An approach to the normalized invariant integral on compact groups

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Abstract. Let X be a compact topological group, starting from the Haar integral on the space of continuous functions on X we give an alternative approach to the normalized invariant integral $\mathcal{J}$ on X and we establish the most results known is this area so that we do not need any knowledge of the measure theory. We show in the last section how one can develop the measure theory using our approach to the invariant integral and we prove that $(X, \mathcal{A}, m)$ is a complete measure space where $\mathcal{A}$ is defined to be the collection of all subsets of X that have integrable characteristic functions and m denotes the measure defined for every $A \in \mathcal{A}$ by $m(A) = \mathcal{J}(\chi_A)$.

Key words: Compact groups, Invariant integral, Haar measure.

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1 Introduction

Through the 20th century serval theories of integration have been developed, for instance, the integral of Riemann, Lebesgue, Denjoy, Perron, McShane, Henstock-Kurzweil and other approaches dealing with functions defined on $\mathbb{R}$ or more generally on $\mathbb{R}^n$ (see [2, 4, 5, 7, 8, 11]).

Dealing with abstract cases, it is remarkable that a considerable amount of measure theory needs to be developed before that the integral can be studied (see [3, 6, 10, 14]). Unlike this conventional way A. Anger and C. Portenier investigated in [1] an abstract theory of integration which is not based on measure theory.

Let X be a compact topological group. Without assuming any knowledge of the measure theory we give in this paper an approach to the invariant integral on X and we obtain using this approach the most results known in this area.

We start from the Haar integral on X, i.e., a positive linear functional $\mathcal{I}$ on $\mathcal{C}(X)$ (= the set of all complex continuous functions on X) which is invariant under left and right translations, and we define the space $\mathcal{L}_1$ of integrable functions as the closure of $\mathcal{C}(X)$ with respect to the semi-norm $\| f \| := \inf \sum_{j=1}^{\infty} \mathcal{I}(\varphi_j)$ where the infimum is taken over all sequences $(\varphi_j) \in \mathcal{C}^+(X)$ satisfying $|f| \leq \sum_{j=1}^{\infty} \varphi_j$. In the case when $\mathcal{I}$ denotes the Riemann integral, this semi-norm was already used by A. Van Daele [12] (see also [13]) to develop the Lebesgue integral starting from the Riemann one without going to measure theory.

As the space $\mathcal{L}_1$ we define $\mathcal{L}_p$ for $p \in [1, \infty]$ to be the set of all functions that can be approximated by continuous functions with respect to a semi-norm $\| \cdot \|_p$ which is defined for every function f by $\| f \|_p^p = \| |f|^p \|$ if $p < \infty$ and $\| f \|_\infty = \lim_{p \to \infty} \| f \|_p$. Next we consider the space $\mathbb{L}_p$ obtained from $\mathcal{L}_p$ by identifying functions $f, g \in \mathcal{L}_p$ such that $\| f - g \| = 0$ and we prove that $\mathbb{L}_p$ equipped with $\| \cdot \|_p$ is a Banach space for every $1 \leq p \leq \infty$.

Given $f \in \mathbb{L}_1$, it will be shown that the the sequence $\mathcal{I}(\varphi_j)$ converges for any sequence $(\varphi_j) \in \mathcal{C}(X)$ satisfying $\lim_{j \to \infty} \| f - \varphi_j \| = 0$. Moreover the limit of $\mathcal{I}(\varphi_j)$ does not depend on the choice of the sequence (φ_j). Define this limit to be the integral of f and denote it by $\mathcal{J}(f)$, we get a natural extention $\mathcal{J}$ of the Haar integral $\mathcal{I}$ to the space $\mathcal{L}_1$ such that $\mathcal{J}$ is invariant under left and right translations. Furthermore, by easy and short proofs we obtain the most known results as the monotone convergence theorem, the dominated convergence theorem, Fatou lemma, Hölder inequality. Thus, our approach can be used as an easy method for initiating students, specially who are not interested in measure theory, to a study of integration on compact groups.

In the last section we show how one can develop measure theory starting from the approach of the integral presented in this paper. After investigating measurability of functions and sets we prove that $\mathbb{L}_p$ has the classical characterization, i.e., $f \in \mathbb{L}_p$ if and only if f is measurable on X and $\| f \|_p < \infty$. Defining $\mathcal{A}$ to be the collection of all subsets of X which have integrable characteristic functions then $\mathcal{A}$ is a σ-algebra and $(X, \mathcal{A}, m)$ is a complete measure space where m denotes the measure defined for every $A \in \mathcal{A}$ by $m(A) = \mathcal{J}(\chi_A)$. If $\mathcal{A}$ contains the Borel $\sigma-$algebra $\mathcal{B}(X)$, which is the case when X has a countable base of open subsets, we prove that m is the completion of the normalized Haar measure μ on X and $\mathcal{A}$ is the completion of $\mathcal{B}(X)$ with respect to μ.

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2 Integrable functions

Let X be a compact topological group and denote by e the neutral element of X. Note that we assume that every compact topological group satisfies the Hausdorff's axiom of separation. Let $\mathcal{F}_K$ (resp. $\mathcal{C}_K$) be the space of all functions (resp. continuous functions) defined on X with values in K where K denotes the field $\mathbb{R}$ or $\mathbb{C}$. For every subset A of X we denote by χ_A the characteristic function of A, i.e., $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \in X \backslash A$. If A is nonempty and finite we write $|A|$ to design the cardinal of A and we define the two following operators R_A and L_A by

$$R_A f(x) := \frac{1}{|A|} \sum_{a \in A} f(xa) \quad \text{and} \quad L_A f(x) := \frac{1}{|A|} \sum_{a \in A} f(ax) \tag{1}$$

for all $x \in X$ and all $f \in \mathcal{F}_{\mathbb{C}}$.

A positive linear form $\mathcal{I}$ on $\mathcal{C}_{\mathbb{C}}$ is called an invariant (or Haar) integral on X if $\mathcal{I}(R_a f) = \mathcal{I}(L_a f) = \mathcal{I}(f)$ for all $f \in \mathcal{C}_{\mathbb{C}}$ and all $a \in X$. The existence of such integral on compact groups is proved (see, e.g., [9]). Furthermore, there is only one normalized (i.e, $\mathcal{I}(\chi_X) = 1$) invariant integral on X; it will be denoted in this paper by $\mathcal{I}$. In [9, Sect. 29] it is shown that $\mathcal{I}(f)$ is the unique $\alpha \in \mathbb{C}$ such that for every $\varepsilon > 0$ we can find two nonempty finite subsets $A, B \subset X$ satisfying $|R_A f - \alpha| \leq \varepsilon$ and $|L_B f - \alpha| \leq \varepsilon$. For every $f \in \mathcal{C}_{\mathbb{C}}$, the complex number $\mathcal{I}(f)$ is said the integral of f and we write $\mathcal{I}(f) = \int_X f(x)\, dx$.

In this section we will define the integral of every function $f \in \mathcal{L}_1$, not necessarily continuous, but let us first remark that for a real continuous function f it is easy to show in view of Dini's theorem (see, e.g., [15]) that

$$\lim_{j \to \infty} \int_X f_j(x)\, dx = \int_X f(x)\, dx$$

for any nondecreasing sequence (f_j) of real continuous functions such that $\lim_{j \to \infty} f_j(x) = f(x)$ for all $x \in X$.

For every function $f \in \mathcal{F}_{\mathbb{C}}$ we define

$$\| f \| := \inf \left\{ \sum_{j=1}^{\infty} \int_X \varphi_j(x)\, dx : (\varphi_j) \in \mathcal{E}_f \right\}, \tag{2}$$

where $\mathcal{E}_f$ denotes the set of all sequences (φ_j) of nonnegative continuous functions on X such that $|f| \leq \sum_{j=1}^{\infty} \varphi_j$. It is clear that $\mathcal{E}_f \neq \emptyset$, in fact it contains the sequence (φ_j) defined by $\varphi_j \equiv 1$ for all $j \geq 1$.

LEMMA 2.1 *Let $f \in \mathcal{F}_{\mathbb{C}}$ and let (f_j) be a sequence in $\mathcal{F}_{\mathbb{C}}$. If $|f| \leq \sum_{j=1}^{\infty} |f_j|$ then*

$$\| f \| \leq \sum_{j=1}^{\infty} \| f_j \| .$$

Proof. without loss of generality we may assume that $\| f_j \| < \infty$ for all $j \geq 1$. Take $\varepsilon > 0$, then for every $j \geq 1$ there exists a sequence $({}^j\varphi_i)_{i \geq 1}$ such that

$$({}^j\varphi_i)_{i \geq 1} \in \mathcal{E}_{f_j}, \tag{3}$$

$$\sum_{i=1}^{\infty} \int_X {}^j\varphi_i(x)\, dx \leq \| f_j \| + \frac{\varepsilon}{2^j}. \tag{4}$$

By (3) the sequence $({}^j\varphi_i)_{i,j\geq 1} \in \mathcal{E}_f$ and by (4)

$$\sum_{i,j=1}^{\infty} \int_X {}^j\varphi_i(x)\,dx \leq \sum_{j=1}^{\infty} \| f_j \| + \varepsilon.$$

Hence $\| f \| \leq \sum_{j=1}^{\infty} \| f_j \| + \varepsilon$. Since $\varepsilon > 0$ is arbitrary small the lemma is proved. $\square$

In virtue of the above lemma we remark that for every $f, g \in \mathcal{F}_{\mathbb{C}}$ we have

$$\| f + g \| \quad \leq \quad \| f \| + \| g \|, \tag{5}$$

$$|f| \leq |g| \quad \Longrightarrow \quad \| f \| \leq \| g \| . \tag{6}$$

In the other hand it is not difficult to show that if $f \in \mathcal{F}_{\mathbb{C}}$ and $\lambda \in \mathbb{C}$ then

$$\| \lambda f \| = |\lambda| \ \| f \| . \tag{7}$$

LEMMA 2.2 *For every $f \in \mathcal{C}_{\mathbb{C}}$ we have*

$$\| f \| = \int_X |f(x)|\,dx.$$

Proof. Consider $f_1 = |f|$, $f_j \equiv 0$ for all $j \geq 2$. Then the sequence $(f_j) \in \mathcal{E}_f$ and by (2) we have

$$\| f \| \leq \int_X |f(x)|\,dx.$$

Let $(\varphi_j) \in \mathcal{E}_f$ and define for every $n \geq 1$

$$\psi_n = \inf \left\{ |f|, \sum_{j=1}^{n} \varphi_j \right\}$$

Then (ψ_n) is a nondecreasing sequence of real continuous functions on X such that $\sup_{n\geq 1} \psi_n(x) = |f(x)|$ for all $x \in X$. Hence

$$\lim_{n\to\infty} \int_X \psi_n(x)\,dx = \int_X |f(x)|\,dx.$$

On the other hand

$$\int_X \psi_n(x)\,dx \leq \sum_{j=1}^{\infty} \int_X \varphi_j(x)\,dx.$$

Therefore

$$\int_X |f(x)|\,dx \leq \sum_{j=1}^{\infty} \int_X \varphi_j(x)\,dx. \tag{8}$$

The fact that the above inequality holds true for every $(\varphi_j) \in \mathcal{E}_f$ implies that

$$\int_X |f(x)|\,dx \leq \| f \| .$$

□

A function $f \in \mathcal{F}_{\mathbb{C}}$ will be said to be *integrable* on X if for each $\varepsilon > 0$ we can find a complex continuous function φ on X such that $\| f - \varphi \| \leq \varepsilon$. We denote by $\mathcal{L}_1$ the set consisting of all integrable functions on X. It is easy to see that $\mathcal{L}_1$ is a complex linear space containing $\mathcal{C}_{\mathbb{C}}$. Moreover, $\Re(f)$, $\Im(f)$, $\overline{f}$, $|f| \in \mathcal{L}_1$ if $f \in \mathcal{L}_1$. Remark that $\| f \|$ is a finite nonnegative real for every $f \in \mathcal{L}_1$, hence in virtue of (5) and (7), the functional $\| \cdot \|$ is a semi-norm on $\mathcal{L}_1$. In the next section we will prove that $\| \cdot \|$ can not define a norm on the space $\mathcal{L}_1$ if the group X is not finite.

THEOREM 2.3 *Let (f_j) be a sequence of integrable functions on X and assume that $\lim_{i,\, j \to \infty} \| f_i - f_j \| = 0$. Then there exists a function $f \in \mathcal{L}_1$ such that $\lim_{j \to \infty} \| f_j - f \| = 0$. (We say that $\mathcal{L}_1$ is complete for the semi-norm $\| \cdot \|$.)*

Proof. Because we may choose a subsequence of (f_j) satisfying the property (9) which follows, we suppose without loss of generality that

$$\| f_{j+1} - f_j \| \leq \frac{1}{2^j}, \text{ for all } j \geq 1. \tag{9}$$

Let f be the function defined on X by

$$f(x) = \lim_{j \to \infty} f_j(x) \quad \text{if this limit exists,} \tag{10}$$

$$f(x) = 0 \quad \text{if not.} \tag{11}$$

Hence without difficulty we can show that for every $i \geq 1$

$$|f(x) - f_i(x)| \leq \sum_{j=i}^{\infty} |f_{j+1}(x) - f_j(x)|. \tag{12}$$

Applying Lemma 2.1, we obtain

$$\| f - f_i \| \leq \sum_{j=i}^{\infty} \| f_{j+1} - f_j \| \leq \frac{1}{2^{i-1}}$$

for every $i \geq 1$, then $\lim_{i \to \infty} \| f_i - f \| = 0$. The theorem will be proved provided we show that f is integrable on X. Since $f_1, f_2, \ldots$ are integrable functions on X we can find $\varphi_1, \varphi_2, \ldots \in \mathcal{C}_{\mathbb{C}}$ such that

$$\| f_j - \varphi_j \| \leq \frac{1}{2^{j-1}}, \;\; j \geq 1.$$

Therefore

$$\| f - \varphi_j \| \leq \| f - f_j \| + \| f_j - \varphi_j \| \leq \frac{1}{2^j},$$

and hence the proof is complete. □

Proposition 2.4 *Let $f \in \mathcal{L}_1$ and let (φ_j) be a sequence of continuous functions on X which converges to f for $\|\cdot\|$. Then $(\int_X \varphi_j \, dx)$ has a limit in $\mathbb{C}$ which does not depend on the sequence (φ_j).*

Proof. By Lemma 2.2 we have

$$|\int_X \varphi_i(x)\, dx - \int_X \varphi_j(x)\, dx| \leq \| \varphi_i - \varphi_j \|$$

for all $i, j \geq 1$. This shows that $(\int_X \varphi_j \, dx)$ is a Cauchy sequence in $\mathbb{C}$ and therefore it has a limit in $\mathbb{C}$. To prove the second part of the proposition, take a sequence $(\psi_j) \in \mathcal{C}_{\mathbb{C}}$ which converges to f for $\|\cdot\|$. For every $j \geq 1$ we have

$$|\int_X \varphi_j(x)\, dx - \int_X \psi_j(x)\, dx| \leq \| \varphi_j - \psi_j \| \leq \| \varphi_j - f \| + \| f - \psi_j \|,$$

and hence

$$\lim_{j\to\infty} \int_X \varphi_j(x)\, dx = \lim_{j\to\infty} \int_X \psi_j(x)\, dx.$$

□

For each $f \in \mathcal{L}_1$ let $\mathcal{J}(f)$ denotes the complex number given by

$$\mathcal{J}(f) = \lim_{j\to\infty} \int_X \varphi_j(x)\, dx, \tag{13}$$

where (φ_j) is a sequence in $\mathcal{C}_{\mathbb{C}}$ which converges to f for $\|\cdot\|$. It is trivial that $\mathcal{J}$ is a positive linear form on $\mathcal{L}_1$ and it satisfies

$$\mathcal{J}(R_a f) = \mathcal{J}(L_a f) = \mathcal{J}(f), \tag{14}$$

for every $f \in \mathcal{L}_1$ and for all $a \in X$. In order to justify (14), it is enough to recall that if $a \in X$ and $\varphi \in \mathcal{C}_{\mathbb{C}}$ then

$$\int_X R_a\varphi(x)\, dx = \int_X L_a\varphi(x)\, dx = \int_X \varphi(x)\, dx.$$

On the other hand by considering the sequence (φ_j) given by $\varphi_j \equiv f$ for every $j \geq 1$, it follows in view of the above Proposition that

$$\mathcal{J}(f) = \mathcal{I}(f) = \int_X f(x)\, dx, \tag{15}$$

for every continuous function f on X. This means that $\mathcal{J}$ is a linear extention of $\mathcal{I}$ to the space $\mathcal{L}_1$. Thus, for each integrable function f on X it becomes natural to call the complex number $\mathcal{J}(f)$ given by (13) the (*invariant*) *integral* of f over X. From Lemma 2.2 and the definition of $\mathcal{J}$ we deduce the following proposition.

Proposition 2.5 *For every $f \in \mathcal{L}_1$ we have $\| f \| = \mathcal{J}(|f|)$.*

Of course $\| f \| < \infty$ if $f \in \mathcal{L}_1$, however the converse statement is in general not true (see Proposition 5.5.*ii*). In fact, as in the classical case, some notion of measurability of the function f needs to be developed. This problem will be investigate in section 5 (see Theorem 5.3). In the remainder of this section we establish some convergences theorems.

THEOREM 2.6 *Let $f \in \mathcal{F}_{\mathbb{R}}$ and let (f_j) be a nondecreasing sequence of real integrable functions on X such that $\sup_{j\geq 1} f_j(x) = f(x)$ for all $x \in X$. If $\sup_{j\geq 1} \mathcal{J}(f_j)$ is finite then $f \in \mathcal{L}_1$ and $\mathcal{J}(f) = \sup_{j\geq 1}(f_j)$.*

Proof. Choose $i \geq 1$; since $|f - f_i| \leq \sum_{j=i}^{\infty} |f_{j+1} - f_j|$ the Lemma 2.1 implies that

$$\| f - f_i \| \leq \sum_{j=i}^{\infty} \| f_{j+1} - f_j \| .$$

On the other hand, $\| f_{j+1} - f_j \| = \mathcal{J}(f_{j+1} - f_j)$ by Proposition 2.5. Then

$$\| f - f_i \| \leq -\mathcal{J}(f_i) + \lim_{j\to\infty} \mathcal{J}(f_j)$$

which shows that $\lim_{i\to\infty} \| f - f_i \| = 0$ and so $f \in \mathcal{L}_1$ by Theorem 2.3. To finish the proof it is enough to remark that $\mathcal{J}(f - f_i) = \| f - f_i \|$ for all $i \geq 1$.□

The fact that $2\max\{f, g\} = f + g + |f - g|$ and $2\min\{f, g\} = f + g - |f - g|$ for every functions $f, g \in \mathcal{F}_{\mathbb{R}}$ shows that $\max\{f, g\}$ and $\min\{f, g\}$ are integrable on X if $f, g \in \mathcal{L}_1 \cap \mathcal{F}_{\mathbb{R}}$. Hence applying the above theorem the following proposition will be obvious.

PROPOSITION 2.7 *Let $f_1, f_2, \ldots$ be real and integrable functions on X such that $f_j \leq g$ (resp. $f_i \geq g$) for all $j \geq 1$ where $g \in \mathcal{L}_1$. Then $\sup_{j\geq 1} f_j$ (resp. $\inf_{j\geq 1} f_j$) is integrable on X.*

THEOREM 2.8 *Let $f \in \mathcal{F}_{\mathbb{R}}$ and let (f_j) be a sequence of nonnegative integrable functions on X such that $f(x) = \liminf_{j\to\infty} f_j(x)$ for all $x \in X$. If $\liminf_{j\to\infty} \mathcal{J}(f_j) < \infty$, then $f \in \mathcal{L}_1$ and $\mathcal{J}(f) \leq \liminf_{j\to\infty} \mathcal{J}(f_j)$.*

Proof. For each $j \geq 1$ consider $h_j = \inf_{i\geq j} f_i$, then (h_j) is a nondecreasing sequence of nonnegative integrable functions on X such that $f = \sup_{j\geq 1} h_j$ and $h_j \leq f_j$ for every $j \geq 1$. This shows that

$$\sup_{j\geq 1} \mathcal{J}(h_j) = \liminf_{j\to\infty} \mathcal{J}(h_j) \leq \liminf_{j\to\infty} \mathcal{J}(f_j) < \infty.$$

So, by Theorem 2.6, $f \in \mathcal{L}_1$ and $\mathcal{J}(f) = \sup_{j\geq 1} \mathcal{J}(h_j)$. Thus the proof is finished. □

THEOREM 2.9 *Let $f \in \mathcal{F}_{\mathbb{C}}$ and let (f_j) be a sequence in $\mathcal{L}_1$ such that $f(x) = \lim_{j\to\infty} f_j(x)$ for all $x \in X$. Assume that there exists $g, g_1, g_2 \ldots \in \mathcal{L}_1$ such that $|f_j| \leq g_j$ for all $j \geq 1$, $g(x) = \lim_{j\to\infty} g_j(x)$ for all $x \in X$, and $\lim_{j\to\infty} \mathcal{J}(g_j) = \mathcal{J}(g)$. Then $f \in \mathcal{L}_1$ and $\mathcal{J}(f) = \lim_{j\to\infty} \mathcal{J}(f_j)$.*

Proof. For each $j \geq 1$ consider $u_j = g_j + f_j$ and $v_j = g_j - f_j$. Assume without loss of generality that all functions f_j are real-valued, then $0 \leq u_j \leq 2g_j$ and $0 \leq v_j \leq 2g_j$. Since $f + g = \liminf_{j\to\infty} u_j$, by applying Theorem 2.8 we show that $f + g \in \mathcal{L}_1$ and

$$\mathcal{J}(f+g) \leq \liminf_{j\to\infty} \mathcal{J}(f_j + g_j).$$

Then $f \in \mathcal{L}_1$ and $\mathcal{J}(f) \leq \liminf_{j\to\infty} \mathcal{J}(f_j)$. Now, by an other application of the same theorem we get

$$\mathcal{J}(\liminf_{j\to\infty} v_j) \leq \liminf_{j\to\infty} \mathcal{J}(v_j)$$

which shows that $\mathcal{J}(f) \geq \limsup_{j\to\infty} \mathcal{J}(f_j)$. □

COROLLARY 2.10 *Let $f \in \mathcal{F}_{\mathbb{C}}$ and let (f_j) be a sequence in $\mathcal{L}_1$ such that $f(x) = \lim_{j\to\infty} f_j(x)$ for all $x \in X$. If there exists $g \in \mathcal{L}_1$ such that $|f_j| \leq g$ for all $j \geq 1$, then $f \in \mathcal{L}_1$ and $\mathcal{J}(f) = \lim_{j\to\infty} \mathcal{J}(f_j)$.*

3 Spaces $\mathcal{L}_p$

Let $f \in \mathcal{F}_{\mathbb{C}}$, for each real $p \geq 1$ we define $\| f \|_p := \| \, |f|^p \, \|^{1/p}$ and we denote by $\mathcal{L}_p$ the set of all functions $f \in \mathcal{F}_{\mathbb{R}}$ such that for every $\varepsilon > 0$ there exists $\varphi \in \mathcal{C}_{\mathbb{C}}$ satisfying $\| f - \varphi \| \leq \varepsilon$. Note that $\| \cdot \|_1 = \| \cdot \|$.

PROPOSITION 3.1 *Let $f \in \mathcal{F}_{\mathbb{C}}$ and let $p, q \geq 1$ be two reals. The following statements hold:*

(i) If $p \leq q$ then $\| f \|_p \leq \| f \|_q$.

(ii) If $\| f \|_p$ and $\| f \|_q$ are finite and $1/p + 1/q = 1$ then $\| fg \| \leq \| f \|_p \| g \|_q$.

Proof. (i) : Suppose that (ii) is proved then it will be clear that for every $r \geq 1$

$$\| f \| \leq \| f \|_r \, . \tag{16}$$

Hence the assertion (i) follows immediately from (16) by considering $r = q/p$.
(ii) : Recall first that for every reals $a, b \geq 0$ we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{17}$$

We claim that $\| fg \| = 0$ if $\| f \|_p = 0$ or $\| g \|_q = 0$. Indeed, (17) shows that

$$\lambda |fg| \leq \frac{\lambda^p}{p} |f|^p + \frac{1}{q} |g|^q,$$

for every $\lambda > 0$. Therefore

$$\lambda \| fg \| \leq \frac{\lambda^p}{p} \| f \|_p^p + \frac{1}{q} \| g \|_q^q .$$

If $\| f \|_p = 0$ then

$$\| fg \| \leq \frac{1}{\lambda p} \| g \|_q .$$

By letting λ tends to ∞ we get $\| fg \| = 0$. Suppose now that $\| f \|_p, \| g \|_q > 0$ then

$$\frac{|fg|}{\| f \|_p \| g \|_q} \leq \frac{|f|^p}{p \| f \|_p^p} + \frac{|g|^q}{q \| g \|_q^q},$$

and therefore

$$\frac{\| fg \|}{\| f \|_p \| g \|_q} \leq \frac{1}{p} + \frac{1}{q} = 1.$$

□

COROLLARY 3.2 *Assume that* $1 < p, q < \infty$ *and* $1/p + 1/q = 1$. *If* $f \in \mathcal{L}_p$ *and* $g \in \mathcal{L}_q$ *then* $fg \in \mathcal{L}_1$ *and* $|\mathcal{J}(fg)| \leq \| f \|_p \| g \|_q$.

Proof. Choose $\varepsilon > 0$ and $\varphi \in \mathcal{C}_{\mathbb{C}}$ satisfying

$$\| f - \varphi \|_p \leq \frac{\varepsilon}{2[1 + \| g \|_q]}.$$

Take $\psi \in \mathcal{C}_{\mathbb{C}}$ such that

$$\| g - \psi \|_q \leq \frac{\varepsilon}{2[1 + \| \varphi \|_p]}.$$

Therefore

$$\begin{aligned} \| fg - \varphi\psi \| &\leq \| (f - \varphi) g \| + \| \varphi (g - \psi) \| \\ &\leq \| f - \varphi \|_p \| g \|_q + \| \varphi \|_p \| g - \psi \|_q \\ &\leq \varepsilon. \end{aligned}$$

To finish the proof recall that $|\mathcal{J}(fg)| \leq \mathcal{J}(|fg|) = \| fg \| \leq \| f \|_p \| g \|_q$. □

In view of Proposition (3.1.*i*) it is trivial that $\lim_{p \to \infty} \| f \|_p$ exists in $[0, \infty]$ for every $f \in \mathcal{F}_{\mathbb{C}}$. We define $\| f \|_\infty$ to be this limit and we denote by $\mathcal{L}_\infty$ the closure of $\mathcal{C}_{\mathbb{C}}$ with respect to $\| \cdot \|_\infty$. It is clear that $\mathcal{L}_\infty$ is a linear space containing all $\mathcal{L}_p$ for $p \geq 1$. Note that Corollary 3.2 and also Proposition (3.1.*ii*) hold true if $p = 1$ and $q = \infty$ (see Proposition 4.3.*iii*). Now let us extend Lemma 2.1 to every $p \in [1, \infty]$.

LEMMA 3.3 *Let $f, f_1, f_2, \ldots \in \mathcal{F}_{\mathbb{C}}$ such that $|f| \leq \sum_{j=1}^{\infty} |f_j|$. Then, for every $1 \leq p \leq \infty$ we have*

$$\| f \|_p \leq \sum_{j=1}^{\infty} \| f_j \|_p . \tag{18}$$

Proof. For $p = 1$ see Lemma 2.1. If (18) holds true for every finite p then it is easy to see that the same inequality holds for $p = \infty$. Let us consider the case where $1 < p < \infty$. If $\| f \|_p = 0$ or $\| f_j \|_p = \infty$ for some $j \geq 1$, the lemma will be trivial. Hence we assume that $\| f \|_p > 0$ and $\| f_j \|_p < \infty$ for all $j \geq 1$. On the other hand, if $g \in \mathcal{F}_{\mathbb{C}}$ such that $|f| \leq |g|$ then by (6) we get $\| f \|_p \leq \| g \|_p$. Therefore, without loss of generality we assume that $|f| = \sum_{j=1}^{\infty} |f_j|$. Let $n \geq 1$ and consider $g_n = \sum_{j=1}^{n} |f_j|$, then

$$g_n^p \leq n^{p-1} \sum_{j=1}^{n} |f_j|^p$$

and hence

$$\| g_n^p \| \leq n^{p-1} \sum_{j=1}^{n} \| |f_j|^p \|,$$

which proves that $\| g_n \|_p < \infty$. Since

$$g_n^p \leq \sum_{j=1}^{n} g_n^{p-1} |f_j|,$$

by Lemma 2.1 we have

$$\| g_n^p \| \leq \sum_{j=1}^{n} \| g_n^{p-1} |f_j| \| .$$

Take $q > 1$ such that $1/p + 1/q = 1$ and apply Proposition (3.1.*ii*), we get

$$\| g_n \|_p \leq \sum_{j=1}^{n} \| f_j \|_p .$$

By letting n tends to ∞ and by applying Proposition 2.5 and Theorem 2.6 we obtain (18). □

As in the case of $p = 1$, it is very easy to see that $\| \cdot \|_p$ defines a semi-norm in $\mathcal{L}_p$ for every $1 \leq p \leq \infty$ and by the same arguments as in the proof of Theorem 2.3 the following result can be shown.

THEOREM 3.4 *For every $1 \leq p \leq \infty$, the space $\mathcal{L}_p$ is complete for $\| \cdot \|_p$.*

4 Spaces $\mathbb{L}_p$

Let $1 \leq p \leq \infty$. If the group X is finite then $\| \cdot \|_p$ is a norm on $\mathcal{L}_p$. In fact, in this case $\mathcal{F}_{\mathbb{C}} = \mathcal{L}_p = \mathcal{C}_{\mathbb{C}}$ and it is not difficult to see that $\| f \|_p$ is given by the formula

$$\| f \|_p = \left(\frac{1}{|X|} \sum_{x \in X} |f(x)|^p \right)^{1/p}$$

if p is finite and

$$\| f \|_\infty = \sup_{x \in X} |f(x)|.$$

Therefore, $\| f \|_p = 0$ if and only if $f(x) = 0$ for all $x \in X$.

Contrary to finite groups we will prove in this section that if X is infinite the condition $\| f \|_p = 0$ is not sufficient to claim that the function f vanishes everywhere on X.

Lemma 4.1 *Assume that X is infinite then for each $\varepsilon > 0$ there exists $f \in \mathcal{C}_{\mathbb{R}}^+$ such that $f(e) = 1$ and $\| f \| < \varepsilon$.*

Proof. Let $\varepsilon > 0$ and choose an integer n such that $n \varepsilon > 1$. Since X is infinite we can find n disjoint elements $a_1, a_2, \ldots, a_n \in X \backslash \{e\}$. For every $j = 1, 2, \ldots, n$, choose an open neighborhood V_j of a_j such that $V_i \cap V_j = \emptyset$ if $i \neq j$. Then

$$U = \bigcap_{j=1}^{n} V_j a_j^{-1}$$

is an open neighborhood of e. Let $K \subset U$ be a compact neighborhood of e, by Urysohn lemma (see, e.g., [9]), there exists a continuous function f on X such that $0 \leq f \leq 1$, $f \equiv 1$ on K and f vanishes on the complement of U. For each $1 \leq j \leq n$, the function $f_j = R_{a_j^{-1}} f$ is real continuous on X, $0 \leq f_j \leq 1$, $f_j \equiv 0$ on $X \backslash U a_j$ and $f_j \equiv 1$ on $K a_j$. Since $U a_i \cap U a_j = \emptyset$ if $i \neq j$, it follows that $0 \leq \sum_{j=1}^{n} f_j \leq 1$ which implies that

$$\sum_{j=1}^{n} \int_X f_j(x)\, dx = \int_X \left(\sum_{j=1}^{n} f_j(x) \right) dx \leq \int_X 1\, dx = 1.$$

On the other hand for every $1 \leq j \leq n$

$$\int_X f_j(x)\, dx = \int_X f(x)\, dx = \| f \|,$$

hence

$$\| f \| \leq \frac{1}{n} < \varepsilon.$$

□

Proposition 4.2 *Let $1 \leq p \leq \infty$. If X is infinite then $\| \chi_A \|_p = 0$ for every countable subset A of X.*

Proof. Let A be a countable subset of X. Since $\chi_A = |\chi_A|^p$ for every real $p \geq 1$ we have only to show that $\| \chi_A \| = 0$. On the other hand, because $\| \chi_A \| \leq \sum_{a \in A} \| \chi_{\{a\}} \|$ it is enough to prove that $\| \chi_{\{x\}} \| = 0$ for every $x \in X$. Hence, choose $x \in X$, $\varepsilon > 0$ and $\varphi_\varepsilon \in \mathcal{C}_{\mathbb{R}}^+$ satisfying $\varphi_\varepsilon(e) = 1$ and $\| \varphi_\varepsilon \| < \varepsilon$. Then

$$\| \chi_{\{x\}} \| \leq \| L_{x^{-1}} \varphi_\varepsilon \| = \| \varphi_\varepsilon \| < \varepsilon.$$

Letting ε tend to zero we get $\| \chi_{\{x\}} \| = 0$. □

A subset A of X will be called a *null set* if $\| \chi_A \| = 0$. Since $0 \leq \chi_A \leq \chi_B$ for every $A \subset B \subset X$ it is clear that all subsets of a null set are also null sets. Moreover, in virtue of Lemma 2.1, it follows that every finite or countable union of null sets is null set. A property $\mathcal{P}$ on X will be said to be true *almost everywhere* (a.e) if $\{x \in X : \mathcal{P}(x) \text{ is not true}\}$ is a null set of X.

Proposition 4.3 *Let $1 \leq p \leq \infty$ and let f, g be two functions of $\mathcal{F}_{\mathbb{C}}$ which coincide almost everywhere on X. The following statements hold:*

(i) $\| f \|_p = 0$ if and only if $f(x) = 0$ a.e on X.

(ii) $\| f \|_p = \| g \|_p$. Moreover $f \in \mathcal{L}_p$ if and only if $g \in \mathcal{L}_p$.

(ii.a) $f \in \mathcal{L}_1$ if and only if $g \in \mathcal{L}_1$ and in this case $\mathcal{J}(f) = \mathcal{J}(g)$.

(iii) If $f \in \mathcal{L}_\infty$ then $N = \{x \in X : |f(x)| > \| f \|_\infty\}$ is a null subset of X and

$$\| f \|_\infty = \sup_{x \in X \setminus N} |f(x)|. \tag{19}$$

Proof. It is not very difficult to deduce (ii) and $(ii.a)$ from (i).
(i): Let $A = \{x \in X : f(x) \neq 0\}$ then $f(x) = 0$ a.e on X iff A is a null set iff $\| \chi_A \|_p = \| \chi_A \| = 0$. Considering $f_j = \chi_A$ for every $j \geq 1$ and applying Lemma 3.3, we obtain $\| f \|_p \leq \sum_{j=1}^\infty \| f_j \|_p$. This shows that $\| f \|_p = 0$ if $f(x) = 0$ a.e on X. Conversely, if $\| f \|_p = 0$ we consider $g_j = f$ for all $j \geq 1$ and we apply again Lemma 3.3 to get $\| \chi_A \|_p = 0$.
(iii): For every $n \geq 1$ let

$$A_n = \left\{ x \in X : |f(x)| \geq \| f \|_\infty + \frac{1}{n} \right\}.$$

In view of Proposition $(3.1.ii)$ and since

$$\left(\| f \|_\infty + \frac{1}{n} \right) \chi_{A_n} \leq |f| \chi_{A_n}$$

we have

$$\left(\| f \|_\infty + \frac{1}{n} \right) \| \chi_{A_n} \| \leq \| f \|_\infty \| \chi_{A_n} \|^{\frac{1}{p}}$$

for every $p > 1$. By letting p tends to 1 we get

$$\left(\| f \|_\infty + \frac{1}{n} \right) \| \chi_{A_n} \| \leq \| f \|_\infty \| \chi_{A_n} \|$$

which shows that A_n is a null subset of X and hence $N = \bigcup_{n=1}^{\infty} A_n$ is also a null subset of X. Finally (19) can be easily shown by absurdity. □

To study the integrability of a function f we do not need to know $f(x)$ at every point $x \in X$, but it will be sufficient to consider f on the complement of a null set. In this direction we will introduce the spaces $\mathbb{L}_p$. In the following, $f \in \overline{\mathcal{F}}_{\mathbb{C}}$ means that f is a complex function defined on the complement of a null subset of X; the space $\overline{\mathcal{F}}_{\mathbb{R}}$ is defined in the same way. Consider the equivalence relation $\sim$ in $\overline{\mathcal{F}}_{\mathbb{C}}$ given by $f \sim g$ iff $f = g$ a.e on X. For every $p \in [1, \infty]$ let

$$\overline{\mathcal{L}}_p := \{ f \in \overline{\mathcal{F}}_{\mathbb{C}} : \text{there exists } g \in \mathcal{L}_p \text{ such that } f \sim g \}.$$

Denote by $\mathbb{L}_p$ the quotient space $\overline{\mathcal{L}}_p / \sim$ and let $\pi_p : \overline{\mathcal{L}}_p \longrightarrow \mathbb{L}_p$ such that $\pi_p(f) = \{ g \in \overline{\mathcal{L}}_p : g \sim f \}$ for every $f \in \overline{\mathcal{L}}_p$. If $\omega \in \mathbb{L}_p$ we define $\| \omega \|_p := \| f \|_p$ where f is a function of $\mathcal{L}_p$ such that $\pi_p(f) = \omega$. Note that $\| \omega \|_p$ is well definite in virtue of Proposition 4.3.

THEOREM 4.4 *For every $1 \leq p \leq \infty$, $\mathbb{L}_p$ equipped with $\| \cdot \|_p$ is a Banach space.*

Proof. Without difficulty we can verify that $\| \cdot \|_p$ is a norm on $\mathbb{L}_p$. The completeness of $\mathbb{L}_p$ follows from Theorem 3.4. □

In the remainder of this paper every function $f \in \overline{\mathcal{F}}_{\mathbb{C}}$ which agrees almost everywhere on X with some function $g \in \mathcal{L}_p$ will be seen as an element of $\mathbb{L}_p$ and $\| f \|_p = \| g \|_p$. In other words we identify each function $f \in \overline{\mathcal{L}}_p$ by its class $\pi_p(f)$. In particular, an element $f \in \mathbb{L}_1$ will be considered as an integrable function on X and by definition $\mathcal{J}(f) = \mathcal{J}(g)$ where $g \in \mathcal{L}_1$ satisfying $\pi_1(g) = f$.

THEOREM 4.5 [monotone convergence theorem] *Let (f_j) be a sequence in $\mathbb{L}_1$ such that for all $j \geq 1$ we have $f_{j+1}(x) \geq f_j(x)$ a.e on X. If $\sup_{j \geq 1} \mathcal{J}(f_j) < \infty$ then $\sup_{j \geq 1} f_j \in \mathbb{L}_1$ and $\mathcal{J}(\sup_{j \geq 1} f_j) = \sup_{j \geq 1} \mathcal{J}(f_j)$.*

Proof. In virtue of Theorem 2.6. □

THEOREM 4.6 [dominated convergence theorem] *Let $f \in \overline{\mathcal{F}}_{\mathbb{C}}$ and let (f_j) be a sequence in $\mathbb{L}_1$ such that $f(x) = \lim_{j \to \infty} f_j(x)$ a.e on X. Assume that there is $g \in \mathbb{L}_1$ such that for all $j \geq 1$ we have $|f_j(x)| \leq g(x)$ a.e on X, then $f \in \mathbb{L}_1$ and $\lim_{j \to \infty} \mathcal{J}(f_j) = \mathcal{J}(f)$.*

Proof. In virtue of Corollary 2.10. □

PROPOSITION 4.7 *Let $1 \leq p \leq \infty$, $f \in \mathbb{L}_p$ and let (f_j) be a sequence in $\mathbb{L}_p$ which converges to f for the norm $\|\cdot\|_p$. Then (f_j) has a subsequence which converges to f almost everywhere on X.*

Proof. By (16), $f \in \mathcal{L}_1$ and (f_j) converges to f with respect to the norm $\|\cdot\|$. Since $\lim_{i,\, j\to\infty} \| f_i - f_j \|= 0$ there exists a subsequence (g_j) of (f_j) such that $\sum_{j=1}^{\infty} \| g_{j+1} - g_j \|< \infty$. Let $A = \bigcup_{j=1}^{\infty}\{x \in X : g_j(x) \text{ is not defined}\}$ and consider for each $i \geq 1$

$$h_i(x) = \begin{cases} |g_1(x)| + \sum_{j=1}^{i} |g_{j+1}(x) - g_j(x)| & \text{if } x \in X\backslash A \\ 0 & \text{if } x \in A. \end{cases}$$

Then (h_i) is a nondecreasing sequence of integrable functions on X satisfying

$$\sup_{i\geq 1} \mathcal{J}(h_i) \leq \| g_1 \| + \sum_{j=1}^{\infty} \| g_{j+1} - g_j \|< \infty.$$

Hence by Theorem 4.5, $h := |g_1| + \sum_{j=1}^{\infty} |g_{j+1} - g_j| \in \mathbb{L}_1$ which yields that $\{x \in X : h(x) = \infty\}$ is a null set. So, $g_1 + \sum_{j=1}^{\infty}(g_{j+1} - g_j)$ converges almost everywhere to a function $g \in \mathcal{F}_{\mathbb{C}}$. Thus $\lim_{j\to\infty} g_j(x) = g(x)$ a.e on X. On the other hand, for every $j > 1$

$$|g_j| \leq |g_1| + \sum_{k=1}^{j-1} |g_{k+1} - g_k| \leq h \;\; a.e \text{ on X}.$$

Then by Theorem 4.6, $g \in \mathbb{L}_1$ and $\lim_{j\to\infty} \| g - g_j \|= 0$. Finally $f(x) = g(x)$ a.e on X because $\lim_{j\to\infty} \| g_j - f \|= 0$. □

5 Return to the measure theory

The lecture has seen in the preceeding sections how the invariant integral on the compact topological group X can be introduced without using the measure theory. The purpose of this section is to show how one can develop the measure theory starting from our approach to the integral on the group X.

DEFINITIONS: A function $f \in \overline{\mathcal{F}}_{\mathbb{C}}$ will be called *measurable* on X if it is almost everywhere a limit of some sequence of complex continuous functions on X. We say that a subset A of X is *measurable* if its characteristic function is measurable on X.

We denote by $\mathcal{M}$ the set of all measurable functions on X. It is easy to see that $\mathcal{M}$ is a linear space which contains $\mathcal{C}_{\mathbb{C}}$ and with every functions f, g also $\overline{f}, \Re(f), \Im(f), |f|$ and $f\, g$. Moreover, if $f, g \in \mathcal{M} \cap \overline{\mathcal{F}}_{\mathbb{R}}$ then $\sup\{f, g\}$ and $\inf\{f, g\}$ are measurable functions on X.

LEMMA 5.1 *The following statements hold:*

(i) Every function $f \in \mathbb{L}_1$ is measurable on X.

(ii) If $f \in \mathcal{M}$ such that $|f| \leq g$ a.e on X for some $g \in \mathbb{L}_1$, then $f \in \mathbb{L}_1$.

Proof. (i) : By Proposition 4.7, f is equal almost everywhere to the limit of some sequence in $\mathcal{C}_{\mathbb{C}}$. (ii) : Assume without loss of generality that $f \in \overline{\mathcal{F}}_{\mathbb{R}}$ and choose a sequence $(\varphi_j) \in \mathcal{C}_{\mathbb{R}}$ which converges almost everywhere to f. For each $j \geq 1$ let $f_j := \sup\{-g, \inf(g, \varphi_j)\}$ then $f_j \in \mathbb{L}_1$, $|f_j| \leq g$ and $\lim_{j\to\infty} f_j(x) = f(x)$ a.e on X. Then Theorem 4.6 shows that $f \in \mathbb{L}_1$. □

PROPOSITION 5.2 *The following statements hold:*

(i) A subset A of X is measurable if and only if χ_A is an integrable function on X.

(ii) If (A_j) is a sequence of measurable sets then $\bigcup_{j=1}^{\infty} A_j$ and $\bigcap_{j=1}^{\infty} A_j$ are also measurable sets.

Proof. Since $\chi_X \in \mathcal{C}_{\mathbb{C}} \subset \mathbb{L}_1$ the assertion (i) follows immediately from the above Lemma. In order to show (ii) it is enough to remark that

$$\chi_{\bigcup_{j=1}^{\infty} A_j} = \sup_{j\geq 1} \chi_{A_j} \text{ and } \chi_{\bigcap_{j=1}^{\infty} A_j} = \inf_{j\geq 1} \chi_{A_j}.$$

□

THEOREM 5.3 *Let $1 \leq p < \infty$, then $f \in \mathbb{L}_p$ if and only if f is a measurable function on X and $\| f \|_p$ is finite.*

Proof. If $f \in \mathbb{L}_p$ it is clear that $f \in \mathcal{M}$ and $\| f \|_p < \infty$. To prove the converse statement it is sufficient to consider the case when f is nonnegative. Let $f \in \mathcal{M}^+$ such that $\| f \|_p < \infty$ and consider $f_n = \inf\{n, f\}$ for every $n \geq 1$. Then by Lemma 5.1, $f_n^p \in \mathbb{L}_1$ and hence $f^p \in \mathbb{L}_1$ in view of the monotone convergence theorem (Theorem 4.5). Choose a sequence $(\varphi_j) \in \mathcal{C}_{\mathbb{R}}^+$ which converges almost everywhere on X to f. Let $n \geq 1$ and define $\psi_j = \inf\{n, \varphi_j\}$ for every $j \geq 1$, then (ψ_j) converges almost everywhere on X to the function f_n. Let $i \geq 1$ then $|\psi_i - \psi_j|^p \in \mathbb{L}_1$ and $|\psi_i - \psi_j|^p \leq 2^p n^p$ for all $j \geq 1$. Therefore applying the dominated convergence theorem (Theorem 4.6) we obtain that $|\psi_i - f_n|^p \in \mathbb{L}_1$. Using the same arguments we show that $\lim_{i\to\infty} \| \psi_i - f_n \|_p = 0$. This and the completeness of the space $\mathbb{L}_p$ imply that $f_n \in \mathbb{L}_p$ for all $n \geq 1$. Again the dominated convergence theorem shows that $|f_n - f|^p \in \mathbb{L}_1$; indeed, $|f_n - f_m|^p \leq 2^p f^p$ and $f^p \in \mathbb{L}_1$. Finally remark that $\lim_{n\to\infty} |f_n(x) - f(x)|^p = 0$ a.e on X which yields that $\lim_{n\to\infty} \| f_n - f \|_p = 0$ seeing that $|f_n - f|^p \leq 2^p f^p$ for all $n \geq 1$. Hence $f \in \mathbb{L}_p$ by Theorem 4.4. □

Since $\chi_{X\setminus A} = 1 - \chi_A$, clearly the complement of any measurable set of X is also a measurable set. Therefore in view of Proposition (5.2), the set $\mathcal{A}$ of all measurable subsets of X is a $\sigma-$algebra which contains all null sets. Define for every $A \in \mathcal{A}$ the nonnegative real number $m(A) = \mathcal{J}(\chi_A)$, hence we obtain the following result.

THEOREM 5.4 *$(X, \mathcal{A}, m)$ is a complete measure space. Moreover m is invariant under left and right translations.*

Proof. By definitions of null sets and measurable sets it is clear that $(X, \mathcal{A})$ is a complete measurable space. To show that m is a measure we have only to show that

$$m(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} m(A_j)$$

for any sequence (A_j) of pairwise disjoint measurable subsets of X. This follows immediately from the monotone convergence theorem and the fact that

$$\chi_{\bigcup_{j=1}^{\infty} A_j} = \sum_{j=1}^{\infty} \chi_{A_j}.$$

Finally, From (14) we deduce that $m(xA) = m(Ax) = m(A)$ for all $A \in \mathcal{A}$ and all $x \in X$. □

Let $\mathcal{B}$ be the $\sigma-$algebra consisting of all Borel subsets of X and let μ be the normalized Haar measure on X. Denote by $\overline{\mu}$ the completion of μ and by $\overline{\mathcal{B}}$ the completion of $\mathcal{B}$ with respect to μ.

PROPOSITION 5.5 *The following statements hold:*

(*i*) $\mathcal{A} \subset \overline{\mathcal{B}}$.

(*ii*) *If X has a countable base of open subsets then $\mathcal{A} = \overline{\mathcal{B}}$ and $m = \overline{\mu}$.*

Proof. (*i*) : Let N be a null subset of X then for every $k \geq 1$ there exists $({}^k\varphi_j) \in \mathcal{C}_{\mathbb{R}}^+$ such that

$$\chi_N \leq \sum_{j=1}^{\infty} {}^k\varphi_j := f_k \quad \text{and} \quad \int_X f_k(x)\, d\mu(x) = \sum_{j=1}^{\infty} \int_X {}^k\varphi_j(x)\, d\mu(x) \leq \frac{1}{k^2}.$$

Let $A_k = \{x \in X : kf_k(x) \geq 1\}$ and $A = \bigcap_{k=1}^{\infty} A_k$ then A is a Borel subset of X which contains N and $\mu(A) \leq \mu(A_k) \leq 1/k$ for every $k \geq 1$. Hence every null subset of X is $\mu-$null which proves that every measurable subset in the sense of this section is in $\overline{\mathcal{B}}$.

(*ii*) : It is enough to prove that $\mathcal{A}$ contains all closed subsets of X. Let F be a closed subset of X, for every $x \in X\backslash F$ then there exists an open neighborhood V_x of x such that $V_x \cap F = \emptyset$. Since X has a countable base of open subsets and since $X\backslash F = \bigcup_{x \notin F} V_x$, we can find $x_1, x_2, \ldots \in X\backslash F$ such that $X\backslash F = \bigcup_{j=1}^{\infty} V_{x_j}$. For each $n \geq 1$ let $U_n = X\backslash \bigcup_{j=1}^{n} \overline{V}_{x_j}$, then (U_n) is a nondecreasing sequence of open subsets of X such that $F = \bigcap_{n=1}^{\infty} U_n$. By Urysohn lemma, for every $n \geq 1$ there exists a function $\varphi_n \in \mathcal{C}_{\mathbb{R}}$ satisfying $0 \leq \varphi_n \leq 1$ on X, $\varphi_n \equiv 1$ on F and $\varphi_n \equiv 0$ on $X\backslash U_n$. Therefore, $\chi_F(x) = \lim_{n\to\infty} \varphi_n(x)$ for all $x \in X$ which proves that $F \in \mathcal{A}$. □

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