}(s, X_s^x))^2 ds.$$

Moreover, with the notation $\tilde{X}_t^x$ for the differentiable flow given in the previous section, we have

$$V_t = \frac{\partial \tilde{X}_t^x}{\partial x}. \tag{40}$$

Remark 25 *Clearly*

$$\tilde{X}_t^x - \tilde{X}_t^y = \int_y^x \frac{\partial \tilde{X}_t^z}{\partial x} dz. \tag{41}$$

Hence, in addition to the representation formula (27), which is restricted to the case $\sigma = 1$*, here we have*

$$\tilde{X}_t^x - \tilde{X}_t^y = \int_y^x \exp\left(A_t^{b,\tilde{X}^z} + B_t^{\sigma,\tilde{X}^z} \right) dz. \tag{42}$$

PROOF (of the theorem). The first part of the theorem (existence, uniqueness and the exponential formula for the solution) is a direct application of theorem 14. Let us prove (40). To this end, let us introduce an approximation of equation (1) with b replaced by $\{b^n\}$, satisfying the assumptions of section 1.2, with the constants in (C1) and (C2) independent of n, b^n converging to b uniformly on compact sets. Fron the differentiability in x of b^n, σ and the corresponding flow $\tilde{X}_t^{n,x}$, it is not difficult to prove that $V_t^n = \frac{\partial \tilde{X}_t^{n,x}}{\partial x}$ (this result is known, see [10]), where V^n is the solution of the equation of the form (38) with b^n in place of b. Moreover (see the previous remark), one get easily

$$\tilde{X}_t^{n,x} - \tilde{X}_t^{n,y} = \int_y^x \exp\left(A_t^{b^n,\tilde{X}^{n,z}} + B_t^{\sigma,\tilde{X}^{n,z}} \right) dz.$$

By theorem 23, $\tilde{X}_t^{n,x} \to \tilde{X}_t^x$ ucp. For x, y and t given we have

$$\int_y^x \exp\left(A_t^{b^n,\tilde{X}^{n,z}} + B_t^{\sigma,\tilde{X}^{n,z}} \right) dz \to \int_y^x \exp\left(A_t^{b,\tilde{X}^z} + B_t^{\sigma,\tilde{X}^z} \right) dz \tag{43}$$

in probability, hence we get (42). From (39) we obtain

$$\tilde{X}_t^x - \tilde{X}_t^y = \int_y^x V_t^z dz$$

where we have denoted by V_t^x the solution to (38) to underline the dependence on x. From the previous identity it follows (40). The proof is complete.

6 Similar results under other assumptions

Part of the theory of the previous sections can be developed under different assumptions on b and σ. In this section we only give an idea of this fact by treating the PDE and the definition of $A^{f,X}$.

Instead of the assumptions of section 1.2, assume here that

$$\begin{gathered} b \in C^{1,0},\ \sigma \in C^{1,1},\ \tfrac{\partial \sigma}{\partial x} \in C^{1,0}, \\ \sigma(t,x) \geq \sigma_0 > 0, \\ b, \sigma \text{ satisfying condition (C1).} \end{gathered}$$

The analysis under these conditions is even easier than the previous one. However, the assumption of time differentiability of b is not so natural. The advantage is that it covers the case of a continuous b independent of time.

For every $t \in [0,T]$ consider the elliptic equation

$$\frac{\sigma^2(t,x)}{2}\frac{\partial^2 F(t,x)}{\partial x^2} = \frac{\partial f(t,x)}{\partial x}, \quad x \in \mathbb{R}. \tag{44}$$

with $f \in C^{1,1}$. We have

$$\frac{\partial^2 F(t,x)}{\partial x^2} = \frac{2}{\sigma^2(t,x)}\frac{\partial f(t,x)}{\partial x}.$$

A canonical solution is given by

$$F(t,x) = \int_0^x \frac{\partial F(t,x')}{\partial x} dx', \tag{45}$$

$$\frac{\partial F(t,x)}{\partial x} = \int_0^x \frac{2}{\sigma^2(t,x')}\frac{\partial f(t,x')}{\partial x} dx'$$

$$= \left[\frac{2f(t,x')}{\sigma^2(t,x')}\right]_0^x + \int_0^x 4f(t,x')\frac{\frac{\partial \sigma(t,x')}{\partial x}}{\sigma^3(t,x')} dx'. \tag{46}$$

With simple basic analysis one can prove:

Lemma 26 *Let $f \in C^{1,0}$ and $f_n \in C^{1,1}$ be given, with f_n and $\frac{\partial f_n}{\partial t}$ converging to f and $\frac{\partial f}{\partial t}$ uniformly in (t,x) over compact sets. Let $F_n \in C^{1,2}$ be the solution corresponding to f_n defined above, and let $F \in C^{1,1}$ be defined by (45)-(46) in terms of f (we also have $\frac{\partial F}{\partial x} \in C^{1,0}$). Then F_n, $\frac{\partial F_n}{\partial x}$, $\frac{\partial F_n}{\partial t}$, and $\frac{\partial^2 F_n}{\partial x \partial t}$, converge to F, $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial t}$, and $\frac{\partial^2 F}{\partial x \partial t}$, uniformly over compact sets.*

Remark 27 *F is a distributional solution of equation (44), when properly interpreted.*

Theorem 28 *Let $f \in C^{1,0}$ be given. For every sequence $f_n \in C^{1,1}$, f_n and $\frac{\partial f_n}{\partial t}$ converging to f and $\frac{\partial f}{\partial t}$ uniformly in (t,x) over compact sets, the integrals*

$$\int_0^t \frac{\partial f_n}{\partial x}(s, X_s)\, ds$$

converge u.c.p. to the ***same*** *limit, denoted by $A_t^{f,X}$, given by*

$$A_t^{f,X} = F(t, X_t) - F(0, X_0) - \int_0^t \frac{\partial F}{\partial x}(s, X_s)\, dX_s - \int_0^t \frac{\partial F}{\partial t}(s, X_s)\, ds$$

where $F \in C^{1,1}$ is the function given by the previous lemma. We denote $A_t^{f,X}$ also by $\int_0^t \frac{\partial f}{\partial x}(s, X_s)\, ds$. It is a Dirichlet process, and it has zero quadratic variation.

PROOF. Let $f_n \in C^{1,1}$ be a sequence as in the statement, and let $F_n \in C^{1,2}$ be the corresponding sequence of solution as in the previous lemma. By Itô formula we have

$$\begin{aligned} F_n(t, X_t) - F_n(0, X_0) &= \int_0^t \frac{\partial F_n}{\partial t}(s, X_s)\, ds + \int_0^t \frac{\partial F_n}{\partial x}(s, X_s)\, dX_s \\ &\quad + \frac{1}{2} \int_0^t \frac{\partial^2 F_n}{\partial x^2}(s, X_s) \sigma^2(s, X_s)\, ds \end{aligned}$$

whence (recall (44))

$$\begin{aligned} \int_0^t \frac{\partial f_n}{\partial x}(s, X_s)\, ds &= F_n(t, X_t) - F_n(0, X_0) \\ &\quad - \int_0^t \frac{\partial F_n}{\partial x}(s, X_s)\, dX_s - \int_0^t \frac{\partial F_n}{\partial t}(s, X_s)\, ds. \end{aligned}$$

By the previous lemma, F_n, $\frac{\partial F_n}{\partial x}$ and $\frac{\partial F_n}{\partial t}$ converge to F, $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial t}$ uniformly on compact sets. Then the right-hand-side converges u.c.p. to the process $A_t^{f,X}$ defined above; hence the left-hand-side also converges to $A_t^{f,X}$ (in particular the limit of the left-hand-side is independent of f_n). The proof can be completed exactly as the proof of theorem 9.

Remark 29 *Define the linear operator*

$$A : C^{1,1} \to L^0 \left(\Omega; C\left([0,T]; \mathbb{R}\right)\right)$$

as

$$(Af)(t) = \int_0^t \frac{\partial f}{\partial x}(s, X_s)\, ds.$$

It can be trivially extended to $C^{0,1}$*. By the previous theorem, it can be uniquely (by density) extended to* $C^{1,0}$*.*

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