Universität Bielefeld

Diplomarbeit

Existence of solutions to Fokker–Planck equations associated to SPDEs driven by cylindrical Lévy noise with bounded jumps

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Introduction

The study of stochastic partial differential equations (abbreviated SPDE) in Hilbert spaces with respect to Wiener type noise has been a very active topic of stochastic analysis for many years (for an overview see [PR07], [DPZ92] or [LR15]). In recent years there has been a growing interest in non-continuous noise (see [PZ07] or [App05]), because of - among other things - its many applications in finance (see [DNØP09] or [App05]).
In addition to the more common (pathwise) approaches to studying solutions of SPDEs (e.g. the variational or the mild approach), it is also possible to study solutions in terms of their laws by considering the corresponding Kolmogorov equations, which are equations on functions or their corresponding Fokker–Planck equations (abbreviated FPE), which are equations on measures.

Aim of this thesis and existing results

The purpose of this thesis is to prove - under certain conditions - existence of a solution $\mu_t(\mathrm{d}x)\mathrm{d}t$ to the following Fokker-Planck equation

$$\int\limits_H \psi(t,x)\mu_t(\mathrm{d}x) = \int\limits_H \psi(s,x)\xi(\mathrm{d}x) + \int\limits_s^t \int\limits_H L_0\psi(s',x)\mu_{s'}(\mathrm{d}x)\mathrm{d}s' \qquad \text{for } \mathrm{d}t\text{-a.e. } t \in [s,T], \tag{FPE}$$

for all $\psi \in D(L_0)$ and an initial condition ξ, which is given by a probability measure on a Hilbert state space H. The Kolmogorov operator L_0 is given by

$$L_0\psi(t,x) = (D_t\psi)(t,x) + \langle (D_x\psi)(t,x), F(t,x)\rangle + U\psi(t,x) \tag{1}$$

for all ψ in a suitable test function space $W_{T,A}$. Furthermore, U is a non-local Ornstein–Uhlenbeck operator to be specified later (see (2.7)) and $F(t,x) : D(F) \subset [0,T] \times H \to H$ is measurable. Hence a solution is given by a path $t \mapsto \mu_t$ in the space of probability measures.

In [Wie11] a very similar case has been studied and parts of the presented framework will be frequently used here. Existence and uniqueness have been shown there for the case where F is Lipschitz continuous, but for merely measurable F only uniqueness has been proven. Existence of a solution has been suggested as a future field of study. Such studies have not been done yet. The aim of this thesis is to realize this future study and prove existence.
The proof will follow the ideas of [BDPR10, Sect. 2], where existence and uniqueness of the solution have been shown in the case of Wiener noise, so we also compliment this work.

The Fokker–Planck equation is completely determined by (1). To explain the origin of the Kolmogorov operator, let us first consider the following stochastic differential equation on a separable Hilbert space H

$$\begin{cases} \mathrm{d}X(t) &= [AX(t) + F(t, X(t))]\mathrm{d}t + \mathrm{d}Y(t), \\ X(s) &= x_0 \in H, 0 \le s \le t \le T \end{cases} \tag{SPDE}$$

with F as above and $A : D(A) \subset H \to H$ being a self-adjoint generator of a C_0-semigroup e^{tA}, $t \ge 0$, in H. The noise term is given by $\mathrm{d}Y(t) = \sqrt{C}\mathrm{d}W(t) + \mathrm{d}J(t)$, where $W(t), t \ge 0$, is a cylindrical Wiener process on H, C is a symmetric positive bounded operator in $L(H)(:= \{\text{bounded linear operators on } H\})$ and $J(t)$ is a Lévy process with bounded jumps.
We will now summarize the results of [Wie11], which are relevant for this work. Let us denote the case, when the nonlinear drift F in (SPDE) is replaced with a Lipschitz continuous F_L, by adding a subscript L, i.e. (SPDE$_L$). By [MPR10, Thm. 2.4] (SPDE$_L$) admits a unique mild solution. Using this and Itô's formula, the Kolmogorov operator corresponding to (SPDE$_L$) has been identified in[Wie11, Chpt. 4] to be

$$L_L\psi(t, x) = (D_t\psi)(t, x) + \langle (D_x\psi)(t, x), F_L(t, x)\rangle + U\psi(t, x) \text{ for all } \psi \in W_{T,A}.$$

In this case he has shown uniqueness and existence of solutions for the FPE related to L_L. In [Wie11, Chpt. 5.2] it has further been proven that the FPE associated to L_0 (i.e. general measurable F) has at most one solution. However, in this case the relation of (SPDE) to L_0 has not been shown.

Let us now give a short overview of the main ideas of the proof in [BDPR10, Sect. 2], where (SPDE) was considered with Wiener noise (from now on (SPDE$_W$)) and existence of a solution to the FPE corresponding to (SPDE$_W$) was shown.
The authors consider (SPDE) with a special bounded F_α instead of F, which has nice approximation properties, and obtain a mild solution to this equation,

which we call (SPDE$_\alpha$). Using this solution, they solve the FPE with F_α replacing F and obtain solution measures $\mu_t^\alpha(\mathrm{d}x)\mathrm{d}t$. The authors proceed to show uniform tightness for these measures, allowing them to apply Prohorov's Theorem to receive a limiting measure. They carry on to show that this limit measure solves the FPE.

Structure and differences in the method of proof

Let us give an overview of the structure of this thesis and on the way explain our approach and own contributions.
This thesis is divided in three chapters. The first chapter gives a short summary of some essential results about semigroups and Lévy processes, including stochastic integration theory with respect to martingale measures.
The second chapter contains our preparations for the main proof, our precise conditions and an explanation for the choice of Kolmogorov operator. It concludes with the formulation of the main theorem.
The last chapter is devoted to the proof of the main theorem, which is structured in three steps.

The first chapter starts with the defintion of a C_0-semigroup, followed by a statement of some of their elementary properties. In particular we define infinitesimal generators and their Yosida approximations. We then proceed to explain fractional powers and exponentials of unbounded positive self-adjoint operators. The main reference in this part is [Paz83].
We then address Lévy processes, beginning with general properties, followed by stochastic integration with respect to martingale measures, which is suitable for type of Lévy processes considered in this thesis. We have opted for the martingale measure framework instead of integration via compensated Poisson measures in the version of [Kno05, Sect. 2.3], because it is more similar to the Wiener case of [PR07, Sect. 2.3.2] in terms of the used elementary functions, which is crucial in Lemma 2.3. We could have used the integration framework with respect to general Lévy processes following [PZ07, Sect. 8.2], but the established Itô-Isometry is not as explicit as the one given in [Sto05, Prop. 3.1.3], whose form will be essential for Lemma 2.3.
We then elaborate on our choice of noise, in particular reviewing general existence conditions for the stochastic convolution and finally defining the notion of a mild solution. Special notice should be taken of the reference [Sto05], which is our main reference in the integration part, and [AR05], where the Lévy–Itô decomposition was shown for Banach spaces.

The second chapter begins with stating our general conditions. Extra attention

should be given to the condition that $(-A)^{2\delta-1}$ is of trace class for some $\delta \in [0, \frac{1}{2}]$, since this condition differs from the original one given in [BDPR10]. The condition there, that $(-A)^{2\delta}$ should be of trace class, seems to be a misprint. We also add special boundedness conditions for the Lévy measure corresponding to the Lévy process (see Condition 2.2), which we will heavily use in Chapter 3, especially in Claim 3.7 and the last approximation in Step 3 of the main proof.
One of the main results of the second chapter is the following bound for the stochastic convolution Y_A from Lemma 2.3:

$$\sup_{t\in[0,T]} \mathbb{E}\big[\big|(-A)^{\delta} Y_A(t)\big|^2\big] < \infty.$$

This inequality is vital for step 2 of the main proof and could not be directly extended from the Wiener case to the Lévy case. This is the reason for the choice of the afore mentioned special integration framework. By showing that $(-A)^{\delta}$ can indeed be taken under the stochastic convolution integral and using the special Itô Isometry for martingale measures to explicitly compute the existence of the resulting stochastic convolution, we were able to prove this bound for our Lévy type noise.
We then explain our choice of the Kolmogorov operator L_0, introduce the Fokker–Planck equation and define the notion of a solution to the FPE.
For this approach it is necessary that L_α corresponds to (SPDE$_\alpha$) and that we have a mild solution to (SPDE$_\alpha$). Therefore, we then present our main assumption that (SPDE$_\alpha$) admits a mild solution and that the solution solves the martingale problem for L_α, which gives us our desired correspondence of L_α to (SPDE$_\alpha$), according to Lemma 2.14.
We then proceed to show the equivalence of (FPE) and

$$\int_0^T \int_H L_0\psi(r,x)\mu_r(\mathrm{d}x)\mathrm{d}r = -\int_H \psi(0,x)\xi(\mathrm{d}x) \text{ for all } t \in [s,T] \text{ and } \psi \in W_{T,A}$$

(see Lemma 2.12). This equivalence was not shown in [Wie11, Rem. 2.2.6]. We finish the chapter with the statement of our main theorem.

The third chapter consists solely of the main proof. Let us now give a more detailed outline of the three steps and the main differences to the proof presented in [BDPR10].
Step 1 starts with the proof of finiteness of the following integral of our approximating solution measure μ_t^α to (FPE$_\alpha$):

$$\int_H |x|^2 \, \mu_t^\alpha(\mathrm{d}x) \leq C < \infty,$$

thus proving tightness with respect to the weak topology for this family of measures. Here we use a different approximation (utilizing the eigenbasis of A), than was suggested in [BDPR10], to deal with the unboundedness of A.
This allows us to apply Prohorov's Theorem repeatedly to obtain the limit measure candidate μ_t. The most notable change in this step - besides extensions and little adaptations - lies in Claim 3.7. In this claim - as in [BDPR10] - we show the equicontinuity of

$$t \mapsto \int_H \psi(t,x)\mu_t^\alpha(\mathrm{d}x) \tag{2}$$

for fixed $\psi \in W_{T,A}$. However, due to the more general noise, we had to make use of an additional boundedness condition (see Condition 2.2) to achieve equicontinuity despite the more general noise.
In Step 2 we begin similarly by computing

$$\int_s^T \left|(-A)^\delta x\right|^2 \mu_t^\alpha(\mathrm{d}x)\mathrm{d}t \leq C < \infty \tag{3}$$

and using this, we deduce that the measures $\mu_t^\alpha(\mathrm{d}x)\mathrm{d}t$ are strongly tight, which enables us to apply Prohorov's Theorem again to conclude that $\mu_t^{\alpha_n}\mathrm{d}t \to \mu_t\mathrm{d}t$ weakly on $[0,t]\times H$ for some (sub-)sequence $\alpha_n \to 0$. This part does not require major changes to accommodate the larger class of noise, except for the change of the test function space.
The third step begins by two lengthy calculations to prove the continuity of

$$H \ni x \mapsto \int_H \psi(t,x+y) - \psi(t,x) - \frac{{}_{H^*}\langle (D_x\psi)(t,x), y\rangle_H}{1+|y|^2} M(\mathrm{d}y), \tag{4}$$

where M is the Lévy measure corresponding to Y, and to prove the continuity of

$$H \ni x \mapsto \frac{1}{2}\int_H \langle \xi, Q\xi\rangle e^{i\langle \xi, x\rangle}\nu_t(\mathrm{d}\xi), \tag{5}$$

where $\nu_t(\mathrm{d}\xi)$ is a certain measure depending on the chosen test function. These calculations are only necessary in the Lévy case and are needed to control the "jump-parts" of L_0.
We then proceed to prove that $\mu_t(\mathrm{d}x)\mathrm{d}t$ solves (FPE). To this end we use the above mentioned equivalent formulation of the Fokker–Planck equation to rewrite our approximating solution (FPE_α) as

$$\int_s^T \int_H L_{\alpha_n}\psi(t,x)\mu_t^{\alpha_n}(\mathrm{d}x)\mathrm{d}t = -\int_H \psi(s,x)\xi(\mathrm{d}\xi).$$

We then proceed to show weak convergence of the left hand side for each summand of the Kolmogorov operator individually, using heavily our previously established weak convergence of the approximation measures.
The convergence of the jump part of L_0 follows by the continuity results (Equations (4) and (5)) and the previous mentioned boundedness condition 2.2.
For the leftover uncontinuous part we had to adjust the approximation approach taken in [BDPR10]. We could leave out an approximation in [BDPR10] dealing with the part of the linear drift in L_0, since our test functions only take non-zero values on a finite subspace of H (see Remark 3.14).
Furthermore, we could omit an approximation by affine linear functions, which was in [BDPR10] realized by one-dimensional Riemann sums. This is in our case not necessary, since the test functions are in our case cylinder functions in space.

Future topics of research

A possible extension of the results presented here could be to allow a broader range of noise, namely dropping the bounded jump condition on the noise part.
It also seems reasonable to hope that it is possible to drop Assumption 2.7 (existence of a mild solution to (SPDE_α)), since it might already be consequence of Assumption 2.15 (existence of a solution to the martingale problem for L_α).

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Chapter 1

Prerequisites

1.1 Semigroups

In this section we are going to define C_0-semigroups, which are important objects in the study of partial differential equations. They can be thought of as an infinite dimensional operator valued generalisation of exponential functions.
We are only going to cover some basic properties and theorems, which we will later need for our proof, such as the Lévy-Itô-decomposition, Yosida approximations and fractional powers. For a very thorough treatment of semigroups we refer to [Paz83] and to [DPZ92] for an introduction to semigroups in the context of SPDE on Hilbert spaces.
Let $(B, |\cdot|_B)$ be a Banach space.

Definition 1.1 (C_0-semigroup)**.** We call a family $S = \{S(t), t \geq 0\}$ of bounded linear operators on a Banach space $(B, |\cdot|_B)$ a strongly continuous semigroup of bounded linear operators (or short C_0-semigroup) iff

i) $S(0)$ is the identity operator I,

ii) $S(t)S(s) = S(t+s)$ for all $t, s \geq 0$,

iii) we have for all $x \in B$ that $|S(t)x - x|_B \to 0$ as $t \downarrow 0$.

If only i) and ii) hold, we call the family simply semigroup.

Theorem 1.2. *Let S be a C_0-semigroup. Then there exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t} \text{ for } 0 \leq t < \infty.$$

We call a C_0-semigroup with $\omega = 0$ uniformly bounded. If additionally $M = 1$ we call it a C_0-semigroup of contractions.

Proof. See [Paz83, Thm. 2.2].

□

Infinitesimal generators

Definition 1.3 (Infinitesimal generator)**.** For a semigroup S we call the linear operator defined by

$$Ax = \lim_{t \downarrow 0} \frac{S(t)x - x}{t} \text{ for all } x \in D(A)$$

on

$$D(A) = \{x \in X | \lim_{t \downarrow 0} \frac{S(t)x - x}{t} \text{ exists } \}$$

the infinitesimal generator A of the semigroup S, where we write $D(A)$ for the domain of A.

Remark 1.4. An infinitesimal generator A of a C_0-semigroup has a dense domain on B and we have that A is closed (see [Paz83, Cor. 2.2]).

The following theorem shows how a semigroup interacts with its infinitesimal generator.

Theorem 1.5. *Let S be a C_0-semigroup on B and let A be its infinitesimal generator.*

i) For $x \in B$ we have

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)x\mathrm{d}s = T(t)x.$$

ii) For $x \in B$ we have

$$\int_0^t T(s)x \,\mathrm{d}s \in D(A)$$

and

$$A\left(\int_0^t T(s)x\mathrm{d}s\right) = T(t)x - x.$$

iii) For $x \in D(A)$ we have $T(t)x \in D(A)$ and

$$\frac{d}{\mathrm{d}t} T(t)x = \mathrm{A\,T}(t)x = T(t)Ax.$$

iv) For $x \in D(A)$ we have

$$T(t)x - T(s)x = \int_s^t T(s')Ax\mathrm{d}s' = \int_s^t AT(s')x\mathrm{d}s'.$$

Proof. See [Paz83, Thm. 2.4]. □

Yosida approximations

Since the infinitesimal generator A of a C_0-semigroup $S(t)$ is not necessarily bounded but only closed, it is often necessary to use approximations A_n of A, which are bounded. These approximations arise naturally out of the formulation of the Hille-Yosida theorem, for which we are going to refer to [DPZ92, P. 371, Thm. A.2].
Recall that the resolvent of a linear operator A is defined as

$$R(\lambda, A) := (\lambda I - A)^{-1}$$

for $\lambda \in p(A)$, where we denote the resolvent set of A as $p(A)$. For closed A this operator is bounded (cf. [DPZ92, P. 371]).

Definition 1.6 (Yosida approximation)**.** For a C_0-semigroup $S(t)$ with infinitesimal generator A and $\|S(t)\| \leq Me^{\omega T}$ for all $t \geq 0$ we call the operators

$$A_n := nAR(n, A) \text{ with } n \geq \omega$$

the Yosida approximations of A.

Remark 1.7. We have that $]\omega, \infty[\subset p(A)$ for the generator A of C_0-semigroup. [DPZ92, p. 372, A.2]

That the Yosida approximations A_n retain some relation to A can be seen in the next two theorems.

Theorem 1.8. *Let A be the infinitesimal generator of a C_0-semigroup $S(t)$ and let A_n be its Yosida approximation. Then we have*

$$\lim_{n\to\infty} A_n x = Ax \textit{ for } x \in D(A).$$

Proof. See [DPZ92, Prop. A.3]. □

Theorem 1.9 (Exponential of Yosida approximations). *For an infinitesimal generator A of a semigroup of contractions $S(t)$ we have for its Yosida approximations A_n that*

$$S(t)x = \lim_{n\to\infty} e^{tA_n x}.$$

Proof. [Paz83, p.11, Cor. 3.5] □

Fractional Powers, exponentials of operators and the spectral theorem

From now on we are going to restrict ourselves to the case of a Hilbert-space H with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $|\cdot|$. Let us explain how to define the operator $h(T)$ for a Borel measurable map $h : H \to H$ and self-adjoint operator $T : H \supset \mathrm{D}(T) \to H$. We are going to translate the spectral theorem for self-adjoint operators from [Wer07, Thm. VII.3.2.], since it includes the definition of the functional calculus in the case of an unbounded A. We then proceed to explain T^α and e^T.

Theorem 1.10 (Spectral theorem for self-adjoint operators). *Let $T : H \supset \mathrm{D}(T) \to H$ be self-adjoint. Then there exists a unique spectral measure E with*

$$\langle Tx, y\rangle = \int\limits_{\mathbb{R}} \lambda \mathrm{d}\langle E_\lambda x, y\rangle \textit{ for all } x \in \mathrm{dom}(T), y \in H.$$

For Borel measurable $h : \mathbb{R} \to \mathbb{R}$ and

$$D_h = \left\{ x \in H | \int\limits_{\mathbb{R}} |h(\lambda)|^2 \,\mathrm{d}\langle E_\lambda x, y\rangle \right\},$$

we have a self-adjoint operator $h(T) : h \supset D_h \to H$ given by

$$\langle h(T)x, y\rangle = \int\limits_{\mathbb{R}} h(\lambda)\mathrm{d}\langle E_\lambda x, y\rangle.$$

Translated from and proof contained in [Wer07, Thm. VII.3.2.].
Thus if A is a positive self-adjoint operator on a Hilbert-space we can use $h(\lambda) = \mathbb{I}_{]0,\infty[}\lambda^\alpha$ for $\alpha \in \mathbb{R}$ to define its fractional power using the spectral decomposition as

$$A^\alpha = \int_0^\infty h(\lambda)\mathrm{d}E_\lambda.$$

where $E(\lambda)$ denotes the corresponding spectral measure. It is defined on

$$D(A^\alpha) := \left\{ x \in H | \int_0^\infty h(\lambda) \mathrm{d}\langle E_\lambda x, y\rangle < \infty \right\}.$$

For details on this case see [Kla08, P. 42].

Let us now take a look at the exponential of operators in the context of operator semigroups. Let $S(t)$ be C_0-semigroup of bounded linear operators on H and A its infinitesimal generator. If A is bounded (i.e. when $S(t)$ is uniformly continuous cf. [Paz83, Thm. 1.2]) we can write

$$T(t) = e^{tA} := \sum_{n=0}^\infty \frac{(tA)^n}{n!}$$

since $\|e^{tA}\| \leq \sum_{n=0}^\infty \frac{t^n \|A\|^n}{n!} = e^{t\|A\|}$ and thus e^{tA} is a bounded operator. Here we denote the standard operator norm by $\|\cdot\|$. We have $T(t) = S(t)$ (see [Paz83, Thm. 1.2 and 1.3]). For C_0-Semigroups A is in general not bounded, but only closed. Thus e^{tA} can not be defined as above, since it would not necessarily be well defined. But there are some ways in which one can give meaning to e^{tA}.
For example, we can define $T(t) = e^{tA} =: \lim_{\lambda\to\infty} e^{tA_\lambda} x$ as the limit of its Yosida approximation as in Theorem 1.9. Another possible interpretation is given in the following theorem.

Theorem 1.11. *We have for a C_0-semigroup $S(t)$ with generator A and*

$$A(h)x := \frac{S(h)x - x}{h}$$

for $h \in \mathbb{R}^+$, that for all $x \in B$

$$T(t)x = \lim_{h\downarrow 0} e^{tA(h)} x$$

uniformly in t on every bounded interval.

Proof. See [Paz83, Thm. 8.1]. □

This gives us a reasonable interpretation of e^{tA} in case of general unboundedness, which is still formally incorrect.
For the special case that A is a self-adjoint operator (not necessarily bounded) we can use the spectral theorem above and define e^{tA} by

$$\langle e^{tA} x, y\rangle = \int_{\mathbb{R}} e^{t\lambda} \mathrm{d}\langle E_\lambda x, y\rangle$$

on $D(e^{tA}) := \{x \in H | \int_{\mathbb{R}} e^{t\lambda} \mathrm{d}\langle E(\lambda)x, x\rangle < \infty\}$.

1.2 Lévy processes

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be stochastic basis and H a separable real Hilbert space with inner product $\langle \cdot , \cdot \rangle$.

Definition 1.12 (Lévy process)**.** We call an H-valued $(\mathcal{F}_t)_{t\geq 0}$ adapted stochastic process $Y(t)$ a Lévy process iff:

i) the increments of Y are independent, i.e. $Y(t_2) - Y(t_1)$ is stochastic independent of $Y(s_2) - Y(s_1)$ for all $0 \leq s_1 \leq s_2 \leq t_1 \leq t_2 \leq \infty$,

ii) the increments of Y are stationary, i.e. for every $s \in \mathbb{R}^+$ the distribution of $Y(t+s) - Y(t)$ is independent of the choice of $t \in \mathbb{R}^+$,

iii) $Y(0) = 0$ holds $\mathbb{P}$-almost surely,

iv) Y is stochastically continuous, i.e. for every $\epsilon > 0$ and any $t \geq 0$ we have that

$$\lim_{h\to 0} \mathbb{P}[|Y(t+h) - Y(t)| \geq \epsilon] = 0 \text{ for all } h \in \mathbb{R}^+.$$

See [PZ07, Def. 4.1] for a slightly more general definition on Banach spaces.

Properties

We will present some general results and theorems about Lévy-Processes in Hilbert spaces in this chapter, without proof, for the convenience of the reader.

Lemma 1.13. *Every Lévy-Process has a cádlág modification.*

Proof. See [PZ07, Thm. 4.1] □

Definition 1.14 (Lévy measure)**.** We call a measure $M \in \mathcal{M}(H)$ a Lévy measure iff $M(\{0\}) = 0$ and

$$\int_H (|y|^2 \wedge 1) M(\mathrm{d}y) < \infty$$

By the Lévy-Khinchin formula, it is possible to specify the type of Lévy process by writing a triple $[a, Q, M]$ where we call a the drift part, corresponding to the linear drift part of the Lévy noise. We refer to Q as the Wiener part, since it specifies the covariance of the Wiener noise contained in a Lévy process. By the noise part we mean M, since it describes the distribution of Jumps, i.e. the number of jumps up to time 1 contained in $B_{x,\epsilon} \subset H$ (Ball around x with radius ϵ) is given by $M(B_{x,\epsilon})$. Since it is also possible to identify the parameter $[a, Q, M]$ of a Lévy-process using the Lévy-Itô decomposition, we will omit stating the Lévy-Khinchine decomposition.

Poisson Processes

In the following we give a short summary of the connection of Lévy processes to Poisson processes and state the famous Lévy-Itô representation for Hilbert space valued Lévy processes. We will follow the framework of [Sto05], which in turn generalizes [App06] using some results from [AR05]. A different approach can be found in [PZ07].
Let us now take a closer look at the discontinuities of Lévy processes. Such a discontinuity at time t can be measured as $\Delta X(t) := X(t) - X(t-)$ where $X(t-) := \lim_{t' \uparrow t} X(t')$. In [AR05] the following theorems have been shown for a Lévy-Process $(X_t)_{t \geq 0}$ on a Banach-space E with its corresponding Borel σ-algebra $\mathcal{B}(E)$. We will only state these theorems for Hilbert spaces.

Theorem 1.15. *Let $A \in \mathcal{B}(H)$ separated from 0, i.e. $0 \in \bar{A}^c$. Then we can define*

$$N(t, A, \omega) = N(t, A) := \sum_{0 \leq s \leq t} \mathbb{I}_A(\Delta X_s) = \#\{0 < s \leq t \mid \Delta X(s) \in A\}$$

and $N(t, A)$ is for $t \in [0, T]$ an adapted counting process without explosion. Additionally it is a Poisson Process.

Proof. See [AR05, Thm. 2.7] □

Theorem 1.16 (Poisson random measure)**.** *For fixed $t \in [0, T]$ and $\omega \in \Omega$ the function*

$$N(t, \cdot, \omega) : \mathcal{B}(H \setminus 0) \to \mathbb{R}^+ \cup \{+\infty\}$$

has for $\mathbb{P}$-almost all ω an unique extension to a σ-finite measure ν_t on $\mathcal{B}(H)$ with $\nu_t(\{0\}) = 0$. From now on we will write $N(t, \mathrm{d}x)$ instead of $\nu_t(\mathrm{d}x)$ and call $N(t, \mathrm{d}x)$ a Poisson random measure. Further, if we set for all $A \in \mathcal{B}(H \setminus \{0\})$, with A seperated from 0, then

$$\begin{aligned} \tilde{\nu} : \mathcal{B}(H \setminus 0) &\to \mathbb{R}^+ \cup \{+\infty\} \\ A &\mapsto \mathbb{E}[N(1, A)]. \end{aligned}$$

The measure $\tilde{\nu}$ can also be extended to a unique σ-finite measure ν on $\mathcal{B}(H)$ with $\nu(\{0\}) = 0$. This measure is a Lévy measures and if the Lévy process X has the characteristic triplet $[a, Q, M]$, it equals the measure M.

Proof. See [AR05, Thm. 2.17, Cor. 2.18, Thm. 2.21]. □

Theorem 1.17 (Compensated Poisson random measure). *Let M be the Lévy measure of the Lévy-Process $(X_t)_{t\geq 0}$. Then we can define the compensated Poisson random measure*

$$\tilde{N}(t, A) := N(t, A) - tM(A).$$

For $A \in \mathcal{B}(H \setminus \{0\})$, with A separated from 0, we have that $(\tilde{N}(t, A))$ is a $(\mathcal{F}_t)$-martingale and that $\mathbb{E}[\tilde{N}(t, A)] = 0$.

Proof. See [Sto05, Lem. 2.3.8]. □

Now we can define integration with respect to these measures.
For $A \in \mathcal{B}(H \setminus 0)$ separated from 0 and $f : A \to H$, $\mathcal{B}(A \cap H)/\mathcal{B}(H)$ measurable, we can define the integration as

$$\int_A f(x)N(t, \mathrm{d}x) := \sum_{0<s\leq t} f(\Delta X(s))\mathbb{I}_A(\Delta X(s)).$$

For M integrable f (i.e. $f \in L^1((A, \mathcal{B}(A), M|_A) \to H))$ we can now define

$$\int_A f(x)\tilde{N}(t, \mathrm{d}x) = \int_A f(x)N(t, \mathrm{d}x) - t\int_A f(x)M(\mathrm{d}x)$$

where the last part is the usual Bochner integration. For a more exhaustive treatment of integration with respect to Poisson random measures in space see [Sto05, 2.4].

Theorem 1.18. *Let $f \in L^2_M := L^2(H \setminus \{0\}, M; H)$ then we have for $A \in \mathcal{B}(H)$ that*

$$M(t) = \int_{A\cap\{|x|<1\}} f(x)\tilde{N}(t, \mathrm{d}x)$$

is a cádlág (i.e. right-continuous with left limits) square integrable martingale with $M(\{0\}) = 0$.

Proof. See [Sto05, Prop. 2.4.6]. □

The next theorem will give very important insight into the structure of the jumps of a Lévy processes.

Theorem 1.19 (Lévy-Itô decomposition)**.** *Let $(Y_t)_{t\geq 0}$ be a Lévy-Process on a Hilbert space H and M the corresponding Lévy measure.*
Suppose $N_t(\omega, \mathrm{d}x)$ is the associated Poisson random measure and $N_t(\omega, \mathrm{d}x) - tM(\mathrm{d}x)$ the compensated Poisson random measure related to the Lévy process $(Y_t)_{t\geq 0}$. Then for all $K \geq 0$, there is $\alpha_K \in H$ such that for all $t \geq 0$

$$Y_t = B_t + \int\limits_{\{|x|<K\}} x(N_t(\mathrm{d}x) - t\nu(\mathrm{d}x)) + \alpha_K t + \int\limits_{\{|x|\geq K\}} xN_t(\mathrm{d}x)$$

where B_t is a H -valued centered Brownian motion. Further, we have that for all $A \in \mathcal{B}(H \setminus 0)$, with A seperated from 0, that $(N_t(A))_{t\geq 0}$ is independent of $(B_t)_{t\geq 0}$.

Proof. [AR05, Thm. 4.1]. □

As we can see every Lévy process consists of a martingale part and a non martingale part.

$$L_t = \underbrace{B_t + \int\limits_{\{|x|<K\}} x(N_t(\mathrm{d}x) - t\nu(\mathrm{d}x))}_{\text{Martingale}} + \underbrace{\alpha_K t + \int\limits_{\{|x|\geq K\}} xN_t(\mathrm{d}x)}_{\text{Non martingale}}$$

In most of this thesis it will be necessary to restrict ourselves to martingale type Lévy noise, since it would be difficult to control the unbounded jump part $\int_{\{|x|\geq K\}} xN_t(\omega)$.

Martingale measures

The following notion is a simplified version of [Sto05] and [App06], who generalize the concept of martingale measures from the finite dimensional version of [Wal86] to Hilbert spaces. Throughout this part we will work on a Ball $S \subset H$ with $S = \{x \in H|\, |x| < K\}$ and use the σ-Algebra $\mathcal{A} := \{A, A \cup \{0\} | A \in \mathcal{B}(S) \text{ and } 0 \notin \bar{A}\}$. Corresponding to this let $S_n := \{x \in S | \frac{1}{n} \leq |x| \leq K\}$ for arbitrarily large $K \in \mathbb{N}$. These unusual spaces are needed to evade working on general Lusin spaces (see [Sto05, p. 42]), which is unnecessary in our case.

Definition 1.20 (Martingale measures)**.** We call a function

$$\mathbb{M} : \mathbb{R}^+ \times \mathcal{A} \times \Omega \to H$$

an $(\mathcal{F}_t)$-martingale measure iff

i) we have $\mathbb{M}(0, A) = \mathbb{M}(t, \emptyset) = 0$ $\mathbb{P}$-almost surely for all $A \in \mathcal{A}$,

ii) we have for all $t > 0$ $\mathbb{M}(t, A \cup B) = \mathbb{M}(t, A) + \mathbb{M}(t, B)$ $\mathbb{P}$-almost surely, for every pair of disjoint $A, B \in \mathcal{A}$,

iii) we have $\sup_{A \in \mathcal{B}(S_n)} \mathbb{E}\,|\mathbb{M}(t, A)|^2 < \infty$ for all $n \in \mathbb{N}$,

iv) $(\mathbb{M}(t, A), t \geq 0)$ is strongly cádlág square integrable $(\mathcal{F}_t)$-martingale for all $A \in \mathcal{A}$,

v) we have for all $n \in \mathbb{N}$ and $(A_j) \in \mathcal{B}(S_n)$ with $A_j \downarrow \emptyset$ that $\lim_{j \to \infty} \mathbb{E}[|\mathbb{M}(t, A_j)|^2] = 0$.

To connect this definition to our setting we will state the following lemma.

Lemma 1.21. *The martingale part of the Lévy-Itô decomposition of a Lévy process*

$$\mathbb{M}(t, A) = B_Q(t)\delta_0(A) + \int\limits_{A \setminus \{0\}} x\tilde{N}(t, \mathrm{d}x) \tag{1.1}$$

is a martingale measure for all $t \geq 0$ and $A \in \mathcal{A}$.

Proof. See [Sto05, Thm. 2.5.2] where also orthogonality and independence of increments were proven. □

We call $\mathbb{M}(t, A)$ in (1.1) a Lévy martingale measure.
We have to introduce a bit more theory for martingale measures, to start with stochastic integration.

Definition 1.22 (Positive operator valued measure)**.** We call a family of positive operators $T = \{T_A, A \in \mathcal{A}\}$ on H, which are bounded, non-negative and self adjoint, a positive operator valued measure iff $T_\emptyset = 0$ and $T_{A \cup B} = T_A + T_B$ for all disjoint $A, B \in \mathcal{A}$.
We call T decomposable iff we can find σ-finite measure ν on $(H, \mathcal{A})$ and a family $\{T_x, x \in H\}$ of bounded non negative self-adjoint operators on H with

$$T_A y = \int\limits_A T_x y \nu(\mathrm{d}x) \text{ with } A \in \mathcal{A}, y \in H,$$

for which we additionally need that $x \mapsto T_x y$ is measurable.

This property is connected to martingale measures in the following way: We say that a martingale measure $\mathbb{M}$ is nuclear with respect to a radon measure τ on $\mathbb{R}^+$

and a nuclear positive operator valued measure T_A (i.e. T_A is a nuclear operator for every $A \in \mathcal{A}$) iff

$$E[\langle \mathbb{M}(]s,t],A),x\rangle \cdot \langle \mathbb{M}(]s,t],A),y\rangle\rangle] = \langle x, T_A y\rangle \cdot \tau(]s,t]) \qquad \text{for } 0 \le s < t < \infty, A \in \mathcal{A}, x, y \in H.$$

In the context of the Lévy martingale measure we get the following theorem about decomposability.

Lemma 1.23 (Decomposition of Lévy martingale measure)**.** *Let $\mathbb{M}(t,A)$ be Lévy martingale measure. Then $\mathbb{M}$ is nuclear with (T,λ), with λ being the Lebesgue measure on $\mathbb{R}^+$ and*

$$T_A y = Qy\delta_0(A) + \int\limits_{A\setminus\{0\}} \langle x,y\rangle x M(\mathrm{d}x)$$

We further have that T is decomposable in $\nu = M + \delta_0$ and

$$T_x = \begin{cases} Q & \text{for } x = 0. \\ \langle x, \cdot\rangle x & \text{else.} \end{cases}$$

A version of this has been shown in [Sto05, p.46 Prop 2.5.4].

Stochastic integral for martingale measures

We are now giving a rough overview of the construction of stochastic integral with respect to martingale measures following [Sto05, Chpt. 3]. As mentioned in the introduction, we have chosen the stochastic integration using martingale measures in this version over [PZ07], since it gives us a very convenient Itô-isometry, which would not be the case otherwise. See [PZ07, 8.7] for a different version of integration with respect to compensated Poisson measures.
The aim of this chapter is to identify a large space of integrands which can be stochastically integrated with respect to a nuclear martingale measure.

For a decomposable operator T corresponding to a nuclear martingale measure $\mathbb{M}(t,A)$ and $0 \le t \le \tilde{T}$, $\mathcal{N}_2(T,t) := \mathcal{N}_2^T(\nu,\tau;t)$ denotes the space of all operator valued mappings

$$R : [0,t] \times \Omega \times S \to \{(A : H \to H) | A \text{ is linear }\},$$

such that $(s,\omega,x) \mapsto R(s,\omega,x)y$ is $\mathcal{F} \times \mathcal{B}(S)$ measurable for every $y \in H$ and we further have

$$\|R\|_{\mathcal{N}_2(T,t)} := \mathbb{E}\left[\int\limits_0^t \int\limits_S \|R(s,x)\sqrt{T_x}\|_2^2 \nu(\mathrm{d}x)\tau(\mathrm{d}t)\right]^{\frac{1}{2}} < \infty.$$

If we endow $\mathcal{N}_2^T(t)$ with the inner product

$$(R_1, R_2)_{\mathcal{N}_2(T;t)} := E\left[\int\limits_0^t \int\limits_S \operatorname{tr}(R_1(s,x)T_x R_2(s,x)^*)\nu(\mathrm{d}x)\tau(\mathrm{d}t)\right],$$

$\mathcal{N}_2(T;t)$ is a Hilbert space (see [Sto05, Lem. 3.1.1]). Here $\|\cdot\|_2 = \|\cdot\|_{L^2}$ denotes the Hilbert–Schmidt norm for operators. From now on we will write $\mathcal{N}_2(T)$ for $\mathcal{N}_2^T(\nu, \tau; \tilde{T})$.
We now define a set of simple functions which we will use to define the integral by approximation. Let

$$\mathcal{N}_2(T) \supset \mathcal{S}_2(T) := \left\{ \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} R_{ij} \mathbb{I}_{]t_i, t_{i+1}]} \mathbb{I}_{A_j} \right\}$$

where $N_1, N_2 \in \mathbb{N}$, $(t_i)_{i \geq 0}$ is a partition of $[0, \tilde{T}]$, $A_0, \ldots, A_{N_2+1} \in \mathcal{A}$ and $R_{ij}h$ is F_{t_i}-measurable.
We have that $\mathcal{S}_2(T)$ is dense in $\mathcal{N}_2(T)$ (cf. [Sto05, Lem. 3.1.2]).
We now define the stochastic integral of a function $R \in \mathcal{S}_2(T)$ as

$$J_t(R) := \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} R_{ij} \mathbb{M}(]\tilde{T} \wedge t_i, t \wedge t_{i+1}], A_j)$$

for $0 \leq t \leq \tilde{T}$.

Theorem 1.24 (Extension of the integral)**.** *The mapping $J_t : \mathcal{S}_2(T) \to L^2((\Omega, \mathcal{F}, P) \to H)$ can be extended to an isometry*

$$J_t : \mathcal{N}_2(T) \to L^2((\Omega, \mathcal{F}, P) \to H).$$

For $R \in \mathcal{N}_2(T)$ we call $J_t(R) =: \int_0^t \int_S R(s,x)\mathbb{M}(\mathrm{d}s, \mathrm{d}x)$ the stochastic integral of R with respect to the nuclear martingale measure $\mathbb{M}$.

Proof. See again [Sto05, Prop. 3.1.3]. □

Properties

Let us state some properties for the just defined Integral.

Theorem 1.25 (Itô-Isometry)**.** *For $R \in \mathcal{N}_2(T)$ the process*

$$\left(\int\limits_0^t \int\limits_S R(s,x)\mathbb{M}(\mathrm{d}s, \mathrm{d}x)\right)_{t \geq 0}$$

is square integrable martingale with values in H.
Further we have this form of the Itô-isometry:

$$\mathbb{E}\left[\int_0^T\int_S \|\mathbb{I}_{[0,t]}R(s,x)\sqrt{T_x}\|_2^2\nu(\mathrm{d}x)\tau(\mathrm{d}x)\right] = \mathbb{E}\left[\left|\int_0^T\int_S R(s,x)\mathbb{M}(\mathrm{d}s,\mathrm{d}x)\right|^2\right]$$

Proof. See [Sto05, Thm. 3.1.5]. □

Similar to the case of stochastic integration with respect to cylindrical Wiener processes we can pull bounded linear operators under the integral.

Theorem 1.26. *For $C \in L(H)$ and $R \in \mathcal{N}_2(T)$. Then $CR \in \mathcal{N}_2(T)$ and we have for all $t \in [0,T]$*

$$C\int_0^t\int_S R(s,x)\mathbb{M}(\mathrm{d}s,\mathrm{d}x) = \int_0^t\int_S CR(s,x)\mathbb{M}(\mathrm{d}s,\mathrm{d}x) \quad P\text{-almost surely}$$

Proof. See [Sto05, 3.3.1]. □

We will later need a more general version of this for closed operators. The following theorem is similar to [DPZ92, 4.15] in the case of Wiener noise. The limit condition might be superfluous.

Lemma 1.27. *Let A be a closed operator, $\mathcal{S}_2(T) \ni R_n \to R \in \mathcal{N}_2(T)$ in $\mathcal{N}_2(T)$. Assume*

$$\int_0^t\int_S AR_n(s,x)\mathbb{M}(\mathrm{d}s,\mathrm{d}x) \to \int_0^t\int_S AR(s,x)\mathbb{M}(\mathrm{d}s,\mathrm{d}x) \text{ in } L^2((\Omega,\mathcal{F},\mathbb{P}) \to H)$$

holds and the integrals are well defined (i.e. $AR, AR_n \in \mathcal{N}_2(T)$ and $R_n, R \in D(A)$). Then we have for $0 \le t \le T$

$$A\int_0^t\int_S R(s,x)\mathbb{M}(\mathrm{d}s,\mathrm{d}x) = \int_0^t\int_S AR(s,x)\mathbb{M}(\mathrm{d}s,\mathrm{d}x) \qquad \mathbb{P}\text{-a.s.}$$

Proof. We need

$$\int_0^t\int_S R_n(s,x)\mathbb{M}(\mathrm{d}s,\mathrm{d}x) \to \int_0^t\int_S R(s,x)\mathbb{M}(\mathrm{d}s,\mathrm{d}x)$$

as $n \to \infty$, which we have by construction and

$$\int_0^t \int_S AR_n(s,x)\mathbb{M}(\mathrm{d}s, \mathrm{d}x) \to \int_0^t \int_S AR(s,x)\mathbb{M}(\mathrm{d}s, \mathrm{d}x),$$

which we have by assumption. Further we have

$$\begin{aligned} A\int_0^t \int_S R_n(s,x)\mathbb{M}(\mathrm{d}s, \mathrm{d}x) &= A\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} R_{ij}\mathbb{M}(]t\wedge t_i, t\wedge t_{i+1}], A_j) \\ &= \int_0^t \int_S AR_n(s,x)\mathbb{M}(\mathrm{d}s, \mathrm{d}x). \end{aligned}$$

Therefore we have

$$\begin{aligned} & A\int_0^t \int_S R_n(s,x)\mathbb{M}(\mathrm{d}s, \mathrm{d}x) \\ = & \int_0^t \int_S AR_n(s,x)\mathbb{M}(\mathrm{d}s, \mathrm{d}x) \qquad \overset{n\to\infty}{\to} \int_0^t \int_S AR(s,x)\mathbb{M}(\mathrm{d}s, \mathrm{d}x) \\ & \qquad\qquad = A\int_0^t \int_S R(s,x)\mathbb{M}(\mathrm{d}s, \mathrm{d}x) \end{aligned}$$

where the last step holds by the closedness of A. $\square$

Lévy martingale processes with cylindrical Wiener noise

As we have seen in Theorem 1.19 a Lévy process Y on a Hilbert-space H can be written as

$$Y(t) = B_t + \int_{\{|x|<K\}} x(N_t(\mathrm{d}x) - t\nu(\mathrm{d}x)) + \alpha_K t + \int_{\{|x|\geq K\}} xN_t(\mathrm{d}x)$$

where B_t is a Q-Wiener process.
In our case we want to consider cylindrical Wiener processes as the Brownian part, i.e. the case where Q is no longer of trace class. To achieve this we can look at processes of the form

$$Y(t) = W(t) + \int_{\{|x|<K\}} x(N_t(\mathrm{d}x) - t\nu(\mathrm{d}x)) + \alpha_K t + \int_{\{|x|\geq K\}} xN_t(\omega) \qquad (1.2)$$

where $W(t)$ is cylindrical wiener process as in [PR07, Sect. 2.5].
Of course this process can no longer be uniquely identified by its characteristic triplet $[\alpha, Q, M]$. If we define J to be Lévy-process with triple $[\alpha, 0, M]$ and W as above we can write $X(t) = J(t) + W(t)$. For simplicity we are going to refer to this process as a cylindrical Lévy process.
In our case we are going to restrict the class of noise further, by assuming that the jumps of our Lévy process are bounded by K, i.e. $|\Delta Y| < K$ and thus reducing our process to

$$Y(t) = W(t) + \int\limits_{\{|x|<K\}} x(N_t(\mathrm{d}x) - t\nu(\mathrm{d}x)). \tag{1.3}$$

Integration with respect cylindrical Lévy process

We are only going to define integration with respect to cylindrical Lévy process in the case where the Lévy part is a martingale, i.e. we have the case (1.3) and our jump part is

$$J(t) = \int\limits_{\{|x|<K\}} x(N_t(\mathrm{d}x) - t\nu(\mathrm{d}x)).$$

Now let $X(t) = J(t) + W(t)$. We want to define

$$\int_s^t \phi(t,x)\mathrm{d}X(t) = \int_s^t \phi(t,x)\mathrm{d}W(t) + \int_s^t \phi(t,x)\mathrm{d}J(t),$$

for a reasonable class of integrands ϕ.
A large class of integrands for the first part on the right hand side has been identified in [PR07, 2.5] as

$$\mathcal{N}_W := \left\{ \Phi : [0,T] \times \Omega \to L_2^0 | \Phi \text{ predictable and } P\left(\int_0^T \|\Phi(s) \circ \sqrt{Q}\|_{L_2}^2 \mathrm{d}s < \infty \right) = 1 \right\}.$$

For the second part the space of potential integrands is $\mathcal{N}_2(T)$ in our framework. Thus the integral makes sense as the sum of two H valued square integrable martingales for all $\phi \in \mathcal{N}_W \cap \mathcal{N}_2(T)$.

Existence of the stochastic convolution

We will now define an object which is essential in the study of mild solutions.

Definition 1.28 (Stochastic convolution)**.** We call the following integral the stochastic convolution corresponding to an H-valued Lévy process Y and a semigroup S

$$Y_{S,C}(t) := \int_0^t S(t-s)C\mathrm{d}Y(s),$$

for some linear operator C.

On notation: In cases where A is the infinitesimal generator of S and the choice of C is obvious we write $Y_A := Y_{A,C} := Y_{S,C}$.
Let us now give conditions for existence of the stochastic convolution.

Theorem 1.29 (Existence of stochastic convolution)**.** *For a Lévy-process Y, $C \in L(H)$, and S a C_0-semigroup with infinitesimal generator A the stochastic convolution $Y_{A,C} : [0,T] \to L^2(H)$ given by*

$$Y_{A,C}(t) = \int_0^t S(t-s)C\mathrm{d}X(s) \text{ for every } t \geq 0$$

exists.

Proof. See [Sto05, Thm. 4.1.7]. □

Existence of the stochastic convolution for "cylindrical lévy noise"

In the case of Y being a cylindrical Wiener process in the sense of (1.3) the stochastic convolution splits into two integrals

$$Y_{A,C} = \int_0^t S(t-s)C\mathrm{d}W(s) + \int_0^t S(t-s)Cx\tilde{N}(\mathrm{d}s, \mathrm{d}x).$$

Let us look at the parts individually. For the Wiener part we have to check

$$P\left(\int_0^t \|S(t-s)C\sqrt{Q}\|_2^2 \mathrm{d}s < \infty\right) = 1.$$

For the jump part we rewrite using Lemma 1.21 (where we have no brownian part)

$$\int_0^t \int_S S(t-s)Cx\tilde{N}(\mathrm{d}s, \mathrm{d}x) = \int_0^t \int_S S(t-s)C\mathbb{M}(\mathrm{d}s, \mathrm{d}x).$$

Thus we have to see if $S(t-s)C \in \mathcal{N}_2^T$. Utilizing the Itô isometry we compute:

$$\begin{aligned}\int\limits_s^t \int\limits_S \|S(t-s)C\sqrt{T_x}\|_2^2 M(\mathrm{d}x)\mathrm{d}s &\leq \int\limits_s^t \int\limits_S \|S(t-s)C\|^2 \|\sqrt{T_x}\|_2^2 M(\mathrm{d}x)\mathrm{d}s \\ &\leq \int\limits_s^t \|S(t-s)C\|^2 \mathrm{d}s \int\limits_S \|\sqrt{T_x}\|_2^2 M(\mathrm{d}x) \\ &\leq \int\limits_s^t \|S(t-s)C\|^2 \mathrm{d}s \int\limits_S |x|^2 M(\mathrm{d}x).\end{aligned}$$

This give us reasonable conditions to check, for the existence of the stochastic convolution.

Mild solution to SPDE with Lévy noise

We are now considering the SPDE on H

$$\begin{aligned}\mathrm{d}X(t) &= (AX(t) + F(t, X(t)))\mathrm{d}t + \mathrm{d}Y(t), \\ X(t_0, t_0, x) &= X_0,\end{aligned} \tag{SPDE}$$

for $t_0 \leq t \leq T$. With $A : H \supset D(A) \to H$ being a linear operator and $F : \mathbb{R}^+ \times H \supset D(F) \to H$ Borel measurable and Y a Lévy martingale in the sense of 1.3. Further let X_0 be square integrable $\mathcal{F}_{t_0}$-measurable random variable with values in H. As usual we denote $X(t) := X(t, t_0, x_0)$.

Definition 1.30. We call a predictable process $X : [t_0, \infty) \times \Omega \to H$ a mild solution to (SPDE) with initial condition $X(t_0) = X_0$ if

$$\sup_{t \in [t_0, T]} \mathbb{E}[|X(t)|_H^2] < \infty \text{ for all } T \in (t_0, \infty)$$

and if we have for all $t \geq t_0$

$$X(t) = S(t-t_0)X_0 + \int_{t_0}^t S(t-r)F(r, X(r,s,x))\mathrm{d}r + \int_{t_0}^t S(t-r)\mathrm{d}Y(r) \quad \mathbb{P}\text{-a.s. }.$$

Furthermore all integrals involved should be well defined.

Transition evolution operator

If we have a mild solution to a SPDE, we are often interested in the law of this solution. One way of studying the law is via transition evolution operators.

Definition 1.31 (Transition evolution operator)**.** If we have a solution $X(t,s,x)$ to (SPDE) in the sense of Definition 1.30 we will call the family of linear operators $P_{s,t}$ on $\mathcal{B}_b(H)$ (i.e. on bounded measurable function on H) given as

$$P_{s,t}\phi(x) := \mathbb{E}[\phi(X(t,s,x))] \text{ for all } \phi \in \mathcal{B}_b(H) \tag{1.4}$$

the family of transition evolution operators corresponding to the mild solution X.

In the context of the Fokker-Planck equation we will be frequently considering the following transformation for $\xi \in \mathcal{M}_1(H)$, i.e. a Borel measure of mass 1 on H,

$$\begin{aligned}
\int_H (P_{s,t}\phi)(x)\xi(\mathrm{d}x) &= \int_H \mathbb{E}[\phi(X(t,s,x))]\xi(\mathrm{d}x) \\
&= \int_H \int_H \phi(y)\ \mathbb{P} \circ X^{-1}(t,s,x)(\mathrm{d}y)\xi(\mathrm{d}x) \\
&= \int_H \int_H \phi(y)\ p_t(x,\mathrm{d}y)\xi(\mathrm{d}x) \\
&= \int_H \phi(y)\ \mu_t(\mathrm{d}y)
\end{aligned}$$

where we set $p_t(x,\mathrm{d}y) := \mathbb{P} \circ X^{-1}(t,s,x)(\mathrm{d}y)$ and $\mu_t(\mathrm{d}x) := p_t(x,\mathrm{d}y)\xi(\mathrm{d}x)$. Sometimes we will write for this transformation $\mu_t(\mathrm{d}x) := P^*_{s,t}\xi(\mathrm{d}x)$.
For the case, where the initial condition measure ξ is a dirac measure, these measures give us the evolution in time of the distribution of the solution to the SPDE. We have that

$$\int_H \phi(y)\mu_t(\mathrm{d}y) = \int_H P_{s,t}\phi(y)\xi(\mathrm{d}y) \text{ for all } \phi \in \mathcal{B}(H). \tag{1.5}$$

Chapter 2

Conditions, preparations and formulation of the main theorem

We start this chapter by introducing our basic conditions on (SPDE), then using these conditions we show an essential bound for the main proof in Lemma 2.3. This leads us to the statement of our approximation conditions, which gives rise to a family of SPDEs called (SPDE_α). We will see in the main proof that this family approximates (SPDE) in a certain sense, for which we need to assume that each (SPDE_α) admits a mild solution. We then define the space of test function on which we then illustrate our specific Fokker-Planck equation and its solution. Following this we go into detail about the choice of the Kolmogorov operator. We then establish an equivalent formulation to (FPE) and discuss the relationship between the solution of a martingale problem and a solution the Fokker-Planck equation.

Let H be a seperable real Hilbert space with $\langle \cdot\, , \cdot \rangle$ as the inner product and let $|\cdot|$ be the corresponding norm. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis. From now on we are considering the following stochastic partial differential equation

$$\begin{aligned} \mathrm{d}X(t) &= [AX(t) + F(t, X(t))]\,\mathrm{d}t + \mathrm{d}Y(t), \qquad \text{(SPDE)} \\ X(s) &= x \in H \text{ with } t \geq s. \end{aligned}$$

Condition 2.1 (General conditions)**.** We have the following conditions on $A : D(A) \subset H \to H$:

a) A is self adjoint.

b) We have an orthonormal basis $(\xi_i)_{i\in\mathbb{N}}$ of H with ξ_i being eigenvectors of A.

c) We have that $\langle Ax, x\rangle < 0$ for every $x \in D(A)$.

d) A is the infinitesimal generator of a C_0-semigroup e^{tA} in H for $t \geq 0$.

e) We have for a $\delta \in]0, \frac{1}{2}[$ that $(-A)^{2\delta-1}$ is a trace class operator.

We assume that $F : D(F) \subset [0,T] \times H \to H$ is a Borel measurable map.

By $\mathrm{d}Y(t) = \sqrt{C}\mathrm{d}W(t) + \mathrm{d}J(t)$ we denote a cylindrical Lévy process in the sense of (1.2) with W being a cylindrical Wiener process and $C : H \to H$ a bounded symmetric positive operator.

Condition 2.2 (Conditions on the jump part)**.** $J(t)$ is a Lévy process on H with characteristic triplet $[0, 0, M]$, where M is Lévy measure with the following properties:

a) $J(t)$ can for some $0 < K < \infty$ be written as

$$J(t) = \int\limits_{\{|x|<K\}} x(N_t(\mathrm{d}x) - tM(\mathrm{d}x))$$

where N_t is the Poisson measure and M the Lévy measure related to Y, i.e. $\tilde{N}(t, \mathrm{d}x) := N_t(\mathrm{d}x) - tM(\mathrm{d}x)$ is a compensated Poisson measure.

b) We have

$$\sup_{\{x \in H\}} \int\limits_H (f_m(P_m(x+y)) - f_m(P_m x)) - \frac{{}_{H^*}\langle (D_x f_m \circ P_m)(x), y \rangle_H}{1 + |y|^2} M(\mathrm{d}y) < \infty$$

for all $f_m \in \mathcal{S}(\mathbb{R}^m, \mathbb{R})$ and $P_m : x \mapsto (\langle x, \xi_{k_1} \rangle, \ldots, \langle x, \xi_{k_m} \rangle)^T$, where $(k_i)_{1 \leq i \leq m}$ is a sequence in $\mathbb{N}$ without duplicates.

c) We have finite second moments of M, i.e.

$$\int\limits_H |x|^2 \, M(\mathrm{d}x) < \infty.$$

Under this conditions we have for the stochastic convolution the following lemma, which will be central in the proof of the main theorem (see Step 2, Claim 3.8)

Lemma 2.3. *Under the above conditions we have*

$$\sup_{t \in [0,T]} \mathbb{E}[\left|(-A)^\delta Y_A(t)\right|^2] \leq c_\delta < \infty.$$

We will proof this lemma by splitting the stochastic convolution in a Wiener part and a pure jump part. We will show for each convolution, that $(-A)^\delta$ can be taken into the integral. This will be more involved for the jump part, since we have to deal with a double integral and we have no reference theorem that allows us to take $(-A)^\delta$ in the convolution integral. Once we have taken $(-A)^\delta$ in the convolution intgeral we compute finiteness explicitly for each part.

Proof. Recall that we can write $\mathrm{d}Y(t)$ as

$$\mathrm{d}Y(t) = \sqrt{C}\mathrm{d}W(t) + \int\limits_{\{|x|<K\}} x\tilde{N}(\mathrm{d}t, \mathrm{d}x),$$

where we set $\tilde{N}(t, \mathrm{d}x) = N_t(\mathrm{d}x) - t \cdot M(\mathrm{d}x)$.
Thus our stochastic convolution becomes

$$Y_A(t) = \underbrace{\int\limits_s^t S(t-s)\sqrt{C}\mathrm{d}W(s)}_{:=W_A(t)} + \underbrace{\int\limits_s^t \int\limits_{\{|x|<K\}} S(t-s)x\tilde{N}(\mathrm{d}s, \mathrm{d}x)}_{:=J_A(t)}.$$

Now we consider our equation and apply the elementary inequality $(a+b)^2 \leq 2(a^2+b^2)$ for $a, b \in \mathbb{R}$:

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left|(-A)^\delta Y_A(t)\right|^2\right] \leq 2(\sup_{t\in[0,T]} \mathbb{E}\left[\left|(-A)^\delta W_A(t)\right|^2\right] + \sup_{t\in[0,T]} \mathbb{E}\left[\left|(-A)^\delta J_A(t)\right|^2\right]).$$

We can thus consider the parts separately. We start with $W_A(t)$:

$$\begin{aligned}\sup_{t\in[0,T]} \mathbb{E}\left[\left|(-A)^\delta W_A(t)\right|^2\right] &= \sup_{t\in[0,T]} \mathbb{E}\left[\left|\int\limits_0^t (-A)^\delta S(t-s)\sqrt{C}\mathrm{d}W(s)\right|^2\right] \\ &= \mathbb{E}\left[\int\limits_0^T \|(-A)^\delta S(T-s)\sqrt{C} \circ \sqrt{Q}\|_2 \mathrm{d}s\right] < \infty,\end{aligned}$$

where we first used [DPZ92, 4.15] to take $(-A)^\delta$ into the integral and then Itô-isometry(compare [PR07, Prop. 2.3.5]). To apply [DPZ92, 4.15] we have computed

$$
\begin{aligned}
\int_0^T \|(-A)^\delta S(T-s)\sqrt{C}\circ\sqrt{Q}\|_{L^2}^2 \mathrm{d}s &\le \int_0^T \|(-A)^\delta S(T-s)\|_{L^2}^2 \overbrace{\|\sqrt{C}\circ\sqrt{Q}\|^2}^{:=\eta} \mathrm{d}s \\
&= \eta \int_0^T \sum_{i=1}^\infty \left|(-A)^\delta S(T-s)\xi_i\right|^2 \mathrm{d}s \\
&= \eta \int_0^T \sum_{i=1}^\infty \left|(-A)^\delta e^{(T-s)A}\xi_i\right|^2 \mathrm{d}s \\
&= \eta \int_0^T \sum_{i=1}^\infty \left|(-\lambda_i)^\delta e^{(T-s)\lambda_i}\xi_i\right|^2 \mathrm{d}s \\
&= \eta \int_0^T \sum_{i=1}^\infty \left|(-\lambda_i)^{2\delta} e^{2(T-s)\lambda_i}\right|_{\mathbb{R}} |\xi_i|^2 \mathrm{d}s \\
&= \eta \sum_{i=1}^\infty |\lambda_i|_{\mathbb{R}}^{2\delta} \int_0^T \left|e^{2(T-s)\lambda_i}\right|_{\mathbb{R}} \mathrm{d}s \\
&= \eta \sum_{i=1}^\infty |\lambda_i|_{\mathbb{R}}^{2\delta} \left(\frac{1}{-2\lambda_i} e^{2(T-s)\lambda_i}\right)\Bigg|_0^T \\
&= \eta \sum_{i=1}^\infty |\lambda_i|_{\mathbb{R}}^{2\delta} \left|\frac{1}{2\lambda_i}\right| \left|(1-e^{2T\lambda_i})\right| \\
&= \eta \frac{1}{2} \sum_{i=1}^\infty |\lambda_i|_{\mathbb{R}}^{2\delta-1} \left(1-(e^{2T\lambda_i})\right) \\
&\le \eta \frac{1}{2} \sup_{i\in\mathbb{N}} (1-e^{2T\lambda_i}) \sum_{i=1}^\infty \left|\lambda_i^{2\delta-1}\right|_{\mathbb{R}} < \infty,
\end{aligned}
\tag{2.1}
$$

where we call λ_i the eigenvalues of A corresponding to ξ_i, which are negative, since $-A$ is positive. Here we have used the inequality $\|\cdot\|_2 \le \|\cdot\| \cdot \|\cdot\|_2$ and the spectral theorem.

Remark 2.4. In the second to last step we can see, that we at this point actually need a slightly weaker condition than that $(-A)^{2\delta-1}$ should be trace-class.
At this point we precisely need $\sum_{i=1}^\infty |\lambda_i|_{\mathbb{R}}^{2\delta-1}\left(1-e^{2T\lambda_i}\right) < \infty$.

Now we proceed with the jump part:

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left|(-A)^\delta J_A(t)\right|^2\right]) = \sup_{t\in[0,T]} \mathbb{E}\left[\left|(-A)^\delta \int\limits_s^t \int\limits_{\{|x|<K\}} S(t-s)x\tilde{N}(\mathrm{d}s,\mathrm{d}x)\right|^2\right]).$$

We want to apply Lemma 1.27 to take $(-A)^\delta$ under the integral sign. Therefore we need to check, if

$$\int\limits_0^T \int\limits_{\{|x|<K\}} (-A)^\delta R_n(s,x)\tilde{N}(\mathrm{d}s,\mathrm{d}x) \overset{n\to\infty}{\to} \int\limits_0^T \int\limits_{\{|x|<K\}} (-A)^\delta S(T-s)\tilde{N}(\mathrm{d}s,\mathrm{d}x) \quad (2.2)$$

in $L^2((\Omega,\mathcal{F},\mathbb{P})\to H)$ for a sequence R_n of our choice with $R_n\in\mathcal{S}_2$ and $R_n(s)\to S(T-s)$ in $\mathcal{N}_2(T)$.
The existence of the right hand side and the limit property will be proven later. Let us first check the existence of the left hand side by using the Itô isometry. We choose

$$R_n(s)x := \sum_{t_i,t_{i+1}\in\tau_n} \sum_{j=0}^{n} e^{(T-t_{i+1})\lambda_j} \langle \xi_j, x\rangle \mathbb{I}_{]t_i,t_{i+1}]}(s)\xi_j$$

for some partition τ_n with fineness approaching 0 as $n\to\infty$. Thus we calculate:

$$\mathbb{E}\left[\int\limits_0^T \int\limits_{\{|x|<K\}} (-A)^\delta R_n(s)x\tilde{N}(\mathrm{d}s,\mathrm{d}x)\right] = \mathbb{E}\left[\int\limits_0^T \int\limits_{\{|x|<K\}} \|(-A)^\delta R_n(s)\sqrt{T_x}\|_2^2 M(\mathrm{d}x)\mathrm{d}s\right]$$

and for the right hand side

$$\int_0^T \int_{\{|x|<K\}} \left\|(-A)^\delta R_n(s)\sqrt{T_x}\right\|_2^2 M(\mathrm{d}x)\mathrm{d}s \leq \int_0^T \left\|(-A)^\delta R_n(s)\right\|_2^2 \mathrm{d}s \overbrace{\int_{\{|x|<K\}} \left\|\sqrt{T_x}\right\|_2^2 M(\mathrm{d}x)}^{:=\tilde{M}}$$

$$\leq \int_0^T \Bigg\|(-A)^\delta \Big(\sum_{t_i,t_{i+1}\in\tau_n} \sum_{j=0}^n e^{(T-t_{i+1})\lambda_j} \cdot \langle \xi_j, x\rangle \mathbb{I}_{]t_i,t_{i+1}]}(s)\xi_j\Big)\Bigg\|_2^2 \mathrm{d}s \cdot \tilde{M}$$

$$\leq \int_0^T \sum_{k=1}^n \left|(-\lambda_k)^\delta \Big(\sum_{t_i\in\tau_n} e^{(T-t_{i+1})\lambda_k}\mathbb{I}_{]t_i,t_{i+1}]}(s)\Big)\right|^2 \mathrm{d}s \cdot \tilde{M}$$

$$\leq \int_0^T \sum_{k=1}^\infty \left|(-\lambda_k)^\delta e^{(T-s)\lambda_k}\right|^2 \mathrm{d}s \cdot \tilde{M},$$

where we proceed as in (2.1). The finiteness of $\tilde{M}$ is shown later in the proof (see (2.3) below).
Next we need to show the existence of

$$\mathbb{E}\left[\left|\int_s^T \int_S (-A)^\delta S(T-s)x\tilde{N}(\mathrm{d}s,\mathrm{d}x)\right|^2\right]$$
$$=\mathbb{E}\left[\int_0^T \int_S \mathbb{I}_{]0,t]}(s)\|(-A)^\delta S(T-s)\sqrt{(}T_x)\|_2^2 M(\mathrm{d}x)\mathrm{d}s\right],$$

where we again used the Itô isometry. Recall that $T_x = (x,\cdot)x$. We can compute

$$\mathbb{E}[\int_0^T \int_S \mathbb{I}_{]0,t]}(s)\|(-A)^\delta S(T-s)\sqrt{(}T_x)\|_2^2 M(\mathrm{d}x)\mathrm{d}t] \leq \int_S \|\sqrt{(}T_x)\|_2^2 M(\mathrm{d}x) \cdot \int_0^T \mathbb{I}_{]0,t]}(s)\|(-A)^\delta S(T-s)\|_2^2 \mathrm{d}s.$$

We relax this to

$$\int_S \|\sqrt(T_x)\|_2^2 M(\mathrm{d}x) \cdot \int_0^T \mathbb{I}_{]0,t]}(s) \|(-A)^\delta S(T-s)\|_2^2 \mathrm{d}s \tag{2.3}$$

$$\leq \int_S |x|^2 \, M(\mathrm{d}x) \cdot \int_0^T \mathbb{I}_{]0,t]}(s) \|(-A)^\delta S(T-s)\|_2^2 \mathrm{d}s < \infty, \tag{2.4}$$

since $\sqrt{T_x}$ is self adjoint and we have

$$\|\sqrt{T_x}\|_2^2 = \mathrm{Tr}(\sqrt{T_x}\sqrt{T_x}^*) = \mathrm{Tr}(T_x) = \sum_{i=1}^\infty \langle\langle x, e_k\rangle x, e_k\rangle = |x|^2 .$$

For the second multiplicand we use again (2.1). Now we show that the integrals converge. Using Itô isometry and linearity we have to consider:

$$\begin{aligned}
&\int_0^T \int_S \|(-A)^\delta (R_n(s) - S(T-s))\sqrt{T_x}\|_2^2 M(\mathrm{d}x)\mathrm{d}s \\
&\leq \int_0^T \|(-A)^\delta (R_n(s) - S(T-s))\|_2^2 \mathrm{d}s \int_S \|\sqrt{T_x}\|_2^2 M(\mathrm{d}x) \\
&\leq \int_0^T \|(-A)^\delta (\sum_{t_i, t_{i+1} \in \tau_l} \sum_{j=0}^n e^{(T-t_{i+1})\lambda_j} \langle \xi_j, \cdot \rangle \mathbb{I}_{]t_i, t_{i+1}]}(s)\xi_j - S(T-s))\|_2^2 \mathrm{d}s \cdot \tilde{M} \\
&\leq \int_0^T \sum_{k=1}^\infty \left| (-\lambda_k)^\delta (\sum_{t_i, t_{i+1} \in \tau_l} \sum_{j=0}^n e^{(T-t_{i+1})\lambda_j} \langle \xi_j, \xi_k \rangle \ \mathbb{I}_{]t_i, t_{i+1}]}(s) - e^{(T-s)\lambda_k}) \right|^2 \mathrm{d}s \cdot \tilde{M} \\
&\leq \sum_{k=1}^n \int_0^T \left| (-\lambda_k)^\delta (\sum_{t_i, t_{i+1} \in \tau_l} e^{(T-t_{i+1})\lambda_k} \mathbb{I}_{]t_i, t_{i+1}]}(s) - e^{(T-s)\lambda_k}) \mathrm{d}s \right|^2 \cdot \tilde{M} \\
&\quad + \sum_{k>n}^\infty \int_0^T \left| (-\lambda_k)^\delta e^{(T-s)\lambda_k} \right|^2 \mathrm{d}s \cdot \tilde{M},
\end{aligned}$$

where we have that

$$\sum_{k=1}^n \int_0^T \left| (-\lambda_k)^\delta (\sum_{t_i, t_{i+1} \in \tau_l} e^{(T-t_{i+1})\lambda_k} \mathbb{I}_{]t_i, t_{i+1}]}(s) - e^{(T-s)\lambda_k}) \mathrm{d}s \right|^2 \overset{n \to \infty}{\to} 0,$$

since the Riemann sum of a uniformly continuous function converges uniformly to the function. And

$$\sum_{k>n}^{\infty} \int_0^T \left|(-\lambda_k)^{\delta} e^{(T-s)\lambda_k}\right|^2 \mathrm{d}s \overset{n\to\infty}{\to} 0,$$

because it is by (2.1) a convergent sum and thus the tail sum converges to 0. So by letting first $l \to \infty$ and then $n \to \infty$ we get our desired convergence.

Thus in this case we have:

$$\sup_{t\leq T} \mathbb{E}[\left|(-A)^{\delta} Y_A(t)\right|^2] \leq 2(\sup_{t\leq T} \mathbb{E}[\left|(-A)^{\delta} W_A(t)\right|^2 + \sup_{t\leq T} \left|(-A^{\delta}) J_A(t)\right|^2]) < \infty.$$

□

Further we will need the following conditions for our result, which are mainly necessary for the approximation.

Approximation conditions

For our proof it is essential that we have a mild solution to a version of our SPDE, which satisfies stronger conditions on the non-linear drift part F. We will now consider SPDE with F_α instead of F, which is not only measurable, but bounded and has further properties useful for approximation. We will refer to this SPDE as (SPDE$_\alpha$). The precise conditions on F and F_α are:

Condition 2.5. There exist bounded measurable maps $F_\alpha : [0,T] \times H \to H, \alpha \in (0,1]$ such that for all $(t,x) \in D(F)$ and all $h \in D(A)$ the following conditions are fulfilled:

a) $\lim_{\alpha\to 0}\langle h, F_\alpha(t,x)\rangle = \langle h, F(t,x)\rangle$.

b) $|F_\alpha(t,x)| \leq |F(t,x)|$.

c) $|\langle h, F(t,x) - F_\alpha(t,x)\rangle| \leq \alpha c(h) \left|F(t,x)\right|^2$ for some $c(h) > 0$.

Condition 2.6. There exists a $K \geq 0$ and a lower semi-continuous function $V : [s,t] \times H \to [1,\infty]$ such that $|F| \leq V$ on $[s,T] \times H$, where we set $|F| = \infty$ on $([s,T] \times H) \setminus D(F)$ and we have

$$P^{\alpha}_{s,t} V^2(t,\,\cdot\,)(x) \leq K V^2(t,x) < \infty \text{ for all } (t,x) \in D(F), t \in [s,T], \alpha \in]0,1]$$

For this F_α we assume the following existence result.

Assumption 2.7. (SPDE$_\alpha$) admits a cádlág mild solution.

Space of test functions

As in [Wie11], we use a time dependent version of test function space $\mathcal{W}_0$ from [LR02]. We refer to [Wie11] for a more explaination of the properties of the test function space.
Let $\mathcal{S}(\mathbb{R}^m, \mathbb{R})$ be the space of m-dimensional real Schwartz functions and let $(\xi_i)_{i\in\mathbb{N}}$ be a orthonormal basis of H consisting of eigenvectors of A (as in Condition 2.1). We define $W_{T,A}$ as the linear span of all functions $\psi : [0,T] \times H \to \mathbb{R}$, for which we can find an $m \in \mathbb{R}$ such that

$$\psi(t,x) = \phi(t) \cdot f_m(\langle \xi_1, x\rangle, \ldots, \langle \xi_m, x\rangle) \text{ for all } (t,x) \in [0,T] \times H,$$

where $\phi \in \mathcal{C}^2([0,T])$, i.e. a twice differentiable $\mathbb{R}$-valued function on $[0,T]$, with $\phi(T) = 0$ and $f_m \in \mathcal{S}(\mathbb{R}^m, \mathbb{R})$. Further we will use W_A which is given by the linear span of functions which can be written as

$$\psi(x) = f_m(\langle \xi_1, x\rangle, \ldots, \langle \xi_m, x\rangle) \text{ for all } x \in H.$$

From now on let $P_m : H \to H$ be the to f_m corresponding projection $P_m x := \langle \xi_1, x\rangle \xi_1 + \ldots + \langle \xi_m, x\rangle \xi_m$. We identify $P_m H$ with $\mathbb{R}^m$ by $\tilde{P}_m : H \to \mathbb{R}^m$ where $\tilde{P}_m := (\langle \xi_1, x\rangle, \ldots, \langle \xi_m, x\rangle)^T$. Sometimes we will write $\tilde{x} := \tilde{P}_m x$.
The inverse Fourier transform of ψ, which is denoted by $g_m : \mathbb{R}^m \to \mathbb{C}$, is uniquely determined by

$$f_m(y) = \int_{\mathbb{R}^m} e^{i\langle r,y\rangle_{\mathbb{R}^m}} g_m(r) \mathrm{d}r$$

and we have $g_m \in \mathcal{S}(\mathbb{R}^m, \mathbb{C})$.
Now set $\nu_m(\mathrm{d}r) := g_m(r)\mathrm{d}r$ and define

$$\nu_t := \phi(t)\nu_m \cdot \Pi_m^{-1}$$

where $\Pi_m := \mathbb{R}^m \ni r \mapsto \sum_{j=1}^m r_j \xi_j \in H$ for some subset $\xi_1, \ldots, \xi_m$ of $(\xi_i)_{i\in\mathbb{N}}$.

Remark 2.8. Note, that for each $\psi \in W_{T,A}$ we can identify a finite dimensional subspace H_m of H, where ψ exclusively takes non zero values. As essentially finite dimensional functions, these functions are weakly continuous (i.e. continuous with respect to the weak topology).

2.1 Fokker-Planck equation

As we have seen it is possible to consider laws of (mild) solutions to (SPDE). This gives rise to the question if it is possible to find an (integral-) equation, which varies on measures, to study these laws of solutions corresponding to (SPDE). Essential for this approach is finding a concrete version of the infinitesimal generator corresponding to the transition evolution operator of our SPDE. This concrete operator is usually defined on a much smaller test function space, which coincides with the abstract operator on this smaller test function space. Let us assume that this smaller test-function space is the earlier introduced $W_{T,A}$ and let us call the infinitesimal generator in its abstract form L and the concrete one L_0. It is also possible to consider the Fokker-Planck equation without direct connection to a SPDE, which is what we do in our case.

Choice of Kolmogorov operator and formulation of FPE

In this section we give a short overview of the results, which led to the choice of the Kolmogorov operator in this setting.
In [LR04] the equation

$$\mathrm{d}X_t = (AX_t + b(X_t))\mathrm{d}t + \mathrm{d}Y_t \tag{2.5}$$

with $(Y_t)_{t\geq 0}$ being a Lévy process on a Hilbert space H, A generating a C_0-semigroup on H and b being an uncontinuous drift has been considered (for precise conditions see [LR04]).
In this case a concrete version of the infinitesimal generator corresponding to the (2.5) was identified (see [LR04, Prop. 3.5]) as

$$\begin{aligned} L_0 u(x) = \langle A^*(u'(x)), x\rangle + \langle u'(x), \alpha\rangle + \frac{1}{2} - \int\limits_{H'} \langle \xi, R\xi\rangle e^{i\langle \xi, x\rangle} \nu(\mathrm{d}\xi) \\ + \int\limits_{H} \left(u(x+y) - u(x) - \frac{\langle u'(x), y\rangle}{1+|y|^2}\right) M(\mathrm{d}y) \end{aligned}$$

for all $x \in H$ and $u \in W_A$, the time independent version of our test function space $W_{T,A}$. Here the authors set for simplicity $\langle\,\cdot\,,\,\cdot\,\rangle := {}_{H^*}\langle\,\cdot\,,\,\cdot\,\rangle_H$.
More similar to our case, in [Wie11, Rem. 3.2.2] the Kolmogorov operator for (SPDE) has been given for the case that F_L is Lipschitz in t. The concrete operator L_L takes for all $\psi \in W_{T,A}$ the form of

$$\begin{aligned} (L_L\psi)(t,x) &= V_0\psi(t,x) + \langle D_x\psi(t,x), F_L\rangle \\ &= D_t\psi(t,x) + U\psi(t,x) + \langle D_x\psi(t,x), F\rangle, \end{aligned} \tag{2.6}$$

where D_x is the Fréchet derivative, D_t the partial derivative in t and with the Ornstein-Uhlenbeck part being

$$
\begin{aligned}
U\psi(t,\cdot)(x) = \langle AD_x\psi(t,x), x\rangle + \langle D_x\psi(t,x), b\rangle - \frac{1}{2}\int_H \langle \xi Q, \xi\rangle e^{i\langle \xi, x\rangle}\nu_t \\
+ \int_H \psi(t,x+y) - \psi(t,x) - \frac{\langle D_x\psi(t,x), y\rangle}{1+|y|^2} M(\mathrm{d}y).
\end{aligned}
\tag{2.7}
$$

Since in our case we have a very similar situation, we are going to consider the Kolmogorov operator L_L for merely measurable F, i.e.

$$
L_0 := V_0\psi(t,x) + \langle D\psi(t,x), F\rangle.
$$

This was also done in [Wie11, 5.2], but without showing the relation to (SPDE). Further it has there been shown that the Fokker-Planck equation corresponding to L_0 has at most one solution for a reasonable initial condition.
Thus the Fokker-Planck equation of interests is

$$
\begin{aligned}
\int_H \psi(t,x)\mu_t(\mathrm{d}x) = \int_H \psi(s,x)\xi(\mathrm{d}x) \\
+ \int_s^t \int_H L_0\psi(r,x)\mu_r(\mathrm{d}x)\mathrm{d}r \text{ for almost all } t \in [s,T]
\end{aligned}
\tag{FPE}
$$

Remark 2.9. Let us stress again that we do not claim, that L_0 is directly related to (SPDE), for example in the sense that it is the generator of a semigroup of transition operators corresponding to a mild solution of (SPDE).

Definition 2.10 (Solution to Fokker-Planck equation)**.** We call a probability kernel $\mu_t(\mathrm{d}x)\mathrm{d}t$ with $t \in [s,T]$ solution to the Fokker-Planck equation with initial condition $\xi \in \mathcal{M}_1(H)$ iff for all $\psi \in W_{T,A}$

$$
\int_H \psi(t,x)\mu_t(\mathrm{d}x) = \int_H \psi(s,x)\xi(\mathrm{d}x) + \int_s^t \int_H L_0\psi(r,x)\mu_r(\mathrm{d}x)\mathrm{d}r \text{ for almost all } t \in [s,T]
\tag{FPE}
$$

and all integrals above exist.

Remark 2.11. Intuitively the Fokker-Planck equation might seem reminiscent of Theorem 1.5.iv), which can be thought of as the case where the transition evolution operators form C_0-semigroup.

$$P_{s,t}x = P_{s,s}x + \int\limits_s^t P_{s,s'}Ax\mathrm{d}s'$$

$$\int\limits_H P_{s,t}xd\xi = \int\limits_H P_{s,s}xd\xi + \int\limits_H \int\limits_s^t P_{s,s'}Ax\mathrm{d}s'd\xi$$

$$\int\limits_H x(P_{s,t})^* d\xi = \int\limits_H \mathrm{Id}\, xd\xi + \int\limits_s^t \int\limits_H Ax(P_{s,s'})^* d\xi\mathrm{d}s'$$

$$\int\limits_H x\mu_t(\mathrm{d}x) = \int\limits_H xd\xi + \int\limits_s^t \int\limits_H Ax\mu_{s'}(\mathrm{d}x)\mathrm{d}s'$$

Equivalent formulations of the Fokker-Planck equation

There are several different formulations for Fokker-Planck equations in different settings (for an overview in the finite-dimensional case see [BKRS15]). We want to present one formulation which is equivalent to (FPE), which will be essential for the main proof. The essence of this formulation is, that it is possible to write the FPE with a single integration of the solution measure.

Lemma 2.12. *If we consider (FPE) which holds for almost all $t \in [s,T]$ on $W_{T,A}$ (where $\psi \in W_{T,A}$ implies $\psi(T,\,\cdot\,) = 0$), then (FPE) is equivalent to*

$$\int\limits_s^T \int\limits_H L_0\psi(s',y)\mu_{s'}(\mathrm{d}y)\mathrm{d}s' = -\int\limits_H \psi(s,y)\xi(\mathrm{d}y) \tag{FPE2}$$

for all $\psi \in W_{T,A}$.

Proof. (FPE)$\Rightarrow$ (FPE2):

As we know $\psi \in W_{T,A}$ implies $\psi(T,\,\cdot\,) = 0$ and therefore if (FPE) holds for T

$$\begin{aligned}\int_H \psi(T,x)\mu_T(\mathrm{d}x) &= \int_H \psi(s,x)\xi(\mathrm{d}x) + \int_s^T \int_H L_0\psi(r,x)\mu_r(\mathrm{d}x)\mathrm{d}r \\ \Rightarrow 0 &= \int_H \psi(s,x)\xi(\mathrm{d}x) + \int_s^T \int_H L_0\psi(r,x)\mu_r(\mathrm{d}x)\mathrm{d}r \\ \Rightarrow -\int_H \psi(s,x)\xi(\mathrm{d}x) &= \int_s^T \int_H L_0\psi(r,x)\mu_r(\mathrm{d}x)\mathrm{d}r.\end{aligned}$$

If (FPE) does not hold for T we can find a sequence $t_n \to T$, for which (FPE) holds for every t_n and we have

$$\lim_{n\to\infty} \int_H \psi(t_n,x)\mu_{t_n}(\mathrm{d}x) = 0.$$

Now we can proceed as above.

(FPE2)$\Rightarrow$ (FPE):
Let $L_0 = \frac{\partial}{\partial t} + L_0'$, $\chi \in C_0^\infty(s,T)$, then $(\chi\cdot\psi) \in W_{T,A}$ and we have

$$\begin{aligned}&\overbrace{-\int_H \chi(s)\psi(s,x)\xi(\mathrm{d}x)}^{=0} = \int_s^T \int_H L_0(\chi\cdot\psi)(r,x)\mu_r(\mathrm{d}x)\mathrm{d}r \\ &\Rightarrow 0 = \int_s^T \int_H \frac{\partial}{\partial r}(\chi(r)\psi(r,x))\mu_r(\mathrm{d}x)\mathrm{d}r + \int_s^T \chi(r) \int_H (L_0'\psi)(r,x)\mu_r(\mathrm{d}x)\mathrm{d}r \\ &\Rightarrow 0 = \int_s^T \int_H (\frac{\partial}{\partial r}\chi)(r)\psi(r,x) + \chi(r)(\frac{\partial}{\partial r}\psi)(r,x)\mu_r(\mathrm{d}x)\mathrm{d}r + \int_s^T \chi(r) \int_H L_0'\psi(r,x)\mu_r(\mathrm{d}x)\mathrm{d}r \\ &\Rightarrow 0 = \int_s^T \frac{\partial}{\partial r}\chi(r) \int_H \psi(r,x)\mu_r(\mathrm{d}x)\mathrm{d}r + \int_s^T \chi(r) \int_H L_0\psi(r,x)\mu_r(\mathrm{d}x)\mathrm{d}r\end{aligned}$$

$$\Rightarrow 0 = \underbrace{[\chi(r) \int_H \psi(r,x)\mu_r(\mathrm{d}x)]\big|_s^T}_{=0} - \int_s^T \chi(r) \frac{\partial}{\partial r} \int_H \psi(r,x)\mu_r(\mathrm{d}x)\mathrm{d}r$$

$$+ \int_s^T \chi(r) \int_H L_0\psi(r,x)\mu_r(\mathrm{d}x)\mathrm{d}r,$$

where we used the product rule, integration by parts and the definition of L_0. Letting $\chi(r) \to \mathbb{I}_{[s,t]}(r)$, using that $\int_H \psi(r,x)\mu_r(\mathrm{d}x)$ is weak differentiable (see the second to last step) and that the integral is absolutely continuous in r, as a one dimensional weak differentiable function([Alt12, U1.6, P. 71]), we can apply the fundamental theorem of calculus for almost all $s', t \in [s,T]$

$$\Rightarrow 0 = -\int_{s'}^t \frac{\partial}{\partial r} \int_H \psi(r,x)\mu_r(\mathrm{d}x)\mathrm{d}r + \int_{s'}^t \int_H L_0\psi(r,x)\mu_r(\mathrm{d}x)\mathrm{d}r$$

$$\Rightarrow 0 = -\int_H \psi(t,x)\mu_t(\mathrm{d}x) + \int_H \psi(s,x)\mu_{s'}(\mathrm{d}x) + \int_{s'}^t \int_H L_0\psi(r,x)\mu_r(\mathrm{d}x)\mathrm{d}r$$

where the last equation holds for almost all t. □

Solution to the martingale problem

Recall that for the main proof we consider L_0 with F_α instead of F. We are going to refer to these Kolmogorov operators as

$$L_\alpha := V_0\psi(t,x) + \langle D\psi(t,x), F_\alpha \rangle$$

or longer

$$L_\alpha\psi(t,x) = D_t\psi(t,x) + \langle D\psi(t,x), F_\alpha(t,x)\rangle + U\psi(t,x) \quad \text{for all } \psi \in W_{T,A}$$

and set $X_t := X_\alpha(t,s,x)$.

Definition 2.13 (Solution to martingale problem)**.** We call a cádlág, progressively measurable stochastic process $(X_t)_{0\leq t\leq T}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in H, a solution to the martingale problem for L_α iff

$$\psi(X_t) - \psi(X_s) - \int_s^t (L_\alpha\psi)(s', X_{s'})\mathrm{d}s' \text{ is an } \mathcal{F}_t\text{-martingale, for all } \psi \in W_A.$$

Martingale problems are closely related to martingale solutions and Fokker-Planck equations. For the latter we proof a short lemma.

Lemma 2.14. *If X_α is a solution to the martingale problem for L_α then the measures given by $\mu_t^\alpha := (P_{s,t}^\alpha)^*\xi(\mathrm{d}x)$ solve the Fokker-Planck equation:*

$$\int_H \psi(x)\mu_t^\alpha(\mathrm{d}x) = \int_H \psi(x)\xi(\mathrm{d}x) + \int_s^t \mathrm{d}s' \int_H L_\alpha\psi(x)\mu_{s'}^\alpha(\mathrm{d}x) \text{ for all } t \in [s,T]$$

with initial condition $\xi \in \mathcal{M}_1(H)$ and $\psi \in W_A$.

For a finite dimensional version of this statement see [BRS11, Ex. 1.6].

Proof. Set $X_\alpha(t) := X_\alpha(t,s,x)$ and recall $P_{s,t}^\alpha(\psi)(x) = \mathbb{E}[\psi(X_\alpha(t,s,x))]$. By assumption we have that for all $\psi \in W_A$ and for all $t \in [s,T]$

$$\psi(X_\alpha(t)) - \psi(X_\alpha(s)) - \int_s^t (L_\alpha\psi)(X_\alpha(s'))\mathrm{d}s'$$

is a martingale. We compute

$$\begin{aligned}
& \mathbb{E}\left[\psi(X_\alpha(t)) - \psi(X_\alpha(0)) - \int_s^t (L_\alpha\psi)(X_\alpha(s'))\mathrm{d}s'\right] = 0 \\
\Rightarrow \quad & \mathbb{E}[\psi(X_\alpha(t))] = \psi(x) + \mathbb{E}\left[\int_s^t (L_0\psi)(s', X_\alpha(s'))\mathrm{d}s'\right] \\
\Rightarrow \quad & P_{s,t}^\alpha(\psi)(x) = \psi(x) + \int_s^t \mathbb{E}\left[(L_0\psi)(X_\alpha(s'))\right]\mathrm{d}s' \\
\Rightarrow \quad & P_{s,t}^\alpha(\psi)(x) = \psi(x) + \int_s^t P_{s,s'}^\alpha(L_0\psi)(x)\mathrm{d}s' \\
\Rightarrow \quad & \int_H P_{s,t}^\alpha(\psi)(x)\mathrm{d}\xi = \int_H \psi(x)\mathrm{d}\xi + \int_s^t \int_H P_{s,s'}^\alpha(L_0\psi)(x)\xi(\mathrm{d}x)\mathrm{d}s' \\
\Rightarrow \quad & \int_H \psi(x)\mu_t^\alpha(\mathrm{d}x) = \int_H \psi(x)\mathrm{d}\xi + \int_s^t \int_H (L_0\psi)(x)\mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s'.
\end{aligned}$$

□

This Lemma, by itself, shows only that a solution to the martingale problem for L_α provides us with a solution to a version of (FPE) for functions in the time independent test function space W_A. This already implies that the solution also solves (FPE) for functions in the time dependent test function space $W_{T,A}$, which is proven for finite-dimensions in [BKRS15, Prop. 6.1.2] and can be shown with a similar proof on Hilbert spaces.
Let us now introduce our last assumption.

Assumption 2.15 (Kolmogorov operator of approximation)**.** The unique cádlág mild solution to (SPDE) is a solution of martingale problem for L_α.

By Lemma 2.14 these assumptions give us measures $\mu_t^\alpha := (P_{s,t}^\alpha)^*\xi$ which solve

$$\int_H \psi(t,x)\mu_t^\alpha(\mathrm{d}x) = \int_H \psi(s,x)\xi(\mathrm{d}x) + \int_s^t \int_H L_\alpha\psi(r,x)\mu_r^\alpha(\mathrm{d}x)\mathrm{d}r \text{ for almost all } t \in [s,T]. \tag{FPE$_\alpha$}$$

Thus we see that L_α is directly related to [SPDE$_\alpha$].

Main theorem

Since we have formulated all our conditions and assumptions we can state the main theorem.

Theorem 2.16. *(Main Theorem) Assuming Conditions 2.1, 2.2, 2.5, 2.6 hold and we further have Assumption 2.7 and 2.15. If we have continuity of*

$$(t,x) \to \langle h, F^\alpha(t,x)\rangle \qquad \forall h \in D(A), \alpha \in (0,1) \tag{2.8}$$

on $[s,T] \times H$ and an initial condition $\xi \in \mathcal{M}_1(H)$ satisfying

$$\int_s^T \int_H (V^2(s',x) + |x|^2)\xi(\mathrm{d}x)\mathrm{d}s' < \infty, \tag{2.9}$$

then there exists a solution $\mu_t(\mathrm{d}x)\mathrm{d}t$ to (FPE)*. It satisfies the following properties:*

i) $\sup_{t\in[s,T]} \int_H |x|^2 \mu_t(\mathrm{d}x) < \infty$,

ii) $t \to \int_H \psi(t,x)\mu_t(\mathrm{d}x)$ *is continuous on $[s,T]$ for all $\psi \in W_{T,A}$,*

iii) *We have for a* $C > 0$

$$\begin{aligned}\int_s^T \int_H \left[V^2(s',x) + \left|(-A)^\delta x\right|^2 + |x|^2\right] \mu_{s'}(\mathrm{d}x)\mathrm{d}s' \\ \leq C \int_s^T \int_H (V^2(s',x) + |x|^2)\xi(\mathrm{d}x)\mathrm{d}s',\end{aligned} \tag{2.10}$$

iv) (FPE) *holds for all* $t \in [s, T]$.

Chapter 3

Proof of the main theorem

First let us give a short overview of the proof of the main theorem, i.e. Theorem 2.16. The proof has three steps. The results in each step are structured in claims.
The main idea of the proof is to first show tightness for the solution measures of the approximating (SPDE). Then we use the better convergence properties of measures via Prohorov's theorem to identify a candidate for a solution to (FPE) (Step 1 and Step 2).
Finally we will show that this candidate indeed solves (FPE) (Step 3).

Step 1: Identifying limit measures

Let us recall our definition of the stochastic convolution in the case of Levy-noise:

$$Y_A(t,s) := \int_s^t S(t-r)dY(r).$$

Remark 3.1. In the following we will use weak convergence with respect to the standard $|\cdot|$-topology which we will refer to as simply weak convergence. Additionally we will need to use weak convergence with respect to the weak topology for better compactness properties, which we will denote as τ_ω-weak convergence.

Claim 3.2. *(Tightness) The solution measures μ_t^α to our approximating* (FPE) *are τ_ω-tight (i.e. tight with respect to the weak topology).*

Proof. For $\alpha \in (0,1]$, $t \geq 0$ let $X_\alpha(t) := X_\alpha(t,s,x_0)$ be the mild solution to (SPDE_α) (see Assumption 2.7) and let

$$\tilde{X}_\alpha(t) = X_\alpha(t) - Y_A(t,s). \tag{3.1}$$

Let us set for fixed $\omega \in \Omega$

$$\tilde{F}_\alpha(t,x) := F_\alpha(t, x + Y_A(t)(\omega)) \text{ for } x \in H$$

Thus we have

$$\tilde{X}_\alpha(t) = S(t-s)x_0 + \int\limits_s^t S(t-r)\tilde{F}_\alpha(r, \tilde{X}_\alpha(r,s,x))\mathrm{d}r,$$

where $\tilde{X}_\alpha(t)$ is continuous by [LR15, Lem. 6.2.9].
We define for this process an approximating process using $P_n : x \mapsto \sum_{i=1}^n \langle x, \xi_i \rangle \xi_i$, with $(\xi_i)_{i\in\mathbb{N}}$ being the eigenbasis of A, by

$$\tilde{X}^n_\alpha(t) := P_n\tilde{X}_\alpha = e^{(t-s)A}\underbrace{P_n x_0}_{=:x_n} + \int\limits_s^t e^{(t-r)A}\underbrace{P_n F_\alpha}_{=:F^n}(r, \tilde{X}_\alpha(r,s,x))\mathrm{d}r.$$

For $s \leq t \leq T$ we have using the definition of $\tilde{X}^n_\alpha$

$$\begin{aligned}
\mathrm{d}t\tilde{X}^n_\alpha(t) &= \frac{\mathrm{d}}{\mathrm{d}\,t}\left(e^{(t-s)A}x_n + \int\limits_s^t e^{(t-r)A}F^n(r, \tilde{X}_\alpha(r,s,x))\mathrm{d}r\right) \\
&= Ae^{(t-s)A}x_n + \frac{\mathrm{d}}{\mathrm{d}\,t}\int\limits_s^t e^{(t-r)A}F^n(r, \tilde{X}_\alpha(r,s,x_0))\mathrm{d}r \\
&= Ae^{(t-s)A}x_n + A\int\limits_s^t e^{(t-r)A}F^n(r, \tilde{X}_\alpha(r))\mathrm{d}r + F^n(t, \tilde{X}_\alpha(t)) \\
&= A\tilde{X}^n_\alpha(t) + F^n(t, \tilde{X}_\alpha(t)).
\end{aligned}$$

The third step holds because of the Leibniz integral rule, which gives us

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t}\int\limits_s^t e^{(t-r)A}F^n(r,\tilde{X}_\alpha(r,s,x_0))\mathrm{d}r &= \int\limits_s^t \frac{\mathrm{d}}{\mathrm{d}\,t}(e^{(t-r)A}F^n(r,\tilde{X}_\alpha(r,s,x_0)))\mathrm{d}r \\ &\quad + e^{(t-t)A}F^n(t,\tilde{X}_\alpha(t,s,x_0))\cdot 1 \\ &\quad - e^{(t-s)A}F^n(t,\tilde{X}_\alpha(s,s,x_0))\cdot 0 \\ &= \int\limits_s^t Ae^{(t-r)A}F^n(r,\tilde{X}_\alpha(r,s,x_0))\mathrm{d}r \\ &\quad + e^{(t-t)A}F^n(t,\tilde{X}_\alpha(t,s,x_0)) \\ &= A\int\limits_s^t e^{(t-r)A}F^n(r,\tilde{X}_\alpha(r,s,x_0))\mathrm{d}r \\ &\quad + F^n(t,\tilde{X}_\alpha(t,s,x_0)).\end{aligned}$$

The derivation above is valid, since we work on the finite dimensional subspace $P_nH \subset H$ and thus the unboundedness of A and the domain $D(A) \subset H$ of A pose no problem.
Taking $\langle\,\cdot\,,\tilde{X}^n_\alpha(t)\rangle$ on both sides, using product rule for inner products ($\frac{\mathrm{d}}{\mathrm{d}\,t}\langle\alpha(t),\alpha(t)\rangle = 2\langle\alpha(t),\alpha'(t)\rangle$) and using that $(-A)^{\frac{1}{2}}$ is again self-adjoint, we compute:

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{X}^n_\alpha(t) = A\tilde{X}^n_\alpha(t) + F^n(t,\tilde{X}_\alpha(t)) \\ \Rightarrow\quad & \langle\frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{X}^n_\alpha(t),\tilde{X}^n_\alpha(t)\rangle = \langle A\tilde{X}^n_\alpha(t),\tilde{X}^n_\alpha(t)\rangle + \langle F^n(t,\tilde{X}_\alpha(t)),\tilde{X}^n_\alpha(t)\rangle \\ \Rightarrow\quad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}\left|\tilde{X}^n_\alpha(t)\right|^2 + \left|(-A)^{\frac{1}{2}}\tilde{X}^n_\alpha(t)\right|^2 = \langle F^n(t,X_\alpha(t)),\tilde{X}^n_\alpha(t)\rangle. \end{aligned} \tag{3.2}$$

Integrating over (t,s) we get to

$$\frac{1}{2}(\left|\tilde{X}^n_\alpha(t)\right|^2 - |x_0|^2) + \int\limits_s^t \left|(-A)^{\frac{1}{2}}\tilde{X}^n_\alpha(s')\right|^2\mathrm{d}s' = \int\limits_s^t \langle F^n(t,\tilde{X}_\alpha(t)),\tilde{X}^n_\alpha(t)\rangle\mathrm{d}t,$$

So we have

$$\left|\tilde{X}^n_\alpha(t)\right|^2 + 2\int\limits_s^t \left|(-A)^{\frac{1}{2}}\tilde{X}^n_\alpha(s')\right|^2\mathrm{d}s' \le |x_0|^2 + \int\limits_s^t \left|(\tilde{X}^n_\alpha(s')\right|^2 + \left|F_\alpha(s',\tilde{X}_\alpha(s'))\right|^2)\mathrm{d}s'. \tag{3.3}$$

Also starting from (3.2), we get using integration by parts

$$e^{-t}\left|\tilde{X}_\alpha^n(t)\right| - e^{s}\left|\tilde{X}_\alpha^n(s)\right| = -\int\limits_s^t e^{-s'}\left|\tilde{X}_\alpha^n(s')\right|^2 \mathrm{d}s' + \int\limits_s^t 2e^{-s'}\left(\langle F^n(s',\tilde{X}_\alpha(s')),\tilde{X}_\alpha^n(s')\rangle - \left|(-A)^{\frac{1}{2}}\tilde{X}_\alpha^n(s')\right|^2\right)\mathrm{d}s'.$$

Applying Young's inequality yields

$$\begin{aligned} e^{-t}\left|\tilde{X}_\alpha^n(t)\right| + 2\int\limits_s^t e^{-s'}\left|(-A)^{\frac{1}{2}}\tilde{X}_\alpha^n(s')\right|^2 \mathrm{d}s' &= e^{-s}x_n + 2\int\limits_s^t e^{-s'}\langle F^n(s',\tilde{X}_\alpha(s')),\tilde{X}_\alpha^n(s')\rangle \mathrm{d}s' \\ &\quad - \int\limits_s^t e^{-s'}\left|\tilde{X}_\alpha^n(s')\right|^2 \mathrm{d}s' \\ &\le e^{-s}x_n + \int\limits_s^t e^{-s'}\left|F^n(s',\tilde{X}_\alpha(s'))\right|^2 \mathrm{d}s'. \end{aligned}$$

We obtain for all $0 \le s \le t \le T$

$$\left|\tilde{X}_\alpha^n(t)\right| + 2\int\limits_s^t \left|(-A)^{\frac{1}{2}}\tilde{X}_\alpha^n(s')\right|^2 \mathrm{d}s' \le \frac{e^{-s}}{e^{-t}}\left(x_n + \int\limits_s^t \left|F^n(s',\tilde{X}_\alpha(s'))\right|^2 \mathrm{d}s'\right)$$

Now we can let $n \to \infty$ and using Fatou's lemma we see

$$\begin{aligned} \left|\tilde{X}_\alpha(t)\right| + 2\int\limits_s^t \left|(-A)^{\frac{1}{2}}\tilde{X}_\alpha(s')\right|^2 \mathrm{d}s' &\le \lim_{n\to\infty}\left|\tilde{X}_\alpha^n(t)\right| + 2\liminf_{n\to\infty}\int\limits_s^t \left|(-A)^{\frac{1}{2}}\tilde{X}_\alpha^n(s')\right|^2 \mathrm{d}s' \\ &\le e^{T}\left(x_0 + \int\limits_s^t \left|\tilde{F}_\alpha(s',\tilde{X}_\alpha(s'))\right|^2 \mathrm{d}s'\right) < \infty, \end{aligned}$$

where we used that F_α is bounded. Resubstituting $\tilde{F}_\alpha$ we can now conclude our approximation and see

$$\left|\tilde{X}_\alpha(t)\right|^2 + 2\int\limits_s^t \left|(-A)^{\frac{1}{2}}\tilde{X}_\alpha(s')\right|^2 \mathrm{d}s' \le |x_0|^2 + \int\limits_s^t \left|(\tilde{X}_\alpha(s')\right|^2 + \left|F_\alpha(s',\tilde{X}_\alpha(s'))\right|^2)\mathrm{d}s'. \tag{3.4}$$

Leaving out the second term on the left hand side and applying Gronwall's lemma (Theorem 3.16, setting $f(t) = \left|\tilde{X}_\alpha(t)\right|^2$ and $\epsilon = |x_0|^2 + \int_s^t |F_\alpha(s', X_\alpha(s'))|^2 \, \mathrm{d}s'$) we get to

$$\left|\tilde{X}_\alpha(t)\right|^2 \leq e^{t-s} |x_0|^2 + e^{t-s} \int_s^t |F_\alpha(s', X_\alpha(s'))|^2 \, \mathrm{d}s'. \tag{3.5}$$

For the last summand on the right hand side we can calculate

$$\begin{aligned}
\mathbb{E}\left[e^{t-s} \int_s^t |F_\alpha(s', X_\alpha(s'))|^2 \, \mathrm{d}s'\right] &= e^{t-s} \int_s^t \mathbb{E}\left[|F_\alpha(s', X_\alpha(s'))|^2\right] \mathrm{d}s' \\
&= e^{t-s} \int_s^t P^\alpha_{s,s'} \, |F_\alpha(s', x_0)|^2 \, \mathrm{d}s' \\
&\leq e^{t-s} \int_s^t P^\alpha_{s,s'} V^2(s', x_0) \mathrm{d}s' \\
&\leq e^{t-s} K \int_s^t V^2(s', x_0) \mathrm{d}s'.
\end{aligned}$$

Here we used the definition of the transition evolution operator (see (1.4)) as well as Condition 2.5b) and Condition 2.6.
Thus by taking expectation of (3.5) we get:

$$\mathbb{E}\left[\left|\tilde{X}(t)\right|^2\right] \leq e^{t-s} |x_0|^2 + e^{t-s} K \int_s^t V^2(s', x_0) \mathrm{d}s'. \tag{3.6}$$

Resubstituting $\tilde{X}$ and using the elementary inequality $a^2 - 2b^2 \leq 2|a-b|^2$ (see Lemma (3.20)) we get:

$$\mathbb{E}\left[|X_\alpha(t, s, x_0)|^2\right] - 2\mathbb{E}\left[|Y_A(t)|^2\right] \leq 2\mathbb{E}\left[|X_\alpha(t) - Y_A(t)|^2\right].$$

Applying this to (3.6) we arrive at

$$\mathbb{E}\left[|X_\alpha(t, s, x_0)|^2\right] \leq 2e^{(T-s)} |x_0|^2 + 2K e^{T-s} \int_s^T V^2(s', x_0) \mathrm{d}s' + 2\kappa$$

for $s \leq t \leq T$ and $\kappa := \sup_{t\in[s,T]} \mathbb{E}\left[|Y_A(t)|^2\right]$.
Integrating with the initial condition ξ over H and using (1.5) we get

$$
\begin{aligned}
\int_H |x|^2 \mu_t^\alpha(\mathrm{d}x) &= \int_H \mathbb{E}\left[|X_\alpha(t,s,x)|^2\right] \xi(\mathrm{d}x) \\
&\leq 2\kappa + \int_H 2e^{T-s} |x|^2 \,\mathrm{d}\xi + \int_H 2\kappa e^{T-s} \int_s^T V^2(s',x)\mathrm{d}s'\xi(\mathrm{d}x) \\
&\leq C_1 + \int_s^T \int_H C_2 |x|^2 + C_3 V^2(s',x)\xi(\mathrm{d}x)\mathrm{d}s' \\
&\leq \max(C_1,C_2,C_3)(1 + \int_s^T \int_H |x|^2 + V^2(s',x)\xi(\mathrm{d}x)\mathrm{d}s') \\
&\leq C < \infty.
\end{aligned} \tag{3.7}
$$

Here we used the integrability condition on the initial condition (see (2.10)).
Now we have shown τ_ω-tightness for the measures μ_t^α, since for every $\epsilon > 0$ we may choose $\delta = \frac{C}{\epsilon}$ and have

$$
C \geq \int_{\{|x|^2 > \delta\}} |x|^2 \mu_t^\alpha(\mathrm{d}x) \geq \int_{\{|x|^2 > \delta\}} \delta \mu_t^\alpha(\mathrm{d}x) = \delta \cdot \mu_t^\alpha(\{|x|^2 > \delta\}).
$$

which implies $\mu_t^\alpha(\{|x|^2 \leq \delta\}) \leq 1 - \frac{C}{\delta}$. This shows τ_ω-tightness since balls in H are τ_ω-compact (see [Meg98, 2.6.19]). $\square$

Claim 3.3. *(Prohorov I) For any given sequence in* $(0,1]$ *convergent to zero there exists a subsequence* $\alpha_n \to 0$ *and measures* μ_t *for all* $t \in [0,T]$*, such that the measures* $\mu_t^{\alpha_n}$ *converge* τ_ω*-weakly to* μ_t *for all* $t \in [0,T]$.

Let us first give a short overview over the approach taken in this claim. We first use Prohorov's theorem to get a subsequence for every $t \in [s,T]$, for which the measures converge to a limit measure. By repeatedly applying Prohorov's theorem (see 3.15) and using diagonalisation we can find a subsequence such that our sequence of measures converges to $\tilde{\mu}_t$ for every $t \in \mathbb{Q} \cap [s,T]$. By a further application of Prohorov's theorem, but this time in t, we can assign a limit measure μ_t for every $t \in [s,T] \setminus \mathbb{Q}$.
A last application of Prohorov' theorem will give us that the initial sequence of measures converges for a sequence $\alpha_n \downarrow 0$ towards μ_t independently of t. Below

we have a diagram showing the relationship between the families of measures and the associated sequences.

$$n \qquad \supseteq \qquad \forall t\ \exists n_k^t \underset{t\in\mathbb{Q}\cap[s,T]}{\overset{\text{Diagonalization}}{\rightsquigarrow}} \tilde{n}_k \qquad\qquad \overset{\in\mathbb{Q}\cap[s,T]}{\exists\ r_n(t)} \ \to t \in [s,T]\setminus\mathbb{Q}$$

$$\begin{tikzcd}
\mu_t^{\alpha_n} \arrow[r, dotted, "\text{Prohorov}"] & \mu_t^{\alpha_{n_k^t}} \arrow[d, "\tau_\omega\text{-w.}"'] & \mu_t^{\alpha_{\tilde{n}_k}} \arrow[d, "\tau_\omega\text{-w.}"'] & \\
& \tilde{\mu}_t & \tilde{\mu}_t \arrow[r, dotted, "\text{Prohorov}"] & \tilde{\mu}_{r_n(t)} \arrow[d, "\tau_\omega\text{-w.}"'] \\
& & & \mu_t
\end{tikzcd}$$

Proof. Using that closed balls are also metrizable (see [Meg98, 2.6.20]), we may apply the second part of Prohorov's theorem.
By applying Prohorov's theorem to μ_t^α with parameter α, we get that for any sequence $(\alpha_n) \to 0$ we have for every t a subsequence $(\alpha_{n_k^t})$ such that $\mu_t^{\alpha_{n_k^t}} \to \tilde{\mu}_t \in \mathcal{M}_1(H)$ τ_ω-weakly.
Now we use a diagonalisation argument to get an surjective enumeration $q : \mathbb{N} \to Q$ of $\mathbb{Q}$ and apply Prohorov's theorem, so that $(\alpha_{n_k}^{q(n+1)})$ is a subsequence of $(\alpha_{n_k}^{q(n)})$ and call the result $\alpha_{\tilde{n}_k}$. Thus we have that $\mu_t^{\alpha_{\tilde{n}_k}} \to \tilde{\mu}_t$ τ_ω-weakly for every $t \in Q \cap [s,T]$ with each $\tilde{\mu}_t \in \mathcal{M}_1(H)$.
We apply Prohorov to the family of limits $(\tilde{\mu}_t)_{t\in\mathbb{Q}\cap[s,T]}$, which is tight by Claim 3.4, with parameter t to get for each $t \in [s,T]\setminus\mathbb{Q}$ a family of sequences $r_n(t) \in [s,T]\cap\mathbb{Q}$ each converging to t, satisfying $\tilde{\mu}_{r_n(t)} \to \mu_t$ τ_ω-weakly.
Now we have a limit candidate family and the sequence $\alpha_k := \alpha_{\tilde{n}_k}$

$$\mu_t := \begin{cases} \mu_t & \text{for } t \in [s,T]\setminus\mathbb{Q} \\ \tilde{\mu}_t & \text{else} \end{cases}$$

and we can claim one of the central results of Step 1:

$$\mu_t^{\alpha_n} \to \mu_t \ \tau_\omega\text{-weakly } \forall t \in [s,T]\setminus\mathbb{Q}.$$

Suppose the claim is wrong and fix t. We apply Prohorov to $\mu_t^{\alpha_n}$ using α_n as the parameter and get that there is a subsequence (α_{n_k}) such that $\mu_t^{\alpha_{n_k}} \to \nu$ τ_ω-weakly for some $\nu \in \mathcal{M}_1(H)\setminus\{\mu_t\}$. Thus for some $\psi \in \mathcal{W}_{T,A}$, since it is by definition measure separating, we have $\mu_t(\psi) \neq \nu(\psi)$, where we define

$$\mu_t(\psi) := \int_H \psi(t,x)\mu_t(\mathrm{d}x).$$

But we also have:

$$
\begin{aligned}
|\nu(\psi) - \mu_t(\psi)| \leq &\left|\nu(\psi) - \mu_t^{\alpha_{n_k}}\right| + \left|\mu_t^{\alpha_{n_l}}(\psi) - \mu_{r_n(t)}^{\alpha_{n_l}}(\psi)\right| \\
&+ \left|\mu_{r_n(t)}^{\alpha_{n_k}}(\psi) - \tilde{\mu}_{r_n(t)}(\psi)\right| + \left|\tilde{\mu}_{r_n(t)}(\psi) - \mu_t(\psi)\right|
\end{aligned}
$$

If we first let $k \to \infty$ and then $n \to \infty$ the first, third and forth terms converge by τ_ω- weak convergence and the second by equicontinuity, which we show in Claim 3.7, to zero. Thus we have $\mu_t(\psi) = \nu(\psi)$, which is a contradiction to the assumption and proves the above given central result of Step 1. So, now we can conclude that we have $\mu_t^{\alpha_n} \to \mu_t$ τ_ω-weakly for all $t \in [s, T]$.

□

Claim 3.4. *We have* $\sup_{t\in[s,T]} \int_H |x|^2 \mu_t(\mathrm{d}x) < \infty$

Proof. By equation (3.7) we have the inequality for μ_t^α, but since $|\cdot|^2$ is a double limit of bounded weakly continuous functions, we get the equation for μ_t. □

Claim 3.5. *For all* $u \in D(L_0)$ *the map* $t \mapsto \int_H u(t,x)\mu_t(\mathrm{d}x)$ *is continuous*

Proof. The continuity follows directly from 3.7, since $\psi \in W_{T,A}$ is weakly continuous. Thus we get for every $\psi \in W_{T,A}$ and sufficiently small $\epsilon > 0$

$$
\begin{aligned}
&\left(\int_H \psi(t,x)\mu_t(\mathrm{d}x) - \int_H \psi(t,x)\mu_{t+\epsilon}(\mathrm{d}x)\right) \\
= &\lim_{n\to\infty} \left(\int_H \psi(t,x)\mu_t^{\alpha_n}(\mathrm{d}x) - \int_H \psi(t,x)\mu_{t+\epsilon}^{\alpha_n}(\mathrm{d}x)\right) < \delta
\end{aligned}
$$

□

Lemma 3.6. *The measure* $\mu_t(\mathrm{d}x)$ *are probability kernels from* $([s,T], \mathcal{B}([s,T]))$ *to* $(H, \mathcal{B}(H))$

Proof. We need to show that $t \mapsto \mu_t(A)$ is measurable on $[s,T]$ for all $A \in \mathcal{B}(H)$. We will proof this by using a monotone class argument according to Theorem 3.19 to extent the continuity of $t \mapsto \int_H u(t,x)\mu_t(\mathrm{d}x)$ to a larger space of functions including indicator functions, such that we will see, that even $t \to \mu_t(A)$ is continuous for all $A \in \mathcal{B}(A)$.
Define

$$\mathcal{H} := \{u : [s,T] \times H \mapsto \mathbb{R} \text{ bd.} \mid t \mapsto \int_H u(t,x)\mu_t(\mathrm{d}x) \text{ continuous and well defined}\}$$

We want to show that $\mathcal{H}$ is a monotone vector space according to Definition 3.17. We see that we have $1_{\mathcal{H}} = 1 \in \mathcal{H}$ since it can be approximated by $\psi(t,x) =$

$\phi(t) f_m^n(P_m x) = 1 \cdot f_m^n(P_m x)$, with $f_m^n \uparrow 1$ for which the continuity holds, and thus we have the statement by Lebesgue's dominated convergence theorem.
If we take $(h_i) \in \mathcal{H}$ with $h_i \uparrow h < \infty$. We can apply Lebesgue and see that $h \in \mathcal{H}$. Now we have to check if $W_{T,A}$ forms an algebra. So let us take $\psi_1, \psi_2 \in W_{T,A}$

$$\psi_1 \cdot \psi_2 = (\phi_1(t) \cdot \phi_2(t))(f_m^1(P_m x) \cdot f_n^2(P_n x)) \in W_{T,A}$$

If we knew $\sigma(W_{T,A}) \supseteq \mathcal{B}([s,T] \times H)$ (clear, since we can approximate all continuous functions by Schwartz functions pointwise), we would have that $t \mapsto \int_H u(t,x)\mu_t(\mathrm{d}x)$ is continuous for all bounded $\sigma(W_{T,A})$-measurable functions and especially the needed measurability, since we have the continuity for indicator functions. □

To conclude step 1, we have to prove the equicontinuity used in Claim 3.2.

Claim 3.7 (Equicontinuity)**.** *For $\mu_t^\alpha(\psi) := \int_H \psi(t,x)\mu_t^\alpha(\mathrm{d}x), t \in [s.T], \alpha \in [0,1], \psi \in W_{T,A}$ we have that*

$$t \mapsto \mu_t^\alpha(\psi)$$

is equicontinuous on $[s,T]$.

Proof. To see this fix $\psi \in W_{T,A}$ and consider:

$$\begin{aligned}
\left|\mu_{t_2}^\alpha(\psi) - \mu_{t_1}^\alpha(\psi)\right| &= \left| \int_s^{t_1} \int_H L_\alpha \psi(s',x)\mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s' - \int_s^{t_2} \int_H L_\alpha \psi(s',x)\mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s' \right| \\
&= \int_{t_1}^{t_2} \int_H \Bigg\{ D_t\psi(s',x) + \langle AD_x\psi(s',x), x\rangle + \langle D_x\psi(s',x), F_\alpha\rangle \\
&\quad + \int_H \left[\psi(s',x+y) - \psi(s',x) - \frac{\langle D_x\psi(s',x), y\rangle}{1+|y|^2}\right] M(\mathrm{d}y) \\
&\quad - \frac{1}{2}\int_H \langle \xi, Q\xi\rangle e^{i\langle \xi, x\rangle}\nu_{s'}(d\xi) \Bigg\} \mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s'.
\end{aligned}$$

Here we used that μ_t^α solves (FPE) and the definition of L_0. We are now going to look at the individual parts separately. For the first summand we get

$$\int_{t_1}^{t_2} \int_H D_t\psi(s',x)\mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s' \le |t_2 - t_1| \, \|D_t\psi\|_\infty,$$

where we denote $\|\cdot\|_\infty := \sup_{[s,T]\times H} |\cdot|$. For the linear drift part we compute:

$$\int_{t_1}^{t_2}\int_H \langle AD\psi(t,x),x\rangle \mu_{s'}^\alpha \mathrm{d}s' \leq \left(\int_{t_1}^{t_2}\int_H |x|\,|A^*D_x\psi|\,\mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s'\right)$$

$$\leq \left(\int_{t_1}^{t_2}\int_H |x|^2\,\mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s'\right)^{\frac{1}{2}} \cdot \left(\int_{t_1}^{t_2}\int_H |A^*D_x(s',x)\psi|^2\,\mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s'\right)^{\frac{1}{2}}$$

$$\leq \left(\int_{t_1}^{t_2}\int_H |x|^2\,\mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s'\right)^{\frac{1}{2}} \cdot \left(|t_2-t_1|\,\|A^*D_x\psi\|_\infty^2\right)^{\frac{1}{2}}$$

$$\leq \underbrace{\left(\int_{t_1}^{t_2}\int_H |x|^2\,\mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s'\right)^{\frac{1}{2}}}_{<\infty,\text{ by Claim 3.7}} |t_2-t_1|^{\frac{1}{2}}\,\|A^*D_x\psi\|_\infty,$$

where we first use the Cauchy-Schwartz' inequality, then Hölder's inequality (in $L(([0,T],\mathrm{d}s)\times(H,\mu_{s'}^\alpha);(H,|\cdot|)))$.
We have $\|A^*D_x\psi\|_\infty < \infty$ since $D_x\psi$ only takes non zero values in a finite subspace of H. Now we look at the non-linear drift part:

$$\int_{t_1}^{t_2}\int_H \langle D_x\psi(t,x),F_\alpha\rangle \mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s' \leq \int_{t_1}^{t_2}\int_H |D_x\psi(t,x)|\,|F_\alpha(s',x)|\,\mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s'$$

$$\leq \left(\int_{t_1}^{t_2}\int_H |F_\alpha(s',x)|^2\,\mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s'\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2}\int_H |D_x\psi(s',x)|^2\,\mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s'\right)^{\frac{1}{2}}$$

$$\leq \left(\int_{t_1}^{t_2}\int_H P_{s,s'}^\alpha V^2(s',x)\xi(\mathrm{d}x)\mathrm{d}s'\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2}\int_H |D_x\psi(s',x)|^2\,\mathrm{d}\mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s'\right)^{\frac{1}{2}}$$

$$\leq \left(\int_{t_1}^{t_2}\int_H KV^2(s',x)\xi(\mathrm{d}x)\mathrm{d}s'\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2}\int_H |D_x\psi(s',x)|^2\,\mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s'\right)^{\frac{1}{2}}$$

$$\leq \underbrace{\left(\int_{t_1}^{t_2}\int_H V^2(s',x)\xi(\mathrm{d}x)\mathrm{d}s'\right)^{\frac{1}{2}}}_{<\infty,\text{ by assumption for initial condition}} K\,\|D_x\psi\|_\infty\,|t_2-t_1|^{\frac{1}{2}},$$

where we again first used Cauchy-Schwartz' inequality, then Hölder's inequality in $L(([0,T],\mathrm{d}s)\times(H,\mu^\alpha_{s'});(H,|\cdot|))$ and later Condition 2.6. For the first jump-part we can compute:

$$
\begin{aligned}
&\int_{t_1}^{t_2}\int_H\int_H \psi(s',x+y)-\psi(s',x)-\frac{\langle\psi(s',x),y\rangle}{1+|y|^2}M(\mathrm{d}y)\mu^\alpha_{s'}(\mathrm{d}x)\mathrm{d}s' \\
=&\int_{t_1}^{t_2}\int_H \phi(s')\int_H(f_m(\tilde{x}+\tilde{y})-f_m(\tilde{x}))-\frac{\langle D_x f_m(\tilde{x}),y\rangle}{1+|y|^2}M(\mathrm{d}y)d\mu^\alpha_{s'}(\mathrm{d}x)\mathrm{d}s' \\
\leq&(\sup_{[t_1,t_2]}\phi(s')\cdot\sup_{\{x\in H\}}\int_H(f_m(\tilde{x}+\tilde{y})-f_m(\tilde{x}))-\frac{\langle D_x f_m(\tilde{x}),y\rangle}{1+|y|^2}M(\mathrm{d}y))\int_{t_1}^{t_2}\int_H \mathrm{d}\mu^\alpha_{s'}(\mathrm{d}x)\mathrm{d}s' \\
\leq&(\sup_{[t_1,t_2]}\phi(s')\cdot\sup_{\{x\in H\}}\int_H(f_m(\tilde{x}+\tilde{y})-f_m(\tilde{x}))-\frac{\langle D_x f_m(\tilde{x}),y\rangle}{1+|y|^2}M(\mathrm{d}y))\,|t_2-t_1|\,,
\end{aligned}
\tag{3.8}
$$

since we assumed for fixed $f_m\in\mathcal{S}(\mathbb{R}^m,\mathbb{R})$

$$
\sup_{\{x\in H\}}\int_H(f_m(\tilde{x}+\tilde{y})-f_m(\tilde{x}))-\frac{\langle D_x f_m(\tilde{x}),y\rangle}{1+|y|^2}M(\mathrm{d}y):=\Lambda<\infty. \tag{3.9}
$$

And the second yields, using $\nu_t=\phi(t)\cdot\nu_m\circ\Pi_m^{-1}$ and $\nu_m(\mathrm{d}r)=g_m(r)\mathrm{d}r$ with $g_m\in\mathcal{S}(\mathbb{R},\mathbb{C})$,

$$
\begin{aligned}
&\left|-\int_{t_1}^{t_2}\frac{1}{2}\int_H\langle\xi,Q\xi\rangle e^{i\langle\xi,x\rangle}\nu_{s'}(\mathrm{d}\xi)\mathrm{d}\mu^\alpha_{s'}(\mathrm{d}x)\mathrm{d}s'\right| \\
&\qquad=\left|\int_{t_1}^{t_2}\int_H-\frac{1}{2}\int_H\langle\xi,Q\xi\rangle e^{i\langle\xi,x\rangle}\phi(s')\nu_m\circ\Pi_m^{-1}(\mathrm{d}\xi)\mathrm{d}\mu^\alpha_{s'}(\mathrm{d}x)\mathrm{d}s'\right| \\
&\qquad\leq|\int_{t_1}^{t_2}\int_H-\frac{1}{2}\underbrace{\sup_{t\in[s,T]}\phi(t)}_{:=C_1}\sup_{x,\xi\in H}e^{i\langle\xi,x\rangle}\underbrace{\int_{\mathbb{R}^m}|\langle\Pi_m r,Q\Pi_m r\rangle g_m(r)|\,\mathrm{d}r}_{:=C_2}\,\mathrm{d}\mu^\alpha_{s'}(\mathrm{d}x)\mathrm{d}s'| \\
&\qquad\leq\left||t_2-t_1|\,(-\frac{1}{2})C_1 1 C_2\right|
\end{aligned}
\tag{3.10}
$$

Putting all the parts together we arrive at:

$$
\begin{aligned}
\left|\mu^{\alpha}_{t_2}(\psi) - \mu^{\alpha}_{t_1}(\psi)\right| &\leq |t_2 - t_1| \, \|D_t\psi\|_\infty \\
&+ \left(\int\limits_{t_1}^{t_2} \int\limits_{H} |x|^2 \, \mathrm{d}\mu^{\alpha}_{s'}(\mathrm{d}x)\mathrm{d}s' \right)^{\frac{1}{2}} |t_2 - t_1|^{\frac{1}{2}} \, \|A^* D_X \psi\|_\infty \\
&+ \left(\int\limits_{t_1}^{t_2} \int\limits_{H} V^2(s', x)\mathrm{d}\xi\mathrm{d}s' \right)^{\frac{1}{2}} K \, \|D_x\psi\|_\infty \, |t_2 - t_1|^{\frac{1}{2}} \\
&+ \sup_{[t_1,t_2]} \phi(s') \cdot \sup_{\{x \in H\}} \int\limits_{H} (f_m(x+y) - f_m(x)) - \frac{\langle D_x f_m(x), y\rangle}{1 + |y|^2} M(\mathrm{d}y) \, |t_2 - t_1| \\
&+ \left| |t_2 - t_1| \, (-\frac{1}{2}) C_1 1 C_2 \right| \leq \Pi \, |t_1 - t_2|^{\frac{1}{2}} \, ,
\end{aligned}
$$

for some $\Pi > 0$, since $|t_2 - t_1| \leq k \, |t_2 - t_1|^{\frac{1}{2}}$ on $[s, T]$ for some constant $k > 0$. □

Step 2: Weak convergence of the measures

Claim 3.8. *By choosing a further subsequence denoted again by α_n the measures $\mu^{\alpha_n}_t(\mathrm{d}x)\mathrm{d}t$ converge weakly to $\mu_t(\mathrm{d}x)\mathrm{d}t$ on $[0, t] \times H$ with $\mu_t(\mathrm{d}x)$ as before.*

Proof. By (3.4) we have

$$
\left|\tilde{X}(t)\right|^2 + 2 \int\limits_{s}^{T} \left|(-A)^{\frac{1}{2}} \tilde{X}(s')\right|^2 \mathrm{d}s' \leq |x_0|^2 + \int\limits_{s}^{T} \left|\tilde{X}(s')\right|^2 + |F_\alpha(s', X_\alpha(s'))|^2 \, \mathrm{d}s'.
$$

Leaving out $\left|\tilde{X}(t)\right|^2$ on the left hand side and using (3.5) we get to

$$\begin{aligned}
\int\limits_s^T \left|(-A)^{\frac{1}{2}}\tilde{X}(s')\right|^2 \mathrm{d}s' &\leq |x_0|^2 + \int\limits_s^T (e^{s'-s}|x|^2 + e^{s'-s}\int\limits_s^{s'} |F_\alpha(s'',X_\alpha(s''))|^2 \mathrm{d}s'' \\
&\quad + |F_\alpha(s',X_\alpha(s'))|^2)\mathrm{d}s' \\
&\leq |x_0|^2 (1 + \int\limits_s^T e^{s'-s}\mathrm{d}s') + (T-s)e^{T-s}\int\limits_s^T |F_\alpha(s'',X_\alpha(s''))|^2 \mathrm{d}s'' \\
&\quad + \int\limits_s^T |F_\alpha(s',X_\alpha(s'))|^2 \mathrm{d}s \\
&\leq C_1 |x_0|^2 + C_2 \int\limits_s^T |F_\alpha(s',X_\alpha(s'))|^2 \mathrm{d}s'.
\end{aligned}$$

And by multiplying with $\|(-A)^{-\frac{1}{2}+\delta}\|^2$, which exists since $-\frac{1}{2}+\delta < 0$ and $(-A)^{2\delta-1}$ is of trace class, we arrive at

$$\begin{aligned}
\|(-A)^{-\frac{1}{2}+\delta}\|^2 C(|x|^2 + \int\limits_s^T |F_\alpha(s',X_\alpha(s'))|^2 \mathrm{d}s') &\geq \int\limits_s^T \left|\|(-A)^{-\frac{1}{2}+\delta}\|(-A)^{\frac{1}{2}}\tilde{X}(s')\right|^2 \mathrm{d}s \\
&\geq \int\limits_s^T \left|(-A)^{-\frac{1}{2}+\delta}(-A)^{\frac{1}{2}}\tilde{X}(s')\right|^2 \mathrm{d}s' \\
&\geq \int\limits_s^T \left|(-A)^{\delta}\tilde{X}(s')\right|^2 \mathrm{d}s'.
\end{aligned}$$

Taking expectation, then using Fubini's theorem for positive functions, resubstituting according to (3.1) and using the elementary inequality $2|a-b|^2 \geq |a|^2 - 2|b|^2$(Lemma 3.20) we see that

$$\begin{aligned}
\int\limits_s^T \mathbb{E}\left|(-A)^\delta X_\alpha(s')\right|^2 \mathrm{d}s' - 2\int\limits_s^T \overbrace{\mathbb{E}\left[\left|(-A)^\delta Y_A(s',s)\right|^2\right]}^{<c_\delta,\text{ by Lemma 2.3}} \mathrm{d}s' &\leq 2\mathbb{E}\int\limits_s^T \left|(-A)^\delta \tilde{X}(s')\right|^2 \mathrm{d}s' \\
&\leq 2C\|(-A)^{-\frac{1}{2}+\delta}\|^2 \mathbb{E}\left[|x|^2 + \int\limits_s^T |F_\alpha(s',X_\alpha(x,s'))|^2 \mathrm{d}s'\right].
\end{aligned}$$

This leads us, using (1.4), to

$$\int_s^T \mathbb{E}\left|(-A)^{\delta} X_{\alpha}(t)\right|^2 \mathrm{d}t \leq 2C\|(-A)^{-\frac{1}{2}+\delta}\|^2 \left(|x|^2 + \int_s^T P^{\alpha}_{s,s'}\left|F_{\alpha}(s',x)\right|^2 \mathrm{d}s'\right) + 2c_{\delta}T.$$

Further integrating with ξ over H yields

$$\begin{aligned}
\int_H \int_s^T \mathbb{E}\left|(-A)^{\delta} X_{\alpha}(t,s,x)\right|^2 \mathrm{d}t\xi(\mathrm{d}x) &= \int_s^T \int_H \left|(-A)^{\delta} x\right|^2 \mu^{\alpha}_t(\mathrm{d}x)\mathrm{d}t \\
&\leq \int_H 2C\|(-A)^{-\frac{1}{2}+\delta}\| \left(|x|^2 + \int_s^T P^{\alpha}_{s,s'}\left|F_{\alpha}(s',x)\right|^2 \mathrm{d}s'\right)\xi(\mathrm{d}x) + 2c_{\delta}T \\
&\leq 2C\|(-A)^{-\frac{1}{2}+\delta}\|(\int_H |x|^2 \xi(\mathrm{d}x) \\
&\quad + \int_s^T \int_H P^{\alpha}_{s,s'}\left|F(s',x)\right|^2 \mathrm{d}s'\xi(\mathrm{d}x)) + 2c_{\delta}T \\
&\leq 2C\|(-A)^{-\frac{1}{2}+\delta}\|(\int_H |x|^2 \xi(\mathrm{d}x) \\
&\quad + C\int_H \int_s^T KV^2(s',x)\mathrm{d}s'\xi(\mathrm{d}x)) + 2c_{\delta}T \\
&\leq \|(-A)^{-\frac{1}{2}+\delta}\|C2K(\int_H |x|^2 \xi(\mathrm{d}x) \\
&\quad + \int_H \int_s^T V^2(s',x)\mathrm{d}s'\xi(\mathrm{d}x)) + 2c_{\delta}T := \kappa,
\end{aligned} \tag{3.11}$$

where we used assumption 2.6 and Fubini's theorem.
Now we can check, if we can apply Prohorov's Theorem one last time. We set $K_{\epsilon} = \left\{\left|(-A)^{\delta}x\right| \leq c\right\}$. Since $(-A)^{-2\delta}$ is of trace class, it follows that $(-A)^{-\delta}$ is compact (since the eigenvalues converge to 0). We compute

$$(-A)^{-\delta}\{x \in H | x \leq c\} = (-A)^{-\delta}(-A)^{\delta}\{x \in H | \left|(-A)^{\delta}x\right| \leq c\} = K_{\epsilon}.$$

Thus K_{ϵ} is compact as the image of a bounded set under a compact operator, and we see

$$\int_s^T \int_{K_\epsilon^c} c\mu_t^\alpha(\mathrm{d}x)\mathrm{d}t \leq \kappa.$$

Thus $\mu_t^\alpha(\mathrm{d}x)dt$ is tight. Note that we indeed have shown tightness and not τ_ω-tightness.

An application of Prohorov's Theorem yields a new convergent subsequence of $\mu_t^{\alpha_n}\mathrm{d}t$, which we again simply denote by $\mu_t^{\alpha_{n_k}}\mathrm{d}t$. We call the limit of this sequence $\mu(dt, \mathrm{d}x)$.
Now we want to see if $\mu(\mathrm{d}t, \mathrm{d}x) = \mu_t(\mathrm{d}x)\mathrm{d}t$. For this purpose fix $f \in C_b([0,T];\mathbb{R})$ and $\psi \in W_{T,A}$ and look at

$$\begin{aligned}\int_s^T \int_H f(t)\psi(t,x)\mu_t(\mathrm{d}x)\mathrm{d}t &= \int_s^T f(t) \lim_{n\to\infty} \int_H \psi(t,x)\mu_t^{\alpha_n}(\mathrm{d}x)\mathrm{d}t \\ &= \lim_{n\to\infty} \int_s^T \int_H f(t)\psi(t,x)\mu_t^{\alpha_n}(\mathrm{d}x)\mathrm{d}t \\ &= \int_s^T \int_H \psi(t,x)\phi(x)\mu_t(\mathrm{d}x, \mathrm{d}t)\end{aligned}$$

The first equality follows by Claim 1 (the weak continuity of $\mu_t^{\alpha_n}$) combined with the weak continuity of ψ, the second by Lebesgue's dominated convergence theorem and the third by the above usage of Prohorov's theorem. □

Claim 3.9. *We have the bound from Equation* (2.10)*.*

Proof. Using Condition 2.6 we see

$$\int_{t_1}^{t_2} \int_H V^2(s',x)\mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s' \leq K \int_{t_1}^{t_2} \int_H V^2(s',x)\xi(\mathrm{d}x)\mathrm{d}s'.$$

Combining this with (3.7) and (3.11) we get

$$\int_s^T \int_H V^2(s',x)\mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s' + \int_s^T \int_H |x|^2 \mu_{s'}^\alpha(\mathrm{d}x)\mathrm{d}s' + \int_s^T \int_H \left|(-A)^\delta x\right|^2 \mu_t^\alpha(\mathrm{d}x)\mathrm{d}t$$

$$\le \max(C_1,C_2,C_3)\left(1+\int_s^T \int_H |x|^2 + V^2(s',x)\mathrm{d}\xi\mathrm{d}s'\right)$$

$$+ K\int_s^T \int_H V^2(s',x)\xi(\mathrm{d}x)\mathrm{d}s' + \int_H \int_s^T V^2(s',x)\mathrm{d}s'\mathrm{d}\xi(x) + 2c_\delta T$$

$$\le \tilde{C}\left(\int_s^T \int_H V^2(s',x) + |x|^2 \xi(\mathrm{d}x)\mathrm{d}s' + 1\right).$$

Note that this implies that $\mu_t(D(F(t,\cdot)) = 1$ for almost all $t \in [s,T]$. $\square$

Step 3: The measure solves the FPE

Recall that for fixed $\psi \in \mathcal{W}_{T,A}$ we can write

$$\psi(t,x) = \phi(t)\cdot f_m(\langle \xi_1, x\rangle, \ldots, \langle \xi_m, x\rangle) = \phi(t)\cdot f_m(\tilde{P}_m x).$$

Claim 3.10. *Fix $\psi \in \mathcal{W}_{T,A}$. The function*

$$J(t,x) := \int_H \left[(\psi(t,x+y)-\psi(t,x)) - \frac{\langle D_x\psi(t,x), y\rangle}{1+|y|^2}\right] M(\mathrm{d}y)$$

is continuous in x.

The main idea of the proof is, to split the integral in two parts and then control each part separately. The first one is an integral over a small ball around 0, where we need to control possible huge mass of M, by showing that our integrand decreases quadratically as zero is approached. We then use the existence of the second moments of M.
The second part is an integral over the rest of the space, where we have to use the structure of the test functions to control the integral, especially the uniform continuity in x.

Proof. Recall that we write $\xi_1,\ldots,\xi_m$ for the basis of the finite subspace on which f_m is defined and the corresponding projection $P_m x =: \sum_{i=1}^m \langle x, \xi_i\rangle \xi_i$. Further we will write $\tilde{\xi}_1,\ldots,\tilde{\xi}_m$ for the projection of the basis to $\mathbb{R}^m$ and $\tilde{P}_m x =$

$(\langle x, \xi_1\rangle, \ldots, \langle x, \xi_m\rangle)^T = \tilde{x}$. Because of the structure of our test-functions we can reduce much of the computations to a finite dimensional subspace.

$$\int_H \left[(\psi(t, x+y) - \psi(t,x)) - \frac{\langle D_x \psi(t,x), y\rangle}{1+|y|^2} \right] M(\mathrm{d}y)$$
$$= \phi(t) \int_H \left[(f_m(\tilde{x}+\tilde{y}) - f_m(\tilde{x})) - \frac{\langle D_x f_m(\tilde{x}), y\rangle}{1+|y|^2} \right] M(\mathrm{d}y)$$

We will now split the integral in two parts($\{|y| < \epsilon\}(J_<)$ and $\{|y| \geq \epsilon\}(J_\geq)$), to control the possibly huge mass of M around 0.
First we consider the set around 0 . We will now use a second order Taylor expansion of $f_m(\tilde{x}+\tilde{y})$ with remainder, to rewrite the integral in terms of second order dependency of $y \in H$. We have

$$f_m(\tilde{x}+\tilde{y}) = f_m(\tilde{x}) + \sum_{i=1}^m (D_{\xi_i} f_m)(\tilde{x})\langle \xi_i, y\rangle + \sum_{i,j \leq m} (D_{\xi_i} D_{\xi_j} f_m)(\tilde{x})\langle \xi_i, y\rangle\langle \xi_j, y\rangle + R_f^x(y),$$

where we denote the partial derivative in direction ξ as D_ξ and the second order remainder term as R_f^x, for which we have $\frac{R_f^x(y)}{|y|^2} \underset{y\to 0}{\longrightarrow} 0$.
Recall that in our case the inner product $\langle (D_x\psi)(t,x), y\rangle$, is in fact the dualization bracket ${}_{H^*}\langle (D_x\psi)(t,x), y\rangle_H$. Thus in this case we have

$${}_{H^*}\langle (D_x f_m)(\tilde{x}), \sum_{i=1}^m \langle y, \xi_i\rangle \xi_i\rangle_H = \sum_{i=1}^m (D_{\xi_i} f_m)(\tilde{x})\langle y, \xi_i\rangle.$$

Now our initial equation simplifies to

$$
\begin{aligned}
&\int\limits_{\{|y|<\epsilon\}} \left[(\psi(t,x+y)-\psi(t,x)) - \frac{\langle (D_x\psi)(t,x),y\rangle}{1+|y|^2}\right] M(\mathrm{d}y) \\
=&\phi(t) \int\limits_{\{|y|<\epsilon\}} \Bigg[\left(\sum_{i=1}^{m}(D_{\xi_i}f_m)(\tilde{x})\langle \xi_i,y\rangle + \sum_{i,j\leq m}(D_{\xi_i}D_{\xi_j}f_m)(\tilde{x})\langle \xi_i,y\rangle\langle \xi_j,y\rangle + R_f^x(y)\right) \\
&\qquad\qquad - \frac{{}_{H^*}\langle (D_x f_m(\tilde{x}))),y\rangle_H}{1+|y|^2}\Bigg] M(\mathrm{d}y) \\
=&\phi(t) \int\limits_{\{|y|<\epsilon\}} \Bigg[\left(1-\frac{1}{1+|y|^2}\right)\cdot\sum_{i=1}^{m}(D_{\xi_i}f_m)(\tilde{x})\langle \xi_i,y\rangle \\
&\qquad\qquad + \sum_{i,j\leq m}(D_{\xi_i}D_{\xi_j}f_m)(\tilde{x})\langle \xi_i,y\rangle\langle \xi_j,y\rangle + R_f^x(y)\Bigg] M(\mathrm{d}y) \\
\leq&\phi(t) \int\limits_{\{|y|<\epsilon\}} \left[\left(1-\frac{1}{1+|y|^2}\right)k_0 + \sum_{i,j\leq m}(D_{\xi_i}D_{\xi_j}f_m)(\tilde{x})\langle \xi_i,y\rangle\langle \xi_j,y\rangle + R_f^x(y)\right] M(\mathrm{d}y) \\
\leq&\phi(t) \int\limits_{\{0<|y|<\epsilon\}} \left[\left(1-\frac{1}{1+|y|^2}\right)k_0 + k_1\langle \xi_i,y\rangle\langle \xi_j,y\rangle + \frac{R_f^x(y)}{|y|^2}|y|^2\right] M(\mathrm{d}y) \\
=&\phi(t) \int\limits_{\{0<|y|<\epsilon\}} \left[\left(1-\frac{1}{1+|y|^2}\right)k_0 + k_1\langle \xi_i,y\rangle\langle \xi_j,y\rangle + k_2|y|^2\right] M(\mathrm{d}y) \\
\leq&\phi(t)(k_1+k_2+k_0) \int\limits_{\{0<|y|<\epsilon\}} \left[\left(1-\frac{1}{1+|y|^2}\right) + \langle \xi_i,y\rangle\langle \xi_j,y\rangle + |y|^2\right] M(\mathrm{d}y).
\end{aligned}
$$

Here we have existence of the integral since we have for small y

$$\left(\left(1-\frac{1}{1+|y|^2}\right) + \langle \xi_i,y\rangle\langle \xi_j,y\rangle + |y|^2\right)\frac{1}{|y|^2} \leq c, \tag{3.12}$$

for some $c>0$ and thus

$$\int\limits_{\{0<|y|<\epsilon\}} \left(1-\frac{1}{1+|y|^2}\right) + \langle \xi_i,y\rangle\langle \xi_j,y\rangle + |y|^2 M(\mathrm{d}y) \leq c \int\limits_{\{0<|y|<\epsilon\}} |y|^2 M(\mathrm{d}y) < \infty.$$

Let us now write the differences for x and x'

$$\begin{aligned}
J_<(t,x) - J_<(t,x') = \phi(t) \int\limits_{\{0<|y|<\epsilon\}} & \Bigg[(1 - \frac{1}{1+|y|^2}) \cdot \sum_{i=1}^{m}((D_{\xi_i} f_m)(\tilde{x}) - (D_{\xi_i} f_m)(\tilde{x}'))\langle \xi_i, y\rangle \\
& + \sum_{i,j\leq m} \left((D_{\xi_i} D_{\xi_j} f_m)(\tilde{x}) - (D_{\xi_i} D_{\xi_j} f_m)(\tilde{x}')\right) \langle \xi_i, y\rangle\langle \xi_j, y\rangle \\
& + \frac{(R_f^x(y) - R_f^{x'}(y))}{|y|^2}|y|^2 \Bigg] M(\mathrm{d}y) \\
& \leq \phi(t) \int\limits_{\{0<|y|<\epsilon\}} \left[C_6(1 - \frac{1}{1+|y|^2}) + C_4\langle \xi_i, y\rangle\langle \xi_j, y\rangle + C_5|y|^2\right] M(\mathrm{d}y)
\end{aligned}$$

where we choose ϵ such that

$$\frac{(R_f^x(y) - R_f^{x'}(y))}{|y|^2} < C_5 = \frac{\delta}{6\phi(t)k_1},$$

which holds since the Taylor remainder R_f^x has the property $\lim_{y\to 0} \frac{R_f^x(y)}{y^2} = 0$ for both x and x'. Further we can choose x' and x close enough to have

$$\left(\sum_{i,j\leq m} (D_{\xi_i} D_{\xi_j} f_m)(\tilde{x}) - (D_{\xi_i} D_{\xi_j} f_m)(\tilde{x}')\right) < C_4 = \frac{\delta}{6\phi(t)k_1},$$

since all the partial derivatives of f_m are continuous. We set

$$k_1 := \int\limits_{\{0<|y|<\epsilon\}} \langle \xi_i, y\rangle\langle \xi_j, y\rangle + |y|^2 M(\mathrm{d}y).$$

Again by continuity of the derivatives of f_m we choose x' and x close enough to have

$$\sum_{i=1}^{m}((D_{\xi_i} f_m)(\tilde{x}) - (D_{\xi_i} f_m)(\tilde{x}'))\langle \xi_i, y\rangle \leq C_6 = C_4.$$

Now let us look at the other part

$$
\begin{aligned}
J_{\geq}(t,x) - J_{\geq}(t,x') = \phi(t) \int\limits_{\{\epsilon \leq |y|\}} & \Big[f_m(\tilde{x}+\tilde{y}) - f_m(\tilde{x}'+\tilde{y}) - (f_m(\tilde{x}) - f_m(\tilde{x}')) \\
& - \frac{\langle (D_x f_m)(\tilde{x}) - (D_x f_m)(\tilde{x}'), y \rangle}{1+|y|^2} \Big] M(\mathrm{d}y) \\
\leq \phi(t) \int\limits_{\{\epsilon \leq |y|\}} & (C_3 + C_2 + C_1)\, P_M^{-1} \circ M(\mathrm{d}y)
\end{aligned}
$$

where we can choose $|x - x'|$ small enough, such that for all $y \in H$

$$
\begin{aligned}
|f_m(\tilde{x}+\tilde{y}) - f_m(\tilde{x}'+\tilde{y})| < C_3 = \frac{\delta}{\phi(t) k_2 6}, \\
|(f_m(\tilde{x}) - f_m(\tilde{x}'))| < C_4 = C_3, \\
\frac{\langle (D_x f_m)(\tilde{x}) - (D_x f_m)(\tilde{x}'), y \rangle}{1+|y|^2} < C_1 = C_3,
\end{aligned}
$$

since f_m and its partial derivatives are uniformly continuous. We set $k_2 := M(\{\epsilon < |x|\})$.
Finally we get to

$$
\begin{aligned}
& |J(t,x) - J(t,x')| = |J_{<}(t,x) - J_{<}(t,x') + J_{\geq}(t,x) - J_{\geq}(t,x')| \\
& \leq \left| \phi(t) \int\limits_{\{0<|y|<\epsilon\}} \left(C_6 (1 - \frac{1}{1+|y|^2}) + C_4 \langle \xi_i, y \rangle \langle \xi_j, y \rangle + C_5 |y|^2 \right) M(\mathrm{d}y) \right| \\
& + \left| \phi(t) \int\limits_{\{\epsilon<|x|\}} (C_3 + C_2 + C_1) M(\mathrm{d}y) \right| \\
& \leq |\phi(t)| ((C_4 + C_5 + C_6) \int\limits_{\{0<|y|<\epsilon\}} \left(|\langle \xi_i, y \rangle \langle \xi_j, y \rangle| + |y|^2 \right) M(\mathrm{d}y) \\
& \qquad + (C_3 + C_2 + C_1) M(\{\epsilon < |x|\})) \\
& \leq \phi(t) ((C_4 + C_5 + C_6) K + (C_3 + C_2 + C_1) M(\{\epsilon < |x|\})) < \delta.
\end{aligned}
$$

□

Claim 3.11. *The function*

$$
J_2 : x \to \frac{1}{2} \int_H \langle \xi, Q\xi \rangle e^{i\langle \xi, x \rangle} \nu_t(d\xi)
$$

is continuous.

Proof. Let us first simplify our equation by using the structure of our test functions to reduce the problem to a finite subspace:

$$\begin{aligned}\frac{1}{2}\int_H \langle \xi, Q\xi\rangle e^{i\langle \xi,x\rangle}\nu_t(\mathrm{d}\xi) &= \phi(t)\frac{1}{2}\int_H \langle \xi, Q\xi\rangle e^{i\langle \xi,x\rangle}\nu_m(\mathrm{d}\xi)\\ &= \phi(t)\frac{1}{2}\int_{\mathbb{R}^m} \langle \Pi_m\xi, Q\Pi_m\xi\rangle e^{i\langle \Pi_m\xi,x\rangle}g_m(\xi)\mathrm{d}\xi.\end{aligned}$$

Now we can write

$$\begin{aligned}|J_2(x)-J_2(x')| &= \left|\phi(t)\frac{1}{2}\int_{\{|\xi|>k\}} \langle \Pi_m\xi, Q\Pi_m\xi\rangle (e^{i\langle \Pi_m\xi,x\rangle}-e^{i\langle \Pi_m\xi,x'\rangle})g_m(\xi)\mathrm{d}\xi\right|\\ &+\left|\phi(t)\frac{1}{2}\int_{\{|\xi|\le k\}} \langle \Pi_m\xi, Q\Pi_m\xi\rangle (e^{i\langle \Pi_m\xi,x\rangle}-e^{i\langle \Pi_m\xi,x'\rangle})g_m(\xi)d\xi\right|\\ &\le \frac{\delta}{2}+\frac{\delta}{2},\end{aligned}$$

where the first part of the inequality is true for a sufficiently large k, since g_m is a Schwartz function. For the second part we are restricted to a compact set and $e^{i\langle\,\cdot\,,x\rangle}$ is continuous, therefore we have uniform continuity, which allows us to choose $|\tilde{x}-\tilde{x}'|\le\epsilon$ such that for all $\xi\in\{|\xi|\le k\}$

$$\left|e^{i\langle\xi,\tilde{x}\rangle}-e^{i\langle\xi,\tilde{x}'\rangle}\right| \le \frac{\delta}{2\cdot\pi k^m \sup_{\{|\xi|\le k\}}\langle \Pi_m\xi, Q\Pi_m\xi\rangle \sup_{\{r\le k\}} g_m(r)\sup_{t\in[s,T]}\phi(t)}.$$

□

Claim 3.12. *The measure* $\mu_t(\mathrm{d}x)\mathrm{d}t$ *solves* (FPE).

Proof. We can use the equivalent formulation (FPE2) of the Fokker-Planck equation to restate (FPE_α) using the sequence α_n from Step 2

$$\int_s^T\int_H L_{\alpha_n}\psi(t,x)\mu_t^{\alpha_n}(\mathrm{d}x)\mathrm{d}t = -\int_H \psi(s,x)\xi(\mathrm{d}x) \text{ for all } n\in\mathbb{N}.$$

Thus if we could show that

$$\lim_{n\to\infty}\int_s^T\int_H L_{\alpha_n}\psi(t,x)\mu_t^{\alpha_n}(\mathrm{d}x)\mathrm{d}t = \int_s^T\int_H L_0\psi(t,x)\mu_t(\mathrm{d}x)\mathrm{d}t$$

for all $\psi \in W_{T,A}$, we would have shown that $\mu_t(\mathrm{d}x)\mathrm{d}t$ solves our initial (FPE). Therefore our focus lies on proving:

$$\int_s^T \int_H L_0\psi(t,x)\mu_t(\mathrm{d}x)\mathrm{d}t \stackrel{!}{=} \lim_{n\to\infty} \int_s^T \int_H \Bigg[D_t\psi(t,x) + -\frac{1}{2}\int_H \langle \xi, Q\xi\rangle e^{i\langle \xi,x\rangle}\nu_t(\mathrm{d}\xi)$$
$$+ \int_H \left(\psi(t,x+y) - \psi(t,x) - \frac{\langle D_x\psi(t,x), y\rangle}{1+|y|^2}\right) M(\mathrm{d}y)$$
$$+ \langle x, A^* D_x\psi(t,x)\rangle + \langle F_\alpha(t,x), D_x\psi(t,x)\rangle \Bigg] \mu_t^{\alpha_n}(\mathrm{d}x)\mathrm{d}t.$$

We now show the convergence part by part. We begin with:

$$\lim_{n\to\infty} \int_s^T \int_H D_t\psi(t,x)\mu_t^{\alpha_n}(\mathrm{d}x)\mathrm{d}t = \int_s^T \int_H D_t\psi(t,x)\mu_t(\mathrm{d}x)\mathrm{d}t.$$

This convergence holds, since $D_t\psi(t,x)$ is a bounded continuous function on $(s,T)\times H$ and thus the weak convergence of $\mu_t^{\alpha_n}\mathrm{d}t$ yields this equality.
For the "jump parts" of the Kolmogorov operator we have already shown continuity(see Claim 3.10, Claim 3.11) in space and continuity in time follows by linearity of the integrals and by continuity of $\psi(\,\cdot\,,x) = \phi(\,\cdot\,)f_m(\tilde{x})$ on $[s,T]$, with $\phi \in \mathcal{C}^2([s,T])$. In Step 2 we have shown boundednes for both parts (see (3.10) and (3.8)). So we have by weak convergence:

$$\lim_{n\to\infty} \int_s^T \int_H -\frac{1}{2}\int_H \langle \xi, Q\xi\rangle e^{i\langle \xi,x\rangle}\nu_t(\mathrm{d}\xi)\mu_t^{\alpha_n}(\mathrm{d}x)\mathrm{d}t = \int_s^T \int_H -\frac{1}{2}\int_H \langle \xi, Q\xi\rangle e^{i\langle \xi,x\rangle}\nu_t(\mathrm{d}\xi)\mu_t(\mathrm{d}x)\mathrm{d}t$$

and

$$\lim_{n\to\infty} \int_s^T \int_H \int_H \left[\psi(t,x+y) - \psi(t,x) - \frac{\langle D\psi(t,x), y\rangle}{1+|y|^2}\right] M(\mathrm{d}y)\mu_t^{\alpha_n}(\mathrm{d}x)\mathrm{d}t$$
$$= \int_s^T \int_H \int_H \left[\psi(t,x+y) - \psi(t,x) - \frac{\langle D\psi(t,x), y\rangle}{1+|y|^2}\right] M(\mathrm{d}y)\mu_t(\mathrm{d}x)\mathrm{d}t$$

Finally we come to the drift part, which will be more complicated.

Remark 3.13. This part will follow the approach of [BDPR10] with a changed test function space and changed approximation steps. The changes in the test function space make it necessary to change the approximation approach of the test functions. In [BDPR10] it was necessary to approximate the spatial parts of the test functions by affine linear functions. This approximation is not necessary in our case, since the spatial depence is restricted to a finite dimensional subspace of H and not as in [BDPR10] depending on the whole of H.

We have to show the following equality

$$\begin{aligned}\lim_{n\to\infty}\int_s^T\int_H &\left[\langle x, A^* D_x\psi(t,x)\rangle + \langle F_\alpha(t,x), D_x\psi(t,x)\rangle\right]\mu_t^{\alpha_n}(\mathrm{d}x)\mathrm{d}t\\ &\stackrel{!}{=}\int_s^T\int_H \left[\langle x, A^* D_x\psi(t,x)\rangle + \langle F(t,x), D_x\psi(t,x)\rangle\right]\mu_t(\mathrm{d}x)\mathrm{d}t\end{aligned}$$

Let us first elaborate on the role of the test functions. Fix $\psi \in W_{T,A}$ and recall that the bracket $\langle F(t,x), D_x\psi(t,x)\rangle$ from L_0 is not an inner product, but in this case the duality bracket

$$\langle F(t,x), D_x\psi(t,x)\rangle := {}_H\langle F(t,x), D_x\psi(t,x)\rangle_{H^*}.$$

Let us take a closer look at $D_x\psi$. We first compute

$$(D_x\psi)(t,x) = \phi(t)(D_x(f_m\circ\tilde{P}_m))(x)\cdot = \phi(t)(\mathrm{D}\, f_m)(\tilde{P}_m(x))\circ\tilde{P}_m(\,\cdot\,),$$

where $(\mathrm{D}\, f_m)(\tilde{P}_m(x))$ is the total derivative of $f_m : \mathbb{R}^m \to \mathbb{R}$ at $\tilde{P}_m(x)$. As we can see $D_x\psi(t,x) \in H^*$ for each $x \in H$.
We rewrite the duality bracket

$$\begin{aligned}{}_H\langle F(t,x), D_x\psi(t,x)\rangle_{H^*} &= D_x\psi(t,x)(F(t,x)) = D_x\psi(t,x)(\sum_{i=1}^{\infty}\langle F(t,x),\xi_i\rangle\xi_i)\\ &= \sum_{i=1}^{m}\langle F(t,x),\xi_i\rangle\underbrace{{}_{H^*}\langle D_x\psi(t,x),\xi_i\rangle_H}_{=(D_\xi\psi)(t,x)}.\end{aligned} \tag{3.13}$$

We can see that ${}_H\langle F_\alpha(t,x)D_x\psi(t,x)\rangle_{H^*}$ is continuous and bounded, sine we have that $\langle F_\alpha(t,x),\xi_i\rangle$ is continuous and bounded by condition 2.5(c) and that $(D_\xi\psi)(t,x)$

is continuous due to ψ being a Schwartz function. Similarly we get that

$$\begin{aligned}
{}_H\langle x, AD_x\psi(t,x)\rangle_{H^*} &= {}_H\langle \sum_{i=1}^{\infty}\langle x,\xi_i\rangle\xi_i, AD_x\psi(t,x)\rangle_{H^*} \\
&= \sum_{i=1}^{m}\langle x,\xi_i\rangle\, {}_H\langle A\xi_i, D_x\psi(t,x)\rangle_{H^*} \\
&= \sum_{i=1}^{m}\langle x,\xi_i\rangle\lambda_i D_{\xi_i}\psi(t,x)
\end{aligned}$$

is continuous, where λ_i are the eigenvalues of A corresponding to ξ_i. We can see that we have no problem with the discontinuity of A, since we only need to evaluate it on a small subspace. Thus ${}_H\langle Ax, D_x\psi(t,x)\rangle_{H^*}$ is also continuous and bounded, which we have since $D_{\xi_i}\psi(t,x)$ is a Schwartz function and therefore rapidly decreases as x approaches infinity.
We can now prove our equation by showing for all $g \in C_b([s,T]\times H)$

$$\begin{aligned}
&\lim_{n\to\infty}\int\limits_s^T\int\limits_H {}_H\langle x, AD_x\psi(t,x)\rangle_{H^*}g(t,x) + {}_H\langle F_{\alpha_n}(t,x), D_x\psi(t,x)\rangle_{H^*}g(t,x)\mu_t^{\alpha_n}(\mathrm{d}x)\mathrm{d}t \\
&\quad= \int_s^T\int_H {}_H\langle x, AD_x\psi(t,x)\rangle_{H^*}g(t,x) + {}_H\langle F(t,x), D_x\psi(t,x)\rangle_{H^*}g(t,x)\mu_t(\mathrm{d}x)\mathrm{d}t,
\end{aligned}$$

since this would establish weak convergence for these integrals.
Now we set for better readability

$$\begin{aligned}
F_\alpha^\psi(t,x) &:= {}_H\langle F_\alpha(t,x), D_x\psi(t,x)\rangle_{H^*} + {}_H\langle x, AD_x\psi(t,x)\rangle_{H^*} \\
F^\psi(t,x) &:= {}_H\langle F(t,x), D_x\psi(t,x)\rangle_{H^*} + {}_H\langle x, AD_x\psi(t,x)\rangle_{H^*}
\end{aligned}$$

Remark 3.14. In [BDPR10] $F_\alpha^\psi(t,x)$ was defined quite different to manage the unboundedness of A, which is not necessary in our case since each test-function ψ takes only values on a finite subspace of H and thus the unboundedness of A poses no problem while showing continuity of $F_\alpha^\psi(t,x)$
We rewrite the above equation to

$$\begin{aligned}
&\lim_{n\to\infty}\int_s^T\int\limits_H F_{\alpha_n}^\psi(t,x)g(t,x)\mu_t^{\alpha_n}(\mathrm{d}x)\mathrm{d}t \\
&= \int\limits_s^T\int\limits_H F^\psi(t,x)g(t,x)\mu_t(\mathrm{d}x)\mathrm{d}t
\end{aligned}$$

We proceed to compute

$$
\begin{aligned}
&\int_s^T \int_H \left| F_\beta^\psi(t,x) - F^\psi(t,x) \right| \mu_t^\alpha(\mathrm{d}x)\mathrm{d}t \\
&\quad \leq \int_s^T \int_H |\langle D_x\psi(t,x), F_\beta(t,x) - F(t,x)\rangle| \\
&\qquad\qquad\qquad + |\langle AD_x\psi(t,x), x\rangle - \langle AD_x\psi(t,x), x\rangle| \, \mu_t^\alpha(\mathrm{d}x)\mathrm{d}t \\
&\quad \leq \int_s^T \int_H |\langle D_x\psi(t,x), F_\beta(t,x) - F(t,x)\rangle| \, \mu_t^\alpha(\mathrm{d}x)\mathrm{d}t \\
&\quad \leq \int_s^T \int_H \left| \phi(t) \sum_{i=1}^m \langle F_\beta(t,x) - F(t,x), \xi_i\rangle \cdot {}_H\langle \xi_i, Df_m \circ P_m(x)\rangle_{H^*} \right| \mu_t^\alpha(\mathrm{d}x)\mathrm{d}t \\
&\quad \leq \beta\gamma(\psi) \int_s^T \int_H |F(t,x)|^2 \, \mu_t^\alpha(\mathrm{d}x)\mathrm{d}t \leq \beta\gamma(\psi) \int_s^T \int_H V^2(t,x)\xi(\mathrm{d}x)\mathrm{d}t,
\end{aligned}
\tag{3.14}
$$

where we used Condition 2.6 and that by Condition 2.5(c) we have that

$$
\begin{aligned}
&\left| \phi(t) \sum_{i=1}^m \langle F_\beta(t,x) - F(t,x), \xi_i\rangle \cdot {}_H\langle \xi_i, Df_m \circ P_m(x)\rangle_{H^*} \right| \\
&\qquad \leq \underbrace{\sup_{[s,T]} \phi(t) \cdot m \cdot \max_{1\leq i\leq m} \sup_{x\in H} D_{\xi_i} f_m(\tilde{x}) \cdot m \cdot \max_{1\leq i\leq m} c(\xi_i)}_{:=\gamma(\psi)} \beta \, |F(t,x)|^2 < \infty.
\end{aligned}
$$

Where $\sup_{x\in H} D_{\xi_i} f_m(\tilde{x}) < \infty$ by virtue of f_m being a Schwartz function. We see that the right-hand side of (3.14) tends to 0 independently of α as $\beta \to 0$.
By Lemma 3.9 we can infer from (3.14) that we also have

$$
\begin{aligned}
&\int_s^T \int_H \left| F_\beta^\psi(t,x) - F^\psi(t,x) \right| \mu_t(\mathrm{d}x)\mathrm{d}t \\
&\qquad \leq \beta\gamma(\psi) \int_s^T \int_H V^2(t,x)\xi(\mathrm{d}x)\mathrm{d}t
\end{aligned}
\tag{3.15}
$$

Thus we now conclude that

$$
\begin{aligned}
&\left|\int_s^T\int_H F_{\alpha_n}^{\psi}(t,x)g(t,x)\mu_t^{\alpha_n}(\mathrm{d}x)\mathrm{d}t-\int_s^T\int_H F^{\psi}(t,x)g(t,x)\mu_t(\mathrm{d}x)\mathrm{d}t\right|\\
&\le \|g\|_\infty\int_s^T\int_H \left|F_{\alpha_n}^{\psi}(t,x)-F^{\psi}(t,x)\right|\mu_t^{\alpha_n}(\mathrm{d}x)\mathrm{d}t\to 0 \text{ by (3.14) as } n\to\infty\\
&+\|g\|_\infty\int_s^T\int_H \left|F^{\psi}(t,x)-F_{\delta}^{\psi}(t,x)\right|\mu_t^{\alpha_n}(\mathrm{d}x)\mathrm{d}t\to 0 \text{ by (3.14) as } \delta\to 0\\
&+\|g\|_\infty\int_s^T\int_H \left|F^{\psi}(t,x)-F_{\delta}^{\psi}(t,x)\right|\mu_t(\mathrm{d}x)\mathrm{d}t\to 0 \text{ by (3.15) as } \delta\to 0\\
&+\underbrace{\left|\int_s^T\int_H F_{\delta}^{\psi}(t,x)g(t,x)\mu_t^{\alpha_n}(\mathrm{d}x)\mathrm{d}t-\int_s^T\int_H F_{\delta}^{\psi}(t,x)g(t,x)\mu_t(\mathrm{d}x)\mathrm{d}t\right|}_{\to 0 \text{ by weak convergence and the boundedness and continuity of } F_{\delta}^{\psi}},
\end{aligned}
$$

which is precisely what we wanted to show. $\square$

Appendix

Special Theorems

Theorem 3.15 (Prohorov's theorem). *Let $\mathcal{K} \subset \mathcal{M}_r(X)$ be a uniformly bounded in the variation norm and uniformly tight family of measures on a completely regular space X. Then $\mathcal{X}$ has a compact closure in the weak topology.*
If, in addition, for every $\epsilon \geq 0$, there exists a metrizable compact set K_ϵ such that $|\mu|(X \setminus K_\epsilon) \leq \epsilon$ for all $\mu \in \mathcal{K}$(which is the case if all compact subsets of X are metrizable), the every sequence in $\mathcal{K}$ contains a weakly convergent subsequence.

Taken from and proof contained in [Bogachev, 8.6.7].

Theorem 3.16 (Gronwall's inequality). *Let μ be a Borel measure on $[0, \infty[$, let $\epsilon \geq 0$, and let f be a Borel measurable function that is bounded on bounded intervals and satisfies*

$$0 \leq f(t) \leq \epsilon + \int\limits_{[0,t[} f(s)\mu(\mathrm{d}s), \text{ for } t \geq 0$$

Then

$$f(t) \leq \epsilon e^{\mu([0,t[)}, t \geq 0$$

Taken from and proof contained in [EK86, App. 5.1].

Definition 3.17 (Monotone vector space). A monotone vector space $\mathcal{H}$ on space Ω is defined to be a collection of bounded, real-valued functions f on Ω satisfying the three conditions:

(i) $\mathcal{H}$ is a vector space over $\mathbb{R}$;

(ii) $1 \in \mathcal{H}$ (i.e., constant functions are in $\mathcal{H}$); and

(iii) it $(f_n)_{n\geq 1} \subset \mathcal{H}$, and $0 \leq f_1 \leq f_2 \leq \ldots \leq f_n \leq \ldots$, and $\lim_{n\to\infty} f_n = f$ and f is bounded, then $f \in \mathcal{H}$.

Taken from [Pro05, P. 7].

Theorem 3.18 (Monotone class theoerem (funtional version))**.** *Let $\mathcal{H}$ be class of bounded functions from a set S into $\mathbb{R}$ satisfying the following conditions:*

(i) $\mathcal{H}$ is a vector space over $\mathcal{H}$;

(ii) the constant function 1 is an element of $\mathcal{H}$;

(iii) if (f_n) is a sequence of non-negative functions in $\mathcal{H}$ such that $f_n \uparrow f$ where f is bounded function on S, then $f \in \mathcal{H}$.

Then if $\mathcal{H}$ contains the indicator function of every set in some π-system $\mathcal{I}$, then $\mathcal{H}$ contains every bounded $\sigma(\mathcal{I})$-measurable function on S.

Taken from and proof contained in [Wil94, Thm. 3.14].

Theorem 3.19 (Monotone class theorem (functional-algebra version))**.** *Let $\mathcal{M}$ be a multiplicative class of bounded real-valued functions defined on a space Ω and let $\mathcal{A} = \sigma\{\mathcal{M}\}$. If $\mathcal{H}$ is a monotone vector space containing $\mathcal{M}$, then $\mathcal{H}$ contains all bounded $\mathcal{A}$ measurable functions.*

Taken from and proof contained in [Pro05, Thm. 8].

Elementary inequalities

Lemma 3.20. *For $a, b \in \mathbb{R}$ we have*

$$a^2 - 2b^2 \leq 2\left|a-b\right|^2$$

Proof. Easily seen by $(a+b)^2 \leq 2a^2 + 2b^2$(equiv. to Hölder) applied to $|a-b+b|$. □

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