GROUND STATE EUCLIDEAN MEASURES FOR QUANTUM LATTICE SYSTEMS ON COMPACT MANIFOLDS

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ABSTRACT. Quantum lattice systems with compact spins and nearest-neighbour interactions are considered. Existence and uniqueness of the corresponding ground state Euclidean measures are proved for sufficiently small mass of the particles.

Mathematics Subject Classifications (1991): Primary: 82B10, 82B20;
Secondary: 60J65.

KEY WORDS: Quantum Lattice Systems, Ground State Euclidean Measures, Uniqueness Problem, Cluster Expansions.

1. Introduction

Let (M, σ) be a compact Riemannian manifold with the corresponding normalized Riemannian volume measure σ, and let Δ be the Laplace–Beltrami operator on M. We consider a quantum lattice system of particles with a mass m described by the following formal Hamiltonian

$$H = -\frac{\hbar^2}{2m} \sum_{i \in \mathbb{Z}^d} \Delta_i + \sum_{\substack{i,j \in \mathbb{Z}^d \\ |i-j|=1}} v(x_i, x_j) \tag{1}$$

in the Hilbert space

$$\mathcal{H} = L_2(M^{\mathbb{Z}^d}, \underset{i \in \mathbb{Z}^d}{\times} \sigma(dx_i)),$$

where Δ_i is the Laplace-Beltrami operator with respect to the variable x_i, v is a symmetric real-valued continuous function over $M \times M$, and $\hbar$ is Planck's constant.

Remark 1. From now on we put $\hbar = 1$ or, what is the same, re-denote $m/\hbar^2$ by m, understanding a small mass as a "large quantumness" of the system.

In [4] we proved uniqueness of the corresponding Euclidean Gibbs states uniformly with respect to the inverse temperature $\beta < \infty$ in the case where the particles have a sufficiently small mass.

Date: 8 December 1999.

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The aim of the present work is to cover the case $\beta = \infty$, i.e., to prove existence and uniqueness of the ground state Euclidean measures in small mass region.

The scheme of the present work is as follows. In Section 2 we introduce the notion of ground state Euclidean measures and formulate our main result about existence and uniqueness of the ground state Euclidean measures corresponding to the quantum lattice systems with a sufficiently small mass m. Sections 3 is devoted to the proof of this result. Namely, modifying the approach from [11] and [4] we construct cluster expansions for the finite volume ground state Euclidean measures and prove their convergence in the thermodynamic limit, whence follows our result about existence and uniqueness of the corresponding ground state Euclidean measures.

2. Ground state Euclidean measures for quantum lattice systems

Denote by $\Omega := C(\mathbb{R} \mapsto M)$ the set of all continuous functions from $\mathbb{R}$ into M and let $\mathfrak{B}_T(\Omega)$, $T \in \overline{\mathbb{R}}_+ := [0, +\infty]$, be the family of Borel σ-algebras generated by all cylinder sets of the form $\{\xi \in \Omega \mid \xi(t_1) \in B_1, \dots, \xi(t_{n-1}) \in B_{n-1}\}$, where $B_1, \dots, B_{n-1} \in \mathfrak{B}(M)$ and $-T < t_1 < t_2 < \dots < t_{n-1} < T$, $n \in \mathbb{N}$. For short, we also denote $\mathfrak{B}(\Omega) := \mathfrak{B}_\infty(\Omega)$.

Next, for any $T \in \mathbb{R}_+ := [0, +\infty[$ and $\xi \in \Omega$ we define

$$\Omega_{\xi,T} := \Big\{\omega \in \Omega \mid \omega(t) = \xi(t) \text{ for all } t \in]-\infty, -T] \cup [T, +\infty[\Big\}.$$

Let

$$p_t^{(m)}(x|y) = \exp\left[\frac{\Delta}{2m} t\right](x, y)$$

be the kernel of the conservative Markov semigroup

$$\exp\left[\frac{\Delta}{2m} t\right]. \tag{2}$$

It follows from the ultracontractivity of the heat semigroup corresponding to the operator (2), see, e.g., [6], that for any $\kappa > 0$ and $t/m \in [\kappa, +\infty[$

$$\sup_{x,y \in M} \left|p_t^{(m)}(x|y) - 1\right| \leqslant \psi(\kappa) e^{-\lambda_1 \frac{t}{m}}, \tag{3}$$

where λ_1 is the first positive eigenvalue of the operator $\frac{1}{2}\Delta$ and ψ is some non-increasing function of κ.

Remark 2. In the following we will systematically write $p_t(x|y)$ instead of $p_t^{(m)}(x|y)$.

We define a Wiener bridge measure $W_{\xi,T}$ on the Borel σ-algebra $\mathfrak{B}(\Omega_{\xi,T})$ generated by all cylinder sets of the form $\{\omega \in \Omega_{\xi,T} \mid \omega(t_1) \in B_1, \dots, \omega(t_{n-1}) \in B_{n-1}\}$, where $B_1, \dots, B_{n-1} \in \mathfrak{B}(M)$ and $-T = t_0 < t_1 < t_2 < \dots < t_n = T$, by the following formula

$$W_{\xi,T}\big(\{\omega \in \Omega_{\xi,T} \mid \omega(t_1) \in B_1, \dots, \omega(t_n) \in B_n\}\big) =$$
$$= \int_{B_1} \sigma(dx_1) \cdots \int_{B_n} \sigma(dx_n) p_{t_1 - t_0}\big(x_1|\xi(t_0)\big) p_{t_2-t_1}(x_2|x_1) \dots p_{t_n - t_{n-1}}\big(\xi(t_n)|x_n\big). \tag{4}$$

For any $\Lambda \subset \mathbb{Z}^d$, $T \in \mathbb{R}_+$ and $\xi_{\mathbb{Z}^d} \in \Omega^{\mathbb{Z}^d}$ we introduce

$$\Omega_{\xi_\Lambda,T} := \mathop{\times}_{i \in \Lambda} \Omega_{\xi_i,T}$$

with the corresponding product σ-algebra

$$\mathcal{G}(\Omega_{\xi_\Lambda,T}) := \mathop{\times}_{i \in \Lambda} \mathfrak{B}(\Omega_{\xi_i,T}).$$

Note that by $\xi_\Lambda := \{\xi_i\}_{i\in\Lambda}$ we usually denote the natural restriction of the configuration $\xi_{\mathbb{Z}^d} = \{\xi_i\}_{i\in\mathbb{Z}^d} \in \Omega^{\mathbb{Z}^d}$ to the set Λ.

In the case of a finite $\Lambda \subset \mathbb{Z}^d$ it is possible to introduce the following product measure on $\mathcal{G}(\Omega_{\xi_\Lambda,T})$

$$W_{\xi_\Lambda,T}(d\omega_\Lambda) := \mathop{\times}_{i\in\Lambda} W_{\xi_i,T}(d\omega_i). \tag{5}$$

Next, for a given real-valued continuous potential v, configurations $\omega_{\mathbb{Z}^d}, \xi_{\mathbb{Z}^d} \in \Omega^{\mathbb{Z}^d}$, a finite set $\Lambda \subset \mathbb{Z}^d$ and $T \in \mathbb{R}_+$, we define a finite volume conditional potential energy of the configuration $\omega_{\mathbb{Z}^d}$ with respect to the configuration $\xi_{\mathbb{Z}^d}$ by

$$U_{\Lambda,T}(\omega_{\mathbb{Z}^d}|\xi_{\mathbb{Z}^d}) := \sum_{\substack{\{i,j\}\subset\Lambda \\ |i-j|=1}} V_T(\omega_i,\omega_j) + \sum_{\substack{i\in\Lambda,\ j\in\Lambda^c \\ |i-j|=1}} V_T(\omega_i,\xi_j), \tag{6}$$

where for any $\omega, \omega' \in \Omega$

$$V_T(\omega,\omega') = \int_{-T}^{T} v\big(\omega(t),\omega'(t)\big)dt, \tag{7}$$

and $\Lambda^c := \mathbb{Z}^d \setminus \Lambda$.

Let us fix $T \in \mathbb{R}_+$, a finite set $\Lambda \subset \mathbb{Z}^d$ and a "boundary condition" $\xi_{\mathbb{Z}^d} \in \Omega^{\mathbb{Z}^d}$. A finite volume ground state Euclidean measure $G_{\Lambda,T}(\cdot|\xi_{\mathbb{Z}^d})$ corresponding to our compact spin quantum lattice system (1) is a measure on

$$\mathcal{G}(\Omega^{\mathbb{Z}^d}) := \mathop{\times}_{i\in\mathbb{Z}} \mathfrak{B}(\Omega)$$

such that for any $B \in \mathcal{G}(\Omega^{\mathbb{Z}^d})$

$$G_{\Lambda,T}(B|\xi_{\mathbb{Z}^d}) = \frac{1}{Z_{\Lambda,T}(\xi_{\mathbb{Z}^d})} \int_{\Omega_{\xi_\Lambda,T}} \chi_B(\omega_\Lambda \times \xi_{\Lambda^c}) \exp\Big[-U_{\Lambda,T}(\omega_\Lambda|\xi_{\mathbb{Z}^d})\Big] W_{\xi_\Lambda,T}(d\omega_\Lambda),$$

where

$$Z_{\Lambda,T}(\xi_{\mathbb{Z}^d}) = \int_{\Omega_{\xi_\Lambda,T}} \exp\Big[-U_{\Lambda,T}(\omega_\Lambda|\xi_{\mathbb{Z}^d})\Big] W_{\xi_\Lambda,T}(d\omega_\Lambda) \tag{8}$$

and $\chi_B(\cdot)$ is the characteristic function of the set B.

Definition 1. A probability measure G on $\mathcal{G}(\Omega^{\mathbb{Z}^d})$ is called ground state Euclidean measure corresponding to the quantum lattice system (1) if for any $B \in \mathcal{G}(\Omega^{\mathbb{Z}^d})$, finite set $\Lambda \subset \mathbb{Z}^d$ and $T \in \mathbb{R}_+$

$$G(B) = \int_{\Omega^{\mathbb{Z}^d}} G_{\Lambda,T}(B \mid \xi_{\mathbb{Z}^d}) G(d\xi_{\mathbb{Z}^d}). \tag{9}$$

Remark 3. Equation (9) is an analogue of the Dobrushin–Lanford–Ruelle (DLR) equation for our situation. It expresses the fact that for any finite set $\Lambda \subset \mathbb{Z}^d$ and $T \in \mathbb{R}_+$ the conditional expectation of the ground state Euclidean measure G with respect to

$$\mathcal{G}^c_{\Lambda,T}(\Omega^{\mathbb{Z}^d}) := \Big(\mathop{\times}_{i\in\Lambda} \mathfrak{B}^c_T(\Omega)\Big) \times \Big(\mathop{\times}_{i\in\Lambda^c} \mathfrak{B}(\Omega)\Big) \tag{10}$$

should G-almost everywhere coincide with the corresponding finite volume Gibbs measure $G_{\Lambda,T}$. By $\mathfrak{B}^{c}_{T}(\Omega)$ in (10) we denoted the σ-algebra generated by all cylinder sets of the form $\{\xi \in \Omega \mid \xi(t_1) \in B_1, \dots, \xi(t_{n-1}) \in B_{n-1}\}$, where $T \in \mathbb{R}_+$, $B_1, \dots, B_{n-1} \in \mathfrak{B}(M)$, $t_1, t_2, \dots, t_{n-1} \in]-\infty, -T] \cup [T, +\infty[$ and $n \in \mathbb{N}$.

The main result of the present paper can now be formulated as follows:

Theorem 1. *There exists a positive constant m_0 such that for any mass $m \leqslant m_0$ there exists a unique ground state Euclidean measure corresponding to the compact spin quantum lattice system with a continuous nearest-neighbour interaction.*

3. Cluster expansions

In this section we prove our main result formulated in Theorem 1. Note that, as far as we are going to obtain the ground state Euclidean measure as a limit of the finite volume ground state Euclidean measures when $\Lambda \nearrow \mathbb{Z}^d$ and $T \to +\infty$, from the very beginning we can suppose that T is big enough, say $T \geqslant T_0$, where T_0 is some positive constant.

Let us for any $m \in [0, m_0]$ choose $N \in \mathbb{N}$ in such a way that

$$\frac{T}{\sqrt{m}} \leqslant N \leqslant \frac{2T}{\sqrt{m}}.$$

This is possible provided that

$$\frac{T_0}{\sqrt{m_0}} > \frac{1}{2},$$

which can be achieved by a proper choice of m_0.

Next, let $a := T/N$ and $W^k_{x|x'}(\cdot)$, $x, x' \in M$, $k \in \mathbb{Z}_N := \{-N, -N+1 \dots, N-1\}$, be the normalized conditional Wiener measure corresponding to the operator (2) and defined on the cylinder σ-algebra over the set

$$\Omega^k_{x|x'} = \Big\{\omega \in C\big([ka, (k+1)a] \mapsto M\big) \mid \omega(ka) = x,\ \omega\big((k+1)a\big) = x'\Big\}$$

of all continuous trajectories from $x \in M$ to $x' \in M$.

As it follows from formulae (8), (6), (7) and (5), (4), the expression for the partition function $Z_{\Lambda,T}(\xi_{\mathbb{Z}^d})$ can be written in the following form:

$$Z_{\Lambda,T}(\xi_{\mathbb{Z}^d}) = \Big\langle \prod_{\substack{i\in\Lambda \\ k\in\mathbb{Z}_N}} p_a(x_{i,k+1}|x_{i,k}) \prod_{k\in\mathbb{Z}_N} \exp\Big[- \sum_{\substack{\{i,j\}\subset\Lambda \\ |i-j|=1}} V_k(\omega_{i,k}, \omega_{j,k}) + {} + \sum_{\substack{i\in\Lambda,\ j\in\Lambda^c \\ |i-j|=1}} V_k(\omega_{i,k}, \xi_{j,k})\Big]\Big\rangle, \quad (11)$$

where for all $k \in \mathbb{Z}_N$, $\omega, \omega' \in \Omega$,

$$V_k(\omega, \omega') := \int_{ka}^{(k+1)a} v\big(\omega(t), \omega'(t)\big)dt,$$

$\langle \dots \rangle$ denotes expectation with respect to the measure

$$\prod_{(i,k)\in\mathbb{Z}^d\times(\mathbb{Z}_N\setminus\{-N\})} \sigma(dx_{i,k}) W^k_{x_{i,k+1}|x_{i,k}}(d\omega_{i,k}),$$

$\xi_{i,k}$ is the restriction of $\xi_i \in \Omega$ to the interval $\big[ka,(k+1)a\big]$, $i \in \Lambda^c$, and

$$x_{i,-N} = \xi_i(-T), \quad x_{i,N} = \xi_i(T) \quad \text{for all} \quad i \in \Lambda.$$

It is easy to see that

$$\prod_{k\in\mathbb{Z}_N} p_a(x_{i,k+1}|x_{i,k}) = \prod_{k\in\mathbb{Z}_N} \big(p_a(x_{i,k+1}|x_{i,k}) - 1 + 1\big) = \sum_{\tau\subset\mathbb{Z}_N}\prod_{k\in\tau} \big(p_a(x_{i,k+1}|x_{i,k}) - 1\big), \tag{12}$$

and analogously,

$$\prod_{\substack{\{i,j\}\subset\Lambda\\|i-j|=1}} e^{-V_k(\omega_{i,k},\omega_{j,k})} \prod_{\substack{i\in\Lambda,\, j\in\Lambda^c\\|i-j|=1}} e^{-V_k(\omega_{i,k},\xi_{j,k})} = \sum_{\rho\subset R_\Lambda}\prod_{\{i,j\}\in\rho}\big(e^{-V_k(\tilde\omega_{i,k},\tilde\omega_{j,k})} - 1\big), \tag{13}$$

where $R_\Lambda := \big\{\{i,j\} \subset \mathbb{Z}^d \mid \ |i-j| = 1 \text{ and } \{i,j\}\cap\Lambda \neq \emptyset\big\}$ is the set of all ribs of the lattice $\mathbb{Z}^d$ having non-empty intersection with the set $\Lambda \subset \mathbb{Z}^d$ and

$$\tilde\omega_{i,k} := \begin{cases}\omega_{i,k}, & \text{if} \quad i \in \Lambda;\\ \xi_{i,k}, & \text{if} \quad i\in\Lambda^c.\end{cases} \tag{14}$$

Definition 2. Any non-empty collection of sets is called connected if the corresponding graph having these sets as vertices and with ribs connecting all the sets with non-empty intersection is connected.

For any $\Lambda \subset \mathbb{Z}^d$ let us define

$$\Gamma_{\Lambda,\mathbb{Z}_N} := \big\{\{i\}\times\{k,k+1\} \mid i\in\Lambda,\, k\in\mathbb{Z}_N\big\} \cup \big\{\{i,j\}\times\{k,k+1\} \mid \{i,j\}\in R_\Lambda,\, k\in\mathbb{Z}_N\big\}.$$

We will call a cluster any finite and connected subset of $\Gamma_{\Lambda,\mathbb{Z}_N}$.

Next, let

$$\mathcal{S}_n(\Gamma_{\Lambda,\mathbb{Z}_N}) := \big\{\gamma = \{\gamma_l\}_{l=1}^n \mid \forall l = 1,\dots,n:\ \gamma_l \subset \Gamma_{\Lambda,\mathbb{Z}_N} \text{ is finite and connected}\big\}$$

be the set of all collections of n (possibly coinciding) clusters from $\Gamma_{\Lambda,\mathbb{Z}_N}$ and

$$\mathcal{D}_n(\Gamma_{\Lambda,\mathbb{Z}_N}) := \big\{\gamma \in \mathcal{S}_n(\Gamma_{\Lambda,\mathbb{Z}_N}) \mid \forall l,l' = 1,\dots,n,\, l\neq l':\ \tilde\gamma_l \cap \tilde\gamma_{l'} = \emptyset\big\},$$

where $\tilde\gamma_0$ denotes the set of vertices, or points from $\mathbb{Z}^d\times\mathbb{Z}_N$, belonging to an arbitrary $\gamma_0 \subset \Gamma_{\Lambda,\mathbb{Z}_N}$. In other words, $\mathcal{D}_n(\Gamma_{\Lambda,\mathbb{Z}_N})$ is the set of n clusters with pairwise non-intersecting set of vertices.

Substituting (12) and (13) into (11), we get that

$$Z_{\Lambda,T}(\xi_{\mathbb{Z}^d}) = 1 + \sum_{n=1}^{\infty}\sum_{\gamma\in\mathcal{D}_n(\Gamma_{\Lambda,\mathbb{Z}_N})}\prod_{l=1}^{n} K_{\Lambda,T}(\gamma_l,\xi_{\mathbb{Z}^d}), \tag{15}$$

where for any finite $\gamma_0 \subset \Gamma_{\Lambda,\mathbb{Z}_N}$

$$K_{\Lambda,T}(\gamma_0,\xi_{\mathbb{Z}^d}) = \Big\langle \prod_{\{i\}\times\{k,k+1\}\in\gamma_0}\big(p_a(x_{i,k+1}|x_{i,k}) - 1\big)\prod_{\{i,j\}\times\{k,k+1\}\in\gamma_0}\big(e^{-V_k(\tilde\omega_{i,k},\tilde\omega_{j,k})} - 1\big)\Big\rangle. \tag{16}$$

Remark 4. Note that if $\tilde\gamma_0 \cap (\Lambda^c\times\mathbb{Z}_N) = \emptyset$, then $K_{\Lambda,T}(\gamma_0,\xi_{\mathbb{Z}^d})$ does not depend on $\xi_{\mathbb{Z}^d}$, Λ and T. In this case it will be denoted simply by $K(\gamma_0)$.

It is easy to see that under our conditions on T, m and N

$$a \leqslant \sqrt{m_0},$$

and so

$$\sup_{\omega,\omega'\in\Omega} \left| e^{-V_k(\omega,\omega')} - 1 \right| \leqslant a\bar{v}e^{a\bar{v}} \leqslant \sqrt{m_0}\bar{v}e^{\sqrt{m_0}\bar{v}}, \tag{17}$$

where

$$\bar{v} = \sup_{x,y\in M} |v(x,y)| < \infty.$$

Therefore, as it follows from formulae (16), (3), (17) and the fact that

$$\frac{a}{m} \geqslant \frac{1}{2\sqrt{m_0}},$$

for any $\Lambda \subset \mathbb{Z}^d$, $T \in [T_0, +\infty]$ and finite $\gamma_0 \subset \Gamma_{\Lambda,\mathbb{Z}_N}$

$$\left| K_{\Lambda,T}(\gamma_0, \xi_{\mathbb{Z}^d}) \right| \leqslant \varepsilon^{\#\gamma_0}, \tag{18}$$

where $\#\gamma_0$ denotes the number of elements from the set γ_0, and the constant ε is equal to

$$\max\left\{ \sqrt{m_0}\bar{v}\exp[\sqrt{m_0}\bar{v}], \psi\big(1/(2\sqrt{m_0})\big)\exp\left[-\lambda_1/(2\sqrt{m_0}) \right] \right\}$$

and can be made arbitrary small by a proper choice of a sufficiently small positive constant m_0.

Remark 5. In the following we will always assume that m_0 is chosen in such a way that $\varepsilon < 1$.

Lemma 1. *For any $\Lambda \subset \mathbb{Z}^d$, $T \in [T_0, +\infty]$, $(i,k) \in \Lambda \times \mathbb{Z}_N$ and $n \geqslant 2$*

$$\sum_{\substack{\gamma_0 \subset \Gamma_{\Lambda,\mathbb{Z}_N} :\, \gamma_0 \text{ -- conn.} \\ \tilde{\gamma}_0 \ni (i,k),\ \#\tilde{\gamma}_0 = n}} \left| K_{\Lambda,T}(\gamma_0, \xi_{\mathbb{Z}^d}) \right| < C_d (c_d \varepsilon^{1/4})^n, \tag{19}$$

where C_d and c_d are some positive constants depending on d.

Proof. Obviously, for any $n \geqslant 2$

$$\min_{\substack{\gamma_0 \subset \Gamma_{\mathbb{Z}^d,\mathbb{Z}} \\ \#\tilde{\gamma}_0 = n}} \#\gamma_0 \geqslant \frac{n}{4},$$

so, from (18) and Remark 5 we get

$$\sum_{\substack{\gamma_0 \subset \Gamma_{\Lambda,\mathbb{Z}_N} :\, \gamma_0 \text{ -- conn.} \\ \tilde{\gamma}_0 \ni (i,k),\ \#\tilde{\gamma}_0 = n}} \left| K_{\Lambda,T}(\gamma_0, \xi_{\mathbb{Z}^d}) \right| \leqslant \varepsilon^{n/4} \sum_{\substack{\gamma_0 \subset \Gamma_{\mathbb{Z}^d,\mathbb{Z}} :\, \gamma_0 \text{ -- conn.} \\ \tilde{\gamma}_0 \ni (i,k),\ \#\tilde{\gamma}_0 = n}} 1. \tag{20}$$

The sum on the right-hand side of (20) equals to the number of clusters containing the point (i,k). As far as for any point $(j,l) \in \mathbb{Z}^d \times \mathbb{Z}$ there are at most $2(2d+1)$ different elements of $\Gamma_{\mathbb{Z}^d,\mathbb{Z}}$ containing it and cardinality of an arbitrary element from $\Gamma_{\mathbb{Z}^d,\mathbb{Z}}$ does not exceed 4, by [10, Lemma 1, § 4, Chapter 2] there exist such constants C_d and c_d that

$$\sum_{\substack{\gamma_0 \subset \Gamma_{\mathbb{Z}^d,\mathbb{Z}} :\, \gamma_0 \text{ -- conn.} \\ \tilde{\gamma}_0 \ni (i,k),\ \#\tilde{\gamma}_0 = n}} 1 < C_d\, c_d^n, \tag{21}$$

which proves the statement of Lemma 1. □

We recall that $\widetilde{\Gamma}_{\Lambda,\mathbb{Z}_N} = (\Lambda \cup \partial\Lambda) \times \mathbb{Z}_N$ denotes the set of points contained in all elements from $\Gamma_{\Lambda,\mathbb{Z}_N}$. Here $\partial\Lambda := \{i \in \mathbb{Z}^d \mid \exists j \in \Lambda : |i-j| = 1\}$. For any $X \subset \widetilde{\Gamma}_{\Lambda,\mathbb{Z}_N}$, $\#\Lambda < \infty$, we introduce

$$Z_{\Lambda,T}(X;\xi_{\mathbb{Z}^d}) = 1 + \sum_{n=1}^{\infty} \sum_{\substack{\gamma \in \mathcal{D}_n(\Gamma_{\Lambda,\mathbb{Z}_N}) \\ \widetilde{\gamma} \subseteq X}} \prod_{l=1}^{n} K_{\Lambda,T}(\gamma_l, \xi_{\mathbb{Z}^d}),$$

where $\widetilde{\gamma} := \tilde{\gamma}_1 \cup \cdots \cup \tilde{\gamma}_n$ and set

$$f_{\Lambda,T}(X;\xi_{\mathbb{Z}^d}) := \frac{Z_{\Lambda,T}(\widetilde{\Gamma}_{\Lambda,\mathbb{Z}_N} \setminus X; \xi_{\mathbb{Z}^d})}{Z_{\Lambda,T}(\widetilde{\Gamma}_{\Lambda,\mathbb{Z}_N}; \xi_{\mathbb{Z}^d})}.$$

Then, by a standard technique (see *eg* [10, §§ 1–3, Chapter 3]) from the cluster expansion (15) and estimate (19) follows

Lemma 2. *For sufficiently small ε, $X \subset \widetilde{\Gamma}_{\Lambda,\mathbb{Z}_N}$, $\#\Lambda < \infty$, $T \in \mathbb{R}_+$ and any $\xi_{\mathbb{Z}^d} \in \Omega^{\mathbb{Z}^d}$*

$$\left| f_{\Lambda,T}(X;\xi_{\mathbb{Z}^d}) \right| \leqslant C\, 2^{\#X}, \tag{22}$$

where C is some positive constant.

Moreover, $f_{\Lambda,T}(X;\xi_{\mathbb{Z}^d})$ can be represented by the following absolutely convergent series

$$f_{\Lambda,T}(X;\xi_{\mathbb{Z}^d}) = 1 + \sum_{n=1}^{\infty} \sum_{\gamma \in \mathcal{S}_n(\Gamma_{\Lambda,\mathbb{Z}_N})}^{(X)} D_X(\gamma) \prod_{l=1}^{n} K_{\Lambda,T}(\gamma_l, \xi_{\mathbb{Z}^d}), \tag{23}$$

where (X) attached to the sum over γ means that the sum runs over all $\gamma \in \mathcal{S}_n(\Gamma_{\Lambda,\mathbb{Z}_N})$ such that every connected component of $\bar{\gamma} := \gamma_1 \cup \cdots \cup \gamma_n$ contains as a vertex at least one element from the set X, and the functions $D_X(\cdot)$ do not depend on Λ and T.

As a consequence, the limit of $f_{\Lambda,T}(X;\xi_{\mathbb{Z}^d})$, as $\Lambda \nearrow \mathbb{Z}^d$ and $T \to +\infty$ exists and is equal to

$$f(X) := 1 + \sum_{n=1}^{\infty} \sum_{\gamma \in \mathcal{S}_n(\Gamma_{\mathbb{Z}^d,\mathbb{Z}})}^{(X)} D_X(\gamma) \prod_{l=1}^{n} K(\gamma_l). \tag{24}$$

This limit does not depend on the "boundary condition" $\xi_{\mathbb{Z}^d}$ and the series on the right-hand side of (24) *is absolutely convergent.*

Remark 6. In the proof of Lemma 2 the principal role is played by the compactness of the spin space, which gives a negligible influence of the boundary conditions onto the specifications if the corresponding space-time volume is big enough. This, of course, is not the case for the models with non-bounded spins. That is why the convergence of cluster expansions for small masses and empty boundary conditions obtained in [11] for the model of quantum anharmonic crystal does not imply uniqueness of the corresponding ground state Euclidean measure.

Denote now by $\mathcal{G}_{\Lambda,T}(\Omega^{\mathbb{Z}^d})$ the σ-algebra generated by all cylinder sets of the form

$$\{\omega_{\mathbb{Z}^d} \in \Omega^{\mathbb{Z}^d} \mid \omega_\Lambda \in A\}, \quad A \in \mathop{\times}_{i\in\Lambda} \mathfrak{B}_T(\Omega)$$

and consider

$$\int_{\Omega^{\mathbb{Z}^d}} F(\omega_{\mathbb{Z}^d}) G_{\Lambda,T}(d\omega_{\mathbb{Z}^d}|\xi_{\mathbb{Z}^d}) = \frac{1}{Z_{\Lambda,T}(\xi_{\mathbb{Z}^d})} \int_{\Omega_{\xi_\Lambda,T}} F(\omega_\Lambda) \exp\Big[- U_{\Lambda,T}(\omega_\Lambda, \xi_{\mathbb{Z}^d})\Big] W_{\xi_\Lambda,T}(d\omega_\Lambda), \tag{25}$$

where $\Lambda \subset \mathbb{Z}^d$ and $T > 0$ are finite and F is an arbitrary bounded $\mathcal{G}_{\Lambda',T'}(\Omega^{\mathbb{Z}^d})$-measurable function, $\Lambda' \subset \Lambda$ and $T' < T$.

In a way similar to our previous considerations, we obtain that

$$\int\limits_{\Omega^{\mathbb{Z}^d}} F(\omega_{\mathbb{Z}^d}) G_{\Lambda,T}(d\omega_{\mathbb{Z}^d}|\xi_{\mathbb{Z}^d}) = \langle F(\omega_{\mathbb{Z}^d})\rangle + \sum_{n=1}^{\infty} \sum_{\gamma \in \mathcal{D}_n(\Gamma_{\Lambda,\mathbb{Z}_N})}^{(\Lambda' \times \mathbb{Z}_{N'})} K^F_{\Lambda,T}(\bar{\gamma}, \xi_{\mathbb{Z}^d}) \times \\ \times f_{\Lambda,T}\big(\Lambda' \times \mathbb{Z}_{N'} \cup \widetilde{\gamma}; \xi_{\mathbb{Z}^d}\big), \quad (26)$$

where $N' := T'/a$ and for any $\gamma_0 \subset \Gamma_{\Lambda,N}$

$$K^F_{\Lambda,T}(\gamma_0, \xi_{\mathbb{Z}^d}) = \Big\langle F(\omega_{\mathbb{Z}^d}) \prod_{\{i\}\times\{k,k+1\}\in\gamma_0} p_a(x_{i,k+1}|x_{i,k}) \prod_{\{i,j\}\times\{k,k+1\}\in\gamma_0} \big(e^{-V_k(\tilde{\omega}_{i,k},\tilde{\omega}_{j,k})} - 1\big) \Big\rangle.$$

From Lemmas 1 and 2 it follows that the series on the right-hand side of (26) is absolutely convergent, and moreover, there exist its limit for $\Lambda \nearrow \mathbb{Z}^d$, $T \to \infty$ and it is equal to the following absolutely convergent series

$$\langle F(\omega_{\mathbb{Z}^d})\rangle + \sum_{n=1}^{\infty} \sum_{\gamma \in \mathcal{D}_n(\Gamma_{\mathbb{Z}^d,\mathbb{Z}})}^{(\Lambda' \times \mathbb{Z}_{N'})} K^F(\widetilde{\gamma}) f\big(\Lambda' \times \mathbb{Z}_{N'} \cup \widetilde{\gamma}\big), \quad (27)$$

where $K^F(\cdot)$ is defined similar to $K(\cdot)$, see Remark 4.

From the convergence of (25) for any bounded $\mathcal{G}_{\Lambda',T'}(\Omega_\beta)$-measurable function F, where $\#\Lambda'$ and $T > 0$ are finite, follows existence of the following limiting measure:

$$G(\cdot) := \lim_{\substack{\Lambda \nearrow \mathbb{Z}^d \\ T \to +\infty}} G_{\Lambda,T}(\cdot|\xi_{\mathbb{Z}^d}).$$

This measure evidently satisfies (9). Its uniqueness follows from the fact that the explicit expression (27) for

$$\int\limits_{\Omega^{\mathbb{Z}^d}} F(\omega_{\mathbb{Z}^d}) G_\beta(d\omega_{\mathbb{Z}^d})$$

does not depend on $\xi_{\mathbb{Z}^d}$.

Acknowledgments.

The third author, R.M., thanks RFFI (grant no. 97–01–00714 and 99–01–00284) and SNSF (grant no. 7SUPJ048214) for financial support. The fourth author, G.S., would like to thank SFB 343 by DFG for finantial support during his stay in the Research Center BiBoS, Bielefeld University in 1999. The support of SFB 256, Universität Bonn, is also gratefully acknowledged.

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