Generalized Integration and Stochastic ODEs. Part I

Franco Flandoli
Università di Pisa, Dipartimento di Matematica Applicata,
Via Bonanno 25B, 56126 Pisa, Italy

Francesco Russo
Université de Paris 13, Institut Galilée, Mathématiques,
99, av. J-B. Clément, 93430 Villetaneuse, France

March 2000

Abstract

Stochastic forward integrals for processes more general than semimartingales are shown to exist, generalized forms of Itô-Wentzell formula and covariation formula are proved, and one-dimensional stochastic equations driven by finite quadratic variation processes and semimartingales are solved. This generalized stochastic calculus is motivated by applications to uniqueness and dependence on parameters for stochastic equations with non regular drift.

Key words:
Forward stochastic integration, Generalized Itô-Wentsell formula, finite quadratic variation process.

AMS-classification: 60H05, 60H10

1 Introduction

The aim of this paper (part I) is to develop a generalized stochastic calculus for a class of finite quadratic variation processes. It is well-known that the stochastic calculus for semimartingales cannot be extended to more general processes unless restrictions on the integrands are imposed, see [4]. Föllmer [7] defines integrals of the form $\int_0^t f(A_s)\, dA_s$ for finite quadratic variation processes (A_t) and C^1 functions $f :\to \mathbb{R}$. Other directions have been explored by Lyons-Zhang [12] (see also [16], [13]), Lyons [11], Bertoin [2], Zähle [27, 28], Russo-Vallois [19], Wolf ([24], [25]).

Here, given a finite quadratic variation processes (A_t), we introduce a class of processes $\mathcal{A}_2$ constructed from (A_t), that we may call A-Itô processes, such that the forward integral (see [19], [20], [21])

$$\int_0^t V_s d^- U_s \tag{1}$$

exists for all $U, V \in \mathcal{A}_2$, and belongs to $\mathcal{A}_2$. In fact we can take the integrand V in a larger class $\mathcal{A}_1 \supset \mathcal{A}_2$. The class $\mathcal{A}_2$ contains for instance the semimartingales and the C^2-functions of A (and $\mathcal{A}_1$ the C^1-functions of A). The set $\mathcal{A}_k$ will be defined as the set of all processes of the form

$$V_t = X_t(A_t)$$

where $(X_t(x), t \in [0,T], x \in \mathbb{R})$ is a C^k Itô field driven by some semimartingale $N = (N^1, ..., N^m)$, such that (A, N) has all its mutual brackets (see more detailed definitions below). The procedure to define the forward integral (1) is a refinement of the method of Föllmer [7]. We make use of a Itô-Wentzell type formula (section 3), and of various formulas for the brackets of certain processes (sections 2.3 and 4). These tools are extentions of previous results of Russo and Vallois [19], [20], [21].

When the basic stochastic calculus for A-Itô processes is developed, we solve non-linear SDEs of the form

$$X_t = x + \int_0^t \sigma(X_s)\, d^- A_s + \int_0^t b(s, X_s)\, dN_s. \tag{2}$$

We prove that, under proper conditions on the coefficients, this equation has a unique solution in $\mathcal{A}_2$.

A part from the intrinsic interest in a generalization of the classical calculus for semimartingales, our primary motivation has been to develop a new approach, based on these tools to the analysis of the SDE

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t & \text{for } t \in [0, T] \\ X_0 = x \end{cases} \tag{3}$$

with $b, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$, and $(W_t)_{t\geq 0}$ a one-dimensional Brownian motion. We are interested in the case when b is not (locally) Lipschitz continuous in x.

Of course on this subject there is a large literature, we recall in particular [1], [3], [5], [10], [15], [17], [22], [23], [26] or [29].

Our final aim, reached in part II [6], is to show that it is possible to perform computations on equation (3), and prove results of uniqueness and regular dependence on parameters, along lines close to the case of Lipschitz continuous drift. In some cases we simply recover well known results, by a method which is closer to the Lipschitz case (in contrast to approaches based on Girsanov transformation, Kolmogorov equation, monotonicity arguments, transformations of drift or diffusion); in other cases we obtain new results. The material developed in the present first part is necessary to reach this purpose. When we say that our approach is close to the Lipschitz case, we mean that we try to prove uniqueness and regular dependence on parameters by estimating the difference of two solutions. The motivation for this approach to stochastic equations originated from certain open problems of uniqueness for nonlinear stochastic partial differential equations (where approaches to uniqueness like the Girsanov transform, etc., do not seem suitable), but the results obtained here are still very far from that final objective. In particular, we still find obstacles to extend this approach to the multidimensional case, even if some progresses made in this direction give us hope.

To examplify the kind of applications developed in part II, consider the *variational equation* corresponding to (3) (i.e. the equation satisfied, at least formally, by the derivative with respect to the initial condition x):

$$\begin{cases} dV_t = \frac{\partial b}{\partial x}(t, X_t)V_t dt + \frac{\partial \sigma}{\partial x}(t, X_t)V_t dW_t & \text{for } t \in [0, T] \\ V_0 = 1. \end{cases} \tag{4}$$

Here $\frac{\partial b}{\partial x}$ is a derivative in the sense of distributions, so the meaning of the integral

$$\int_0^t \frac{\partial b}{\partial x}(s, X_s) V_s \, ds$$

is not clear. By means of the integrals of the class (1) we give a meaning to this stochastic integral, after a proper definition of the finite quadratic variation process

$$A_t^{b,X} = \int_0^t \frac{\partial b}{\partial x}(s, X_s)\, ds.$$

Moreover, we interpret equation (4) in the form

$$V_t = 1 + \int_0^t V_s d^- A_s + \int_0^t \frac{\partial \sigma}{\partial x}(s, X_s)\, dW_s,$$

that is a particular case of equation (2). In this way we are able to give a meaning to equation (4) and to solve it. The calculus involved is, eventually, similar to the case of regular drift.

2 Forward, backward integrals and covariation

2.1 Preliminaries

We first recall some basic concepts. For simplicity all the processes will be supposed to be continuous. Let $(X_t)_{t \geq 0}$, $(Y_t)_{t \geq 0}$, be real stochastic processes. We recall the definition of respectively the forward, backward integrals and brackets:

$$\int_0^{\cdot} Y d^- X = \lim_{\varepsilon \to 0^+} \int_0^{\cdot} Y_s \frac{X_{s+\varepsilon} - X_s}{\varepsilon}\, ds$$

$$\int_0^{\cdot} Y d^+ X = \lim_{\varepsilon \to 0^+} \int_0^{\cdot} Y_s \frac{X_s - X_{(s-\varepsilon)\vee 0}}{\varepsilon}\, ds$$

$$[X, Y] = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^{\cdot} (X_{s+\varepsilon} - X_s)\,(Y_{s+\varepsilon} - Y_s)\, ds$$

provided the limits, taken in the uniform convergence in probability on each compact interval (ucp) sense, exist. We recall that a sequence X^n of real processes converge ucp to some process X if $\sup_{t\leq T}|X_t^n - X_t|$ converges to zero in probability for all $T>0$. We denote $[X,X]$ also by $[X]$. We recall some basic concepts from [19], [20].

If X is such that $[X,X]$ exists, it is called a finite quadratic variation process. We denote $[X,X]$ also by $[X]$. A vector process $(X^1,...,X^m)$ is said to have all its mutual brackets if $[X^i,X^j]$ exists for every $1\leq i\leq j\leq m$. In this case $[X^i,X^j]$ has bounded variation. We recall that, without the previous condition, a bracket $[X^1,X^2]$ may exist and not have bounded variation: if $f\in C^0$, B is a classical Brownian motion, $[f(B),B]$ is well defined but it does not have bounded variation. For this sake, we can consult for instance [20], [8] and [14] which develop Itô formula for functions $f\in C^1$, or less regular, of the Brownian motion or more general semimartingale.

Remark 1

$$[X,Y]=\int_0^{\cdot} Y d^+X - \int_0^{\cdot} Y d^-X$$

provided that two of the three previous terms exist.

Remark 2 *If X is a finite quadratic variation process and A is such that $[A,A]=0$, then the bracket $[X,A]$ exists and $[X,A]=0$.*

Remark 3 *Let $X=\left(X^1,...,X^d\right)$ be a vector of real processes having all its mutual brackets, $F,G\in C^1(\mathbb{R}^d)$. Then the bracket $[F(X),G(X)]$ exists and is given by*

$$[F(X),G(X)]=\sum_{i,j=1}^{d}\int_0^{\cdot}\partial_i F(X)\partial_j G(X)\, d\left[X^i,X^j\right].$$

This includes the case $d=2$, $F(x_1,x_2)=f(x_1)$, $G(x_1,x_2)=g(x_2)$, $f,g\in C^1(\mathbb{R})$.

Remark 4 *(Classical Itô formula) Let $X=\left(X^1,...,X^d\right)$ be a vector of real processes having all its mutual brackets such that X^i, $2\leq i\leq d$, are either semimartingales, or finite quadratic variation processes such that the forward integrals $\int_0^t \partial_i F(X)\, d^-X^i$ exist (resp. bounded variation processes), X^1 is a* *finite quadratic variation process,* $F \in C^2(\mathbb{R}^d)$ *(resp.* $F \in C^{2,1}(\mathbb{R} \times \mathbb{R}^{d-1})$*). Then the forward integral* $\int_0^t \partial_1 F(X)\, d^- X^1$ *exists and we have*

$$\begin{aligned} F(X_t) &= F(X_0) + \sum_{i=2}^{d} \int_0^t \partial_i F(X)\, d^- X^i + \int_0^t \partial_1 F(X)\, d^- X^1 \qquad (5) \\ &+ \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \partial^2_{i,j} F(X)\, d\left[X^i, X^j\right] \end{aligned}$$

(resp.

$$\begin{aligned} F(X_t) &= F(X_0) + \sum_{i=2}^{d} \int_0^t \partial_i F(X)\, dX^i + \int_0^t \partial_1 F(X)\, d^- X^1 \qquad (6) \\ &+ \frac{1}{2} \int_0^t \partial^2_{1,1} F(X)\, d\left[X^1, X^1\right] \text{).} \end{aligned}$$

A similar formula with the additional term $\int_0^t \frac{\partial F}{\partial t}(s, X_s)\, ds$ *is true if* F *depends also on* t*, of class* C^1*.*

2.2 Itô-Dirichlet fields

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration. Let $N = (N^1, ..., N^m)$ be a continuous $\mathcal{F}$-semimartingale (resp. $\mathcal{F}$-local martingale). We shall say that a random field $\left(X_t(x),\, t \in [0,T],\, x \in \mathbb{R}^d\right)$ is a C^0 *Itô field (*resp. a C^0 *Itô martingale field) driven by* N if it is a.s. continuous, there are $f : \mathbb{R}^d \times \Omega \to \mathbb{R}$, $a^i : [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ a.s. continuous, f being $\mathcal{F}_0$-measurable and a^i being $\mathcal{F}$-adapted for every x, such that

$$X_t(x) = f(x) + \sum_{i=1}^{m} \int_0^t a^i(s,x)\, dN^i_s. \qquad (7)$$

Given a continuous $\mathcal{F}$-local martingale $N = (N^1, ..., N^m)$, a random field $\left(X_t(x),\, t \in [0,T],\, x \in \mathbb{R}^d\right)$ is said to be a C^0 *Itô-Dirichlet field driven by* N if it can be written as

$$X_t(x) = M_t(x) + Z_t(x)$$

where $(M_t(x))$ is a C^0 Itô martingale field driven by N and $(Z_t(x))$ is a *strict zero quadratic variation process* in the following sense:

$$\sup_{|x|\leq R} \int_0^T \left(Z_{t+\varepsilon}(x) - Z_t(x)\right)^2 \frac{dt}{\varepsilon} \to 0 \quad ucp \tag{8}$$

for all $R > 0$. We also assume that $(Z_t(x))$ is a.s. continuous in (t, x), and it is $\mathcal{F}$-adapted for every x.

Example 5 *If Z is a Hölder continuous process in $t \in [0, T]$ of parameter $\alpha > \frac{1}{2}$, uniformly with respect to x on compact sets, then (8) is verified.*

Example 6 *If*

$$Z_t(x) = \sum_{i=1}^{l} \int_0^t b^i(s, x)\, dV_s^i$$

where b^i are continuous fields and $(V_t^i)_{t\in[0,T]}$, $i = 1, ..., l$, are bounded variation processes, then (8) is verified. In particular, a C^0 Itô filed is also a C^0 Itô-Dirichlet field.

Finally, we shall be interested in more regular Itô fields. Given a positive integer k, a C^0 Itô field $\left(X_t(x),\, t \in [0, T],\, x \in \mathbb{R}^d\right)$ driven by N of the form (7) will be called a *C^k Itô field driven by N* if:

1) $(X_t(x))$ is a.s. of class C^k in x, with derivatives in x up to order k a.s. continuous in (t, x),

2) a^i is a.s. of class C^k in x, with derivatives in x up to order k a.s. continuous in (t, x),

3) f is a.s. of class C^k,

4) for every multi-index $\alpha = (\alpha_1, ..., \alpha_d)$ with $|\alpha| \leq k$ we have

$$\frac{\partial^\alpha X_t(x)}{\partial x^\alpha} = \frac{\partial^\alpha f(x)}{\partial x^\alpha} + \sum_{i=1}^{m} \int_0^t \frac{\partial^\alpha a^i(s, x)}{\partial x^\alpha}\, dN_s^i.$$

In fact, in the definition of C^k Itô field driven by N we could assume only that a^i and f are C^{k-1} in x with Hölder continuous $(k-1)$-derivatives: the k-derivatives of the coefficients do not appear explicitly in the statements and proofs, and the Hölder continuity of the $(k-1)$-derivatives is used to apply substitution arguments in some proofs.

The definition of C^k Itô martingale field is similar, with the only difference that N is a local martingale.

2.3 Brackets of Itô-Dirichlet fields

The following proposition extends the result of remark 3. This is the only result of this paper involving the concept of Itô-Dirichlet fields. In most of the work we only need Itô fields. However, on one side the next result on brackets is more properly posed in the most general framework of Itô-Dirichlet fields. On the other side, in part II [6] we need to compute the bracket of an Itô-Dirichlet field which is not an Itô field.

Proposition 7 *Let* $X = \left(X_t(x),\, t \in [0,T]\,,\, x \in \mathbb{R}^d\right)$*,* $Y = \left(Y_t(x),\, t \in [0,T]\,,\, x \in \mathbb{R}^d\right)$ *be two* C^0 *Itô-Dirichlet fields driven by a local martingale* $N = (N^1, ..., N^m)$*, with decompositions* $M^1 + Z^1$*,* $M^2 + Z^2$*. We denote by* a_j^i*, the coefficients of* M^j*,* $j = 1, 2$*,* $1 \leq i \leq m$*. We suppose moreover that* M^1*, and* M^2 *are* C^1 *Itô martingale fields. Let* $A = \left(A^1, ..., A^d\right)$ *be a process such that* (A, N) *has all its mutual brackets. Then the bracket* $[X_t(A_t^1), Y_t(A_t^2)]$ *exists and it is given by*

$$\begin{aligned}
[X(A), Y(A)]_t &= \sum_{i,j=1}^{m} \int_0^t a_1^i\,(s, A_s) \cdot a_2^j\,(s, A_s)\; d\left[N^i, N^j\right]_s \qquad (9)\\
&+ \sum_{i,j=1}^{d} \int_0^t \frac{\partial X_s}{\partial x_i}\,(A_s) \cdot \frac{\partial Y_s}{\partial x_j}\,(A_s)\; d\left[A^i, A^j\right]_s\\
&+ \sum_{i=1}^{d} \sum_{j=1}^{m} \int_0^t \frac{\partial X_s}{\partial x_i}\,(A_s) \cdot a_2^j\,(s, A_s)\; d\left[A^i, N^j\right]_s\\
&+ \sum_{i=1}^{m} \sum_{j=1}^{d} \int_0^t a_1^i\,(s, A_s) \cdot \frac{\partial Y_s}{\partial x_j}\,(A_s)\; d\left[N^i, A^j\right]_s.
\end{aligned}$$

Example 8 *If* $(N^1, ..., N^m)$ *is a standard Brownian motion* $(W^1, ..., W^m)$ *then the integral sum in the first line can be replaced by*

$$\sum_{i=1}^{m} \int_0^t a_1^i\,(s, A_s) \cdot a_2^i\,(s, A_s)\; ds.$$

PROOF (of the proposition) For simplicity of notations we give the proof in the case $d = m = 1$, $X = Y$. The proof in the general case does not contain essential differences. We have to prove

$$
\begin{aligned}
[X(A), X(A)]_t &= \int_0^t (a(s, A_s))^2 \, d[N,N]_s \\
&\quad + \int_0^t \left(\frac{\partial X_s}{\partial x}(A_s) \right)^2 d[A,A]_s \\
&\quad + 2\int_0^t \frac{\partial X_s}{\partial x}(A_s) \cdot a(s, A_s) \, d[N,A]_s .
\end{aligned}
\tag{10}
$$

We can write

$$
\begin{aligned}
X_{s+\varepsilon}(A_{s+\varepsilon}) - X_s(A_s) &= X_{s+\varepsilon}(A_{s+\varepsilon}) - X_{s+\varepsilon}(A_s) \\
&\quad + X_{s+\varepsilon}(A_s) - X_s(A_s)
\end{aligned}
$$

so that for $t \in [0,T]$

$$
\frac{1}{\varepsilon}\int_0^t (X_{s+\varepsilon}(A_{s+\varepsilon}) - X_s(A_s))^2 \, ds = I_1 + I_2 + I_3
$$

where

$$
I_1 = \frac{1}{\varepsilon}\int_0^t (X_{s+\varepsilon}(A_{s+\varepsilon}) - X_{s+\varepsilon}(A_s))^2 \, ds
$$

$$
I_2 = \frac{2}{\varepsilon}\int_0^t (X_{s+\varepsilon}(A_{s+\varepsilon}) - X_{s+\varepsilon}(A_s)) \cdot (X_{s+\varepsilon}(A_s) - X_s(A_s)) \, ds
$$

$$
I_3 = \frac{1}{\varepsilon}\int_0^t (X_{s+\varepsilon}(A_s) - X_s(A_s))^2 \, ds.
$$

Now I_1 does not create any problem, since it equals

$$
\frac{1}{\varepsilon}\int_0^t \left(\int_0^1 \frac{\partial X_{s+\varepsilon}}{\partial x}(A_s + \alpha(A_{s+\varepsilon} - A_s)) \, d\alpha \right)^2 (A_{s+\varepsilon} - A_s)^2 \, ds
$$

and by usual uniform continuity arguments, see for instance the proof of proposition 2.1 in [20], it converges to

$$
\int_0^t \left(\frac{\partial X_s}{\partial x}(A_s) \right)^2 d[A,A]_s .
$$

Concerning I_3, assumption (8) implies that it converges if and only if it converges

$$\frac{1}{\varepsilon}\int_0^t \left(M_{s+\varepsilon}\left(A_s\right) - M_s\left(A_s\right)\right)^2 ds.$$

This equals

$$\int_0^t \left(\frac{1}{\sqrt{\varepsilon}}\int_s^{s+\varepsilon} a(u,x)dN_u\right)^2 \Bigg|_{x=A_s} ds.$$

Since $a(u,x)$ is Hölder continuous in u, we can apply a sostitution argument and have

$$\int_0^t \left(\frac{1}{\sqrt{\varepsilon}}\int_s^{s+\varepsilon} a(u,A_s)dN_u\right)^2 ds.$$

It remains to show that

$$\int_0^t \left\{\left(\frac{1}{\sqrt{\varepsilon}}\int_s^{s+\varepsilon} a(u,A_s)dN_u\right)^2 - \frac{1}{\varepsilon}\left(N_{s+\varepsilon}-N_s\right)^2\left(a(s,A_s)\right)^2\right\} ds \qquad (11)$$

converges ucp to zero.

Using localization techniques, we can suppose that

$$\sup_{s\in[0,T],\, x\in\mathbb{R}} |a(s,x)|, \quad \sup_{s\in[0,T]} |A_s|, \quad \sup_{s\in[0,T]} \left|[N,N]_s\right|, \qquad (12)$$

are bounded random variables. For classical typical localization arguments the reader can consult [19], [20], [21]. Now (11) is equal to

$$\begin{aligned}
&\int_0^t \left\{\frac{1}{\sqrt{\varepsilon}}\int_s^{s+\varepsilon} a(u,A_s)dN_u - \frac{1}{\sqrt{\varepsilon}}\left(N_{s+\varepsilon}-N_s\right)a(s,A_s)\right\} \\
&\quad\times\left\{\left(\tfrac{1}{\sqrt{\varepsilon}}\textstyle\int_s^{s+\varepsilon} a(u,A_s)dN_u - \tfrac{1}{\sqrt{\varepsilon}}\left(N_{s+\varepsilon}-N_s\right)a(s,A_s)\right)\right. \\
&\qquad\left. +\tfrac{2}{\sqrt{\varepsilon}}\left(N_{s+\varepsilon}-N_s\right)a(s,A_s)\right\} ds \\
&\le \left(\int_0^T \left|\frac{1}{\sqrt{\varepsilon}}\int_s^{s+\varepsilon}\left(a(u,A_s)-a(s,A_s)\right)dN_u\right|^2 ds\right)^{\frac{1}{2}} \\
&\quad\times\left(2\textstyle\int_0^T \left|\tfrac{1}{\sqrt{\varepsilon}}\int_s^{s+\varepsilon}\left(a(u,A_s)-a(s,A_s)\right)dN_u\right|^2 ds\right. \\
&\qquad\left. +\tfrac{8}{\varepsilon}\textstyle\int_0^T \left(N_{s+\varepsilon}-N_s\right)^2 a^2(s,A_s)ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Now

$$\int_0^T \frac{(N_{s+\varepsilon}-N_s)^2}{\varepsilon} a^2(s,A_s)ds$$
$$\leq \ C\int_0^T \frac{(N_{s+\varepsilon}-N_s)^2}{\varepsilon} ds \to [N,N]_{T.}$$

In order to conclude the proof that (11) converges to zero it remains to prove the following result:

$$E\int_0^T \left|\frac{1}{\sqrt{\varepsilon}}\int_s^{s+\varepsilon}(a(u,A_s)-a(s,A_s))\,dN_u\right|^2 ds \to 0 \tag{13}$$

To this purpose, recall (12). We have

$$E\int_0^T \left|\frac{1}{\sqrt{\varepsilon}}\int_s^{s+\varepsilon}(a(u,A_s)-a(s,A_s))\,dN_u\right|^2 ds$$
$$= \ \frac{1}{\varepsilon}\int_0^T E\left[\int_s^{s+\varepsilon}((a(u,A_s)-a(s,A_s)))^2\,d\,[N,N]_u\right] ds$$

$$= E\int_0^T \left(\int_{u-\varepsilon}^{u}((a(u,A_s)-a(s,A_s)))^2\,\frac{ds}{\varepsilon}\right) d\,[N,N]_u$$

which clearly converges to zero because of Lebesgue dominated convergence theorem.

At this level we have proved the convergence of I_3 to the first term of (10).

We still have to prove the convergence of I_2. It can be written as

$$\frac{2}{\varepsilon}\int_0^t\left(\int_0^1 \frac{\partial X_{s+\varepsilon}}{\partial x}\left(A_s+\alpha\left(A_{s+\varepsilon}-A_s\right)\right)d\alpha\right)$$
$$\cdot\left(A_{s+\varepsilon}-A_s\right)\cdot\left(X_{s+\varepsilon}\left(A_s\right)-X_s\left(A_s\right)\right)\,ds.$$

By usual uniform continuity arguments, it is sufficient to prove that

$$\frac{1}{\varepsilon}\int_0^{\cdot}\left(A_{s+\varepsilon}-A_s\right)\cdot\left(X_{s+\varepsilon}\left(A_s\right)-X_s\left(A_s\right)\right)\,ds \to \int_0^{\cdot} a\left(s,A_s\right)d\left[A,N\right]_s \quad ucp. \tag{14}$$

Moreover, it is easy to see that

$$\frac{1}{\varepsilon}\int_0^{\cdot} (A_{s+\varepsilon} - A_s) \cdot (Z_{s+\varepsilon}(A_s) - Z_s(A_s))\, ds \to 0$$

so in (14) we can replace X by M.

Similarly as for the proof of the convergence of I_3, we have

$$\begin{aligned}
& \frac{1}{\varepsilon}\int_0^t (A_{s+\varepsilon} - A_s)(M_{s+\varepsilon}(A_s) - M_s(A_s))\, ds \\
= & \frac{1}{\varepsilon}\int_0^t (A_{s+\varepsilon} - A_s)\left(\int_s^{s+\varepsilon} a(u,x) dN_u\right)\Big|_{x=A_s} ds \\
= & \frac{1}{\varepsilon}\int_0^t (A_{s+\varepsilon} - A_s)\, a(s,A_s)\,(N_{s+\varepsilon} - N_s)\, ds \\
& + \frac{1}{\varepsilon}\int_0^t (A_{s+\varepsilon} - A_s)\left(\int_s^{s+\varepsilon} a(u,x) dN_u - a(s,x)(N_{s+\varepsilon} - N_s)\right)\Big|_{x=A_s} ds \\
= & J_1 + J_2
\end{aligned}$$

where

$$J_1 \to \int_0^{\cdot} a(s,A_s) d[A,N]_s \quad ucp$$

and by substitution as above

$$\begin{aligned}
J_2 & = \int_0^t \left(\frac{1}{\sqrt{\varepsilon}}\int_s^{s+\varepsilon} (a(u,A_s) - a(s,A_s))\, dN_u\right)\frac{1}{\sqrt{\varepsilon}}(A_{s+\varepsilon} - A_s)\, ds \\
& \leq \left(\int_0^T \left(\frac{1}{\sqrt{\varepsilon}}\int_s^{s+\varepsilon} (a(u,A_s) - a(s,A_s))\, dN_u\right)^2 ds\right)^{\frac{1}{2}} \left(\int_0^T \frac{1}{\varepsilon}(A_{s+\varepsilon} - A_s)^2\, ds\right)^{\frac{1}{2}}.
\end{aligned}$$

This converges to zero in accordance with (13). The proof is complete.

3 Itô-Wentzell formula

In our applications we need in fact a generalization of the classical Itô formula stated in remark 4. It will be of Itô-Wentzell type, see e.g. [9].

Proposition 9 *Let* $F=\left(F(t,x),\, t\in[0,T]\,,\, x\in\mathbb{R}^d\right)$ *be a* C^2 *Itô field driven by a semimartingale* N*, of the form (7). Let* $A=\left(A^1,...,A^d\right)$ *be a* $\mathcal{F}$*-adapted process such that* (A,N) *has all its mutual brackets. Assume that the forward integrals* $\int_0^t \frac{\partial F}{\partial x_i}(s,A_s)\, d^-A_s^i$ *exist for* $i=2,...,d$*. Then we have*

$$F(t,A_t)=F(0,A_0)+\sum_{i=1}^m\int_0^t a^i(s,A_s)\,dN_s^i+\sum_{i=1}^d\int_0^t\frac{\partial F}{\partial x_i}(s,A_s)\,d^-A_s^i$$

$$+\frac{1}{2}\sum_{i,j=1}^d\int_0^t\frac{\partial^2 F}{\partial x_i\partial x_j}(s,A_s)\,d\left[A^i,A^j\right]_s+\sum_{i=1}^m\sum_{j=1}^d\int_0^t\frac{\partial a^i}{\partial x_j}(s,A_s)\,d\left[N^i,A^j\right]_s. \tag{15}$$

Remark 10 *In particular* $\int_0^{\cdot}\frac{\partial F}{\partial x_1}(s,A_s)\,d^-A_s^1$ *exists. A similar formula would hold for* $\int_0^{\cdot}\frac{\partial F}{\partial x_1}(s,A_s)\,d^+A_s^1$*.*

PROOF We give the proof for $m=d=1$ for simplicity of notations. Let $s\in\,]0,T[$; we have

$$F\left(s+\varepsilon,A_{s+\varepsilon}\right)-F\left(s,A_s\right)=I_1(s,\varepsilon)+I_2(s,\varepsilon)$$

where

$$I_1(s,\varepsilon)=F\left(s+\varepsilon,A_{s+\varepsilon}\right)-F\left(s+\varepsilon,A_s\right)$$

$$I_2(s,\varepsilon)=F\left(s+\varepsilon,A_s\right)-F\left(s,A_s\right).$$

On one side,

$$\int_0^t\frac{F\left(s+\varepsilon,A_{s+\varepsilon}\right)-F\left(s,A_s\right)}{\varepsilon}ds$$

converges ucp to $F\left(t,A_t\right)-F\left(0,A_0\right)$. On the other side, since

$$I_2(s,\varepsilon)=\int_s^{s+\varepsilon}a\left(u,A_s\right)\,dN_u$$

(because A is adapted and because a is Hölder continuous in x to apply a substitution argument), it is not difficult to show that

$$\int_0^t \frac{I_2(s,\varepsilon)}{\varepsilon} ds$$

converges ucp to

$$\int_0^t a\left(u, A_u\right)\, dN_u.$$

It remains to discuss $I_1(s,\varepsilon)$. This gives

$$I_1(s,\varepsilon) = J_1(s,\varepsilon) + J_2(s,\varepsilon) + J_3(s,\varepsilon) + J_4(s,\varepsilon)$$

where

$$J_1(s,\varepsilon) = \frac{\partial F}{\partial x}\left(s, A_s\right)\left(A_{s+\varepsilon} - A_s\right)$$

$$J_2(s,\varepsilon) = \left(\frac{\partial F}{\partial x}\left(s+\varepsilon, A_s\right) - \frac{\partial F}{\partial x}\left(s, A_s\right)\right)\left(A_{s+\varepsilon} - A_s\right)$$

$$J_3(s,\varepsilon) = \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\left(s, A_s\right)\left(A_{s+\varepsilon} - A_s\right)^2$$

$$\begin{aligned} J_4(s,\varepsilon) &= \int_0^1 (1-\alpha)\left(\frac{\partial^2 F}{\partial x^2}\left(s+\varepsilon, A_s + \alpha\left(A_{s+\varepsilon} - A_s\right)\right) - \frac{\partial^2 F}{\partial x^2}\left(s, A_s\right)\right) d\alpha \\ &\quad \times \left(A_{s+\varepsilon} - A_s\right)^2. \end{aligned}$$

Integrating from 0 to t the previous expressions and using similar arguments to those of the proof of proposition 7, we ucp get

$$\lim_{\varepsilon\to 0}\int_0^t J_2(s,\varepsilon)\,\frac{ds}{\varepsilon} = \int_0^t \frac{\partial a}{\partial x}(s, A_s)\, d\left[N, A\right]_s.$$

Using classical uniform continuity arguments for $\frac{\partial^2 F}{\partial x^2}$ and pathwise weak convergence of $\int_0^t \left(A_{s+\varepsilon} - A_s\right)^2 \frac{ds}{\varepsilon}$ on $[0,T]$, we also obtain

$$\lim_{\varepsilon\to 0}\int_0^t J_3(s,\varepsilon)\,\frac{ds}{\varepsilon} = \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s, A_s)\, d\left[A, A\right]_s$$

$$\lim_{\varepsilon \to 0} \int_0^t J_4(s,\varepsilon) \frac{ds}{\varepsilon} = 0.$$

This forces in particular the convergence of

$$\int_0^t J_1(s,\varepsilon) \frac{ds}{\varepsilon}$$

whose limit is by definition $\int_0^t \frac{\partial F}{\partial x}(s, A_s)\, d^- A_s$. The proof is complete.

4 Existence of forward integrals

From this section we restrict ourselves to $d = 1$. Thus A will be a scalar process with finite quadratic variation, and the Itô fields $X_t(x)$ will have $x \in \mathbb{R}$.

With the help of the Itô-Wentzell formula, in this section we first show the existence of the forward integral

$$\int_0^t X_s(A_s)\, d^- A_s$$

for suitable random fields $(X_t(x),\, t \in [0,T],\, x \in \mathbb{R})$; then with the aid of this forward integral we show the existence of a more general forward integral

$$\int_0^t X_s(A_s)\, d^- Y_s(A_s) \tag{16}$$

for suitable random fields $(X_s(x))$ and $(Y_s(x))$. The only assumption on A imposed below is that the process (A, N) has all its mutual bracket, N being the semimartingale driving X and Y. Since A in general will not be a semimartingale, the class of integrand $X_s(A_s)$ is a restricted one, but still including semimartingales and functions of A itself. Apparently, the integration defined in this section is not a particular case of the known stochastic integration theories.

4.1 Forward integral with integrator A

In this section it is always understood that (A_t) is a continuous $\mathcal{F}$-adapted process.

Definition 11 *Given* (A_t)*, we denote by* $\mathcal{A}_k$ *the set of processes* V_t *of the form*

$$V_t = X_t\left(A_t\right)$$

where $\left(X_t\left(x\right),\, t \in [0,T]\,,\, x \in \mathbb{R}\right)$ *is a* C^k *Itô field driven by some continuous* $\mathcal{F}$*-semimartingale* $N = (N^1, ..., N^m)$ *such that* (A, N) *has all its mutual brackets.*

Remark 12 *The set* $\mathcal{A}_k$ *is an algebra, since the space of all* C^k *Itô fields* $(X_t(x))$ *is an algebra (by classical stochastic calculus for semimartingales).*

Remark 13 *If* $(X_t(x))$ *is a* C^2 *Itô field of the form (7), so* $V_t = X_t\left(A_t\right)$ *belongs to* $\mathcal{A}_2$*, we can use Itô-Wentzell formula to express* V_t *as*

$$\begin{aligned} V_t &= V_0 + \sum_{i=1}^{m} \int_0^t a^i(s, A_s)\, dN_s^i + \int_0^t \frac{\partial X_s}{\partial x}(A_s)\, d^- A_s \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial^2 X_s}{\partial x^2}(A_s)\, d\left[A, A\right]_s + \sum_{i=1}^{m} \int_0^t \frac{\partial a^i}{\partial x}(s, A_s)\, d\left[N^i, A\right]_s . \end{aligned} \tag{17}$$

In particular, the forward integral $\int_0^t \frac{\partial X_s}{\partial x}(A_s)\, d^- A_s$ *exists.*

Definition 14 *Given a* C^k *Itô field* $X_t(x)$ *of the form (7), we denote by* $X_t^*(x)$ *the* C^{k+1} *Itô field defined as*

$$X_t^*(x) = \int_0^x X_t(x')\, dx'$$

having the representation

$$X_t^*(x) = f^*(x) + \sum_{i=1}^{m} \int_0^t a^{i*}(s, x)\, dN_s^i$$

with

$$f^*(x) = \int_0^x f(x')\, dx', \quad a^{i*}(t, x) = \int_0^x a^i(t, x')\, dx'.$$

Proposition 15 *The forward integral*

$$\int_0^t V_s\, d^- A_s$$

exists for all $V \in \mathcal{A}_1$. If $V_t = X_t\,(A_t)$, $(X_t(x))$ of the form (7), then we have the formula

$$\int_0^t V_s\, d^- A_s = X_t^*(A_t) - f^*(A_0) - \sum_{i=1}^m \int_0^t a^{i*}(s, A_s)\, dN_s^i$$

$$-\frac{1}{2}\int_0^t \frac{\partial X_s}{\partial x}(A_s)\, d\,[A, A]_s - \frac{1}{2}\sum_{i=1}^m \int_0^t a^i(s, A_s)\, d\left[N^i, A\right]_s. \tag{18}$$

In particular, we have

$$\int_0^\cdot V_s\, d^- A_s \in \mathcal{A}_2.$$

PROOF: It is a direct consequence of the Itô-Wentzell formula and the previous definitions and remarks.

Remark 16 *As announced, the previous proposition proves the existence of a forward integral for processes more general than semimartingales, which does not seem a particular case of other theories of generalized integration. Let us notice that by similar reasonings we can define the backward integral*

$$\int_0^t V_s\, d^+ A_s. \tag{19}$$

Remark 17 *Let $\mathcal{I}$ be the linear operator*

$$\mathcal{I} : \mathcal{A}_1 \to L^0\left(\Omega; C\left([0, T]\,; \mathbb{R}\right)\right)$$

defined as

$$\left(\mathcal{I}V\right)_t = \int_0^t V_s\, d^- A_s.$$

Assume that A has zero quadratic variation. Then

$$(\mathcal{I}V)_t = X_t^*(A_t) - f^*(A_0) - \sum_{i=1}^m \int_0^t a^{i*}(s, A_s)\, dN_s^i.$$

Therefore, the mapping $\mathcal{I}$ can be extended to $\mathcal{A}_0$, and even to processes $V_t = X_t(A_t)$ where $X_t(x)$ is the derivative in x of a C^0 Itô field (thus $X_t(.)$ is a distribution). However in this case we do not know whether $(\mathcal{I}V)_t$ is a forward integral.

We complete this subsection with the computation of the bracket of the previous integrals.

Lemma 18 *Let $U, V \in \mathcal{A}_1$. Then*

$$\left[\int_0^{\cdot} U_s\, d^- A_s, \int_0^{\cdot} V_s\, d^- A_s\right]_t = \int_0^t U_s V_s\, d\,[A, A]_s\,. \tag{20}$$

PROOF. We can write $V_t = X_t(A_t)$, $U_t = Y_t(A_t)$, where $X_t(x)$ and $Y_t(x)$ are C^1-Itô fields given by

$$X_t(x) = f(x) + \sum_{i=1}^m \int_0^t a^i(s, x)\, dN_s^i, \tag{21}$$

$$Y_t(x) = g(x) + \sum_{i=1}^m \int_0^t \alpha^i(s, x)\, dN_s^i \tag{22}$$

(it is not restrictive to take the same driving semimartingales). First of all we recall that bounded variation processes give zero contributions in brackets calculations. Formula (18) implies that the left-hand-side of (20) equals

$$\left[X^*(A) - \sum_{i=1}^m \int_0^{\cdot} a^{i*}\left(s, A_s\right)\, dN_s^i,\ Y^*(A) - \sum_{i=1}^m \int_0^{\cdot} \alpha^{i*}\left(s, A_s\right)\, dN_s^i\right]_t.$$

Proposition 7 and the bilinearity of covariation gives the right member of (20), after a few easy calculations (see next lemma for similar ones).

Lemma 19 *Let $V \in \mathcal{A}_1$, M a $\mathcal{F}$-local martingale, R a continuous $\mathcal{F}$-adapted process. Assume that $[A, M]$ exists. Then*

$$\left[\int_0^{\cdot} V_s d^- A_s, \int_0^{\cdot} R_s dM_s\right]_t = \int_0^t V_s R_s d\,[A, M]_s\,. \tag{23}$$

PROOF. We assume the representation (21), with $V_t = X_t\,(A_t)$. From (18) we have that the left-hand-side of (23) equals

$$\left[X^*\,(A) - \sum_{i=1}^{m} \int_0^{\cdot} a^{i*}\,(s, A_s)\, dN_s^i, \int_0^{\cdot} R_s dM_s\right]_t.$$

To $\left[X^*\,(A)\,, \int_0^{\cdot} R_s dM_s\right]_t$ we apply proposition 7 and have

$$\sum_{i=1}^{m} \int_0^t a^{i*}\,(s, A_s)\, R_s d\left[N^i, M\right]_s + \int_0^t X_s\,(A_s)\, R_s d\,[A, M]_s\,.$$

The sum is

$$\int_0^t X_s\,(A_s)\, R_s d\,[A, M]_s$$

completing the proof (since $V = X(A)$).

From the previous facts we can deduce an interesting result.

Corollary 20 *If A is a Dirichlet process, i.e. $A = M + \tilde{A}$, with M a local martingale and $\tilde{A}$ a process with zero quadratic variation, then for all $V \in \mathcal{A}_1$ the forward integral $\int_0^{\cdot} V_s d^- A_s$ is a Dirichlet process.*

PROOF. From the definition of forward integral, the forward integral $\int_0^{\cdot} V_s d^- \tilde{A}_s$ exists (by linearity), equal to

$$\int_0^{\cdot} V_s d^- A_s - \int_0^{\cdot} V_s dM_s.$$

Then it is sufficient to apply the bilinearity of the covariation and the previous two lemmata, to show that $\int_0^{\cdot} V_s d^- \tilde{A}_s$ has zero quadratic variation. Since $\int_0^{\cdot} V_s dM_s$ is a local martingale, the corollary is proved.

4.2 Forward integral with more general integrator

The following proposition states simultaneously the existence of the announced forward integral $\int_0^t X_s(A_s)\, d^- Y_s(A_s)$ and a generalization of Itô-Wentzell formula to a chain rule. Setting $V = 1$, in the next formula, we obtain the previous Itô-Wentzell formula.

Proposition 21 *(Itô-Wentzell chain rule) For every $V \in \mathcal{A}_1$ and $U \in \mathcal{A}_2$, the forward integral*

$$\int_0^t V_s \, d^- U_s \tag{24}$$

exists and belongs to $\mathcal{A}_2$. If $V_t = X_t(A_t)$, $U_t = Y_t(A_t)$, where $(X_t(x))$ and $(Y_t(x))$ have the representations (21), (22), we have the formula

$$\int_0^t V_s \, d^- U_s = \sum_{i=1}^m \int_0^t V_s \alpha^i(s, A_s)\, dN_s^i + \int_0^t V_s \frac{\partial Y_s}{\partial x}(A_s)\, d^- A_s$$

$$+\frac{1}{2}\int_0^t V_s \frac{\partial^2 Y_s}{\partial x^2}(A_s)\, d[A,A]_s + \sum_{i=1}^m \int_0^t V_s \frac{\partial \alpha^i}{\partial x}(s, A_s)\, d\left[N^i, A\right]_s. \tag{25}$$

Remark 22 *We already know from the previous subsection that the forward integral*

$$\int_0^t R_s \, d^- A_s$$

exists if $R \in \mathcal{A}_1$. In particular here

$$\int_0^t V_s \frac{\partial Y_s}{\partial x}(A_s)\, d^- A_s$$

exists because $V \cdot \frac{\partial Y}{\partial x}(A) \in \mathcal{A}_1$.

PROOF. The proof is very similar to the case $V = 1$. One has to prove the ucp convergence of

$$\int_0^t V_s \frac{U_{s+\varepsilon}(A_{s+\varepsilon}) - U_s(A_s)}{\varepsilon} ds.$$

The existence of $\int_0^t V_s \frac{\partial Y_s}{\partial x}(A_s)\, d^- A_s$, known a priori from the previous section, is needed in the proof. We omit the details.

The next corollary expresses a nontrivial substitution property.

Corollary 23 *Let $V, Z \in \mathcal{A}_1$. We set $U_t = \int_0^t Z_s d^- A_s$ (which belongs to $\mathcal{A}_2$, so $\int_0^t V_s d^- U_s$ is well defined). Then*

$$\int_0^t V_s d^- U_s = \int_0^t V_s Z_s d^- A_s. \tag{26}$$

PROOF. We have to use (25), so we need a representation of U as $U_t = Y_t(A_t)$, Y having a representation of the form (22). However, we have the additional information $U_t = \int_0^t Z_s d^- A_s$, which provides formulas for Y and α^i in terms of the representation of $Z \in \mathcal{A}_1$. Substituting these formulas into (25) we shall find the result.

Assume $Z_t = X_t(A_t)$, X with representation

$$X_t(x) = f(x) + \sum_{i=1}^{m_0} \int_0^t a^i(s,x)\, dN_s^{0,i}.$$

Then we have (18), with $m = m_0$ and $N^i = N^{0,i}$, so $U_t = Y_t(A_t)$ with

$$Y_t(x) = X_t^*(x) - f^*(A_0) - \sum_{i=1}^{m_0} \int_0^t a^{i*}(s, A_s)\, dN_s^{0,i}$$

$$-\frac{1}{2}\int_0^t \frac{\partial X_s}{\partial x}(A_s)\, d[A,A]_s - \frac{1}{2}\sum_{i=1}^{m_0} \int_0^t a^i(s,A_s)\, d\left[N^{0,i}, A\right]_s.$$

To compare this representation of $Y_t(x)$ with (22) we have to take $m = 2m_0 + 1$,

$$N^i = N^{0,i} \text{ and } \alpha^i(s,x) = a^{i*}(s,x) - a^{i*}(s,A_s) \text{ for } i = 1, ..., m_0,$$

$$N^{m_0+1} = [A,A],\ \alpha^{m_0+1}(s,x) = -\frac{1}{2}\frac{\partial X_s}{\partial x}(A_s),$$

$$N^i = \left[N^{0,i}, A\right] \text{ and } \alpha^i(s,x) = -\frac{1}{2}a^i(s,A_s) \text{ for } i = m_0+2, ..., 2m_0+1,$$

$$g(x) = f^*(x) - f^*(A_0).$$

Therefore in (25) we have to substitute

$$\alpha^i (s,x) = 0 \text{ for } i = 1, ..., m_0,$$

$$\frac{\partial Y_s}{\partial x} (A_s) = X_s (A_s) = Z_s;$$

and notice that in (25)

$$\int_0^t V_s \alpha^{m_0+1} (s, A_s)\, dN_s^{m_0+1} + \frac{1}{2} \int_0^t V_s \frac{\partial^2 Y_s}{\partial x^2} (A_s)\, d[A, A]_s = 0$$

$$\sum_{i=m_0+2}^{2m_0+1} \int_0^t V_s \alpha^i (s, A_s)\, dN_s^i + \sum_{i=1}^{m} \int_0^t \frac{\partial \alpha^i}{\partial x} (s, A_s)\, d\left[N^i, A\right]_s = 0.$$

Therefore (25) reduces to (26). The proof is complete.

Finally, let us compute the covariation of these new integrals.

Proposition 24 *Let $U, V \in \mathcal{A}_1$, $\Phi, \Psi \in \mathcal{A}_2$. Then*

$$\left[\int_0^{\cdot} U_s d^- \Phi_s, \int_0^{\cdot} V_s d^- \Psi_s\right]_t = \int_0^t U_s V_s d\, [\Phi, \Psi]_s\, .$$

The proof is a not difficult exercise based on the bilinearity of the covariation, the representation (25), lemma 18, lemma 19 and proposition 7.

4.3 A-Itô processes

We have introduced processes of the form $V_t = X_t(A_t)$ and we have developed some rules of calculus for these processes. In comparison with the classical Itô calculus, it would be interesting to have a representation of these new processes as sum of stochastic integrals and to have rules of calculus based on this representation.

Lemma 25 *A process V is of class $\mathcal{A}_2$ if and only if it has a representation of the form*

$$V_t = V_0 + \sum_{i=1}^{m} \int_0^t \alpha^i(s)\, dN_s^i + \int_0^t \gamma(s)\, d^- A_s \tag{27}$$

where α^i and γ are continuous $\mathcal{F}$-adapted processes, with $\gamma \in \mathcal{A}_1$, V_0 is $\mathcal{F}_0$-measurable and $N = (N^1, ..., N^m)$ is a continuous $\mathcal{F}$-semimartingale, (A, N) with all its mutual brackets.

This result is a rewriting of formula (17) in one direction, and of proposition 15 in the opposite direction (the semimartingales may change from one representation to the other). The elements of $\mathcal{A}_2$ may be called *A-Itô processes.*

Remark 26 *The decomposition of an A-Itô process V in general is not unique.*

Remark 27 *If A is a Dirichlet process, then every A-Itô process is Dirichlet too. This follows from Corollary 20 and the representation (27).*

Some basic rules of calculus based on this representation are expressed by the following proposition. The assumption $F \in C^{1,3}$ is stronger than the classical one. It is needed, in view of the following remark, to have that $\frac{\partial F}{\partial x_i}(X_t^1(A_t), ..., X_t^n(A_t))$, where $X_t^i(A_t) = V_t^i$, is of the form $Y_t(A_t)$ with $(Y_t(x))$ an Itô field (we need to apply Itô formula to the composition $\frac{\partial F}{\partial x_i}(X_t^1(x), ..., X_t^n(x)))$.

Remark 28 *If $X^1(A), ..., X^n(A) \in \mathcal{A}_2$, and $\varphi \in C^2(\mathbb{R}^n)$, then*

$$\varphi\left(X^1(A), ..., X^n(A)\right) \in \mathcal{A}_2.$$

Indeed, by Itô formula, $(t, x) \mapsto \varphi(X_t^1(x), ..., X_t^n(x))$ is a C^2 Itô field. Unfortunately, even if in certain applications we need only

$$\varphi\left(X^1(A), ..., X^n(A)\right) \in \mathcal{A}_1,$$

we have to impose C^2 regularity on φ and the $X_t^i(x)$. In the next proposition we apply this remark to $\varphi = \frac{\partial F}{\partial x_i}$.

Proposition 29 *Let $V^1, ..., V^n \in \mathcal{A}_2$ and $F(t, x_1, ..., x_n)$ of class $C^{1,3}$. Then*

$$F\left(V_t^1, ..., V_t^n\right) = F\left(V_0^1, ..., V_0^n\right) + \sum_{i=1}^n \int_0^t \frac{\partial F}{\partial x_i}\left(V_s^1, ..., V_s^n\right) d^- V_s^i$$

$$+\frac{1}{2}\sum_{i,j=1}^{n}\int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}\left(V_s^1,...,V_s^n\right) d\left[V^i,V^j\right]_s \tag{28}$$

where the brackets exist by proposition 7, and the forward integrals exist by proposition 21 (the integrators belong to $\mathcal{A}_2$, the integrands to $\mathcal{A}_2$ by the previous remark). Moreover, as in the case of semimartingales, we can make the substitution in the forward integrals:

$$\int_0^t \frac{\partial F}{\partial x_i}\left(V_s^1,...,V_s^n\right) d^- V_s^k = \sum_{i=1}^{m}\int_0^t \frac{\partial F}{\partial x_i}\left(V_s^1,...,V_s^n\right) \alpha^{i,k}(s)\, dN_s^i$$

$$+\int_0^t \frac{\partial F}{\partial x_i}\left(V_s^1,...,V_s^n\right) \gamma^k(s)\, d^- A_s. \tag{29}$$

where $\alpha^{i,k}$ and γ^k are the coefficients of V^k in a representation of the form (27).

The Itô formula (28) is an application of the classical Itô formula of remark 4: the process $(V^1,...,V^n)$ has all its mutual brackets and the forward integrals exist. Equation (29) is a consequence of (27) and corollary 23.

Example 30 *If $F(x_1,x_2) = x_1 x_2$, we have*

$$V_t^1 V_t^2 = V_0^1 V_0^2 + \int_0^t V_s^2 d^- V_s^1 + \int_0^t V_s^1 d^- V_s^2 + \left[V^1,V^2\right]_t. \tag{30}$$

5 On a SDE driven by a finite quadratic variation process and a semimartingale

We are interested in the well posedness of the following SDE:

$$X_t = x + \int_0^t \sigma(X_s)\, d^- A_s + \int_0^t b(s,X_s)\, dN_s \tag{31}$$

where N is a $\mathcal{F}$-semimartingale, A ia a continuous $\mathcal{F}$-adapted process such that (A,N) has all its mutual brackets. We make also the following assumptios on σ and b:

a) $\sigma \in C^3(\mathbb{R})$, σ', σ'' bounded,

b) $b : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ being a.s. continuous and Lipschitz continuous in the second argument, and $\mathcal{F}$-adapted.

The main result of this section is the following theorem.

Theorem 31 *There is a unique $\mathcal{F}$-adapted solution in $\mathcal{A}_2$ to equation (31).*

Before proving the theorem we need to establish some preliminaries. The main tool in the proof is the so called Doss-Sussman transform as in [21]. Let $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be defined as the unique solution to

$$\begin{cases} \frac{\partial F}{\partial r}(r,x) & = & \sigma\left(F\left(r,x\right)\right) \\ F(0,x) & = & x. \end{cases}$$

The classical theory of ordinary differential equations tells us that F defines a flow. σ being of class C^3, $F\left(r,.\right)$ is a C^3-diffeomorphism on $\mathbb{R}$. We set

$$H\left(r,x\right) = F^{-1}\left(r,x\right),$$

the inverse being taken with respect to the second variable x. H is again of class C^3.

In order to prove uniqueness we need to state some properties of the functions F and H.

Lemma 32

$$a)\ \frac{\partial F}{\partial x}\left(r,x\right) = \exp\left(\int_0^r \sigma'\left(F\left(s,x\right)\right) ds\right)$$

$$b)\ \frac{\partial H}{\partial r}\left(r,x\right) = -\sigma\left(x\right)\frac{\partial H}{\partial x}\left(r,x\right)$$

$$c)\ \frac{\partial^2 H}{\partial r^2}\left(r,x\right) = -\sigma\left(x\right)\frac{\partial^2 H}{\partial r \partial x}\left(r,x\right)$$

$$d)\ \frac{\partial H}{\partial x}\left(r,F\left(r,y\right)\right) = \frac{1}{\frac{\partial F}{\partial x}\left(r,y\right)}$$

$$\begin{aligned} e)\ \frac{\partial^2 H}{\partial r \partial x}\left(r,x\right) &= -\sigma'\left(x\right)\frac{\partial H}{\partial x}\left(r,x\right) \\ &\quad -\sigma\left(x\right)\left(\frac{\partial H}{\partial x}\left(r,x\right)\right)^2 \int_0^r \sigma''\left(F\left(s,H\left(r,x\right)\right)\right) ds \end{aligned}$$

$$f)\ \frac{\partial^2 H}{\partial r^2}(r,x) = \sigma(x)\sigma'(x)\frac{\partial H}{\partial x}(r,x) + \sigma^2(x)\left(\frac{\partial H}{\partial x}(r,x)\right)^2 \int_0^r \sigma''(F(s,H(r,x)))\,ds$$

$$g)\ \frac{\partial^2 H}{\partial x^2}(r,x) = -\left(\frac{\partial H}{\partial x}(r,x)\right)^2 \int_0^r \sigma''(F(s,H(r,x)))\,ds.$$

PROOF. All these properties follow essentially from the identity

$$F(r,H(r,x)) \equiv x \text{ for every } r$$

and further derivation.

The basic idea of the proof of uniqueness in theorem 1 is the application of Itô formula to $H(A_t, X_t)$. This will involve mutual brackets between A and X where X is a solution to (31).

Lemma 33 *If $X \in \mathcal{A}_2$ satisfies (31), then*

$$a)\ [X,X]_t = \int_0^t \sigma^2(X_s)\,d[A,A]_s + 2\int_0^t \sigma(X_s)\,b(s,X_s)\,d[A,N]_s + \int_0^t b^2(s,X_s)\,d[N,N]_s$$

$$a)\ [X,A]_t = \int_0^t \sigma(X_s)\,d[A,A]_s + \int_0^t b(s,X_s)\,d[A,N]_s.$$

PROOF. In section 4.3 we showed that X admits a suitable decomposition (27). In particular, since X solves equation (31), we use the decomposition given by the right-hand-side of (31). Using lemma 18 and lemma 19, a) and b) follow easily.

PROOF of the theorem. We start with uniqueness. Let X be a solution to (31). Setting $Y_t = H(A_t, X_t)$ and applying classical Itô formula (remark 4) we get

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \frac{\partial H}{\partial r}(A,X)\, d^- A + \int_0^t \frac{\partial H}{\partial x}(A,X)\, d^- X \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 H}{\partial r^2}(A,X)\, d[A,A] + \int_0^t \frac{\partial^2 H}{\partial r \partial x}(A,X)\, d[A,X] \\ &\qquad + \frac{1}{2}\int_0^t \frac{\partial^2 H}{\partial x^2}(A,X)\, d[X,X]. \end{aligned} \tag{32}$$

By the previous lemma, (X,A) has all its mutual brackets and $\frac{\partial H}{\partial r}(A,X)$, $\frac{\partial H}{\partial x}(A,X)$ belong to $\mathcal{A}_1$ because $\frac{\partial H}{\partial r}$, $\frac{\partial H}{\partial x} \in C^2(\mathbb{R}^2)$ and $A, X \in \mathcal{A}_2$, see remark 28, so that (32) is justified.

We will now express partial derivatives of H with respect to $\frac{\partial H}{\partial x}$, $\frac{\partial^2 H}{\partial x^2}$. So, taking into account respectively lemma 32.b and the previous remark (to treat the first two terms of (32)), lemma 32.f-2.e and lemma 33.b (to treat the second two terms of (32)), lemma 32.g and lemma 33.a (for the last term of (32)), we get

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \frac{\partial H}{\partial x}(A_s, X_s)\, b(s, X_s)\, dN_s \\ &+ \frac{1}{2}\int_0^t \left\{ \sigma\sigma' \frac{\partial H}{\partial x} + \sigma^2 \left(\frac{\partial H}{\partial x}\right)^2 \int_0^A \sigma''(F(u,H))\, du \right\}(A,X)\, d[A,A] \\ &+ \int_0^t \left\{ \sigma\left(-\sigma' \frac{\partial H}{\partial x}\right) - \sigma^2 \left(\frac{\partial H}{\partial x}\right)^2 \int_0^A \sigma''(F(u,H))\, du \right\}(A,X)\, d[A,A] \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 H}{\partial x^2}(A,X)\, \sigma^2(X)\, d[A,A] \end{aligned}$$

$$\begin{aligned} &- \int_0^t b(s,X_s) \left\{ \sigma' \frac{\partial H}{\partial x} - \sigma \left(\frac{\partial H}{\partial x}\right)^2 \int_0^{A_s} \sigma''(F(u,H))\, du \right\}(A_s, X_s)\, d[A,N]_s \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 H}{\partial x^2}(A_s, X_s)\, 2b(s,X_s)\, \sigma(X_s)\, d[A,N]_s \\ &- \frac{1}{2}\int_0^t \left\{ \left(\frac{\partial H}{\partial x}\right)^2 \int_0^{A_s} \sigma''(F(u,H))\, du \right\}(A_s, X_s)\, b^2(s,X_s)\, d[N,N]_s. \end{aligned}$$

We substitute

$$\frac{\partial^2 H}{\partial x^2}(A,X) = \left\{ \left(\frac{\partial H}{\partial x} \right)^2 \int_0^A \sigma''(F(s,H))\, ds \right\} (A,X)$$

and we perform several simplification. Finally we get

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \frac{\partial H}{\partial x}(A_s, X_s)\, b(s, X_s)\, dN_s \\ &\quad -\frac{1}{2} \int_0^t \left\{ \sigma\sigma' \frac{\partial H}{\partial x} \right\} (A,X)\, d[A,A] \\ &\quad - \int_0^t b(s,X_s) \left\{ \sigma' \frac{\partial H}{\partial x} \right\} (A_s, X_s)\, d[A,N]_s \\ &\quad -\frac{1}{2} \int_0^t \left\{ \left(\frac{\partial H}{\partial x} \right)^2 \int_0^{A_s} \sigma''(F(u,H))\, du \right\} (A_s, X_s)\, b^2(s, X_s)\, d[N,N]_s . \end{aligned}$$

Lemma 32.d gives us

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \frac{b(s, F(A_s, Y_s))}{\frac{\partial F}{\partial x}(A_s, Y_s)} dN_s \\ &\quad -\frac{1}{2} \int_0^t \frac{\sigma\sigma'(F(A,Y))}{\frac{\partial F}{\partial x}(A,Y)} d[A,A] \end{aligned}$$

$$\begin{aligned} & - \int_0^t b(s, F(A_s, Y_s)) \frac{\sigma'(F(A_s, Y_s))}{\frac{\partial F}{\partial x}(A_s, Y_s)} d[A,N]_s \\ & -\frac{1}{2} \int_0^t \left(\int_0^{A_s} \sigma''(F(u, Y_s))\, du \right) \frac{b^2(s, F(A_s, Y_s))}{\left(\frac{\partial F}{\partial x}(A_s, Y_s) \right)^2} d[N,N]_s . \end{aligned} \tag{33}$$

At this level we can say that Y solves a classical semimartingale driven SDE. For uniqueness and existence of that equation we need the following lemma.

Let $T > 0$. A function $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ will be said to belong to the class LL if h is locally Lipschitz with linear growth, which means

$$|h(t,x) - h(t,y)| \leq C_R |x-y|, \text{ for all } t \in [0,T],\ x,y \in B(0,R),$$

$$|h(t,x)| \leq C(1+|x|), \text{ for all } t \in [0,T],\ x \in \mathbb{R}.$$

Lemma 34 *Let $T>0$, $x\in\mathbb{R}$, $f_1,...,f_n:[0,T]\times\mathbb{R}\times\Omega\to\mathbb{R}$, a.s. continuous, $\mathcal{F}$-adapted, belonging a.s. to the class LL. Let $N^1,...,N^n$ be continuous $\mathcal{F}$-semimartingales. Then there exists a unique $\mathcal{F}$-adapted continuous solution (U_t) of the equation*

$$U_t = x + \sum_{i=1}^{n} \int_0^t f_i\left(s, U_s\right) dN_s^i. \tag{34}$$

PROOF. It follows from general results contained in [18].

We apply lemma 34 to equation (34) with $N^1=N$, $N^2=[A,A]$, $N^3=[A,N]$, $N^4=[N,N]$,

$$f_1\left(t,y\right) = \frac{b\left(t, F\left(A_t,y\right)\right)}{\frac{\partial F}{\partial x}\left(A_t,y\right)}$$

$$f_2\left(t,y\right) = -\frac{1}{2}\frac{\sigma\sigma'\left(F\left(A_t,y\right)\right)}{\frac{\partial F}{\partial x}\left(A_t,y\right)}$$

$$f_3\left(t,y\right) = -b\left(t, F\left(A_t,y\right)\right)\frac{\sigma'\left(F\left(A_t,y\right)\right)}{\frac{\partial F}{\partial x}\left(A_t,y\right)}$$

$$f_4\left(t,y\right) = \frac{1}{2}\left(\int_0^{A_t}\sigma''\left(F\left(u,y\right)\right)du\right)\frac{b^2\left(t,F\left(A_t,y\right)\right)}{\left(\frac{\partial F}{\partial x}\left(A_t,y\right)\right)^2}.$$

These fields are clearly continuous $\mathcal{F}$-adapted. It remains to prove that they belong a.s. to the LL class. For this the basic tools are the explicit expression of $\frac{\partial F}{\partial x}(r,x)$ given in lemma 32.a, and the following property: if $\sigma_1,\sigma_2:[0,T]\times\mathbb{R}\to\mathbb{R}$ belong to the class LL, and σ_1 is bounded, then $\sigma_1\cdot\sigma_2$ also belongs to the class LL.

At this level equation (33) admits a unique adapted solution and therefore uniqueness for (31) is established.

Existence follows through similar arguments applying Itô formula to $X_t = F\left(A_t,Y_t\right)$, Y being the unique solution to equation (33).

Remark 35 *The result of theorem 31 can be extended to equation (31) generalized to the case when there is a finite number of semimartingales, say*

$$X_t = x + \int_0^t \sigma(X_s)\, d^- A_s + \sum_{i=1}^n \int_0^t b^i(s, X_s)\, dN_s^i.$$

An important particular case is given in the affine case: $\sigma(x) = x$, $N^1 = N$, $N_t^2 = t$, $b^1(s,x) = x$, $b^2(s,x) = \gamma_s$, *where* γ *is a given continuous* $\mathcal{F}$*-adapted process.*

Next result is directly used in part II [6], and it can be derived from theorem 31, according to the previous remark. However, we shall provide a simpler independent proof, interesting also because it is close to the classical one for semimartingales (after the stochastic calculus of the previous sections has been developed).

Theorem 36 *Consider the linear (or more precisely affine) SDE*

$$X_t = x + \int_0^t X_s\, d^- A_s + \int_0^t X_s\, dN_s + \int_0^t g_s\, ds \tag{35}$$

where N *is a continuous* $\mathcal{F}$*-semimartingale and* A *is a continuous* $\mathcal{F}$*-adapted process such that* (N, A) *has all its mutual brackets, and* g *is an adapted process with integrable paths. Then there is a unique solution in* $\mathcal{A}_2$*, given by*

$$X_t = \mathcal{U}_t \left(x + \int_0^t \mathcal{U}_s^{-1} g_s\, ds \right) \tag{36}$$

where

$$\mathcal{U}_t = \exp\left(A_t + N_t - A_0 - N_0 - \frac{1}{2}[A+N]_t \right). \tag{37}$$

STEP 1. First notice that $\mathcal{U}_t$ and $\mathcal{U}_t^{-1}$ belong to $\mathcal{A}_2$, since $\exp(\pm A) \in \mathcal{A}_2$, semimartingales belong to $\mathcal{A}_2$, and $\mathcal{A}_2$ is an algebra. For the same reason $\mathcal{U}_t^{\pm 1} \in \mathcal{A}_k$ for every k. Let us prove that $\mathcal{U}_t$ solves equation (35) with $g \equiv 0$ and $x = 1$. The process

$$Y_t = A_t + N_t - A_0 - N_0 - \frac{1}{2}[A+N]_t$$

has finite quadratic variation, since (A,N) has all its mutual brackets, so by the classical Itô formula (remark 4) we have

$$\mathcal{U}_t = 1 + \int_0^t \mathcal{U}_s d^- Y_s + \frac{1}{2}\int_0^t \mathcal{U}_s d\,[Y,Y]_s$$

$$= 1 + \int_0^t \mathcal{U}_s d^- A_s + \int_0^t \mathcal{U}_s dN_s. \tag{38}$$

Similarly,

$$\mathcal{U}_t^{-1} = 1 - \int_0^t \mathcal{U}_s^{-1} d^- Y_s + \frac{1}{2}\int_0^t \mathcal{U}_s^{-1} d\,[Y,Y]_s$$

$$= 1 - \int_0^t \mathcal{U}_s^{-1} d^- A_s - \int_0^t \mathcal{U}_s^{-1} dN_s + \int_0^t \mathcal{U}_s^{-1} d\,[A+N]_s\,. \tag{39}$$

STEP 2. Let us prove that (36) provides a solution to equation (35). By linearity, it is sufficient to prove that

$$X_t = \mathcal{U}_t Z_t, \quad Z_t = \int_0^t \mathcal{U}_s^{-1} g_s ds,$$

is a solution of (35) with $x = 0$. Let us apply (30) to X_t (notice that Z has bounded variation):

$$X_t = \int_0^t Z_s d^- \mathcal{U}_s + \int_0^t \mathcal{U}_s dZ_s.$$

By the substitution argument (29) and by (38), we have

$$X_t = \int_0^t Z_s \mathcal{U}_s d^- A_s + \int_0^t Z_s \mathcal{U}_s dN_s + \int_0^t \mathcal{U}_s \mathcal{U}_s^{-1} g_s ds,$$

proving the claim.

STEP 3. Let us finally prove that a solution $X \in \mathcal{A}_2$ of (35) is necessarily given by (36). If $X \in \mathcal{A}_2$, we can apply (30) to get

$$\mathcal{U}_t^{-1} X_t = X_0 + \int_0^t X_s d^- \mathcal{U}_s^{-1} + \int_0^t \mathcal{U}_s^{-1} d^- X_s + \left[\mathcal{U}^{-1}, X\right]_t.$$

Assuming now that X solves (35), and using (39), by means of the substitution argument (29) we get

$$\mathcal{U}_t^{-1} X_t = x + \int_0^t \mathcal{U}_s^{-1} g_s ds + \int_0^t X_s \mathcal{U}_s^{-1} d\,[A+N]_s + \left[\mathcal{U}^{-1}, X\right]_t .$$

Moreover, again from (39) and (35) we have

$$\left[\mathcal{U}^{-1}, X\right]_t = -\int_0^t X_s \mathcal{U}_s^{-1} d\,[A+N]_s\,,$$

where we have to use lemmata lemma 18, lemma 19 and the bilinearity of the covariation. Therefore we have proved

$$\mathcal{U}_t^{-1} X_t = x + \int_0^t \mathcal{U}_s^{-1} g_s ds,$$

which implies (36). The proof is complete.

Acknowledgements The second author is grateful to Prof. Giuseppe Da Prato for the kind invitation to the Scuola Normale Superiore of Pisa and for the stimulating atmosphere created. The first author is grateful to Université de Paris 13 for the kind hospitality.

References

[1] M. T. Barlow, Skew Brownian motion and a one dimensional stochastic differential equation, *Stochastics* **25**, 1-2 (1988).

[2] J. Bertoin, Sur une integrale pour les processus a α-variation bornee, *Ann. Probab.* **17**, no. 4, 1521-1535 (1989).

[3] R. J. Chitashvili, T. A. Toronjadze, On one-dimensional stochastic differential equations with unit diffusion coefficient. Structure of solutions, *Stochastics* **4**, 281-315 (1981).

[4] C. Dellacherie, P.-A. Meyer, *Probabilités et Potentiels. Théorie des Martingales.* Chapter 8. Hermann, Paris 1980.

[5] H. J. Engelbert, On the theorem of T. Yamada and S. Watanabe, *Stochastics and Stoch. Reports* **36**, 205-216 (1991).

[6] F. Flandoli, F. Russo, Generalized Integration and Stochastic ODEs. Part II, preprint.

[7] H. Föllmer, Calcul d'Itô sans probabilités, *Seminaire de Probabilites* **XV**, 143-150, LNM 850, Springer-Verlag, Berlin, 1981.

[8] H. Föllmer, P. Protter, A. N. Shiryayev, Quadratic covariation and an extension of Itô's formula,*Bernoulli* **1**, 149-169 (1995).

[9] H. Kunita, *Stochastic Differential Equations and Stochastic Flows of Diffeomorphisms*, Ecole d'été de Saint-Flour XII, 1982, LNM 1097, P.L. Hennequin Ed., Springer-Verlag, Berlin 1984.

[10] J. F. Le Gall, Applications du temps local aux equations differentielles stochastiques unidimensionelles, *Seminaire de Probabilites* **XVII**, 15-31, LNM 986, Springer-Verlag, Berlin, 1983.

[11] T.J. Lyons, Differential equations driven by rough signals, to appear on *Revista Math. Iberoamericana.*

[12] T.J. Lyons, T.S. Zhang, Decomposition of Dirichlet processes and its applications, *Ann. Probab.* **22**, 494-524 (1994).

[13] T.J. Lyons, W. Zheng, A crossing estimate for the canonical process on a Dirichlet space and tightness result, Colloque Paul Lévy sur les processus stochastiques, *Astérisque* **157-158**, 249-271 (1988).

[14] S. Moret, D. Nualart, Quadratic covariation and Itô formula, to appear on *Journal of Theoretical Probability.*

[15] S. Nakao, On the pathwise uniqueness of solutions of one-dimensional stochastic differential equations, *Osaka J. Math.* **9**, 513-518 (1972).

[16] S. Nakao, Stochastic calculus for continuous additive functionals of zero energy, *Wahrscheinlichkeitstheorie verw. Gebiete* **68**, 557-578 (1985).

[17] E. Perkins, Local times and pathwise uniqueness for stochastic differential equations, *Seminaire de Probabilites* **XVI**, 201-208, LNM 920, Springer-Verlag, Berlin, 1982.

[18] P. Protter, *Stochastic Integration and Differential Equations*, Springer-Verlag, New York 1992.

[19] F. Russo, P. Vallois, The generalized covariation process and Itô formula, *Stoch. Proc. Their Appl.* **59**, 81-104 (1995).

[20] F. Russo, P. Vallois, Itô formula for C^1-functions of semimartingales, *Probab. Theory Relat. Fields* **104**, 27-41 (1996).

[21] F. Russo, P. Vallois, Stochastic calculus with respect to a finite quadratic variation process, Preprint Université Paris 13, 1998. To appear: Stochastics and Stoch. Reports.

[22] D. Revuz, M. Yor, *Continuous Martingales and Brownian Motion*, 2^{nd} ed., Springer-Verlag, Berlin, 1994.

[23] A. Yu. Veretennikov, On strong solutions and explicit formulas for solutions of stochastic integral equations, *Math. USSR Sbornik* **39**, no. 3, 387-403 (1981).

[24] J. Wolf, Transformations of semimartingales and local Dirichlet processes, *Stochastics and Stoch. Rep.* **62**, 95-101 (1997).

[25] J. Wolf, An Itô formula for continuous local Dirichlet processes, *Stochastics and Stoch. Rep.* **62**, 103-115 (1997).

[26] T. Yamada, S. Watanabe, On the uniqueness of solutions of stochastic differential equations, *J. Math. Kyoto Univ.* **11**, 155-167 (1971).

[27] M. Zähle, Integration with respect to fractal functions and stochastic calculus. I, *Probab. Theory and Relat. Fields* **111**, 333-374 (1998).

[28] M. Zähle, On the link between fractional and stochastic calculus. Part II. *To appear: Mathematische Nachrichten.*

[29] A. K. Zvonkin, A transformation of the phase space of a diffusion process which annihilates the drift, *Math. USSR Sbornik* **22**, no. 1, 129-149 (1975).