Thesis

Stochastic Analysis related to Gamma measures

- Gibbs perturbations and associated Diffusions

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Dennis Hagedorn aus Minden (Westf.)
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Summary

In this thesis (consisting of Parts I - III) we study Gamma measures located on the cone $\mathbb{K}(\mathbb{R}^d)$ of discrete Radon measures. They form, as well as the Gaussian and Poisson measures, an important class of measures on infinite dimensional spaces and appeared in the representation theory of groups. In the present thesis, the following topics of Gamma analysis are developed:

- Construction of Gibbs perturbations for the Gamma measures
- Differential structure on the cone $\mathbb{K}(\mathbb{R}^d)$
- Integration by parts formulas for Gamma and Gibbs measures
- Construction of associated diffusions

In Part I, we define a homeomorphism $\mathbb{T}$ between the cone $\mathbb{K}(\mathbb{R}^d)$ and a subset of the configuration space $\Gamma(\hat{\mathbb{R}}^d)$ over the product space $\hat{\mathbb{R}}^d$ of marks in $\mathbb{R}_+ := (0,\infty)$ and positions in $\mathbb{R}^d$. This subset consists of pinpointing configurations with finite local mass. Then we construct Gamma measures on $\mathbb{K}(\mathbb{R}^d)$ as image measures, under $\mathbb{T}$, of proper Poisson measures on $\Gamma(\hat{\mathbb{R}}^d)$.

In Part II, we establish Gibbs perturbations of Gamma measures w.r.t. a pair potential that describes the interaction of particles and satisfies certain stability properties: We follow the Dobrushin-Lanford-Ruelle approach to Gibbs random fields in classical statistical mechanics and introduce the corresponding Gibbs formalism on the cone. Proving the existence of the Gibbs measures on the cone $\mathbb{K}(\mathbb{R}^d)$ is a non-trivial problem, even for a non-negative potential. We know about the cone $\mathbb{K}(\mathbb{R}^d)$ less than about the configuration space $\Gamma(\hat{\mathbb{R}}^d)$, hence we transfer the problem to $\Gamma(\hat{\mathbb{R}}^d)$ via the homeomorphism $\mathbb{T}^{-1}$. Even on $\Gamma(\hat{\mathbb{R}}^d)$, the transfered potential with infinite range does not fit the standard framework because of the high concentration close to 0 of the underlying intensity measure on $\mathbb{R}_+$. We develop analytic techniques, involving Lyapunov functionals and weak dependence on boundary conditions, to construct Gibbs measures on $\Gamma(\hat{\mathbb{R}}^d)$ and characterize sets supporting them. Using the homeomorphism $\mathbb{T}$, we establish the existence of Gibbs perturbations on the cone.

To obtain diffusions on the cone, in Part III, we introduce a gradient which consists of extrinsic and intrinsic parts. They correspond to the motion of marks and positions of particles, respectively. An important result here (and a new issue in infinite dimensions) is an integration by parts formula without an underlying quasi-invariance property of the involved Gamma measure. Next, we study conservative gradient Dirichlet forms of Gibbs measures constructed in Part II. To check their quasi-regularity, we define a Polish space, in which we embed the cone. Therefore, we study *a priori* diffusions on the Polish space. A crucial issue here is that the diffusions are actually located on a subset of $\mathbb{T}^{-1}(\mathbb{K}(\mathbb{R}^d))$. Using this fact and the homeomorphism $\mathbb{T}$, we construct diffusions on the cone. In particular, we get an example of diffusions describing the motion of densely distributed particles.

Contents

Chapter 1

Introduction

Handling and modeling complex systems have become an essential part of modern science. For a long time, complex systems have been treated in physics, where e.g. methods of probability theory are used to determine their macroscopic behavior by their microscopic properties. Nowadays complex systems, ranging from e.g. ecosystems to the climate, biological populations, societies and financial markets, play an important role in various fields such as biology, chemistry, robotics, computer science and even social science.

A mathematical tool to study these systems is infinite dimensional analysis. Widely applied, e.g. in financial mathematics and mathematical physics, are Gaussian and Poissonian analysis corresponding to Gauss, resp. Poisson measures. We develop some Gamma analysis related to Gamma measures, which may serve to model biological systems.

The mentioned measures are infinite dimensional analogues of measures classified by Meixner. A first step in the related analysis is to study sets supporting them. In particular, Gaussian measures are located on linear spaces; Poisson measures are supported by the space of locally finite configurations. And Gamma measures have full mass on the cone of locally finite, discrete Radon measures. The analysis developed for the Gaussian and Poissonian measures includes chaos decompositions, differential structures on the underlying spaces, corresponding Dirichlet forms and associated diffusions, whereas for the Gamma measures a chaos decomposition and a quasi-invariance property w.r.t. multiplication of marks is known. One of our aims is to introduce a differential structure on the cone, construct Dirichlet forms and get associated diffusions on the cone.

An important feature of complex systems is the interaction of their components. Let us exemplify this with a prominent physical example, namely a gas: To model a free gas, Poisson measures are used. 'Free' means that any interaction of the molecules is absent. But, the molecules of real gases interact with each other. To model this, the notion of Gibbs perturbations of Poisson measures has been introduced and studied. Considering Gamma measure as states of free systems, we will also study Gibbs perturbations of Gamma measures.

A mathematical model for the above mentioned many-particle systems, namely spaces of locally finite configurations, appeared first in statistical mechanics. Such a configuration space describes the positions of *identical* particles in a phase space, e.g. $\mathbb{R}^d$. Here, *locally finite* means that there are only finitely many particles in any compact area. To describe the allocation of particles a Poisson measure with a certain intensity measure can be used. It distributes the particles independently of each other (cf. e.g. Subsection 1.1.2 or Chapter 2).

As mentioned above, particles may interact and influence each other. Gibbs measures are suitable to describe this phenomena. In the late 1960s Dobrushin, Lanford and Ruelle introduced the mathematical setting for Gibbs measures that are used to describe equilibrium states of infinitely large systems (cf. [Dob68, Dob70b, LR69, Rue69]), which strongly encouraged the development of the theory of Markov random fields (cf. [Geo88, Pre76]). Generally speaking, one distinguishes between two main classes of Gibbs measures, namely, *spin systems on graphs* or *discrete metric spaces* (cf., e.g., [Lan20, Isi25, Geo88]) and *particle systems in continuum*, e.g., in $\mathbb{R}^d$ (cf., e.g., [AKR98b, Kun99, AKPR06]).

We will treat Gibbs measures for particle systems in the continuum $\mathbb{R}^d$, $d \in \mathbb{N}$, and an attached space of marks $\mathbb{R}_+ := [0, \infty)$ with an *infinite* measure on the marks. Thus, we extend models treated in the second class.

Standard references for the theory of Gibbs measures are [Geo88, Pre76]. More recent ones are [AKPR06] (an overview) and [KPR10] (an analytic approach).

Aims

We will develop some structure of the Gamma analysis (cf. also Section 1.2):

- We study Gibbs perturbations of the Gamma measures,
- introduce a differential structure on the cone of discrete measures,
- establish integration by parts formulas and
- construct diffusions corresponding to associated Dirichlet forms.

The differential structure on the cone will turn out to be richer than the one on the linear spaces on which the Gaussian measures are located and the one on the configuration spaces which support the Poisson measures. It is richer in the sense that there exists an extrinsic differential structure related to the marks (or values) of the discrete measures and an intrinsic one corresponding to their support or positions.[1]

1.1 Infinite dimensional analysis

We will repeat very briefly some of the known results of Gaussian, Poissonian and Gamma analysis. Although, we focus on the Gamma analysis, for the convenience of the reader, we also repeat some of the known results for the Gaussian and Poisson case. This allows us to compare the different situations more easily. But before, we describe in which sense they are similar.

Meixner classification

In 1933 Meixner studied functions of the type $\exp[xu(t)]/f(t)$ where $t \mapsto u(t)$ and $t \mapsto f(t)$ are analytic functions. He found that there only exist five different systems of orthogonal polynomials whose generating functions for related orthogonal polynomials are of this type. The obtained classification yields (cf. [Mei34]) the Gaussian (normal), binomial, Gamma, Poisson and a fifth class. For the Gaussian, Gamma and Poisson measure there exists a generalization to infinite dimensional spaces.

There exists a famous characterization of Fourier transformations for probability measures on Hilbert spaces given by the following theorem:

Theorem 1.1.1 (Minlos, see e.g. [GV64, Section II.3.1])**.** *Let H be a real separable Hilbert space and $k : H \to \mathbb{C}$ be a continuous function with $k(0) = 1$ and which is positive definite on H. The latter means that $\forall\, h_1, \ldots, h_N \in H$, $\forall\, c_1, \ldots, c_N \in \mathbb{C}$:*

$$\sum_{i,j=1}^{N} k(h_i - h_j) c_i \overline{c_j} \geq 0.$$

[1] A point $x \in \mathbb{R}^d$ is called a *position* of such a locally finite, discrete Radon measure η on $\mathbb{R}^d$, if $s_x := \eta(\{x\}) \neq 0$. The set of all positions is called the *support* $\tau(\eta)$. We refer to the value s_x as a *mark*.

Then for every Hilbert-Schmidt extension H_- of H, there exists exactly one probability measure μ on H_- such that

$$\forall \phi \in H_+ : \qquad \tilde{\mu}(\phi) := \int_{H_-} e^{i\langle x,\phi\rangle} \mu(dx) = k(\phi),$$

where $H_+ \subset H \subset H_-$ and H_+ is the dual space of H_-.

For particular measures the support can be classified more precisely, as we will see in the following.

1.1.1 Gaussian analysis

The Gaussian analysis has been studied quite extensively. As a general reference see [BK95a], [Bog98] or [Hid05, pp. 46ff] (a historical overview by Izumi Kubo).

Via the Minlos theorem, one obtains a probability measure μ_0 that corresponds to the functional

$$k(\phi) = e^{-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R}^d,dx)}}, \quad \phi \in L^2(\mathbb{R}^d, dx) = H,$$

It is called the *Gaussian White Noise measure.* The corresponding triple is $\mathcal{S}'(\mathbb{R}^d) \supset L^2(\mathbb{R}^d, dx) \supset \mathcal{S}(\mathbb{R}^d)$ and $\mu_0(\mathcal{S}'(\mathbb{R}^d)) = 1$. Here, $\mathcal{S}(\mathbb{R}^d)$ denotes the space of Schwartz test functions.

Chaos decomposition

Theorem 1.1.2 (Itô-Segal decomposition, cf. [BK95a, Section 2.2f])**.** *Any $F \in L^2(\mathcal{S}', \mu_0)$ can be written as an orthogonal decomposition*

$$F(\omega) = \sum_{n=0}^{\infty} \langle f^{(n)}, :\omega^{\hat{\otimes} n}: \rangle,$$

where $\{\langle f^{(n)}, :\omega^{\hat{\otimes} n}:\rangle\}_{n\in\mathbb{N}}$ denotes the system of generalized Hermite polynomials. The latter may be expressed as multiple stochastic integrals w.r.t. to a Wiener process.

Differential structure

Let us introduce a set of cylindrical functions

$$\begin{aligned} \mathcal{F}C_b^\infty(\mathcal{S}', \mathcal{S}) \; := \; & \{f(\langle\rho_1, \cdot\rangle, \dots, \langle\rho_N, \cdot\rangle) \,|\, N \in \mathbb{N},\, f \in C_b^\infty(\mathbb{R}^N), \\ & \rho_1, \dots, \rho_N \in \mathcal{S}\} \subset L^2(\mathcal{S}', \mu_0). \end{aligned}$$

We fix $F = f(\langle \rho_1, \cdot \rangle, \ldots, \langle \rho_N, \cdot \rangle) \in \mathcal{F}C_b^\infty(\mathcal{S}', \mathcal{S})$ with $N \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^N)$ and $\rho_1, \ldots, \rho_N \in \mathcal{S}$.

Definition 1.1.3. *For $h \in L^2(\mathbb{R}^d, dx)$ define the* directional derivative *as*

$$\nabla_h^{\mathcal{S}'} F(\omega) := \sum_{j=1}^N \frac{\partial}{\partial x_j} f(\langle \rho_1, \omega \rangle, \ldots, \langle \rho_N, \omega \rangle) \langle \rho_j, h \rangle_{T_\omega(\mathcal{S}')}, \quad \omega \in \mathcal{S}',$$

where we set $T_\omega(\mathcal{S}') := L^2(\mathbb{R}^d, dx)$ for each $\omega \in \mathcal{S}'$. The gradient *is*

$$\nabla^{\mathcal{S}'} F(\omega) := \sum_{j=1}^N \frac{\partial}{\partial x_j} f(\langle \rho_1, \omega \rangle, \ldots, \langle \rho_N, \omega \rangle)) \rho_j, \quad \omega \in \mathcal{S}'.$$

As usual,

$$\nabla_h^{\mathcal{S}'} F(\omega) = \langle \nabla^{\mathcal{S}'} F(\omega), h \rangle_{T_\omega(\mathcal{S}')}, \quad \omega \in \mathcal{S}'.$$

The geometry of the space $\mathcal{S}'$ is *flat* because the tangent space at each point is the same.

Quasi-invariance property

Proposition 1.1.4 (Cameron-Martin, see, e.g., [BK95a, Theorem II.2.3]). *For all $h \in L^2(\mathbb{R}^d, dx)$ we have*

$$\frac{d\mu_0(\omega - h)}{d\mu_0(\omega)} = e^{\langle \omega, h \rangle_{T_\omega(\mathcal{S}')} - \frac{1}{2}\|h\|^2_{T_\omega(\mathcal{S}')}}, \quad \omega \in \mathcal{S}'.$$

Here, $\langle \omega, h \rangle$, $\omega \in \mathcal{S}'$ is defined as a measurable linear function μ_0-a.s..

Integration by parts

Via the quasi-invariance formula one gets an integration by parts formula.

Theorem 1.1.5 (cf., e.g., [BK95b, Section IV.3]). *For all $h \in L^2(\mathbb{R}^d, dx)$ and F, $G \in \mathcal{F}C_b^\infty(\mathcal{S}', \mathcal{S})$ we have*

$$\begin{aligned}\int_{\mathcal{S}'} \nabla_h^{\mathcal{S}'} F(\omega)\, G(\omega) \mu_0(d\omega) = &- \int_{\mathcal{S}'} F(\omega) \nabla_h^{\mathcal{S}'} G(\omega)\, \mu_0(d\omega) \\ &+ \int_{\mathcal{S}'} F(\omega) G(\omega) \beta^{\mu_0}(h, \omega) \mu_0(d\omega),\end{aligned}$$

where the logarithmic derivative *is*

$$\omega \mapsto \beta^{\mu_0}(h, \omega) := \langle \omega, h \rangle_{T_\omega(\mathcal{S}')}.$$

Dirichlet forms and associated stochastic dynamics

We define the corresponding gradient bilinear form for all $F, G \in \mathcal{F}C_b^\infty(\mathcal{S}', \mathcal{S})$ as

$$\mathcal{E}^{\mu_0}(F,G) := \int_{\mathcal{S}'} \langle \nabla^{\mathcal{S}'} F(\omega), \nabla^{\mathcal{S}'} G(\omega)\rangle_{T_\omega(\mathcal{S}')} \mu_0(d\omega).$$

It is closable in $L^2(\mathcal{S}', \mu_0)$ and its closure is a conservative symmetric Dirichlet form which is quasi-regular (cf. e.g. [MR92, Corollary II.3.3 and IV.4a)]).

1.1.2 Poisson measure

Already in [VGG75] Poisson measures are given as examples for quasi-invariant ergodic measures on configuration spaces. The Poissonian white noise analysis was developed in [CP90, IK88, NV95, Pri95]. In the late 1990s started some new development in stochastic analysis and differential geometry on configuration spaces (cf. e.g. [AKR98a, AKR98b]), when integration by parts formulas and Dirichlet forms were derived for these measures. The chaos decompositions of Poisson measures is presented in, e.g., [KdSSU98].

Let $m(dx) = \rho(x)dx$ with $\rho \in H^{1,2}_{\mathrm{loc}}(\mathbb{R}^d)$, where dx denotes the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Via the Minlos theorem we obtain for the functional

$$C_0(\mathbb{R}^d) \ni f \mapsto e^{\langle e^{if}-1, m\rangle} = e^{\int (e^{if(x)}-1)m(dx)}$$

the existence of a Poisson measure π_m (with intensity measure m). It is located a priori in a linear space of generalized functions, but has de facto full support on the configuration space over $\mathbb{R}^d$, which we present below.

The configuration space

The configuration space $\Gamma(\mathbb{R}^d)$ over $\mathbb{R}^d$ is defined to be the collection of locally finite subsets of $\mathbb{R}^d$:

$$\Gamma(\mathbb{R}^d) := \{\gamma \subset \mathbb{R}^d \mid |\gamma \cap K| < \infty \text{ for all } K \subset \mathbb{R}^d \text{ compact}\},$$

where $|A|$ denotes the number of elements of a set A.

A configuration can be viewed as a positive measure, i.e.,

$$\Gamma(\mathbb{R}^d) \ni \gamma \mapsto \sum_{x\in\gamma} \delta_x(dy) \in \mathcal{M}(\mathbb{R}^d),$$

where δ_x denotes the Dirac measure at $x \in \mathbb{R}^d$ and $\mathcal{M}(\mathbb{R}^d)$ the set of positive Radon measures on $\mathbb{R}^d$. We equip the configuration space $\Gamma(\mathbb{R}^d)$ with the relative topology as a subset of the space $\mathcal{M}(\mathbb{R}^d)$ with the vague topology: It is the smallest topology such that the following functions are continuous

$$\mathcal{M}(\mathbb{R}^d) \ni \gamma \mapsto \langle f, \gamma \rangle = \int_{\mathbb{R}^d} f(x)\gamma(dx) = \sum_{x \in \gamma} f(x) \in \mathbb{R}, \quad f \in C_0(\mathbb{R}^d).$$

We equip $\Gamma(\mathbb{R}^d)$ with the corresponding Borel σ-algebra $\mathcal{B}(\Gamma(\mathbb{R}^d))$. (For further details cf. Section 2.1 resp. the papers mentioned above).

Remark 1.1.6. *The* Mecke identity *is a useful characterization of the Poisson measure: Let $f : \mathbb{R}^d \times \Gamma(\mathbb{R}^d) \to [0, \infty)$ be a $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\Gamma(\mathbb{R}^d))$-measurable function. Then*

$$\int_{\Gamma(\mathbb{R}^d)} \int_{\mathbb{R}^d} f(x, \gamma)\gamma(dx)\pi_m(d\gamma) = \int_{\mathbb{R}^d} \int_{\Gamma(\mathbb{R}^d)} f(x, \gamma + \delta_x)\pi_m(d\gamma)m(dx).$$

Chaos decomposition

Similarly to the Gaussian case, we have an orthogonal chaos decomposition:

Theorem 1.1.7 (see [KdSSU98, Subsection 2.3])**.** *For all $F \in L^2(\Gamma(\mathbb{R}^d), \pi_m)$, we have*

$$F(\gamma) = \sum_{n=0}^{\infty} \langle C_n^m(\gamma), f^{(n)} \rangle,$$

where the system $\{C_n^m(f^{(n)})(\gamma) = \langle C_n^m(\gamma), f^{(n)} \rangle\}_{n \in \mathbb{N}}$ is called the system of generalized Charlier polynomials for the Poisson measure π_m.

Differential structure

We define the set of cylindrical functions:

$$\begin{aligned} \mathcal{F}C_b^\infty(\Gamma(\mathbb{R}^d), C_0^\infty(\mathbb{R}^d)) \;\; := \;\; & \{g_F(\langle \rho_1, \cdot \rangle, \ldots, \langle \rho_N, \cdot \rangle) | N \in \mathbb{N}, \\ & \qquad g_F \in C_b^\infty(\mathbb{R}^N), \rho_1, \ldots, \rho_N \in C_0^\infty(\mathbb{R}^d)\}. \end{aligned}$$

Let $V_0(\mathbb{R}^d)$ denote the set of all smooth vector fields with compact support.

Definition 1.1.8. *The* directional derivative *of $F \in \mathcal{F}C_b^\infty(\Gamma(\mathbb{R}^d), C_0(\mathbb{R}^d))$ w.r.t. $v \in V_0(\mathbb{R}^d)$ is defined as*

$$\nabla_v^\Gamma F(\gamma) := \sum_{j=1}^{N} \frac{\partial g_F}{\partial s_j}(\langle \varphi_1, \gamma \rangle, \ldots, \langle \varphi_N, \gamma \rangle) \, \langle \nabla_v^{\mathbb{R}^d} \varphi_j, \gamma \rangle_{T_\gamma(\Gamma)}, \tag{1.1.1}$$

where the tangent space *at a configuration* $\gamma \in \Gamma(\mathbb{R}^d)$ *is*

$$T_\gamma(\Gamma) := L^2(\mathbb{R}^d, \gamma).$$

The gradient *is*

$$\nabla^\Gamma F(\gamma) := \sum_{j=1}^N \frac{\partial g_F}{\partial s_j}(\langle \varphi_1, \gamma\rangle, \ldots, \langle \varphi_N, \gamma\rangle)\, \nabla^{\mathbb{R}^d} \varphi_j \qquad \in T_\gamma(\Gamma).$$

Comparing with the Gaussian case, the geometry of $\Gamma(\mathbb{R}^d)$ is *non-flat* because the tangent spaces are different at each point. The gradient and the directional derivative satisfy

$$\nabla_v^\Gamma F(\gamma) = \left\langle \nabla^\Gamma F(\gamma), v\right\rangle_{T_\gamma(\Gamma)}.$$

Quasi-invariance property

Let $\mathrm{Diff}_0(\mathbb{R}^d)$ denote the set of all diffeomorphisms $\phi : \mathbb{R}^d \to \mathbb{R}^d$ with compact support, i.e., there exists a compact set $\Delta \in \mathcal{B}_c(X)$ such that for all $x \in \Delta^c$ we have $\phi(x) = x$.

For each $\phi \in \mathrm{Diff}_0(\mathbb{R}^d)$, we define its lifting

$$\phi : \Gamma(\mathbb{R}^d) \ni \gamma \mapsto \phi(\gamma) := \{\phi(x) | x \in \gamma\} \in \Gamma(\mathbb{R}^d).$$

Proposition 1.1.9 (see [AKR98a, Prop. 2.2.]). *The Poisson measure* π_m *is (quasi-)invariant w.r.t the group* $\mathit{Diff}_0(\mathbb{R}^d)$, *and for any* $\phi \in \mathit{Diff}_0(\mathbb{R}^d)$ *we have*

$$\frac{d(\phi^*\pi_m)}{d\pi_m}(\gamma) = \prod_{x\in\gamma} p_\phi^m(x) \exp\left(\int_{\mathbb{R}^d} \big(1 - p_\phi^m(x)\big) m(dx)\right),$$

where $*$ *indicates that we take the image measure and the Radon-Nikodym density* p_ϕ^m *is defined as*

$$\begin{aligned} p_\phi^m(x) :=& \frac{d\big(\phi^* m\big)(x)}{dm(x)} = \frac{\rho(\phi^{-1}(x))}{\rho(x)} \frac{d\mathfrak{v}\big(\phi^{-1}(x)\big)}{d\mathfrak{v}(x)} = \frac{\rho(\phi^{-1}(x))}{\rho(x)} J_{\mathfrak{v}}(\phi)(x), \\ & \text{if } x \in \{0 < \rho < \infty\} \cap \{0 < \rho \circ \phi^{-1} < \infty\}; \\ p_\phi^m(x) :=& 1, \ \text{otherwise.} \end{aligned}$$

Here, $J_{\mathfrak{v}}(\phi)$ *denotes the Jacobian determinant of* ϕ.

Integration by parts

Theorem 1.1.10 (see [AKR98a, Thm 3.1]). *We get an integration by parts formula for all* F, $G \in \mathcal{F}C_b^\infty(\Gamma(\mathbb{R}^d), C_0^\infty(X))$ *and any* $v \in V_0(\mathbb{R}^d)$*:*

$$\begin{aligned}\int_{\Gamma(\mathbb{R}^d)} (\nabla_v^\Gamma F)(\gamma) G(\gamma) \pi_m(d\gamma) &= -\int_{\Gamma(\mathbb{R}^d)} F(\gamma)(\nabla_v^\Gamma G)(\gamma)\pi_m(d\gamma) \\ &- \int_{\Gamma(\mathbb{R}^d)} F(\gamma)G(\gamma)\beta^{\pi_m}(v,\gamma)\pi_m(d\gamma),\end{aligned}$$

where

$$\beta^{\pi_m}(v,\gamma) := \int_{\mathbb{R}^d} \Big(\langle \beta^m(x), v(x)\rangle_{T_x(\mathbb{R}^d)} + div^{\mathbb{R}^d} v(x) \Big) \gamma(dx) \quad \textit{and}$$

$$\beta^m(x) := \frac{\nabla^{\mathbb{R}^d}\rho(x)}{\rho(x)} \in T_x(X) \ \textit{with, as usual, } \beta^m := 0 \ \textit{on } \{\rho = 0\}.$$

As for the Gaussian measure, this integration by parts formula is derived via the quasi-invariance property of the Poisson measure.

Dirichlet forms and associated stochastic dynamics

We define a gradient bilinear form for all $F, G \in \mathcal{F}C_b^\infty(\Gamma(\mathbb{R}^d), C_0^\infty(\mathbb{R}^d))$ by

$$\mathcal{E}^{\pi_m,\Gamma}(F,G) := \int_{\Gamma(\mathbb{R}^d)} \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\rangle_{T_\gamma(\Gamma)} \mu_0(d\gamma).$$

It is closable in $L^2(\Gamma(\mathbb{R}^d), \pi_m)$ and its closure is a conservative symmetric gradient Dirichlet form which is quasi-regular and local (cf. e.g. [AKR98b, Proposition 5.1., Theorem 5.1. and Corollary 5.1]).

Theorem 1.1.11 (see [AKR98b, Theorem 5.2] and [RS98, Proposition 1]). *There exists a conservative diffusion process*[2]

$$\mathbf{M}^{\Gamma(\mathbb{R}^d)} = \Big(\mathbf{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t\geq 0}, (\mathbf{\Theta}_t)_{t\geq 0}, (\mathbf{X}_t)_{t\geq 0} (\mathbf{P}_\gamma)_{\gamma\in\Gamma(\mathbb{R}^d)} \Big)$$

on $\Gamma(\mathbb{R}^d)$ *which is properly associated with* $\big(\mathcal{E}^{\pi_m,\Gamma}, \mathcal{D}\big(\mathcal{E}^{\pi_m,\Gamma}\big)\big)$*, i.e., for all* ($\pi_m$*-versions) of* $F \in L^2(\Gamma(\mathbb{R}^d), \pi_m)$ *and all* $t > 0$ *the function*

$$\Gamma(\mathbb{R}^d) \ni \gamma \mapsto p_t F(\gamma) := \int_\Omega F(\mathbf{X}(t)) d\mathbf{P}_\gamma$$

[2] This is a conservative strong Markov process with continuous sample paths

is an $\mathcal{E}^{\pi_m,\Gamma}$-quasi-continuous version of $\exp(-tL^{\pi_m,\Gamma})F$*, where* $L^{\pi_m,\Gamma}$ *is the generator of* $\big(\mathcal{E}^{\pi_m,\Gamma}, \mathcal{D}(\mathcal{E}^{\pi_m,\Gamma})\big)$ *(cf. [MR92, Section I.2]).* $\mathbf{M}^{\Gamma(\mathbb{R}^d)}$ *is up to π_m-equivalence unique (cf. [MR92, Theorem VI.6.4]). In particular,* $\mathbf{M}^{\Gamma(\mathbb{R}^d)}$ *is π_m-symmetric (i.e., $\int G p_t F d\pi_m = \int F p_t G d\pi_m$ for all $F, G : \Gamma(\mathbb{R}^d) \to \mathbb{R}$, $\mathcal{B}(\Gamma(\mathbb{R}^d))$-measurable) and has π_m as an invariant measure.*

1.1.3 Gamma measure

Let us quickly summarize some known results regarding the Gamma measures (cf. also Chapter 3 for details). The Gamma measures over a compact set are strongly related to (multiplicative) Lebesgue measures (cf. [TVY01]). The latter are considered in [VGG83] from a point of view of group representation. In [KdSS98, Subsection 4.2] a chaos decomposition for the Gamma measures is presented.

The cone of positive discrete measures

The *cone of locally finite discrete measures* is defined as

$$\mathbb{K}(\mathbb{R}^d) := \Big\{ \eta = \sum s_i \delta_{x_i} \Big| s_i \in \mathbb{R}_+, x_i \in \mathbb{R}^d, x_i \neq x_j \forall\ i, j \in \mathbb{N},\ i \neq j, \\ \forall \Lambda \in \mathcal{B}_c(\mathbb{R}^d) : \eta(\Lambda) < \infty \Big\} \subset \mathcal{M}(\mathbb{R}^d).$$

Here, $\mathcal{B}_c(\mathbb{R}^d)$ denotes the collection of Borel sets in $\mathbb{R}^d$ with compact closure.

The Gamma measure $\mathcal{G}_\theta$, $\theta > 0$ being a shape parameter, is characterized via (compare [TVY01, P.279], [KdSSU98, Definition 4.1])

$$\mathbb{E}_{\mathcal{G}_\theta}\left[\exp\left(-\langle a, \cdot \rangle\right)\right] = \exp\left(-\theta \int_{\mathbb{R}^d} \log(1 + a(x)) m(dx)\right),$$

where $a : \mathbb{R}^d \to [0, \infty)$, is a bounded, compactly supported Borel function.

Chaos decomposition

Similarly to the Gaussian and the Poissonian case, we have an orthogonal chaos decomposition:

Theorem 1.1.12 ([KdSSU98, Subsection 4.2])**.** *Any $F \in L^2(\mathbb{K}(\mathbb{R}^d), \mathcal{G}_\theta)$ can be written as*

$$F(\eta) = \sum_{n=0}^{\infty} \langle L_n^\theta(\eta), f^{(n)} \rangle,$$

where the system $\{L_n^\theta(f^{(n)})(\eta) = \langle L_n^\theta(\eta), f^{(n)} \rangle\}_{n \in \mathbb{N}}$ is called the system of generalized Laguerre polynomials for the Gamma measure $\mathcal{G}_\theta$.

Quasi-invariance

In [TVY01] the (extrinsic) quasi-invariance formula w.r.t. multiplication is outlined for the case that $\mathbb{R}^d$ is replaced by $[0,1]$.

Theorem 1.1.13 (cf. [TVY01, Theorem 3.1])**.** *The Gamma measure $\mathcal{G}_\theta$ on $\mathbb{K}(\mathbb{R}^d)$ is quasi-invariant under the multiplication $M_h : \mathbb{K}(\mathbb{R}^d) \ni \eta \mapsto e^h\eta \in \mathbb{K}(\mathbb{R}^d)$ for each $h \in C_0(\mathbb{R}^d)$. The corresponding density is*

$$\frac{d(M_h\mathcal{G}_\theta)}{d\mathcal{G}_\theta}(\eta) = \exp\left(-\theta\int_{\mathbb{R}^d} h(x)m(dx)\right)\exp\left(-\int_{\mathbb{R}^d}\left(e^{-h(x)}-1\right)d\eta(x)\right).$$

This quasi-invariance is an essential property of Gamma measures. It is deeply related with the structure of Gamma measure (cf. [LS01]).

1.2 Content

This thesis is divided into three parts: In Part I we introduce our basic object, namely the *Gamma measure* $\mathcal{G}_\theta$, $\theta > 0$ being a shape parameter, resp. (as a related Poisson space model) the *Gamma-Poisson measure* $\mathcal{P}_\theta$. In Part II we construct *Gibbs perturbations* of the Gamma measure $\mathcal{G}_\theta$; and in Part III we outline some *differential structure* on the cone $\mathbb{K}(\mathbb{R}^d)$, Dirichlet forms related to Gamma and Gibbs measures and associated diffusions.

Here, we only give a brief insight. For a more detailed overview of the content, more motivation and relations to existing literature, we refer to the beginning of the respective chapters. For the convenience of the reader, we included an index of the most important notations, definitions and results.

1.2.1 Gamma measures

Let m be a non-atomic Radon measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ-algebra on $\mathbb{R}^d$. On $\mathbb{R}_+ := (0,\infty)$ being equipped with the metric $d_{\mathbb{R}_+}(s_1,s_2) := |\ln\frac{s_1}{s_2}|$, s_1 and $s_2 \in \mathbb{R}_+$, we consider the measure

$$\lambda_\theta := \theta\frac{1}{t}e^{-t}dt, \quad \theta > 0 \text{ being a shape parameter.}$$

Each $\hat{x} = (s_x, x) \in \hat{\mathbb{R}}^d := \mathbb{R}_+ \times \mathbb{R}^d$ may describe a particle with a mark s_x being located at a position $x \in \mathbb{R}^d$.

In our considerations the configurations space $\Gamma(\hat{\mathbb{R}}^d)$ over $\hat{\mathbb{R}}^d$ will play a central role. It is defined as (cf. also Section 2.1)

$$\Gamma(\hat{\mathbb{R}}^d) := \left\{\gamma \subset \hat{\mathbb{R}}^d \;\middle|\; |\gamma_\Lambda| < \infty, \quad \forall\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)\right\},$$

where $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$ is a Borel set with compact closure, $\gamma_\Lambda := \gamma \cap \Lambda$ is the restriction of γ to Λ and $|\gamma_\Lambda|$ denotes the set cardinality.

Let $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$. The Poisson measure $\mathcal{P}_\theta^\Lambda$ on $\Gamma(\Lambda) := \{\gamma \in \Gamma(\hat{\mathbb{R}}^d) | \gamma \subset \Lambda\} = \bigsqcup_{n \geq 0} \{\gamma \in \Gamma(\Lambda) \,|\, |\gamma| = n\}$ with intensity measure $\lambda_\theta \otimes m$ is given by

$$\mathcal{P}_\theta^\Lambda := e^{-\lambda_\theta \otimes m(\Lambda)} \sum_{n \geq 0} \frac{1}{n!} \big(\lambda_\theta \otimes m\big)^{\hat{\otimes} n}.$$

Because of the consistency of $\{\mathcal{P}_\theta^\Lambda | \Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)\}$ by Kolmogorov's theorem there exists a unique probability measure $\mathcal{P}_\theta$ such that

$$\mathcal{P}_\theta \circ \mathbb{P}^{-1}_{\hat{\mathbb{R}}^d, \Lambda} = \mathcal{P}_\theta^\Lambda,$$

where $\mathbb{P}_{\hat{\mathbb{R}}^d, \Lambda}$ is the projection from $\Gamma(\hat{\mathbb{R}}^d)$ to $\Gamma(\Lambda)$: $\mathbb{P}_{\hat{\mathbb{R}}^d, \Lambda}(\gamma) = \gamma \cap \Lambda$.

We identify a smaller set $\Gamma_f(\hat{\mathbb{R}}^d) \subset \Gamma(\hat{\mathbb{R}}^d)$ that supports $\mathcal{P}_\theta$ (cf. Subsection 2.2.1): To that end, we introduce the set of *pinpointing* configurations (cf. Definition 2.2.2)

$$\Gamma_p(\hat{\mathbb{R}}^d) := \big\{\gamma \in \Gamma(\hat{\mathbb{R}}^d) \big| \text{ for all } (s_1, x_1), (s_2, x_2) \in \gamma \text{ we have } \\ x_1 = x_2 \Rightarrow s_1 = s_2 \big\}.$$

For all $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ and $\gamma \in \Gamma_p(\hat{\mathbb{R}}^d)$ we define a local mass via (cf. Definition 2.2.6)

$$\mathfrak{m}_\Delta(\gamma) := \sum_{\hat{x} = (s_x, x) \in \gamma} s \mathbb{1}_\Delta(x) = \int_{\hat{\mathbb{R}}^d} s \mathbb{1}_\Delta(x) \gamma(d\hat{x}).$$

Combining these two definitions, we specify the set of pinpointing configurations with *finite local mass* as (cf. Definition 2.2.7)

$$\Gamma_f(\hat{\mathbb{R}}^d) := \big\{\gamma \in \Gamma_p(\hat{\mathbb{R}}^d) \,\big|\, \mathfrak{m}_\Delta(\gamma) < \infty, \ \forall \Delta \in \mathcal{B}_c(\mathbb{R}^d)\big\}.$$

The following result, which is the main one of Chapter 2, will enable us to construct the Gamma measure. We have

$$\mathcal{P}_\theta(\Gamma_f(\hat{\mathbb{R}}^d)) = 1.$$

For our further considerations it is important that there exists a bijection between $\Gamma_f(\hat{\mathbb{R}}^d)$ and the *cone of discrete Radon measures*

$$\mathbb{K}(\mathbb{R}^d) := \Big\{\eta = \sum s_i \delta_{x_i} \Big| s_i \in \mathbb{R}_+, x_i \in \mathbb{R}^d, x_i \neq x_j \forall\, i, j \in \mathbb{N},\ i \neq j, \\ \forall \Delta \in \mathcal{B}_c(\mathbb{R}^d) : \eta(\Delta) < \infty \Big\}.$$

The mentioned bijection is even a homeomorphism and is defined by (cf. (3.1.2))

$$\begin{array}{rrcl} \mathbb{T}: & \Gamma_f(\hat{\mathbb{R}}^d) & \to & \mathbb{K}(\mathbb{R}^d) \\ & \gamma = \{(s_x, x)\} & \mapsto & \eta := \sum_{(s_x,x)\in\gamma} s_x \delta_x. \end{array}$$

The image measure of $\mathcal{P}_\theta$ under $\mathbb{T}$ is denoted by $\mathcal{G}_\theta$ and called *Gamma* measure with shape parameter $\theta > 0$. We call $\mathcal{P}_\theta$ *Gamma-Poisson* measure.

These two objects, namely the cone $\mathbb{K}(\mathbb{R}^d)$ and the Gamma measure $\mathcal{G}_\theta$, are studied in more detail in Chapter 3. We present two important results:

- *All moments exist* (cf. Theorem 3.2.6): For $n \in \mathbb{N}$ and for each bounded Borel function $a : \mathbb{R}^d \to \mathbb{R}$ that is supported by $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ we have

$$\mathbb{E}_{\mathcal{G}_\theta}[\langle a, \cdot\rangle^n] \leq n!\|a\|_\infty \theta^n m(\Delta)^n < \infty.$$

- *Quasi-invariance property* (cf. Theorem 3.3.3): The Gamma measure $\mathcal{G}_\theta$ is quasi-invariant under the multiplications $M_h : \mathbb{K}(\mathbb{R}^d) \ni \eta \mapsto e^h\eta \in \mathbb{K}(\mathbb{R}^d)$, where $h \in C_0(\mathbb{R}^d)$.

These two properties are essentially used in Part III to establish integration by parts formulas and study related Dirichlet forms.

The construction is extended to the case of a locally compact Polish space X.

1.2.2 Gibbs perturbations

In Part II, we construct Gibbs perturbations of the Gamma measure $\mathcal{G}_\theta$ on $\mathbb{K}(\mathbb{R}^d)$ by means of a pair potential $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ describing the interaction between particles. So far, we considered the "free case" of a Gamma measure $\mathcal{G}_\theta$, where $\phi = 0$. Here, we give a heuristic description. The precise definitions and results are presented in Chapters 4 and 5.

We introduce a Gibbs formalism on $\mathbb{K}(\mathbb{R}^d)$ following the *Dobrushin-Lanford-Ruelle* (**DLR**) approach to Gibbs random fields in classical statistical mechanics (cf. Section 5.3): As an example, we consider a basic model with

$$\phi(x, y) = a(x - y), \quad x, y \in \mathbb{R}^d,$$

where $a \geq 0$ is bounded, even and compactly supported.

For each $\eta \in \mathbb{K}(\mathbb{R}^d)$ and a *tempered boundary condition* $\xi \in \mathbb{K}^{\mathtt{t}}(\mathbb{R}^d)$ (cf. (5.3.5)), the *relative energy* $H_\Delta(\eta|\xi)$ in a bounded area $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ is given by (cf. (5.3.4))

$$H_\Delta(\eta|\xi) := \int_\Delta \int_\Delta \phi(x,y)\eta(dx)\eta(dy) + \int_{\Delta^c} \int_\Delta \phi(x,y)\eta(dx)\xi(dy).$$

Fix an inverse temperature $\beta = 1/T > 0$. A *local Gibbs measure in volume* Δ is the probability measure on $\mathbb{K}(\Delta)$ defined as

$$\mu_\Delta(d\eta|\xi) = \frac{1}{Z_\Delta(\xi)} e^{-\beta H_\Delta(\eta|\xi)} \mathcal{G}_\theta^\Delta(d\eta),$$

where $\mathcal{G}_\theta^\Delta$ is the Gamma measure on $\mathbb{K}(\Delta)$. The *probability kernels*

$$\pi_\Delta(B|\xi) = \mu_\Delta(\{\eta \in \mathbb{K}(\Delta) | \eta \cup \xi_{\Delta^c} \in B\}|\xi), \quad B \in \mathcal{B}(\mathbb{K}(\mathbb{R}^d)),$$

indexed by $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ and $\xi \in \mathbb{K}^{\mathtt{t}}(\mathbb{R}^d)$, constitute the *Gibbs specification* on $\mathbb{K}(\mathbb{R}^d)$. It determines corresponding *tempered Gibbs measures* $\mu_{\mathbb{K}}$ on $\mathbb{K}(\mathbb{R}^d)$ via the (**DLR**) equation (cf. (6.3.18))

$$\int_{\mathbb{K}(\mathbb{R}^d)} \pi_\Delta(B|\eta)\mu_{\mathbb{K}}(d\eta) = \mu_{\mathbb{K}}(B),$$

valid for all $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ and $B \in \mathcal{B}(\mathbb{K}(\mathbb{R}^d))$ (cf. Definition 5.3.9).[3] The set of all tempered Gibbs measures related to the specification $\pi = \{\pi_\Delta\}_{\Delta \in \mathcal{B}_c(\mathbb{R}^d)}$ will be denoted by $\mathcal{G}\mathrm{ibbs}_\phi^{\mathtt{t}}(\mathbb{K}(\mathbb{R}^d))$ (cf. Definition 5.3.5). By the construction, all $\mu \in \mathcal{G}\mathrm{ibbs}_\phi^{\mathtt{t}}(\mathbb{K}(\mathbb{R}^d))$ are supported by $\mathbb{K}^{\mathtt{t}}(\mathbb{R}^d)$.

The first step of our considerations is to show the existence of such Gibbs measures $\mu_{\mathbb{K}} \in \mathcal{G}\mathrm{ibbs}_\phi^{\mathtt{t}}(\mathbb{K}(X))$, which is a non-trivial problem (cf. also below). But before going into details, we will formulate some results. To this end, we have to be more specific about the conditions on the symmetric function ϕ. We distinguish two cases:

1. The potential ϕ is *non-negative*, i.e. $\phi \geq 0$, and has finite range, i.e.,

$$\forall \Delta \in \mathcal{B}_c(\mathbb{R}^d)\ \exists\, \mathcal{U}_\Delta : \quad \phi(x,y) = \phi(y,x) = 0 \quad \text{if } x \in \Delta \text{ and } y \in \mathcal{U}_\Delta^c.$$

2. The potential ϕ, which may take possibly negative values, satisfies some *stability* properties. Merely speaking the repulsion part ϕ^+ of the potential shall dominate its attraction part ϕ^- (for the precise formulation see Subsection 5.3.1).

[3]Here, we set $\pi(d\gamma|\xi) = 0$ if $\xi \notin \mathbb{K}^{\mathtt{t}}(\mathbb{R}^d)$.

We obtain the following main results describing the set $\mathcal{G}\mathrm{ibbs}^{\mathsf{t}}_{\phi}(\mathbb{K}(\mathbb{R}^d))$:

- *Existence* (cf. Theorem 5.3.7): For each ϕ as above, there exists a corresponding Gibbs measure, i.e.,
$$\mathcal{G}\mathrm{ibbs}^{\mathsf{t}}_{\phi}(\mathbb{K}(\mathbb{R}^d)) \neq \varnothing.$$
- *Uniform moment bounds* (cf. Theorems 5.3.10 and 5.3.11): For $\Delta \in \mathcal{B}_c(X)$ and $N \in \mathbb{N}$ there exists $C_N(\Delta) > 0$ such that for all $\mu_{\mathbb{K}} \in \mathcal{G}\mathrm{ibbs}^{\mathsf{t}}_{\phi}(\mathbb{K}(X))$
$$\int_{\mathbb{K}(X)} \eta(\Delta)^N \mu_{\mathbb{K}}(d\eta) < C_N(\Delta).$$

Our strategy is to reformulate the existence problem in terms of related Gibbs measures on the configuration space $\Gamma(\hat{\mathbb{R}}^d)$. To some extent, this approach is similar to the free case, where we first define the Gamma-Poisson measure on $\Gamma(\hat{\mathbb{R}}^d)$ and then use the bijective mapping $\mathbb{T}$ between the cone $\mathbb{K}(\mathbb{R}^d)$ and the set $\Gamma_f(\hat{\mathbb{R}}^d)$ of pinpointing configurations with finite local mass. Recall that

$$\mathbb{T}^{-1}: \quad \mathbb{K}(\mathbb{R}^d) \ni \eta = \sum s_x \delta_x \mapsto \gamma = \{(s_x, x)\} \in \Gamma_f(\hat{\mathbb{R}}^d) \subset \Gamma(\hat{\mathbb{R}}^d).$$

Using $\mathbb{T}^{-1}$, we can transfer the corresponding objects (like the potential, relative energy and local specification) to the configuration space $\Gamma(\hat{\mathbb{R}}^d)$, e.g.,

$$\begin{array}{lll} \phi(x,y),\ x,y \in \mathbb{R}^d & \text{to} & V(\hat{x},\hat{y}) = s_x s_y \phi(x,y), \quad \hat{x},\hat{y} \in \hat{X}; \\ \mu_\Delta(d\eta|\xi) & \text{to} & \mu_{\mathbb{R}_+\times\Delta}(d\gamma|\mathbb{T}^{-1}(\xi)); \\ \pi_\Delta(d\eta|\xi) & \text{to} & \pi_{\mathbb{R}_+\times\Delta}(d\gamma|\mathbb{T}^{-1}(\xi)). \end{array}$$

We will call the specification kernels $\pi_{\mathbb{R}_+\times\Delta}(d\gamma|\mathbb{T}^{-1}(\xi)) \in \mathcal{M}^1(\Gamma(\hat{\mathbb{R}}^d))$ *semi-local* because they are indexed by "stripes" $\mathbb{R}_+ \times \Delta \subset \hat{\mathbb{R}}^d$. Using the (**DLR**) formalism on $\Gamma(\hat{\mathbb{R}}^d)$, we then define the associated Gibbs measures on $\Gamma(\hat{\mathbb{R}}^d)$ corresponding to the *semi-local specification*

$$\pi_\Gamma = \{\pi_{\mathbb{R}_+\times\Delta}(d\gamma|\tilde{\xi}) | \Delta \in \mathcal{B}_c(\mathbb{R}^d), \tilde{\xi} \in \Gamma_f(\hat{\mathbb{R}}^d)\}.$$

The set of all of such Gibbs measures will be denoted by $\mathcal{G}\mathrm{ibbs}^{\mathsf{t}}_V(\Gamma(\hat{\mathbb{R}}^d))$ (cf. also Subsection 4.5.1 and Section 5.1). Actually, each $\mu_\Gamma \in \mathcal{G}\mathrm{ibbs}^{\mathsf{t}}_V(\Gamma(\hat{\mathbb{R}}^d))$ is supported by the subset $\Gamma^{\mathsf{t}}(\hat{\mathbb{R}}^d) \subset \Gamma_f(\hat{\mathbb{R}}^d)$ of tempered configurations, which is defined by (5.1.35).

The one-to-one correspondence between the local specification kernels π_Δ on $\mathbb{K}(\mathbb{R}^d)$ and the *semi-local* ones, $\pi_{\mathbb{R}_+\times\Delta}$, on $\Gamma(\hat{\mathbb{R}}^d)$ (cf. Subsection 5.3.2) implies the one-to-one correspondence between the classes of Gibbs measures $\mathcal{G}\mathrm{ibbs}^{\mathtt{t}}_{\phi}(\mathbb{K}(\mathbb{R}^d))$ and $\mathcal{G}\mathrm{ibbs}^{\mathtt{t}}_{V}(\Gamma(\hat{\mathbb{R}}^d))$ (cf. Theorem 5.3.6). By the above construction, the set $\mathcal{G}\mathrm{ibbs}^{\mathtt{t}}_{V}(\Gamma(\hat{\mathbb{R}}^d))$ consists of those $\mu_\Gamma \in \mathcal{M}^1(\Gamma(\hat{\mathbb{R}}^d))$ which solve the (**DLR**) equation and have full measure on $\Gamma_f(\hat{\mathbb{R}}^d)$ (cf. Theorem 4.3.31, resp. Corollary 5.2.11). Hence, we will first construct and study the Gibbs measures μ_Γ on $\Gamma(\hat{\mathbb{R}}^d)$ and then reformulate the corresponding results for the Gibbs measures $\mu_{\mathbb{K}}$ on $\mathbb{K}(\mathbb{R}^d)$.

Even on the configuration space $\Gamma(\hat{\mathbb{R}}^d)$, neither the potential, nor the *semi*-local specification kernels are standard (cf. e.g. [Rue69, Rue70] or [AKR98b, KLU99, Kun99]).

Concerning the semi-local specification, usually one considers *local* specification kernels on $\Gamma(\hat{\mathbb{R}}^d)$, where instead of $\mathbb{R}_+ \times \Delta$, $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$, one chooses $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$ (cf. Section 4.1). An important issue is that the classes of measures $\mu \in \mathcal{M}^1(\Gamma(\hat{\mathbb{R}}^d))$ that solve the (**DLR**) equation w.r.t. to the local Gibbs specification $(\pi_\Lambda)_{\Lambda\in\mathcal{B}_c(\hat{\mathbb{R}}^d)}$ respectively semi-local one $(\pi_{\mathbb{R}_+\times\Delta})_{\Delta\in\mathcal{B}_c(\mathbb{R}^d)}$ indeed coincide, as we show in Theorem 4.5.9 (see also Remark 5.1.24). This immediately implies the one-to-one correspondence between the two classes of Gibbs measures on $\mathbb{K}(\mathbb{R}^d)$ and $\Gamma(\hat{\mathbb{R}}^d)$, i.e., between $\mathcal{G}\mathrm{ibbs}^{\mathtt{t}}_{V}(\Gamma(\hat{\mathbb{R}}^d))$ and $\mathcal{G}\mathrm{ibbs}^{\mathtt{t}}_{\phi}(\mathbb{K}(\mathbb{R}^d))$. So, we will first study the set $\mathcal{G}\mathrm{ibbs}^{\mathtt{t}}_{V}(\Gamma(\hat{\mathbb{R}}^d))$ and then reformulate the main results for $\mu \in \mathcal{G}\mathrm{ibbs}^{\mathtt{t}}_{\phi}(\mathbb{K}(\mathbb{R}^d))$.

Furthermore, the potential $V(\hat{x},\hat{y}) = s_x s_y \phi(x,y)$ does not fit the standard framework on $\Gamma(\hat{\mathbb{R}}^d)$ (cf. Section 4.2 for more details), in so far as:

1. V is in general not translation invariant in $\hat{\mathbb{R}}^d$.

2. V may have an infinite range in $\hat{\mathbb{R}}^d$.

3. If $\phi(x,y) = a(x-y) \geq 0$, where $x, y \in \mathbb{R}^d$ and $a \in C_0^1(\mathbb{R}^d)$ not identical to zero, then (cf. Lemma 4.2.1)

$$C(\beta) := \operatorname*{ess\,sup}_{\hat{x}\in\hat{\mathbb{R}}^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}_+} \left| e^{-\beta s_x s_y a(x-y)} - 1 \right| \lambda_\theta(ds_x)dx = \infty.$$

   So, V voids the uniform integrability condition, which is $C(\beta) < \infty$ (cf. e.g. [AKR98b, Kun99]).

4. The intensity measure λ_θ on the marks is infinite, i.e., $\lambda_\theta(\mathbb{R}_+) = \infty$, and, moreover, it has a high concentration as $s \searrow 0$. (In marked configuration spaces the intensity measure on $\mathbb{R}_+$ is usually assumed to be finite, cf. [KLU99].)

We overcome these difficulties in Chapters 4 and 5, where we study in detail the related Gibbs measures on $\Gamma(\hat{\mathbb{R}}^d)$. In particular, we establish their existence and uniform moment bounds (cf. Theorems 4.2.7 and 4.3.34, resp. 5.2.8 and 5.2.10).

In Chapter 4 we concentrate on the case of a non-negative potential $V \geq 0$ and construct a Gibbs measure being specified by the *local* specification kernel π_Λ, $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$ (cf. Section 4.2). To establish the existence of $\mu_\Gamma \in \mathcal{G}\mathrm{ibbs}_V^{\mathtt{t}}(\Gamma(\hat{\mathbb{R}}^d))$, we derive uniform moment bounds for $\mu_\Lambda(d\gamma|\xi)$ (cf. Proposition 4.2.3). These bounds imply that each net of local specification kernels $\pi_\Lambda(d\gamma|\xi)$ with a fixed boundary condition ξ is *locally equicontinuous* (cf. Proposition 4.2.6). This yields the existence of a certain $\mu_\Gamma \in \mathcal{M}^1(\Gamma(\hat{\mathbb{R}}^d))$ being a limit point of such a net as $\Lambda \nearrow \hat{\mathbb{R}}^d$, for which we then show that it satisfies the (**DLR**) equation, i.e., $\mu_\Gamma \in \mathcal{G}\mathrm{ibbs}_V^{\mathtt{t}}(\Gamma(\hat{\mathbb{R}}^d))$ (cf. Theorem 4.2.7). Therefore, the set $\mathcal{G}\mathrm{ibbs}_V^{\mathtt{t}}(\Gamma(\hat{\mathbb{R}}^d))$ is non-void. After establishing the existence, we deduce certain moment estimates being *uniform* for all Gibbs measures $\mu_\Gamma \in \mathcal{G}\mathrm{ibbs}_V^{\mathtt{t}}(\Gamma(\hat{\mathbb{R}}^d))$ (cf. Theorems 4.3.31 and 4.3.34). These estimates allow us to identify an ‹exponentially tempered› subset $\Gamma^{\mathtt{t}}_{\mathrm{ex}}(\hat{\mathbb{R}}^d) \subset \Gamma_f(\hat{\mathbb{R}}^d)$ on which each $\mu_\Gamma \in \mathcal{G}\mathrm{ibbs}_V^{\mathtt{t}}(\Gamma(\hat{\mathbb{R}}^d))$ has full measure (cf. Remark 4.3.32 and Corollary 4.4.2).

In our general considerations in Chapter 5, we remove the assumption that $V \geq 0$ and work directly with the semi-local specification kernels $\pi_{\mathbb{R}_+ \times \Delta}$, $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$. As we already mentioned above, both specifications lead to the same set $\mathcal{G}\mathrm{ibbs}_V^{\mathtt{t}}(\Gamma(\hat{\mathbb{R}}^d))$. To construct such Gibbs measures (cf. Theorem 5.2.8), we need more advanced analytic techniques than in Chapter 4. These involve introducing certain *Lyapunov functionals* and establishing the *weak dependence* of Gibbs specification kernels on boundary conditions. A key issue in the existence proof is Proposition 5.2.4, where we get a uniform bound (as $\Delta \nearrow \mathbb{R}^d$) for the exponential integral of a Lypunov functional w.r.t. the local specification kernels $\pi_{\mathbb{R}_+ \times \Delta}$. For a large class of boundary conditions $\xi \in \Gamma(\hat{\mathbb{R}}^d)$, this allows us to prove the local equicontinuity of the specification kernels $(\pi_{\mathbb{R}_+ \times \Delta}(d\gamma|\xi))_{\Delta \in \mathcal{B}_c(\mathbb{R}^d)}$ (cf. Definition 5.1.16), which implies their tightness in a proper topology (cf. Proposition 5.2.7). Finally, we check that all cluster points μ_Γ of the Gibbs specification $\{\pi_{\mathbb{R}_+ \times \Delta}\}$ (as $\Delta \nearrow \hat{\mathbb{R}}^d$) are surely Gibbs. The properties of $\mu \in \mathcal{G}\mathrm{ibbs}_V^{\mathtt{t}}(\Gamma(\hat{\mathbb{R}}^d))$, including moment bounds and a characterization of supporting sets, are summarized in Theorem 5.2.10 and Corollary 5.2.11.

In Section 5.3 we make a transition to the cone $\mathbb{K}(\mathbb{R}^d)$. Using the canonical homeomorphism $\mathbb{T}$ (cf. (3.1.2)), we obtain Gibbs measures $\mu_\mathbb{K}$ on $\mathbb{K}(\mathbb{R}^d)$ as image measures of Gibbs measures μ_Γ on $\Gamma(\hat{\mathbb{R}}^d)$, i.e., $\mu_\mathbb{K} = \mathbb{T}^*\mu_\Gamma$. Then we can directly reformulate the main results obtained in Chapter 4 and 5 in the Gibbsian framework on $\mathbb{K}(\mathbb{R}^d)$. Among them, we mention the existence result (cf. Theorem 5.3.7) and the uniform moment bounds (cf. Theorem 5.3.10).

These results will be of particular importance in Part III: There we establish integration by parts formulas for Gibbs measures on $\mathbb{K}(\mathbb{R}^d)$ and study related Dirichlet forms and operators (cf. Chapter 6), which are then used to construct associated diffusions on $\mathbb{K}(\mathbb{R}^d)$ in Chapter 7.

Let us point out that the technique developed in Sections 5.1 and 5.2 also covers more general potentials than in the basic model with $V(\hat{x}, \hat{y}) = s_x \phi(x, y) s_y$ (see the general setup fixed in Subsection 5.1.3 and the corresponding results in Theorem 5.2.8 and 5.2.10). Moreover, the results of Chapter 4 and 5 are extended from $\mathbb{R}^d$ to general locally compact Polish spaces X.

1.2.3 Differential calculus over $\mathbb{K}(\mathbb{R}^d)$

In Part III, we incorporate movement of the marks (extrinsic) and positions (intrinsic). In fact, we construct diffusions

- for *extrinsic*, *intrinsic* and *joint motion* that
- are located in $\mathbb{K}(\mathbb{R}^d)$ and
- are related to $\mathcal{G}_\theta$, as well as to some class of Gibbs perturbations.
- In particular, we get a diffusion of a dense set in $\mathbb{R}^d$.

Our approach is based on using Dirichlet forms: Roughly speaking, for each quasi-regular and local Dirichlet form, there exists an associated diffusion. In Chapter 7 we treat the question of quasi-regularity (and locality) of gradient Dirichlet forms to obtain associated diffusions.

Prominent examples for diffusions over spaces of measures are Fleming-Viot processes, which are motivated by biological considerations (cf. [Hoc91, EK93] and Chapter 7). Diffusions constructed via Dirichlet forms in the configuration space framework are considered, e.g., in [AKR98a, AKR98b] and [KLU99]. The theory of Dirichlet forms is explained, for example, in [MR92] or, the symmetric case, in [FOT94].

Differential calculus and Dirichlet forms

We first introduce some gradients on functions over $\mathbb{K}(\mathbb{R}^d)$ (cf. Section 6.1) and then deduce integration by parts formulas and construct related Dirichlet forms (cf. Section 6.3).

Fix $h \in C_0(\mathbb{R}^d)$, $v \in V_0(\mathbb{R}^d)$ (being the set of smooth sections of the tangent space $T(X)$ with compact support) and consider cylinder functions

$$F(\eta) = g_F(\langle \phi, \eta \rangle), \tag{1.2.1}$$

where $g_F \in C_b^\infty(\mathbb{R}^d)$ and $\phi \in C_0^\infty(\mathbb{R}^d)$.[4] We construct an *extrinsic* ($\nabla^{\mathbb{K}}_{\text{ext},h} F$), an *intrinsic* ($\nabla^{\mathbb{K}}_{\text{int},v} F$) and a joint ($\nabla^{\mathbb{K}}_{h,v} F$) *directional derivative* (cf. Subsections 6.1.2, 6.1.3 and 6.1.4): Let $F : \mathbb{K}(\mathbb{R}^d) \to \mathbb{R}$. We define an *extrinsic derivative* in direction h via the multiplications $M_{th} : \mathbb{K}(\mathbb{R}^d) \ni \eta \mapsto e^{th}\eta \in \mathbb{K}(\mathbb{R}^d)$ with $t \in \mathbb{R}$:

$$\nabla^{\mathbb{K}}_{\text{ext},h} F(\eta) := \frac{d}{dt} F(e^{th}\eta),$$

whenever the right-hand side exists (cf. Definition 6.1.1). For the particular cylinder function specified in (1.2.1), we have (cf. Proposition 6.1.7)

$$\nabla^{\mathbb{K}}_{\text{ext},h}\ g_F(\langle \phi, \eta \rangle) = g_F'(\langle \phi, \eta \rangle)\langle \phi h, \eta \rangle$$

Let $\mathbb{R} \ni t \mapsto \phi_t^v(x)$ be the solution to the Cauchy problem $\frac{d}{dt}(\phi_0^v)(x) = v(\phi_t^v(x))$ and $\phi_0^v(x) = x$ for $x \in \mathbb{R}^d$. Then we define the *intrinsic derivative* along v as

$$\nabla^{\mathbb{K}}_{\text{int},v} F(\eta) := \frac{d}{dt} F\left((\phi_t^v)^* \eta\right)\bigg|_{t=0},$$

whenever the right-hand side exists (cf. Definition 6.1.9). In particular, for the above cylinder functions (cf. (1.2.1)) we have (cf. Proposition 6.1.14)

$$\nabla^{\mathbb{K}}_{\text{int},v}\ g_F(\langle \phi, \eta \rangle) = g_F'(\langle \phi, \eta \rangle)\langle \nabla^X_v \phi, \eta \rangle.$$

Combining the extrinsic and intrinsic directional derivative, we get the *joint* one (cf. Subsection 6.1.4)

$$\nabla^{\mathbb{K}}_{h,v} F(\eta) = \nabla^{\mathbb{K}}_{\text{ext},h} F(\eta) + \nabla^{\mathbb{K}}_{\text{int},v} F(\eta).$$

[4] $C_b^\infty(\mathbb{R}^d)$ is the set of all arbitrarily many times differentiable bounded functions from $\mathbb{R}^d$ and $C_0^\infty(\mathbb{R}^d)$ that subset whose functions are compactly supported.

For the corresponding gradients, we choose the following *tangent spaces*: The *extrinsic* one is $T_\eta^{\text{ext}}(\mathbb{K}) := L^2(\mathbb{K}(\mathbb{R}^d),\eta)$ (cf. Definition 6.1.3). The *extrinsic gradient* is defined by

$$\left\langle \nabla_{\text{ext}}^{\mathbb{K}} F(\eta), h \right\rangle_{T_\eta^{\text{ext}}(\mathbb{K})} = \nabla_{\text{ext},h}^{\mathbb{K}} F(\eta),$$

whenever the right-hand side exists. In particular (cf. Proposition 6.1.14),

$$\nabla_{\text{ext}}^{\mathbb{K}}\ g_F(\langle \phi, \eta \rangle) = g_F'(\langle \phi,\eta\rangle)\phi \quad \in T_\eta^{\text{ext}}(\mathbb{K}).$$

Furthermore, we define the *intrinsic tangent space* $T_\eta^{\text{int}}(\mathbb{K})$ at $\eta \in \mathbb{K}(\mathbb{R}^d)$ as the Hilbert space $L^2(\mathbb{R}^d \to T\,(\mathbb{R}^d),\eta)$ of measurable η-square integrable sections (measurable vector fields) $V_\eta : X \to T\,(\mathbb{R}^d)$ with the scalar product

$$\langle V_\eta^1, V_\eta^2 \rangle_{T_\eta^{\text{int}}(\mathbb{K})} := \int_{\mathbb{R}^d} \langle V_\eta^1(x), V_\eta^2(x) \rangle_{T_x(\mathbb{R}^d)} \eta(dx),$$

where $V_\eta^1, V_\eta^2 \in T_\eta^{\text{int}}(\mathbb{K})$. The *intrinsic gradient* is defined via

$$\langle (\nabla_{\text{int}}^{\mathbb{K}} F)(\eta), v \rangle_{T_\eta^{\text{int}}(\mathbb{K})} = (\nabla_{\text{int},v}^{\mathbb{K}} F)(\eta),$$

whenever the right-hand side exists. In particular for F as in (1.2.1)

$$\nabla_{\text{int}}^{\mathbb{K}}\ g_F(\langle \phi, \eta \rangle) = g_F'(\langle \phi,\eta\rangle)\nabla^X\phi \quad \in T_\eta^{\text{int}}(\mathbb{K}).$$

Combining the extrinsic and intrinsic part, we get the *tangent space* at $\eta \in \mathbb{K}(X)$

$$T_\eta(\mathbb{K}) = T_\eta^{\text{ext}}(\mathbb{K}) \oplus T_\eta^{\text{int}}(\mathbb{K})$$

and the *gradient*

$$(\nabla^{\mathbb{K}} F)(\eta) = \big((\nabla_{\text{ext}}^{\mathbb{K}} F)(\eta), (\nabla_{\text{int}}^{\mathbb{K}} F)(\eta) \big) \quad \in T_\eta(\mathbb{K}).$$

To obtain corresponding extrinsic, intrinsic and joint *integration by parts formulas*, we fix a measure $m(dx) = \rho(x)\mathfrak{v}(dx)$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\rho \in H_{\text{loc}}^{1,2}(\mathbb{R}^d, \mathfrak{v})$. We define the *extrinsic*, *intrinsic* and *joint logarithmic derivatives* (cf. Definitions 6.3.17 and 6.3.30)

$$\begin{aligned}
\langle \beta_{\text{ext}}^{\mathcal{G}_\theta}(\eta), h \rangle_{T_\eta^{\text{ext}}(\mathbb{K})} &:= \theta\langle h, m\rangle - \langle h,\eta\rangle,\\
\langle \beta_{\text{int}}^{\mathcal{G}_\theta}(\eta), v \rangle_{T_\eta^{\text{int}}(\mathbb{K})} &:= \int_{\mathbb{R}^d} \langle \beta^m(x), v(x)\rangle_{T_x(\mathbb{R}^d)} + div^{\mathbb{R}^d} v(x)\eta(dx),\\
\langle \beta^{\mathcal{G}_\theta}(\eta), (h,v) \rangle_{T_\eta(\mathbb{K})} &:= \langle \beta_{\text{ext}}^{\mathcal{G}_\theta}(\eta), h \rangle_{T_\eta^{\text{ext}}(\mathbb{K})} + \langle \beta_{\text{int}}^{\mathcal{G}_\theta}(\eta), v \rangle_{T_\eta^{\text{int}}(\mathbb{K})}.
\end{aligned}$$

We establish the following *integration by parts* formula (cf. Theorem 6.3.39):

$$\int_{\mathbb{K}(\mathbb{R}^d)} \nabla^{\mathbb{K}}_{h,v} F(\eta) \mathcal{G}_\theta(d\eta) = - \int_{\mathbb{K}(\mathbb{R}^d)} F(\eta) \langle \beta^{\mathcal{G}_\theta}(\eta), (h,v) \rangle_{T_\eta(\mathbb{K})} \mathcal{G}_\theta(d\eta).$$

It yields for $h = 0$ an *intrinsic* and for $v = \mathrm{id}_{\mathbb{R}^d}$ an *extrinsic* integration by parts formula.

The integration by parts formula is a key tool to study corresponding *Dirichlet forms*. The above results hold for more general functions, in particular for each *cylinder function*

$$F \in \mathcal{F}C_b^\infty(\mathbb{K}(\mathbb{R}^d), C_0^\infty(\mathbb{R}^d)) =: \mathbb{S}_{\mathbb{K}(\mathbb{R}^d)}$$

being of the form (cf. Definition 6.1.5)

$$F(\eta) = g_F\big(\langle \phi_1, \eta \rangle, \dots, \langle \phi_N, \eta \rangle\big),$$

where $g_F \in C_b^\infty(\mathbb{R}^N)$ and $\phi_i \in C_0^\infty(\mathbb{R}^d)$ for $i = 1, \dots, N$, $N \in \mathbb{N}$.

The next step is to study the joint Dirichlet form related to $\mathcal{G}_\theta$ (cf. Subsection 6.3.4). We present a main result of Chapter 6:

- *Dirichlet form* (cf. Proposition 6.3.47 and Theorem 6.3.48): The *joint bilinear form* (cf. (6.3.46))

$$\mathcal{E}^{\mathcal{G}_\theta}(F,G) := \int_{\mathbb{K}(\mathbb{R}^d)} \big\langle \nabla^{\mathbb{K}} F(\eta), \nabla^{\mathbb{K}} G(\eta) \big\rangle_{T_\eta(\mathbb{K})} \mathcal{G}_\theta(d\eta), \quad F, G \in \mathbb{S}_{\mathbb{K}(\mathbb{R}^d)},$$

  is closable and its closure is a conservative Dirichlet form.

Analogous results hold in the intrinsic and extrinsic case (cf. Theorems 6.3.29 and 6.3.38).

We deduce the above results for more general measures on $\mathbb{K}(\mathbb{R}^d)$:

1. Let $\mathcal{G}_\lambda$ on $\mathbb{K}(\mathbb{R}^d)$ be the image measure of a Poisson measure $\mathcal{P}_\lambda$ on $\Gamma(\hat{\mathbb{R}}^d)$ whose intensity measure λ on $\mathbb{R}_+$ has first and second moments, i.e.,

$$m_1(\lambda) + m_2(\lambda) = \int_{\mathbb{R}_+} \big(s + s^2\big)\, \lambda(ds) < \infty.$$

   Then the intrinsic results hold for $\mathcal{G}_\lambda$ (cf. Theorems 6.3.8, 6.3.14 and Proposition 6.3.12 in Subsection 6.3.1). [5]

[5]Each $\mathcal{G}_\lambda$ is a Levy measures (cf. Definition 3.1.5).

2. The extrinsic, intrinsic and joint results are extended to the Gibbsian case (cf. Theorems 6.3.19, 6.3.33, 6.3.39; 6.3.29, 6.3.38 and 6.3.48).

The results of Chapter 6 hold in a more general setting, where $\mathbb{R}^d$ is replaced by an arbitrary connected, orientated, separable C^∞-Riemannian manifold with Riemannian metric d_X.

Equilibrium processes

A main difficulty to obtain diffusions on $\mathbb{K}(\mathbb{R}^d)$ is to find the correct underlying Polish space for the quasi-regular property.[6]

For simplicity, let m be the Lebesgue measure dx on $\mathbb{R}^d$. We use the *configuration space of multiple configurations* in $\hat{\mathbb{R}}^d$ (cf. (7.2.1))

$$\ddot{\Gamma}(\hat{\mathbb{R}}^d) := \left\{ \gamma = \sum_{y\in\gamma} m_y \delta_y \,\middle|\, m_y \in \mathbb{N} \text{ and } \gamma(\Lambda) < \infty,\ \forall\ \Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d) \right\}.$$

We define a functional (cf. Definition 7.2.7)

$$d_{\ddot{\Gamma}_f}(\gamma,\gamma') : \ddot{\Gamma}(\hat{\mathbb{R}}^d) \times \ddot{\Gamma}(\hat{\mathbb{R}}^d) \ni (\gamma,\gamma') \mapsto d_{\ddot{\Gamma}_f}(\gamma,\gamma') \in [0,\infty],$$

which is a metric on (cf. Definition 7.2.9)

$$\ddot{\Gamma}_f(\hat{\mathbb{R}}^d) := \left\{ \gamma \in \ddot{\Gamma}(\hat{\mathbb{R}}^d) \;\middle|\; d_{\ddot{\Gamma}_f}(\gamma,\emptyset) < \infty \right\}.$$

The space $(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), d_{\ddot{\Gamma}_f})$ is Polish (cf. Theorem 7.2.11); and it is the space on which we will work.

Consider the pre-Dirichlet form defined for all $F \in \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), C_0(\hat{\mathbb{R}}^d))$ via (cf. Definition 7.2.18) [7]

$$\mathcal{E}^{\mathcal{P}_\theta,\ddot{\Gamma}_f}(F,F) := \int_{\ddot{\Gamma}_f(\hat{\mathbb{R}}^d)} \int_{\hat{\mathbb{R}}^d} \left(\sqrt{s}\frac{d}{ds}F(\gamma)\right)^2 + \left(\frac{1}{\sqrt{s}}\frac{d}{dx}F(\gamma)\right)^2 \gamma(ds,dx)\mathcal{P}_\theta(d\gamma),$$

[6] Since we do not know whether $\mathbb{K}(\mathbb{R}^d)$ is a Lusin space, we cannot apply the abstract results [BBR06, BBR08].

[7] The set $\mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{X}), C_0^\infty(\hat{X}))$ consists of all functions F which can be represented as

$$\ddot{\Gamma}(\hat{X}) \ni \gamma \mapsto F(\gamma) = g_F(\langle\varphi_1,\gamma\rangle,\ldots,\langle\varphi_N,\gamma\rangle),$$

with some $N \in \mathbb{N}$, $g_F \in C_b^\infty(\mathbb{R}^N)$ and $\varphi_i \in C_0^\infty(\hat{X})$, $1 \le i \le N$.

where $\mathcal{P}_\theta$ denotes the extension of the Gamma-Poisson measure to $\ddot{\Gamma}_f(\hat{\mathbb{R}}^d)$ by 0. The closure $(\mathcal{E}^{\mathcal{P}_\theta,\ddot{\Gamma}_f}, \mathcal{D}(\mathcal{E}^{\mathcal{P}_\theta,\ddot{\Gamma}_f}))$ is a conservative Dirichlet form (cf. Theorem 7.2.22). As an essential step to obtain an associated diffusion, we prove that this Dirichlet form is quasi-regular (cf. Theorem 7.2.39).

We get a conservative diffusion[8] $\mathbf{M}^{\ddot{\Gamma}_f}$ that is properly associated with $(\mathcal{E}^{\mathcal{P}_\theta,\ddot{\Gamma}_f}, \mathcal{D}(\mathcal{E}^{\mathcal{P}_\theta,\ddot{\Gamma}_f}))$ (cf. Theorem 7.3.7). One drawback is that the process is only constructed on $\ddot{\Gamma}(\hat{\mathbb{R}}^d)$. We prove that it is actually a diffusion on the set of *pinpointing configurations with finite local mass* $d_{\ddot{\Gamma}_f}(\emptyset,\cdot)$, i.e., on

$$\Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d) := \Big\{\gamma \in \ddot{\Gamma}_f(\hat{\mathbb{R}}^d) \big| \gamma(\mathbb{R}_+ \times \{x\}) \le 1 \qquad \forall x \in \mathbb{R}^d\Big\}$$

(cf. Theorem 7.3.12). For the proof we show that $\ddot{\Gamma}(\hat{\mathbb{R}}^d)\backslash\Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d)$ is $\mathcal{E}^{\mathcal{P}_\theta,\ddot{\Gamma}_f}$-exceptional by extending a technique presented in [RS98]. Then we get a main result of Chapter 7:

- *Existence of a conservative diffusion on* $\mathbb{K}(\mathbb{R}^d)$ (cf. Theorem 7.4.4): Let $d \ge 2$. Then there exists a conservative diffusion process

  $$\mathbf{M} := \mathbf{M}^{\mathbb{K}(\mathbb{R}^d)} = \Big(\mathbf{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t\ge0}, (\mathbf{\Theta}_t)_{t\ge0}, (\mathbf{X}(t))_{t\ge0}, (\mathbf{P}_\eta)_{\eta\in\mathbb{K}(\mathbb{R}^d)}\Big)$$

  on $\mathbb{K}(\mathbb{R}^d)$ which is properly associated with $\big(\mathcal{E}^{\mathcal{G}_\theta,\ddot{\Gamma}_f}, \mathcal{D}\big(\mathcal{E}^{\mathcal{G}_\theta,\ddot{\Gamma}_f}\big)\big)$,[9] i.e., for all ($\mathcal{G}_\theta$-versions) of $F \in L^2(\mathbb{K}(\mathbb{R}^d), \mathcal{G}_\theta)$ and all $t > 0$ the function

  $$\mathbb{K}(\mathbb{R}^d) \ni \eta \mapsto p_t F(\eta) := \int_\Omega F(\mathbf{X}(t)) d\mathbf{P}_\eta$$

  is an $\mathcal{E}^{\mathcal{G}_\theta,\ddot{\Gamma}_f}$-quasi-continuous version of $\exp(-tL^{\mathcal{G}_\theta,\ddot{\Gamma}_f})F$, where $L^{\mathcal{G}_\theta,\ddot{\Gamma}_f}$ is the generator of $\big(\mathcal{E}^{\mathcal{G}_\theta,\ddot{\Gamma}_f}, \mathcal{D}\big(\mathcal{E}^{\mathcal{G}_\theta,\ddot{\Gamma}_f}\big)\big)$ (cf. also Theorem 7.2.20). $\mathbf{M}$ is up to $\mathcal{G}_\theta$-equivalence unique. In particular, $\mathbf{M}$ is $\mathcal{G}_\theta$-symmetric (i.e., $\int G p_t F d\mathcal{G}_\theta = \int F p_t G d\mathcal{G}_\theta$ for all $F, G : \mathbb{K}(\mathbb{R}^d) \to \mathbb{R}$, $\mathcal{B}(\mathbb{K}(\mathbb{R}^d))$-measurable) and has $\mathcal{G}_\theta$ as an invariant measure.

- *Existence of extrinsic and extrinsic diffusions on* $\mathbb{K}(\mathbb{R}^d)$ (cf. Corollary 7.4.5): There exist extrinsic, intrinsic and joint diffusions on $\mathbb{K}(\mathbb{R}^d)$, $d \ge 2$, describing the motion of marks and positions.[10]

  In particular, there exists a diffusion describing the motion of the dense set $\tau(\eta_t) \in \mathbb{R}^d$, where $\eta_t \in \mathbb{K}(\mathbb{R}^d)$ for all $t \ge 0$.

[8] A diffusion is a strong Markov process with continuous sample paths

[9] This is the Dirichlet form on $\mathbb{K}(\mathbb{R}^d)$ that corresponds to $\big(\mathcal{E}^{\mathcal{P}_\theta,\ddot{\Gamma}_f}, \mathcal{D}\big(\mathcal{E}^{\mathcal{P}_\theta,\ddot{\Gamma}_f}\big)\big)$.

[10] The extrinsic motion exists also for $d = 1$.

The above results extend to more general situations (cf. Theorem 7.4.4):

1. We can equip the Lebesgue measure dx with a density $\rho \in H^{1,2}_{\mathrm{loc}}(\mathbb{R}^d, dx)$ such that $m(dx) = \rho(x)dx$ fulfills (cf. (7.2.12))

$$\exists M,\ C \geq 1 : \quad m\left(\left\{x \in \mathbb{R}^d \;\middle|\; |x| \leq k\right\}\right) \leq MC^k.$$

2. In addition to the first extension, $\mathcal{G}_\theta$ can be replaced by a Gibbs perturbation of $\mathcal{G}_\theta$ w.r.t. some non-negative potential $\phi \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$.

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Part I

Gamma and Gamma-Poisson measures

Chapter 2

Poisson measures

In this chapter, we recall properties of the space of (locally finite) configurations, construct Poisson measures on it (cf. Section 2.1) and establish the concrete setting in which we will work (cf. Section 2.2).

The notion of a configuration space as a model for describing many-particle systems appeared first in statistical mechanics. There the state of an ideal gas is described by a *Poisson* random point field. Nowadays configuration spaces are also widely applied in computer science and biology.

One of the early papers treating configuration spaces is [VGG75]. Others are e.g. [AKR98a, AKR98b], where the geometry of configuration spaces is studied, or [Geo88, Pre05, KPR10], where the existence of Gibbs measures is treated. For a more detailed account, we refer to e.g. [Kun99].

In Section 2.1, we fix a locally compact Polish space Y with the Borel σ-algebra $\mathcal{B}(Y)$ and define the configurations space $\Gamma(Y)$ over Y as (cf. also Subsection 2.1.1)

$$\Gamma(Y) := \big\{\gamma \subset Y \big| |\gamma_\Lambda| < \infty, \quad \forall \Lambda \in \mathcal{B}_c(Y)\big\},$$

where $\Lambda \in \mathcal{B}_c(Y)$ is a Borel set with compact closure, $\gamma_\Lambda := \gamma \cap \Lambda$ denotes the restriction of γ to Λ and $|\gamma_\Lambda|$ denotes the set cardinality. Fixing a *non-atomic* intensity measure σ on $(Y, \mathcal{B}(Y))$, we construct the Poisson measure π_σ on $\Gamma(Y)$ (cf. Subsection 2.1.2).

Then (cf. Section 2.2), we outline the concrete setting in which we will work:

1. We introduce a space of marks, $\mathbb{R}_+ :=]0, \infty[$, and a space of positions, $\mathbb{R}^d$ with $d \in \mathbb{N}$.

2. Let m be a non-atomic Radon measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ be equipped with
$$\lambda_\theta := \theta \frac{1}{s} e^{-s} ds, \quad \theta > 0 \text{ being a fixed parameter.}$$
Note that λ_θ has a high concentration close to 0. Specifying $Y := \hat{\mathbb{R}}^d := \mathbb{R}_+ \times \mathbb{R}^d$ and $\sigma = \lambda_\theta \otimes m$, we get the *Gamma*-Poisson measure $\mathcal{P}_\theta$, which will be an import measure for our considerations.

3. We identify a smaller set $\Gamma_f(\hat{\mathbb{R}}^d) \subset \Gamma(\hat{\mathbb{R}}^d)$ that supports $\mathcal{P}_\theta$ (cf. Subsection 2.2.1):

   (a) We define the set of *pinpointing* configurations (cf. Definition 2.2.2)
   $$\Gamma_p(\hat{\mathbb{R}}^d) := \big\{\gamma \in \Gamma(\hat{\mathbb{R}}^d) \big| \text{ for all } (s_1, x_1), (s_2, x_2) \in \gamma \text{ we have } x_1 = x_2 \Rightarrow s_1 = s_2\big\}.$$

   (b) For all $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ and $\gamma \in \Gamma_p(\hat{\mathbb{R}}^d)$ we set the local mass as (cf. Definition 2.2.6)
   $$\mathfrak{m}_\Delta(\gamma) := \sum_{\hat{x}=(s_x,x)\in\gamma} s \mathbb{1}_\Delta(x).$$
   We define the set of pinpointing configurations with *finite local mass* as (cf. Definition 2.2.7)
   $$\Gamma_f(\hat{\mathbb{R}}^d) := \big\{\gamma \in \Gamma_p(\hat{\mathbb{R}}^d) \big| \mathfrak{m}_\Delta(\gamma) < \infty, \ \forall \Delta \in \mathcal{B}_c(\mathbb{R}^d)\big\}.$$

   (c) We show that $\mathcal{P}_\theta\big(\Gamma_f(\hat{\mathbb{R}}^d)\big) = 1$ (cf. Theorems 2.2.4 and 2.2.9), which is basically due to the following relation between $\mathfrak{m}_\Delta$ and $\lambda_\theta \otimes m$:
   $$\int_{\Gamma(\hat{\mathbb{R}}^d)} \mathfrak{m}_\Delta(\gamma) \mathcal{P}_\theta = \int_\Delta \int_{\mathbb{R}_+} s_x \lambda_\theta(ds) m(dx) < \infty, \ \forall \Delta \in \mathcal{B}_c(\mathbb{R}^d).$$

   By the last step, we can consider the image measure of $\mathcal{P}_\theta$ on the cone of discrete Radon measures $\mathbb{K}(\mathbb{R}^d)$, which is bijective to $\Gamma_f(\hat{\mathbb{R}}^d)$ (cf. Chapter 3).

2.1 A short introduction to configuration spaces

We recall some facts about the configuration space $\Gamma(Y)$ related to the topology and the measurable structure and construct the Poisson measure. This outline is based on [Kun99, Section 2.1] and [KPR10, Section 2.1].

2.1.1 Configuration space

The local compact Polish space Y is called *phase space*. It is equipped with the Borel σ-algebra $\mathcal{B}(Y)$ generated by the family $\mathcal{O}(Y)$ of open sets in Y. The system $\mathcal{B}_c(Y)$ consists of all Borel sets with compact closure. Let $C_0(Y)$ denote the set of continuous functions $f : Y \to \mathbb{R}$ with compact support.

Configurations $\gamma \in \Gamma(Y)$

For each $\Lambda \in \mathcal{B}(Y)$, the *configuration space* $\Gamma(\Lambda)$ is the system of all locally finite subsets of Λ:

$$\Gamma(\Lambda) := \{\gamma \subset \Lambda \, || \gamma_{\Lambda'}| < \infty, \ \forall \, \Lambda' \in \mathcal{B}_c(\Lambda)\} \,. \tag{2.1.1}$$

Each $\gamma \in \Gamma(\Lambda)$ can be identified with the corresponding counting measure $\sum_{y\in\gamma} \delta_y$, where δ_y denotes the Dirac measure with mass 1 at point y. For example, to the configuration $\gamma = \emptyset$ there corresponds the zero measure on Λ. Therefore we have a natural embedding

$$\Gamma(\Lambda) \subset \mathcal{M}(\Lambda),$$

where $\mathcal{M}(\Lambda)$ denotes the linear space of all Radon measures on Λ.

Topology on the configuration space $\Gamma(Y)$ Fix $\Lambda \in \mathcal{B}(Y)$. We equip $\Gamma(\Lambda)$ with the *vague topology* inherited from $\mathcal{M}(\Lambda)$, i.e. with the coarsest topology on $\mathcal{M}(\Lambda)$ such that each of the following maps is continuous

$$\mathcal{M}(\Lambda) \ni \nu \mapsto \langle f, \nu \rangle := \int_\Lambda f(y)\nu(dy), \quad f \in C_0(\Lambda). \tag{2.1.2}$$

The vague topology on $\Gamma(\Lambda)$ will be denoted by $\mathcal{O}(\Gamma(\Lambda))$.

We remark (cf. [KPR10, P.5]) that $\Gamma(\Lambda)$ equipped with the vague topology is a Polish space (cf. [Kal83, 15.7.7] and, for a concret metric, [KK06]).

The Borel σ-algebra on $\Gamma(Y)$ Let $\Lambda \in \mathcal{B}(Y)$. By $\mathcal{B}(\Gamma(Y))$ we denote the Borel σ-algebra associated to the vague topology on $\Gamma(Y)$. An equivalent definition of $\mathcal{B}(\Gamma(\Lambda))$ can be given via the counting mappings defined for all $\Lambda', \tilde{\Lambda} \in \mathcal{B}(Y)$ as

$$\begin{aligned} N_{\Lambda',\tilde{\Lambda}} : \quad \Gamma(\Lambda') &\to \mathbb{N}_0 \cup \{\infty\} \\ \gamma &\mapsto |\gamma \cap \tilde{\Lambda}|. \end{aligned} \tag{2.1.3}$$

Namely,

$$\mathcal{B}(\Gamma(\Lambda)) = \sigma\big(N_{\Lambda,\tilde{\Lambda}} | \tilde{\Lambda} \in \mathcal{B}_c(\Lambda)\big). \tag{2.1.4}$$

Lemma 2.1.1 ([Pre05, Lemma 2.13, p. 24])**.** *Each function*

$$\Gamma(Y) \ni \gamma \mapsto \langle f, \gamma \rangle$$

is $\mathcal{B}(\Gamma(Y))$-measurable for $f : Y \to \mathbb{R}$ being $\mathcal{B}(Y)$-measurable and supported by a compact set.

Finite Configuration

Fix $\Lambda \in \mathcal{B}(Y)$. We define for $n \in \mathbb{N}_0$ the *space of n-point configurations over* Λ

$$\Gamma_0^{(n)}(\Lambda) := \{\gamma \in \Gamma(\Lambda) \,|\, |\gamma| = n\}, \text{ for } n \in \mathbb{N} \text{ and } \quad \Gamma_0^{(0)}(\Lambda) := \{\emptyset\}. \tag{2.1.5}$$

The *space of finite configurations located in a set* Λ is defined as the disjoint union

$$\Gamma_0(\Lambda) := \bigsqcup_{n \in \mathbb{N}_0} \Gamma_0^{(n)}(\Lambda) \tag{2.1.6}$$

We call $\Gamma_0(Y)$ the *space of finite configurations.*

Note that $\Gamma_0(\Lambda) = \big\{\gamma \in \Gamma(Y) \,\big|\, |\gamma| < \infty,\ \gamma_{Y \setminus \Lambda} = \emptyset\big\}$ and $\Gamma_0(\Lambda) = \Gamma(\Lambda)$ for $\Lambda \in \mathcal{B}_c(Y)$.

Borel σ-albgebra on $\Gamma_0(Y)$ We will define more structure on $\Gamma_0^{(n)}(Y)$. For each $n \in \mathbb{N}_0$ and $\Lambda \in \mathcal{B}(Y)$ let

$$\widetilde{\Lambda^n} := \big\{(y_1, \dots, y_n) \big| y_k \in \Lambda \text{ for } 1 \le k \le n,\ \ y_k \ne y_j \text{ if } k \ne j\big\}$$

and equip $\Gamma_0^{(n)}(\Lambda)$ with the weakest topology, denoted by $\mathcal{O}(\Gamma_0^{(n)}(\Lambda))$, such that each of the following natural ("symmetrizing") mappings is continuous

$$\begin{array}{rccl} \mathrm{sym}_\Lambda^n : & \widetilde{\Lambda^n} & \to & \Gamma_0^{(n)}(\Lambda) \\ & (y_1, \dots, y_n) & \mapsto & \{y_1, \dots, y_n\}. \end{array} \tag{2.1.7}$$

Then we equip $\Gamma_0(\Lambda)$ with the topology $\mathcal{O}\big(\Gamma_0(\Lambda)\big)$ being the topology of disjoint unions of $\mathcal{O}\big(\Gamma_0^{(n)}(\Lambda)\big)$ on $\Gamma_0^{(n)}(\Lambda)$.

Let $\mathcal{B}\big(\Gamma_0^{(n)}(\Lambda)\big)$ denote the Borel σ-algebra on $\Gamma_0^{(n)}(\Lambda)$ which is generated by $\mathcal{O}\big(\Gamma_0^{(n)}(\Lambda)\big)$ and $\mathcal{B}\big(\Gamma_0(\Lambda)\big)$ the one on $\Gamma_0(\Lambda)$. Using (2.1.3), we have (cf. e.g. [Len75])

$$\mathcal{B}\big(\Gamma_0^{(n)}(\Lambda)\big) = \sigma\big(N_{\Lambda,\tilde{\Lambda}} \big| \tilde{\Lambda} \in \mathcal{B}_c(\Lambda)\big). \tag{2.1.8}$$

For $\Lambda \in \mathcal{B}_c(Y)$, we note that not only $\Gamma_0(\Lambda) = \Gamma(\Lambda)$, but also

$$\mathcal{B}\big(\Gamma_0(\Lambda)\big) = \mathcal{B}\big(\Gamma(\Lambda)\big) = \mathcal{B}(\Gamma(Y)) \cap \Gamma(\Lambda), \quad \forall \Lambda \in \mathcal{B}_c(Y).$$

Algebra of cylindrical sets

Fix $\Lambda, \tilde{\Lambda} \in \mathcal{B}(Y)$ with $\tilde{\Lambda} \subset \Lambda$. We define

$$\mathcal{B}_{\tilde{\Lambda}}\big(\Gamma(\Lambda)\big) := \sigma\left(\left\{ N_{\Lambda,\Lambda'} \,\middle|\, \Lambda' \in \mathcal{B}(\Lambda),\ \Lambda' \subset \tilde{\Lambda} \right\}\right) \ \big(\subset \mathcal{B}(\Gamma(\Lambda))\big). \tag{2.1.9}$$

It is only sensitive to sets in $\tilde{\Lambda}$. We consider the natural projections

$$\begin{array}{rccc} \mathbb{P}_{\Lambda,\tilde{\Lambda}}: & \Gamma(\Lambda) & \to & \Gamma(\tilde{\Lambda}) \\ & \gamma & \mapsto & \gamma_{\tilde{\Lambda}} \end{array} \tag{2.1.10}$$

and note that

$$\mathcal{B}_{\tilde{\Lambda}}(\Gamma(\Lambda)) = \mathbb{P}^{-1}_{\Lambda,\tilde{\Lambda}} \circ \mathcal{B}(\Gamma(\tilde{\Lambda})). \tag{2.1.11}$$

In order to avoid confusions we will not use the same notation for both of them because $\mathcal{B}(\Gamma(\tilde{\Lambda}))$ and $\mathcal{B}_{\tilde{\Lambda}}(\Gamma(\Lambda))$ are σ-algebras on different spaces, $\Gamma(\tilde{\Lambda})$ resp. $\Gamma(\Lambda)$. Using the later σ-algebra we define the *algebra of cylindrical sets*

$$\mathcal{B}_{\mathrm{cyl}}(\Gamma(\Lambda)) := \bigcup_{\tilde{\Lambda} \in \mathcal{B}_c(\Lambda)} \mathcal{B}_{\tilde{\Lambda}}(\Gamma(\Lambda)) \tag{2.1.12}$$

Note that $\mathcal{B}_{\mathrm{cyl}}(\Gamma(\Lambda))$ is a subsystem of $\mathcal{B}(\Gamma(\Lambda))$.

2.1.2 Poisson measure

We perform the well-known explicit construction of π_σ (see e.g. [AKR98b, Section 2.1] or [DVJ03, Section 2.4]). On the underlying phase space Y, we fix an *intensity* measure σ being a *non-atomic* Radon measure on $(Y, \mathcal{B}(Y))$, for which

$$\sigma(\{x\}) = 0 \ \text{ for all } x \in Y.$$

Typically, we have $\sigma(Y) = \infty$.

By $B(\Gamma(Y))$ we denote the set of all bounded $\mathcal{B}(\Gamma(Y))$-measurable functions $F : \Gamma(Y) \to \mathbb{R}$. For each $\Lambda \in \mathcal{B}_c(Y)$, the corresponding *Lebesgue-Poisson* measure P^Λ_σ with intensity measure σ on $(\Gamma(\Lambda), \mathcal{B}(\Gamma(\Lambda)))$ is defined by the identity

$$\begin{aligned} &\int_{\Gamma(\Lambda)} F(\gamma_\Lambda) P^\Lambda_\sigma(d\gamma_\Lambda) \\ :=&F(\{\varnothing\}) + \sum_{n\in\mathbb{N}} \frac{1}{n!} \int_{\Lambda^n} F(\{x_1, \dots, x_n\}) \mathrm{d}\sigma(x_1) \dots \mathrm{d}\sigma(x_n), \end{aligned} \tag{2.1.13}$$

which should hold for all bounded measurable functions $F \in B(\Gamma(\Lambda))$. Taking into account that $P_\sigma^\Lambda(\Gamma(\Lambda)) = e^{\sigma(\Lambda)} < \infty$, we introduce the probability measures

$$\pi_\sigma^\Lambda := e^{-\sigma(\Lambda)} P_\sigma^\Lambda. \tag{2.1.14}$$

Note that the family $\left\{\pi_\sigma^\Lambda \mid \Lambda \in \mathcal{B}_c(Y)\right\}$ is consistent, which means (using the projections defined in (2.1.15))

$$\pi_\sigma^\Lambda \circ \mathbb{P}_{\Lambda,\Lambda'}^{-1} = \pi_\sigma^{\Lambda'} \quad \text{whenever } \Lambda' \subset \Lambda.$$

By Kolmogorov's theorem (cf. [Pat67, Theorem V.3.2]), the *Poisson* measure π_σ is the unique probability measure on $(\mathit{\Gamma}, \mathcal{B}(\mathit{\Gamma}))$ such that

$$\pi_\sigma^\Lambda = \pi_\sigma \circ \mathbb{P}_\Lambda^{-1} \quad \text{for all } \Lambda \in \mathcal{B}_c(Y). \tag{2.1.15}$$

An equivalent way of defining π_σ is to claim that, for any collection of disjoint domains $(\Lambda_j)_{j=1}^N \subset \mathcal{B}_c(Y)$, the random variables $N_{\hat{X},\Lambda_j}(\gamma)$ (cf. (2.1.3)) should be mutually independent and distributed by the Poissonian law with parameters $\sigma(\Lambda_j)$, i.e.,

$$\pi_\sigma\left(\left\{\gamma \in \mathit{\Gamma} \mid N_{\Lambda_j}(\gamma) = n\right\}\right) = \frac{\sigma^n(\Lambda_j)}{n!} e^{-\sigma(\Lambda_j)}, \quad n \in \mathbb{Z}_+. \tag{2.1.16}$$

Another well-known analytic characterization of π_σ is given through its Laplace transform, see e.g. [GV64],

$$\int_{\mathit{\Gamma}(Y)} \exp\langle f, \gamma\rangle \mathrm{d}\pi_\sigma(\gamma) = \exp\left\{\int_Y \left(e^{f(x)} - 1\right) \mathrm{d}\sigma(x)\right\}, \quad f \in C_0(Y). \tag{2.1.17}$$

Topologies on spaces of measures over $\Gamma(Y)$

Let $\mathcal{M}^1(\Gamma(Y))$ denote the space of all probability measures on $\Gamma(Y)$.

Definition 2.1.2 ([KPR10, Subsection 2.4])**.** *On the space of all probability measures $\mathcal{M}^1(\Gamma(Y))$ we introduce the topology of* local setwise convergence*. This topology, which we denote by $\mathcal{T}_{\mathrm{loc}}$, is defined as the coarsest topology making the maps $\mu \mapsto \mu(B)$ continuous for all sets B from the algebra*

$$\mathcal{B}_{cyl}(\Gamma(Y)) = \bigcup_{|\Lambda|<\infty} \mathcal{B}_\Lambda(\Gamma(Y)).$$

Equivalently, $\mathcal{T}_{\mathrm{loc}}$ is the coarsest topology such that $\mu \mapsto \mu(F)$ is continuous for all bounded $\mathcal{B}_{\mathrm{cyl}}(\mathit{\Gamma}(Y))$-measurable functions $F : \mathit{\Gamma}(Y) \to \mathbb{R}$. Since the topology $\mathcal{T}_{\mathrm{loc}}$ is not metrizable (cf. [Geo88, p.57]), the notions of convergence and sequential convergence in $\mathcal{T}_{\mathrm{loc}}$ do not coincide.

2.2 Gamma-Poisson measures

We specify the abstract setting of Section 2.1 to our needs: In particular, we introduce $\mathbb{R}_+$ as a space of marks and a locally compact Polish space X as a space of positions. Then we get the Gamma-Poisson measure $\mathcal{P}_\theta$ on $\Gamma(\hat{X})$, where $\hat{X} = \mathbb{R}_+ \times X$. The main aim is to prove that $\mathcal{P}_\theta\big(\Gamma_f(\hat{X})\big) = 1$ (cf. Theorem 2.2.9).

The space $\mathbb{R}_+$ of marks

We equip $\mathbb{R}_+ =]0, \infty[$ with the *logarithmic metric*[1]

$$d_{\mathbb{R}_+}(x, y) := \left| \ln \frac{x}{y} \right| \quad \forall\, x, y \in \mathbb{R}_+. \tag{2.2.1}$$

This metric is invariant under multiplication. This means that for all positive functions $g : \mathbb{R}_+ \ni x \mapsto gx := g(x) \in \mathbb{R}_+$, we have $d_{\mathbb{R}_+}(gx, gy) = d_{\mathbb{R}_+}(x, y)$. Furthermore, the metric $d_{\mathbb{R}_+}$ is locally equivalent to the usual one on $\mathbb{R}$ being restricted to $\mathbb{R}_+$. Hence, $(\mathbb{R}_+, d_{\mathbb{R}_+})$ is a locally compact Polish space.

Let $\mathcal{B}(\mathbb{R}_+)$ denote the Borel σ-algebra on $\mathbb{R}_+$. Observe that $\mathcal{B}(\mathbb{R}_+) = \mathcal{B}(\mathbb{R}) \cap \mathbb{R}_+$. We consider a Radon measure λ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)$ such that

$$\int_{\mathbb{R}_+} s\lambda(ds) < \infty \tag{2.2.2}$$

A typical example is, letting ds denote the Lesgue measure on $\mathbb{R}_+$ and $\theta > 0$ being a fixed parameter,

$$\lambda_\theta(ds) = \theta \frac{e^{-s}}{s} ds. \tag{2.2.3}$$

This measure has a high concentration close to zero.

We like to point out a particular property to familiarize ourselves with the space $(\mathbb{R}_+, d_{\mathbb{R}_+})$: In the metric $d_{\mathbb{R}_+}$ let us consider the ball $B_R(1)$ centered at 1 with radius $R > 0$,

$$\begin{aligned} B_R(1) =& \{r \in \mathbb{R}_+ | d_{\mathbb{R}_+}(r, 1) \leq R\} = \left\{ r \in \mathbb{R}_+ | \left| \ln \frac{r}{1} \right| \leq R \right\} \\ =& \left\{ r \in \mathbb{R}_+ | \left(r \geq 1 : \, r \leq e^R \right) \text{or} \, (0 < r \leq 1 : \, -\ln r \leq R) \right\} \\ =& \left[e^{-R}, e^R \right], \end{aligned} \tag{2.2.4}$$

which shows that the distance from 1 to e^{-R}, as well as the one from 1 to e^R, is R. Therefore, the distance to 0, as well as the one to ∞, is infinite.

[1] The logarithmic metric is quite useful to construct later a corresponding Gibbs measure in Chapter 4.

The space X of positions

We denote for the locally compact Polish space X by $\mathcal{B}(X)$ its Borel σ-algebra and by $\mathcal{B}_c(X)$ that subset consisting only of the Borel sets with compact closure. Moreover, we pick a non-atomic Radon measure m on X.

Remark 2.2.1. 1. *Our* standard example *is $X = \mathbb{R}^d$, $d \in \mathbb{N}$ fixed, being the d-dimensional vector space over the real numbers with the usual Euclidean metric and $m(dx) = dx$ being the Lebesgue measure on $\mathbb{R}^d$.*

2. *We stress that we have a broken symmetry between the spaces of marks and the space of positions: For the standard example, we see that for all $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$*

$$\int_{\mathbb{R}_+} \int_{\Delta} s\, dx \lambda_\theta(ds) < \infty. \tag{2.2.5}$$

Configuration space over $\mathbb{R}_+ \times X$

The next step is to combine the space of marks $\mathbb{R}_+$ and the space of positions X to one space, so that we can define measures on $\Gamma(\mathbb{R}_+ \times X)$.

Let $\hat{X} := \mathbb{R}_+ \times X$ be the product space of the space $\mathbb{R}_+$ of marks and the space X of positions (or locations). We refer to $\hat{X}$ as the *phase space*. From now on without further notice, we denote the elements of $\hat{X}$ by $\hat{x} = (s, x)$ where $s \in \mathbb{R}_+$, $x \in X$. The same is true, if these elements are indexed, i.e. $\hat{x}_i = (s_i, x_i) \in \hat{X}$ for all $i \in \mathbb{N}$. $\hat{X}$ is equipped with the metric

$$d_{\hat{X}}(\hat{x}_1, \hat{x}_2) := d_{\mathbb{R}_+}(s_1, s_2) + d_X(x_1, x_2), \quad \forall \hat{x}_1, \hat{x}_2 \in \hat{X} \tag{2.2.6}$$

and the corresponding Borel σ-algebra $\mathcal{B}(\hat{X})$. Moreover, for notational convenience we define for each $\Lambda \subset \hat{X}$ the projection to $\mathbb{R}_+$ and X by

$$\Lambda_{\mathbb{R}_+} := \{s \in \mathbb{R}_+ \,|\, \exists\, x \in X : \; (s, x) \in \Lambda\} \quad \text{and} \tag{2.2.7}$$

$$\Lambda_X := \{x \in X \,|\, \exists\, s \in \mathbb{R}_+ : \; (s, x) \in \Lambda\}. \tag{2.2.8}$$

We refer to $\Lambda_{\mathbb{R}_+}$ as the set of types, *marks* or species, whereas Λ_X can be considered as the *support*.

2.2.1 Poisson measure on $\Gamma(\hat{X})$

Our next task is to introduce the Gamma-Poisson measure $\mathcal{P}_\theta$ on $\Gamma(\hat{X})$ and show $\mathcal{P}_\theta(\Gamma_f(\hat{X})) = 1$.

Setting $Y = \hat{X}$, we apply Section 2.1 to our setting: $\Gamma(\hat{X})$ is the *configuration space over* $\hat{X}$. Each $\gamma \in \Gamma(\hat{X})$ can be represented as

$$\gamma = \{\hat{x}_i \mid i \in \mathbb{N}\},$$

where $\hat{x}_i = (s_i, x_i) \in \hat{X}$, $s_i \in \mathbb{R}_+$ and $x_i \in X$. As usual, $\mathcal{M}^1(\Gamma(\hat{X}))$ denote the space of probability measures on $(\Gamma(\hat{X}), \mathcal{B}(\Gamma(\hat{X})))$. On $\Gamma(\hat{X})$ we consider the Poisson measure $\pi_{\lambda\otimes m}$ with intensity measure $\lambda \otimes m$ on $\hat{X}$ (cf. (2.1.15)) and denote it by

$$\mathcal{P}_\lambda(d\gamma) := \pi_{\lambda\otimes m}(d\gamma). \tag{2.2.9}$$

In particular, we fix $\theta > 0$ and consider (cf. (2.2.3))

$$\mathcal{P}_\theta(d\gamma) := \pi_{\lambda_\theta\otimes m}(d\gamma). \tag{2.2.10}$$

We call it *Gamma-Poisson measure* on $\Gamma(\hat{X})$.[2]

For each $\Lambda \in \mathcal{B}_c(\hat{X})$ denote by

$$\begin{array}{llll} P_\lambda^\Lambda := P_{\lambda\otimes m}^\Lambda & \text{and} & P_\theta^\Lambda := P_{\lambda_\theta\otimes m}^\Lambda, & \text{resp.} \\ \mathcal{P}_\lambda^\Lambda := \pi_{\lambda\otimes m}^\Lambda(d\gamma) & \text{and} & \mathcal{P}_\theta^\Lambda := \pi_{\lambda_\theta\otimes m}^\Lambda(d\gamma). & \end{array} \tag{2.2.11}$$

the corresponding Lebesgue-Poisson resp. Poisson measures on $\Gamma(\Lambda)$ (cf. (2.1.14), (2.1.1)).

Support of the Gamma-Poisson measure $\mathcal{P}_\theta$

The support of each configuration $\gamma \in \Gamma(\hat{X})$ is given by the projection

$$\tau(\gamma) := \gamma_X. \tag{2.2.12}$$

It represents the positions of all particles. We note that typically the support $\tau(\gamma)$ does not have to be a configuration on X, i.e., $\tau(\gamma) \cap \Delta$ is in general dense in Δ for any $\varnothing \neq \Delta \in \mathcal{B}_c(X)$ open. Since $\gamma \in \Gamma(\hat{X})$ is countable, also its support $\tau(\gamma)$ is countable.

Pinpointing configurations

Definition 2.2.2. *The set of* pinpointing configurations *in* $\Lambda \in \mathcal{B}(\hat{X})$ *is given by*

$$\Gamma_p(\hat{X}) := \Big\{\gamma = \{\hat{x}_i\} \in \Gamma(\Lambda) \Big| \ \forall \hat{x}_1 = (s_1, x_1),\ \hat{x}_2 = (s_2, x_2) \in \gamma : \\ x_i = x_j \Rightarrow s_i = s_j \Big\}. \tag{2.2.13}$$

[2] $\mathcal{P}_\lambda$ is a marked Poisson measure, if $\lambda(\mathbb{R}_+) < \infty$ (cf. [KLU99, Section 2.1]). But, $\mathcal{P}_\theta$ on $\Gamma(\hat{\mathbb{R}}^d)$ is a compound Poisson measure (cf. [KdSSU98, Definition 3.1]).

Each pinpointing configuration $\gamma \in \Gamma_p(\hat{X})$ has one mark s_x associated to a position $x \in \tau(\gamma)$. Therefore, we may denote

$$\gamma = \big\{(s_x, x)\big| x \in \tau(\gamma)\big\} \qquad \forall \gamma \in \Gamma_p(\hat{X}).$$

Here, we use the visual image of a pin: The pinpoint indicates the location x of the particle, whereas the pinhead represents its mark s_x.

Lemma 2.2.3. *The set of pinpointing configurations on each $\Lambda \in \mathcal{B}(\hat{X})$ is measurable, i.e. $\Gamma_p(\hat{X}) \in \mathcal{B}(\Gamma(\Lambda))$. In particular, $\Gamma_p(\hat{X}) \in \mathcal{B}(\Gamma(\hat{X}))$.*

Proof. First of all, we assume that $\Lambda \in \mathcal{B}_c(\hat{X})$. Let $D \in \mathcal{B}(X \times X)$ denotes the diagonal of $X \times X$, i.e., $D := \{(x, x)|x \in X\}$. The complement to $\Gamma_p(\hat{X})$ can be represented as

$$\Gamma_p(\hat{X})^c = \{\gamma \in \Gamma(\Lambda)\,|\exists\ \{\hat{x}_1, \hat{x}_2\} \subset \gamma :\ \{x_1, x_2\} \in D\} = \bigcup_{k\in\mathbb{N}} A_k, \tag{2.2.14}$$

where $A_k \subset \Gamma^{(k)}(\Lambda)$ is defined by

$$A_k := \big\{\gamma \in \Gamma^{(k)}(\Lambda)\,||\gamma| = k, \exists\ \{\hat{x}_1, \hat{x}_2\} \subset \gamma :\ \{x_1, x_2\} \in D\big\}. \tag{2.2.15}$$

For each set A_k, there exists

$$\tilde{A}_k = {\operatorname{sym}_{\hat{X}}^{k}}^{-1}(A_k) \in \mathcal{B}\left(\hat{X}^k\right).$$

This associated set is symmetric in each component and, actually, $\tilde{A}_k \in \mathcal{B}(\Lambda^k)$. Hence, $A_k \in \mathcal{B}(\Gamma(\Lambda)) \subset \mathcal{B}(\Gamma(\hat{X}))$.

Now we treat the general case of a fixed $\Lambda \in \mathcal{B}(\hat{X})$. We choose a countable covering $\{\Lambda_n\}_{n\in\mathbb{N}}$ of $\hat{X}$(and thus of Λ) consisting of increasing compact sets $\Lambda_n \in \mathcal{B}_c(\hat{X})$.[3] We have that

$$\Gamma_p(\hat{X}) = \bigcap_{n\in\mathbb{N}} \{\gamma \in \Gamma(\Lambda)\,|\gamma_{\Lambda_n} \in \Gamma_p(\Lambda_n \cap \Lambda)\} = \bigcap_{n\in\mathbb{N}} \mathbb{P}^{-1}_{\Lambda,\Lambda_n\cap\Lambda}(\Gamma_p(\Lambda_n \cap \Lambda)), \tag{2.2.16}$$

where $\mathbb{P}_{\Lambda,\Lambda_n\cap\Lambda}$ is the $\mathcal{B}(\Gamma(\Lambda))/\mathcal{B}(\Gamma(\Lambda_n \cap \Lambda))$-measurable projection from $\Gamma(\Lambda)$ to $\Gamma(\Lambda_n \cap \Lambda)$. Thus,

$$\Gamma_p(\hat{X})^c \in \mathcal{B}(\Gamma(\Lambda)).$$

□

[3]Let us show this: Because $\hat{X}$ is locally compact, we find for each $\hat{x} \in \hat{X}$ an open set $B_{\hat{x}} \in \mathcal{B}_c(\hat{X})$ containing $\hat{x}$. Then

$$\bigcup_{\hat{x}\in\hat{X}} B_{\hat{x}} \subset \hat{X}$$

is an open covering of $\hat{X}$. Since $\hat{X}$ has the Lindelöf property (cf. [Sie00, Theorems 65 and 49]), we can choose a countable subsection. Hence, we find such an increasing sequence.

Theorem 2.2.4. *We have $\mathcal{P}_\lambda(\Gamma_p(\hat{X})) = 1$. In particular, the Gamma-Poisson measure $\mathcal{P}_\theta$ is supported by $\Gamma_p(\hat{X})$.*

Remark 2.2.5. *$\mathcal{P}_\theta(\Gamma_p(\hat{\mathbb{R}}^d)) = 1$ is given in [KdSS98, Proposition 3.2] for whose proof one may also compare [Kal74, Kal83, MKM78].*

Proof of Theorem 2.2.4. The basic idea of the proof is that $m \otimes m(D) = 0$, where D denotes the diagonal in $X \times X$. To apply this, we rewrite $\Gamma_p(\hat{X})^c$ using an increasing compact sequence $\{A_n\}_{n\in\mathbb{N}}$ covering $\hat{X}$ (similarly to the last proof) and estimate its measure to be 0.

We will show that $\mathcal{P}_\lambda(\Gamma_p(\hat{X})) = 1$. To this end, we choose a countable covering $\{\Lambda_n\}_{n\in\mathbb{N}}$ of $\hat{X}$ consisting of increasing compact sets $\Lambda_n \in \mathcal{B}_c(\hat{X})$.[3] For simplicity, these sets are chosen to be of product type, i.e. $\Lambda_n = \Lambda_{n,\mathbb{R}_+} \times \Lambda_{n,X}$. Taking the complement of the intersection given in (2.2.16) (with $\Lambda = \hat{X}$) we have

$$\mathcal{P}_\lambda\left(\Gamma_p^c(\hat{X})\right) = \mathcal{P}_\lambda\left(\bigcup_{n\in\mathbb{N}} \mathbb{P}^{-1}_{\hat{X},\Lambda_n}\Gamma_p(\Lambda_n)^c\right) \le \sum_{n=1}^{\infty} \mathcal{P}_\lambda\left(\mathbb{P}^{-1}_{\hat{X},\Lambda_n}\Gamma_p(\Lambda_n)^c\right),$$

where $\mathbb{P}_{\hat{X},\Lambda_n}$ is the projection of $\Gamma(\hat{X})$ onto $\Gamma(\Lambda_n)$ (cf. (2.1.15)). Hence, it is enough to prove that for all $\Lambda \in \mathcal{B}_c(\hat{X})$ being of product type[4]

$$0 \overset{!}{=} \mathcal{P}_\lambda\left(\mathbb{P}^{-1}_{\hat{X},\Lambda}\left(\Gamma_p(\hat{X})^c\right)\right) = e^{-\lambda\otimes m(\Lambda)} P^{\Lambda}_{\lambda\otimes m}\left(\Gamma_p(\hat{X})^c\right),$$

which is shown if

$$0 \overset{!}{=} P^{\Lambda}_{\lambda\otimes m}\left(\Gamma_p(\hat{X})^c\right). \tag{2.2.17}$$

In order to show (2.2.17), we fix such a Λ. Using (2.2.15) we calculate

$$P^{\Lambda}_{\lambda\otimes m}(\Gamma_p(\hat{X})^c) \le \sum_{k=0}^{\infty} P^{\Lambda}_{\lambda\otimes m}\left(\left\{\gamma \in \Gamma(\Lambda)^{(k)} \,|\, \exists\ \{\hat{x}_1, \hat{x}_2\} \subset \gamma,\ \{x_1, x_2\} \in D\right\}\right)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}(\lambda\otimes m)^{\otimes k}\left(\left\{\ \{\hat{x}_1, \dots, \hat{x}_k\} \subset \Lambda \,\middle|\, \exists\, i, j \in \{1, \dots, k\}, i \ne j : \{x_i, x_j\} \in D\right\}\right)$$

$$\le \sum_{k=2}^{\infty} \frac{1}{k!}\binom{k}{2}(\lambda\otimes m)^{\otimes(k-2)}(\Lambda^{k-2})\ \underbrace{(m\otimes m)(D)}_{=0} \cdot \lambda(\Lambda_{\mathbb{R}_+})^2 = 0$$

□

Our next task is to prove $\mathcal{P}_\theta(\Gamma_f(\hat{X})) = 1$.

[4] By "!" we indicate a property that is to be shown.

A local mass map By definition, the support $\tau(\gamma)$ of each $\gamma \in \Gamma(\hat{X})$ is a countable set. Thus, for each $\gamma \in \Gamma_p(\hat{X})$ we can sum up the associated marks s_x, $x \in \tau(\gamma)$.

Definition 2.2.6. *For each* $\gamma \in \Gamma(\hat{X})$ *we define its* local mass[5] *on* $\Delta \in \mathcal{B}(X)$ *by*

$$\mathfrak{m}_\Delta(\gamma) := \int_{\mathbb{R}_+ \times \Delta} s\gamma(ds, dx) = \langle s \otimes \mathbb{1}_\Delta(x), \gamma\rangle \in [0, \infty]. \tag{2.2.18}$$

In particular, for $\gamma \in \Gamma_p(\hat{X})$ and $\Delta \in \mathcal{B}(X)$

$$\mathfrak{m}_\Delta(\gamma) = \sum_{x \in \tau(\gamma) \cap \Delta} s_x \in [0, \infty]. \tag{2.2.19}$$

Definition 2.2.7. *The* set of pinpointing configurations with finite local mass $\Gamma_f(\hat{X})$ *is defined by*

$$\Gamma_f(\hat{X}) := \Gamma_{f,\mathfrak{m}}(\hat{X}) := \left\{\gamma \in \Gamma_p(\hat{X}) \middle| \ \forall \Delta \in \mathcal{B}_c(X): \ \ \mathfrak{m}_\Delta(\gamma) < \infty\right\}. \tag{2.2.20}$$

Remark 2.2.8. *By Lemma 2.1.1, the map* $\Gamma(\hat{X}) \ni \gamma \to \mathfrak{m}_\Delta(\gamma) \in \mathbb{R}$ *is* $\mathcal{B}(\Gamma(\hat{X}))$*-measurable for all* $\Delta \in \mathcal{B}(X)$ *as the limit of measurable functions. Moreover,* $\Gamma_f(\hat{X}) \in \mathcal{B}(\Gamma(\hat{X}))$.

Theorem 2.2.9. *We have* $\mathcal{P}_\lambda(\Gamma_f(\hat{X})) = 1$. *In particular, the Gamma-Poisson measure* $\mathcal{P}_\theta$ *is supported by* $\Gamma_f(\hat{X})$.

Proof. Fix $\Delta \in \mathcal{B}_c(X)$. Recall that by Remark 2.2.8 the function $\Gamma(\hat{X}) \ni \gamma \to \mathfrak{m}_\Delta(\gamma)$ is measurable. We have

$$\begin{aligned}\int_{\Gamma(\hat{X})} \mathfrak{m}_\Delta(\gamma)\mathcal{P}_\lambda(d\gamma) &= \int_\Gamma \langle s \otimes \mathbb{1}_\Delta(x), \gamma\rangle \mathcal{P}_\lambda(d\gamma) \\ = \int_X \int_{\mathbb{R}_+} s\mathbb{1}_\Delta \lambda(ds) m(ds) &= m(\Delta) \int_{\mathbb{R}_+} s\lambda(ds) < \infty.\end{aligned} \tag{2.2.21}$$

Hence, for all $\Delta \in \mathcal{B}_c(X)$ we have

$$\mathfrak{m}_\Delta(\gamma) < \infty, \qquad \text{for } \gamma \in \Gamma \ (\mathcal{P}_\lambda\text{-a.e.}).$$

□

[5] The general concept will be given in Definition 4.3.4.

Chapter 3

Gamma measures

In this chapter, we will present the main objects of our considerations, namely the *cone of discrete Radon measures* and the *Gamma measures*. Moreover, we outline a quasi-invariance property of the Gamma measure.

The Gamma measures on infinite dimensional spaces appeared in the representation theory of groups (cf. [VGG75]). They are closely related to multiplicative Lebesgue measures. In [KdSSU98] Gamma measures are treated as a particular case of *compound Poisson measures* and the related chaos decomposition and annilation and creation operators are studied. In [TVY01], a constructive approach for the Gamma measures is given. In [Sta03], the Gamma measures appear as examples of "invariant probability measure for a class of continuous state branching processes with immigration."[1]

Our first aim (cf. Section 3.1) is to introduce the underlying topological space, namely the *cone of discrete measures* (cf. Definition 3.1.1)

$$\begin{aligned}\mathbb{K}(\mathbb{R}^d) := \Big\{\eta = \sum s_i\delta_{x_i}\Big| s_i \in \mathbb{R}_+, x_i \in \mathbb{R}^d, x_i \neq x_j \forall\ i,j \in \mathbb{N},\ i \neq j,\\ \forall \Lambda \in \mathcal{B}_c(\mathbb{R}^d) : \eta(\Lambda) < \infty\Big\}.\end{aligned}$$

Then (cf. Subsection 3.2.1) we get the Gamma measure $\mathcal{G}_\theta$, $\theta > 0$ a fixed parameter, on $\mathbb{K}(\mathbb{R}^d)$ as an image measure of the Gamma-Poisson measure $\mathcal{P}_\theta$ on $\Gamma(\hat{\mathbb{R}}^d)$ (cf. Chapter 2) and prove that (cf. Theorems 3.1.7 and 3.2.2)

$$\mathbb{E}_{\mathcal{G}_\theta}\left[\exp\left(-\langle a,\cdot\rangle\right)\right] = \exp\left(-\theta\int_{\mathbb{R}^d}\log(1+a(x))dm(x)\right), \tag{3.0.1}$$

[1]This is cited from [Sta03, Abstract].

where $a : \mathbb{R}^d \to [0, \infty)$, is a bounded, compactly supported, non-negative Borel function such that $\log(1 + a) \in L^1(\mathbb{R}^d, m)$ and m has been fixed to be a non-atomic Radon measure on $\mathbb{R}^d$.

From (3.0.1) we deduce two important properties of $\mathcal{G}_\theta$:

- All finite *moments* exist, i.e., for $n \in \mathbb{N}$ and for each bounded Borel function $a : \mathbb{R}^d \to \mathbb{R}$ that is supported by $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ we have (cf. Theorem 3.2.6)
$$\mathbb{E}_{\mathcal{G}_\theta}[\langle a, \cdot \rangle^n] \leq n! \, \langle \|a\|_\infty \mathbb{1}_\Delta, \theta m \rangle^n < \infty. \tag{3.0.2}$$
- The Gamma measure $\mathcal{G}_\theta$ is quasi-invariant under $\mathbb{K}(\mathbb{R}^d) \ni \eta \mapsto e^h \eta \in \mathbb{K}(\mathbb{R}^d)$, where $h \in C_0(\mathbb{R}^d)$ (cf. Theorem 3.3.3).

In Part III, the later two properties are heavily used to do differential calculus related to $\mathcal{G}_\theta$ on $\mathbb{K}(\mathbb{R}^d)$.

These results extend to the case that $\mathbb{R}^d$ is replaced by an arbitrary locally compact Polish space X, which we fix from now on together with a non-atomic Radon measure m on X.

3.1 Levy measures on the cone of discrete Radon measures

The main aim of this section is to introduce the cone $\mathbb{K}(X)$ and Levy measures on $\mathbb{K}(X)$, which include Gamma measures.

3.1.1 The cone $\mathbb{K}(X)$

Definition 3.1.1. *The* cone of locally finite discrete measures *over X is defined as*
$$\mathbb{K}(X) := \Big\{ \eta = \sum s_i \delta_{x_i} \Big| s_i \in \mathbb{R}_+, x_i \in X, x_i \neq x_j \forall\ i, j \in \mathbb{N},\ i \neq j, \\ \forall \Lambda \in \mathcal{B}_c(X) : \eta(\Lambda) < \infty \Big\} \subset \mathcal{M}(X), \tag{3.1.1}$$
where $\mathcal{M}(X)$ denotes the set of all Radon measures over X.

Remark 3.1.2. *Heuristically spoken, $\eta \in \mathbb{K}(X)$ means that on each position there should only be one particle with its specific mark and that the mass (=the sum of marks) does not explode locally.*

Definition 3.1.3. *For each $\eta \in \mathbb{K}(X)$ we denote its* support *by*
$$\tau(\eta) := \{x \in X | \, \eta(x) := \eta(\{x\}) \neq 0\}.$$

3.1.2 Levy measures

Definition 3.1.4. *A Radon measure λ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ satisfying*

$$\lambda(\mathbb{R}_+) = \infty \qquad \text{and} \qquad m_1(\lambda) := \int_{\mathbb{R}_+} s\lambda(ds) < \infty$$

is called a Levy measure on $\mathbb{R}_+$. *The* first moment *of λ is $m_1(\lambda)$.*

The Borel σ-algebra on $\mathbb{K}(X)$

The topology on $\mathbb{K}(X)$ is inherited from $\Gamma_f(X)$ being equipped with the subspace topology of $(\Gamma(\hat{X}), \mathcal{O}(\Gamma(\hat{X})))$ (cf. Section 2.1): That is, we equip $\mathbb{K}(X)$ with the strongest topology $\mathcal{O}(\mathbb{K}(X))$ such that the following bijective map is continuous

$$\begin{array}{rrcl} \mathbb{T}: & \Gamma_f(\hat{X}) & \to & \mathbb{K}(X) \\ & \gamma = \{(s_x, x)\} & \mapsto & \eta := \sum_{(s_x,x)\in\gamma} s_x\delta_x. \end{array} \tag{3.1.2}$$

Then we equip $\mathbb{K}(X)$ with the corresponding Borel σ-algebra $\mathcal{B}(\mathbb{K}(X))$.

Definition 3.1.5 (compare [TVY01, Definition 2.1])**.** *A* Levy measure $\mathcal{G}_\lambda$ *on $(\mathbb{K}(X), B(\mathbb{K}(X)))$ over the space $(X, \mathcal{B}(X), m)$ with Levy (intensity) measure λ on $\mathbb{R}_+$ is a Poisson process on $\mathbb{K}(X)$ such that its Laplace transform fulfills*

$$\mathbb{E}_{\mathcal{G}_\lambda}\left[\exp\left(-\langle a, \cdot\rangle\right)\right] = \exp\left(-\int_{\mathbb{R}_+\times X} (1 - e^{-a(x)s})\lambda(ds)m(dx)\right), \tag{3.1.3}$$

where $\langle a, \eta\rangle := \int_X a(x)d\eta(x)$ and $a : X \to \mathbb{R}$ is a compactly supported, bounded, non-negative Borel function.

Remark 3.1.6. *This definition differs from the more general one, where one assumes for an intensity measure λ on $\mathbb{R}_+$ just $\int_0^1 s^2\lambda(ds) < \infty$ (cf. [App09, P.29, (1.10)]). The additional integrability insures that $\mathcal{G}_\lambda(\mathbb{K}(X)) = 1$.*

Theorem 3.1.7 (compare [AKR98a], [TVY01])**.** *Let λ be Levy measure on $\mathbb{R}_+$ with $m_1(\lambda) < \infty$. Then there exists a corresponding Levy measure $\mathcal{G}_\lambda$ on $(\mathbb{K}(X), \mathcal{B}(\mathbb{K}(X)))$ which has λ as an intensity measure on $\mathbb{R}_+$.*

Proof. This follows using the results of Section 2.2, especially Theorem 2.2.9. Namely, we consider the Poisson measure $\mathcal{P}_\lambda$ on $\Gamma(\hat{X})$ with intensity measure $\lambda \otimes m$. By Theorem 2.2.9, $\mathcal{P}_\lambda\big(\Gamma_f(\hat{X})\big) = 1$. Note that $\Gamma_f(\hat{X}) \subset \mathcal{B}(\Gamma(\hat{X}))$ is bijective to $\mathbb{K}(X)$ under the map $\mathbb{T}$ given in (3.1.2). Hence, we define $\mathcal{G}_\lambda$ as the image measure of $\mathcal{P}_\lambda$ under $\mathbb{T}$:

$$\mathcal{G}_\lambda = \mathbb{T}^*\mathcal{P}_\lambda.$$

Then $\mathcal{G}_\lambda$ has the required properties.[2] □

Remark 3.1.8. *We emphasize that for $\mathcal{G}_\lambda$-a.e. $\eta \in \mathbb{K}(X)$*

$$|\tau(\eta) \cap \Delta| = \infty \qquad \forall \Delta \in \mathcal{B}_c(X) : m(\Delta) \neq 0.$$

3.2 Gamma measures

We have a look at two important classes of examples of measures being supported by the cone $\mathbb{K}(X)$. The first one consists of special Levy measures, namely of all Gamma measures $\mathcal{G}_\theta$, $\theta > 0$, (cf. Definition 3.2.1). If $m(X) < \infty$, we can consider a second class consisting of infinite measures, namely of multiplicative Lebesgue measures (cf. Definition 3.2.7).

3.2.1 Gamma measures

This subsection is based on [TVY01].[3]

Definition 3.2.1 (compare [TVY01, Definition 2.2][3])**.** *A* Gamma measure *with shape parameter $\theta > 0$ is the Levy measure on $(\mathbb{K}(X), \mathcal{B}(\mathbb{K}(X)))$ with the following Levy intensity measure on $\mathbb{R}_+$ (cf. (2.2.3))*

$$\lambda_\theta(ds) = \theta \frac{e^{-t}}{t} dt.$$

Theorem 3.2.2 (compare [TVY01, P.279] [3])**.** *The measure λ_θ is indeed a Levy measure on $\mathbb{R}_+$ and $\mathcal{G}_\theta$ exists. Moreover,*

$$\mathbb{E}_{\mathcal{G}_\theta}\left[\exp\left(-\langle a, \cdot\rangle\right)\right] = \exp\left(-\theta \int_X log(1 + a(x))dm(x)\right), \tag{3.2.1}$$

where $a : X \to (-1, \infty)$, is a bounded, compactly supported Borel function such that $\log(1 + a) \in L^1(X, m)$*.*

[2] The existence of $\mathcal{G}_\lambda$ can also be shown by using the existence and the mapping theorem in [Kin93, P.23, resp. p.18], and Campbell's theorem (cf. [Kin93, P.28]) to deduce the Laplace formula. Then one still has to check that it is supported by the cone $\mathbb{K}(X)$.

[3] In [TVY01] the case X being $[0, 1]$ is considered.

Proof. One checks that λ_θ is a Levy measure.[4] Hence, by Theorem 3.1.7 there exists a corresponding Levy measure. The formula for the Laplace transform follows by using (3.1.3) and Lemma 3.2.3, i.e.,

$$\int_0^\infty \frac{e^{-a(x)t}-1}{t}e^{-t}dt = -\log(1+a(x)) \leq -\log(1-1+\varepsilon_x),$$

where $\varepsilon_x > 0$. Moreover, the mentioned bound is sharp because, otherwise, there would exists a non null set such that the logarithm is not finite.

□

Lemma 3.2.3. *For $\delta > 0$, $c \geq -1+\delta$ and $\varepsilon \geq 0$ we get*

$$\int_\varepsilon^\infty \frac{e^{-ct}-1}{t}e^{-t}dt = -\log(1+c) - F(\varepsilon, c),$$

where

$$F(\varepsilon, c) := \sum_{k=1}^\infty \frac{(-\varepsilon)^k((1+c)^k-1)}{k!k}$$

converges absolutely. In particular,

$$\int_0^\infty \frac{e^{-ct}-1}{t}e^{-t}dt = -\log(1+c).$$

Proof. The following integral exists because we find, using the mean value theorem for $t \leq 1$, a dominating integrable function for the integrand. Using the transformation rule with $t' = (c+1)t$ we derive

$$\int_0^\infty \frac{e^{-ct}-1}{t}e^{-t}dt = \lim_{\varepsilon\to 0}\Big(\underbrace{\int_{\varepsilon(c+1)}^\infty \frac{e^{-t'}}{t'}dt'}_{=:E_1(\varepsilon(c+1))} - \underbrace{\int_\varepsilon^\infty \frac{e^{-t}}{t}dt}_{=:E_1(\varepsilon)}\Big).$$

Taking the derivative of $E_1(\varepsilon)$ w.r.t. ε and then finding a primitive we obtain

$$\left.\frac{dE_1(x)}{dx}\right|_{x=\varepsilon} = -\frac{e^{-\varepsilon}}{\varepsilon} = \sum_{k=1}^\infty \frac{(-\varepsilon)^{k-1}}{k!} - \frac{1}{\varepsilon}$$

and

$$E_1(\varepsilon) = -\sum_{k=1}^\infty \frac{(-\varepsilon)^k}{k!k} - \log\varepsilon + C,$$

[4] Indeed, $\lambda_\theta((1,\infty)) \leq \theta\int_1^\infty 1\cdot e^{-t}dt < \infty$, $\int_{\mathbb{R}_+} tt^{-1}e^{-t}dt = 1$ and $\lambda_\theta((0,\infty)) \geq \theta\int_0^1 e^{-1}t^{-1}dt = \infty$.

where $C > 0$. Note

$$\left|\frac{(-\varepsilon)^{k+1}k!k}{(-\varepsilon)^k(k+1)!(k+1)}\right| = \frac{|-\varepsilon|}{k+1}\cdot(1-\frac{1}{k}) < \frac{1}{2} \quad \text{for } k > 2\varepsilon.$$

Thus the sum exists, converges absolutely and is bounded by $e^{\varepsilon}-1$. Using this formula for ε and $\varepsilon(c+1)$, we conclude

$$\begin{aligned}\int_0^\infty \frac{e^{-ct}-1}{t}e^{-t}dt &= \lim_{\varepsilon\to 0} -\sum_{k=1}^\infty \frac{(-\varepsilon)^k((1+c)^k-1)}{k!k} - \log\frac{\varepsilon(c+1)}{\varepsilon} \\ &= -\log(c+1).\end{aligned}$$

Furthermore,

$$\begin{aligned}|F(\varepsilon,c)| &\leq \sum_{k=1}^\infty \left|\frac{(-\varepsilon)^k((1+c)^k-1)}{k!k}\right| \\ &\leq \sum_{k=0}^\infty \frac{(\varepsilon)^k(1+c)^k}{k!} - 1 + e^{\varepsilon} - 1 \leq e^{\varepsilon(1+c)} + e^{\varepsilon} - 2,\end{aligned}$$

i.e., the sums converges absolutely. □

3.2.2 Moments of Gamma measures

Definition 3.2.4. *Let μ be a measure on $(\mathbb{K}(X), \mathcal{B}(\mathbb{K}(X)))$, $a : X \to \mathbb{R}$ be a bounded, compactly supported Borel function and $n \in \mathbb{N}$. Then*

$$\mathbb{E}_\mu\left[\langle a, \cdot\rangle^n\right]$$

is called an n^{th} moment *of μ.*

Lemma 3.2.5 (see [GR80, P.19, eq. 0.430 2])**.** *For $n \in \mathbb{N}_0$, $f, g \in C^n(\mathbb{R})$*

$$\frac{d^n}{dt^n}(g\circ f(t)) = \sum_{\substack{\sum l i_l = n \\ i_1,\dots i_k\in\mathbb{N}_0}} \frac{n!}{i_1!\dots i_k!}\left(\frac{f^{(1)}(t)}{1!}\right)^{i_1}\cdots\left(\frac{f^{(k)}(t)}{k!}\right)^{i_k}\left(\frac{d^n}{dy^n}g\right)\circ f(t),$$

where $\frac{d^n}{dy^n}g$ denotes the n^{th} derivative of $y \mapsto g(y)$.

Theorem 3.2.6. *Let $\theta > 0$ and $a : X \to \mathbb{R}$ be a bounded, compactly supported Borel function. Then for all $n \in \mathbb{N}$*

$$\mathbb{E}_{\mathcal{G}_\theta}[\langle a, \cdot\rangle^n] = \sum_{\substack{\sum l i_l = n \\ i_1,\dots i_k\in\mathbb{N}_0}} \frac{n!}{i_1!\dots i_k!}\left\langle\frac{a^1}{1}, \theta m\right\rangle^{i_1}\cdots\left\langle\frac{a^k}{k}, \theta m\right\rangle^{i_k} \tag{3.2.2}$$

If a is supported by $\Delta \in \mathcal{B}_c(X)$, then

$$\mathbb{E}_{\mathcal{G}_\theta}[\langle a, \cdot\rangle^n] \leq n!\|a\|_\infty\theta^n m(\Delta)^n. \tag{3.2.3}$$

Proof. The proof is done by deriving the Laplace transformation of λa w.r.t. $\lambda \in \mathbb{R}$ and evaluating it at 0: For sufficiently small λ, we have $\lambda a > -\frac{1}{2}$ and (cf. Theorem 3.2.2)

$$\begin{aligned}\mathbb{E}_{\mathcal{G}_\theta}\left[(-\langle a,\cdot\rangle)^n\right] &= \left(\frac{d}{d\lambda}\right)^n \mathbb{E}_{\mathcal{G}_\theta}[\exp(-\lambda\langle a,\cdot\rangle)]\Big|_{\lambda=0} \\ &= \left(\frac{d}{d\lambda}\right)^n \exp(-\theta\langle\log(1+\lambda a), m\rangle)\Big|_{\lambda=0}.\end{aligned}$$

It equals (cf. Lemma 3.2.5)

$$\begin{aligned}&\sum_{\substack{\sum l i_l = n \\ i_1,\dots i_k\in\mathbb{N}_0}} \frac{n!}{i_1!\dots i_k!}\left(\frac{\left\langle \frac{(-1)^1 0!}{(1+\lambda a)^1}a^1, \theta m\right\rangle}{1!}\right)^{i_1}\cdots\left(\frac{\left\langle \frac{(-1)^k (k-1)!}{(1+\lambda a)^k}a^k, \theta m\right\rangle}{k!}\right)^{i_k} \\ &\qquad (\exp)\circ(\langle -\log(1+\lambda a), \theta m\rangle)\Big|_{\lambda=0} \\ =& \sum_{\substack{\sum l i_l = n \\ i_1,\dots i_k\in\mathbb{N}_0}} \frac{n!}{i_1!\dots i_k!}\underbrace{(-1)^{\sum l i_l}}_{(-1)^n}\left\langle\frac{a^1}{1}, \theta m\right\rangle^{i_1}\cdots\left\langle\frac{a^k}{k}, \theta m\right\rangle^{i_k}.\end{aligned}$$

Hence, we obtain the desired formula by multiplying with $(-1)^n$. $\square$

3.2.3 Multiplicative Lebesgue measures

Definition 3.2.7 (cf. [TVY01, Definition 4.1])**.** *Let $m(X) < \infty$ and $\theta > 0$. A* multiplicative Lebesgue measure *$\mathcal{L}_\theta$ with parameter $\theta > 0$ on $\mathbb{K}(X)$ is defined by*

$$\mathcal{L}_\theta(d\eta) = e^{\langle 1,\eta\rangle}\mathcal{G}_\theta(d\eta).$$

Lemma 3.2.8. *Its Laplace transform is*

$$\mathbb{E}_{\mathcal{L}_\theta}\left[e^{-\langle a,\cdot\rangle}\right] = \exp\left(-\theta\int_X \log a(x) m(dx)\right),$$

where $a : X \to \mathbb{R}_+$ is a strictly positive, bounded, compactly supported Borel function with $\log a \in L^1(X, m)$.

Proof. This follows by

$$\mathbb{E}_{\mathcal{L}_\theta}\left[e^{-\langle a,\cdot\rangle}\right] = \mathbb{E}_{\mathcal{G}_\theta}\left[e^{-\langle a-1,\cdot\rangle}\right] = \exp\left(-\theta\int_X \log(1+a(x)-1) m(dx)\right).$$

$\square$

3.2.4 Additive Lebesgue measures

Let

$$E(X) := \left\{ \sum s_i \delta_{x_i} \middle| \{(s_i, x_i) | i \in \mathbb{N}\} \subset \mathbb{R} \times X : \quad \eta := \sum_{i \in \mathbb{N}} e^{s_i} \delta_{x_i} \in \mathbb{K}(X) \right\}.$$

To define the additive Lebesgue measure we use the mapping

$$\begin{array}{rccc} Log\colon & \mathbb{K}(X) & \to & E(X) \\ & \sum_i s_i \delta_{x_i} & \mapsto & \sum_i \log(s_i) \delta_{x_i}. \end{array}$$

Definition 3.2.9 (cf. [Ver07, Section 4.4])**.** *Let $\theta > 0$ and $m(X) < \infty$. The* additive Lebesgue measure $\mathcal{L}_\theta^{add}$ *on $E(X)$ is defined as the image measure of $\mathcal{L}_\theta$ under Log:*

$$\mathcal{L}_\theta^{add} := Log^* \mathcal{L}_\theta. \tag{3.2.4}$$

3.3 Basic properties of $\mathcal{G}_\theta$, $\mathcal{L}_\theta$ and $\mathcal{L}_\theta^{add}$

Fix $\theta > 0$, we present a quasi-invariance and extremality property of the Gamma measure $\mathcal{G}_\theta$ and the Lebesgue measure $\mathcal{L}_\theta$. Moreover, we show an ergodicity property of the Gamma measures.

3.3.1 Quasi-invariance, ergodicity and extremality of $\mathcal{G}_\theta$

Analogously to [TVY01] we outline a transformation rule and the ergodicity of the Gamma measure $\mathcal{G}_\theta$. In addition we show its extremality.

Multiplicator operator on the cone

Definition 3.3.1. *For each $h \in C_0(X)$ the* multiplicator *is given by*

$$\begin{array}{rccc} M_h\colon & \mathbb{K}(X) & \to & \mathbb{K}(X) \\ & \eta & \mapsto & e^h \eta. \end{array}$$

That is $(M_h \eta)(x) = e^{h(x)} \eta(x)$.

Remark 3.3.2. *The multiplier M_h is well defined because h is continous and bounded.*

Theorem 3.3.3 (compare [TVY01, Theorem 3.1][3]). *The Gamma measure $\mathcal{G}_\theta$ is quasi-invariant under M_h for each $h \in C_0(X)$. The corresponding density is*

$$\frac{d(M_h\mathcal{G}_\theta)}{d\mathcal{G}_\theta}(\eta) = \exp\left(-\theta \int_X h(x)m(dx)\right)\exp\left(-\int_X \left(e^{-h(x)}-1\right)d\eta(x)\right).$$

Proof. We split the Gamma measure and consider the action of the multiplication on both parts independently. On one part the action is trivial and on the other one the formula follows by the Laplace transform.

Fix $h \in C_0(X)$ and $\Lambda \in \mathcal{B}_c(X) : h = \mathbb{1}_\Lambda h$. Then M_h leaves all marks outside of Λ invariant. Hence, by the multiplicative property, it is enough to calculate the Radon-Nikodyn derivative for $\mathcal{G}_\theta^\Lambda$.[5] Let $\xi = M_h\eta$. Consider an arbitrary function $l \in C_0(X)$. Then

$$f_l(\xi) := \int_\Lambda e^{l(x)}\xi(dx) = \int_\Lambda e^{h(x)}e^{l(x)}\eta(dx) = f_{hl}(\eta).$$

Therefore, using (3.2.1), the Laplace transform equals

$$\begin{aligned}
&\mathbb{E}_{\mathcal{G}_\theta^\Lambda}[\exp(-f_l(M_h(\cdot)))] = \mathbb{E}_{\mathcal{G}_\theta^\Lambda}[\exp(-f_{hl}(\cdot)]\\
=\ &\exp\left(-\theta\int_\Lambda \log\left(1+e^{l(x)}e^{h(x)}\right)m(dx)\right)\\
=\ &\exp\left(-\theta\int_\Lambda h(x)dm(x)\right)\exp\left(-\theta\int_\Lambda log\left(e^{-h(x)}+e^{l(x)}\right)m(dx)\right)\\
=\ &\exp\left(-\theta\int_\Lambda h(x)dm(x)\right)\mathbb{E}_{\mathcal{G}_\theta^\Lambda}\left[\exp\left\langle e^{-h}-1+e^l,\cdot\right\rangle\right].
\end{aligned}$$

For the last equation, the Laplace transform (cf. (3.2.1)) is applicable because $h \in C_0(h)$ (Hence, $\exists\delta > 0 :\ e^{-h} - 1 \geq -1 + \delta$). This concludes the proof. □

Theorem 3.3.4 (cf. [TVY01, Theorem 3.2][3]). *The action of the group $C_0(X)$ on the space $(\mathbb{K}(X), \mathcal{G}_\theta)$ is ergodic.*

Proof. This follows by straightforwardly adapting the proof of [TVY01, Theorem 3.2].[6] □

[5] On $\mathcal{G}_\theta^{\Lambda^C}$ it is $1 = \exp\left(-\theta\int_{\Lambda^C} h(x)m(dx)\right)\exp\left(-\int_{\Lambda^C}\left(e^{-h(x)}-1\right)\eta(dx)\right)$.

[6] For the convenience of the reader, we give the details: Let $G : \mathbb{K}(X) \to \mathbb{R}$ be a measurable function on $\mathbb{K}(X)$, which is for all $h \in C_0(X)$ invariant under M_h, i.e. $G(M_h\eta) = G(\eta)$ $\mathcal{G}_\theta$-a.e.. Consider an arbitrary Borel function $k : \mathbb{R} \to \mathbb{R}$. Then for each $h \in C_0(X)$ by

Lemma 3.3.5. *Let $\theta_1, \theta_2 > 0$. Then there exists no non trivial convex combination of Gamma measures $\mathcal{G}_{\theta_1}$ and $\mathcal{G}_{\theta_2}$ which equals $\mathcal{G}_\theta$.*

Proof. If there existed θ_1, $\theta_2 \in \mathbb{R}^+\backslash\theta$ and $c \in [0,1]$ fixed such that

$$\mathcal{G}_\theta = c\mathcal{G}_{\theta_1} + (1-c)\mathcal{G}_{\theta_2},$$

we would obtain by the Laplace transform for any feasible function a:

$$\begin{aligned} & \exp\left(-\theta\langle\log(1+a), m\rangle\right) \\ = \; & c\exp\left(-\theta_1\langle\log(1+a), m\rangle\right) + (1-c)\exp\left(-\theta_2\langle\log(1+a), m\rangle\right) \\ = \; & c\left(\exp\left(-\langle\log(1+a), m\rangle\right)\right)^{\theta_1} + (1-c)\left(\exp\left(-\langle\log(1+a), m\rangle\right)\right)^{\theta_2}. \end{aligned}$$

Denoting $x := \exp\left(-\langle\log(1+a), m\rangle\right) \in \mathbb{R}^+$ we see

$$x^\theta = cx^{\theta_1} + (1-c)x^{\theta_2} \quad \Leftrightarrow \quad 1 = cx^{\theta_1-\theta} + (1-c)x^{\theta_2-\theta}.$$

Considering the different cases implies a contradiction for x sufficiently large: Assume that $\theta_1 > \theta$ then for x tending to infinity the r.h.s of the last equation tends to infinity, which leads to a contradiction for the function a being sufficiently small. The same is true for $\theta_2 > \theta$. On the other hand, if θ is bigger than θ_1 and θ_2, the r.h.s tends to 0 for a being sufficiently small.

Hence, no Gamma measure $\mathcal{G}_\theta$ can be represented by any non-trivial convex combination of finitely many Gamma measures. □

3.3.2 Projective invariance and convex combinations of $\mathcal{L}_\theta$ and $\mathcal{L}_\theta^{add}$

Theorem 3.3.6 (cf. [TVY01, Theorem 4.1][3])**.** *Let $m(X) < \infty$. For each $h \in C_0(X)$, the multiplicative Lebesgue measure $\mathcal{L}_\theta$ is projective-invariant under M_h, and the corresponding density is given by*

$$\frac{d(M_h\mathcal{L}_\theta)}{d\mathcal{L}_\theta}(\eta) = \exp\left(-\theta\int_X h(x)m(dx)\right).$$

Theorem 3.3.3

$$\mathbb{E}_{\mathcal{G}_\theta}[k(G(\cdot)] = \mathbb{E}_{\mathcal{G}_\theta}[k(G(M_h\cdot))] = \mathbb{E}_{\mathcal{G}_\theta}\left[k(G(\cdot))\exp\left(-\langle(e^{-h}-1),\cdot\rangle\right)\right]\exp\left(-\theta\langle h, m\rangle\right).$$

We deduce

$$\begin{aligned} & \mathbb{E}_{\mathcal{G}_\theta}\left[k(G(\cdot))\exp\left(-\langle(e^{-h}-1),\cdot\rangle\right)\right] = \mathbb{E}_{\mathcal{G}_\theta}\left[k(G(\cdot)]\cdot\exp\left(-\theta\langle -h, m\rangle\right)\right] \\ = \; & \mathbb{E}_{\mathcal{G}_\theta}\left[k(G(\cdot)]\cdot\mathbb{E}_{\mathcal{G}_\theta}\left[\exp\left(-\langle e^{-h(x)}-1,\cdot\rangle\right)\right]\right], \end{aligned}$$

where we used the Laplace transform (cf. Definition 3.2.1) and that h is compactly supported. Since for any G the last equation holds for any arbitrary Borel function k and any $h \in C_0(X)$, $\mathcal{G}_\theta$ is ergodic.

Proof. As before we set for $l \in C_0(X)$ $f_l(\eta) := \int_X e^{l(x)}\eta(dx)$ and obtain

$$\mathbb{E}_{\mathcal{L}_\theta}[e^{-f_l(M_h(\cdot))}] = \mathbb{E}_{\mathcal{L}_\theta}[e^{f_{l+h}(\cdot)}] = e^{-\theta\langle \log(e^{l+h}),m\rangle} = e^{-\theta\langle h,m\rangle}\mathbb{E}_{\mathcal{L}_\theta}[e^{-f_l(\cdot)}].$$

□

Lemma 3.3.7. *Let $\theta_1, \theta_2 > 0$. There exists no non-trivial convex combination of multiplicative "Lebesgue" measures $\mathcal{L}_{\theta_1}$ and $\mathcal{L}_{\theta_2}$ which equals $\mathcal{L}_\theta$.*

Proof. Arguing similarly as in the proof of Lemma 3.3.5, we get the assertion. (We just replace $\log(1+a)$ by $\log a$.) □

Definition 3.3.8. *For each $h \in \mathcal{M}_0 := \{h \in C_0(X) \mid \langle h, m\rangle = 0\}$ we define*

$$\begin{array}{rccl} A_h\colon & E(X) & \to & E(X) \\ & \eta = \sum_i s_i\delta_{x_i} & \mapsto & \sum_i (s_i + h(x_i))\delta_{x_i} =: h + \eta. \end{array}$$

Theorem 3.3.9 (cf. [Ver07, Theorem 6])**.** *Let $m(X) < \infty$. For each $h \in \mathcal{M}_0$, the additive Lebesgue measure $\mathcal{L}_\theta^{add}$ is invariant under A_h.*

Proof. Theorem 3.3.6 yields

$$\begin{aligned} d(A_h\mathcal{L}_\theta^{add})(\eta) &= d(A_h Log\mathcal{L}_\theta)(\eta) = d(Log M_h\mathcal{L}_\theta)(\eta) \\ &= d(Log\mathcal{L}_\theta)(\eta) = d\mathcal{L}_\theta^{add}(\eta). \end{aligned}$$

□

Part II

Gibbs perturbations

Gibbs perturbations

In Part II, we study Gibbs perturbations of Gamma measures on $\mathbb{K}(\mathbb{R}^d)$. We will describe heuristically our approach, the detailed definitions and results are presented in Chapters 4 and 5.

Let $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a pair potential which describes the interaction between particles. To introduce a Gibbs formalism on $\mathbb{K}(\mathbb{R}^d)$ (cf. Section 5.3), we follow the lines of the *Dobrushin-Lanford-Ruelle* (**DLR**) approach to Gibbs random fields in classical statistical mechanics:

For each $\eta \in \mathbb{K}(\mathbb{R}^d)$ and a *boundary condition* $\xi \in \mathbb{K}(\mathbb{R}^d)$, the *relative energy* $H_\Delta(\eta|\xi)$ in a bounded area $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ is defined by (cf. (5.3.4))

$$H_\Delta(\eta|\xi) := \int_\Delta \int_\Delta \phi(x,y)\eta(dx)\eta(dy) + \int_{\Delta^c} \int_\Delta \phi(x,y)\eta(dx)\xi(dy).$$

We fix an inverse temperature $\beta = 1/T > 0$. A *local Gibbs measure* in volume Δ is given by

$$\mu_\Delta(d\eta|\xi) = \frac{1}{Z_\Delta(\xi)} e^{-\beta H_\Delta(\eta|\xi)} \mathcal{G}_\theta^\Delta(d\eta) \in \mathcal{M}^1(\mathbb{K}(\mathbb{R}^d)),$$

where $\mathcal{G}_\theta^\Delta$ is the Gamma measure on $\mathbb{K}(\Delta)$. The *probability kernels*

$$\pi_\Delta(B|\xi) = \mu_\Delta(\{\eta \in \mathbb{K}(\Delta) | \eta \cup \xi_{\Delta^c} \in B\}|\xi), \quad B \in \mathcal{B}(\mathbb{K}(\mathbb{R}^d)),$$

indexed by $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ and $\xi \in \mathbb{K}(\mathbb{R}^d)$, form the *Gibbs specification* on $\mathbb{K}(\mathbb{R}^d)$. It describes corresponding Gibbs measures μ on $\mathbb{K}(\mathbb{R}^d)$ via the (**DLR**) equation (cf. (6.3.18))

$$\int_{\mathbb{K}(\mathbb{R}^d)} \pi_\Delta(B|\eta)\mu(d\eta) = \mu(B),$$

valid for all $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ and $B \in \mathcal{B}(\mathbb{K}(\mathbb{R}^d))$ (cf. Definition 5.3.9). The set of all Gibbs measures related to the specification $\pi = \{\pi_\Delta\}_{\Delta \in \mathcal{B}_c(\mathbb{R}^d)}$ will be denoted by $\mathcal{G}\mathrm{ibbs}_\phi(\mathbb{K}(\mathbb{R}^d))$ (cf. Definition 5.3.5).

The existence of such Gibbs measures is far from being trivial. We will reformulate the existence problem in terms of the related Gibbs measures on the configuration space $\Gamma(\hat{\mathbb{R}}^d)$. Also in the free case, we have first defined the Gamma-Poisson measure on $\Gamma(\hat{\mathbb{R}}^d)$ and then use the bijective mapping $\mathbb{T}$ between the cone $\mathbb{K}(\mathbb{R}^d)$ and the set $\Gamma_f(\hat{\mathbb{R}}^d)$ of pinpointing configurations with finite local mass, i.e.,

$$\mathbb{T}^{-1}: \quad \mathbb{K}(\mathbb{R}^d) \ni \eta = \sum s_x \delta_x \mapsto \gamma = \{(s_x, x)\} \in \Gamma_f(\hat{\mathbb{R}}^d) \subset \Gamma(\hat{\mathbb{R}}^d).$$

Using $\mathbb{T}^{-1}$, we map the involved objects (like the potential, relative energy and local specification) to the configuration space $\Gamma(\hat{\mathbb{R}}^d)$, e.g.,

$$\begin{array}{lll} \phi(x,y),\ x,y \in \mathbb{R}^d & \text{to} & V(\hat{x},\hat{y}) = s_x s_y \phi(x,y), \quad \hat{x},\hat{y} \in \hat{X}; \\ \mu_\Delta(d\eta|\xi) & \text{to} & \mu_{\mathbb{R}_+\times\Delta}(d\gamma|\mathbb{T}^{-1}(\xi)); \\ \pi_\Delta(d\eta|\xi) & \text{to} & \pi_{\mathbb{R}_+\times\Delta}(d\gamma|\mathbb{T}^{-1}(\xi)). \end{array}$$

Since the corresponding specification kernels $\pi_{\mathbb{R}_+\times\Delta}(d\gamma|\mathbb{T}^{-1}(\xi)) \in \mathcal{M}^1(\Gamma(\hat{\mathbb{R}}^d))$ are indexed by "stripes" $\mathbb{R}_+ \times \Delta \subset \hat{\mathbb{R}}^d$, they are called *semi-local*. The associated Gibbs measures on $\Gamma(\hat{\mathbb{R}}^d)$ corresponding to the *semi-local specification*

$$\pi_\Gamma = \{\pi_{\mathbb{R}_+\times\Delta}(d\gamma|\tilde{\xi})|\Delta \in \mathcal{B}_c(\mathbb{R}^d), \tilde{\xi} \in \Gamma_f(\hat{\mathbb{R}}^d)\}$$

are also specified via a (**DLR**) equation (cf. also (4.5.14)). Such Gibbs measures constitute the set $\mathcal{G}\mathrm{ibbs}_V(\Gamma(\hat{\mathbb{R}}^d))$ (cf. also Subsection 4.5.1 and Section 5.1).

Due to the one-to-one correspondence between the local specification kernels π_Δ on $\mathbb{K}(\mathbb{R}^d)$ and the *semi-local* ones, $\pi_{\mathbb{R}_+\times\Delta}$, on $\Gamma(\hat{\mathbb{R}}^d)$ (cf. Subsection 5.3.2), we get the one-to-one correspondence between the classes of Gibbs measures $\mathcal{G}\mathrm{ibbs}_\phi(\mathbb{K}(\mathbb{R}^d))$ and $\mathcal{G}\mathrm{ibbs}_V(\Gamma(\hat{\mathbb{R}}^d))$. By the above construction, the set $\mathcal{G}\mathrm{ibbs}_V(\Gamma(\hat{\mathbb{R}}^d))$ consists of those $\mu_\Gamma \in \mathcal{M}^1(\Gamma(\hat{\mathbb{R}}^d))$ which solve the (**DLR**) equation and have full measure on $\Gamma_f(\hat{\mathbb{R}}^d)$ (cf. Theorem 4.3.31, resp. Corollary 5.2.11). This is our motivation to first study the Gibbs measures μ_Γ on $\Gamma(\hat{\mathbb{R}}^d)$ and then transfer the corresponding results to the Gibbs measures $\mu_\mathbb{K}$ on $\mathbb{K}(\mathbb{R}^d)$.

Even though we are on the configuration space $\Gamma(\hat{\mathbb{R}}^d)$, neither the potential, nor the *semi*-local specification kernels are standard. Usually on $\Gamma(\hat{\mathbb{R}}^d)$ one considers *local* specification kernels π_Λ, which are indexed by bounded sets $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$ (cf. Section 4.1). Both specifications (the local and the semi-local one) determine the same set of Gibbs measures (cf. Theorem 4.5.9 resp.

Remark 5.1.24). On the other hand, the potential $V(\hat{x},\hat{y}) = s_x s_y \phi(x,y)$ does not fit the standard framework on $\Gamma(\hat{\mathbb{R}}^d)$. One of the reasons is that the intensity measure λ_θ has a high concentration for s close to 0 and that V is not translation-invariant and has an infinite range in $\hat{\mathbb{R}}^d$ (cf. Section 4.2 for details).

In Chapters 4 and 5, we establish the existence of Gibbs measures $\mu \in \mathcal{G}\mathrm{ibbs}_V(\Gamma(\hat{X}))$ and show uniform moment bounds (cf. Theorems 4.2.7 and 4.3.34, resp. 5.2.8 and 5.2.10). In Chapter 4 we consider non-negative potentials $V \geq 0$ and construct a Gibbs measure being specified by the *local* specification kernels π_Λ, $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$ (cf. Section 4.2). In Chapter 5 we remove the assumption that $V \geq 0$ and work with the semi-local specification kernels $\pi_{\mathbb{R}_+ \times \Delta}$, $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$. As we already mentioned above, both specifications lead to the same set $\mathcal{G}\mathrm{ibbs}_V(\Gamma(\hat{\mathbb{R}}^d))$.

As we will see in Chapter 5, our existence and moment results for Gibbs measures on $\Gamma(\hat{\mathbb{R}}^d)$ extend to more general potentials than in the basic model with $V(\hat{x},\hat{y}) = s_x s_y \phi(x,y)$, $\hat{x},\hat{y} \in \hat{\mathbb{R}}^d$.

Finally, in Section 5.3, we come back to the initial setting of Gibbs measures on the cone $\mathbb{K}(\mathbb{R}^d)$. A main result of this section (and also of Part II) is the existence of $\mu \in \mathcal{G}\mathrm{ibbs}_\phi(\mathbb{K}(\mathbb{R}^d))$ (cf. Theorem 5.3.6), i.e. that

$$\mathcal{G}\mathrm{ibbs}_\phi(\mathbb{K}(\mathbb{R}^d)) \neq \varnothing.$$

Furthermore, the properties of $\mu \in \mathcal{G}\mathrm{ibbs}_\phi(\mathbb{K}(\mathbb{R}^d))$, including moment bounds and a characterization of supporting sets, are summarized in Theorems 5.3.10 and 5.3.11.

Chapter 4

Gibbs measures on $\Gamma(\hat{X})$ with non-negative potential

Our aim is to construct Gibbs perturbations of the Gamma-Poisson measure $\mathcal{P}_\theta$ by means of a pair potential V. It describes the interaction between particles and may depend on their positions and marks. In Chapters 2 and 3 we considered the "free case", where $V = 0$.

In this chapter we study the technically easier case of $V \geq 0$, for which elementary probability techniques are sufficient. Then (Chapter 5), we handle more general potentials that satisfy certain stability conditions (cf. Section 5.1). For the later case we need more advanced methods. We stress that even for non-negative potentials V, we are not in a standard framework of Gibbs measures on (marked) configuration spaces because of the irregularity properties of the intensity measure $\lambda_\theta \otimes m$ on the underlying space $\hat{\mathbb{R}}^d$ (cf. e.g. [Rue69, Rue70] or [AKR98b, KLU99, Kun99]; for details cf. Section 4.2). [1]

The Gibbsian formalism is presented in Section 4.1. We start from a pair potential V of the form:[2]

$$V(\hat{x}, \hat{y}) = s_x s_y a(x - y), \qquad \hat{x}, \hat{y} \in \hat{\mathbb{R}}^d,$$

where $a \geq 0$ is a bounded, even and compactly supported $\mathcal{B}(\mathbb{R}^d)$-measurable function. For each $\gamma \in \Gamma(\hat{\mathbb{R}}^d)$ and a *boundary condition* $\xi \in \Gamma_f(\hat{X})$, we define

[1] For a general review on the construction of Gibbs measures, we refer to [AKPR06].

[2] Although, we start with a translation invariant potential $\phi(x, y) = a(x - y)$, our considerations are not limited to this case.

the *relative energy* $H_\Lambda(\gamma|\xi)$ in a bounded area $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$:

$$H_\Lambda(\gamma|\xi) := \sum_{\hat{x},\hat{y}\in\gamma_\Lambda} V(\hat{x},\hat{y}) + 2 \sum_{\substack{\hat{x}\in\gamma_\Lambda \\ \hat{y}\in\xi_{\Lambda^c}}} V(\hat{x},\hat{y}).$$

Let $\beta = 1/T > 0$ denote the inverse temperature. A *local Gibbs measure in volume* Λ is defined by

$$\mu_\Lambda(d\gamma|\xi) = \frac{1}{Z_\Lambda(\xi)} e^{-\beta H_\Lambda(\gamma|\xi)} P_\theta^\Lambda(d\gamma) \in \mathcal{M}^1\big(\Gamma(\Lambda)\big),$$

where P_θ^Λ is the Lebesgue-Poisson measure on $\Gamma(\Lambda)$ (cf. (2.2.11)). The family of *local specification kernels* $\pi_\Lambda(d\gamma|\xi)$, $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$ and $\xi \in \Gamma_f(\hat{\mathbb{R}}^d)$, which are defined by

$$\pi_\Lambda(B|\xi) = \mu_\Lambda\big(\{\eta \in \Lambda \big| \eta \cup \xi_{\Lambda^c} \in B\}|\xi\big), \quad B \in \mathcal{B}(\Gamma(\hat{\mathbb{R}}^d)),$$

determines the conditional distributions in finite volumes of a *Gibbs measure* $\mu \in \mathcal{M}^1(\Gamma(\hat{\mathbb{R}}^d))$. Analytically this relation is described by the so-called DLR equation given in (4.1.13). The set of all Gibbs measures that correspond to $\mathcal{P}_\theta$ and the pair potential V is denoted by $\mathcal{G}\text{ibbs}_V(\Gamma(\hat{\mathbb{R}}^d))$.

Already on $\Gamma(\hat{\mathbb{R}}^d)$, being a non-linear infinite dimensional manifold, the existence of a Gibbs measure is non-trivial. Furthermore, because of the specific features of our interaction potential and intensity measure, we are far from the standard framework known in the literature for particle systems in continuum (cf. [Rue69, Rue70] or [AKR98b, KLU99, Kun99]).

To establish the existence of $\mu \in \mathcal{G}\text{ibbs}_V(\Gamma(\hat{\mathbb{R}}^d))$, we derive uniform moment bounds for $\mu_\Lambda(d\gamma|\xi)$ (cf. Proposition 4.2.3). These bounds imply that each net of local specification kernels $\pi_\Lambda(d\gamma|\xi)$ with a fixed boundary condition ξ is locally equicontinuous (cf. Proposition 4.2.6). This implies the existence of a certain $\mu \in \mathcal{G}\text{ibbs}_V(\Gamma(\hat{\mathbb{R}}^d))$ being a limit point of such a net as $\Lambda \nearrow \hat{\mathbb{R}}^d$, for which we then check that it satisfies the (**DLR**) equation (cf. Theorem 4.2.7). Therefore, the set $\mathcal{G}\text{ibbs}_V(\Gamma(\hat{\mathbb{R}}^d))$ is non-void.

After establishing the existence, we deduce certain moment estimates being *uniform* for all Gibbs measures $\mu \in \mathcal{G}\text{ibbs}_V(\Gamma(\hat{\mathbb{R}}^d))$ (cf. Theorems 4.3.31 and 4.3.34). These estimates allow us to identify an ‹exponentially tempered› subset $\Gamma_{\text{ex}}^{\mathsf{t}}(\hat{\mathbb{R}}^d) \subset \Gamma_f(\hat{\mathbb{R}}^d)$ on which each $\mu \in \mathcal{G}\text{ibbs}_V(\Gamma(\hat{\mathbb{R}}^d))$ has full measure (cf. Remark 4.3.32 and Corollary 4.4.2).

Similarly as in the free case, one may map each $\mu \in \mathcal{G}\mathrm{ibbs}_V(\Gamma(\hat{\mathbb{R}}^d))$ to $\mathbb{K}(\hat{\mathbb{R}}^d)$ via the bijective map (cf. (3.1.2))

$$\mathbb{T} : \Gamma_f(\hat{\mathbb{R}}^d) \ni \gamma = \{(s_x, x)\} \mapsto \eta = \sum s_x \delta_x \in \mathbb{K}(\mathbb{R}^d).$$

The question arises whether we can introduce an intrinsic Gibbs formalism on $\mathbb{K}(\mathbb{R}^d)$ such that for each $\mu_\Gamma \in \mathcal{G}\mathrm{ibbs}_V(\Gamma(\hat{\mathbb{R}}^d))$ the image measure

$$\mu_{\mathbb{K}} := \mathbb{T}^* \mu_\Gamma \in \mathcal{M}^1(\mathbb{K}(\mathbb{R}^d))$$

will be a Gibbs measure on $\mathbb{K}(\mathbb{R}^d)$. This question will be answered in Section 5.3, where we will give a direct construction of the corresponding Gibbs specification $\pi_{\mathbb{K}} = (\pi_\Delta)_{\Delta \subset \mathbb{K}}$ on $\mathbb{K}(\mathbb{R}^d)$.

As a preliminary step in Subsection 4.5.1, we introduce the family of *semi-local* specification kernels $\{\pi_{\mathbb{R}_+ \times \Delta}(d\gamma|\xi) \mid \Delta \in \mathcal{B}_c(\mathbb{R}^d),\ \xi \in \Gamma_f(\hat{\mathbb{R}}^d)\}$ on $\Gamma(\hat{\mathbb{R}}^d)$, which will satisfy

$$\mathbb{T}^* \pi_{\mathbb{R}_+ \times \Delta}(d\gamma|\xi) = \pi_\Delta(d\eta|\mathbb{T}(\xi)), \quad \Delta \in \mathcal{B}_c(\mathbb{R}^d),\ \xi \in \Gamma_f(\hat{\mathbb{R}}^d).$$

As we prove in Theorem 4.5.9, both specifications (the local and the semi-local one) determine the same class of Gibbs measures.

The results of this chapter hold for more general potentials and underlying spaces (cf. Theorems 4.3.26, 4.3.34 and 4.5.9).

4.1 Gibbsian formalism on $\Gamma(\hat{X})$

Let X be a locally finite Polish space equipped with a non-atomic measure m. Given a non-negative pair potential $V : \hat{X} \times \hat{X} \to [0, \infty)$, we construct the Gibbs perturbation of the Gamma-Poisson measure $\mathcal{P}_\theta$. To this end, we will follow the standard DLR-approach. Below, we briefly recall the definition of the corresponding Gibbs measures on $\Gamma(\hat{X})$.

4.1.1 Potential

We assume V to be $\mathcal{B}(\hat{X} \times \hat{X})$-measurable, symmetric and non-negative.

Example 4.1.1. *A typical choice (and a basic model setting, cf. Section 4.2) is $X = \mathbb{R}^d$ equipped with the Lebesgue measure $m(dx) = dx$ and the pair potential*

$$V(\hat{x}, \hat{y}) = s_x s_y a(x - y) \qquad \forall \hat{x}, \hat{y} \in \hat{\mathbb{R}}^d, \tag{4.1.1}$$

where the $\mathcal{B}(\mathbb{R}^d)$-measurable function $a : \mathbb{R}^d \to [0, \infty)$ has a compact support and is bounded, non-negative and even, *i.e., there exist $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ and $M < \infty$:*

$$0 \leq a(x) = a(-x) \leq M \mathbb{1}_\Delta(x), \qquad \forall x \in \mathbb{R}^d.$$

4.1.2 Relative energy

For each *boundary condition* $\xi \in \Gamma(\hat{X})$ (cf. (2.2.20)) and a bounded area $\Lambda \in \mathcal{B}_c(\hat{X})$ we define the *relative energy* $\Gamma(\hat{X}) \ni \gamma \mapsto H_\Lambda(\gamma|\xi) \in \mathbb{R} \cup \{\infty\}$ by

$$H_\Lambda(\gamma|\xi) := \sum_{\hat{x},\hat{y} \in \gamma_\Lambda} V(\hat{x}, \hat{y}) + 2 \sum_{\substack{\hat{x} \in \gamma_\Lambda \\ \hat{y} \in \xi_{\Lambda^c}}} V(\hat{x}, \hat{y}) \tag{4.1.2}$$

Here, the first sum is taken over all *ordered* pairs $(\hat{x}, \hat{y}) \in \gamma_\Lambda \times \gamma_\Lambda$. This includes also the summand corresponding to $(\hat{x}, \hat{x})$.

Finiteness of the relative energy

Lemma 4.1.2. *Let $\gamma \in \Gamma(\hat{\mathbb{R}}^d)$ and $\xi \in \Gamma_f(\hat{X})$ (cf. (2.2.20)). Then the relative energy (cf. (4.1.2)) that corresponds to the above example is finite:*[3]

$$H_\Lambda(\gamma|\xi) = \sum_{\hat{x},\hat{y} \in \gamma \cap \Lambda} a(x-y) s_x s_y + 2 \sum_{\substack{\hat{x} \in \gamma \cap \Lambda \\ \hat{y} \in \xi \cap \Lambda^C}} a(x-y) s_x s_y < \infty.$$

Proof. For the first sum this follows by the definition of local finite configurations $\gamma \in \Gamma(\hat{\mathbb{R}}^d)$ ((2.1.1)) and the boundedness assumption on a. For the second one we use in addition that a has a bounded support. Thus only points $y \in \tau(\xi \cap \Lambda^c)$ lying in the set

$$\mathcal{U}_\Lambda := \left\{ x \in \mathbb{R}^d \,\middle|\, \operatorname{dist}(x, \Lambda) := \inf_{\hat{y} \in \Lambda} \{|x - y|\} \leq R \right\} \in \mathcal{B}_c(\mathbb{R}^d)$$

are taken in the sum. Here, we fix some $R > 0$ such that

$$a(x) = 0 \text{ whenever } |x| > R.$$

Thus, by the local mass property of the space $\Gamma_f(\hat{X})$ (cf. (2.2.20) in Definition 2.2.7) the sum is finite. □

For the general case see Lemma 4.3.16 and Theorem 4.3.13.

[3] More general potentials are treated in Lemma 4.3.16.

Measurability

Since we have not found an explicit reference for the measurability of the relative energy in both components, we give the proof here (also, to keep the exposition self contained).

Lemma 4.1.3. *The relative energy $H_\Lambda(\gamma|\xi)$ is $\mathcal{B}(\Gamma(\hat{X})\times\Gamma(\hat{X}))$-measurable in $(\gamma,\xi)\in\Gamma(\hat{X})\times\Gamma(\hat{X})$.*

Proof. Here, we do not exclude the possibility that $H_\Lambda(\xi,\gamma)$ is infinite. We have for any $\gamma,\xi\in\Gamma(\hat{X})$

$$H_\Delta(\gamma|\xi) = \lim_{\substack{\Lambda\nearrow\hat{X}\\ \Lambda\in\mathcal{B}_c(\hat{X})}} H_\Delta(\gamma_\Lambda|\xi_\Lambda), \qquad \forall\Delta\in\mathcal{B}_c(\hat{X}).$$

It is enough to show for each fixed Λ that the function $\Gamma(\hat{X})\times\Gamma(\hat{X})\ni(\gamma,\xi)\mapsto H_\Delta(\gamma_\Lambda,\xi_\Lambda)$ obeys the required measurability. The later is implied (cf. (2.1.7) ff) by the claim that for any $n_1,n_2\in\mathbb{N}$ the following function is $\mathcal{B}\left(\bigotimes_{i=1}^{n_1}\hat{X}\times\bigotimes_{i=1}^{n_2}\hat{X}\right)$-measurable:

$$\begin{aligned}
&\Lambda^{n_1+n_2}\ni(\hat{x}_1,\dots,\hat{x}_{n_1})\times(\hat{y}_1,\dots,\hat{y}_{n_2})\mapsto H_\Delta(\{\hat{x}_1,\dots,\hat{x}_{n_1}\}|\{\hat{y}_1,\dots,\hat{y}_{n_2}\})\\
&=\sum_{i,j=1}^{n_1}\mathbb{1}_{\Lambda\cap\Delta}(\hat{x}_i)\mathbb{1}_{\Lambda\cap\Delta}(\hat{x}_j)V(\hat{x}_i,\hat{x}_j)\\
&\qquad+\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\mathbb{1}_{\Lambda\cap\Delta}(\hat{x}_i)\mathbb{1}_{\Lambda\cap\Delta^c}(\hat{y}_j)\big(V(\hat{x}_i,\hat{y}_j)+V(\hat{y}_j,\hat{x}_i)\big).
\end{aligned}\tag{4.1.3}$$

Each summand is $\mathcal{B}\left(\times_{i=1}^{n_1}\hat{X}\times\times_{i=1}^{n_2}\hat{X}\right)$-measurable because V is $\mathcal{B}(\hat{X}\times\hat{X})$ measurable. Since the sums in (4.1.3) are symmetrizing, the claim follows (cf. (2.1.7)). □

4.1.3 Local specification

We fix V to be of the type described in (4.1.1). (More general potentials are considered in Section 4.3).

Let us fix an *inverse temperature* $\beta>0$. For each set $\Lambda\in\mathcal{B}_c(\hat{X})$ and each boundary condition $\xi\in\Gamma(\hat{X})$, we define the *local Gibbs measures*

$$\mu_\Lambda(d\gamma|\xi) := \begin{cases}\frac{1}{Z_\Lambda(\xi)}e^{-\beta H_\Lambda(\gamma|\xi)}P_\theta^\Lambda(d\gamma), & \text{if } \xi\in\Gamma_f(\hat{X})\\ 0, & \text{else}\end{cases}\tag{4.1.4}$$

on $\Gamma(\Lambda)$, where $H_\Lambda(\gamma|\xi)$ is the relative energy introduced in (4.1.2) above. Here, P_θ^Λ is the Lebesgue-Poisson measure on $\Gamma(\Lambda)$ with intensity measure $\lambda_\theta \otimes m$ (cf. (2.1.13)), and $Z_\Lambda(\xi)$ is the *normalizing factor*

$$\begin{aligned} Z_\Lambda(\xi) &:= \int_{\Gamma(\Lambda)} e^{-\beta H_\Lambda(\gamma|\xi)} P_\theta^\Lambda(d\gamma) \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \exp\left(-\beta H_\Lambda \left\{ \left(\bigcup_{i=1}^{n} (s_i, x_i) \middle| \xi \right) \right\} \right) \prod_{i=1}^{n} \lambda_\theta \otimes m(d\hat{x}_i). \end{aligned} \tag{4.1.5}$$

Note that for any $\xi \in \Gamma_f(\hat{X})$

$$1 \leq Z_\Lambda(\xi) \leq e^{\lambda_\theta \otimes m(\Lambda)} < \infty. \tag{4.1.6}$$

By assumption, all $\mu_\Lambda(\cdot|\xi)$ are probability measures for $\xi \in \Gamma_f(\hat{X})$.

Remark 4.1.4. *If $V = 0$ then $\mu_\Lambda(\cdot|\xi) = \mathcal{P}_\theta^\Lambda$ for all $\xi \in \Gamma(\hat{X})$ (cf. (2.2.11), (2.1.14) and (2.1.13) vs. (4.1.4) and (4.1.5)).*

Definition 4.1.5. *The* local specification

$$\pi = \{\pi_\Lambda(\cdot|\xi) | \Lambda \in \mathcal{B}_c(\hat{X}),\ \xi \in \Gamma(\hat{X})\}$$

is a family of stochastic kernels

$$\mathcal{B}(\Gamma(\hat{X})) \times \Gamma(\hat{X}) \ni (B, \xi) \mapsto \pi_\Lambda(B|\xi) \in [0,1] \tag{4.1.7}$$

given by

$$\begin{gathered} \pi_\Lambda(B|\xi) := \mu_\Lambda(B_{\Lambda,\xi}|\xi), \\ B_{\Lambda,\xi} := \{\gamma_\Lambda \in \Gamma(\Lambda) \,|\, \gamma_\Lambda \cup \xi_{\Lambda^c} \in B\} \in \mathcal{B}(\Gamma(\Lambda)). \end{gathered} \tag{4.1.8}$$

This means that for each $F \in B(\Gamma(\hat{X}))$ we have

$$\int_{\Gamma(\hat{X})} F(\gamma) \pi_\Lambda(B|\xi) = \int_{\Gamma(\Lambda)} F(\gamma_\Lambda \cup \xi_{\Lambda^c}) \mu_\Lambda(d\gamma_\Lambda|\xi).$$

Consistency property

Fix $\Lambda \in B_c(\hat{X})$. By construction (cf. [Pre76, Proposition 6.3] or [Pre05, Proposition 2.7, p. 20]), the family (4.1.8) obeys the *consistency property*, which means that for all $B \in \mathcal{B}(\Gamma(\hat{X}))$ and $\xi \in \Gamma(\hat{X})$

$$\int_{\Gamma(\hat{X})} \pi_\Delta(B|\gamma) \pi_\Lambda(\mathrm{d}\gamma|\xi) = \pi_\Lambda(B|\xi), \quad \Delta \subseteq \Lambda. \tag{4.1.9}$$

For $\xi \in \Gamma_f(\hat{X})$, each specification kernel kernel $\pi_\Lambda(\mathrm{d}\gamma|\xi)$ is a *probability measure* on $\Big(\Gamma(\hat{X}), \mathcal{B}(\Gamma(\hat{X}))\Big)$. Given $F \in B(\Gamma(\hat{X}))$ and $\mu \in \mathcal{M}^1(\Gamma(\hat{X}))$ let us define $\pi_\Lambda F \in B(\Gamma(\hat{X}))$ and $\pi_\Lambda \mu \in \mathcal{M}^1(\Gamma(\hat{X}))$ by

$$(\pi_\Lambda F)(\xi) := \int_{\Gamma(\hat{X})} F(\eta)\pi_\Lambda(\mathrm{d}\eta|\xi), \quad \xi \in \Gamma(\hat{X}), \tag{4.1.10}$$

$$(\pi_\Lambda \mu)(B) := \int_{\Gamma(\hat{X})} \pi_\Lambda(B|\eta)\mu(\mathrm{d}\eta), \quad B \in \mathcal{B}(\Gamma(\hat{X})), \tag{4.1.11}$$

which are obviously related by the duality $\langle \pi_\Lambda F, \mu \rangle = \langle F, \pi_\Lambda \mu \rangle$. Here and elsewhere, we use the following shorthand for expectations

$$\langle F, \mu \rangle := \mu\,(F) := \int_{\Gamma(\hat{X})} F \mathrm{d}\mu. \tag{4.1.12}$$

4.1.4 Gibbs measures

Definition 4.1.6. *A probability measure $\mu \in \mathcal{M}^1(\Gamma(\hat{X}))$ is called a* grand canonical **Gibbs measure** *(or* **state***) with pair potential V and inverse temperature $\beta > 0$ if it satisfies the* Dobrushin-Lanford-Ruelle *(**DLR**) equilibrium equation*

$$\int_{\Gamma(\hat{X})} \pi_\Lambda(B|\xi)\mu(\mathrm{d}\xi) = \mu(B) \tag{4.1.13}$$

valid for all $\Lambda \in \mathcal{B}_c(\hat{X})$ and $B \in \mathcal{B}(\Gamma(\hat{X}))$. Fixed an inverse temperature β, the associated set of all Gibbs states will be denoted by $\mathcal{G}\mathrm{ibbs}_V(\Gamma(\hat{X}))$.

Remark 4.1.7. *From the definition of the local specification, we have that any solution of the* (**DLR**) *equation is supported by $\Gamma_f(\hat{X})$.*

To obtain the (**DLR**) *equation it is enough to check (4.1.13) only for $B \in \mathcal{B}_{\mathrm{cyl}}(\Gamma(\hat{X}))$. Indeed, using Caratheodory's theorem, we deduce that $\mu\big|_{\mathcal{B}_{\mathrm{cyl}}(\Gamma(\hat{X}))}$ extends uniquely to a measure on $\sigma(\mathcal{B}_{cyl}(\Gamma(\hat{X}))) = \mathcal{B}(\Gamma(\hat{X}))$. Hence, (4.1.13) holds for all $B \in \mathcal{B}(\Gamma(\hat{X}))$.*

Whenever it is clear on which underlying space we consider the Gibbs measures, we simply write $\mathcal{G}\mathrm{ibbs}_V$ instead of $\mathcal{G}\mathrm{ibbs}_V(\Gamma(\hat{X}))$.

The existence of such a measure under suitable conditions is shown in Section 4.2 for the non-negative potential defined by (4.1.1) (cf. Theorem 4.2.7). In Section 4.3 we handle more general potentials (see Theorem 4.3.26).

4.2 Existence of Gibbs measures: Basic model

We show the existence of a Gibbs measure corresponding to the Gamma-Poisson measure $\mathcal{P}_\theta$ and the non-negative symmetric potential given in Example 4.1.1, i.e.,

$$V(\hat{x}, \hat{y}) := s_x s_y a(x-y), \quad \hat{x}, \hat{y} \in \hat{\mathbb{R}}^d. \tag{4.2.1}$$

After explaining why this type of potential does in general not fit the standard framework, we outline the scheme that we use for proving the existence (cf. below). An important step for the existence proof is the uniform bound given in Proposition 4.2.3. Here, we stick to the basic setting, to clearly state some new essential issues concerning the existence of a Gibbs measure. (The more general setting is handled in Section 4.3.)

Remarks on the potential

The potential specified above (cf. (4.2.1)) has an finite range of interaction regarding its $\mathbb{R}^d$ component, i.e.,

$$\exists R > 0 : \quad V(\hat{x}, \hat{y}) = 0, \qquad \text{if } |x-y| > R.$$

If $a = 0$, we are in the free case, which we have treated in Section 2.2. Thus, without loss of generality, $a \neq 0$ (m-a.e.). Let us fix such a potential for this section.

Lemma 4.2.1. *Fix $\beta > 0$ and consider a non-negative, even function $a \in C_0(\mathbb{R}^d)$ ($a \neq 0$). Then*

$$C(\beta) := \operatorname*{ess\,sup}_{\hat{x} \in \hat{\mathbb{R}}^d} \int_{\hat{\mathbb{R}}^d} \left| e^{-\beta s_x s_y a(x-y)} - 1 \right| \lambda_\theta \otimes m(d\hat{x}) = \infty. \tag{4.2.2}$$

Proof. For all $s \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$

$$\begin{aligned}
&\int_{\hat{\mathbb{R}}^d} \left| e^{-\beta a(y-x)st} - 1 \right| \lambda_\theta \otimes m(d(t,y)) \\
=& \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} \frac{1 - e^{-\beta a(y-x)st}}{t} \theta e^{-t} dt \, m(dy) \\
=& \theta \int_{\mathbb{R}^d} \log(1 + \beta a(y-x)s) m(dy) \\
=& \theta \int_{\mathbb{R}^d} \log(1 + \beta a(y)s) m(dy), \qquad (4.2.3)
\end{aligned}$$

where we used Lemma 3.2.3. There exists $x_0 \in \mathbb{R}^d$ such that $a(x_0) > 0$. Since a is continuous, there exists $\varepsilon > 0$ such that

$$a_m := \min_{x \in \overline{B_\varepsilon}(x_0)} a(x) > 0,$$

where $\overline{B_\varepsilon}(x_0)$ denotes the closed ball centered at x_0 with radius ε. Thus,

$$\int_{\mathbb{R}^d} \log(1 + \beta a(y)s)m(dy) \geq \log(1 + \beta a_m s)m\big(B_\varepsilon(x_0)\big) \xrightarrow[s\to\infty]{} \infty.$$

$\square$

A uniform integrability condition is that $C(\beta) < \infty$ (cf. [AKR98b, Kun99]), but we have $C(\beta) = \infty$ for $0 \neq a \in C_0(\mathbb{R}^d)$ (cf. Lemma 4.2.1). Moreover, another principal difference to the existing literature on marked configuration spaces is that there are infinitely many particles located in any compact $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ set with non-empty interior, i.e.,

$$\int_{\Gamma(\hat{\mathbb{R}}^d)} |\tau(\gamma) \cap \Delta|\, \mathcal{P}_\theta(d\gamma) = \infty, \qquad \forall \varnothing \neq \Delta \in \mathcal{B}_c(\mathbb{R}^d).$$

Usually, this quantity is assumed to be finite (cf. [Kun99, KLU99, KdSS98, AKR98a]).

Remark 4.2.2. *Summing up, we emphasize that this basic model does not fit the classical framework of classical statistical mechanics. Thus it cannot be covered, e.g., by [Rue70, Theorem 5.5], where translation invariance is used, nor by the more recent ones [AKR98a, Kun99], where $C(\beta) < \infty$ is still present. Also the infinite measure λ_θ on $\mathbb{R}_+$ does not fit the abstract scheme of marked configuration spaces, where the intensity measure on the space of marks is assumed to be normalized or finite (cf. [KLU99]).*

The general scheme of the existence result

According to [Geo88], it is important to introduce a correct topology on $\mathcal{M}^1(\Gamma(\hat{X}))$ to obtain the existence of a Gibbs measure μ. The basic aim is to show that

1. "the net $\pi_\Lambda(\cdot|\xi)_{\Lambda\in\mathcal{B}_c(\hat{X})}$ has a cluster point (in the chosen topology)"[4]

2. "each cluster point of $\pi_\Lambda(\cdot|\xi)_{\Lambda\in\mathcal{B}_c(\hat{X})}$ belongs to $\mathcal{G}\mathrm{ibbs}_V$."[4]

[4] This is cited from [Geo88, Chapter 4].

Here, a coarse topology is useful to show the first property, whereas a sufficiently fine topology is needed for the second one. Thus, one has to balance these requirements to find an appropriate topology.

The existence problem has been solved for several models by various authors. We give briefly comments on two classical approaches and refer to e.g. [Pas08] for other ones and more details.

1. **General Dobrushin's criterion for existence of Gibbs measures** It is a standard approach for showing the existence of Gibbs measures was presented in [Dob70b, Theorem 1] (cf. also [Dob70a]). For systems in continuum, it relies on the reduction to a lattice system, which allows to apply the general Dorbrushin criterium for Gibbs fields on $\mathbb{Z}^d$. For classical particle systems in $\mathbb{R}^d$, this method has been further developed in [KKP04] and [PZ99].[5]

2. **Ruelle's technique of superstability estimates** This technique goes back to the celebrated paper [Rue69] (cf. also [LP76]). *Translation invariance* of the interaction potential is used for this approach, which yields the existence of a superstable Gibbs measure for a certain class of boundary conditions.[5]

These approaches do not apply directly in our setting. It is a challenging problem to extend these techniques. Such extensions involve the choice of an appropriate partition of $\mathbb{R}^d$ and the local mass map $\mathfrak{m}$, which allows us to control the (super) stability properties of the interaction. Instead of this, we realize the following (analytic) approach to the existence problem:

Scheme of the existence proof

To construct Gibbs measures, we perform the following steps:[6]

1. Identify a set of boundary conditions, denoted by $\Gamma_f(\hat{X})$, such that

$$H_\Lambda(\gamma|\xi) < \infty \text{ for all } \gamma, \xi \in \Gamma_f(\hat{X}) \text{ and } \Lambda \in \mathcal{B}_c(\hat{X})$$

and $\mathcal{G}_\theta(\Gamma_f(\hat{X})) = 1$ (cf. Lemma 4.1.2).

[5]This is taken from [Pas08, Chapter 1, (iii),I] and [KPR10], to which we also refer for further details.

[6]In our scheme, we appropriately alter the basic idea presented in [Geo88] for the lattice case (and adapted to the continuous model in [KPR10]) to fit our framework. This includes, e.g., to incorporate the concept of a local mass map and handling that $\tau(\gamma)$ is dense.

2. Derive the following support property of the local Gibbs measures (cf. Proposition 4.2.3): For all $\Delta \in \mathcal{B}_c(X)$, it holds

$$\limsup_{\Lambda \in \mathcal{B}_c(\hat{X})} \int_{\Gamma(\Lambda)} \mathfrak{m}_\Delta(\gamma_\Lambda) \mu_\Lambda(d\gamma_\Lambda|\xi) \leq C(\xi) < \infty.$$

3. Check the local equicontinuity for each net $\{\pi_\Lambda(\cdot|\xi)|\ \Lambda \in \mathcal{B}_c(\hat{X})\}$ for all $\xi \in \Gamma_f(\hat{X})$ by the support property.[7]

4. Conclude the existence of a Gibbs measure μ as a cluster point of the net $\{\pi_\Lambda(d\gamma|\xi)\}_{\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)}$, $\xi \in \Gamma_f(\hat{\mathbb{R}}^d)$ fixed, as $\Lambda \nearrow \hat{X}$ by the local equicontinuity.

In detail, we show that the above net has a cluster point μ in $\mathcal{T}_{\mathrm{loc}}$, which is supported by $\Gamma_f(\hat{X})$. Using the consistency of the local specification, we deduce that any such μ satisfies the **(DLR)** equation (cf. Theorem 4.2.7).

4.2.1 Support of the local specification kernels

For all $\xi \in \Gamma_f(\hat{\mathbb{R}}^d)$, The local Gibbs states $\mu(\cdot|\xi)$ are probability measures on $\Gamma(\Lambda)$ because $H_\Lambda(\gamma|\xi) < \infty$ for all $\gamma \in \Gamma(\hat{\mathbb{R}}^d)$, $\xi \in \Gamma_f(\hat{\mathbb{R}}^d)$ (cf. Lemma 4.1.3). Recall that

$$\mathfrak{m}_\Delta(\gamma) := \sum_{\hat{x} \in \gamma} s_x \mathbb{1}_\Delta(x) \qquad \text{for all } \Delta \in B_c(\mathbb{R}^d) \text{ and all } \gamma \in \Gamma(\hat{\mathbb{R}}^d).$$

Proposition 4.2.3. *Let $\xi \in \Gamma_f(\hat{\mathbb{R}}^d)$. Then for all $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$ the probability measures $\mu_\Lambda(\cdot|\xi)$ have full support on $\Gamma(\Lambda) \cap \Gamma_f(\hat{\mathbb{R}}^d)$ and $\pi_\Lambda(\cdot|\xi)$ on $\Gamma_f(\hat{\mathbb{R}}^d)$, respectively.*

In detail, we have for each $\Delta \in \mathcal{B}(\mathbb{R}^d)$ and each $\xi \in \Gamma(\hat{X})$

$$\begin{aligned} \int_{\Gamma(\Lambda)} \mathfrak{m}_\Delta(\gamma_\Lambda) \mu_\Lambda(d\gamma_\Lambda|\xi) &\leq \int_\Lambda \mathfrak{m}_\Delta(\{\hat{x}\})(\lambda_\theta \otimes m)(d\hat{x}) \\ &\leq \int_\Lambda s_x \mathbb{1}_\Delta(x)(\lambda_\theta \otimes m)(d\hat{x}) \leq \theta m(\Delta \cap \Lambda_X). \end{aligned} \tag{4.2.4}$$

If $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ then this integral is finite and the right-hand side in (4.2.4) can be bounded uniformly in Λ.

[7] For this and the following step, we adapt ideas presented in [KPR10, Subsection 3.2], cf. the proof of Proposition 4.2.6.

Remark 4.2.4. *The later estimate, is quite powerful (cf. also Theorem 4.3.31 and Remark 4.3.32) and it seems that it has not been exploited so far. For further details, we refer also to Remarks 4.3.19 and 4.3.20.*

Proof of Proposition 4.2.3. We explicitly estimate the integral by separating one particle from the others. The additional task is to handle that the points are coupled via the relative energy.

For each $\xi \notin \Gamma_f(\hat{X})$, by definition (cf. (4.1.4)) $\mu_\Lambda(\cdot|\xi) = 0$. Thus, the bound holds trivially. Hence, w.l.o.g. $\gamma \in \Gamma_f(\hat{X})$.

By Remark 2.2.8 $\mathfrak{m}_\Delta$ is $\mathcal{B}(\Gamma(\hat{\mathbb{R}}^d))$-measurable. For any $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$, $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ and $\xi \in \Gamma_f(\hat{\mathbb{R}}^d)$ we have

$$\int_{\Gamma(\Lambda)} \mathfrak{m}_\Delta(\gamma_\Lambda)\mu_\Lambda(d\gamma_\Lambda|\xi) = \frac{1}{Z_\Lambda(\xi)} \int_{\Gamma(\Lambda)} \mathfrak{m}_\Delta(\gamma_\Lambda) e^{-\beta H_\Lambda(\gamma_\Lambda|\xi)} P_\theta^\Lambda(d\gamma_\Lambda)$$
$$= \frac{1}{Z_\Lambda(\xi)} \sum_{n\geq 1} \frac{1}{n!} \sum_{k=1}^{n} \int_{\Lambda^n} \mathfrak{m}_\Delta(\{\hat{x}_k\}) e^{-\beta H_\Lambda(\{\hat{x}_1,\ldots,\hat{x}_n\}|\xi)} \prod_{i=1}^{n} (\lambda_\theta \otimes m)(d\hat{x}_i), \quad (4.2.5)$$

where P_θ^Λ denotes the Lebesgue-Poisson measure with intensity measure $\lambda_\theta \otimes m$ (cf. (2.2.9)).

Let $k \in \{1, \ldots, n\}$ be fixed. We are able to decouple the k^{th} particle from the others:

$$H_\Lambda(\{\hat{x}_1, \ldots, \hat{x}_n\}|\xi) = \sum_{\substack{i,j=1,\ldots,n \\ \hat{x}_i,\hat{x}_j \in \Lambda}} V(\hat{x}_i, \hat{x}_j) + \sum_{\substack{\hat{x}_i \in \Lambda,\ i=1,\ldots,n \\ \hat{y} \in \xi_{\Lambda^C}}} 2V(\hat{x}_i, \hat{y})$$
$$= \sum_{\substack{i,j=1 \\ i\neq k\neq j}}^{n} V(\hat{x}_i, \hat{x}_j) + 2\sum_{\substack{j=1 \\ j\neq k}}^{n} V(\hat{x}_k, \hat{x}_j) + V(\hat{x}_k, \hat{x}_k)$$
$$+ \sum_{\substack{i=1,\ldots,n;\ i\neq k \\ \hat{x}_i \in \Lambda, \hat{y} \in \xi_{\Lambda^C}}} 2V(\hat{x}_i, \hat{y}) + \sum_{\substack{\hat{x}_k \in \Lambda \\ \hat{y} \in \xi_{\Lambda^C}}} 2V(\hat{x}_k, \hat{y}). \quad (4.2.6)$$

Since $V \geq 0$, we get

$$H_\Lambda(\{\hat{x}_1, \ldots, \hat{x}_n\}|\xi) \geq \sum_{\substack{i,j=1 \\ i\neq k\neq j}}^{n} V(\hat{x}_i, \hat{x}_j) + \sum_{\substack{i=1,\ldots,n \\ i\neq k,\ \hat{x}_i \in \Lambda \\ \hat{y} \in \xi_{\Lambda^C}}} 2V(\hat{x}_i, \hat{y})$$
$$+ V(\hat{x}_k, \hat{x}_k) + \sum_{\substack{\hat{x}_k \in \Lambda \\ \hat{y} \in \xi_{\Lambda^C}}} 2V(\hat{x}_k, \hat{y}). \quad (4.2.7)$$

Plugging (4.2.7) into (4.2.5), we get

$$\begin{aligned}
&\int_{\Gamma(\Lambda)} \mathfrak{m}_\Delta(\gamma_\Lambda)\mu_\Lambda(d\gamma_\Lambda|\xi) \\
&\le \frac{1}{Z_\Lambda(\xi)} \sum_{n\ge 1} \frac{1}{n!} \sum_{k=1}^{n} \int_{\Lambda^n} \mathfrak{m}_\Delta(\{\hat{x}_k\}) e^{-\beta H_\Lambda(\{\hat{x}_k\}|\xi\})} \\
&\qquad\qquad \times e^{-\beta H_\Lambda(\{\hat{x}_1,\dots,\hat{x}_{k-1},\hat{x}_{k+1},\dots,\hat{x}_n\}|\xi)} \prod_{i=1}^{n} (\lambda_\theta \otimes m)(d\hat{x}_i) \\
&= \frac{1}{Z_\Lambda(\xi)} \Bigg(\int_\Lambda \mathfrak{m}_\Delta(\{\hat{x}_1\}) e^{-\beta H_\Lambda(\{\hat{x}_k\}|\xi\})} (\lambda_\theta \otimes m)(d\hat{x}_1) \\
&\quad + \sum_{n\ge 2} \frac{1}{n!} \sum_{k=1}^{n} \int_\Lambda \mathfrak{m}_\Delta(\{\hat{x}_k\}) e^{-\beta H_\Lambda(\{\hat{x}_k\}|\xi\})} (\lambda_\theta \otimes m)(d\hat{x}_k) \\
&\quad \times \int_{\Lambda^{n-1}} e^{-\beta H_\Lambda(\{\hat{x}_1,\dots,\hat{x}_{k-1},\hat{x}_{k+1},\dots,\hat{x}_n\}|\xi)} \prod_{i=1, i\neq k}^{n} (\lambda_\theta \otimes m)(d\hat{x}_i) \Bigg)
\end{aligned}$$

Taking the common integral out, this equals

$$\begin{aligned}
&\int_\Lambda \mathfrak{m}_\Delta(\{\hat{x}\}) e^{-\beta H_\Lambda(\{\hat{x}_k\}|\xi)} (\lambda_\theta \otimes m)(d\hat{x}) \frac{1}{Z_\Lambda(\xi)} \\
&\quad \times \int_{\Lambda^{N-1}} \left(1 + \sum_{n\ge 2} \frac{n}{n!} \int_{\Lambda^{n-1}} e^{-\beta H_\Lambda(\{\hat{x}_1,\dots,\hat{x}_{n-1}\}|\xi)} \prod_{i=1}^{n-1} (\lambda_\theta \otimes m)(d\hat{x}_i) \right) \\
&= \int_\Lambda \mathfrak{m}_\Delta(\{\hat{x}\}) e^{-\beta H_\Lambda(\{\hat{x}_k\}|\xi\})} (\lambda_\theta \otimes m)(d\hat{x}) \frac{1}{Z_\Lambda(\xi)} Z_\Lambda(\xi). \qquad (4.2.8)
\end{aligned}$$

The later is dominated by

$$\int_{\hat{\mathbb{R}}^d} s_x \mathbb{1}_\Delta(x)(\lambda_\theta \otimes m)(d\hat{x}) = \theta m(\Delta) \int_{\mathbb{R}_+} t t^{-1} e^{-t} dt = \theta m(\Delta). \qquad (4.2.9)$$

If $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ then

$$\begin{aligned}
&\sup_{\Lambda\in\mathcal{B}_c(\hat{\mathbb{R}}^d)} \int_{\Gamma(\hat{\mathbb{R}}^d)} \mathfrak{m}_\Delta(\gamma)\pi_\Lambda(d\gamma|\xi) \\
&= \sup_{\Lambda\in\mathcal{B}_c(\hat{\mathbb{R}}^d)} \left(\int_{\Gamma(\hat{\mathbb{R}}^d)} \mathfrak{m}_\Delta(\gamma)\mu_\Lambda(d\gamma|\xi) + \mathfrak{m}_\Delta(\xi_{\Lambda^c}) \right) \\
&\quad \le \theta m(\Delta) + \sup_{\Lambda\in\mathcal{B}_c(\hat{\mathbb{R}}^d)} \mathfrak{m}_\Delta(\xi_{\Lambda^c}) \le \theta m(\Delta) + \mathfrak{m}_\Delta(\xi) < \infty, \qquad (4.2.10)
\end{aligned}$$

where $\xi \in \Gamma_f(\hat{\mathbb{R}}^d)$. This implies the support properties of $\pi_\Lambda(\cdot|\xi)$ resp. $\mu_\Lambda(\cdot|\xi)$ stated in the assertion. □

4.2.2 Local equicontinuity

The local equicontinuity allows us to construct a Gibbs measure as a limit point of a net consisting of the local specification kernels.

Definition 4.2.5 (cf. [Geo88, (4.6) Definition])**.** *Let Y be a locally compact Polish space. The net of probability measures $\{\nu_\Lambda |\ \Lambda \in \mathcal{B}_c(Y)\}$ on $\Gamma(Y)$ is called* locally equicontinuous *if for all $\tilde{\Lambda} \in \mathcal{B}_c(Y)$ and each sequence $\{B_N\}_{N\in\mathbb{N}} \subset \mathcal{B}(\Gamma(\tilde{\Lambda}))$ with $B_N \downarrow \varnothing$*

$$\lim_{N\to\infty} \limsup_{\substack{\Lambda\in\mathcal{B}_c(Y)\\ \Lambda\nearrow Y}} \nu_\Lambda(B_N) = 0. \tag{4.2.11}$$

Proposition 4.2.6. *The net $\{\pi_\Lambda(\mathrm{d}\gamma|\xi)|\ \Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)\}$ is locally equicontinuous for each fixed $\xi \in \Gamma_f(\hat{\mathbb{R}}^d)$.*

Proof. As in [KPR10], we adapt to the configuration space $\Gamma(\hat{\mathbb{R}}^d)$ the arguments used for proving Theorem 4.12 and Corollary 4.13 in [Geo88].

Fix an arbitrary compact set $\tilde{\Lambda} = \tilde{\Lambda}_{\mathbb{R}_+} \times \tilde{\Lambda}_{\mathbb{R}^d} \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$ and let $\{B_N\}_{N\in\mathbb{N}}$ be any sequence of sets from $\mathcal{B}(\Gamma(\tilde{\Lambda}))$ such that $B_N \downarrow \varnothing$ as $N \to \infty$. We choose $R > 0$ such that $a\big|_{B^c_R(0)} = 0$ and set

$$\begin{aligned}\mathcal{U} := \mathcal{U}_R(\tilde{\Lambda}) := &\left\{x \in \mathbb{R}^d \;\middle|\; \mathrm{dist}_{\mathbb{R}^d}(x, \tilde{\Lambda}_{\mathbb{R}^d}) := \inf_{y\in\tilde{\Lambda}_{\mathbb{R}^d}} |x-y| \leq R\right\}\\ &\in \mathcal{B}_c(\mathbb{R}^d).\end{aligned} \tag{4.2.12}$$

Consider the following Borel subsets of configurations whose local masses at $\mathcal{U}$ are bounded by $T > 0$,

$$\Gamma(\mathcal{U}, T) := \left\{\gamma \in \Gamma(\hat{\mathbb{R}}^d)\ \mid (\mathfrak{m}_{\mathcal{U}}(\gamma)) \leq T\right\}. \tag{4.2.13}$$

Note that the map $\hat{y} \mapsto V(\hat{x}, \hat{y}) = s_x s_y a(x-y)$, $\hat{x} \in \tilde{\Lambda}$, is surely zero outside of $\mathbb{R}_+ \times \mathcal{U}$. For each $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$ and $\xi \in \Gamma_f(\hat{\mathbb{R}}^d)$, by the definition of a local specification (cf. (4.1.7), (4.1.8)) and the consistency property (cf. (4.1.9)), we have

$$
\begin{aligned}
\pi_\Lambda(B_N|\xi) &= \pi_\Lambda(B_N \cap [\Gamma(\mathcal{U},T)^c|\xi) \\
&\qquad + \int_{\Gamma(\hat{\mathbb{R}}^d)} \pi_{\tilde{\Lambda}\cap\Lambda}(B_N \cap \Gamma(\mathcal{U},T)|\eta)\pi_\Lambda(d\eta|\xi) \qquad (4.2.14) \\
&= \pi_\Lambda(B_N \cap [\Gamma(\mathcal{U},T)]^c|\xi) \\
&\qquad + \int_{\Gamma(\hat{\mathbb{R}}^d)} \frac{1}{Z_{\tilde{\Lambda}\cap\Lambda}(\eta)} \int_{\Gamma(\tilde{\Lambda}\cap\Lambda)} \mathbb{1}_{B_N\cap\Gamma(\mathcal{U},T)}(\gamma_{\tilde{\Lambda}\cap\Lambda} \cup \eta_{(\tilde{\Lambda}\cap\Lambda)^c}) \\
&\qquad\quad \times \exp\{-\beta H_{\tilde{\Lambda}\cap\Lambda}(\gamma_{\tilde{\Lambda}\cap\Lambda}|\eta)\}\, P_\theta^{\tilde{\Lambda}\cap\Lambda}(\mathrm{d}\gamma_{\tilde{\Lambda}\cap\Lambda})\pi_\Lambda(\mathrm{d}\eta|\xi).
\end{aligned}
$$

Using that $Z_{\tilde{\Lambda}\cap\Lambda} \geq 1$, the later is dominated by

$$
\begin{aligned}
&\pi_\Lambda([\Gamma(\mathcal{U},T)]^c|\xi) \qquad (4.2.15) \\
&\qquad + \int_{\Gamma(\hat{\mathbb{R}}^d)} \int_{\Gamma(\tilde{\Lambda}\cap\Lambda)} \mathbb{1}_{B_N\cap\Gamma(\mathcal{U},T)}(\gamma_{\tilde{\Lambda}\cap\Lambda} \cup \eta_{(\tilde{\Lambda}\cap\Lambda)^c}) \\
&\qquad\quad \times \exp\{-\beta H_{\tilde{\Lambda}\cap\Lambda}(\gamma_{\tilde{\Lambda}\cap\Lambda}|\eta)\}\, P_\theta^{\tilde{\Lambda}\cap\Lambda}(\mathrm{d}\gamma)\pi_\Lambda(\mathrm{d}\eta|\xi). \qquad (4.2.16)
\end{aligned}
$$

Chebyshev's inequality insures that for each $\varepsilon > 0$ there exists $T(\varepsilon,\xi) > 0$ such that

$$
\pi_\Lambda([\Gamma(\mathcal{U},T)]^c|\xi) \leq \varepsilon \quad \text{for all } T \geq T(\varepsilon,\xi) \text{ and } \Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d). \qquad (4.2.17)
$$

Indeed, using Proposition 4.2.3, we get

$$
\begin{aligned}
&\pi_\Lambda\left(\left\{\gamma \in \Gamma(\hat{\mathbb{R}}^d) : \mathfrak{m}_{\mathcal{U}}(\gamma) > T\right\}\Big|\, \xi\right) \\
&\qquad \leq \int_{\Gamma(\hat{\mathbb{R}}^d)} \frac{\mathfrak{m}_{\mathcal{U}}(\gamma)}{T} \pi_\Lambda(d\gamma|\xi) \qquad (4.2.18) \\
&\qquad \leq \frac{1}{T}\left(\theta m\big(\mathcal{U}_R(\tilde{\Lambda})\big) + \mathfrak{m}_{\mathcal{U}}(\xi_{\Lambda^c})\right) =: \frac{1}{T} C_{\mathcal{U},\xi} < \infty.
\end{aligned}
$$

As $T \to \infty$, the whole term becomes arbitrary small.

Since $H_{\tilde{\Lambda}\cap\Lambda}\big(\gamma_{\tilde{\Lambda}\cap\Lambda}\big|\eta\big) \geq 0$ for all $\gamma \in \Gamma(\hat{\mathbb{R}}^d)$ and $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$, the inner integral in (4.2.16) is dominated by

$$
\begin{aligned}
&\int_{\Gamma(\tilde{\Lambda}\cap\Lambda)} \mathbb{1}_{B_N\cap\Gamma(\mathcal{U},T)}(\gamma_{\tilde{\Lambda}\cap\Lambda} \cup \eta_{\theta,(\tilde{\Lambda}\cap\Lambda)^c}) P_\theta^{\tilde{\Lambda}\cap\Lambda}(d\gamma_{\tilde{\Lambda}}) \\
&\qquad \leq P_\theta^{\tilde{\Lambda}}(B_N \cap \Gamma(\tilde{\Lambda})) \leq \epsilon, \qquad \text{as soon as } N \geq N(\epsilon). \qquad (4.2.19)
\end{aligned}
$$

Plugging (4.2.17) and (4.2.19) back into (4.2.15) and letting $T \nearrow \infty$ and then $B_N \downarrow \varnothing$, we get the required equicontinuity of the net $\{\pi_\Lambda(\mathrm{d}\gamma|\xi)|\, \Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)\}$ (cf. (4.2.11)). $\square$

4.2.3 Existence of Gibbs measures: Basic model

We show that each limit measure that we obtain by the local equicontinuity is indeed a Gibbs measure.[8]

Theorem 4.2.7. *Let $a \in L^1(\mathbb{R}^d, m)$ be as described in Example 4.1.1, i.e., $\exists M_a < \infty$ and $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$: $0 \leq a(x) = a(-x) \leq M \mathbb{1}_\Delta(x)$ for all $x \in \mathbb{R}^d$. Assume that*

$$V(\hat{x}, \hat{y}) = s_x s_y a(x - y), \quad \forall \hat{x}, \hat{y} \in \hat{\mathbb{R}}^d. \tag{4.2.20}$$

Then there exists a (non-zero) Gibbs measure μ corresponding to the potential V and the Gamma-Poisson measure $\mathcal{P}_\theta$, $\theta > 0$ being the fixed parameter. It is supported by $\Gamma_f(\hat{\mathbb{R}}^d)$.

Proof. We will show that the local equicontinuous specification has a cluster point in $\mathcal{T}_{\text{loc}}$ (cf. Definition 2.1.2). For this candidate (for being a Gibbs measure) we show that it is supported by the ‹tempered› configurations $\gamma \in \Gamma_f(\hat{\mathbb{R}}^d)$. The final step is to prove the **(DLR)** equation (cf. (4.1.13)), for which we use the monotonicity property of the local specification.

We first observe that the relative energy $H_\Lambda(\eta|\xi)$ is finite for all $\eta \in \Gamma(\hat{\mathbb{R}}^d)$, $\xi \in \Gamma_f(\hat{\mathbb{R}}^d)$ and $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$ (cf. Lemma 4.1.2).

Let $\xi \in \Gamma_f(\hat{\mathbb{R}}^d)$ be fixed from now on. By Proposition 4.2.6 the net $\{\pi_\Lambda(d\gamma|\xi) |\ \Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)\}$ is locally equicontinuous. By [Geo88, Proposition 4.9] combined with [Pat67, Theorem V.3.2], any locally equicontinuous net in $\mathcal{M}^1(\Gamma(\hat{\mathbb{R}}^d))$ has at least one $\mathcal{T}_{\text{loc}}$-cluster point. Thus, there exists a limit point

$$\mu := \lim_{N \to \infty} \pi_{\Lambda_N}(\cdot|\xi) \in \mathcal{M}^1(\Gamma(\hat{\mathbb{R}}^d))$$

taken along some order generating sequence $\Lambda_N \nearrow \hat{\mathbb{R}}^d$, such that for all local sets $B \in \mathcal{B}_{\text{cyl}}(\Gamma(\hat{\mathbb{R}}^d))$:

$$\pi_{\Lambda_N}(B|\xi) \to \mu(B) \quad \text{as } N \to \infty. \tag{4.2.21}$$

To check the support property, we take advantage of $\pi_\Lambda(\cdot|\xi)$ being supported by $\Gamma_f(\hat{\mathbb{R}}^d)$ for each $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$ (cf. Proposition 4.2.3).

Indeed, fix an arbitrary $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$. Since $\mathfrak{m}_\Delta(\cdot)$ is not $\mathcal{B}_{\text{cyl}}(\Gamma(\hat{\mathbb{R}}^d))$-measurable, we use a cut-off procedure. For all $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$ the local mass $\gamma \mapsto \mathfrak{m}_\Delta(\gamma_\Lambda)$ is $\mathcal{B}_{\text{cyl}}(\Gamma(\hat{\mathbb{R}}^d))$-measurable; and

$$\lim_{\Lambda \nearrow \hat{\mathbb{R}}^d} \mathfrak{m}_\Delta(\gamma \cap \Lambda) \nearrow \mathfrak{m}_\Delta(\gamma), \qquad \forall \gamma \in \Gamma(\hat{\mathbb{R}}^d).$$

[8] This idea also used in [KPR10, Subsection 3.2].

Hence, by Beppo Levi,

$$\begin{aligned}
\int_{\Gamma(\hat{\mathbb{R}}^d)} \mathfrak{m}_\Delta(\gamma)\mu(d\gamma) &= \lim_{\substack{\Lambda\nearrow\hat{\mathbb{R}}^d\\ \Lambda\in B_c(\hat{\mathbb{R}}^d)}} \int_{\Gamma(\hat{\mathbb{R}}^d)} \mathfrak{m}_\Delta(\gamma_\Lambda)\mu(d\gamma)\\
&= \lim_{\substack{\Lambda\nearrow\hat{\mathbb{R}}^d\\ \Lambda\in B_c(\hat{\mathbb{R}}^d)}} \lim_{N\to\infty} \int_{\Gamma(\hat{\mathbb{R}}^d)} \mathfrak{m}_\Delta(\gamma_\Lambda)\pi_{\Lambda_N}(d\gamma|\xi)\\
&\le \lim_{\substack{\Lambda\nearrow\hat{\mathbb{R}}^d\\ \Lambda\in B_c(\hat{\mathbb{R}}^d)}} \left(\lim_{N\to\infty} \int_{\Gamma(\hat{\mathbb{R}}^d)} \mathfrak{m}_\Delta(\gamma_{\Lambda\cap\Lambda_N})\mu_{\Lambda_N}(d\gamma_{\Lambda_N}|\xi) + \limsup_{N\to\infty} \mathfrak{m}_\Delta(\Lambda\cap\xi_{\Lambda_N^c})\right)\\
&\le \lim_{\substack{\Lambda\nearrow\hat{\mathbb{R}}^d\\ \Lambda\in B_c(\hat{\mathbb{R}}^d)}} \int_{\hat{\mathbb{R}}^d} \mathfrak{m}_\Delta(\{\hat{x}\})\mathbb{1}_\Lambda(\hat{x})\lambda_\theta\otimes m(d\hat{x}) \le \theta m(\Delta) =: M_\Delta < \infty, \qquad (4.2.22)
\end{aligned}$$

where we used the uniform bound given in Proposition 4.2.3 (cf. (4.2.4)).

To prove that μ is a Gibbs measure, it remains to check the **(DLR)** equation. To this end, we would like to apply the consistency property of the specification kernels π_{Λ_N}.

Fix $\tilde{\Lambda} \in \mathcal{B}_c(\mathbb{R}_+ \times \mathbb{R}^d)$ and $B \in \mathcal{B}_{\mathrm{cyl}}(\Gamma(\hat{\mathbb{R}}^d))$. We take care that the function $\Gamma(\hat{\mathbb{R}}^d) \ni \gamma \mapsto \pi_{\tilde{\Lambda}}(B|\gamma)$ is in general only $\mathcal{B}(\Gamma(\hat{\mathbb{R}}^d))$- and not $\mathcal{B}_{\mathrm{cyl}}(\Gamma(\hat{\mathbb{R}}^d))$-measurable. We overcome this problem by using a cut-off procedure and the fact that the measure μ is defined on the σ-algebra generated by the algebra $\mathcal{B}_{\mathrm{cyl}}(\Gamma(\hat{\mathbb{R}}^d))$. In fact, for each $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$, we consider the truncated function $\Gamma(\hat{\mathbb{R}}^d) \ni \gamma \mapsto \pi_{\tilde{\Lambda}}(B|\gamma_\Lambda)$, which is (obviously) $\mathcal{B}_\Lambda(\Gamma(\hat{\mathbb{R}}^d))$- and therefore $\mathcal{B}_{\mathrm{cyl}}(\Gamma(\hat{\mathbb{R}}^d))$-measurable.

We will justify the following equations to obtain the **(DLR)** one:

$$\begin{aligned}
\int_{\Gamma(\hat{\mathbb{R}}^d)} \pi_{\tilde{\Lambda}}(B|\gamma)\mu(d\gamma) &\overset{1.}{=} \lim_{\substack{\Lambda\nearrow\hat{\mathbb{R}}^d\\ \Lambda\in\mathcal{B}_c(\hat{\mathbb{R}}^d)}} \int_{\Gamma(\hat{\mathbb{R}}^d)} \pi_{\tilde{\Lambda}}(B|\gamma_\Lambda)\mu(d\gamma)\\
&\overset{2.}{=} \lim_{\substack{\Lambda\nearrow\hat{\mathbb{R}}^d\\ \Lambda\in\mathcal{B}_c(\hat{\mathbb{R}}^d)}} \lim_{N\to\infty} \int_{\Gamma(\hat{\mathbb{R}}^d)} \pi_{\tilde{\Lambda}}(B|\gamma_\Lambda)\pi_{\Lambda_N}(d\gamma|\xi),\\
&\overset{3.}{=} \lim_{N\to\infty} \int_{\Gamma(\hat{\mathbb{R}}^d)} \pi_{\tilde{\Lambda}}(B|\gamma)\pi_{\Lambda_N}(d\gamma|\xi)\\
&\overset{4.}{=} \lim_{N\to\infty} \pi_{\Lambda_N}(B|\xi) \overset{5.}{=} \mu(B). \qquad (4.2.23)
\end{aligned}$$

The second and fifth equation follow by the definition of μ (cf. (4.2.21)). The fourth holds by the consistency of the local specification (cf. (4.1.9)).

We have to argue a little more to obtain the other ones (cf. (4.2.25) for the first and (4.2.28) for the third one).

Denote $\tilde{\Delta} := \tilde{\Lambda}_{\hat{\mathbb{R}}^d}$. In order to show that the the first equation in (4.2.23) holds, we use that by Lemma 4.3.28 (cf. (4.3.29)) below there exists $M_{\Lambda'} < \infty$ such that

$$\begin{aligned}
&\int_{\Gamma(\hat{\mathbb{R}}^d)} |\pi_{\tilde{\Delta}}(B|\gamma_\Lambda) - \pi_{\tilde{\Lambda}}(B|\gamma)|\, \mu(d\gamma) \\
\leq & 2\beta M_a \Big(1 + e^{\lambda_\theta \otimes m(\tilde{\Lambda})}\Big) M_{\Lambda'} \int_{\Gamma(\hat{\mathbb{R}}^d)} \mathfrak{m}_{\tilde{\Lambda}}(\gamma_{\Lambda^c}) \mu(d\gamma) \\
\leq & 2\beta M_a \Big(1 + e^{\lambda_\theta \otimes m(\tilde{\Lambda})}\Big) M_{\Lambda'} \int_{\hat{\mathbb{R}}^d} \mathfrak{m}_{\tilde{\Lambda} \cap \Lambda^c_{\mathbb{R}^d}}(\{\hat{x}\}) \lambda_\theta \otimes m(d\hat{x}),
\end{aligned} \tag{4.2.24}$$

where we used (4.2.22) to deduce the last inequality. The integrand in the last line in (4.2.24) converges pointwisely to 0. Thus, using Lebesgue's dominated convergence theorem, we have justified the first equality in (4.2.23), i.e.,

$$\int_{\Gamma(\hat{\mathbb{R}}^d)} \pi_{\tilde{\Lambda}}(B|\gamma)\mu(d\gamma) = \lim_{\substack{\Lambda \nearrow \hat{\mathbb{R}}^d \\ \Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)}} \int_{\Gamma(\hat{\mathbb{R}}^d)} \pi_{\tilde{\Lambda}}(B|\gamma_\Lambda)\mu(d\gamma). \tag{4.2.25}$$

It remains to prove the third equality in (4.2.23): Using (4.3.29) we have

$$\begin{aligned}
&\left| \int_{\Gamma(\hat{\mathbb{R}}^d)} \big[\pi_{\tilde{\Lambda}}(B|\gamma_\Lambda) - \pi_{\tilde{\Lambda}}(B|\gamma)\big] \pi_{\Lambda_N}(d\gamma|\xi) \right| \\
\leq & \Big(1 + e^{\lambda_\theta \otimes m(\tilde{\Lambda})}\Big) \beta M_a M_{\Lambda'} \int_{\Gamma(\hat{\mathbb{R}}^d)} \mathfrak{m}_{\tilde{\Delta}}(\gamma_{\Lambda^c}) \pi_{\Lambda_N}(d\gamma|\xi).
\end{aligned} \tag{4.2.26}$$

Proposition 4.2.3 yields the following bound for the last integral

$$\begin{aligned}
&\int_{\Lambda_N} \mathfrak{m}_{(\tilde{\Lambda} \cap \Lambda^c)_{\mathbb{R}^d}}(\{\hat{x}\}) \lambda_\theta \otimes m(d\hat{x}) + \mathfrak{m}_{\tilde{\Delta}}(\xi_{\Lambda^c \cap \Lambda_N{}^c}) \\
\leq & \int_{\hat{\mathbb{R}}^d} \mathfrak{m}_{(\tilde{\Lambda} \cap \Lambda^c)_{\mathbb{R}^d}}(\{\hat{x}\}) \lambda_\theta \otimes m(d\hat{x}) + \mathfrak{m}_{\tilde{\Delta}}(\xi_{\Lambda^c}) \\
\leq & \int_{\hat{\mathbb{R}}^d} \mathfrak{m}_{(\tilde{\Lambda} \cap \Lambda^c)_{\mathbb{R}^d}}(\{\hat{x}\}) \lambda_\theta \otimes m(d\hat{x}) + \sum_{\hat{x} \in \xi_{\Lambda^c \cap \Lambda_N{}^c}} \mathfrak{m}_{\tilde{\Delta}}(\{\hat{x}\}).
\end{aligned} \tag{4.2.27}$$

As before, by Lebesgue's dominated convergence theorem the integral becomes arbitrary small for $\Lambda \nearrow \hat{X}$. The same is true for the second sum.

Thus, the third equation in (4.2.23) holds, i.e.,

$$\lim_{\substack{\Lambda\nearrow\hat{\mathbb{R}}^d\\ \Lambda\in\mathcal{B}_c(\hat{\mathbb{R}}^d)}} \lim_{N\to\infty} \int_{\Gamma(\hat{\mathbb{R}}^d)} \pi_{\tilde{\Lambda}}(B|\gamma_\Lambda)\pi_{\Lambda_N}(d\gamma|\xi)$$
$$= \lim_{N\to\infty} \int_{\Gamma(\hat{\mathbb{R}}^d)} \pi_{\tilde{\Lambda}}(B|\gamma)\pi_{\Lambda_N}(d\gamma|\xi). \tag{4.2.28}$$

Hence, μ is a Gibbs measure (being supported by $\Gamma_f(\hat{\mathbb{R}}^d)$). □

4.3 Existence of Gibbs measures on $\Gamma(\hat{X})$: General case

We come back to the general case of X being a locally compact Polish space with non-atomic Radon measure m. We generalize the concept of a local (w.r.t. $\lambda_\theta \otimes m$) mass map on $\Gamma(\hat{X})$ to include more general potentials. Then we follow the scheme presented above (cf. P. 66) to show the existence of a Gibbs measure. The uniform bound for the local Gibbs measures (cf. Proposition 4.3.18) is a key issue, not only for showing the existence, but also for proving the uniform bound for all Gibbs measures (cf. Theorem 4.3.31).[9]

We define the set $L_s(X \times X)$ of symmetric functions which are only supported on a ‹strip around the diagonal›:

Definition 4.3.1. *By $L_s(X \times X)$ we denote the set of* ***bounded*** *symmetric $\mathcal{B}(X)$-measurable functions ϕ over $X \times X$ which fulfill*

(**FR**) ***Finite range :*** *For any $\Delta \in \mathcal{B}_c(X)$ there exists $\mathcal{U}_\Delta \in \mathcal{B}_c(X)$ such that*

$$\phi(x, y) = \phi(y, x) = 0, \quad \forall x \in \Delta,\ y \in \mathcal{U}_\Delta^c. \tag{4.3.1}$$

Remark 4.3.2. *In the basic model setting the finite range condition is fulfilled if*

$$\exists R \in [0, \infty): \quad \forall x, y \in \mathbb{R}^d: \quad |x - y| > R \ \Rightarrow\ \phi(x, y) = 0. \tag{4.3.2}$$

In this case, we set for $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$

$$\mathcal{U}_\Delta := \left\{ x \in \mathbb{R}^d \ \middle|\ \mathrm{dist}_{\mathbb{R}^d}(x, \Delta) := \inf_{y\in\Delta} |x - y| \le R \right\} \in \mathcal{B}_c(\mathbb{R}^d). \tag{4.3.3}$$

[9] We point out that the result of this section hold not only for the Gamma-Poisson measure $\mathcal{P}_\theta$, but even for more general Poisson measures. For this compare Section 5.1 and especially the setting and (5.1.25) therein. But for the sake of simplicity, we stick to our main motivation, namely the Gamma-Poisson measure $\mathcal{P}_\theta$.

Admissible potentials

The class of admissible potentials depends strongly on the behavior of the Levy measure λ_θ (and on m). In general, we can handle all symmetric potentials $V(\hat{x}, \hat{y})$ obeying

$$0 \leq V(\hat{x}, \hat{y}) \leq \mathfrak{l}(\hat{x})\mathfrak{l}(\hat{y})\phi(x, y),$$

where $\phi \in L_s(X \times X)$ and $\mathfrak{l} : \hat{X} \to \mathbb{R}_+$ is such that

$$\hat{X} \ni \hat{x} \mapsto \mathfrak{l}(\hat{x})1\!\!1_\Delta(x) \in L^1(\hat{X}, \lambda_\theta \otimes m), \qquad \text{for all } \Delta \in \mathcal{B}_c(X),$$

i.e., $\mathfrak{l}$ is a *semi-local* function (cf. Definitions 4.3.1 and 4.3.4). Respectively, the set of admissible boundary configuration is

$$\Gamma_{f,\mathfrak{l}}(\hat{X}) := \left\{ \xi \in \Gamma_p(\hat{X}) \middle| \sum_{x \in \tau(\xi) \cap \Delta} \mathfrak{l}(\hat{x}) < \infty, \qquad \forall \Delta \in \mathcal{B}_c(X) \right\}.$$

Whenever it is clear which semi-local function $\mathfrak{l}$ is involved, we omit the index $\mathfrak{l}$. Below, we will show that $\mathcal{P}_\theta\big(\Gamma_{f,\mathfrak{l}}(\hat{X})\big) = 1$.

Example 4.3.3. *These admissible potentials include*

$$V_p(\hat{x}, \hat{y}) = s_x^p \cdot s_y^p\, a(x - y) \quad \hat{x}, \hat{y} \in \hat{\mathbb{R}}^d,$$

where $0 \leq a \in L^1(\mathbb{R}^d, dx)$ is bounded, even and compactly supported. More generally, by choosing $\phi \in L_s(X \times X)$ (cf. Definition 4.3.1), we can treat the following one

$$V_p(\hat{x}, \hat{y}) = s_x^p \cdot s_y^p\, \phi(x, y) \quad \hat{x}, \hat{y} \in \hat{X}.$$

Another type of potentials is the 1-particle potential:

$$V(\hat{x}, \hat{y}) = s_x^{1/2} s_y^{1/2} 1\!\!1_{\{x=y\}}(x, y) b(x),$$

where $b \in L^\infty(\mathbb{R}^d, dx)$ is bounded.

4.3.1 A (general) local (w.r.t. $\lambda_\theta \otimes m$) mass map

Here, we introduce the concepts of a semi-local functions and of a local mass map. Using them, we can extend the potentials of the basic model (cf. Section 4.2).

Definition 4.3.4. *Let $\mathfrak{l} : \hat{X} \to \mathbb{R}^+$ be $\mathcal{B}(\hat{X})$-measurable. The function $\mathfrak{l}$ is called* semi-local *(w.r.t. $\lambda_\theta \otimes m$) if it is integrable on $\mathbb{R}_+ \times \Delta$, for all $\Delta \in \mathcal{B}_c(X)$. This means*

$$\int_{\mathbb{R}_+ \times \Delta} \mathfrak{l}(\hat{x}) \lambda_\theta \otimes m(d\hat{x}) \leq C_\Delta < \infty, \quad \forall \Delta \in \mathcal{B}_c(X). \tag{4.3.4}$$

Example 4.3.5. *Let $X = \mathbb{R}^d$ with $d \in \mathbb{N}$ and let $m(dx) = dx$ be the Lebesgue measure on $\mathbb{R}^d$. For $p > 0$, the function*

$$\mathfrak{l}_p : \hat{\mathbb{R}}^d \ni \hat{x} \mapsto \mathfrak{l}_p(\hat{x}) := s_x^p \in \mathbb{R}^+ \tag{4.3.5}$$

is semi-local (w.r.t. $\lambda_\theta \otimes m$). An upper bound for the corresponding integral (cf. (4.3.4)) is given for each $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ by

$$\int_{\mathbb{R}_+ \times \Delta} \mathfrak{l}_p(\hat{x}) \lambda_\theta \otimes m(d\hat{x}) = m(\Delta) \cdot \int_{\mathbb{R}_+} s^p \theta \frac{e^{-s}}{s} ds = m(\Delta) \theta \Gamma(p) < \infty,$$

where $\Gamma(p)$ denotes the classical Gamma function.

Local mass map

We introduce the second important concept of the so-called local mass map.

Definition 4.3.6. *Let $\mathfrak{l}$ be a semi-local function in the sense of Definition 4.3.4. The* associated (semi-)local *(w.r.t. $\lambda_\theta \otimes m$)* mass map

$$\mathfrak{m} := \mathfrak{m}_\mathfrak{l} : \mathcal{B}(\hat{X}) \times \Gamma(\hat{X}) \ni (\Lambda, \gamma) \mapsto \mathfrak{m}_\Lambda(\gamma) \in \overline{\mathbb{R}^+}, \tag{4.3.6}$$

is defined via

$$\mathfrak{m}_\Lambda(\gamma) := \langle \mathfrak{m}_\Lambda, \gamma \rangle = \sum_{\hat{x} \in \gamma} \mathfrak{l}(\hat{x}) 1\!\!1_\Lambda \qquad \forall \gamma \in \Gamma(\hat{X}). \tag{4.3.7}$$

Remark 4.3.7. *Let $\mathfrak{l}$ be a semi-local function. Its associated local mass map is additive, i.e.*

$$\mathfrak{m}_\Lambda(\textstyle\bigsqcup_{i=1}^\infty \gamma_i) = \sum_{i=1}^\infty \mathfrak{m}_\Lambda(\gamma_i), \quad \forall \Lambda \in \mathcal{B}(\hat{X}),\ \forall \left(\gamma_i\right)_{i \in \mathbb{N}} \subset \Gamma(\hat{X}) : \ \gamma_i \cap \gamma_j = \emptyset \text{ whenever } i \neq j, \tag{4.3.8}$$

and monotone, i.e.

$$\forall \Lambda, \tilde{\Lambda} \in \mathcal{B}(\hat{X}) : \ \Lambda \subseteq \tilde{\Lambda} \text{ implies } \mathfrak{m}_\Lambda \leq \mathfrak{m}_{\tilde{\Lambda}}. \tag{4.3.9}$$

Moreover, for any $\Lambda \in \mathcal{B}(\hat{X})$,

$$\text{the map } \mathcal{B}(\Gamma) \ni \gamma \mapsto \mathfrak{m}_\Lambda(\gamma) \quad \text{is} \quad \mathcal{B}(\Gamma)\text{-measurable} \tag{4.3.10}$$

as the limit of measurable functions (cf. also Lemma 2.1.1).

From the context it will be clear whether we treat the semi-local function $\mathfrak{l}$ or its associated local mass map $\mathfrak{m}$, which is defined in (4.3.7). Thus, from now on we shall denote them by the same symbol $\mathfrak{m}$, if no confusion seems to be possible.

The previous considerations motivate the following

Definition 4.3.8. *A map* $\mathfrak{m}$

$$\mathfrak{m} \;: \mathcal{B}(\hat{X}) \times \Gamma(\hat{X}) \ni (\Lambda, \gamma) \mapsto \mathfrak{m}_\Lambda(\gamma) \in \overline{\mathbb{R}^+}, \tag{4.3.11}$$

is called a (semi-)local (additive) mass map *(w.r.t.* $\lambda_\theta \otimes m$*) (or for short* local mass map*), if it fulfills the additivity (cf. (4.3.8)), the monotonicity (cf. (4.3.9)), the measurability (cf. (4.3.10)) and the (semi-local) integrability, i.e.,*

$$\int_{\hat{X}} \mathfrak{m}_{\mathbb{R}_+ \times \Delta}(\hat{x}) \lambda_\theta \otimes m(d\hat{x}) \leq C_\Delta < \infty, \quad \forall\, \Delta \in \mathcal{B}_c(X). \tag{4.3.12}$$

If Λ *is fixed,* $\mathfrak{m}_\Lambda$ *is called a* local *(w.r.t.* $\lambda_\theta \otimes m$*)* mass *at* $\Lambda \in \mathcal{B}(\hat{X})$*. If* $\mathfrak{m}_\Lambda = \mathfrak{m}_{\hat{X}}$ *for all* $\Lambda \in \mathcal{B}(\hat{X})$ *then we call* $\mathfrak{m} \;:= \mathfrak{m}_{\hat{X}}$ *a* global *(w.r.t.* $\lambda_\theta \otimes m$*)* mass*.*

If instead of (4.3.8) only the (weaker) subadditive *property*

$$\begin{array}{c} \mathfrak{m}_\Lambda(\varnothing) = 0 \quad \text{and} \quad \mathfrak{m}_\Lambda(\bigcup_{i=1}^\infty \gamma_i) \leq \sum_{i=1}^\infty \mathfrak{m}_\Lambda(\gamma_i), \\ \forall \Lambda \in \mathcal{B}(\hat{X}),\ \forall\ \left(\gamma_i\right)_{i \in \mathbb{N}} \subset \Gamma(\hat{X}) \end{array}, \tag{4.3.13}$$

holds, we call $\mathfrak{m}$ *a* subadditive local (w.r.t. $\lambda_\theta \otimes m$) mass map*. We will omit "(w.r.t.* $\lambda_\theta \otimes m$*)" most of the time, when it is clear which intensity measures on* $\hat{X}$ *are meant.*

Depending on the question of interest, we will make one or another choice of local resp. global masses related to $\lambda_\theta \otimes m$.

Remark 4.3.9.

1. *As we will see later (cf. Example 4.4.3) there exist global mass maps which cannot be constructed by means of any semi-local function. Therefore, it is reasonable to introduce the general concept of a local mass map* $\mathfrak{m}$*.*

2. *The term* (semi-)local *reflects the fact that the integrability condition has only to hold w.r.t. bounded sets* $\Delta \in \mathcal{B}_c(X)$*. Whereas the term* global *mass is motivated by the claim that (4.3.12) holds for* $\Delta = X$*.*

3. Due to the additivity, each local mass map $\mathfrak{m}$ can be written as

$$\mathfrak{m}_\Lambda(\gamma) = \langle \mathfrak{m}_\Lambda(\{\cdot\}), \gamma\rangle, \qquad \forall \gamma \in \Gamma(\hat{X}),\ \Lambda \in \mathcal{B}(\hat{\mathbb{R}}^d).$$

Remark 4.3.10. *We can easily include the local (w.r.t. $\lambda_\theta \otimes m$) mass map considered in Section 4.2 in this new concept. This also explains the following abbreviation, which we will use without further advice: For ‹stripes› $\Lambda = \mathbb{R}_+ \times \Delta$ with $\Delta \in \mathcal{B}(X)$ we set*

$$\mathfrak{m}_\Delta := \mathfrak{m}_{\mathbb{R}_+\times\Delta}.$$

Fix an arbitrary map $\tilde{\mathfrak{m}} : \mathcal{B}(X) \times \Gamma(\hat{X}) \ni (\Delta, \gamma) \mapsto \tilde{\mathfrak{m}}_\Delta(\gamma) \in [0,\infty]$. We set for all $\Lambda \in \mathcal{B}(\hat{X})$

$$\mathfrak{m}_\Lambda := \begin{cases} \tilde{\mathfrak{m}}_\Delta, & \text{if } \Lambda = \mathbb{R}_+ \times \Delta, \text{ where } \Delta \in \mathcal{B}(X), \\ 0, & \text{else.} \end{cases} \tag{4.3.14}$$

Thus, we may use the same name and symbol for both objects defined in (4.3.13) and (4.3.14) respectively.

Example 4.3.11. *It is easy to check that for any $p > 0$ the map*

$$\mathfrak{m}_p : \mathcal{B}(X) \times \Gamma(\hat{X}) \ni \Delta \times \gamma \mapsto \mathfrak{m}_{p,\Delta}(\gamma) := \sum_{x\in\tau(\gamma)\cap\Delta} s_x^p \tag{4.3.15}$$

is a local mass map in the sense of Definition 4.3.8. The required additivity, monotonicity and measurability (cf. (4.3.8), (4.3.9) and (4.3.10)) are clear and the integrability w.r.t. $\lambda_\theta \otimes m$ follows by Example 4.3.5. It will be used to construct Gibbs measures corresponding to the potential V_p discussed in Example 4.3.3.

4.3.2 Support of the Gamma-Poisson measure

Using local mass maps, we describe sets on which the Gamma-Poisson measure $\mathcal{P}_\theta$ has full mass.

Definition 4.3.12. *Let $\mathfrak{m}$ be a local (w.r.t. $\lambda_\theta \otimes m$) mass map. We define the* set of (pinpointing) configurations with finite local mass *by*

$$\Gamma_f(\hat{X}) := \Gamma_{f,\mathfrak{m}}(\hat{X}) := \left\{\gamma \in \Gamma_p(\hat{X}) \middle|\ \mathfrak{m}_\Lambda(\gamma) < \infty,\ \forall \Lambda \in \mathbb{R}_+ \times \mathcal{B}_c(X)\right\}.$$

If it is clear which local mass map is meant, we omit the index $\mathfrak{m}$.

The following result generalizes Theorem 2.2.9.

Theorem 4.3.13. *For any (semi-)local (w.r.t. $\lambda_\theta \otimes m$) mass map $\mathfrak{m}$, the Gamma-Poisson measure $\mathcal{P}_\theta$ is supported by $\Gamma_f(\hat{X}) \in \mathcal{B}(\Gamma(\hat{X}))$, i.e.*

$$\mathcal{P}_\theta\big(\Gamma_f(\hat{X})\big) = 1.$$

Proof. The claim is proved similarly to Theorem 2.2.9. □

The later result is also a special case of Theorem 4.3.34 below for $V \equiv 0$.

Corollary 4.3.14. *The Gamma-Poisson measure $\mathcal{P}_\theta$ is supported by*

$$\bigcap_{p>0} \Big\{ \gamma \in \Gamma_p(\hat{X}) \,|\forall \Delta \in \mathcal{B}_c(X) : \mathfrak{m}_{p,\Delta}(\gamma) < \infty \Big\}. \tag{4.3.16}$$

Proof. This follows by Theorem 4.3.13 and the fact that the set in (4.3.16) can be written as a countable intersection over $p_n \searrow 0$ as $n \nearrow \infty$ (cf. also Example 4.3.11). □

4.3.3 Finiteness of the relative energy

We describe admissible boundary configurations $\xi \in \Gamma(\hat{X})$ such that all local Hamiltonians $\Gamma(\hat{X}) \ni \gamma \mapsto H_\Lambda(d\gamma|\xi)$ are well-defined for $\Lambda \in \mathcal{B}_c(\hat{X})$.

Example 4.3.15. *For the following potentials $V(\hat{x}, \hat{y})$, $\hat{x}, \hat{y} \in \hat{X}$, the relative energy is finite, if the boundary condition ξ is chosen from the mentioned set:*

1. *$\xi \in \Gamma_{f,\mathfrak{m}_1}(\hat{X})$, for $V(\hat{x}, \hat{y}) = s_x s_y \phi(x, y)$, and*

2. *$\xi \in \Gamma_{f,\mathfrak{m}_p}(\hat{X})$, for $V_p(\hat{x}, \hat{y}) = s_x{}^p s_y{}^p \phi(x, y)$ with $p > 0$,*

where $\phi \in L_s(X \times X)$ (cf. Definition 4.3.1).

Fix a non-negative, symmetric potential $V : X \times X \to [0, \infty]$ and a semi-local function or local mass map $\mathfrak{m}$ such that

$$V(\hat{x}, \hat{y}) \leq \mathfrak{m}_{\hat{X}}(\{\hat{x}\}) \cdot \mathfrak{m}_{\hat{X}}(\{\hat{y}\})\phi(x, y), \quad \hat{x}, \hat{y} \in \hat{X}, \tag{4.3.17}$$

where $\phi \in L_s(X \times X)$.

Lemma 4.3.16. *Let V be as in (4.3.17). Let $\gamma \in \Gamma(\hat{X})$ and $\xi \in \Gamma_f(\hat{X})$ (cf. (2.2.20)), then the relative energy (cf. (4.1.2))*

$$H_\Lambda(\gamma|\xi) = \sum_{\hat{x},\hat{y} \in \gamma \cap \Lambda} V(\hat{x}, \hat{y}) + 2 \sum_{\substack{\hat{x} \in \gamma \cap \Lambda \\ \hat{y} \in \xi \cap \Lambda^C}} V(\hat{x}, \hat{y})$$

is finite for all $\Lambda \in \mathcal{B}_c(X)$. If $\mathfrak{m}$ is a global mass, this result even holds for $\phi \equiv 1$.

Proof. We proceed similarly to Lemma 4.1.2: For the first sum this follows immediately by the definition of local finite configurations $\gamma \in \Gamma(\hat{X})$ (cf. (2.1.1)). The finiteness of the second sum is guaranteed by the choice of $\xi \in \Gamma_f(\hat{X})$. □

Remark 4.3.17. *The potentials, which we treat in Lemma 4.3.16 can have an infinite range w.r.t.* $\mathbb{R}_+ \times X$*, as long as the proper global mass exists.*

4.3.4 Support of the local specification kernels

Similar as in Section 4.1, we define for each $\Lambda \in \mathcal{B}_c(\hat{X})$ and $\xi \in \Gamma(\hat{X})$ the local Gibbs state $\mu_\Lambda(d\gamma|\xi)$ (cf. (4.1.4)) and the local specification kernel $\pi_\Lambda(d\gamma|\xi)$ (cf. (4.1.8)). Let $\mathcal{G}\text{ibbs}_V$ denote the set of corresponding Gibbs measures defined by Definition 4.1.6, where we use instead of $\Gamma_f(\hat{X})$ the set $\Gamma_{f,\mathfrak{m}}(\hat{X})$ (cf. Definition 4.3.12).

The following proposition is a key estimate for proving the existence and the uniform moment bounds of the Gibbs measures.

Proposition 4.3.18. *Let V be as in (4.3.17). Then, for each $\xi \in \Gamma_f(\hat{X})$ and each $\Lambda \in \mathcal{B}_c(\hat{X})$, the probability measure $\mu_\Lambda(\cdot|\xi)$ is supported by $\Gamma_f(\Lambda)$ and $\pi_\Lambda(\cdot|\xi)$ respectively by $\Gamma_f(\hat{X})$.*

In detail, for any subadditive *local (w.r.t. $\lambda_\theta \otimes m$) mass map $\tilde{\mathfrak{m}}$, for all $\tilde{\Lambda} \in \mathcal{B}(\hat{X})$ and each $\xi \in \Gamma(\Gamma(\hat{X}))$*

$$\int_{\Gamma(\Lambda)} \tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\gamma_\Lambda)\mu_\Lambda(d\gamma_\Lambda|\xi) \leq \int_\Lambda \tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\{\hat{x}\})(\lambda_\theta \otimes m)(d\hat{x}). \tag{4.3.18}$$

If $\tilde{\Lambda} = \mathbb{R}_+ \times \Delta$ with $\Delta \in \mathcal{B}_c(X)$ then this integral is dominated by θC_Δ, being the constant corresponding to (4.3.12). If $\tilde{\mathfrak{m}}$ is a local mass map and $V = 0$, then the above estimate is optimal, i.e., (4.3.18) becomes an equality.

Proof. The proof is quite similar to that of Proposition 4.2.3, where one uses the idea of separating a single particle. □

Remark 4.3.19. *Having a closer look at the proof of Proposition 4.2.3, we see that if we separate (instead of* 1 *particle) a group of n particles from the others we obtain the weaker result*

$$\int_{\Gamma(\Lambda)} \tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\gamma_\Lambda)\mu_\Lambda(d\gamma_\Lambda|\xi) \leq \sum_{i=1}^{n} \int_{\Gamma^{(i)}(\Lambda)} \tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\gamma) P_\theta^\Lambda(d\gamma),$$

where P_θ^Λ is the Lebesgue-Poisson measure on $\Gamma(\Lambda)$ and $\Gamma^{(i)}(\Lambda)$ is the (i)-particle configuration space over Λ.

Remark 4.3.20. *We point out a similar result, which one can obtain by Ruelle's equation (cf. [Rue70, (5.12), (E)]). Let* $\tilde{\mathfrak{m}}_{\tilde{\Lambda}}$ *be supported by* $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$*, then*

$$\begin{aligned}
\int_{\Gamma(\hat{X})} \tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\gamma)\pi_\Lambda(d\gamma|\xi) &= \int_{\Gamma(\Lambda)} \left(\int_{\Gamma(\Lambda^c)} \tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\eta \cup \gamma) e^{-\beta H_\Lambda(\gamma|\xi)} \pi_\Lambda(d\gamma|\xi) \right) P_\theta^\Lambda(d\eta) \\
&= \int_{\Gamma(\Lambda)} \left(\int_{\Gamma(\Lambda^c)} \left[\tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\eta) + \underbrace{\tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\gamma)}_{=0} \right] \underbrace{e^{-\beta H_\Lambda(\gamma|\xi)} \pi_\Lambda(d\gamma|\xi)}_{=1} \right) P_\theta^\Lambda(d\eta) \\
&\leq \int_{\Gamma(\Lambda)} \tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\eta) P_\theta^\Lambda(d\eta). \qquad (4.3.19)
\end{aligned}$$

We see that this estimate does not allow to get a uniform bound for $\Lambda \nearrow \mathbb{R}_+ \times \Delta$ *with* $\Delta \in \mathcal{B}_c(X)$*. On the other hand, we obtain by (4.3.18) for* $\tilde{\Lambda} = \mathbb{R}_+ \times \Delta$ *that*

$$\begin{aligned}
\sup_{\Lambda \nearrow \hat{X},\ \Lambda \in \mathcal{B}_c(\hat{X})} \int_{\Gamma(\hat{X})} \tilde{\mathfrak{m}}_{\mathbb{R}_+ \times \Delta}(\gamma)\pi_\Lambda(d\gamma|\xi) &\leq \int_{\Gamma(\hat{X})} \tilde{\mathfrak{m}}_{\mathbb{R}_+ \times \Delta}(\gamma)\mathcal{P}_\theta(d\gamma) \\
&\leq \int_{\hat{X}} \tilde{\mathfrak{m}}_{\mathbb{R}_+ \times \Delta}(\{\hat{x}\})(\lambda \otimes m)(d\hat{x}) \leq \theta C_\Delta < \infty. \qquad (4.3.20)
\end{aligned}$$

So in our applications, we use instead of (4.3.19) the preciser estimate (4.3.18).

Estimates for higher moments for the specification kernels

Proposition 4.3.21. *Let V be as in (4.3.17). We have for any* subadditive *local (w.r.t.* $\lambda_\theta \otimes m$*) mass map* $\tilde{\mathfrak{m}}$ *that for all* $\tilde{\Lambda} \in \mathcal{B}(\hat{X})$

$$\begin{aligned}
\int_{\Gamma(\Lambda)} \left[\tilde{\mathfrak{m}}_{\hat{X}}(\gamma_{\tilde{\Lambda}\cap\Lambda})\right]^2 \mu_\Lambda(d\gamma_\Lambda|\xi) &\leq \int_{\Gamma(\hat{X})} \left[\tilde{\mathfrak{m}}_{\hat{X}}(\gamma_{\tilde{\Lambda}})\right]^2 \mathcal{P}_\theta(d\gamma) \\
&\leq \int_{\hat{X}} \left[\, \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}\}_{\tilde{\Lambda}})\right]^2 (\lambda_\theta \otimes m)(d\hat{x}) + \left(\int_{\hat{X}} \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}\}_{\tilde{\Lambda}})(\lambda_\theta \otimes m)(d\hat{x}) \right)^2, \qquad (4.3.21)
\end{aligned}$$

uniformly for all $\Lambda \in \mathcal{B}_c(\hat{X})$*. In particular, if* $\tilde{\Lambda} \in \mathcal{B}_c(\hat{X})$*, then the last summand in the r.h.s of (4.3.21) is finite. If we additionally assume, that the first summand is finite for all* $\tilde{\Lambda} \in \mathcal{B}_c(\hat{X})$*, then for each* $\tilde{\Lambda} \in \mathcal{B}_c(\hat{X})$ *there exists a positive constant* $C(\tilde{\Lambda}, \tilde{\mathfrak{m}})$ *such that*

$$\int_{\Gamma(\Lambda)} \left[\tilde{\mathfrak{m}}_{\hat{X}}(\gamma_{\tilde{\Lambda}\cap\Lambda})\right]^2 \mu_\Lambda(d\gamma_\Lambda|\xi) \leq C(\tilde{\Lambda}, \tilde{\mathfrak{m}}) \qquad (4.3.22)$$

uniformly for all $\xi \in \Gamma_f(\hat{X})$*.*

Proof. By the subadditivity (cf. (4.3.13)), we get for each $\Lambda \in \mathcal{B}_c(\hat{X})$ and $\xi \in \Gamma_f(\hat{X})$

$$\begin{aligned}&\int_{\Gamma(\Lambda)} \tilde{\mathfrak{m}}_{\hat{X}}\gamma_{\Lambda\cap\tilde{\Lambda}}{}^2 \mu_\Lambda(d\gamma_\Lambda|\xi)\\ \le&\frac{1}{Z_\Lambda(\xi)}\sum_{n\ge 0}\frac{1}{n!}\int_{\Lambda^n}\left(\sum_{k=1}^n \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}_k\}_{\tilde{\Lambda}})\right)^2 e^{-\beta H_\Lambda(\{\hat{x}_1,\dots,\hat{x}_n\}|\xi)}\prod_{i=1}^n(\lambda_\theta\otimes m)(d\hat{x}_i).\end{aligned} \tag{4.3.23}$$

Using

$$\left(\sum_{k=1}^n \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}_k\}_{\tilde{\Lambda}})\right)^2 = \sum_{\substack{1\le k,j\le n\\ k\ne j}} \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}_k\}_{\tilde{\Lambda}})\, \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}_j\}_{\tilde{\Lambda}}) + \sum_{1\le k\le n} \tilde{\mathfrak{m}}_{\hat{X}}(\{x_k\}_{\tilde{\Lambda}})^2 \tag{4.3.24}$$

and performing the calculation analogously to (4.2.7) and (4.2.9), we get

$$\begin{aligned}\int_{\Gamma(\Lambda)}\left[\tilde{\mathfrak{m}}_{\hat{X}}(\gamma_{\tilde{\Lambda}\cap\Lambda})\right]^2\mu_\Lambda(d\gamma_\Lambda|\xi) &\le \int_{\Gamma^{(1)}(\Lambda)}\left[\, \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}\}_{\tilde{\Lambda}})\right]^2 e^{-\beta H_\Lambda(\{\hat{x}\}|\xi\})} P_\theta^\Lambda(d\{\hat{x}\})\\ &+\int_{\Gamma^{(2)}(\Lambda)} \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}_1\}_{\tilde{\Lambda}})\, \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}_2\}_{\tilde{\Lambda}}) e^{-\beta H_\Lambda(\{\hat{x}_1,\hat{x}_2\}|\xi\})} P_\theta^\Lambda(d\{\hat{x}_1,\hat{x}_2\}).\end{aligned} \tag{4.3.25}$$

Using again that $V \ge 0$, we estimate the last term by

$$\int_{\hat{X}}\left[\, \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}\}_{\tilde{\Lambda}})\right]^2(\lambda_\theta\otimes m)(d\hat{x}) + \left(\int_{\hat{X}} \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}\}_{\tilde{\Lambda}})(\lambda_\theta\otimes m)(d\hat{x})\right)^2.$$

Using also Theorem 3.2.6, we conclude the proof of the assertion. □

Remark 4.3.22. *1. We even have that (4.3.22) holds for higher moments, but then the formula is quite involved. For $N \in \mathbb{N}$, it takes the form*

$$\begin{aligned}\int_{\Gamma(\Lambda)}\left[\tilde{\mathfrak{m}}_{\hat{X}}(\gamma_{\tilde{\Lambda}\cap\Lambda})\right]^N\mu_\Lambda(d\gamma_\Lambda|\xi) &\le \int_{\Gamma(\hat{X})}\left[\tilde{\mathfrak{m}}_{\hat{X}}(\gamma_{\tilde{\Lambda}})\right]^N\mathcal{P}_\theta(d\gamma)\\ &\le \sum_{\substack{k_1+\dots k_n=N\\ 1\le k_i\le N,\ n\in\mathbb{N}}} C_{k_1,\dots,k_n}\prod_{i=1}^n\int_{\hat{X}}\left[\, \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}\}_{\tilde{\Lambda}})\right]^{k_i}\lambda_\theta\otimes m(d\hat{x})\end{aligned} \tag{4.3.26}$$

provided the integrals in the right-hand side are finite. This is surely the case, if we assume that

$$\int \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}\}_\Delta)^n\lambda_\theta\otimes m(d\hat{x}) < \infty \quad \textit{for } 1\le n\le N.$$

To prove this result, we first get a formula analogously to (4.3.24), which we plug in (4.5.9) and calculate the integral over the different summands individually. Here, we have to split up to N particles from the others. This yields an estimate that has a similar structure as (4.3.21), in so far it depends on $\tilde{\Lambda}$ and N, but is independent of Λ.

2. *In a similar way one can consider positive k-body potentials ($k \geq 2$). Here, a sufficient condition for (4.3.26) with $N \geq 1$ will be*

$$\int_{\bigsqcup_{i=1}^{k} \Gamma^{(i)}(\Lambda)} \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}_1\}_\Delta)^n P_\theta^\Lambda(d\{\hat{x}_1, \ldots, \hat{x}_i\}) < \infty, \quad \textit{for all } 1 \leq n \leq kN.$$

4.3.5 Local equicontinuity

The following step is essential for proving the existence of Gibbs measures, i.e., that $\mathcal{G}\mathrm{ibbs}_V \neq \varnothing$.

Proposition 4.3.23. *Let V be as in (4.3.17). Then, for each fixed $\xi \in \Gamma_f(\hat{X})$, the net $\{\pi_\Lambda(\mathrm{d}\gamma|\xi)|\ \Lambda \in \mathcal{B}_c(\hat{X})\}$ is locally equicontinuous.*

If $\mathfrak{m}$ is a global mass map, then we may drop the finite interaction range condition and still obtain the local equicontinuity stated above.

Proof. The results follows by adapting the arguments used in proving Proposition 4.2.6. Namely, e.g., we set

$$\mathcal{U} := \begin{cases} \mathbb{R}_+ \times X, & \text{if } \mathfrak{m} \text{ is a global mass,} \\ \mathbb{R}_+ \times \mathcal{U}_\Delta, & \text{otherwise (cf. (4.3.3)),} \end{cases}$$

where $\Delta \in \mathcal{B}_c(\hat{X})$ is such that $\Lambda \subset \mathbb{R}_+ \times \Delta$. Then we repeat the estimates (4.2.14) to (4.2.19) for the specification kernels defined by (4.1.8) (with the general potential V as in (4.3.17) and $\Gamma_f(\hat{X})$ defined by Definition 4.3.12) and deduce the assertion. □

4.3.6 Existence

Now we are in position to prove one of the main results of Chapter 4, which ensures the existence of $\mu \in \mathcal{G}\mathrm{ibbs}_V$. We start with

Definition 4.3.24. *A local (w.r.t. $\lambda_\theta \otimes m$) mass map $\mathfrak{m} : \mathcal{B}(\hat{X}) \times \Gamma(\hat{X}) \to [0, \infty]$ is called*

(QL) ***quasi local:*** *There exists $c_\mathfrak{m} \in (0, \infty)$ such that for each $\Lambda \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}_c(X)$ one finds $\Lambda' \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}_c(X)$ with*

$$\mathfrak{m}_{\hat{X}}(\gamma_\Lambda) \leq c_\mathfrak{m} \mathfrak{m}_{\Lambda'}(\gamma). \tag{4.3.27}$$

Remark 4.3.25. *Given a semi-local function $\tilde{\mathfrak{l}}$, then its associated local mass map (cf. Definition 4.3.6) is quasi local. In particular, this holds for all examples discussed in Example 4.3.11.*

Moreover, this property holds obviously for any global mass map.

Theorem 4.3.26. *Let $V : \hat{X} \times \hat{X} \to [0, \infty]$ be a non-negative potential and $\mathfrak{m}$ a semi-local (w.r.t. $\lambda_\theta \otimes m$) function or a local (w.r.t. $\lambda_\theta \otimes m$) mass map such that*

$$V(\hat{x}, \hat{y}) \leq \mathfrak{m}_{\hat{X}}(\{\hat{x}\}) \cdot \mathfrak{m}_{\hat{X}}(\{\hat{y}\})\phi(x, y) \quad \forall \hat{x}, \hat{y} \in \hat{X}, \tag{4.3.28}$$

*where we assume that $\phi \in L_s(X \times X)$ and that **(QL)** holds.*

Then for any $\beta, \theta > 0$, there exists a Gibbs measures corresponding to $\mathcal{P}_\theta$ and V, i.e.,

$$\mathcal{G}\mathrm{ibbs}_V \neq \varnothing.$$

Each $\mu \in \mathcal{G}\mathrm{ibbs}_V$ is supported by $\Gamma_f(\hat{X})$.

If $\mathfrak{m}$ is a global mass, we may even take $\phi \equiv 1$ in (4.3.28).

Remark 4.3.27. *Theorem 4.3.26 holds for each potential given in Example 4.3.15.*

Proof of Theorem 4.3.26. We proceed along the lines for proving the existence result in the basic model in Theorem 4.2.7 with obvious modifications. Note that the specification kernels $\pi_\Lambda(d\gamma|\xi)$, $\xi \in \Gamma_f(\hat{X})$, are probability measures on $\Gamma(\hat{X})$.

Indeed, similarly as in the mentioned proof, we get the existence of a limit measure μ using Lemma 4.3.16 and Proposition 4.3.23. We deduce that μ is indeed a Gibbs measure through Lemmas 4.3.28, 4.3.29 and Proposition 4.3.30 below. To this end, we crucially use the following estimate which we get by Proposition 4.3.18 (cf. (4.3.18)): Fix an arbitrary $\tilde{\Lambda} \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}_c(X)$. We have for all $\Lambda' \in \mathcal{B}(\hat{X})$ and $\Lambda_N \in \mathcal{B}_c(\hat{X})$

$$\begin{aligned}
&\int_{\Gamma(\hat{X})} \mathfrak{m}_{\tilde{\Lambda}}(\gamma_{\Lambda'})\pi_{\Lambda_N}(d\gamma|\xi) \\
&= \int_{\Gamma(\Lambda_N)} \mathfrak{m}_{\tilde{\Lambda}}(\gamma_{\Lambda_N \cap \Lambda'} \cup \xi_{\Lambda_N^c \cap \Lambda'})\mu_{\Lambda_N}(d\gamma_{\Lambda_N}|\xi) \\
&\leq \int_{\Gamma(\Lambda_N)} \mathfrak{m}_{\tilde{\Lambda}}(\gamma_{\Lambda_N \cap \Lambda'})\mu_{\Lambda_N}(d\gamma_{\Lambda_N \cap \Lambda'}|\xi) + \mathfrak{m}_{\tilde{\Lambda}}(\xi_{\Lambda_N^c \cap \Lambda'})\Big) \\
&\leq \int_{\hat{X}} \mathfrak{m}_{\tilde{\Lambda}}(\{\hat{x}\}_{\Lambda'})\lambda_\theta \otimes m(d\hat{x}) + \mathfrak{m}_{\tilde{\Lambda}}(\xi_{\Lambda'}) \leq \theta C_{\tilde{\Lambda}} + \mathfrak{m}_{\tilde{\Lambda}}(\xi_{\Lambda'}) < \infty.
\end{aligned}$$

Note that this bound is uniform in $N \in \mathbb{N}$.

It remains to show the above mentioned lemmas and proposition:

Lemma 4.3.28. *Suppose we are in the setting of Theorem 4.3.26. Then, for each $\Delta \in \mathcal{B}_c(\hat{X})$, there exist $\Delta' \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}_c(X)$ (being the same as in (4.3.27)) and $M_\phi, M_{\Delta'} \in (0,\infty)$ such that for $B \in \mathcal{B}_\Lambda(\Gamma(\hat{X}))$ and all $\Lambda \in \mathcal{B}_c(\hat{X})$:*[10]

$$\begin{aligned}|\pi_\Delta(B|\gamma_\Lambda) - \pi_\Delta(B|\gamma)| &\leq 2\beta c_{\mathfrak{m}} M_\phi M_{\Delta'}\Big(1 + e^{\lambda_\theta \otimes m(\Delta)}\Big)\mathfrak{m}_{\Delta'}(\gamma_{\Lambda^c}) \\ &=: C(\mathfrak{m}, \Delta, \phi)\mathfrak{m}_{\Delta'}(\gamma_{\Lambda^c}) =: C\mathfrak{m}_{\Delta'}(\gamma_{\Lambda^c}), \end{aligned} \tag{4.3.29}$$

where we chose $c_{\mathfrak{m}} \in (0,\infty)$ such that (**QL**) *holds.*

Proof. Let $B \in \mathcal{B}_\Lambda(\Gamma(\hat{X}))$, then $\mathbb{1}_B(\eta \cup \gamma_{\Lambda\cap\Delta^c}) = \mathbb{1}_B(\eta \cup \gamma_{\Delta^c})$ for all $\eta, \gamma \in \Gamma(\hat{X})$. Hence, for any $\gamma \in \Gamma_f(\hat{X})$ we have

$$\begin{aligned} &|\pi_\Delta(B|\gamma_\Lambda) - \pi_\Delta(B|\gamma)| \\ \leq &\frac{1}{Z_\Delta(\gamma_\Lambda)} \int_{\Gamma_f(\hat{X})} \mathbb{1}_B(\eta_\Delta \cup \gamma_{\Lambda\cap\Delta^c}) e^{-\beta H_\Delta(\eta_\Delta|\gamma_\Lambda)} \\ &\qquad \times \Bigg(1 - \exp\Bigg\{-2\beta \sum_{\substack{\hat{x}\in\eta_\Delta,\\ \hat{y}\in\gamma_{\Lambda^c\cap\Delta^c}}} V(\hat{x},\hat{y})\Bigg\}\Bigg) P_\theta^\Delta(d\eta_\Delta) \\ &\quad + \left|\frac{1}{Z_\Delta(\gamma_\Lambda)} - \frac{1}{Z_\Delta(\gamma)}\right| \int_{\Gamma_f(\hat{X})} \mathbb{1}_B(\eta_\Delta \cup \gamma_{\Delta^c}) e^{-\beta H_\Delta(\eta_\Delta|\gamma)} P_\theta^\Delta(d\eta_\Delta). \end{aligned} \tag{4.3.30}$$

Dropping the first indicator function, we get the following dominator of the first summand

$$\frac{1}{Z_\Delta(\gamma_\Lambda)} \int_{\Gamma_f(\hat{X})} e^{-\beta H_\Delta(\eta_\Delta|\gamma_\Lambda)} \Bigg(1 - \exp\Bigg\{-2\beta \sum_{\substack{\hat{x}\in\eta_\Delta \\ \hat{y}\in\gamma_{\Lambda^c\cap\Delta^c} \\ \hat{y}\in\mathcal{U}_\Delta}} V(\hat{x},\hat{y})\Bigg\}\Bigg) P_\theta^\Delta(d\eta_\Delta), \tag{4.3.31}$$

where we used (cf. also (4.3.1)) that there exists $\mathcal{U}_\Delta \in \mathcal{B}(\hat{X})$ such that

$$V(\hat{x},\hat{y}) = 0 \quad \text{for all } \hat{x} \in \Delta \text{ and } \hat{y} \in \mathcal{U}_\Delta^c.$$

[10] Despite our usual convention to use Δ to denote sets in X, we use $\Delta \in \mathcal{B}_c(\hat{X})$ in Lemmas 4.3.28, 4.3.29 and Proposition 4.3.30 to maintain a better readability.

(If $\mathfrak{m}$ is a global mass, we choose w.l.o.g. $\mathcal{U}_\Delta = \hat{X}$; otherwise, we can choose $\mathcal{U}_\Delta \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}_c(X)$.) Since $P_\theta^\Delta(\Gamma(\Delta)) = e^{\lambda_\theta \otimes m(\Delta)}$, we estimate the second summand in (4.3.30) by

$$\underbrace{\frac{1}{Z_\Delta(\gamma)}}_{\leq 1} \cdot \frac{1}{Z_\Delta(\gamma_\Lambda)} \big| Z_\Delta(\gamma) - Z_\Delta(\gamma_\Lambda) \big| e^{\lambda_\theta \otimes m(\Delta)}. \tag{4.3.32}$$

We use that $Z_\Delta(\gamma) = \int_{\Gamma_f(\hat{X})} e^{-\beta H_\Delta(\eta_\Delta|\gamma)} P_\theta^\Delta(d\eta_\Delta)$ to rewrite the difference in the last line. Using the upper bound established above (cf. (4.3.31) and (4.3.32)) for the summands in (4.3.30), we deduce that uniformly for all $B \in \mathcal{B}_\Lambda(\Gamma(\hat{X}))$

$$\begin{aligned}
&|\pi_\Delta(B|\gamma_\Lambda) - \pi_\Delta(B|\gamma)| \\
\leq & \frac{1}{Z_\Delta(\gamma_\Lambda)} \Big(1 + e^{\lambda_\theta \otimes m(\Delta)}\Big) \int_{\Gamma_f(\hat{X})} e^{-\beta H_\Delta(\eta_\Delta|\gamma_\Lambda)} \\
& \qquad \times \Bigg(1 - \exp\Bigg\{ -2\beta \sum_{\substack{\hat{x} \in \eta_\Delta \\ \hat{y} \in \gamma_{\Lambda^c \cap \Delta^c \cap \mathcal{U}_\Delta}}} V(\hat{x}, \hat{y}) \Bigg\}\Bigg) P_\theta^\Delta(d\eta_\Delta) \\
\leq & \Big(1 + e^{\lambda_\theta \otimes m(\Delta)}\Big) \int_{\Gamma_f(\hat{X})} \Bigg(2\beta \sum_{\substack{\hat{x} \in \eta_\Delta \\ \hat{y} \in \gamma_{\Lambda^c \cap \Delta^c \cap \mathcal{U}_\Delta}}} V(\hat{x}, \hat{y}) \Bigg) \mathcal{P}_\theta^\Delta(d\eta_\Delta) \\
\leq & 2\beta \Big(1 + e^{\lambda_\theta \otimes m(\Delta)}\Big) \\
& \quad \times \int_{\Gamma_f(\hat{X})} \Bigg(\sum_{\substack{\hat{x} \in \eta_\Delta \\ \hat{y} \in \gamma_{\Lambda^c \cap \Delta^c \cap \mathcal{U}_\Delta}}} \mathfrak{m}_{\hat{X}}(\{\hat{x}\}) \mathfrak{m}_{\hat{X}}(\{\hat{y}\}) \phi(x, y) \Bigg) \mathcal{P}_\theta^\Delta(d\eta_\Delta),
\end{aligned} \tag{4.3.33}$$

where we applied (4.3.28) together with the elementary inequality $1 - e^{-\beta\alpha} \leq \beta\alpha$ for all $\alpha, \beta \geq 0$. The next step is to estimate the integral in the last line of (4.3.33). Taking into account that

$$\sup_{x,y \in X} \phi(x, y) =: M_\phi < \infty$$

(cf. our assumptions resp. Definition 4.3.1), we get

$$\begin{aligned}
& \int_{\Gamma_f(\hat{X})} \Bigg(\sum_{\substack{\hat{x} \in \eta_\Delta \\ \hat{y} \in \gamma_{\Lambda^c \cap \Delta^c \cap \mathcal{U}_\Delta}}} \mathfrak{m}_{\hat{X}}(\{\hat{x}\}) \mathfrak{m}_{\hat{X}}(\{\hat{y}\}) \phi(x, y) \Bigg) \mathcal{P}_\theta^\Delta(d\eta_\Delta) \\
\leq & M_\phi \int_{\Gamma(\hat{X})} \mathfrak{m}_{\hat{X}}(\eta_\Delta) \mathfrak{m}_{\hat{X}}(\gamma_{\Lambda^c \cap \mathcal{U}_\Delta}) \mathcal{P}_\theta^\Delta(d\eta_\Delta) \\
\leq & M_\phi \mathfrak{m}_{\hat{X}}(\gamma_{\Lambda^c \cap \mathcal{U}_\Delta}) \int_\Delta \mathfrak{m}_{\hat{X}}(\{\hat{x}\}) (\lambda_\theta \otimes m)(d\hat{x}),
\end{aligned} \tag{4.3.34}$$

where we applied Proposition 4.3.18. Using **(QL)** (cf. (4.3.27)) and the monotonicity of a local mass map (cf. (4.3.10)) we deduce

$$\exists \Delta' \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}_c(X): \quad \begin{array}{ll} \mathfrak{m}_{\mathbb{R}_+\times X}(\gamma_{\mathcal{U}_\Delta}) & \leq c_{\mathfrak{m}} \mathfrak{m}_{\Delta'}(\gamma) \\ \mathfrak{m}_{\mathbb{R}_+\times X}(\gamma_\Delta) & \leq c_{\mathfrak{m}} \mathfrak{m}_{\Delta'}(\gamma) \end{array} \quad \forall \gamma \in \Gamma(\hat{X}). \tag{4.3.35}$$

This yields a bound for (4.3.34), namely

$$M_\phi \mathfrak{m}_{\Delta'}(\gamma_{\Lambda^c}) \int_{\hat{X}} \mathfrak{m}_{\Delta'}(\{\hat{x}\}) \lambda_\theta \otimes m(d\hat{x}) \leq 2\beta c_{\mathfrak{m}} M_\phi \mathfrak{m}_{\Delta'}(\gamma_{\Lambda^c}) M_{\Delta'}, \tag{4.3.36}$$

where we used the integrability of $\mathfrak{m}$ (cf. (4.3.12)) to find that

$$\exists M_{\Delta'} > 0: \quad \int_{\hat{X}} \mathfrak{m}_{\Delta'}(\{\hat{x}\}) \lambda_\theta \otimes m(d\hat{x}) \leq M_{\Delta'}.$$

We summarize the estimates given in (4.3.33), (4.3.34) and (4.3.36):

$$|\pi_\Delta(B|\gamma_\Lambda) - \pi_\Delta(B|\gamma)| \leq 2\beta c_{\mathfrak{m}} M_\phi M_{\Delta'} \Big(1 + e^{\lambda_\theta \otimes m(\Delta)}\Big) \mathfrak{m}_{\Delta'}(\gamma_{\Lambda^c}), \tag{4.3.37}$$

which yields the claim. □

The following lemma will be applied for the local specification kernels (i.e. $\nu(d\gamma) := \pi_\Lambda(d\gamma|\xi)$). That is why in its formulation we assume the dependence on a boundary condition ξ.

Lemma 4.3.29. *Suppose that we are in the setting of Theorem 4.3.26.*[10] *Fix $\Delta \in \mathcal{B}_c(\hat{X})$ and $\xi \in \Gamma_f(\hat{X})$. Assume that for some $\nu \in \mathcal{M}^1(\Gamma(\hat{X}))$ it holds for all $\Lambda' \in \mathcal{B}(\hat{X})$ and $\Delta' \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}_c(X)$*

$$\int_{\Gamma(\hat{X})} \mathfrak{m}_{\Delta'}(\gamma_{\Lambda'}) \nu(d\gamma) \leq \int_{\hat{X}} \mathfrak{m}_{\Delta'}(\{\hat{x}\}_{\Lambda'}) \left(\lambda_\theta \otimes m\right)(d\hat{x}) + \mathfrak{m}_{\Delta'}(\xi_{\Lambda'}). \tag{4.3.38}$$

Then for any $\varepsilon > 0$ there exists $\Lambda = \Lambda(\varepsilon, \mathfrak{m}, \Delta, \phi, \xi) \in \mathcal{B}_c(\hat{X})$ such that, for all $\nu \in \mathcal{M}^1(\Gamma(\hat{X}))$ fulfilling (4.3.38) and for all $\tilde{\Lambda} \in \mathcal{B}_c(\hat{X})$ with $\tilde{\Lambda} \supset \Lambda$, we have

$$\left| \int_{\Gamma(\hat{X})} \left[\pi_\Delta(B|\gamma_{\tilde{\Lambda}}) - \pi_\Delta(B|\gamma)\right] \nu(d\gamma) \right| \leq \varepsilon. \tag{4.3.39}$$

In other words,

$$\lim_{\substack{\Lambda \nearrow \hat{X} \\ \Lambda \in \mathcal{B}_c(\hat{X})}} \int_{\Gamma(\hat{X})} \pi_\Delta(B|\gamma_\Lambda) \nu(d\gamma) = \lim_{\substack{\Lambda \nearrow \hat{X} \\ \Lambda \in \mathcal{B}_c(\hat{X})}} \int_{\Gamma(\hat{X})} \pi_\Delta(B|\gamma) \nu(d\gamma). \tag{4.3.40}$$

Proof. Using (4.3.29) we have

$$\left|\int_{\Gamma(\hat{X})} \left[\pi_\Delta(B|\gamma_\Lambda) - \pi_\Delta(B|\gamma)\right]\nu(d\gamma)\right| \\ \leq C(\mathfrak{m}, \Delta, \phi) \int_{\Gamma(\hat{X})} \mathfrak{m}_{\Delta'}(\gamma_{\Lambda^c})\nu(d\gamma). \tag{4.3.41}$$

Equation (4.3.38) with $\Lambda' = \Lambda^c$ yields the following bound for the last integral

$$\int_{\hat{X}} \mathfrak{m}_{\Delta'}(\{\hat{x}\}_{\Lambda^c})\lambda_\theta \otimes m(d\hat{x}) + \mathfrak{m}_{\Delta'}(\xi_{\Lambda^c}), \xi\rangle. \tag{4.3.42}$$

By Lebesgue's dominated convergence theorem, (4.3.42) becomes arbitrary small for sufficiently large Λ. Thus, the claim holds. □

Proposition 4.3.30. *Suppose we are in the setting of Theorem 4.3.26. We assume that a sequence $(\mu_N)_{N\in\mathbb{N}} \subset \mathcal{M}^1(\Gamma(\hat{X}))$ satisfies (4.3.38). Moreover, let for each $\Delta \in \mathcal{B}_c(\hat{X})$ and $B \in \mathcal{B}_{cyl}(\Gamma(\hat{X}))$ there exists $N_0 \in \mathbb{N}$ such that*[10]

$$\text{for all } N \geq N_0 \qquad \int_{\Gamma(\hat{X})} \pi_\Delta(B|\gamma)\mu_N(d\gamma) = \mu_N(B). \tag{4.3.43}$$

If $\mu \in \mathcal{M}^1(\Gamma(\hat{X}))$ is the τ_{loc}-limit of the sequence $\left(\mu_N\right)_{N\in\mathbb{N}}$, then:

1. *μ obeys the estimate (4.3.43) and hence is supported by $\Gamma_f(\hat{X})$).*
2. *μ satiesfies the* ***(DLR)*** *equation (4.1.13).*

Proof. Fix an arbitrary $\Delta \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}_c(X)$. Since in general $\mathfrak{m}_\Delta(\cdot)$ is not $\mathcal{B}_{\text{cyl}}(\Gamma(\hat{X}))$-measurable, we use the following cut-off procedure. For all $\Lambda \in \mathcal{B}_c(\hat{X})$ the local mass

$$\Gamma(\hat{X}) \ni \gamma \mapsto \mathfrak{m}_\Delta(\gamma_\Lambda)$$

is $\mathcal{B}_{\text{cyl}}(\Gamma(\hat{X}))$-measurable. Beppo Levi yields that for all $\Lambda' \in \mathcal{B}(\hat{X})$

$$\int_{\Gamma(\hat{X})} \mathfrak{m}_\Delta(\gamma_{\Lambda'})\mu(d\gamma) = \lim_{\substack{\Lambda\nearrow\hat{X} \\ \Lambda\in B_c(\hat{X})}} \int_{\Gamma(\hat{X})} \mathfrak{m}_\Delta(\gamma_{\Lambda'\cap\Lambda})\mu(d\gamma)$$

$$= \lim_{\substack{\Lambda\nearrow\hat{X} \\ \Lambda\in B_c(\hat{X})}} \lim_{N\to\infty} \int_{\Gamma(\hat{X})} \mathfrak{m}_\Delta(\gamma_{\Lambda'\cap\Lambda})\mu_N(\gamma)$$

Using (4.3.38), the last line is dominated by

$$\lim_{\substack{\Lambda \nearrow \hat{X} \\ \Lambda \in B_c(\hat{X})}} \lim_{N\to\infty} \left(\int_{\hat{X}} \mathfrak{m}_{\Delta'}(\{\hat{x}\}_{\Lambda'\cap\Lambda})\lambda_\theta \otimes m(d\hat{x}) + \mathfrak{m}_{\Delta'}(\xi_{\Lambda'\cap\Lambda}) \right)$$
$$\leq \int_{\hat{X}} \mathfrak{m}_{\Delta'}(\{\hat{x}\}_{\Lambda'})\lambda_\theta \otimes m(d\hat{x}) + \mathfrak{m}_{\Delta'}(\xi_{\Lambda'}) < \infty, \tag{4.3.44}$$

where we used for the finiteness (4.3.12) and that $\xi \in \Gamma_f(\hat{X})$.

We justify the following relations to obtain the (**DLR**) equation: Let $\Delta \in \mathcal{B}_c(\hat{X})$. Using Beppo Levi, we get

$$\int_{\Gamma(\hat{X})} \pi_\Delta(B|\gamma)\mu(d\gamma) \overset{1.}{=} \lim_{\substack{\Lambda \nearrow \hat{X} \\ \Lambda \in \mathcal{B}_c(\hat{X})}} \int_{\Gamma(\hat{X})} \pi_\Delta(B|\gamma_\Lambda)\mu(d\gamma)$$
$$\overset{2.}{=} \lim_{\substack{\Lambda \nearrow \hat{X} \\ \Lambda \in \mathcal{B}_c(\hat{X})}} \lim_{N\to\infty} \int_{\Gamma(\hat{X})} \pi_\Delta(B|\gamma_\Lambda)\mu_N(d\gamma)$$
$$\overset{3.}{=} \lim_{N\to\infty} \int_{\Gamma(\hat{X})} \pi_\Delta(B|\gamma)\mu_N(d\gamma) \overset{4.}{=} \lim_{N\to\infty} \mu_N(d\gamma) \overset{5.}{=} \mu(B). \tag{4.3.45}$$

The second and fifth equation follow by the definition of μ . The fourth holds by the consistency assumed in (4.3.43). For the first and third equality we use Lemma 4.3.29. Hence, we have shown that μ is a Gibbs measure (being supported by $\Gamma_f(\hat{X})$). □

This completes the proof of Theorem 4.3.26. □

4.3.7 Support of Gibbs measures

Let us have a closer look on the support properties of $\mu \in \mathcal{G}\mathrm{ibbs}_V$.

Theorem 4.3.31. *Let $\tilde{\mathfrak{m}}$ be an arbitrary local (w.r.t. $\lambda_\theta \otimes m$) mass map and $\tilde{\Lambda} \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}_c(X)$. Fix a Gibbs measure $\mu \in \mathcal{G}\mathrm{ibbs}_V$. Then*

$$\int_{\Gamma(\hat{X})} \tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\gamma)\mu(d\gamma) \leq \int_{\Gamma(\hat{X})} \tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\gamma)\mathcal{P}_\theta(d\gamma)$$
$$\leq \int_{\hat{X}} \tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\{\hat{x}\})\lambda \otimes m(d\hat{x}) < \infty. \tag{4.3.46}$$

This bound is uniform for all Gibbs measure μ corresponding to V and $\mathcal{P}_\theta$. In particular, $\mu(\Gamma_{f,\tilde{\mathfrak{m}}}(\hat{X})) = 1$.

Proof. Our idea is to exploit the uniform estimate for local Gibbs measures (cf. Proposition 4.3.18) and the **DLR** property.

Fix some $\tilde{\Lambda} \in \mathcal{B}(\hat{X})$. Using Beppo Levi, we have

$$\int_{\Gamma(\hat{X})} \tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\gamma)\mu(d\gamma) = \lim_{\substack{\Lambda \nearrow \hat{X} \\ \Lambda \in B_c(\hat{X})}} \int_{\Gamma(\hat{X})} \tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\gamma_\Lambda)\mu(d\gamma).$$

By the **(DLR)** equation the right-hand side of the later equation equals

$$\lim_{\substack{\Lambda \nearrow \hat{X} \\ \Lambda \in B_c(\hat{X})}} \lim_{\substack{\Lambda_N \nearrow \hat{X} \\ \Lambda_N \in B_c(\hat{X})}} \int_{\Gamma(\hat{X})} \int_{\Gamma(\Lambda_N)} \tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\gamma_\Lambda)\pi_{\Lambda_N}(d\gamma|\xi)\mu(d\xi)$$

By the additivity of $\tilde{\mathfrak{m}}_\Lambda$ and the definition of the local specification kernel (cf. (4.1.8)), it is dominated by

$$\lim_{\substack{\Lambda \nearrow \hat{X} \\ \Lambda \in B_c(\hat{X})}} \left(\lim_{\substack{\Lambda_N \nearrow \hat{X} \\ \Lambda_N \in B_c(\hat{X})}} \int_{\Gamma(\hat{X})} \int_{\Gamma(\Lambda_N)} \tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\gamma_{\Lambda \cap \Lambda_N})\mu_{\Lambda_N}(d\gamma_{\Lambda_N}|\xi)\mu(d\xi) \right.$$
$$\left. + \limsup_{\substack{\Lambda_N \nearrow \hat{X} \\ \Lambda_N \in B_c(\hat{X})}} \int_{\Gamma(\hat{X})} \tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\xi_{\Lambda \cap \Lambda_N^c})\mu(d\xi) \right).$$

Note that $\xi_{\Lambda \cap \Lambda_N^c}$ converges trivially to $\varnothing$. This means, there exists $N_0 > 0$ such that $\Lambda \subset \Lambda_N$ for all $N \geq N_0$, whence $\xi_{\Lambda \cap \Lambda_N^c} = \varnothing$. Hence, the last line equals

$$\lim_{\substack{\Lambda \nearrow \hat{X} \\ \Lambda \in B_c(\hat{X})}} \left(\lim_{\substack{\Lambda_N \nearrow \hat{X} \\ \Lambda_N \in B_c(\hat{X})}} \int_{\Gamma(\hat{X})} \int_{\Gamma(\Lambda_N)} \tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\gamma_{\Lambda \cap \Lambda_N})\mu_{\Lambda_N}(d\gamma_{\Lambda_N}|\xi)\mu(d\xi) \right)$$

Using the uniform bound given in Proposition 4.3.18 (cf. (4.3.18)), the later is dominated by

$$\leq \lim_{\substack{\Lambda \nearrow \hat{X} \\ \Lambda \in B_c(\hat{X})}} \int_{\Gamma(\hat{X})} \int_{\hat{X}} \tilde{\mathfrak{m}}_{\tilde{\Lambda}}(\{\hat{x}\}_\Lambda)\lambda_\theta \otimes m(d\hat{x})\mu(d\xi). \tag{4.3.47}$$

If actually $\tilde{\Lambda} \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}_c(\hat{X})$, then the last line is bounded by $\theta C_{\tilde{\Lambda}} < \infty$. Hence, we have proved that $\mu(\Gamma_{f,\tilde{\mathfrak{m}}}(\hat{X})) = 1$. □

Remark 4.3.32. *To obtain the last result, we crucially used that $V \geq 0$.*

1. *Let* $\mathcal{G}\text{ibbs}_V \neq \varnothing$. *Then Theorem 4.3.31 implies for any local mass map* $\tilde{\mathfrak{m}}$ *that*
$$\forall \mu \in \mathcal{G}\text{ibbs}_V \quad \mu(\Gamma_{f,\tilde{\mathfrak{m}}}(\hat{X})) = 1.$$

2. *The later shows that the a priori assumption of* μ *being supported by* $\Gamma_f(\hat{X})$ *is no restriction; and it is in fact not necessary (cf. also Remark 4.3.33 below).*

The following remark concerns the support property of $\mu \in \mathcal{G}\text{ibbs}_V$:

Remark 4.3.33. *Let* μ *be a Gibbs measure and* $A \subset \mathcal{B}(\Gamma(\hat{X}))$ *be such that there exists* $\Lambda \in \mathcal{B}_c(\hat{X}) : \pi_\Lambda(\Gamma(\hat{X})|\xi) = 0$ *for all* $\xi \in A$. *Then* $\mu(A) = 0$ *because*

$$\begin{aligned} 1 =& \mu(\Gamma(\hat{X})) = \int_{\Gamma(\hat{X})} \pi_\Lambda(\Gamma(\hat{X})|\xi)\mu(d\xi) = \int_{A^c} \pi(\Gamma(\hat{X})|\xi)\mu(d\xi) \\ &\leq \int_{A^c} \mu(d\xi) = 1 - \mu(A). \end{aligned}$$

4.3.8 Higher moments of Gibbs measures

Theorem 4.3.34. *Let* $V : \hat{X} \times \hat{X} \to [0,\infty)$ *be as in Theorem 4.3.26 with the corresponding local (w.r.t.* $\lambda_\theta \otimes m$*) mass map* $\mathfrak{m}$. *Fix a Gibbs measure* $\mu \in \mathcal{G}\text{ibbs}_V$ *and a* subadditive *local mass map* $\tilde{\mathfrak{m}}$. *Then for* $N \in \{1,2\}$ *and* $\Delta \in \mathcal{B}(\hat{X})$

$$\begin{aligned} &\int_{\Gamma(\hat{X})} \tilde{\mathfrak{m}}_{\hat{X}}(\eta_\Delta)^N \mu(d\eta) \leq \int_{\Gamma(\hat{X})} \tilde{\mathfrak{m}}_{\hat{X}}(\eta_\Delta)^N \mathcal{P}_\theta(d\eta) \\ &\leq \int_{\hat{X}} \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}\}_\Delta)^N (\lambda_\theta \otimes m)(d\hat{x}) + \left(\int_{\hat{X}} \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}\}_\Delta)(\lambda_\theta \otimes m)(d\hat{x}) \right)^N. \end{aligned} \tag{4.3.48}$$

For $N = 1$ *we may drop the first summand. In this case the r.h.s. in (4.3.48) is by assumption finite for any* $\Delta \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}_c(X)$. *For* $N = 2$ *the r.h.s. is finite if* $\Delta \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}_c(X)$ *and*

$$\int_{\hat{X}} \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}\}_\Delta)^2 (\lambda_\theta \otimes m)(d\hat{x}) < \infty.$$

Remark 4.3.35. *By the definition of a local mass map,*

$$\tilde{\mathfrak{m}}_{\hat{X}}(\{\cdot\}_\Delta)^1 \in L^1(\hat{X}, \lambda_\theta \otimes m).$$

Proof. Similarly to the proof of Theorem 4.3.31, this fact follows by the (**DLR**) property and Proposition 4.3.18 resp. 4.3.21. Using the monotone convergence theorem, we get

$$\begin{aligned}
\int_{\Gamma(\Lambda)} \tilde{\mathfrak{m}}_{\hat{X}}(\eta_\Delta)^N \mu(d\eta) &= \lim_{\substack{\Lambda \nearrow \hat{X} \\ \Lambda \in \mathcal{B}_c(\hat{X})}} \int_{\Gamma(\hat{X})} \tilde{\mathfrak{m}}_{\hat{X}}(\eta_{\Delta\cap\Lambda})^N \mu(d\eta) \\
&= \lim_{\substack{\Lambda \nearrow \hat{X} \\ \Lambda \in \mathcal{B}_c(\hat{X})}} \lim_{\substack{\tilde{\Lambda} \nearrow \hat{X},\ \Lambda \subseteq \tilde{\Lambda}, \\ \tilde{\Lambda} \in \mathcal{B}_c(\hat{X})}} \int_{\Gamma(\Lambda)} \int_{\Gamma(\tilde{\Lambda})} \tilde{\mathfrak{m}}_{\hat{X}}(\eta_{\Delta\cap\Lambda})^N \mu_{\tilde{\Lambda}}(d\eta|\xi)\mu(d\xi) \\
&\leq \lim_{\substack{\Lambda \nearrow \hat{X} \\ \Lambda \in \mathcal{B}_c(\hat{X})}} \lim_{\substack{\tilde{\Lambda} \nearrow \hat{X},\ \Lambda \subseteq \tilde{\Lambda}, \\ \tilde{\Lambda} \in \mathcal{B}_c(\hat{X})}} \int_{\Gamma(\Lambda)} \bigg(\int_{\hat{X}} \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}\}_\Delta)^N (\lambda_\theta \otimes m)(d\hat{x}) \\
&\qquad + \bigg(\int_{\hat{X}} \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}\}_\Delta)(\lambda_\theta \otimes m)(d\hat{x}) \bigg)^N \bigg) \mu(d\xi),
\end{aligned}$$

where we applied Proposition 4.3.21 for $N = 2$, resp. Proposition 4.3.18 for $N = 1$. This yields the claim. □

Remark 4.3.36. *For the moments of higher orders, results analogous to those mentioned in Remark 4.3.22 hold. Here, we only point out that*

$$\int_{\Gamma(\hat{X})} \tilde{\mathfrak{m}}_{\hat{X}}(\eta_\Delta)^N \mu(d\eta) < \infty \quad \textit{for some } N \in \mathbb{N}$$

whenever

$$\int_{\hat{X}} \tilde{\mathfrak{m}}_{\hat{X}}(\{\hat{x}\}_\Delta)^n \lambda_\theta \otimes m(d\hat{x}) \quad \textit{for all } 1 \leq n \leq N.$$

4.4 A closer look at Gibbs measures

In the previous section, we have shown that the (convex set) $\mathcal{G}\mathrm{ibbs}_V \neq \varnothing$ (cf. Theorem 4.3.26). In this section we show that the set $\mathcal{G}\mathrm{ibbs}_V$ is a compact set in $\mathcal{T}_{\mathrm{loc}}$ (cf. Subsection 4.4.4).

For simplicity in this section, we consider the case of $X = \mathbb{R}^d$ (with $d \in \mathbb{N}$) being equipped with the Lebesgue measure $m(dx) = dx$.[11] Using

[11] Actually, we can handle more general spaces which fulfill a certain covering property (cf. Remark 4.4.2 and esp. Definition 5.3.1) to construct an associated global mass and obtain the announced results. But, for the sake of clarity, we stick to the basic model setting.

properties of the underlying phase space $\hat{\mathbb{R}}^d$, we introduce a smaller subset $\Gamma^{\mathfrak{t}}(\hat{\mathbb{R}}^d) \subset \Gamma_f(\hat{\mathbb{R}}^d)$ of the so-called ‹tempered configurations› such that

$$\mu(\Gamma^{\mathfrak{t}}(\hat{\mathbb{R}}^d)) = 1, \quad \text{for all } \mu \in \mathcal{G}\mathrm{ibbs}_V.$$

From now on, the lower index $\mathbb{R}_+$ shall remind us that we treat an object in $\mathbb{R}_+$ or a property related to the space $\mathbb{R}_+$. The same notational convention is true for X, resp. $\mathbb{R}^d$.

4.4.1 Covering of $\hat{\mathbb{R}}^d$

Let $(\mathbb{R}_+, d_{\mathbb{R}_+})$ and $(\mathbb{R}^d, |\cdot|)$ be the components of the underlying phase space $\hat{\mathbb{R}}^d = \mathbb{R}_+ \times \mathbb{R}^d$ described in Section 2.2. We construct a covering of the space $\hat{\mathbb{R}}^d$, which consists of balls whose volumes are uniformly bounded. On $\mathbb{R}^d$ we use the standard covering related to the Euclidean distance, whereas on $\mathbb{R}_+$ we use a special one related to the logarithmic distance.[12]

We fix an arbitrary $g \in (0, 1)$. Let $Q_{\mathbb{R}^d,z}$, $z \in \mathbb{Z}^d$, denote the balls with radius dg, centered at qz, i.e.,

$$Q_{\mathbb{R}^d,z} := \left\{ x \in \mathbb{R}^d \,\middle|\, |gz - x| \leq d \cdot g \right\}.$$

Then $\left(Q_{\mathbb{R}^d,z}\right)_{z\in\mathbb{Z}^d}$ covers $\mathbb{R}^d$ and

$$\sup_{z\in\mathbb{Z}^d} m\left(Q_{\mathbb{R}^d,z}\right) =: M_{\mathbb{R}^d} < \infty \tag{4.4.1}$$

because the Lebesgue measure is translation invariant.[13] Instead of $Q_{\mathbb{R}^d,z}$ we can also consider the hypercubes

$$Q_{k,g} := [-g/2, g/2) + kg, \quad k \in \mathbb{Z}^d.$$

In the later case, we have a disjoint partition of $\mathbb{R}^d = \bigsqcup_{k\in\mathbb{Z}^d} Q_{k,g}$.

For $\mathbb{R}_+$ we have a different type of covering: For all $N \in \mathbb{Z}$ we define

$$Q_{\mathbb{R}_+,N} := [g^{N+1/2}, g^{N-1/2}). \tag{4.4.2}$$

Since the intensity measures λ_θ is non-atomic, it does not matter whether we consider the half open intervals or their closures. The closures are balls in $(\mathbb{R}_+, d_{\mathbb{R}_+})$ that are centered at points g^N and have radius $-\frac{1}{2}\ln g > 0$ (cf. (2.2.1)).

[12] Thus we have to leave the framework of Ruelle, cf. [Rue69, Rue70].

[13] If we take instead of the Euclidean norm the sup-norm, we get the usual choice of cubes (cf. e.g. [Rue69], [AKR98b], [KPR10]).

Lemma 4.4.1. *The half open intervals $Q_{\mathbb{R}_+,N}$ cover $\mathbb{R}_+$ and their volumes are uniformly bounded:*

$$\sup_{N\in\mathbb{Z}} \lambda_\theta\left(Q_{\mathbb{R}_+,N}\right) \leq \theta\left(|\ln g|+1\right) =: M_{\mathbb{R}_+} < \infty \tag{4.4.3}$$

Proof. If $N<0$ then $1 \leq g^N = (\frac{1}{g})^{|N|}$ and

$$\theta\int_{g^{N+1/2}}^{g^{N-1/2}} e^{-t}t^{-1}dt \leq \theta\int_{g^{N+1/2}}^{g^{N-1/2}} e^{-t}g^{|N|-1/2}dt \leq \theta\int_{g}^{\infty} e^{-t}\cdot 1dt = \theta\frac{1}{e^g}.$$

On the other hand, if $N \geq 0$ then $0 < g^N \leq 1$ and for $0 < a \leq b \leq 1$

$$\int_a^b e^{-t}\frac{1}{t}dt \leq \int_a^b \frac{1}{t}dt = \ln\frac{b}{a}.$$

Therefore, for $N \geq 0$ we obtain

$$\theta\int_{g^{N+1/2}}^{g^{N-1/2}} e^{-t}\frac{1}{t}dt \leq \theta(\ln g)(N-1/2-(N+1/2)) = \theta\ln\frac{1}{g}.$$

Hence, the claim follows. $\square$

Setting for all $k := (N,z) \in \mathbb{I} := \mathbb{Z}\times\mathbb{Z}^d$

$$Q_k := Q_{(N,z)} := Q_{\mathbb{R}_+,N}\times Q_{\mathbb{R}^d,z}, \tag{4.4.4}$$

we get sets with uniformly bounded volume covering $\hat{\mathbb{R}}^d$.

Remark 4.4.2. *We summarize the important facts regarding the space that we will use for the construction: Let us suppose that we are in the general framework, i.e., we consider again an arbitrary locally compact Polish space X (cf. Section 4.3). Let $\mathbb{I} \subset \mathbb{Z}\times\mathbb{Z}^d = \{k=(N,z)|N\in\mathbb{Z}, z\in\mathbb{Z}^d\}$. We define for $q,g>0$ and $k=(N,z)\in\mathbb{I}$ the weights*

$$c_{k,q} := e^{-q|z|}e^{-q|N\ln g|} \geq 0. \tag{4.4.5}$$

($\mathbf{AP'}$) ***Admissible partition:*** *Let g, q, $\mathbb{I}$ and $(Q_{g,k})_{k\in\mathbb{I}}$ be chosen such that $Q_{g,k}$ are a partition of $\hat{X}$, i.e. $\hat{X} = \bigsqcup_{k\in\mathbb{I}} Q_{g,k}$, the weights are summable, i.e.,*

$$C_q := \sum_{k\in\mathbb{I}} c_{k,q} < \infty, \qquad \forall\, q>0, \tag{4.4.6}$$

and the volumes of the partition sets $Q_{g,k}$ are uniformly bounded, i.e.,

$$\sup_{k\in\mathbb{I}} \lambda_\theta\otimes m(Q_{g,k}) < \infty. \tag{4.4.7}$$

4.4.2 An associated global mass

We introduce some notation to keep the presentation more easily readable. For $k \in \mathbb{I}$ (cf. (4.4.4)), we denote the restriction of a configuration $\gamma \in \Gamma(\hat{X})$ to the ball $Q_{g,k}$ by

$$\gamma_k := \gamma_{Q_{g,k}}$$

and the configuration space over this ball by

$$\Gamma_k := \Gamma(Q_{g,k}) := \left\{ \gamma_k \,\middle|\, \gamma \in \Gamma(\hat{X}) \right\} \subset \Gamma_0.$$

Furthermore, we define the *projection of* $Q_{g,k}$ *to* $\mathbb{R}_+$ and $\mathbb{R}^d$ by

$$Q_{k,\mathbb{R}_+} := (Q_{g,k})_{\mathbb{R}_+}, \qquad \text{resp.} \qquad Q_{k,\mathbb{R}^d} := (Q_{g,k})_{\mathbb{R}^d}.$$

For any local mass map $\mathfrak{m}$ and for each $k \in \mathbb{I}$, we abbreviate the local mass in $Q_{g,k}$ by

$$\mathfrak{m}_k := \mathfrak{m}_{Q_{g,k}}.$$

Example 4.4.3. *For each $p \geq 0$ and $k \in \mathbb{I}$, we define (cf. Example 4.3.11)*

$$\Gamma(\hat{X}) \ni \gamma \mapsto \mathfrak{m}_{p,k}(\gamma) := \mathfrak{m}_{p,Q_{g,k}}(\gamma) := \sum_{(s,x) \in \gamma \cap Q_{g,k}} s^p, \tag{4.4.8}$$

which is finite for any $\gamma \in \Gamma_p(\hat{X})$ (cf. (2.1.1)). For $p > 0$ each $\mathfrak{m}_{p,Q_{g,k}}$ is a global mass map (cf. also Lemma 4.4.5 below).

We present a second local mass map, which will not be used afterwards in our considerations. It incorporates the following aspects: A configuration describes the allocation of particles in $\hat{\mathbb{R}}^d$. For each area Λ we have some known information about the allocated particles. We assume that information is processed in chunks $Q_{g,k}$ and that no information outside of a given area Λ may be used, i.e., that information is not previsible. The the following local mass maps are in correspondence with this interpretation:

Example 4.4.4. *Let $p > 0$ and define*

$$\begin{aligned} \mathfrak{m}_{p,g}\colon \quad \mathcal{B}(\hat{\mathbb{R}}^d) \times \Gamma(\hat{X}) \;&\to\; \overline{\mathbb{R}^+} \\ (\Lambda, \gamma) \;&\mapsto\; \sum_{\hat{x} \in \gamma \cap \Lambda_{\mathbb{I}}} s^p, \end{aligned} \tag{4.4.9}$$

where

$$\Lambda_{\mathbb{I}} := \bigcup_{k\in\mathbb{I},\ Q_{g,k}\subset\Lambda} Q_{g,k}, \quad \forall\Lambda\in\mathcal{B}(\hat{\mathbb{R}}^d). \tag{4.4.10}$$

Note that $\mathfrak{m}_0 = |\gamma\cap\Lambda_{\mathbb{I}}|$ *is not a local mass map because (4.3.12) is void. For* $p>0$, $\mathfrak{m}_{p,g}$ *is a* local *(w.r.t.* $\lambda_\theta\otimes dx$*) mass at* $Q_{g,k}$ *and* $\mathfrak{m}_p$ *is a corresponding local mass map. We point out that there exists no semi-local function whose associated local mass map coinsides with* $\mathfrak{m}_{p,g}$.

Lemma 4.4.5. *For any* $p\geq 0$,

$$\sup_{\substack{k\in\mathbb{I},\ \xi\in\Gamma_f(\hat{X}),\\ \Lambda\in B_c(\hat{\mathbb{R}}^d)}} \int_{\Gamma_k} \mathfrak{m}_{p,k}(\gamma)\mu_\Lambda(d\gamma|\xi)\leq c_p<\infty, \tag{4.4.11}$$

where

$$c_p := \begin{cases} \theta M_{\mathbb{R}^d}(1+|\ln g|), & \text{if } p=0,\\ \theta M_{\mathbb{R}^d}(1+\frac{1}{p}), & \text{if } 0<p\leq 1,\\ \theta M_{\mathbb{R}^d}(1+(\lceil p-1\rceil)!), & \text{if } 1\leq p,\end{cases}$$

and $M_{\mathbb{R}_+}$ *and* $M_{\mathbb{R}^d}$ *are the same as in (4.4.3), resp. (4.4.1).*

Proof. We apply Proposition 4.3.18 for the subadditive local mass map

$$\mathcal{B}(\hat{\mathbb{R}}^d)\times\Gamma(\hat{X})\ni(\Delta,\gamma)\mapsto \mathbb{1}_{\Gamma(\Delta_{\mathbb{I}})}(\gamma)\cdot\mathfrak{m}_{p,\Delta_{\mathbb{I}}}(\gamma),$$

to deduce that

$$\begin{aligned}\int_{\Gamma_k}\mathfrak{m}_{p,k}(\gamma)\mu_\Lambda(d\gamma) &= \int_{\Gamma(\hat{X})}\mathbb{1}_{\Gamma(Q_{g,k})}(\gamma)\cdot\mathfrak{m}_{p,k}(\gamma)\mu_\Lambda(d\gamma)\\ \leq \int_{\Gamma^{(1)}(\Lambda)}\mathbb{1}_{\Gamma(Q_{g,k})}(\gamma)\cdot\mathfrak{m}_{p,k}(\gamma)P_\theta^\Lambda(d\gamma) &\leq \int_{Q_{g,k}}\mathfrak{m}_{p,k}(\{\hat{x}_i\})\lambda_\theta\otimes m(d\hat{x}_i).\end{aligned} \tag{4.4.12}$$

The integral is finite (cf. (4.3.12)), thus it remains to show that the bound is uniform for all $Q_{g,k}$, $k\in\mathbb{I}$, i.e.,

$$\int_{Q_{g,k}}\mathfrak{m}_{p,k}(\{\hat{x}_i\})\lambda_\theta\otimes m(d\hat{x}_i)\leq c_p. \tag{4.4.13}$$

In the case $p=0$, the claim follows from the uniform bound for the projection of the sets $Q_{g,k}$ into $\mathbb{R}_+$ and $\mathbb{R}^d$ given by Lemma 4.4.1, resp. (4.4.1). If $p>0$, it follows by Example 4.3.5 and the following calculations: For $0<p\leq 1$

$$\Gamma(p)\leq\int_0^1 s^{p-1}\cdot 1ds+\int_1^\infty 1\cdot e^{-s}ds\leq\frac{1}{p}+1.$$

For $1 \leq p$

$$\Gamma(p) \leq \int_0^1 1 \cdot e^{-s} ds + \int_1^\infty s^{\lceil p-1 \rceil} e^{-s} ds \leq 1 + \int_0^\infty s^{\lceil p-1 \rceil} e^{-s} ds$$
$$\leq 1 + \left[-s^{\lceil p-1 \rceil} e^{-s} \right]_0^\infty - \int_0^\infty -e^{-s} \cdot \lceil p-1 \rceil s^{\lceil p-1 \rceil - 1} ds = 1 + (\lceil p-1 \rceil)!.$$

□

Remark 4.4.6. *We can generalize the results of Lemma 4.4.5, also to the situation described in Remark 4.4.2:*

1. *From the proof of the lemma, namely by (4.4.13), we see that (4.4.11) holds for any local mass map $\mathfrak{m}$ obeying*

$$\sup_{k \in \mathbb{I}} \int_{\hat{X}} \mathfrak{m}_k(\{\hat{x}\})(\lambda_\theta \otimes m)(d\hat{x}) \leq C < \infty, \tag{4.4.14}$$

2. *Furthermore, if we have a* finite local *(w.r.t. $\lambda_\theta \otimes m$)* mass map *obeying*

$$\int_{\hat{X}} \mathfrak{m}_{\hat{X}}(\{\hat{x}\}) \lambda_\theta \otimes m(d\hat{x}) \leq C < \infty, \tag{4.4.15}$$

then we even obtain (cf. Proposition 4.3.18) that

$$\sup_{\substack{\xi \in \Gamma_f(\hat{X}) \\ \Lambda \in \mathcal{B}_c(\hat{X})}} \int_{\Gamma(\hat{X})} \mathfrak{m}_{\hat{X}}(\gamma) \mu_\Lambda(\gamma|\xi) < \infty. \tag{4.4.16}$$

Definition 4.4.7. *Let $\tilde{\mathfrak{m}}$ be a local (w.r.t. $\lambda_\theta \otimes m$) mass map. If it fulfills (4.4.14), then we call $\tilde{\mathfrak{m}}$ a local mass map* with uniform integrability *(w.r.t. $\lambda_\theta \otimes m$). If even the stronger condition (4.4.15) holds, we call $\tilde{\mathfrak{m}}$ a* finite *local (w.r.t. $\lambda_\theta \otimes m$) mass map.*

We call a semi-local function uniform integrable, *resp.* finite, *if the associated local mass map has this property.*

Example 4.4.8. *The local mass maps $\mathfrak{m}_p$, $p > 0$, defined by (4.4.9) in Example 4.4.3 fulfill the uniform integrability, as one can check using (4.4.13) and*

$$\int_{\hat{\mathbb{R}}^d} \mathfrak{m}_{p,k}(\{\hat{x}\}) \lambda_\theta \otimes m(d\hat{x}) = \int_{Q_k} \mathfrak{m}_{p,k}(\{\hat{x}\}) \lambda_\theta \otimes m(d\hat{x}).$$

Lemma 4.4.9. *Under the assumptions of Proposition 4.3.18, we have for any local mass map $\tilde{\mathfrak{m}}$ that*

$$\int_{\Gamma(\hat{X})} \tilde{\mathfrak{m}}_k(\gamma)\pi_\Lambda(d\gamma_\Lambda|\xi) \leq \int_\Lambda \tilde{\mathfrak{m}}_k(\{\hat{x}\})(\lambda_\theta \otimes m)(d\hat{x}) + \tilde{\mathfrak{m}}_k(\xi_{\Lambda^c}). \tag{4.4.17}$$

If (4.4.14) is fulfilled with the constant $C > 0$, then the first summand is (uniformly for all Q_k) bounded by this constant C.

Proof. The claim follows directly by Proposition 4.3.18. □

Global mass

The next step is to find a suitable global mass on $\Gamma(\hat{X})$ to $\overline{\mathbb{R}^+}$. With its help one can get a better insight into sets on which the Gamma-Poisson measure $\mathcal{P}_\theta$ is supported.

Definition 4.4.10. *Let $\tilde{\mathfrak{m}} : \mathcal{B}(\hat{\mathbb{R}}^d) \times \Gamma(\hat{X}) \to \overline{\mathbb{R}^+}$ be a map. For $q > 0$, let us set*

$$\|\gamma\|_{\tilde{\mathfrak{m}},q} := \sum_{k\in\mathbb{I}} c_{k,q}\, \tilde{\mathfrak{m}}_k(\gamma), \qquad \gamma \in \Gamma(\hat{X}), \tag{4.4.18}$$

where, for each $k = (N, z) \in \mathbb{I}$, we define the weight (cf. (4.4.5))

$$c_{k,q} := \exp\{-q\,(|N\ln(g)| + |z|)\} \geq 0.$$

If $\|\cdot\|_{q,\tilde{\mathfrak{m}}}$ is a global (w.r.t. $\lambda_\theta \otimes m$) mass map for all $q > 0$, we call each $\|\cdot\|_{q,\tilde{\mathfrak{m}}}$ the associated q-weighted global *(w.r.t. $\lambda_\theta \otimes m$)* mass *with parameter $q > 0$. We refer to the whole family $\{\|\cdot\|_{q,\tilde{\mathfrak{m}}} \big| q > 0\}$ as the associated global mass (map).*

Example 4.4.11. *An associated global mass exists for 1) $\mathfrak{m}_0$; 2) each semi-local function that is uniformly integrable; 3) mass map with uniform integrability, and in particular 4) for each $\mathfrak{m}_p$ and $\mathfrak{m}_{p,g}$, $p > 0$.*

Proof. The additivity and monotonicity (see (4.3.8) and (4.3.9)) are clear. The measurability (cf. (4.3.10)) is obvious because the corresponding map that is defined by (4.4.18) is the limit of $\mathcal{B}(\Gamma(\hat{X}))$-measurable functions (cf. Theorem 4.3.13). We check the integrability (cf. (4.3.12)) via Fubini-Tonelli

$$\int_{\hat{\mathbb{R}}^d} \|\{\hat{x}\}\|_{\mathfrak{m},q} \lambda_\theta \otimes m(d\hat{x}) = \sum_{k\in\mathbb{I}} c_{k,q} \int_{\hat{\mathbb{R}}^d} \mathfrak{m}_k(\{\hat{x}\})\lambda_\theta \otimes m(d\hat{x})$$
$$\leq C_q \cdot C < \infty, \tag{4.4.19}$$

where C is chosen appropriately. Here, in the first two cases we used the uniform integrability (cf. (4.4.14)) to estimate the integrals and (4.4.6) to check the summability; and in the third case we used (4.4.7) saying that

$$\int_{\hat{\mathbb{R}}^d} \mathfrak{m}_k(\{\hat{x}\})\lambda_\theta \otimes m(d\hat{x}) \leq C.$$

$\square$

Proposition 4.4.12. *Let $q > 0$ and $\|\cdot\|_{q,\tilde{\mathfrak{m}}}$ be an associated q-weighted global mass. Under the assumptions of Proposition 4.3.18, we have*

$$\int_{\Gamma(\Lambda)} \|\gamma\|_{\tilde{\mathfrak{m}},q}\ \mu_\Lambda(d\gamma|\xi) \leq C \cdot C_q < \infty. \tag{4.4.20}$$

This implies that, for each $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$, $\pi_\Lambda(\cdot|\xi)$ is supported by $\Gamma_{f,\|\ \cdot\|_{\tilde{\mathfrak{m}},q}}(\hat{\mathbb{R}}^d)$ whenever $\xi \in \Gamma_{f,\|\ \cdot\|_{\tilde{\mathfrak{m}},q}}(\hat{\mathbb{R}}^d)$.

Proof. Since each global mass is a local mass map and hence it fulfills (4.4.14) with the constant CC_q (cf. (4.4.19)), the result follows by Lemma 4.4.9. $\square$

The support property implied by Proposition 4.4.12 motivates the following definition:

Definition 4.4.13. *Let $\tilde{\mathfrak{m}}$ be a local mass map with uniform integrability (w.r.t. $\lambda_\theta \otimes m$) and $q > 0$. We define the following set of q-*tempered *configurations*

$$\Gamma_{f,\tilde{\mathfrak{m}},\mathcal{Q}}(\hat{\mathbb{R}}^d) := \Gamma_{f,\|\cdot\|_{q,\tilde{\mathfrak{m}}}}(\hat{\mathbb{R}}^d) \cap \Gamma_{f,\tilde{\mathfrak{m}}}(\hat{\mathbb{R}}^d). \tag{4.4.21}$$

If it is clear which local mass map is meant, we omit the index $\tilde{\mathfrak{m}}$.

Remark 4.4.14. *The associated global mass is measurable, $\Gamma_{f,\tilde{\mathfrak{m}},q}(\hat{X}) \in \mathcal{B}(\Gamma(\hat{X}))$ and $\mathcal{P}_\theta(\Gamma_{f,\tilde{\mathfrak{m}},q}(\hat{\mathbb{R}}^d)) = 1$ (cf. Remark 4.3.7 and Theorem 4.3.13).*

Corollary 4.4.15. *Let $q > 0$ and $\|\cdot\|_{q,\tilde{\mathfrak{m}}}$ be an associated q-weighted global mass. If we set*

$$V(\hat{x},\hat{y}) = \|\{\hat{x}\}\|_{q,\tilde{\mathfrak{m}}} \cdot \|\{\hat{y}\}\|_{q,\tilde{\mathfrak{m}}}, \qquad \forall \hat{x},\hat{y} \in \hat{\mathbb{R}}^d,$$

then there exists a Gibbs measure corresponding to V and $\mathcal{P}_\theta$.

If $\tilde{\mathfrak{m}}$ is a semi-local function that is uniformly integrable, there exists an associated $q-$weighted global mass. In particular, for $\tilde{\mathfrak{m}} = \mathfrak{m}_p$ with $p > 0$ we have constructed a Gibbs measure for a pair potential that vanishes nowhere.

Proof. Applying Theorem 4.3.26 for the local mass map $\Gamma(\hat{X}) \ni \gamma \mapsto \|\gamma\|_{q,\tilde{\mathfrak{m}}}$ and the above potential, yields the result. $\square$

4.4.3 Support properties of Gibbs measures

In this subsection we show that each $\mu \in \mathcal{G}\mathrm{ibbs}_V$ is supported by the following set $\Gamma^{\mathbf{t}}_{\tilde{\mathfrak{m}}}(\hat{\mathbb{R}}^d)$ consisting of *tempered configurations*:

Definition 4.4.16. *Let $\tilde{\mathfrak{m}}$ be a local mass map with uniform integrability (w.r.t. $\lambda_\theta \otimes m$). We define the set of* tempered *configurations by*

$$\Gamma^{\boldsymbol{t}}_{\tilde{\mathfrak{m}}}(\hat{\mathbb{R}}^d) := \bigcap_{q>0} \Gamma_{f,\tilde{\mathfrak{m}},q}(\hat{\mathbb{R}}^d), \tag{4.4.22}$$

where $\Gamma_{f,\tilde{\mathfrak{m}},\mathcal{Q}}(\hat{\mathbb{R}}^d) = \Big\{\gamma \in \Gamma_{f,\tilde{\mathfrak{m}}}(\hat{\mathbb{R}}^d) \ \mid \|\gamma\|_{\tilde{\mathfrak{m}},q} < \infty\Big\}$ (cf. Definition 4.4.13) and $\|\cdot\|_{\tilde{\mathfrak{m}},q}$ is the q-weighted global mass associated to the local mass map $\tilde{\mathfrak{m}}$ and $q > 0$. More precisely (cf. (4.4.18)),

$$\|\gamma\|_{\tilde{\mathfrak{m}},q} = \sum_{k\in\mathbb{I}} \tilde{\mathfrak{m}}_k(\gamma) c_{k,q},$$

where for each $k = (N, z) \in \mathbb{I}$ we defined (cf. (4.4.5))

$$c_{k,q} := \exp\big\{-q\big(|N \ln g| + |z|\big)\big\}.$$

As before, if it is clear which is the involved local mass map, we may drop the index of the local mass map, i.e.,

$$\Gamma^{\boldsymbol{t}}(\hat{\mathbb{R}}^d) = \Gamma^{\boldsymbol{t}}_{\tilde{\mathfrak{m}}}(\hat{\mathbb{R}}^d).$$

Remark 4.4.17. *Since $\Gamma_{f,\tilde{\mathfrak{m}},\tilde{q}}(\hat{\mathbb{R}}^d) \subset \Gamma_{f,\tilde{\mathfrak{m}},q}(\hat{\mathbb{R}}^d)$ for $0 < \tilde{q} < q$, we obtain that*

$$\Gamma^{\boldsymbol{t}}_{\tilde{\mathfrak{m}}}(\hat{\mathbb{R}}^d) = \bigcap_{q>0} \Gamma_{f,\tilde{\mathfrak{m}},q}(\hat{\mathbb{R}}^d) \in \mathcal{B}(\Gamma(\hat{\mathbb{R}}^d))$$

and thus $\mathcal{P}_\theta(\Gamma^{\boldsymbol{t}}_{\tilde{\mathfrak{m}}}(\hat{\mathbb{R}}^d)) = 1$.

Corollary 4.4.18. *Let a potential V fulfill the assumptions of Theorem 4.3.26 (with corresponding local mass map $\mathfrak{m}$). Then each Gibbs measure μ that corresponds to the potential V and the Gamma-Poisson measure $\mathcal{P}_\theta$ is supported by the set of tempered configurations $\Gamma^{\boldsymbol{t}}_{\tilde{\mathfrak{m}}}(\hat{\mathbb{R}}^d)$, i.e.,*

$$\mu(\Gamma^{\boldsymbol{t}}_{\tilde{\mathfrak{m}}}(\hat{\mathbb{R}}^d)) = 1. \tag{4.4.23}$$

Here we may choose an arbitrary (semi-local function or) local (w.r.t. $\lambda_\theta \otimes m$) mass map $\tilde{\mathfrak{m}}$ that admits an associated global mass map. In particular, such a Gibbs measure exists and is supported by $\Gamma^{\boldsymbol{t}}_{\mathfrak{m}_p}(\hat{\mathbb{R}}^d)$, $p > 0$.

Proof. The existence follows by Theorem 4.3.26. The support property follows by Theorem 4.3.31, which holds for any local mass map. Since we consider the global mass $\|\cdot\|_{\tilde{\mathfrak{m}},q}$, $q>0$, as a constant local mass map, this implies the support property (cf. Definition 4.4.16).

The existence and the support properties in the particular case of $\mathfrak{m}_p$, $p>0$, follow by Theorem 4.3.26 and Example 4.4.11.

□

Example 4.4.19. *Let $V(\hat{x},\hat{y}) = s_x s_y a(x-y)$ be as in Example 4.1.1. Then any Gibbs measure corresponding to V is supported by*

$$\Gamma^{t}_{ex}(\hat{\mathbb{R}}^d) := \bigcap_{\substack{p>0\\0<g<1}} \left\{ \gamma \in \Gamma(\hat{\mathbb{R}}^d) \middle| \sum_{k=(N,z)\in\mathbb{Z}^{d+1}} e^{-q(|z|+N|\ln(g)|)} \sum_{\hat{x}\in\gamma\cap Q_{g,k}} s_x^p < \infty \right\}.$$

where $Q_{g,k} := [g^{N+1/2}, g^{N-1/2}) \times \left([-1/2,1/2)^d + z\right)$ and $\hat{\mathbb{R}}^d = \bigsqcup_{k\in\mathbb{Z}^{d+1}} Q_{g,k}$.

4.4.4 Compactness of the set of Gibbs measure

We return to the general framework of (X,d_X) being a locally compact Polish space.

Theorem 4.4.20. *Let V fulfill the assumptions of Theorem 4.3.26. Then, for any local mass maps $\tilde{\mathfrak{m}}$ and for all $\Lambda \in \mathcal{B}(\mathbb{R}_+)\times\mathcal{B}_c(X)$, there exists $C_{\Lambda,\tilde{\mathfrak{m}}} > 0$ such that*

$$\sup_{\mu\in\mathcal{G}\mathrm{ibbs}_V} \int_{\Gamma(\hat{X})} \tilde{\mathfrak{m}}_\Lambda(\gamma)\mu(d\gamma) \leq C_{\Lambda,\tilde{\mathfrak{m}}}. \tag{4.4.24}$$

Moreover, $\mathcal{G}\mathrm{ibbs}_V$ is a compact set in the topology $\mathcal{T}_{\mathrm{loc}}$.

Corollary 4.4.21. *Let $\left\{\tilde{\mathfrak{m}}_i \middle| i\in\mathbb{N}\right\}$ be a countable family of local mass maps and the potential V fulfill the assumptions of Theorem 4.3.26. Then*

$$\mu\left(\bigcap_{i\in\mathbb{N}} \Gamma_{f,\tilde{\mathfrak{m}}_i}(\hat{X})\right) = 1, \qquad \textit{for all } \mu\in\mathcal{G}\mathrm{ibbs}_V. \tag{4.4.25}$$

Proof of Theorem 4.4.20. The proof of Theorem 4.4.20 extends the arguments used for proving the corresponding result in [KPR10]. The a-priori bound (4.4.24) follows by Theorem 4.3.31.

The next step is to prove compactness. To this end, we note that it is enough to prove the local equicontinuity of each net consisting of points of

$\mathcal{G}\mathrm{ibbs}_V$. Then every net in $\mathcal{G}\mathrm{ibbs}_V$ has a $\mathcal{T}_{\mathrm{loc}}$-cluster point in $\mathcal{M}^1(\Gamma(\hat{X}))$, which is equivalent to the relative compactness of $\mathcal{G}\mathrm{ibbs}_V$ in the topology $\mathcal{T}_{\mathrm{loc}}$. Finally, by Proposition 4.3.30, any of the $\mathcal{T}_{\mathrm{loc}}$-limit measures is surely a Gibbs measure which is supported by $\Gamma_f(\hat{X})$ (cf. Theorem 4.3.31).

For any $\mu \in \mathcal{G}\mathrm{ibbs}_V$, all $\Delta, \Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$ and each sequence $\{B_N\}_{N\in\mathbb{N}} \subset \mathcal{B}(\Gamma(\hat{X})_\Delta)$ with $B_N \downarrow \varnothing$, we have (cf. (4.2.14) - (4.2.19))

$$\begin{aligned}
&\mu(B_N) = \int_{\Gamma(\hat{X})} \pi_\Lambda(B_N|\eta)\mu(d\eta)\\
\leq &\int_{\Gamma(\hat{X})} \frac{1}{T}\bigg(\int_{\hat{\mathbb{R}}^d} \mathfrak{m}_{\mathcal{U}}(\{\hat{x}\})\lambda_\theta \otimes m(d\hat{x}) + \mathfrak{m}_{\mathcal{U}}(\xi_{\Lambda^c})\bigg)\mu(d\xi) + P_\theta^\Lambda(B_N)\\
\leq &\frac{1}{T}\bigg(C_{\mathcal{U}} + \int_{\Gamma(\hat{X})} \mathfrak{m}_{\mathcal{U}}(\xi_{\Lambda^c})\mu(d\xi)\bigg) + P_\theta^\Lambda(B_N), \qquad \forall T>0,\ N\in\mathbb{N}, \quad (4.4.26)
\end{aligned}$$

where $\mathcal{U}$ is chosen as in the proof of Proposition 4.3.23. Because of (4.3.46), the above integral becomes arbitrary small as $B_N \downarrow \varnothing$ and $T \nearrow \infty$. Hence,

$$\lim_{N\to\infty} \sup_{\mu\in\mathcal{G}\mathrm{ibbs}_V} \mu(B_N) = 0,$$

which proves the required local equicontinuity. □

Corollary 4.4.22. *Let $\tilde{\mathfrak{m}}$ be a uniformly integrable (w.r.t. $\lambda_\theta \otimes m$) local mass map, then $\mathcal{G}\mathrm{ibbs}_V$ is a compact set in the topology $\mathcal{T}_{loc}$ and $\mu(\Gamma^{\mathfrak{t}}_{\tilde{\mathfrak{m}}}(\hat{X})) = 1$ for each $\mu \in \mathcal{G}\mathrm{ibbs}_V$.*

Proof. By Theorem 4.4.20 any limit point is again a Gibbs measure. Its support property follows by Corollary 4.4.18 (and Example 4.4.11). □

4.5 A modified description of Gibbs measures on $\Gamma(\hat{X})$

In this section we will present a different point of view on Gibbs measures treated in Section 4.3. The main new issue is that we do not localize in both components $\mathbb{R}_+$ and X. Instead, we define a *semi-local* (Gibbs) specifications

$$\Big\{\pi_{\mathbb{R}_+\times\Delta}(\cdot|\xi)\big|\Delta \in \mathcal{B}_c(X),\ \xi \in \Gamma_f(\hat{X})\Big\},$$

index by the ‹stripes› $\mathbb{R}_+ \times \Delta$ in $\hat{X}$. For the situation of Section 5.3, where we consider Gibbs measures on the cone $\mathbb{K}(X)$, it seems natural to consider the later kind of specification where the component in $\mathbb{R}_+$ becomes ‹invisible›.

We show that both Gibbs specifications (i.e., the local one and the *semi-local*) define via the (**DLR**) equation the same set of Gibbs measure in the basic model (cf. Theorem 4.5.9 and Example 4.1.1).

We assume throughout this section that $V : \hat{X} \times \hat{X} \to [0, \infty)$ is a potential such that Theorem 4.3.26 is satisfied with some fixed local mass map $\mathfrak{m}$.

4.5.1 Semi-local specification

Relative energy

We define the *relative energy* for all $\xi \in \Gamma_f(\hat{X})$ and $\Delta \in \mathcal{B}_c(X)$ as

$$\Gamma_f(\hat{X}) \ni \gamma \to H_{\mathbb{R}_+ \times \Delta}(\gamma|\xi) := \sum_{x,y \in \tau(\gamma)\cap\Delta} V(\hat{x}, \hat{y}) + 2 \sum_{\substack{x \in \tau(\gamma)\cap\Delta, \\ y \in \tau(\xi)\cap\Delta^c}} V(\hat{x}, \hat{y}), \quad (4.5.1)$$

where τ denotes the support of $\eta \in \Gamma_f(\hat{X})$, i.e. $\tau(\eta) := \eta_X$. Here, we take the sum over all ordered pairs $(x, y) \in (\tau(\gamma) \times \tau(\gamma)) \cap (\Delta \times \Delta)$.

Lemma 4.5.1. *For the above potential V we have*

$$H_{\mathbb{R}_+ \times \Delta}(\gamma|\xi) < \infty, \quad \forall \gamma, \xi \in \Gamma_f(\hat{X}) \text{ and } \Delta \in \mathcal{B}_c(X).$$

Proof. The proof is similar to that of Lemma 4.3.16: One crucially uses that $\gamma \in \Gamma_f(\hat{X})$ (instead of $\gamma \in \Gamma(\hat{X})$) for the finiteness of the first sum in (4.5.1) and, in addition, that the sum is taken over $x \in \tau(\gamma) \cap \mathcal{U}_\Delta$ (cf. **(FR)** and (4.3.1)) for the finiteness of the second one in (4.5.1). □

Semi-local Gibbs states

We fix an inverse temperature $\beta = 1/T > 0$ from now on. For each $\Delta \in B_c(X)$ and $\xi \in \Gamma(\hat{X})$, we define the *(semi-local) Gibbs state* with boundary condition ξ as a probability measure on $\Gamma(\mathbb{R}_+ \times \Delta)$

$$\mu_{\mathbb{R}_+ \times \Delta}(d\eta|\xi) := \begin{cases} \frac{1}{Z_{\mathbb{R}_+ \times \Delta}(\xi)} e^{-\beta H_{\mathbb{R}_+ \times \Delta}(\eta|\xi)} \mathcal{P}_\theta^{\mathbb{R}_+ \times \Delta}(d\eta), & \text{if } \xi \in \Gamma_f(\hat{X}) \\ 0, & \text{otherwise,} \end{cases} \quad (4.5.2)$$

where $\mathcal{P}_\theta^{\mathbb{R}_+ \times \Delta}$ is the Gamma-Poisson measure on $\Gamma(\mathbb{R}_+ \times \Delta)$. The normalizing constant

$$Z_{\mathbb{R}_+ \times \Delta}(\xi) := \int_{\Gamma(\mathbb{R}_+ \times \Delta)} e^{-\beta H_{\mathbb{R}_+ \times \Delta}(\eta|\xi)} \mathcal{P}_\theta^{\mathbb{R}_+ \times \Delta}(d\eta) \leq 1. \quad (4.5.3)$$

is called *partition function*. Note that $\mathcal{P}_\theta^{\mathbb{R}_+ \times \Delta}(\Gamma_f(\mathbb{R}_+ \times \Delta)) = 1$, where $\Gamma_f(\mathbb{R}_+ \times \Delta) := \{\gamma \in \Gamma_f(\hat{X}) \mid \gamma \subset \mathbb{R}_+ \times \Delta\}$.

Semi-local specification

Definition 4.5.2. *The* semi-local specification

$$\pi = \{\pi_{\mathbb{R}_+\times\Delta}(\cdot|\xi)|\Delta \in \mathcal{B}_c(X),\, \xi \in \Gamma(\hat{X})\}$$

is a family of stochastic kernels

$$\mathcal{B}(\Gamma(\hat{X})) \times \Gamma(\hat{X}) \ni (B,\xi) \mapsto \pi_{\mathbb{R}_+\times\Delta}(B|\xi) \in [0,1] \tag{4.5.4}$$

given by

$$\pi_{\mathbb{R}_+\times\Delta}(B|\xi) := \mu_{\mathbb{R}_+\times\Delta}\left(\left\{\gamma_{\mathbb{R}_+\times\Delta}\,\middle|\,\gamma_{\mathbb{R}_+\times\Delta} \cup \xi_{\mathbb{R}_+\times\Delta^c} \in B\right\}\middle|\xi\right). \tag{4.5.5}$$

Remark 4.5.3. *In our case, we have no more $Z_\Lambda(\xi) \geq 1$, since in the definition (4.5.3) we used the Poisson measure (instead of the Lebesgue-Poisson one, as in (4.1.5)). This causes an additional technical problem to control $Z_\Lambda(\xi)$ from below (cf. Lemma 4.5.4).*

In Lemma 4.5.4 and Proposition 4.5.5 we show that the semi-local specification kernels $\pi_{\mathbb{R}_+\times\Delta}(\cdot|\xi)$ are well-defined as probability measures on $\Gamma(\hat{X})$ for $\xi \in \Gamma_f(\hat{X})$ and that they can be seen as a limit of the (original) local specification kernels $\pi_{I\times\Delta}(\cdot|\xi)$, $I \in \mathcal{B}_c(\hat{X})$, as $I \nearrow \mathbb{R}_+$ (cf. Definition 4.1.5).

Lemma 4.5.4. *Suppose we are in the setting of Theorem 4.3.26. Let $\mathfrak{m}$ and $\mathfrak{m}^2$ be semi-local, i.e., (4.3.12) holds for $\mathfrak{m}^2$, or $V(\hat{x},\hat{x}) = 0$, for all $\hat{x} \in \hat{\mathbb{R}}^d$. Then the normalizing constant $Z_{\mathbb{R}_+\times\Delta}(\xi)$ is strictly positive for $\xi \in \Gamma_f(\hat{X})$. In detail, we have that for all $\Delta \in \mathcal{B}_c(X)$*

$$0 < C_{\phi,\mathfrak{m},\Delta} \leq Z_{\mathbb{R}_+\times\Delta}(\xi) \leq 1, \tag{4.5.6}$$

where

$$C_{\phi,\mathfrak{m},\Delta} := \exp\left(-2\beta\|\phi\|_\infty\left(\left(C^{(1)}_{\mathfrak{m},\Delta}\right)^2 + C^{(2)}_{\mathfrak{m},\Delta} + \mathfrak{m}_{\hat{X}}(\xi_{\mathbb{R}_+\times\mathcal{U}_\Delta})C^{(1)}_{\mathfrak{m},\Delta}\right)\right),$$

$$C^{(1)}_{\mathfrak{m},\Delta} := \int_{\mathbb{R}_+\times\Delta} \mathfrak{m}_{\hat{X}}(\{\hat{x}\})\lambda_\theta \otimes m(d\hat{x}) < \infty, \tag{4.5.7}$$

$$C^{(2)}_{\mathfrak{m},\Delta} := \int_{\mathbb{R}_+\times\Delta} (\mathfrak{m}_{\hat{X}}(\{\hat{x}\}))^2\lambda_\theta \otimes m(d\hat{x}) < \infty. \tag{4.5.8}$$

If $V(\hat{x},\hat{x}) = 0$ for all $\hat{x} \in \hat{X}$, then $C^{(2)}_{\mathfrak{m},\Delta}$ is replaced by 0. (We point out that this lower bound even holds for negative potentials, cf. Chapter 5.)

Proof. Assumption **(QL)** allows us to find $\mathbb{R}_+ \times \Delta' \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}_c(X)$ to apply the integrability condition (4.3.12). This implies that the integrals in (4.5.7) and (4.5.8) are finite.

Since the function $\mathbb{R} \ni x \to e^{-x}$ is convex, by Jensen's inequality we have a lower bound for $Z_{\mathbb{R}_+\times\Delta}(\xi)$ with any $\Delta \in \mathcal{B}_c(X)$, namely

$$\begin{aligned}
&\int_{\Gamma(\mathbb{R}_+\times\Delta)} \exp\left(-\beta H_{\mathbb{R}_+\times\Delta}(\eta|\xi)\right) \mathcal{P}_\theta^{\mathbb{R}_+\times\Delta}(d\eta) \\
&\geq \exp\left(-\beta \int_{\Gamma(\mathbb{R}_+\times\Delta)} H_{\mathbb{R}_+\times\Delta}(\eta|\xi)\mathcal{P}_\theta^{\mathbb{R}_+\times\Delta}(d\eta)\right) \\
&\geq \exp\Bigg(-\int_{\Gamma(\mathbb{R}_+\times\Delta)} \beta\|\phi\|_\infty \Bigg[\sum_{\hat{x},\hat{y}\in\eta_{\mathbb{R}_+\times\Delta}} \mathfrak{m}_{\hat{X}}(\{\hat{x}\})\mathfrak{m}_{\hat{X}}(\{\hat{y}\}) \\
&\qquad + 2\sum_{\substack{\hat{x}\in\eta_{\mathbb{R}_+\times\Delta}\\ \hat{y}\in\xi_{\mathbb{R}_+\times\mathcal{U}_\Delta}}} \mathfrak{m}_{\hat{X}}(\{\hat{x}\})\mathfrak{m}_{\hat{X}}(\{\hat{y}\})\Bigg]\mathcal{P}_{\theta,\mathbb{R}_+\times\Delta}(d\eta)\Bigg),
\end{aligned}$$

where we used (4.3.28) together with the assumption that $\|\phi\|_\infty < \infty$. So, it is enough to check that the following integral is finite:

$$\int_{\Gamma(\mathbb{R}_+\times\Delta)} (\mathfrak{m}_{\hat{X}}(\eta_{\mathbb{R}_+\times\Delta}))^2 + \mathfrak{m}_{\hat{X}}(\eta_{\mathbb{R}_+\times\Delta})\, \mathfrak{m}_{\hat{X}}(\xi_{\mathbb{R}_+\times\mathcal{U}_\Delta})\mathcal{P}_\theta^{\mathbb{R}_+\times\Delta}(d\eta).$$

This follows by Proposition 4.3.21:

$$\begin{aligned}
&\int_{\Gamma(\mathbb{R}_+\times\Delta)} (\mathfrak{m}_{\hat{X}}(\eta_{\mathbb{R}_+\times\Delta}))^2\mathcal{P}_{\mathbb{R}_+\times\Delta}(d\eta) \leq \left(C^{(1)}_{\mathfrak{m},\Delta}\right)^2 + C^{(2)}_{\mathfrak{m},\Delta} < \infty, \qquad (4.5.9)\\
&\int_{\Gamma(\mathbb{R}_+\times\Delta)} \mathfrak{m}_{\hat{X}}(\eta_{\mathbb{R}_+\times\Delta})\mathcal{P}_{\mathbb{R}_+\times\Delta}(d\eta) \leq C^{(1)}_{\mathfrak{m},\Delta} < \infty, \quad \textit{and}\\
&\mathfrak{m}_{\hat{X}}(\xi_{\mathbb{R}_+\times\mathcal{U}_\Delta}) < \infty \qquad \text{for } \xi \in \Gamma_f(\hat{X}).
\end{aligned}$$

If $V(\hat{x},\hat{x}) = 0$ for all $\hat{x} \in \hat{X}$, then we may drop $C^{(2)}_{\mathfrak{m},\Delta}$ because of (4.3.24). □

Thus, for each $\xi \in \Gamma_f(\hat{X})$, $\pi_{\mathbb{R}_+\times\Delta}(d\eta|\xi)$ is well-defined as a probability measure on $\Gamma(\hat{X})$. Now we show that the semi local specification $\pi_{\mathbb{R}_+\times\Delta}(d\eta|\xi)$ can be constructed as a limit of the local kernels $\pi_\Lambda(d\eta|\xi)$, $\Lambda \in \mathcal{B}_c(\mathbb{R}_+\times\Delta)$

Proposition 4.5.5. *Under the assumptions of Lemma 4.5.4, we have*

$$\pi_{\mathbb{R}_+\times\Delta}(d\eta|\xi) = \tau_{loc} - \lim_{\substack{\Lambda\nearrow\mathbb{R}_+\times\Delta\\ \Lambda\in\mathcal{B}_c(\mathbb{R}_+\times\Delta)}} \pi_\Lambda(d\eta|\xi), \qquad \xi \in \Gamma_f(\hat{X}), \tag{4.5.10}$$

where the limit is independent of the net; and thus it is unique.

Proof. Fix $\xi \in \Gamma_f(\hat{X})$. The net $\{\pi_\Lambda(\cdot|\xi) \mid \Lambda \in \mathcal{B}_c(\mathbb{R}_+ \times \Delta),\ \Lambda \nearrow \mathbb{R}_+ \times \Delta\}$ is local equicontinuous (see (4.2.17), (4.2.19) and (4.2.18) in the proof of Proposition 4.3.23). Hence, there exists a $\mathcal{T}_{\mathrm{loc}}$-limit probability measure

$$\tilde{\pi}_{\mathbb{R}_+ \times \Delta}(\cdot|\xi) := \mathcal{T}_{\mathrm{loc}} - \lim_{\substack{\Lambda_N \nearrow \mathbb{R}_+ \times \Delta \\ \Lambda_N \in \mathcal{B}_c(\mathbb{R}_+ \times \Delta)}} \pi_\Lambda(\cdot|\xi)$$

along some sequence $\Lambda_N \nearrow \mathbb{R}_+ \times \Delta$. It remains to show that $\tilde{\pi}_{\mathbb{R}_+ \times \Delta}(B|\xi) \stackrel{!}{=} \pi_{\mathbb{R}_+ \times \Delta}(B|\xi)$ for all $B \in \mathcal{B}_{\mathrm{cyl}}(\Gamma_f(X))$. Indeed, there exists $\Lambda_0 \in \mathcal{B}_c(\hat{X})$ such that $B \subset \mathcal{B}_{\Lambda_0}(\Gamma(\hat{X}))$. We have

$$\begin{aligned}
&\tilde{\pi}_{\mathbb{R}_+ \times \Delta}(B|\xi) = \lim_{N\to\infty} \pi_{\Lambda_N}(B|\xi) = \lim_{N\to\infty} \mu_{\Lambda_N}(B_{\Lambda_N,\xi}|\xi) \\
&= \lim_{N\to\infty} \frac{e^{\lambda_\theta \otimes m(\Lambda_N)}}{Z_{\Lambda_N}(\xi)} \int_{\Gamma(\Lambda_N)} 1\!\!1_{B_{\Lambda_N,\xi}}(\eta_{\Lambda_N}) e^{-\beta H_{\Lambda_N}(\eta_{\Lambda_N}|\xi)} \underbrace{e^{-\lambda_\theta \otimes m(\Lambda_N)} P_\theta^{\Lambda_N}(d\eta_{\Lambda_N})}_{\mathcal{P}_\theta^{\Lambda_N}(d\eta_{\Lambda_N})} \\
&= \lim_{N\to\infty} \frac{\int_{\Gamma(\hat{X})} 1\!\!1_{B_{\Lambda_N,\xi}}(\eta) e^{-\beta H_{\Lambda_N}(\eta_{\Lambda_N}|\xi)} \mathcal{P}_\theta(d\eta)}{\int_{\Gamma(\hat{X})} 1\!\!1_{\Gamma(\Lambda),\xi}(\eta) e^{-\beta H_{\Lambda_N}(\eta_{\Lambda_N}|\xi)} \mathcal{P}_\theta(d\eta)} \\
&= \frac{\int_{\Gamma(\hat{X})} 1\!\!1_{B_{\mathbb{R}_+ \times \Delta,\xi}}(\eta) e^{-\beta H_{\mathbb{R}_+ \times \Delta}(\eta_{\mathbb{R}_+ \times \Delta}|\xi)} \mathcal{P}_\theta(d\eta)}{\int_{\Gamma(\hat{X})} 1\!\!1_{\Gamma(\Lambda),\xi}(\eta) e^{-\beta H_{\mathbb{R}_+ \times \Delta}(\eta_{\mathbb{R}_+ \times \Delta}|\xi)} \mathcal{P}_\theta(d\eta)}.
\end{aligned} \tag{4.5.11}$$

By Lebesgue's dominated convergence theorem we may take the limit inside the integrals in the last line in (4.5.11). Therefore, the claim is proved, i.e., for any $B \in \mathcal{B}_{\mathrm{cyl}}(\Gamma(\hat{X}))$ and $\xi \in \Gamma_f(\hat{X})$,

$$\tilde{\pi}_{\mathbb{R}_+ \times \Delta}(B|\xi) = \pi_{\mathbb{R}_+ \times \Delta}(B|\xi).$$

□

Proposition 4.5.6. *Under the assumptions of Lemma 4.5.4, the family of semi-local specification kernels* $\{\pi_{\mathbb{R}_+ \times \Delta}\}_{\Delta \in \mathcal{B}_c(X)}$ *(cf. Definition 4.5.2) obeys the* consistency property, *which means that for all* $B \in \mathcal{B}(\Gamma(\hat{X}))$ *and* $\xi \in \Gamma(\hat{X})$

$$\int_{\Gamma(\hat{X})} \pi_{\mathbb{R}_+ \times \Delta}(B|\gamma) \pi_{\mathbb{R}_+ \times \Delta'}(\mathrm{d}\gamma|\xi) = \pi_{\mathbb{R}_+ \times \Delta'}(B|\xi), \tag{4.5.12}$$

where $\Delta, \Delta' \in \mathcal{B}_c(X)$ *are such that* $\Delta \subset \Delta'$.

Proof. If $\xi \notin \Gamma_f(\hat{X})$, then the assertion is clear. Hence, w.l.o.g. $\xi \in \Gamma_f(\hat{X})$. Using Proposition 4.5.5 twice we get

$$\begin{aligned}
&\int_{\Gamma(\hat{X})} \pi_{\mathbb{R}_+\times\Delta}(B|\gamma)\pi_{\mathbb{R}_+\times\Delta'}(\mathrm{d}\gamma|\xi) \\
&= \lim_{\substack{\Lambda\nearrow\mathbb{R}_+\times\Delta \\ \Lambda\in\mathcal{B}_c(\mathbb{R}_+\times\Delta)}} \lim_{\substack{\Lambda'\nearrow\mathbb{R}_+\times\Delta' \\ \Lambda'\in\mathcal{B}_c(\mathbb{R}_+\times\Delta')}} \int_{\Gamma_f(\hat{X})} \pi_\Lambda(B|\gamma)\pi_{\Lambda'}(\mathrm{d}\gamma|\xi) \\
&= \lim_{\substack{\Lambda'\nearrow\mathbb{R}_+\times\Delta' \\ \Lambda'\in\mathcal{B}_c(\mathbb{R}_+\times\Delta')}} \pi_{\Lambda'}(B|\xi) = \pi_{\mathbb{R}_+\times\Delta'}(B|\xi).
\end{aligned} \tag{4.5.13}$$

Here we applied Lebesgue's dominated convergence theorem three times and the consistency of $\{\pi_\Lambda(\cdot|\xi)|\Lambda \in B_c(\hat{X}), \xi \in \Gamma_f(\hat{X})\}$ (cf. (4.1.9)) once. □

4.5.2 A modified concept of Gibbs measures

Now we are in position to define a modified concept of Gibbs measures on $\Gamma(\hat{X})$.

Definition 4.5.7. *Let the assumptions of Lemma 4.5.4 be fulfilled. A probability measure μ on $\Gamma(\hat{X})$ is called a **Gibbs measure** (or **state**) with pair potential V and inverse temperature $\beta > 0$ if it satisfies the* Dobrushin-Lanford-Ruelle ***(DLR)*** *equilibrium equation*

$$\int_{\Gamma(\hat{X})} \pi_{\mathbb{R}_+\times\Delta}(B|\eta)\mu(\mathrm{d}\eta) = \mu(B) \tag{4.5.14}$$

for all $\Delta \in \mathcal{B}_c(X)$ and $B \in \mathcal{B}(\Gamma(\hat{X}))$. Fixed an inverse temperature β, the associated set of all Gibbs states will be denoted by $\mathcal{G}\mathrm{ibbs}_{V,s}(\Gamma(\hat{X}))$.

Remark 4.5.8. *From the definition of the local specification (cf. (4.5.5) and (4.5.2)), we have that any solution of the* (**DLR**) *equation is supported by $\Gamma_f(\hat{X})$.*

To obtain the (**DLR**) *equation it is enough to check (4.5.14) only for $B \in \mathcal{B}_{\mathrm{cyl}}(\Gamma(\hat{X}))$. Indeed, using Caratheodory's theorem, we deduce that $\mu\big|_{\mathcal{B}_{\mathrm{cyl}}(\Gamma(\hat{X}))}$ extends uniquely to a measure on $\sigma(\mathcal{B}_{cyl}\Gamma(\hat{X})) = \mathcal{B}(\Gamma(\hat{X}))$. Hence, (4.5.14) holds for all $B \in \mathcal{B}(\Gamma(\hat{X}))$.*

Whenever it is clear on which space the Gibbs measure is considered, we write $\mathcal{G}\mathrm{ibbs}_{V,s}$.

Theorem 4.5.9. *Under the assumptions of Lemma 4.5.4, we have that*

$$\mu \in \mathcal{G}\text{ibbs}_{V,s} \quad \Longleftrightarrow \quad \mu \in \mathcal{G}\text{ibbs}_V. \tag{4.5.15}$$

In particular, this holds in the basic model.

Proof. If $\mu \in \mathcal{G}\text{ibbs}_V$, then for all $B \in \mathcal{B}_{\text{cyl}}(\Gamma(\hat{X}))$, $\Delta \in \mathcal{B}_c(X)$ and $\Lambda \in \mathcal{B}_c(\hat{X})$

$$\begin{aligned}\mu(B) &= \int_{\Gamma(\hat{X})} \pi_\Lambda(B|\eta)\mu(d\eta) \\ &= \lim_{\Lambda \nearrow \mathbb{R}_+ \times \Delta} \int_{\Gamma_f(\hat{X})} \pi_\Lambda(B|\eta)\mu(d\eta) = \int_{\Gamma(\hat{X})} \pi_{\mathbb{R}_+ \times \Delta}(B|\eta)\mu(d\eta),\end{aligned}$$

where we applied Proposition 4.5.5. This shows that $\mu \in \mathcal{G}\text{ibbs}_{V,s}$.

Conversely, let $\mu \in \mathcal{G}\text{ibbs}_{V,s}$. We have the following chain of equations for all $B \in \mathcal{B}_{\text{cyl}}(\Gamma(\hat{X}))$, $\Lambda, \Lambda' \in \mathcal{B}_c(\hat{X})$ and $\Delta \in \mathcal{B}_c(X)$ with $\Lambda' \subset \Lambda \subset \mathbb{R}_+ \times \Delta$:

$$\begin{aligned}\mu(B) &\overset{1.}{=} \int_{\Gamma(\hat{X})} \pi_{\mathbb{R}_+ \times \Delta}(B|\eta)\mu(d\eta) \\ &\overset{2.}{=} \lim_{\Lambda \nearrow \mathbb{R}_+ \times \Delta} \int_{\Gamma_f(\hat{X})} \pi_\Lambda(B|\eta)\mu(d\eta) \\ &\overset{3.}{=} \lim_{\Lambda \nearrow \mathbb{R}_+ \times \Delta} \int_{\Gamma_f(\hat{X})} \int_{\Gamma_f(\hat{X})} \pi_{\Lambda'}(B|\eta')\pi_\Lambda(d\eta'|\eta)\mu(d\eta) \\ &\overset{2.}{=} \int_{\Gamma_f(\hat{X})} \int_{\Gamma_f(\hat{X})} \pi_{\Lambda'}(B|\eta')\pi_{\mathbb{R}_+ \times \Delta}(d\eta'|\eta)\mu(d\eta) \\ &\overset{4.}{=} \int_{\Gamma(\hat{X})} \pi_{\Lambda'}(B|\eta)\mu(d\eta),\end{aligned}$$

which implies the assertion. Here, the argument labeled: 1. is that $\mu \in \mathcal{G}\text{ibbs}_{V,s}$ (cf. Definition 4.5.7), 2. is Proposition 4.5.5 and 3. is (4.1.9). □

The *semi-local* specification kernels can be considered themselves as Gibbs measures in unbounded volumes.

Definition 4.5.10. *Let $\Lambda' \in \mathcal{B}(\hat{X})$. A probability measure $\mu \in \mathcal{M}^1(\Gamma(\hat{X}))$ (with full measure on $\Gamma_f(\hat{X})$) is a Λ'-**Gibbs measure** (or **state**) with pair potential V if it satisfies the (**DLR**) equation*

$$\int_{\Gamma_f(\hat{X})} \pi_{\Lambda_0}(B|\eta)\mu(\mathrm{d}\eta) = \mu(B) \tag{4.5.16}$$

valid for all $\Lambda_0 \in \mathcal{B}_c(\Lambda')$ and $B \in \mathcal{B}(\Gamma(\hat{X}))$. Fixed an inverse temperature β, the associated set of all Gibbs states will be denoted by $\mathcal{G}\text{ibbs}_{V,\Lambda'}(\Gamma(\hat{X}))$.

Proposition 4.5.11. *Let the assumptions of Lemmas 4.3.29 and 4.5.4 hold and $(\mu_N)_{N\in\mathbb{N}} \in \mathcal{M}^1(\Gamma(\hat{X}))$ be such that $\mu_N(\Gamma_f(\hat{X})) = 1$ and (4.3.39) hold for each μ_N with $\xi \in \Gamma_f(\hat{X})$. Assume that there exists $\Lambda' \in \mathcal{B}(\hat{X})$ such that for all $\Lambda \in \mathcal{B}_c(\Lambda')$ and all $B \in \mathcal{B}_{cyl}(\Gamma_f(X))$ we find $N_0 > 0$ with*

$$\int_{\Gamma(\hat{X})} \pi_\Lambda(B|\gamma)\mu_N(d\gamma) = \mu_N(B) \qquad \forall N > N_0. \tag{4.5.17}$$

If μ is the $\mathcal{T}_{loc}$ limit of the sequence $\left(\mu_N\right)_{N\in\mathbb{N}}$, then:

1. *The estimate (4.3.38) also holds for μ, which implies that μ is supported by $\Gamma_f(\hat{X})$;*

2. *The (**DLR**) equation (4.5.16) holds also for the limit μ.*

Proof. The proof follows that one of Proposition 4.3.30 with obvious modifications. Namely, we fix $\Lambda \in \mathcal{B}_c(\Lambda')$ instead of $\Delta \in \mathcal{B}_c(\hat{X})$ and only obtain equation (4.5.16) instead of the (**DLR**) one. □

Corollary 4.5.12. *Let $\Delta \in \mathcal{B}_c(X)$ and $\xi \in \Gamma_f(\hat{X})$ and let the assumptions of Lemma 4.5.4 be fulfilled. Then the local specification kernel $\pi_{\mathbb{R}_+\times\Delta}(\cdot|\xi)$ is a $(\mathbb{R}_+ \times \Delta)$-Gibbs measure.*

Proof. This follows by Proposition 4.5.11. □

Chapter 5

Gibbsian measure for general potentials

In this chapter we consider the existence problem for Gibbs measures corresponding to general, not necessarily translation invariant or non-negative potentials $V : \hat{\mathbb{R}}^d \times \hat{\mathbb{R}}^d \to \mathbb{R}$ with infinite interaction range in $\hat{\mathbb{R}}^d$. In particular, we construct Gibbs measures μ_Γ on $\Gamma(\hat{\mathbb{R}}^d)$ whose image measure $\mu_{\mathbb{K}}$ on the cone $\mathbb{K}(\mathbb{R}^d)$ are Gibbs perturbations of a Gamma measure $\mathcal{G}_\theta$ (cf. Section 5.3). The later can be seen as a main result of Part II.

A main (technical) achievement of this chapter is to remove the assumption $V \geq 0$ (cf. Chapter 4). Instead of this, we have to impose some stability properties on V (cf. Subsection 5.1.3 for the precise formulation). Merely speaking, we assume that the repulsion part V^+ of the potential V dominates its attractive part V^-.

As before, we use the DLR approach to define the related set $\mathcal{G}\mathrm{ibbs}_V^{\mathfrak{t}}(\Gamma(\hat{\mathbb{R}}^d))$ of ‹tempered› Gibbs measures (cf. Section 5.1). To construct such Gibbs measures (cf. Theorem 5.2.8), we introduce certain Lyapunov functionals and establishing the weak dependence of Gibbs specification kernels on boundary conditions. For the existence proof, Proposition 5.2.4 is essential. There where we get a uniform bound (as $\Lambda \nearrow \hat{\mathbb{R}}^d$) for the exponential integral of a *Lypunov functional* w.r.t. the local specification kernels π_Λ. This enables us to prove, for a large class of boundary conditions $\xi \in \Gamma(\hat{\mathbb{R}}^d)$, the local equicontinuity of the specification kernels $(\pi_\Lambda(d\gamma|\xi))_{\Lambda\in\mathcal{B}_c(\hat{\mathbb{R}}^d)}$ (cf. Definition 5.1.16), which implies their tightness in a proper topology (cf. Proposition 5.2.7). As a last step in the existence proof, we show that all cluster points μ_Γ of the Gibbs specification $\{\pi_\Lambda\}$ (as $\Lambda \nearrow \hat{\mathbb{R}}^d$) are Gibbs, i.e., $\mu_\Gamma \in \mathcal{G}\mathrm{ibbs}_V^{\mathfrak{t}}(\Gamma(\hat{\mathbb{R}}^d))$ (cf. (5.2.20)). A second result is a uniform moment bound (cf. Theorem 5.2.10) for all ‹tempered› Gibbs measures (cf. (5.1.36)).

As in Chapter 4, we first consider a basic model (cf. Subsection 5.1.1), and then extend it to more general potentials with non-trivial interaction of the marks and positions. The most general setup is fixed in Subsection 5.1.3.

In Section 5.3 we come back to the cone $\mathbb{K}(\mathbb{R}^d)$. We obtain Gibbs measures $\mu_{\mathbb{K}}$ on $\mathbb{K}(\mathbb{R}^d)$ as image measures under the homeomorphism $\mathbb{T}$ (cf. (3.1.2)) of a particular class of Gibbs measures μ_Γ on $\Gamma(\hat{\mathbb{R}}^d)$. By this transition, we can easily reformulate the main results obtained in Chapter 4 and 5 in the Gibbsian framework on $\mathbb{K}(\mathbb{R}^d)$. Among them are the existence of $\mu \in \mathcal{G}\mathrm{ibbs}_\phi(\mathbb{K}(X))$ (cf. Theorem 5.3.7) and uniform moment bounds (cf. Theorem 5.3.10).

Using these results in Part III, we derive integration by parts formulas for Gibbs measures on $\mathbb{K}(\mathbb{R}^d)$ and study related Dirichlet forms and operators (cf. Chapter 6), which are then used to construct associated diffusions on $\mathbb{K}(\mathbb{R}^d)$ in Chapter 7.

With the technique developed in Sections 5.1 and 5.2 we handle more general potentials than in the basic model with $V(\hat{x}, \hat{y}) = s_x \phi(x, y) s_y$ (see the corresponding results in Theorem 5.2.8 and 5.2.10).

5.1 Gibbsian formalism on $\Gamma(\hat{X})$

As before, let X be a locally compact Polish space equipped with a non-atomic Radon measure m and let λ be a Radon measure on $\mathbb{R}_+$. Fix a symmetric pair potential $V : \hat{X} \times \hat{X} \to \mathbb{R}$, i.e.,

$$V(\hat{x}, \hat{y}) = V(\hat{y}, \hat{x}), \quad \forall \hat{x}, \hat{y} \in \hat{X}, \tag{5.1.1}$$

that can be written as

$$V(\hat{x}, \hat{y}) = \mathfrak{l}(\hat{x})\mathfrak{l}(\hat{y})\phi(x, y), \quad \forall \hat{x}, \hat{y} \in \hat{X}, \tag{5.1.2}$$

where $\mathfrak{l} : \hat{X} \to [0, \infty)$ is a $\mathcal{B}(\hat{X})$-measurable function and $\phi : \hat{X} \times \hat{X} \to \mathbb{R}$ a *bounded* and $\mathcal{B}(\hat{X} \times \hat{X})$-measurable one. Their exact properties will be specified below.

5.1.1 The potential in the basic model

We impose the following assumptions on the potential V:

($\mathbf{V}'$) *Let the space of positions X be $\mathbb{R}^d$ with $d \in \mathbb{N}$ and m be the Lebesgue measure on $\mathbb{R}^d$. The intensity measure on the marks $\mathbb{R}_+$ is $\lambda = \lambda_\theta$. Moreover, we fix the semi-local function*

$$\mathfrak{l}(\hat{x}) = s_x \qquad \textit{and the potential} \qquad \phi \in L_s(\mathbb{R}^d \times \mathbb{R}^d)$$

such that the following conditions hold:

($\mathbf{FR}'$) ***Finite range:*** *There exists $R \in (0, \infty)$ such that*

$$\phi(x, y) = 0 \quad \text{if } |x - y| > R. \tag{5.1.3}$$

($\mathbf{LB}'$) ***Lower boundedness***:

$$\inf_{x,y \in \mathbb{R}^d} \phi(x, y) =: -M > -\infty. \tag{5.1.4}$$

($\mathbf{RC}'$) ***Repulsion condition**: There exists $\delta > 0$ such that*

$$\inf_{\substack{x,y \in \mathbb{R}^d \\ |x-y| \le \delta}} \phi(x, y) > A_\delta := 4Mm_\delta^{\text{int}}, \tag{5.1.5}$$

with interaction parameter (cf. (5.1.17)) below

$$m_\delta^{\text{int}} := \left(\frac{R}{\delta} + 2\right)^d. \tag{5.1.6}$$

We remark that neither *translation invariance* nor *continuity* of V is assumed.

5.1.2 Partition of the space $\hat{X}$

We will introduce a partition of the space $\hat{X}$ (cf. Example 5.1.1 for $\hat{X} = \hat{\mathbb{R}}^d$) for a better understanding of the above conditions. This partition turns out to be very helpful for the later proofs.

Let us consider a countable index set Z and a *partition*

$$\hat{X} = \bigsqcup_{k \in Z} \hat{Q}_k \tag{5.1.7}$$

of the phase space $\hat{X}$ into ‹elementary› sets $\hat{Q}_k \in \mathcal{B}(\hat{X})$, $k \in Z$.

Example 5.1.1 (Basic model). *Recall that in this case* $V : \hat{\mathbb{R}}^d \times \hat{\mathbb{R}}^d \to \mathbb{R}$ *with*

$$V(\hat{x}, \hat{y}) = s_x s_y \phi(x, y).$$

Let us choose the parameter $g := \delta/\sqrt{d}$ *with some* $\delta > 0$ *satisfying assumption* (**RC**$'$). *For each* k *in the index set* $Z := \mathbb{Z}^d$, *we define the strip*

$$\begin{aligned} \hat{Q}_k = & \hat{Q}_{g,k} := \mathbb{R}_+ \times Q_{g,k}, \ \textit{where} \\ Q_{g,k} := & \left[-\frac{1}{2} g, \frac{1}{2} g \right[^d + kg \subset \mathbb{R}^d. \end{aligned} \tag{5.1.8}$$

The cubes $Q_{g,k}$ *have edge length* $g > 0$, *are centered at the points* gk, $k \in \mathbb{Z}^d$ *and*

$$\operatorname{diam}(Q_{g,k}) := \sup_{\hat{x}, \hat{y} \in \hat{Q}_{g,k}} |x - y|_{\mathbb{R}^d} = \delta. \tag{5.1.9}$$

This implies that $\phi(x, y) \geq A_\delta$ *for all* $\hat{x}, \hat{y} \in \gamma_{\hat{Q}_{g,k}}$. *Moreover,*

$$\sup_{k \in Z} \int_{\hat{Q}_{g,k}} \left(s_x + s_x^2 \right) (\lambda_\theta \otimes m)(d\hat{x}) < \infty. \tag{5.1.10}$$

Let us explain the choice of the constant m_δ in (5.1.6). To this end, let us introduce some more concepts and notation:

For $k \in Z$ and $\gamma \in \Gamma(\hat{X})$, we define

$$\Gamma_k := \Gamma(\hat{Q}_k), \qquad \text{and} \qquad \gamma_k := \gamma \cap \hat{Q}_k. \tag{5.1.11}$$

To each finite index set $\mathcal{K} \Subset Z$ there corresponds[1]

$$\Lambda_{\mathcal{K}} := \bigsqcup_{k \in \mathcal{K}} \hat{Q}_k \in \mathcal{B}(\hat{X}); \tag{5.1.12}$$

the family of all such domains will be denoted by $\mathcal{Q}_c(\hat{X})$. For $\Lambda \in \mathcal{B}(\hat{X})$

$$\mathcal{K}_\Lambda := \{ j \in Z \mid \hat{Q}_j \cap \Lambda \neq \varnothing \}, \tag{5.1.13}$$

i.e. $|\mathcal{K}_\Lambda|$ is the number of partition sets $\hat{Q}_k$ having non-void intersection with Λ. We note that in our example setting

$$|\mathcal{K}_\Lambda| < \infty, \qquad \forall \Lambda \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}_c(X). \tag{5.1.14}$$

[1] Here, $\mathcal{K} \Subset Z$ means that $\mathcal{K}$ is a *finite* subset of Z.

For a given volume $\Lambda \in \mathcal{B}(\hat{X})$ with $|\mathcal{K}_\Lambda| < \infty$ we can construct its *'minimal'* covering

$$\Lambda_{\mathcal{Q}} := \bigsqcup_{k \in \mathcal{K}_\Lambda} \hat{Q}_k \in \mathcal{Q}_c(\hat{X}). \tag{5.1.15}$$

For each $k \in Z$, the family of '*neighbor*' partition sets of $\hat{Q}_k$, i.e., those partition sets $\hat{Q}_j$ having a point $y \in \hat{Q}_j$ that interacts with a point $x \in \hat{Q}_k$, is indexed by

$$\partial^{\text{int}} k := \left\{ j \in Z \mid \exists x \in \hat{Q}_k, \exists y \in \hat{Q}_j : \quad \phi(x,y) \neq 0 \right\}. \tag{5.1.16}$$

The number of interacting '*neighbor*' partition sets for each $\hat{Q}_k$, $k \in Z$, is dominated by

$$m^{\text{int}} := \sup_{k \in Z} |\partial^{\text{int}} k| \leq \infty. \tag{5.1.17}$$

In our example setting, we have roughly estimated them by m_δ^{int} defined in (5.1.6).

5.1.3 A potential V in the general framework

We outline a framework to handle general potentials

$$V : \hat{X} \times \hat{X} \to \mathbb{R}.$$

We denote from now on by $\mathcal{P}_\lambda$ (analogously to (2.2.11)) the Poisson measure with intensity measure $\lambda \otimes m$ on the configuration space $\Gamma(\hat{X})$, respectively $\mathcal{P}_\lambda^\Lambda$ on $\Gamma(\Lambda)$ for all $\Lambda \in \mathcal{Q}_c(\hat{X})$.

Picking a proper partition of X

It is convenient for later references to summarize which properties of the underlying partition and measure space are crucial in the proofs. In particular, in the basic example such a proper partition has been constructed by strips (cf. Example 5.1.1).

Definition 5.1.2. *A partition*

$$\hat{X} = \mathbb{R}_+ \times X = \bigsqcup_{k \in Z} \hat{Q}_k, \tag{5.1.18}$$

with ‹elementary› sets $\hat{Q}_k \in \mathcal{B}(\hat{X})$ indexed by a at most countable set $Z = \{k\}$, is said to be admissible *if the following condition holds:*

(**AP**) ***Admissible partition:***

$$M_1 := \sup_{k\in Z} \int_{\hat{Q}_k} \mathfrak{l}(\hat{x})\lambda\otimes m(d\hat{x}) < \infty, \tag{5.1.19}$$

$$M_2 := \sup_{k\in Z} \int_{\hat{Q}_k} \mathfrak{l}(\hat{x})^2\lambda\otimes m(d\hat{x}) < \infty, \tag{5.1.20}$$

$$m^{\mathrm{int}} = \sup_{k\in Z} |\partial^{\mathrm{int}} k| < \infty. \tag{5.1.21}$$

Moreover, the index set Z can be equipped with a metric $\rho : Z\times Z \to [0,\infty)$ such that

$$m_{\alpha,k_0} := \sum_{k\in Z} e^{-\alpha\rho(k,k_0)} < \infty, \qquad \text{for all } \alpha > 0 \text{ and } k_0 \in Z. \tag{5.1.22}$$

Remark 5.1.3. *1. We point out that (5.1.21) yields that $\mathcal{U}_\Lambda \in \mathcal{Q}_c(\hat{X})$ for all $\Lambda \in \mathcal{Q}_c(\hat{X})$ where $\mathcal{U}_\Lambda := \bigcup_{k\in Z} \left\{ \hat{Q}_k \;\middle|\; \partial^{\mathrm{int}} k \cap \mathcal{K}_\Lambda \neq \varnothing \right\}$.*

2. The classical situation of Gibbs measures over a configuration space $\Gamma(\hat{X})$ is included in this scheme by picking λ to be the Dirac measure in 1 and $\mathfrak{l}(\hat{x}) = 1$.[2]

3. Similarly, the case of a marked configuration space with a finite *measure λ is covered by choosing $\mathfrak{l}(\hat{x}) = s_x$ and the partition sets having the form $\mathbb{R}_+ \times \Delta$ with $\Delta \in \mathcal{B}(X)$.*

4. An idea how condition (5.1.22) could be relaxed is given in [KP07, Subsection 2.4].

A supporting set of the Poisson measure $\mathcal{P}_\lambda$

Definition 5.1.4. *If* (**AP**) *without (5.1.20) holds, then $\mathfrak{l}$ is a $\mathcal{Q}$-*local *function. In this case, we define a $\mathcal{Q}$-*local *(w.r.t. $\lambda\otimes m$)* mass $\mathfrak{m}$ *by*

$$\mathfrak{m}_\Lambda(\gamma) = \sum_{\hat{x}\in\gamma\cap\Lambda} \mathfrak{l}(\hat{x}), \quad \forall \Lambda \in \mathcal{Q}_c(\hat{X}) \cup \{\hat{X}\}, \tag{5.1.23}$$

and the set of pinpointing configurations with $\mathcal{Q}$-local mass $\mathfrak{m}$ *by*

$$\Gamma_f(\hat{X}) := \Gamma_{f,\mathfrak{m},\mathcal{Q}}(\hat{X}) := \left\{ \gamma \in \Gamma_p(\hat{X}) \;\middle|\; \mathfrak{m}_\Lambda(\gamma) < \infty, \quad \forall \Lambda \in \mathcal{Q}_c(\hat{X}) \right\}. \tag{5.1.24}$$

[2]That the measure is supported by $\Gamma(X)$ follows by Corollary 5.1.7.

Remark 5.1.5. *The last definition extends the idea of a semi-local mass map (cf. Definition 4.3.8) and of a pinpointing set with finite local mass (cf. Definition 4.3.12).*

If the partition is fixed, we omit the index $\mathcal{Q}$ in the corresponding notations. Moreover, we call the corresponding objects for short local.

Theorem 5.1.6. *If* (**AP**) *holds, then $\Gamma_f(\hat{X}) \in \mathcal{B}(\Gamma(\hat{X}))$ and*

$$\mathcal{P}_\lambda(\Gamma_f(\hat{X})) = 1. \tag{5.1.25}$$

Proof. The claim follows with obvious changes from the proof of Remark 4.3.7 and Theorem 4.3.13. □

Corollary 5.1.7. *Let $\hat{A} \in \mathcal{B}(\hat{X})$ be negligible for the measure $\lambda \otimes m$. Then the set of configurations* not touching $\hat{A}$ *has full probability w.r.t. the Poisson measure $\mathcal{P}_\lambda$, i.e.,*

$$\mathcal{P}_\lambda\left(\left\{\gamma \in \Gamma(\hat{X}) \,\middle|\, \gamma \subset \hat{A}^c\right\}\right) = 1. \tag{5.1.26}$$

Proof. Applying the appropriate version of Proposition 4.3.18 for the subadditive local mass map $1\!\!1_{\{\gamma\in\Gamma(\hat{X})|\gamma\subset\hat{A}^c\}}$ and performing the usual limit procedure, we obtain the assertion. □

Definition 5.1.8. *Let* (**AP**) *hold. By $L_{s,\mathcal{Q}}(\hat{X} \times \hat{X})$ we denote the set of* bounded *symmetric $\mathcal{B}(\hat{X} \times \hat{X})$-measurable functions ϕ over $\hat{X} \times \hat{X}$ obeying*

(**FR**) ***Finite range:*** *For all $\Lambda \in \mathcal{Q}_c(\hat{X})$, there exists $\mathcal{U}_\Lambda \in \mathcal{Q}_c(\hat{X})$ with*

$$\phi(\hat{x}, \hat{y}) = \phi(y, x) = 0 \quad \forall \hat{x} \in \Lambda,\ \hat{y} \in \mathcal{U}_\Lambda^c. \tag{5.1.27}$$

Remark 5.1.9. *In the basic example, i.e. $X = \mathbb{R}^d$, the* (**FR**) *condition reads as*

$$\exists R \in [0, \infty): \quad \forall \hat{x}, \hat{y} \in \hat{\mathbb{R}}^d: \quad |x - y| > R \ \Rightarrow\ \phi(\hat{x}, \hat{y}) = 0. \tag{5.1.28}$$

In this case, we set for $\Lambda \in \mathcal{Q}_c(\hat{\mathbb{R}}^d)$ (cf. (5.1.15))

$$\mathcal{U}_\Lambda := \bigsqcup_{k\in Z} \{\hat{Q}_k \mid \hat{Q}_k \cap \Lambda_R \neq \varnothing\} \in \mathcal{Q}_c(\hat{\mathbb{R}}^d), \tag{5.1.29}$$

where $\Lambda_R := \left\{\hat{x} \in \hat{\mathbb{R}}^d \ \mid \mathrm{dist}_{\mathbb{R}^d}(x, \Lambda_{\mathbb{R}^d}) := \inf_{\hat{y}\in\Lambda} |x - y| \leq R\right\}.$

Using these concepts, we specify the conditions on the pair interaction potential:

(**V**) ***Potential***: *Let* (**AP**) *hold for the potential V defined in (5.1.1) via*

$$V(\hat{x}, \hat{y}) = \mathfrak{l}(\hat{x}) \cdot \mathfrak{l}(\hat{y})\phi(\hat{x}, \hat{y}), \quad \forall \hat{x}, \hat{y} \in \hat{X}. \tag{5.1.30}$$

Suppose that for the bounded function $\phi \in L_{s,\mathcal{Q}}(\hat{X} \times \hat{X})$ the two conditions hold:

(**LB**) ***Lower boundedness***: *For some $M \geq 0$*

$$\inf_{\hat{x}, \hat{y} \in \hat{X}} \phi(\hat{x}, \hat{y}) \geq -M. \tag{5.1.31}$$

(**RC**) ***Repulsion condition***: *For some $A_0 \geq 0$:*

$$\inf_{k \in Z} \inf_{\hat{x}, \hat{y} \in \hat{Q}_k} \phi(\hat{x}, \hat{y}) =: A_0. \tag{5.1.32}$$

Furthermore, we assume the following relation between the constants in (**LB**) *and in* (**RC**)*:*

$$A := 4Mm^{\text{int}} < A_0 \tag{5.1.33}$$

where m^{int} was defined in (5.1.17). We also include the particular case $M = 0$ by setting $A = 0$ and say that (**V**) *holds.*

The relation (5.1.33) means that the repulsion part ϕ^+ of ϕ dominates its attraction part ϕ^-. Note that neither *translation invariance* nor *continuity* of V is assumed.

Remark 5.1.10. *In Chapter 4, we treated the case $M = 0$ for the particular measure $\lambda = \lambda_\theta$. The results obtained in Chapter 4 extend to general Radon measures λ. (The changes in the corresponding proofs are obvious.) Hence, $M = 0$ is covered (in principle) by Chapter 4. Thus, we may assume from now on that $M > 0$.*

5.1.4 Gibbsian formalism

Everywhere below, we assume that (**V**) holds. We abbreviate the $\mathcal{Q}$-local mass

$$\nmid \gamma \nmid_{\mathfrak{m}} := \mathfrak{m}_{\hat{X}}(\gamma) := \sum_{\hat{x} \in \gamma} \mathfrak{l}(\hat{x}), \qquad \gamma \in \Gamma_f(\hat{X}). \tag{5.1.34}$$

Tempered configurations

We define an appropriate set of *tempered* configurations (cf. also Definition 4.4.16):

$$\Gamma^{\mathfrak{t}}(\hat{X}) := \Gamma^{\mathfrak{t}}_{\mathfrak{l}}(\hat{X}) := \bigcap_{\alpha>0} \Gamma_{\alpha,\mathfrak{m}}(\hat{X}), \tag{5.1.35}$$

$$\Gamma_{\alpha,\mathfrak{m}}(\hat{X}) := \left\{ \gamma \in \Gamma_f(\hat{X}) \middle| |||\gamma|||_{\mathfrak{m},\alpha,k_0} < \infty \right\}, \tag{5.1.36}$$

where

$$|||\gamma|||_{\mathfrak{m},\alpha,k_0} := \left(\sum_{k\in Z} \nmid \gamma_k \nmid^2_{\mathfrak{m}} e^{-\alpha\rho(k,k_0)} \right)^{1/2}. \tag{5.1.37}$$

We may choose an arbitrary $k_0 \in Z$ for the definition of the tempered set in (5.1.36) because for all $k_0, k_1, k \in Z$

$$e^{-\alpha\rho(k,k_1)} e^{-\alpha\rho(k_1,k_0)} \leq e^{-\alpha\rho(k,k_0)} \leq e^{-\alpha\rho(k,k_1)} e^{\alpha\rho(k_1,k_0)}.$$

Therefore, from now on we fix $k_0 \in Z$ and denote

$$|||\cdot|||_{\alpha} = |||\cdot|||_{\mathfrak{m},\alpha,k_0}.$$

Conditional Hamiltonian

For each $\Lambda \in \mathcal{Q}_c(\hat{X})$ and $\gamma, \xi \in \Gamma_f(\hat{X})$ we introduce the *conditional Hamiltonians* $H_\Lambda(\cdot|\xi) : \Gamma(\Lambda) \to \overline{\mathbb{R}}$ by

$$H_\Lambda(\gamma|\xi) := \sum_{\hat{x},\hat{y}\in\gamma\cap\Lambda} V(\hat{x},\hat{y}) + 2 \sum_{\hat{x}\in\gamma\cap\Lambda,\ \hat{y}\in\xi\cap\Lambda^c} V(\hat{x},\hat{y}), \tag{5.1.38}$$

$$H(\gamma_\Lambda) := H_\Lambda(\gamma_\Lambda|\varnothing). \tag{5.1.39}$$

Here, the summation is taken over all *ordered pairs* $(\hat{x},\hat{y}) \in \gamma_\Lambda \times \gamma_\Lambda$. Hence, we sum twice over distinct points $\hat{x},\hat{y} \in \gamma_\Lambda$, i.e., $(\hat{x},\hat{y})$ and $(\hat{y},\hat{x})$, and once over $(\hat{x},\hat{x})$.

Lemma 5.1.11. *If* (**V**) *holds, then*

$$|H_\Lambda(\gamma|\xi)| < \infty, \qquad \textit{for all } \gamma,\xi \in \Gamma_f(\hat{X}) \textit{ and } \Lambda \in \mathcal{Q}_c(\hat{X}).$$

Example 5.1.12. *In the basic example, we have*

$$|H_{\mathbb{R}_+\times Q_{g,k}}(\gamma|\xi)| < \infty, \qquad \textit{for all } \gamma,\xi \in \Gamma_f(\hat{X}) \textit{ and } k \in \mathbb{Z}^d.$$

Proof of Lemma 5.1.11. The proof is similar to that one of Lemmas 4.3.16 and 4.5.1, so we just briefly recall the main idea. Note that

$$H_\Lambda(\gamma|\xi) \le \mathfrak{m}_\Lambda(\gamma)\mathfrak{m}_\Lambda(\gamma)\|\phi\|_\infty + 2\mathfrak{m}_\Lambda(\gamma)\mathfrak{m}_{\mathcal{U}_\Lambda}(\xi)\|\phi\|_\infty,$$

where

$$\mathcal{U}_\Lambda := \bigsqcup_{k\in Z} \left\{ \hat{Q}_k \mid \partial^{\mathrm{int}} k \cap \mathcal{K}_\Lambda \neq \varnothing \right\} \quad \in \mathcal{Q}_c(\hat{X}). \tag{5.1.40}$$

Since $\gamma, \xi \in \Gamma_f(\hat{X})$, the assertion follows. $\square$

Lyapunov functional

For $\lambda \in [0, \infty)$, let us define the map

$$\Gamma_f(\hat{X}) \ni \gamma \mapsto \Phi(\gamma) := \lambda \dagger \gamma \dagger_{\mathfrak{m}}^2 \ge 0, \tag{5.1.41}$$

which will play the role of a *Lyapunov functional.* To show upper and lower bounds for the partition function, the following estimate is essential.

Lemma 5.1.13. *Let* (**V**) *hold. Fix* $k \in Z$, $\xi \in \Gamma_f(\hat{X})$ *and* $\Lambda \in \mathcal{Q}_c(\hat{X})$. *Then*

$$H_\Lambda(\gamma_\Lambda|\xi) \ge \frac{1}{2} A \sum_{j\in\mathcal{K}_\Lambda} \dagger \gamma_{j\cap\Lambda} \dagger_{\mathfrak{m}}^2 - \frac{1}{4} A \sum_{l\in\mathcal{K}_{\mathcal{U}_\Lambda}} \dagger \xi_{l\cap\Lambda^c} \dagger_{\mathfrak{m}}^2 . \tag{5.1.42}$$

In particular, we have for $\xi = \varnothing$,

$$H_{\hat{Q}_k}(\gamma_k) \ge \frac{1}{2} A \dagger \gamma_k \dagger_{\mathfrak{m}}^2 . \tag{5.1.43}$$

Proof. By obvious calculations

$$\begin{aligned} H_\Lambda(\gamma_\Lambda|\xi) &= \sum_{\hat{x},\hat{y}\in\gamma_\Lambda} V(\hat{x},\hat{y}) + 2 \sum_{\substack{\hat{x}\in\gamma_\Lambda \\ \hat{y}\in\xi_{\Lambda^c}}} V(\hat{x},\hat{y}) \\ &= \sum_{\substack{j\in\mathcal{K}_\Lambda \\ l\in\mathcal{K}_\Lambda}} \sum_{\substack{\hat{x}\in\gamma_{j\cap\Lambda} \\ \hat{y}\in\gamma_{l\cap\Lambda}}} V(\hat{x},\hat{y}) + 2 \sum_{\substack{j\in\mathcal{K}_\Lambda \\ l\in\mathcal{K}_{\Lambda^c}}} \sum_{\substack{\hat{x}\in\gamma_{j\cap\Lambda} \\ \hat{y}\in\xi_{l\cap\Lambda^c}}} V(\hat{x},\hat{y}). \end{aligned}$$

By (5.1.13), (**LB**) (cf. (5.1.31)), (**RC**) (cf. (5.1.32)), (5.1.33) and (**AP**), the right-hand side above dominates

$$\begin{aligned} A \sum_{j\in\mathcal{K}_\Lambda} \dagger \gamma_{j\cap\Lambda} \dagger_{\mathfrak{m}}^2 - M \sum_{j\in\mathcal{K}_\Lambda} \sum_{\substack{l\in\mathcal{K}_\Lambda\cap\partial^{\mathrm{int}} j \\ l\neq j}} \dagger \gamma_{j\cap\Lambda} \dagger_{\mathfrak{m}} \dagger \gamma_{l\cap\Lambda} \dagger_{\mathfrak{m}} \\ - 2M \sum_{j\in\mathcal{K}_\Lambda} \dagger \gamma_j \dagger_{\mathfrak{m}} \sum_{l\in\mathcal{K}_{\Lambda^c}\cap\partial^{\mathrm{int}} j} \dagger \xi_l \dagger_{\mathfrak{m}} . \end{aligned}$$

Using (5.1.21) and that $ab \leq 1/2(a^2+b^2)$, for $a, b \geq 0$, we see that the above term dominates

$$
\begin{aligned}
&+ A \sum_{j\in\mathcal{K}_\Lambda} \nmid \gamma_j \nmid_{\mathfrak{m}}^2 - M\Bigg(\sum_{j\in\mathcal{K}_\Lambda} m^{\mathrm{int}} \nmid \gamma_j \nmid_{\mathfrak{m}}^2 \Bigg) \\
&- M\Bigg(\sum_{j\in\mathcal{K}_\Lambda} m^{\mathrm{int}} \nmid \gamma_j \nmid_{\mathfrak{m}}^2 + \sum_{j\in\mathcal{K}_\Lambda} \sum_{l\in\mathcal{K}_{\Lambda^c}\cap\partial^{\mathrm{int}} j} \nmid \xi_{l\cap\Lambda^c} \nmid_{\mathfrak{m}}^2 \Bigg) \\
=& \sum_{j\in\mathcal{K}_\Lambda} \nmid \gamma_j \nmid_{\mathfrak{m}}^2 \left(A - 2Mm^{\mathrm{int}}\right) - M \sum_{j\in\mathcal{K}_\Lambda} \sum_{l\in\mathcal{K}_{\Lambda^c}\cap\partial^{\mathrm{int}} j} \nmid \xi_{l\cap\Lambda^c} \nmid_{\mathfrak{m}}^2 \\
=& \frac{1}{2} A \sum_{j\in\mathcal{K}_\Lambda} \nmid \gamma_j \nmid_{\mathfrak{m}}^2 - M \sum_{j\in\mathcal{K}_\Lambda} \sum_{l\in\mathcal{K}_{\Lambda^c}\cap\partial^{\mathrm{int}} j} \nmid \xi_{l\cap\Lambda^c} \nmid_{\mathfrak{m}}^2,
\end{aligned}
\tag{5.1.44}
$$

where we used that $A = 4Mm^{\mathrm{int}}$ (cf. (5.1.33)). By **(FR)** and (5.1.21), the last of the three summands in (5.1.44) dominates

$$
-Mm^{\mathrm{int}} \sum_{l\in\mathcal{K}_{\mathcal{U}_\Lambda}} \nmid \xi_{l\cap\Lambda^c} \nmid_{\mathfrak{m}}^2 .
$$

The later estimate and (5.1.33) conclude the prove. $\square$

Partition function

Fix an inverse temperature $\beta := 1/T > 0$. For each $\Lambda \in \mathcal{Q}_c(\hat{X})$ and $\xi \in \Gamma_f(\hat{X})$, we define the *partition function*

$$
Z_\Lambda(\xi) := \int_{\Gamma_f(\Lambda)} e^{-\beta H_\Lambda(\gamma_\Lambda|\xi)} \mathcal{P}_\lambda^\Lambda(d\gamma_\Lambda).
$$

Lemma 5.1.14. *Let* (**V**) *hold. For any $\Lambda \in \mathcal{Q}_c(\hat{X})$, there exists a positive constant $C_0(\Lambda, \xi)$ such that*

$$
0 < Z_\Lambda(\xi) := \int_{\Gamma(\Lambda)} \exp\left\{-\beta H_\Lambda(\gamma_\Lambda|\xi)\right\} \mathcal{P}_\lambda^\Lambda(d\gamma_\Lambda) \leq C_0(\Lambda, \xi) < \infty.
$$

Proof. Our idea is to use Lemma 4.5.4 to obtain a lower bound and respectively Lemma 5.1.13 to get an upper one. We define for each $\Lambda \in \mathcal{Q}_c(\hat{X})$ and $\gamma, \xi \in \Gamma_f(\hat{X})$

$$
H_\Lambda^+(\gamma|\xi) := \sum_{\hat{x},\hat{y}\in\gamma\cap\Lambda} \mathfrak{l}(\hat{x})\mathfrak{l}(\hat{y})\phi^+(\hat{x},\hat{y}) + 2 \sum_{\hat{\gamma}\cap\Lambda,\ \hat{y}\in\xi\cap\Lambda^c} \mathfrak{l}(\hat{x})\mathfrak{l}(\hat{y})\phi^+(\hat{x},\hat{y}),
$$

where $\phi^+(\hat{x},\hat{y}) := \max(\phi(x,y),0)$. By Jensen's inequality

$$\begin{aligned} Z_\Lambda(\xi) &\geq \int_{\Gamma(\Lambda)} e^{-\beta H^+_\Lambda(\gamma|\xi)} \mathcal{P}^\Lambda_\lambda(d\gamma) \geq \exp\left(-\beta \int_{\Gamma(\Lambda)} H^+_\Lambda(\gamma|\xi)\mathcal{P}^\Lambda_\lambda(d\gamma)\right) \\ &\geq \exp\Big(-\beta\|\phi\|_\infty \big(\mathfrak{m}_\Lambda(\gamma)^2 + 2\mathfrak{m}_\Lambda(\gamma)\mathfrak{m}_{\mathcal{U}_\Lambda}(\xi)\big)\Big). \end{aligned}$$

We get, using (the proof of) Lemma 4.5.4 (cf. (4.5.6)):

$$Z_\Lambda(\xi) \geq \exp\left\{ -\beta\|\phi\|_\infty \Big[\left(C^{(1)}_{\mathfrak{m},\Lambda}\right)^2 + C^{(2)}_{\mathfrak{m},\Lambda} + \ \mathfrak{m}_{\hat{X}}(\xi_{\mathcal{U}_\Lambda}) C^{(1)}_{\mathfrak{m},\Lambda}\Big]\right\} > 0, \quad (5.1.45)$$

where we set, using (**AP**),

$$\begin{aligned} C^{(1)}_{\mathfrak{m},\Lambda} &:= 2\int_\Lambda \mathfrak{l}(\hat{x})\lambda_\theta \otimes m(\hat{x}) \leq 2\,|\mathcal{K}_\Lambda|\, M_1 < \infty, \\ C^{(2)}_{\mathfrak{m},\Lambda} &:= 2\int_\Lambda (\mathfrak{l}(\hat{x}))^2\lambda_\theta \otimes m(\hat{x}) \leq 2\,|\mathcal{K}_\Lambda|\, M_2 < \infty. \end{aligned}$$

Moreover, by (5.1.42) we deduce an upper bound

$$\begin{aligned} Z_\Lambda(\xi) \leq \int_{\Gamma(\hat{X})} &\exp\left\{ -2\beta A \sum_{j\in\mathcal{K}_\Lambda} \dagger\, \gamma_{j\cap\Lambda} \,\dagger^2_{\mathfrak{m}} \right\} \mathcal{P}^\Lambda_\lambda(d\gamma_\Lambda) \\ &\times \exp\left\{ \beta M m^{\mathrm{int}} \sum_{l\in\mathcal{K}_{\mathcal{U}_\Lambda}} \dagger\, \xi_{l\cap\Lambda^c} \,\dagger^2_{\mathfrak{m}} \right\} =: C_0(\Lambda,\xi) < \infty, \end{aligned}$$

which completes the proof. □

Remark 5.1.15. *As we see from the proof (cf. (5.1.45)), we only use that* $\|\phi\|_\infty < \infty$ *for the lower bound in Lemma 5.1.14.*

For the upper bound, the conditions (**LB**), (**RC**) *(cf. (5.1.31), resp. (5.1.32)) and (5.1.33) are sufficient.*

$\mathcal{Q}$-local specification

For each $\Lambda \in \mathcal{Q}_c(\hat{X})$ we define the *$\mathcal{Q}$-local Gibbs state* (or *local Gibbs state* for short) with boundary condition $\xi \in \Gamma(\hat{X})$ as

$$\mu_\Lambda(d\eta|\xi) := \begin{cases} \frac{1}{Z_\Lambda(\xi)} e^{-\beta H_\Lambda(\eta|\xi)} \mathcal{P}^\Lambda_\lambda(d\eta), & \text{if } \eta,\xi \in \Gamma^{\mathfrak{t}}(\hat{X}), \\ 0, & \text{otherwise,} \end{cases} \quad (5.1.46)$$

where $Z_\Lambda(\xi)$ is the *partition function* considered in Lemma 5.1.14.

Definition 5.1.16. *The* local specification $\pi = \{\pi_\Lambda(\cdot|\xi)|\Lambda \in \mathcal{Q}_c(\hat{X}),\ \xi \in \Gamma(\hat{X})\}$ *is a family of stochastic kernels*

$$\mathcal{B}(\Gamma(\hat{X})) \times \Gamma(\hat{X}) \ni (B, \xi) \mapsto \pi_\Lambda(B|\xi) \in [0, 1] \tag{5.1.47}$$

given by

$$\pi_\Lambda(B|\xi) := \mu_\Lambda\Big(\{\gamma_\Lambda \in \Gamma(\Lambda) \,|\, \gamma_\Lambda \cup \xi_{\Lambda^c} \in B\} \Big| \xi\Big). \tag{5.1.48}$$

Remark 5.1.17.

1. *By Lemma 5.1.14,*

$$0 < Z_\Lambda(\xi) < \infty, \qquad \forall \xi \in \Gamma_f(\hat{X}). \tag{5.1.49}$$

2. *It is easy to see that (5.1.49) holds for all $\Lambda \in \mathcal{B}(\hat{X})$ with $\Lambda_\mathcal{Q} \in \mathcal{Q}_c(\hat{X})$. (This follows by the proof of Lemma 5.1.14, where we replace Λ by $\Lambda_\mathcal{Q}$ in the lower bound in (5.1.45) and the upper bound holds unchanged (cf. (5.1.42).)*

3. *By construction (cf. [Pre76, Proposition 6.3] or [Pre05, Proposition 2.6]), the family (5.1.48) obeys the* consistency property, *which means that for all $\tilde{\Lambda}, \Lambda \in \mathcal{Q}_c(\hat{X})$ with $\tilde{\Lambda} \subseteq \Lambda$*

$$\int_{\Gamma(\hat{X})} \pi_{\tilde{\Lambda}}(B|\gamma)\pi_\Lambda(\mathrm{d}\gamma|\xi) = \pi_\Lambda(B|\xi), \quad \forall B \in \mathcal{B}(\Gamma(\hat{X})) \text{ and } \xi \in \Gamma(\hat{X}). \tag{5.1.50}$$

Remark 5.1.18. *In the basic example this reads as follows: The $\mathcal{Q}$-*local specification $\pi = \{\pi_{\mathbb{R}_+ \times Q_{g,k}}(d\eta|\xi)|k \in \mathbb{Z}^d,\ \xi \in \Gamma(\hat{X})\}$ *is given for $\xi \in \Gamma^{\boldsymbol{t}}(\hat{X})$ and $Q_{g,k}$ (cf. (5.1.8)) by*

$$\pi_{\mathbb{R}_+ \times Q_{g,k}}(B|\xi) := \mu_{\mathbb{R}_+ \times Q_{g,k}}(\{\gamma \in \Gamma^{\boldsymbol{t}}(\mathbb{R}_+ \times Q_{g,k})|\gamma \cup \xi_{\mathbb{R}_+ \times Q_{g,k}{}^c} \in B\}),$$
$$\mu_{\mathbb{R}_+ \times Q_{g,k}}(d\eta|\xi) := \frac{1}{Z_{\mathbb{R}_+ \times Q_{g,k}}(\xi)} e^{-\beta H_{\mathbb{R}_+ \times Q_{g,k}}(\eta|\xi)} \mathcal{P}_\theta^{\mathbb{R}_+ \times Q_{g,k}}(d\eta).$$

Gibbs measures

Definition 5.1.19. *A probability measure μ on $\Gamma(\hat{X})$ is called a* ***Gibbs measure*** *(or* ***state****) with pair potential V and inverse temperature $\beta > 0$ if it satisfies the* Dobrushin-Lanford-Ruelle ***(DLR)*** *equilibrium equation*

$$\int_{\Gamma(\hat{X})} \pi_\Lambda(B|\eta)\mu(\mathrm{d}\eta) = \mu(B) \tag{5.1.51}$$

for all $\Lambda \in \mathcal{Q}_c(\hat{X})$ and $B \in \mathcal{B}(\Gamma(\hat{X}))$. The associated set of all Gibbs states will be denoted by $\mathcal{G}\mathrm{ibbs}_V^{\boldsymbol{t}}(\Gamma(\hat{X}))$.

Remark 5.1.20. *By the construction of the specification (cf. (5.1.46) - (5.1.47)), each Gibbs measure μ is surely supported by $\Gamma^{\mathfrak{t}}(\hat{X})$.*

We denote the *algebra of all local partition events* by

$$\mathcal{B}_{\mathcal{Q}}(\Gamma(\hat{X})) := \bigcup_{\Lambda \in \mathcal{Q}_c(\hat{X})} \mathcal{B}_{\Lambda}(\Gamma(\hat{X})). \tag{5.1.52}$$

Remark 5.1.21. *To obtain the* **DLR** *equation it is enough to check (5.1.51) only for $B \in \mathcal{B}_{\mathcal{Q}}(\Gamma(\hat{X}))$. Indeed, using Caratheodory's theorem, we deduce that $\mu\big|_{\mathcal{B}_{\mathcal{Q}}(\Gamma(\hat{X}))}$ extends uniquely to a measure on $\sigma(\mathcal{B}_{\mathcal{Q}}(\Gamma(\hat{X}))) = \mathcal{B}(\Gamma(\hat{X}))$, where we used Kuratowski's theorem (cf. Theorem A.1.7) to get the last equality. Hence, (5.1.51) holds for all $B \in \mathcal{B}(\Gamma(\hat{X}))$.*

Before we proceed, we define the appropriate topology for convergence of the local Gibbs states:[3]

Definition 5.1.22. *On the space of all probability measures $\mathcal{M}^1(\Gamma(\hat{X}))$ we introduce the topology of $\mathcal{Q}$-*wise convergence*. This topology, which we denote by $\mathcal{T}_{\mathcal{Q}}$, is defined as the coarsest topology making the maps $\mathcal{M}^1(\Gamma(\hat{X})) \ni \mu \mapsto \mu(B)$ continuous for all sets $B \in \mathcal{B}_{\mathcal{Q}}(\Gamma(\hat{X}))$.*

Remark 5.1.23. *If for all $\Lambda \in \mathcal{B}_c(\hat{X})$ the set $\Lambda_{\mathcal{Q}} \in \mathcal{Q}_c(\hat{X})$, then $\mathcal{Q}$-wise convergence implies the local setwise convergence (cf. Definition 2.1.2).*

Remark 5.1.24. *Let us assume that we are in the basic model framework (cf. Subsection 5.1.1). Then the Gibbs measures $\mu \in \mathcal{G}\mathrm{ibbs}^{\mathfrak{t}}_{V,s}(\Gamma(\hat{\mathbb{R}}^d))$ are defined via the* semi-local *specification $\pi = \{\pi_{\mathbb{R}_+ \times \Delta}(d\gamma|\xi) \mid \Delta \in \mathcal{B}_c(\mathbb{R}^d),\ \xi \in \Gamma(\hat{\mathbb{R}}^d)\}$. One the other hand, one can define, similar to Section 4.1, a* local *specification $\pi = \{\pi_\Lambda(d\gamma|\xi) \mid \Lambda \in \mathcal{B}_c(\hat{\mathbb{R}}^d), \xi \in \Gamma(\hat{\mathbb{R}}^d)\}$.*

It can be checked that the analog version of Theorem 4.5.9 holds in this case, namely that both (i.e. the local and the semi-local) specifications determine the same set of Gibbs measures. This follows analogously to the proof of Theorem 4.5.9, where we use Lemma 5.1.13 to see that Lebesgue's dominated convergence theorem is applicable in the proof of Proposition 4.5.5.

5.2 Existence for general potentials

In order to prove the existence an essential step is to check that the net of $\mathcal{Q}$-local Gibbs specification kernels $\{\pi_\Lambda(d\eta|\xi) | \Lambda \in \mathcal{Q}_c(\hat{X})\}$, with a fixed tempered boundary condition $\xi \in \Gamma^{\mathfrak{t}}(\hat{X})$ (cf. (5.1.36)) is locally equicontinuous.

[3] See for the idea of localizing in a different framework e.g. [Geo88, Section 4.1] or [KPR10, Section 2], where in the later $\Lambda \in \mathcal{B}_c(\hat{X})$ is chosen instead of $\Lambda \in \mathcal{Q}_c(\hat{X})$.

To that end, we introduce the map $\Gamma(\hat{X}) \ni \gamma \mapsto \lambda \dagger \gamma_k \dagger_{\mathfrak{m}}^2$, which will play the role of a Lyapunov functional in our theory, and show that it is exponentially integrable (cf. Lemma 5.2.2). After that we deduce a weak dependence on boundary conditions (cf. Proposition 5.2.4): For each $\lambda \in (\frac{1}{4}\beta A, \frac{1}{3}\beta A)$, we find $\mathcal{C}_\lambda > 0$ such that uniformly for all $k \in Z$ and $\xi \in \Gamma^{\mathfrak{t}}(\hat{X})$

$$\limsup_{\mathcal{K} \nearrow Z} \int_{\Gamma(\hat{X})} \exp\left\{\lambda \dagger \gamma_k \dagger_{\mathfrak{m}}^2\right\} \pi_{\mathcal{K}}(d\gamma|\xi) \le \mathcal{C}_\lambda.$$

By the weak dependence, we deduce first the mentioned local equicontinuity (cf. Proposition (5.2.7)) and then the existence of a Gibbs measure (cf. Theorem 5.2.8).

Throughout this section we assume that (**V**) holds. As a preliminary step, we check that $\exp\{\lambda \dagger \gamma_k \dagger_{\mathfrak{m}} \mathfrak{m}^2\}$ is integrable w.r.t. $\pi_k(d\gamma|\xi)$.

Lemma 5.2.1. *Let* (**V**) *hold. Fix* $k \in Z$, $\xi \in \Gamma_f(\hat{X})$, $\Lambda \in \mathcal{Q}_c(\hat{X})$ *and* $\lambda \in [0, \beta\frac{A}{2}]$. *Then*

$$\begin{aligned}&\int_{\Gamma(\Lambda)} \exp\left\{\lambda \dagger \gamma_k \dagger_{\mathfrak{m}}^2\right\} \pi_\Lambda(\mathrm{d}\gamma|\xi)\\ &\le \exp\left(\Upsilon_{\Lambda,\varepsilon} + \left(\frac{\varepsilon}{2}\beta\|\phi\|_\infty M_{1,\Lambda} + \beta\frac{A}{4}\right)\sum_{l \in \mathcal{K}_{\mathcal{U}_\Lambda}} \dagger \xi_l \dagger_{\mathfrak{m}}^2\right) < \infty, \qquad (5.2.1)\end{aligned}$$

where $\varepsilon > 0$ *is arbitrary and*

$$\Upsilon_{\Lambda,\varepsilon} := \beta\|\phi\|_\infty\left(M_{2,\Lambda} + \frac{1}{2\varepsilon}M_{1,\Lambda}|\mathcal{K}_{\mathcal{U}_\Lambda}|\right) < \infty.$$

Proof. By Lemma 5.1.13 the integral in (5.2.1) can be estimated by

$$\begin{aligned}&\frac{1}{Z_\Lambda(\xi)}\int_{\Gamma(\hat{X})} \exp\left\{-(\frac{1}{2}\beta A - \lambda) \dagger \gamma_{k\cap\Lambda} \dagger_{\mathfrak{m}}^2 - \beta\frac{A}{2}\sum_{\substack{j \in \mathcal{K}_\Lambda\\ j \ne k}} \dagger \gamma_{j\cap\Lambda} \dagger_{\mathfrak{m}}^2\right\}\mathcal{P}_\lambda^\Lambda(d\gamma_\Lambda)\\ &\times \exp\left\{\beta\frac{A}{4}\sum_{l \in \mathcal{K}_{\mathcal{U}_\Lambda}} \dagger \xi_{l\cap\Lambda^c} \dagger_{\mathfrak{m}}^2\right\}. \qquad (5.2.2)\end{aligned}$$

Lemma 5.1.14 (cf. (5.1.45)) yields the claim, where we note that

$$\sum_{j \in \mathcal{K}_{\mathcal{U}_\Lambda}} \dagger \xi_j \dagger_{\mathfrak{m}} \le \frac{1}{2\varepsilon}|\mathcal{K}_{\mathcal{U}_\Lambda}| + \frac{1}{2}\varepsilon\sum_{j \in \mathcal{K}_{\mathcal{U}_\Lambda}} \dagger \xi_j \dagger_{\mathfrak{m}}^2. \qquad (5.2.3)$$

□

In Section 5.2.1, we will improve this result by showing that the constant $\Upsilon_{\Lambda,\varepsilon}$ can be chosen uniformly as $\Lambda \nearrow \hat{X}$.

5.2.1 Weak dependence on boundary conditions

In this section we prepare some technical estimates on the local specification kernels which will be crucial to prove the existence of a Gibbs measure $\mu \in \mathcal{G}\mathrm{ibbs}_V^{\mathfrak{t}}(\Gamma(\hat{X}))$. To this end, we use an inductive scheme that is based on the consistency property (5.1.50). We start by deducing the following bound in the ‹elementary› partition sets $\hat{Q}_k$, $k \in Z$.

Lemma 5.2.2. *For all $\lambda \le \frac{1}{2}\beta A$, $k \in Z$ and $\xi \in \Gamma_f(\hat{X})$*

$$\int_{\Gamma_k} \exp\left\{\lambda \nmid \gamma_k \nmid_{\mathfrak{m}}^2\right\} \pi_k(\mathrm{d}\gamma|\xi) \le \exp\left(\Upsilon_\varepsilon + \left(\frac{\beta M}{2} + C^+\varepsilon\right) \sum_{j \in \partial^{\mathrm{int}} k} \nmid \xi_j \nmid_{\mathfrak{m}}^2\right), \tag{5.2.4}$$

where $\varepsilon > 0$ is arbitrary and

$$\begin{aligned} \Upsilon_\varepsilon &:= \beta\|\phi\|_\infty \left(M_2 + \frac{M_1}{\varepsilon} m^{\mathrm{int}}\right) < \infty, \\ C^+ &:= \beta\|\phi\|_\infty \frac{M_1}{2} < \infty. \end{aligned} \tag{5.2.5}$$

Proof. By (5.1.42) for $\Lambda = \hat{Q}_k$ and the proof of Lemma 5.2.1, we get this result. □

Remark 5.2.3. *The estimate (5.2.4) expresses the so-called weak dependence on boundary conditions. Analytically this means that*

$$\left(\frac{\beta M}{2} + C^+\varepsilon\right) m^{\mathrm{int}} < \lambda < \frac{1}{2}\beta A,$$

which is always possible for small enough $\varepsilon > 0$ provided we assume that $\lambda \in (\beta\frac{A}{8}, \beta\frac{A}{2}]$ and $A_0 > 4Mm^{\mathrm{int}}$ (cf. (5.1.33)).

Moment estimates

Consider now arbitrary large domains $\Lambda_{\mathcal{K}} = \bigsqcup_{k\in\mathcal{K}} \hat{Q}_k \in \mathcal{Q}_c(\hat{X})$ indexed by $\mathcal{K} \Subset Z$. Note that $\Lambda_{\mathcal{K}} \nearrow \hat{X}$ as $\mathcal{K} \nearrow Z$. Using the estimate (5.2.4) and the consistency property (5.1.50), our next step will be to get similar moment estimates for all specification kernels $\pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi) := \pi_{\Lambda_{\mathcal{K}}}(\mathrm{d}\gamma|\xi)$.

Proposition 5.2.4. *Let $\lambda \in (\beta\frac{A}{4}, \beta\frac{A}{2}]$. Then there exists $\mathcal{C}_\lambda < \infty$ such that for all $k \in Z$, $\xi \in \Gamma^{\mathfrak{t}}(\hat{X})$*

$$\limsup_{\mathcal{K}\nearrow Z} \int_{\Gamma(\hat{X})} \exp\left\{\lambda \nmid \gamma_k \nmid_{\mathfrak{m}}^2\right\} \pi_{\mathcal{K}}(d\gamma|\xi) \le \mathcal{C}_\lambda. \tag{5.2.6}$$

Moreover, for each $\alpha > 0$ one finds a proper $\nu_\alpha > 0$ such that

$$\limsup_{\mathcal{K} \nearrow Z} \int_{\Gamma(\hat{X})} \exp\left\{\nu_\alpha |||\gamma|||_\alpha^2\right\} \pi_{\mathcal{K}}(d\gamma|\xi) \leq \mathcal{C}_\alpha. \tag{5.2.7}$$

Proof. Let us define

$$0 \leq n_k(\mathcal{K}|\xi) := \log\left\{\int_{\Gamma(\hat{X})} \exp\left\{\lambda \dagger \gamma_k \dagger_{\mathfrak{m}}^2\right\} \pi_{\mathcal{K}}(d\gamma|\xi)\right\}, \quad k \in Z, \tag{5.2.8}$$

which are finite by Lemma 5.2.1. In particular,

$$n_k(\mathcal{K}|\xi) := \Phi(\xi_k) \quad \text{if} \ \ k \notin \mathcal{K}.$$

Next, we will find a global bound for the whole sequence $(n_k(\mathcal{K}|\xi))_{k \in Z}$, which then implies the required estimates on each of its components.

Integrating both sides of (5.2.4) with respect to $\pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi)$ with an arbitrary $\xi \in \Gamma^{\mathfrak{t}}(\hat{X})$ and taking into account the consistency property (5.1.50), we arrive at the following estimate for $k \in \mathcal{K}$

$$\begin{aligned} n_k(\mathcal{K}|\xi) &\leq \Upsilon_\varepsilon + \log\left\{\int_{\Gamma(\hat{X})} \exp\left(\left(\beta M + C^+\varepsilon\right) \sum_{j \in \partial^{\mathrm{int}} k} \dagger \xi_j \dagger_{\mathfrak{m}}^2\right) \pi_{\mathcal{K}}(d\gamma|\xi)\right\} \\ &= \Upsilon_\varepsilon + \left(\left(\beta M + C^+\varepsilon\right) \sum_{j \in \mathcal{K}^C \cap \partial^{\mathrm{int}} k} \dagger \xi_j \dagger_{\mathfrak{m}}^2\right) \\ &+ \log\left\{\int_{\Gamma(\hat{X})} \exp\left(\left(\beta M + C^+\varepsilon\right) \sum_{j \in \mathcal{K} \cap \partial^{\mathrm{int}} k} \dagger \xi_j \dagger_{\mathfrak{m}}^2\right) \pi_{\mathcal{K}}(d\gamma|\xi)\right\}. \end{aligned} \tag{5.2.9}$$

We will apply the multiple Hölder inequality

$$\mu\left(\prod_{j=1}^K f_j^{t_j}\right) \leq \prod_{j=1}^K \mu^{t_j}(f_j), \qquad \mu(f_j) := \int f_j \mathrm{d}\mu, \tag{5.2.10}$$

valid for any probability measure μ, nonnegative functions f_j, and $t_j \geq 0$ such that $\sum_{j=1}^K t_j \leq 1$, $K \in \mathbb{N}$.

Choose $0 < \delta < 1$, $\varepsilon > 0$ such that

$$0 < B^+ := \frac{1}{4}\beta A + \varepsilon m^{\mathrm{int}} C^+ < \delta\lambda \leq \lambda_0 := \beta\frac{A}{2}, \tag{5.2.11}$$

where C^+ is defined by (5.2.5). In our context $f_j := \exp\{\lambda \nmid \gamma_j \nmid_{\mathfrak{m}}^2\}$ and we set

$$t_j := \frac{\beta M}{\lambda} + \frac{\varepsilon C^+}{\lambda}, \qquad \text{for } j \in \mathcal{K} \cap \partial^{\text{int}} k.$$

Using this setting for (5.2.10), we deduce that the last summand in (5.2.9) is dominated by

$$\begin{aligned}
&\sum_{j \in \mathcal{K} \cap \partial^{\text{int}} k} \log \left(\int_{\Gamma(\hat{X})} \exp\left(\lambda \nmid \gamma_j \nmid_{\mathfrak{m}}\right) \pi_{\mathcal{K}}(d\gamma|\xi) \right)^{t_j} \\
&\quad = \sum_{j \in \mathcal{K} \cap \partial^{\text{int}} k} \frac{\beta M + \varepsilon C^+}{\lambda} n_j(\mathcal{K}|\xi). \qquad (5.2.12)
\end{aligned}$$

Let $\mathcal{K} \Subset Z$ contain a fixed point $k_0 \in Z$. Let $\vartheta \geq 0$ such that $\rho_Z(k_0, j) \leq \vartheta$ for all $j \in \partial^{\text{int}} k_0$. Fix $\alpha > 0$. We multiply (5.2.9) with the weight $\exp\{-\alpha\rho(k, k_0)\}$ and take the sum in (5.2.12) over all $j \in \mathcal{K}$ and multiply the sum with the same weight. Then we get

$$\begin{aligned}
n_{k_0}(\mathcal{K}|\xi) &\leq \sum_{k \in \mathcal{K}} \left[n_k(\mathcal{K}|\xi) \exp\{-\alpha \rho_Z(k_0, k)\} \right] \qquad (5.2.13) \\
&\leq \left[1 - \frac{B^+}{\lambda} e^{\alpha\vartheta} \right]^{-1} \left[\Upsilon_\varepsilon + B^+ e^{\alpha\vartheta} ||\xi_{\mathcal{K}^c}||_\alpha^2 \right].
\end{aligned}$$

We have

$$\left[1 - \frac{B^+ e^{\alpha\vartheta}}{\lambda} \right]^{-1} = 1 + \frac{B^+ e^{\alpha\vartheta}}{\lambda - B^+ e^{\alpha\vartheta}} \leq 1 + \frac{B^+ e^{\alpha\vartheta}}{\frac{1}{\delta} B^+ - B^+ e^{\alpha\vartheta}} = 1 + \frac{e^{\alpha\vartheta}}{\frac{1}{\delta} - e^{\alpha\vartheta}},$$

where we used (5.2.11). Plugging this back into (5.2.13), we get

$$\begin{aligned}
n_{k_0}(\mathcal{K}|\xi) &\leq \sum_{k \in \mathcal{K}} \left[n_k(\mathcal{K}|\xi) \exp\{-\alpha \rho(k_0, k)\} \right] \\
&\leq \left[\Upsilon_\varepsilon + B^+ e^{\alpha\vartheta} |||\xi_{\mathcal{K}^c}|||_\alpha^2 \right] \left(\frac{1}{1 - \delta e^{\alpha\vartheta}} \right) =: \mathcal{C}_{1,\delta,\alpha,\mathcal{K},\varepsilon,\xi} =: \mathcal{C}_1. \qquad (5.2.14)
\end{aligned}$$

Since $|||\xi_{\mathcal{K}^c}|||_\alpha$ tends to zero as $\mathcal{K} \nearrow Z$, we obtain for each $k_0 \in Z$

$$\limsup_{\mathcal{K} \nearrow Z} \sum_{k \in \mathcal{K}} \left[n_k(\mathcal{K}|\xi) \exp\{-\alpha \rho(k_0, k)\} \right] \leq \Upsilon_\varepsilon \left(\frac{1}{1 - \delta e^{\alpha\vartheta}} \right), \qquad (5.2.15)$$

and thus, by letting $\alpha \to 0$, we complete the proof of (5.2.6):

$$\limsup_{\mathcal{K} \nearrow Z} n_{k_0}(\mathcal{K}|\xi) \leq \frac{1}{1-\delta} \Upsilon_\varepsilon =: \log \mathcal{C}_\lambda =: \log \mathcal{C}_\lambda. \qquad (5.2.16)$$

So, we have for each $\lambda \le \lambda_0$ fulfilling (5.2.11)

$$\limsup_{\mathcal{K}\nearrow Z} \int_{\Gamma(\hat{X})} \exp\left\{\lambda \dagger \gamma_k \dagger_{\mathfrak{m}}^2\right\} \pi_{\mathcal{K}}(d\gamma|\xi) \le \exp\left\{\frac{1}{1-\delta}\Upsilon_\varepsilon\right\} = \mathcal{C}_0. \qquad (5.2.17)$$

By the Hölder inequality (5.2.10) we see that we can find a $\mu_\alpha > 0$ such that (5.2.7) holds: We choose

$$\nu_\alpha := \lambda_0 \left[\sum_{k\in Z} \exp\{-\alpha\rho(k,k_0)\}\right]^{-1}, \qquad (5.2.18)$$

which is well-defined by (5.1.22). By (5.2.10), we get

$$\begin{aligned}
&\int_{\Gamma(\hat{X})} \exp\left\{\nu_\alpha \sum_{k\in Z} \dagger \gamma_k \dagger_{\mathfrak{m}}^2 \, e^{-\alpha\rho(k,k_0)}\right\}\pi_{\mathcal{K}}(d\gamma|\xi) \\
&\le \prod_{k\in\mathcal{K}} \left(\int_{\Gamma(\hat{X})} \exp\left\{\lambda_0 \dagger \gamma_k \dagger_{\mathfrak{m}}^2\right\} \pi_{\mathcal{K}}(d\gamma|\xi)\right)^{\frac{\nu_\alpha}{\lambda_0}\exp\{-\alpha\rho(k,k_0)\}} e^{|||\xi_{\mathcal{K}^c}|||_\alpha^2} \\
&\le \left(\exp\left\{\sum_{k\in\mathcal{K}} n_k(\mathcal{K}|\xi)\exp\{-\alpha\rho(k,k_0)\}\right\}\right)^{\frac{\nu_\alpha}{\lambda_0}\exp\{-\alpha\rho(k,k_0)\}} e^{|||\xi_{\mathcal{K}^c}|||_\alpha^2}.
\end{aligned}$$

Hence, using also (5.2.15), we deduce

$$\limsup_{\mathcal{K}\nearrow Z} \int_{\Gamma(\hat{X})} \exp\left\{\nu_\alpha |||\alpha|||_\gamma^2\right\} \pi_{\mathcal{K}}(d\gamma|\xi) \le \exp\left\{\Upsilon_\varepsilon\left(\frac{1}{1-\delta e^{\alpha\nu}}\right)\right\} =: C_\alpha.$$

□

5.2.2 Uniform bounds for local Gibbs states

Corollary 5.2.5. *Let* (**V**) *hold. Then for all* $\mathcal{K} \Subset Z$*, there exists* $\mathcal{C}(\mathcal{K}) < \infty$ *such that*

$$\limsup_{\substack{\Lambda\nearrow\hat{X} \\ \Lambda\in\mathcal{Q}_c}} \int_{\Gamma(\Lambda)} \left(\sum_{k\in\mathcal{K}} \dagger \gamma_k \dagger_{\mathfrak{m}} \, \mathfrak{m}\right) \pi_\Lambda(d\gamma|\xi) \le \mathcal{C}(\mathcal{K}) < \infty$$

uniformly for all $\xi \in \Gamma^t(\hat{X})$.

Proof. The result follows immediately by Young's inequality and (5.2.6). □

5.2.3 Local equicontinuity

An important step in proving the existence of a $\mu \in \mathcal{G}\mathrm{ibbs}_V$ is to establish the equicontinuity of the local specification. To this end, we adopt [Geo88, Definition 4.6] to our setting:

Definition 5.2.6. *Fix $\xi \in \Gamma^{\boldsymbol{t}}(\hat{X})$. The net $\{\pi_\Lambda(d\gamma|\xi)|\Lambda \in \mathcal{Q}_c(\hat{X})\}$ is called* $\mathcal{Q}$-locally equicontinuous *iff for all $\tilde{\Lambda} \in \mathcal{Q}_c(\hat{X})$ (cf. (5.1.52)) and each sequence $\{B_N\}_{N\in\mathbb{N}} \subset \mathcal{B}_{\tilde{\Lambda}}(\Gamma(\hat{X}))$ with $B_N \downarrow \varnothing$*

$$\lim_{N\to\infty} \limsup_{\substack{\Lambda\nearrow\hat{X}\\ \Lambda\in\mathcal{Q}_c(\hat{X})}} \pi_\Lambda(B_N|\xi) = 0. \tag{5.2.19}$$

Proposition 5.2.7. *Let* (**V**) *hold. Then for each fixed $\xi \in \Gamma^{\boldsymbol{t}}(\hat{X})$ the net $\{\pi_\Lambda(\mathrm{d}\gamma|\xi)|\ \Lambda \in \mathcal{Q}_c(\hat{X})\}$ is locally equicontinuous. Moreover, each of its limit points is supported by $\Gamma^{\boldsymbol{t}}(\hat{X})$.*

Proof. In principal, the arguments work as in Proposition 4.2.6 using $\mathcal{Q}_c(\hat{X})$ instead of $\mathcal{B}_c(\hat{X})$. Indeed, to get the lim sup of the first summand in (4.2.14) arbitrarily small, we use the support property given by Corollary 5.2.5 (instead of Proposition 4.2.3) for $\mathfrak{m}_{\mathcal{U}}$, where $\mathcal{U} \in \mathcal{Q}_c(\hat{X})$ by (**FR**). Taking into account (5.1.21), (5.1.42) and (5.1.45), we see that the lim sup of the second summand in (4.2.14) also vanishes. Combining these arguments, we get the assertion. □

5.2.4 Existence of Gibbs measures

Now we are in position to deduce a main result of this section. Namely, we show that each limit point that we obtained by the local equicontinuity proved above is indeed a Gibbs measure.

Theorem 5.2.8. *Let $V : \hat{X} \times \hat{X} \to \mathbb{R}$ and $\lambda \otimes m$ be such that* (**V**) *holds.*

Then there exists a Gibbs measure μ corresponding to the potential V and the probability measure $\mathcal{P}_\lambda$. It is supported by $\Gamma^{\boldsymbol{t}}(\hat{X})$. Therefore,

$$\mathcal{G}\mathrm{ibbs}_V^{\boldsymbol{t}}(\Gamma(\hat{X})) \neq 0.$$

Furthermore, the set $\mathcal{G}\mathrm{ibbs}_V^{\boldsymbol{t}}(\Gamma(\hat{X}))$ is compact in the topology $\mathcal{T}_{\mathcal{Q}}$.

Proof. We will follow the proof of Theorem 4.2.7 with the appropriate changes to get this result. But, we emphasize that now the (**DLR**) property can be derived much more easily.

Similar as in the proof of Theorem 4.2.7 (cf. (4.2.21) and (4.2.22)), we get a limit point $\mu \in \mathcal{M}^1(\Gamma(\hat{X}))$. Namely, Proposition 5.2.7 (and [Geo88, Proposition 4.9] combined with [Pat67, Theorem V.3.2]) yield

$$\mu := \mathcal{B}_{\mathcal{Q}} - \lim_{N\to\infty} \pi_{\Lambda_N}(\cdot|\xi) \in \mathcal{M}^1(\Gamma(\hat{X})),$$

where $\Lambda_N \nearrow \hat{X}$, $\Lambda_N \in \mathcal{Q}_c(\hat{X})$, is some order generating sequence. We have for all partition events $B \in \mathcal{B}_{\mathcal{Q}}(\Gamma(\hat{X}))$ that

$$\pi_{\Lambda_N}(B|\xi) \to \mu(B) \quad \text{as } N \to \infty$$

and $\mu(\Gamma^{\mathfrak{t}}(\hat{X})) = 1$.

The limit point μ is surely Gibbs just by the definition of the $\mathcal{Q}$-local specification. Indeed, fix $\Lambda \in \mathcal{Q}_c(\hat{X})$ and $B \in \mathcal{B}_{\mathcal{Q}}(\Gamma(\hat{X}))$ arbitrarily. By (**FR**), we can pick $\mathcal{U}_\Lambda \in \mathcal{Q}_c(\hat{X})$ and justify the following equations to get the (**DLR**) one. Note that the steps 1 and 3 are easier than in Theorem 4.3.26:

$$\begin{aligned}
\int_{\Gamma(\hat{X})} \pi_\Lambda(B|\gamma)\mu(d\gamma) &\overset{1.}{=} \int_{\Gamma(\hat{X})} \pi_\Lambda(B|\gamma_{\mathcal{U}_\Lambda})\pi_{\Lambda_N}(d\gamma|\xi) \\
&\overset{2.}{=} \lim_{N\to\infty} \int_{\Gamma(\hat{X})} \pi_\Lambda(B|\gamma_{\mathcal{U}_\Lambda})\pi_{\Lambda_N}(d\gamma|\xi) \\
&\overset{3.}{=} \lim_{N\to\infty} \int_{\Gamma(\hat{X})} \pi_\Lambda(B|\gamma)\pi_{\Lambda_N}(d\gamma|\xi) \\
&\overset{4.}{=} \lim_{N\to\infty} \pi_{\Lambda_N}(B|\xi) \\
&\overset{5.}{=} \mu(B).
\end{aligned} \tag{5.2.20}$$

The first and third equality follow by the choice of $\mathcal{U}_\Lambda$, the second and fifth one by the definition of μ and the fourth one by the consistency of the local specifications (cf. (5.1.50)).

It remains to show the compactness of $\mathcal{G}\mathrm{ibbs}_V^{\mathfrak{t}}(\Gamma(X))$. We make similar changes as for the proof of Proposition 5.2.7. Using in addition Corollary 5.2.11 below (resp. Theorem 4.3.34 if (5.1.33) is void), we deduce the compactness adopting the arguments in Theorem 4.4.20 (cf. (4.4.26)). Thus, all the assertions are shown. □

Uniqueness of Gibbs measures

Having established the existence of Gibbs measures, the next problem is to show their uniqueness or non-uniqueness, which is more difficult than the existence. The non-uniqueness corresponds to the presence of phase transitions in our model. The answer strongly depends on the interplay between the model parameters like the temperature β, the intensity θ and the stability properties of the potential.

Two basic techniques to show uniqueness of Gibbs measures are *Ruelle's superstability estimates* [Rue69, Rue70] and *Dobrushin's criterion* ([Dob70b]), as well as its modifications [DP81, DP83]. However, they cannot be applied directly in our framework and need essential modifications, which we will not discuss here.

We only point out the following particular example, which however does not cover the most interesting situation of the translation invariant case in $(\mathbb{R}^d, dx)$:

Theorem 5.2.9. *Let V and $\lambda \otimes m$ be such that* $(\mathbf{V})$ *holds. If $\hat{X} \in \mathcal{Q}_c(\hat{X})$, then there exists exactly one Gibbs measure corresponding to V and $\mathcal{P}_\lambda$, i.e.* $\left|\mathcal{G}\mathrm{ibbs}_V^{\boldsymbol{t}}(\Gamma(\hat{X}))\right| = 1$.

Proof. This follows by (5.1.46), Remark 5.1.17, Definition 5.1.19 and Theorem 5.2.8. □

5.2.5 Moment estimates for Gibbs measures

Theorem 5.2.10. *Let* $(\mathbf{V})$ *and (5.1.33) hold. For each $\alpha > 0$ one finds a certain $\nu_\alpha > 0$ such that for all $\mu \in \mathcal{G}\mathrm{ibbs}_V^{\boldsymbol{t}}(\Gamma(\hat{X}))$*

$$\int_{\Gamma(\hat{X})} \exp\left\{\nu_\alpha |||\gamma|||_\alpha^2\right\} \mu(d\gamma) \le \mathcal{C}_\alpha, \tag{5.2.21}$$

where $\mathcal{C}_0 < \infty$ is the same as in Proposition 5.2.4 (cf. (5.2.7)).

Proof. Using Proposition 5.2.4 instead of Proposition 4.3.18 and doing the obvious changes in the proof of Theorem 4.3.34, we establish this result.

Indeed, fix $\tilde{\Lambda} \in \mathcal{B}(\hat{X})$. Using Beppo Levi, we have

$$\int_{\Gamma(\hat{X})} \exp\left\{\nu_\alpha |||\gamma|||_\alpha^2\right\} \mu(d\gamma) = \lim_{M \nearrow \infty} \int \exp\left\{\nu_\alpha |||\gamma|||_\alpha^2 \wedge M\right\} \mu(d\gamma).$$

By the **(DLR)** equation this equals

$$\lim_{M \nearrow \infty} \lim_{\substack{\Lambda_N \nearrow \hat{X} \\ \Lambda_N \in B_c(\hat{X})}} \int_{\Gamma(\hat{X})} \int_{\Gamma(\hat{X})} \exp\left\{\nu_\alpha |||\gamma|||_\alpha^2 \wedge M\right\} \pi_{\Lambda_N}(d\gamma|\xi)\mu(d\xi).$$

By Lebesgue's dominated convergence theorem, the later equals

$$\begin{aligned}
&\lim_{M\nearrow\infty}\int_{\Gamma(\hat{X})}\Bigg(\lim_{\substack{\Lambda_N\nearrow\hat{X}\\ \Lambda_N\in B_c(\hat{X})}}\int_{\Gamma(\hat{X})}\exp\left\{\nu_\alpha|||\gamma_{\Lambda_N}|||_\alpha^2\wedge M\right\}\pi_{\Lambda_N}(d\gamma_{\Lambda_N}|\xi)\Bigg)\mu(d\xi)\\
&\le\lim_{M\nearrow\infty}\int_{\Gamma(\hat{X})}\Bigg(\lim_{\substack{\Lambda_N\nearrow\hat{X}\\ \Lambda_N\in B_c(\hat{X})}}\int_{\Gamma(\hat{X})}\exp\left\{\nu_\alpha|||\gamma_{\Lambda_N}|||_\alpha^2\right\}\pi_{\Lambda_N}(d\gamma_{\Lambda_N}|\xi)\Bigg)\mu(d\xi).
\end{aligned}$$

Since $\mu(\Gamma^{\mathfrak{t}}(\hat{X}))=1$, we may apply the uniform bound proven in Proposition 5.2.4 (cf. (5.2.21)). Thus, the later is dominated by $\mathcal{C}_\alpha$, which was to be shown. □

Corollary 5.2.11. *Let* (**V**) *be fulfilled with the* $\mathcal{Q}$*-local function* $\mathfrak{l}$*. For each* $\Lambda\in\mathcal{Q}_c(\hat{X})$ *and* $N\in\mathbb{N}$*, there exists* $C_N(\Lambda)>0$ *such that for all* $\mu\in\mathcal{G}\mathrm{ibbs}_V^{\mathfrak{t}}(\Gamma(\hat{X}))$

$$\int_{\Gamma(\hat{X})}\langle\mathfrak{l}1\!\!1_\Lambda,\gamma\rangle^N\mu(d\gamma)<C_N(\Lambda). \tag{5.2.22}$$

Proof. This follows by Theorem 5.2.10. Indeed, fix $k\in Z$. By (5.1.37), for all $\Lambda\in\mathcal{Q}_c(\hat{X})$ there exists $\varepsilon:=\varepsilon(\Lambda,k)>0$:

$$\langle\mathfrak{l}1\!\!1_\Lambda,\gamma\rangle\le\varepsilon(\Delta,k)|||\gamma|||_{\mathfrak{m},1,k}.$$

Then choosing ν_1 (cf. Theorem 5.2.10), we obtain the assertion because

$$\langle\mathfrak{l}1\!\!1_\Lambda,\gamma\rangle\le\frac{\varepsilon N!}{(\nu_1)^N}(\nu_1)^N|||\gamma|||_1^N\le\frac{\varepsilon N!}{(\nu_1)^N}\exp\left(\nu_1|||\gamma|||_{\mathfrak{m},1,k}\right).$$

Hence, the assertion follows by Theorem 5.2.10. □

Remark 5.2.12. *If (5.1.33) is void and* $\phi\ge0$*, we get the following result: Fix* $N\in\mathbb{N}$ *and a* $\mathcal{Q}$*-local function* $\tilde{\mathfrak{l}}$*. Assume that*

$$\int_\Lambda\tilde{\mathfrak{l}}(\hat{x})^n\lambda\otimes m(d\hat{x})<\infty,\qquad\textit{for all }\Lambda\in\mathcal{Q}_c(\hat{X})\textit{ and all }1\le n\le N.$$

Then (compare also Remark 4.3.36) we get only that for all $\Lambda\in\mathcal{Q}_c(\hat{X})$ *there exists* $C_N(\Lambda)<\infty$ *such that for each* $\mu\in\mathcal{G}\mathrm{ibbs}_V^{\mathfrak{t}}(\Gamma(\hat{X}))$

$$\int_{\Gamma_f(\Lambda)}\langle\tilde{\mathfrak{l}},\gamma\rangle^N\mu(d\gamma)<C. \tag{5.2.23}$$

5.3 Gibbs measures on the cone

In this section we discuss the transition from $\Gamma(\hat{X})$ to $\mathbb{K}(X)$. Hence, we obtain Gibbs measures on the cone $\mathbb{K}(X)$ corresponding to a sufficiently ‹nice› potential $\phi : X \times X \to \mathbb{R}$. To this end, we first specify the conditions on ϕ.

We fix the following local mass map $\mathfrak{m}$ (cf. Example 4.3.11):

$$\mathfrak{m}(\Lambda, \{(s_x, x)\}) = s_x \mathbb{1}_{\Lambda_X}(x) \quad \forall \Lambda \in \mathcal{B}(\hat{X}),\ \forall (s_x, x) \in \hat{X}.$$

Let us recall some properties of the cone $\mathbb{K}(X)$. Let $F \in \mathcal{F}C_b(\mathbb{K}(X), C_0(X))$ be a cylinder function, i.e., it can be written as

$$F(\eta) = g_F(\langle \varphi_1, \eta\rangle, \dots, \langle \varphi_N, \eta\rangle), \quad \eta \in \mathbb{K}(X) \tag{5.3.1}$$

with some $g_F \in C_b(\mathbb{R}^N)$ and $\varphi_i \in C_0(\hat{X})$, $1 \le i \le N \in \mathbb{N}$. Then for each $\mu \in \mathcal{M}^1(\mathbb{K}(X))$, we have

$$\begin{aligned} &\int_{\mathbb{K}(X)} F(\eta)\mu(d\eta) = \int_{\mathbb{K}(X)} g_F(\langle \varphi_1, \eta\rangle, \dots, \langle \varphi_N, \eta\rangle)\mu(d\eta) \\ &= \int_{\Gamma_f(\hat{X})} g_F(\langle \mathrm{id}_{\mathbb{R}_+} \otimes \varphi_1, \eta\rangle, \dots, \langle \mathrm{id}_{\mathbb{R}_+} \otimes \varphi_N, \eta\rangle)\Big(\big(\mathbb{T}^{-1}\big)^* \mu\Big)(d\gamma). \end{aligned} \tag{5.3.2}$$

We recall that $\mathbb{T}$ is the (bijective) homeomorphism between between $\Gamma_f(\hat{X})$ and $\mathbb{K}(X)$ (cf. (3.1.2)), i.e.,

$$\begin{aligned} \mathbb{T}: \qquad \Gamma_f(\hat{X}) &\to \mathbb{K}(X) \\ \gamma = \{(s_x, x) \,|\, x \in \tau(\gamma)\} &\mapsto \eta := \sum_{x \in \tau(\gamma)} s_x \delta_x. \end{aligned} \tag{5.3.3}$$

5.3.1 Gibbsian formalism on $\mathbb{K}(X)$

We fix a pair potential $\phi : X \times X \to \mathbb{R}$, which is a bounded, symmetric $\mathcal{B}(X \times X)$-measurable function. We define the corresponding pair potential $V_\phi : \hat{X} \times \hat{X} \to \mathbb{R}$ by

$$V_\phi(\hat{x}, \hat{y}) := s_x s_y \phi(x, y), \quad \hat{x}, \hat{y} \in \hat{X}.$$

We fix an admissible partition of X:

Definition 5.3.1 (Admissible partition). *Let Z be a countable index set. A partition*

$$X = \bigsqcup_{k \in Z} Q_k,$$

$Q_k \in \mathcal{B}(X)$ for each $k \in Z$, is admissible, *if the partition $\hat{X} = \bigsqcup_{k \in Z} \hat{Q}_k$, where $\hat{Q}_k := \mathbb{R}_+ \times Q_k$ for all $k \in Z$, is admissible in the sense of Definition 5.1.2.*

We impose the following assumption on ϕ:

$(\phi)_{\mathbb{K}(X)}$ *The* symmetric potential $\phi : X \times X \to \mathbb{R}$ *is such that* $(\mathbf{V})$ *holds for the corresponding V_ϕ with the admissible partition $\hat{X} = \bigsqcup_{k \in Z} \mathbb{R}_+ \times Q_k$, where $Q_k \in \mathcal{B}(X)$ for all $k \in Z$.*

In the case of $X = \mathbb{R}^d$, it is sufficient, e.g., to assume the conditions $(\mathbf{FR}')$, $(\mathbf{LB}')$ and $(\mathbf{RC}')$ (cf. Subsection 5.1.1 and Example 5.1.1).

The relative energy

From here on, we always assume that $(\phi)_{\mathbb{K}(X)}$ holds. We define the algebra of 'local' subsets in X (cf. also Subsection 5.1.2) by

$$Q_c(X) := \{\Delta \mid \mathbb{R}_+ \times \Delta \in Q_c(\hat{X})\}.$$

For each $\eta, \xi \in \mathbb{K}(X)$ and $\Delta \in \mathcal{Q}_c(X)$, we define the *relative energy*

$$H_\Delta(\eta|\xi) := \int_\Delta \int_\Delta \phi(x,y)\eta(dx)\eta(dy) + \int_{\Delta^c} \int_\Delta \phi(x,y)\eta(dx)\xi(dy). \tag{5.3.4}$$

Lemma 5.3.2. *If $(\phi)_{\mathbb{K}(X)}$ holds, then $|H_\Delta(\eta|\xi)| < \infty$ for all $\eta, \xi \in \mathbb{K}(X)$ and $\Delta \in Q_c(X)$.*

Proof. This follows by Lemma 5.1.11 because $\mathbb{T} : \Gamma_f(\hat{X}) \to \mathbb{K}(X)$ is homeomorphic. □

Local specification

By $\mathbb{K}^t(X)$ we denote the *tempered cone* which is the image of $\Gamma^{\mathrm{t}}(\hat{X})$ (cf. Subsection 5.1.4) under $\mathbb{T}$ (cf. (5.3.3), i.e.,

$$\mathbb{K}^t(X) := \mathbb{T}\left(\Gamma^{\mathrm{t}}(\hat{X})\right). \tag{5.3.5}$$

Let us fix an *inverse temperature* $\beta = 1/T > 0$. For each $\Delta \in \mathcal{Q}_c(X)$, the *local Gibbs measure* on $\mathbb{K}(\Delta)$ is defined by

$$\mu_\Delta(d\eta|\xi) := \begin{cases} \frac{1}{Z_\Delta(\xi)} e^{-\beta H_\Delta(\eta|\xi)} \mathcal{G}_\theta^\Delta(d\eta), & \text{if } \eta, \xi \in \mathbb{K}^t(X), \\ 0, & \text{otherwise,} \end{cases} \tag{5.3.6}$$

where $\mathcal{G}_\theta^\Delta$ is the Gamma measure with shape parameter θ on $\mathbb{K}(\Delta)$ and

$$Z_\Delta(\xi) := \int_{\mathbb{K}(\Delta)} e^{-\beta H_\Delta(\eta|\xi)} \mathcal{G}_\theta^\Delta(d\eta). \tag{5.3.7}$$

is the *partition function* (i.e., normalizing factor).

Remark 5.3.3. *For all $\xi \in \mathbb{K}^t(X)$, (5.3.7) is well-defined, i.e.,*

$$0 < Z_\Delta(\xi) < \infty.$$

For $\phi \geq 0$, this follows by Lemma 4.5.4, and furthermore $Z_\Delta(\xi) \leq 1$. For general ϕ obeying (5.1.33), this follows by Lemma 5.1.14.

Definition 5.3.4. *The* local specification $\pi = \{\pi_\Delta\}_{\Delta \in \mathcal{Q}_c(X)}$ *on* $\mathbb{K}(X)$ *is a family of stochastic kernels*

$$\mathcal{B}(\mathbb{K}(X)) \times \mathbb{K}(X) \ni (B, \xi) \mapsto \pi_\Delta(B|\xi) \in [0, 1] \tag{5.3.8}$$

given by

$$\begin{aligned} \pi_\Delta(B|\xi) &:= \mu_\Delta(B_{\Delta,\xi}|\xi), \\ B_{\Delta,\xi} &:= \{\gamma_\Delta \in K(\Delta) \,|\, \gamma_\Delta \cup \xi_{\Delta^c} \in B\} \in \mathcal{B}(K(\Delta)). \end{aligned} \tag{5.3.9}$$

As we will see below, there is a one-to-one correspondence with the semi-local specification kernels $\pi_{\mathbb{R}_+ \times \Delta}(d\gamma|\xi)$ on $\Gamma(\hat{X})$ (cf. Subsection 4.5.1).

Gibbs measure

Definition 5.3.5 (Gibbs measure on the cone $\mathbb{K}(X)$)**.** *A probability measure $\mu \in \mathcal{M}^1(\mathbb{K}(X))$ (that has a-priori full measure on $\mathbb{K}^t(X)$) is called a* tempered **Gibbs measure** *(or* **state***) with pair potential ϕ and inverse temperature $\beta > 0$ if it satisfies the* Dobrushin-Lanford-Ruelle ***(DLR)*** *equilibrium equation*

$$\int_{\mathbb{K}(X)} \pi_\Delta(B|\eta)\mu(\mathrm{d}\eta) = \mu(B) \tag{5.3.10}$$

for all $\Delta \in \mathcal{Q}_c(X)$ and $B \in \mathcal{B}(\mathbb{K}(X))$. The associated set of all Gibbs states will be denoted by $\mathcal{G}\mathrm{ibbs}_\phi^t(\mathbb{K}(X))$.

5.3.2 One-to-one correspondence between between Gibbs measures on $\mathbb{K}(X)$ and $\Gamma(\hat{X})$

The following theorem shows the intrinsic connection between the *semi-local* specification kernels on $\Gamma(\hat{X})$ and the *local* specification kernels on $\mathbb{K}(X)$.

Theorem 5.3.6. *Let $\phi : X \times X \to \mathbb{R}$ be such that $(\phi)_{\mathbb{K}(\mathbf{X})}$ holds. Since $\mathcal{F}C_b(\mathbb{K}(X), C_0(X))$ are a measure defining class, we have a one-to-one correspondence between local Gibbs specifications π_Δ on $\mathbb{K}(X)$ and semi-local ones $\pi_{\mathbb{R}_+\times\Delta}$ on $\Gamma(\hat{X})$. Indeed, for each $F \in \mathcal{F}C_b(\mathbb{K}(X), C_0(X))$, obeying (cf. (5.3.1))*

$$F(\eta) = g_F(\langle\varphi_1, \eta\rangle, \dots, \langle\varphi_N, \eta\rangle), \quad \eta \in \mathbb{K}(X),$$

with some $g_F \in C_b(\mathbb{R}^N)$, $\varphi_1, \dots, \varphi_N \in C_0(\hat{X})$ and $N \in \mathbb{N}$, we have

$$\begin{aligned}&\int_{\mathbb{K}(X)} F(\eta)\pi_\Delta(d\eta|\xi)\\ &= \int_{\Gamma_f(\hat{X})} g_F(\langle \mathrm{id}_{\mathbb{R}_+}\otimes\varphi_1, \gamma\rangle, \dots, \langle \mathrm{id}_{\mathbb{R}_+}\otimes\varphi_N, \eta\rangle)\pi_{\mathbb{R}_+\times\Delta}(d\gamma|\mathbb{T}^{-1}\xi)\end{aligned} \tag{5.3.11}$$

for all $\xi \in \mathbb{K}(X)$ and $\Delta \in \mathcal{Q}_c(X)$.

This implies that there is a one-to-one correspondence between the Gibbs measures on $\Gamma(\hat{X})$ and on $\mathbb{K}(X)$: We have

$$\mu_\mathbb{K} \in \mathcal{G}\mathrm{ibbs}^{\boldsymbol{t}}_\phi(\mathbb{K}(X)) \quad \Longleftrightarrow \quad \mu_\Gamma := \left(\mathbb{T}^{-1}\right)^*\mu_\mathbb{K} \in \mathcal{G}\mathrm{ibbs}^{\boldsymbol{t}}_{V_\phi}(\Gamma(\hat{X})).$$

Using F as above, we have for $\mu_\mathbb{K} \in \mathcal{G}\mathrm{ibbs}^{\boldsymbol{t}}_\phi(\mathbb{K}(X))$:

$$\begin{aligned}&\int_{\mathbb{K}(X)} F(\eta)\mu_\mathbb{K}(d\eta)\\ &\qquad = \int_{\Gamma_f(\hat{X})} g_F(\langle \mathrm{id}_{\mathbb{R}_+}\otimes\varphi_1, \gamma\rangle, \dots, \langle \mathrm{id}_{\mathbb{R}_+}\otimes\varphi_N, \gamma\rangle)\mu_\Gamma(d\gamma),\end{aligned} \tag{5.3.12}$$

where $\mathrm{id}_{\mathbb{R}_+}$ is the identity on $\mathbb{R}_+$, i.e., $\mathrm{id}_{\mathbb{R}_+}(s) = s$ for all $s \in \mathbb{R}_+$.

Proof. Follows by the construction. We give the details: If $\xi \in \Gamma_f(\hat{X})\backslash\Gamma^{\mathsf{t}}(\hat{X})$, then (5.3.11) holds trivially (cf. (5.1.46) and (5.3.6)). Let $\xi \in \Gamma^{\mathsf{t}}(\hat{X})$ and F be as in the assertion. Fix $\Delta \in \mathcal{Q}_c(X)$. Then by (5.3.2)

$$\begin{aligned}&\int_{\mathbb{K}(X)} F(\gamma)e^{-\beta H_\Delta(\eta|P(\xi))}\mathcal{G}^\Delta_\theta(d\eta)\\ &= \int_{\Gamma(\hat{X})} g_F(\langle \mathrm{id}_{\mathbb{R}_+}\otimes\varphi_1, \gamma\rangle, \dots, \langle \mathrm{id}_{\mathbb{R}_+}\varphi_N, \eta\rangle)e^{\beta H_{\mathbb{R}_+\times\Delta}(\gamma|\xi)}\mathcal{P}^\Delta_\theta(d\gamma).\end{aligned}$$

Hence, (5.3.11) follows. Fix $B \in \mathcal{B}(\mathbb{K}(X))$. Note that

$$B \in \mathcal{B}(\mathbb{K}(X)) \Leftrightarrow \tilde{B} := \mathbb{T}^{-1}(B) \in \mathcal{B}(\Gamma(\hat{X})).$$

Let $\mu \in \mathcal{G}\mathrm{ibbs}^{\mathbf{t}}_{\phi}(\mathbb{K}(X))$. Using $\mu(\mathbb{K}^t(X)) = 1$, the (**DLR**) equation and (5.3.2), we get

$$\mu_\Gamma(\tilde{B}) = \mu_{\mathbb{K}}(B) = \int_{\mathbb{K}^t(X)} \pi_\Delta(B|\xi)\mu_{\mathbb{K}}(d\xi) = \int_{\Gamma^{\mathbf{t}}(\hat{X})} \pi_{\mathbb{R}_+\times\Delta}(\tilde{B}|\tilde{\xi}))\mu_\Gamma(d\tilde{\xi}).$$

Hence, $\mu_\Gamma \in \mathcal{G}\mathrm{ibbs}^{\mathbf{t}}_{V_\phi}(\Gamma(\hat{X}))$. On the other hand, if $\mu_\Gamma \in \mathcal{G}\mathrm{ibbs}^{\mathbf{t}}_{V_\phi}(\Gamma(\hat{X}))$, then

$$\mu_{\mathbb{K}}(B) = \mu_\Gamma(\tilde{B}) = \int_{\Gamma^{\mathbf{t}}(\hat{X})} \pi_{\mathbb{R}_+\times\Delta}(\tilde{B}|\tilde{\xi})\mu_\Gamma(d\tilde{\xi}) = \int_{\mathbb{K}^t(X)} \pi_\Delta(B|\xi)\mu_{\mathbb{K}}(d\xi).$$

This implies that $\mu_{\mathbb{K}} \in \mathcal{G}\mathrm{ibbs}^{\mathbf{t}}_{\phi}(\mathbb{K}(X))$. □

5.3.3 Existence of Gibbs measures

Using Theorem 5.3.6, we will transfer the results obtained for Gibbs measures $\mu_\Gamma \in \mathcal{G}\mathrm{ibbs}^{\mathbf{t}}_{V_\phi}(\Gamma(\hat{X}))$ to Gibbs measures $\mu_{\mathbb{K}} \in \mathcal{G}\mathrm{ibbs}_{\phi}(\mathbb{K}(X))$. This yields the existence result and uniform moment estimates.

Theorem 5.3.7. *Let ϕ be such that $(\phi)_{\mathbb{K}(X)}$ is fulfilled. Then there exists a tempered Gibbs measure on $\mathbb{K}(X)$, i.e.,*

$$\mathcal{G}\mathrm{ibbs}^{t}_{\phi}(\mathbb{K}(X)) \neq \varnothing.$$

Proof. Using Theorem 5.3.6, the assertions follow by Theorem 5.2.8. □

Remark 5.3.8. *Furthermore, the set $\mathcal{G}\mathrm{ibbs}^{t}_{\phi}(\mathbb{K}(X))$ is compact in $\mathcal{T}_{str}$ (cf. Definition 5.3.9 below).*

Analogue to $\mathcal{T}_{\mathcal{Q}}(\Gamma(\hat{X}))$ (cf. Definition 5.1.22), we introduce

Definition 5.3.9. *The topology of* local stripwise convergence *on $\mathcal{M}^1(\mathbb{K}(X))$, denoted by $\mathcal{T}_{\mathrm{str}}$, is defined as the coarsest topology making the maps $\mu \mapsto \mu(B)$ continuous for all sets B from the algebra of ‹stripe› events*

$$\mathcal{B}_{str}(\mathbb{K}(X)) = \bigcup_{\Delta\in\mathcal{Q}_c(X)} \mathcal{B}_\Delta(\mathbb{K}(X)),$$

where $\mathcal{B}_\Delta(\mathbb{K}(X)) := \mathbb{P}_\Delta^{-1}\mathcal{B}(\mathbb{K}(\Delta))$. Here, we define the projections $\mathbb{P}_\Delta$ by

$$\mathbb{P}_\Delta : \quad \mathbb{K}(X) \ni \eta \mapsto \eta_\Delta := \sum_{x\in\tau(\eta)\cap\Delta} s_x\delta_x \in \mathbb{K}(\Delta).$$

Moment estimates

Theorem 5.3.10. *Let* $(\phi)_{\mathbb{K}(X)}$ *be fulfilled. For each* $\Delta \in \mathcal{Q}_c(X)$ *and* $N \in \mathbb{N}$ *there exists* $C_N(\Delta) > 0$ *such that for all* $\mu_{\mathbb{K}} \in \mathcal{G}\mathrm{ibbs}_\phi^{\boldsymbol{t}}(\mathbb{K}(X))$

$$\int_{\mathbb{K}(X)} \eta(\Delta)^N \mu_{\mathbb{K}}(d\eta) < C_N(\Delta). \tag{5.3.13}$$

Proof. Using Theorem 5.3.6, this follows by Corollary 5.2.11 (if (5.1.33) holds), resp. by Theorem 4.3.34 and Remark 4.3.36 (if (5.1.33) is void). □

For stable potentials, we have exponential moment bounds:

Theorem 5.3.11. *Let* $(\phi)_{\mathbb{K}(X)}$ *and (5.1.33) hold. For each* $\alpha > 0$ *one finds a certain* $\nu_\alpha > 0$ *such that for all* $\mu_{\mathbb{K}} \in \mathcal{G}\mathrm{ibbs}_\phi^{\boldsymbol{t}}(\mathbb{K}(X))$

$$\int_{\mathbb{K}(X)} \exp\left\{\nu_\alpha |\|\eta\||_\alpha^2\right\} \mu_{\mathbb{K}}(d\eta) \le \mathcal{C}_\alpha,$$

where $\mathcal{C}_\alpha < \infty$ *is the same as in Proposition 5.2.4 (cf. (5.2.7)).*

Proof. Follows by Theorem 5.2.10. □

Part III

Differential calculus

Chapter 6

Differential calculus and Dirichlet forms

In Chapter 3 we saw the static picture, now we will introduce some movement of the marks and positions. Our motivation is to use Dirichlet forms: Roughly speaking, for each quasi-regular and local Dirichlet form, there exists an associated diffusion.

Prominent examples for diffusions over spaces of measures are Fleming-Viot processes, which are motivated by biological considerations (cf. [Hoc91, EK93] and Chapter 7 for details). They are located on a space of probability measures. Dirichlet forms in the configuration space framework are considered, e.g., in [AKR98a, AKR98b] and [KLR99].

Although our construction of Dirichlet forms is related to the one on marked configuration spaces (cf. e.g. [KdSS98, KLU99]), there is an essential difference: Gamma measures can be viewed as Poisson measure on a "marked" configuration space with an infinite measure on the marks, whereas the mentioned references treat the case that the measure on the marks is *finite*. More recently, Wasserstein diffusions and entropic measures have been studied using Dirichlet form in [vRS09, AvR10].

The theory of Dirichlet forms is explained, for example, in [MR92] or, the symmetric case, in [FOT94]. We outline a general scheme how to get an equilibrium process:

1. Identify an appropriate directional derivative (defined via a translation group) and a tangent space T_η at $\eta \in \mathbb{K}(\mathbb{R}^d)$ to get a corresponding gradient ∇ on functions over $\mathbb{K}(\mathbb{R}^d)$.

2. Choose a measure μ on $(\mathbb{K}(\mathbb{R}^d), \mathcal{B}(\mathbb{K}(\mathbb{R}^d)))$ and deduce a quasi-invariance

property of μ w.r.t. the translation group.

3. Use the quasi-invariance property to establish an integration by parts formula.

4. Show that the set of functions $\mathbb{S}$ admitting this integration by parts formula is dense in $L^2(\mathbb{K}(\mathbb{R}^d), \mu)$.

5. Define a corresponding gradient bilinear form

$$\mathcal{E}^\mu(F, G) = \int_\Gamma \langle \nabla F(\eta), \nabla G(\eta)\rangle_{T_\eta}\, \mu(d\eta), \quad \forall F, G \in \mathbb{S}$$

   and deduce via the integration by parts formula that it is closable.

6. Prove that the closure of $(\mathcal{E}^\mu, \mathbb{S})$ w.r.t. the norm that is induced by the bilinear form, namely

$$\mathbb{S} \ni F \mapsto \left(\mathcal{E}_1^\mu(F, F)\right)^{1/2} := \left(\langle F, F\rangle_{L^2(\mathbb{K}(\mathbb{R}^d),\mu)} + \mathcal{E}^\mu(F, F)\right)^{1/2},$$

   is a Dirichlet form. It is denoted by $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$.

7. Deducing that it is quasi-regular, one obtains an associated "nice" Markov process (cf. [MR92, Thm IV.3.5]).

8. If $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is local, one even gets an associated diffusion.

In this chapter, we will study the first six steps, whereas the last two are left to be analyzed in Chapter 7. Our main motivation is to get Dirichlet forms related to the Gibbs perturbations of Gamma measure $\mathcal{G}_\theta$ on $\mathbb{K}(\mathbb{R}^d)$.

Let us briefly indicate, where to find the main steps of the above program: In Section 6.1, we construct an *extrinsic* ($\nabla^{\mathbb{K}}_{\text{ext}}$), an *intrinsic* ($\nabla^{\mathbb{K}}_{\text{int}}$) and a joint ($\nabla^{\mathbb{K}}$) *gradient* (cf. Subsections 6.1.2, 6.1.3 and 6.1.4). They exist for all *cylinder functions*

$$F \in \mathcal{F}C_b^\infty(\mathbb{K}(\mathbb{R}^d), C_0^\infty(\mathbb{R}^d)) =: \mathbb{S}_{\mathbb{K}}$$

(cf. Definition 6.1.5). In Section 6.3, we get *extrinsic*, *intrinsic* and *joint integration by parts* formulas w.r.t. $\mathcal{G}_\theta$ (cf. Theorems 6.3.19, 6.3.33 and 6.3.39). This is the main step to consider corresponding *Dirichlet forms*. In order to apply the Dirichlet form approach (compare the 4$^{\text{th}}$ point), we show in Section 6.2 that (cf. Theorem 6.2.7 and esp. Corollary 6.2.8)

$$\mathbb{S}_{\mathbb{K}} \subset L^2(\mathbb{K}(\mathbb{R}^d), \mathcal{G}_\theta) \text{ is dense.}$$

Let us consider the *bilinear form* (cf. (6.3.46))

$$\mathcal{E}^{\mathcal{G}_\theta}(F,G) := \int_{\mathbb{K}(\mathbb{R}^d)} \langle \nabla^{\mathbb{K}} F(\eta), \nabla^{\mathbb{K}} G(\eta) \rangle_{T_\eta(\mathbb{K})} \, \mathcal{G}_\theta(d\eta), \quad \forall F,G \in \mathbb{S}_{\mathbb{K}}.$$

Using the integration by parts formula, we prove that it is closable and that its closure is a conservative Dirichlet form (cf. Theorem 6.3.48). Analogous results hold in the intrinsic and extrinsic case (cf. Theorems 6.3.29 and 6.3.38).

We deduce the above results for more general measures on $\mathbb{K}(\mathbb{R}^d)$:

1. Let $\mathcal{G}_\lambda$ on $\mathbb{K}(\mathbb{R}^d)$ be a Levy measure (cf. Definition 3.1.5) whose intensity measure λ on $\mathbb{R}_+$ has first and second moments, i.e.,

$$m_1(\lambda) + m_2(\lambda) = \int_{\mathbb{R}_+} \left(s + s^2\right) \lambda(ds) < \infty.$$

   Then the intrinsic results hold for $\mathcal{G}_\lambda$ (cf. Theorems 6.3.8, 6.3.14 and Proposition 6.3.12 in Subsection 6.3.1).

2. The extrinsic, intrinsic and joint results are extended to the Gibbsian case (cf. Theorems 6.3.19, 6.3.33, 6.3.39; 6.3.29, 6.3.38 and 6.3.48).

The above results are extended to a connected, orientated C^∞ Riemannian manifold X, which we fix from now on. Particular results of this chapter are published in [Hag11].

6.1 Differential geometry on the cone $\mathbb{K}(X)$

Let us start our scheme by introducing a differential geometry on $\mathbb{K}(X)$. We construct a gradient on functions over $\mathbb{K}(X)$ consisting of an *extrinsic* part related to the motion of marks (cf. Subsections 6.1.2) and an *intrinsic* one related to the motion of positions (cf. Subsections 6.1.3). They are joined to get the gradient w.r.t. changing marks and positions (cf. Subsections 6.1.4).[1]

[1]For Subsection 6.1.2, it is sufficient that X is a locally compact Polish space. But, in order to keep a clearer structure, we already fixed X to be a connected, orientated C^∞ Riemannian manifold at the beginning of this chapter.

6.1.1 Group of motions

Similar as in [KLU99, Section 2], we consider the group of continuous currents, i.e., all continuous mappings

$$X \ni x \mapsto a(x) \in \mathbb{R}_+$$

being equal to 1 outside a compact set. We define a multiplication $a_1 a_2$ in this group as the pointwise multiplication of the mappings a_1 and a_2 and denote this group by $\mathbb{R}_+^X$.

Let $\mathrm{Diff}_0(X)$ denote the group of diffeomorphism on X with compact support, i.e., which equal the identity map id_X outside of a compact set. It acts in $\mathbb{R}_+^X$ by automorphisms: for each $\varphi \in \mathrm{Diff}_0(X)$, we have

$$\mathbb{R}_+^X \ni a \mapsto a \circ \varphi^{-1} \in \mathbb{R}_+^X.$$

Thus, we can endow the Cartesian product of $\mathrm{Diff}_0(X)$ and $\mathbb{R}_+^X$ with the following multiplication: for $g_1 = (\varphi_1, a_1)$, $g_2 = (\varphi_2, a_2) \in \mathrm{Diff}_0(X) \times \mathbb{R}_+^X$

$$g_1 g_2 = (\varphi_1 \circ \varphi_2, a_1(a_2 \circ \varphi_1^{-1}))$$

and obtain a semidirect product

$$\mathfrak{G} := \mathrm{Diff}_0(X) \rightthreetimes \mathbb{R}_+^X$$

of the groups $\mathrm{Diff}_0(X)$ and $\mathbb{R}_+^X$. The group $\mathfrak{G}$ acts on $\mathbb{R}_+ \times X$ for each $g = (a, \psi) \in \mathfrak{G}$ via

$$\mathbb{R}_+ \times X \ni (s, x) \mapsto g(s, x) = (a(\psi(x))s, \psi(x)) \in \mathbb{R}_+ \times X.$$

6.1.2 Extrinsic Gradient

We define for each $t \in \mathbb{R}_+$ and $h \in C_0(X)$ the *translation*

$$\begin{aligned} M_{th} : \mathbb{K}(X) &\rightarrow \mathbb{K}(X) \\ \eta &\mapsto \eta_t^h := e^{th}\eta = \sum_{x \in \tau(\eta)} e^{th(x)} s_x \delta_x. \end{aligned}$$

Definition 6.1.1. *The* extrinsic directional derivative *of a function $F : \mathbb{K}(X) \to \mathbb{R}$ in direction $h \in C_0(X)$ is defined as*

$$\nabla^{\mathbb{K}}_{ext,h} F(\eta) := \left. \frac{d}{dt} F(\eta_t^h) \right|_{t=0},$$

whenever the expression on the right-hand side exists.

Remark 6.1.2. *1. The transformation only changes the marks of the discrete measure η. Therefore, we call them* extrinsic.

2. By definition the directional derivate fulfills the product rule, i.e., for $F, G : \mathbb{K}(X) \to \mathbb{R}$ for which the directional derivative exists we get

$$\nabla^{\mathbb{K}}_{ext,h}(F \cdot G)(\eta) = \nabla^{\mathbb{K}}_{ext,h}F(\eta) \cdot G(\eta) + F(\eta) \cdot \nabla^{\mathbb{K}}_{ext,h}G(\eta).$$

Definition 6.1.3. *We choose the* extrinsic tangent space *of $\mathbb{K}(X)$ at $\eta \in \mathbb{K}(X)$ to be*

$$T^{\mathrm{ext}}_{\eta}(\mathbb{K}) := L^2(X, \eta). \tag{6.1.1}$$

Definition 6.1.4. *The* extrinsic gradient $\nabla^{\mathbb{K}}_{ext}$ *of a function $F : \mathbb{K}(X) \to \mathbb{R}$ is defined as*

$$\begin{aligned} \nabla^{\mathbb{K}}_{ext}F : \quad & \mathbb{K}(X) \to T^{\mathrm{ext}}_{\cdot}(\mathbb{K}) \\ & \eta \to (\nabla^{\mathbb{K}}_{ext}F)(\eta) \quad \in T^{\mathrm{ext}}_{\eta}(\mathbb{K}), \end{aligned}$$

whenever the extrinsic directional derivative of that function in each direction $h \in C_0(X)$ exists and it holds that for all $h \in C_0(X)$

$$\nabla^{\mathbb{K}}_{ext,h}F(\eta) = \big\langle \nabla^{\mathbb{K}}_{ext}F(\eta), h \big\rangle_{T^{\mathrm{ext}}_{\eta}(\mathbb{K})} =: \langle \nabla^{\mathbb{K}}_{ext}F, h \rangle(\eta). \tag{6.1.2}$$

Existence for cylinder functions

We show the existence of the extrinsic gradient for some cylinder functions:

Definition 6.1.5 (Cylinder functions)**.** *Let $k, l \in \mathbb{N}_0^{\infty}$ and*

$$C_0^k(X) := C^k(X) \cap C_0(X).$$

By $\mathcal{F}C_b^k(\mathbb{K}(X), C_0^l(X))$ we denote the set of those cylinder functions *$F : \mathbb{K}(X) \to \mathbb{R}$ that can be represented as*

$$F(\eta) = g_F\big(\langle \phi_1, \eta \rangle, \ldots, \langle \phi_N, \eta \rangle\big), \tag{6.1.3}$$

where $N \in \mathbb{N}$, $g_F \in C_b^k(\mathbb{R}^N)$ and $\phi_i \in C_0^l(X)$ for $i = 1, \ldots, N$.

Remark 6.1.6. *The set of cylinder functions that we consider differs from the usual one that one might expect to use (cf. e.g. [AKR98a, KLR99]): For the configuration space $\Gamma(\hat{\mathbb{R}}^d)$ one normally calculates the gradient for $\mathcal{F}C_b^{\infty}(\Gamma(\hat{\mathbb{R}}^d), C_0(\mathbb{R}_+ \times \mathbb{R}^d))$. But, this does not fit our geometrical structure. However, the set $\mathcal{F}C_b^1(\mathbb{K}(X), C_0(X))$, which we consider, is sufficient to define the Dirichlet forms because it is dense in $L^2(\mathbb{K}(X), \mathcal{G}_\theta)$ (cf. Corollary 6.2.8).*

Fix $F = g_F\big(\langle\phi_1,\cdot\rangle,\dots,\langle\phi_N,\cdot\rangle\big) \in \mathcal{F}C_b^1(\mathbb{K}(X),C_0(X))$, where $N \in \mathbb{N}$, $g_F \in C_b^1(\mathbb{R}^N)$ and $\phi_i \in C_0(X)$ for $i = 1,\dots,N$.

Proposition 6.1.7. *For each $h \in C_0(X)$ and each $\eta \in \mathbb{K}(X)$, we get*

$$(\nabla^{\mathbb{K}}_{ext,h}F)(\eta) \;=\; \sum_{i=1}^{N}\partial_i g_F(\langle\rho_1,\eta\rangle,\dots,\langle\rho_n,\eta\rangle)\langle h,\rho_i\rangle_{T^{\mathrm{ext}}_\eta(\mathbb{K})}.$$

Proof. The formula for the directional derivative follows by the chain rule: Fix $h \in C_0(X)$ and $\eta \in \mathbb{K}(X)$. Then

$$\begin{aligned}\nabla^{\mathbb{K}}_{\mathrm{ext},h}F(\eta) &= \frac{d}{dt}F(e^{th}\eta)\bigg|_{t=0} = \frac{d}{dt}g_F(\langle\rho_1,e^{th}\eta\rangle,\dots,\langle\rho_N,e^{th}\eta\rangle)\bigg|_{t=0}\\ &= \sum_{i=1}^{N}\partial_i g_F(\langle\rho_1,e^{th}\eta\rangle,\dots,\langle\rho_N,e^{th}\eta\rangle)\frac{d}{dt}\langle\rho_i,e^{th}\eta\rangle)\bigg|_{t=0}.\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{d}{dt}\langle\rho_i,e^{th}\eta\rangle &= \frac{d}{dt}\sum_{x\in\tau(\eta)}\rho_i(x)e^{th(x)}\eta(x) = \sum_{x\in\tau(\eta)}h(x)\rho_i(x)e^{th(x)}\eta(x)\\ &= \langle h\rho_i e^{th},\eta\rangle. \qquad (6.1.4)\end{aligned}$$

(The sums are finite because $\rho_i \in C_0(X)$, cf. (3.1.1)). With (6.1.1) we get

$$\frac{d}{dt}\langle\rho_i,e^{th}\eta\rangle = \langle h,\rho_i e^{th}\rangle_{T^{\mathrm{ext}}_\eta(\mathbb{K})}$$

and conclude (t=0)

$$(\nabla^{\mathbb{K}}_{\mathrm{ext},h}F)(\eta) = \sum_{i=1}^{N}\partial_i g_F(\langle\rho_1,\eta\rangle,\dots,\langle\rho_n,\eta\rangle)\langle h,\rho_i\rangle_{T^{\mathrm{ext}}_\eta(\mathbb{K})}.$$

$\square$

Proposition 6.1.8. *The gradient exists and is*

$$(\nabla^{\mathbb{K}}_{ext}F)(\eta) = \sum_{i=1}^{N}\partial_i g_F(\langle\rho_1,\eta\rangle,\dots,\langle\rho_N,\eta\rangle)\rho_i, \qquad \textit{for all } \eta \in \mathbb{K}(X).$$

Explicitly writing the argument $x \in X$ we see

$$(\nabla^{\mathbb{K}}_{ext}F)(\eta,x) = \sum_{i=1}^{N}\partial_i g_F(\langle\rho_1,\eta\rangle,\dots,\langle\rho_N,\eta\rangle)\rho_i(x).$$

Proof. By Proposition 6.1.7 and Definition 6.1.3 we have for all $h \in C_0(X)$ that

$$\begin{aligned}\langle \nabla^{\mathbb{K}}_{\text{ext}} F(\eta), h\rangle_{T^{\text{ext}}_\eta(\mathbb{K})} =& \sum_{i=1}^N \partial_i g_F(\langle \rho_1, \eta\rangle, \dots, \langle \rho_N, \eta\rangle)\langle h, \rho_i\rangle_{T^{\text{ext}}_\eta(\mathbb{K})}\\ =&(\nabla^{\mathbb{K}}_{\text{ext},h} F)(\eta).\end{aligned}$$

□

6.1.3 Intrinsic Gradient

In Subsection 6.1.2 we constructed a gradient for the motion of the marks, now we define one for the motion of the positions in the state space X. Here, we proceed similarly as in the case of a marked configuration space (cf. [KdSSU98, Section 3]).[2]

Let $V(X)$ denote the set of all C^∞-vector fields on X (i.e. smooth sections of the tangent space $T(X)$). We use that subset $V_0(X) \subset V(X)$ which consists of all those vector fields with compact support.

For any $x \in X$, $v \in V_0(X)$ the curve $\mathbb{R} \ni t \mapsto \phi^v_t(x) \in X$ is defined as the solution to the following Cauchy problem

$$\begin{cases} \frac{d}{dt}\phi^v_t(x) & = v(\phi^v_t(x)) \\ \phi^v_0(x) & = x \end{cases}.$$

Since $v \in V_0(X)$ has compact support, there exists a solution to this Cauchy problem. Furthermore, the mappings $\{\phi^v_t, t \in \mathbb{R}\}$ form a one-parameter subgroup in $\mathrm{Diff}_0(X)$.

We fix $v \in V_0(X)$. Having the group ϕ^v_t, $t \in \mathbb{R}$, we can consider for any $\eta \in \mathbb{K}(X)$ the curve $\mathbb{R} \ni t \mapsto \phi^{v*}_t(\eta) \in \mathbb{K}(X)$, i.e., for all $f \in C_0(X)$ we have

$$\int_X f(y)\big(\phi^{v*}_t\eta\big)(dy) = \int_X f\big(\phi^v_t(x)\big)\eta(dx) = \sum_{x\in\tau(\eta)} s_x f\big(\phi^v_t(x)\big).$$

Hence,

$$\mathbb{R} \ni t \mapsto \phi^{v*}_t \eta = \sum_{x\in\tau(\eta)} s_x \delta_{\phi^v_t(x)} \in \mathbb{K}(X). \tag{6.1.5}$$

[2] In this subsection we use more of the structure of X, namely the fact that X is a connected, orientated C^∞ (non-compact) Riemannian manifold (cf. Definition 6.1.11 and Proposition 6.1.14).

Definition 6.1.9. *For a function $F : \mathbb{K}(X) \to \mathbb{R}$ we define the* intrinsic directional derivative *along the vector field $v \in V_0(X)$ as*

$$(\nabla^{\mathbb{K}}_{int,v} F)(\eta) := \frac{d}{dt} F(\phi_t^{v*}\eta)\Big|_{t=0},$$

provided the right-hand side exists.

Remark 6.1.10. *It is called* intrinsic *because only the positions change while the marks are fixed (cf. (6.1.5)).*

Definition 6.1.11. *We define the* intrinsic tangent space $T_\eta^{\mathrm{int}}(\mathbb{K})$ *to the cone $\mathbb{K}(X)$ at $\eta \in \mathbb{K}(X)$ to be the Hilbert space $L^2(X \to T(X), \eta)$ of measurable η-square integrable sections (measurable vector fields) $V_\eta : X \to T(X)$ with the scalar product*

$$\langle V_\eta^1, V_\eta^2 \rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})} := \int_X \langle V_\eta^1(x), V_\eta^2(x) \rangle_{T_x(X)} \eta(dx),$$

where $V_\eta^1, V_\eta^2 \in T_\eta^{\mathrm{int}}(\mathbb{K})$.

Remark 6.1.12. *If $\rho \in C_0^\infty(X)$, then $\nabla^X \rho \in T_\eta^{\mathrm{int}}(\mathbb{K})$ for all $\eta \in \mathbb{K}(X)$.*

Definition 6.1.13. *Let $F : \mathbb{K}(X) \to \mathbb{R}$ be such that the intrinsic directional derivative $\nabla^{\mathbb{K}}_{int,v} F$ exists for all $v \in V_0(X)$. The* intrinsic gradient $\nabla^{\mathbb{K}}_{int}$ *of F is defined as the mapping $\mathbb{K}(X) \ni \eta \mapsto (\nabla^{\mathbb{K}}_{int} F)(\eta) \in T_\eta^{\mathrm{int}}(\mathbb{K})$ such that*

$$(\nabla^{\mathbb{K}}_{int,v} F)(\eta) = \langle (\nabla^{\mathbb{K}}_{int} F)(\eta), v \rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})} \qquad \text{for all } v \in V_0(X). \quad (6.1.6)$$

The intrinsic gradient $\nabla^{\mathbb{K}}_{int}$ is defined for all those functions for which the above holds.

Existence for cylinder functions

Fix $F = g_F(\langle \rho_1, \eta \rangle, \dots, \langle \rho_N, \eta \rangle) \in \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X))$.

Proposition 6.1.14. *For all $v \in V_0(X)$ the intrinsic directional derivative $\nabla^{\mathbb{K}}_{int,v} F$ exists:*

$$(\nabla^{\mathbb{K}}_{int,v} F)(\eta) = \sum_{i=1}^N \partial_i g_F(\langle \rho_1, \eta \rangle, \dots, \langle \rho_N, \eta \rangle) \langle \nabla_v^X \rho_i, \eta \rangle.$$

Here, $\nabla_v^X \rho$ is the directional derivative *of $\rho \in C_0^\infty(X)$ along the vector field $v \in V_0(X)$, i.e.,*

$$(\nabla_v^X \rho)(x) = \langle \nabla^X \rho(x), v(x) \rangle_{T_x(X)},$$

where ∇^X *denotes the* gradient *over* X. *Furthermore,*

$$(\nabla^{\mathbb{K}}_{int,v}F)(\eta) = \langle \sum_{i=1}^{N} \partial_i g_F(\langle \rho_1, \eta\rangle, \ldots, \langle \rho_N, \eta\rangle)\nabla^X \rho_i, v\rangle_{T^{\mathrm{int}}_\eta(\mathbb{K})}.$$

Proof. For any $\eta \in \mathbb{K}(X)$, $v \in V_0(X)$ we have

$$\begin{aligned} F(\phi_t^{v*}\eta) &= g_F(\langle \rho_1, \phi_t^{v*}\eta\rangle, \ldots, \langle \rho_N, \phi_t^{v*}\eta\rangle) \\ &= g_F(\langle \rho_1 \circ \phi_t^v, \eta\rangle, \ldots, \langle \rho_N \circ \phi_t^v, \eta\rangle). \end{aligned}$$

This implies the first assertion. The second one follows by the linearity of the pairing and using that $(\nabla^X_v \rho)(x) = \langle \nabla^X \rho(x), v(x)\rangle_{T_x(X)}$. $\square$

6.1.4 Joint gradient

After having defined an extrinsic and an intrinsic gradient, we join them to obtain a gradient over the cone $\mathbb{K}(X)$ w.r.t. both components.

Definition 6.1.15. *Let* $h \in C_0(X)$ *and* $v \in V_0(X)$, *then the* directional derivative *of a function* $F : \mathbb{K}(X) \to \mathbb{R}$ *at the point* $\eta \in \mathbb{K}(X)$ *is defined to be*

$$(\nabla^{\mathbb{K}}_{h,v}F)(\eta) := (\nabla^{\mathbb{K}}_{ext,h}F)(\eta) + (\nabla^{\mathbb{K}}_{int,v}F)(\eta).$$

We set the tangent space *of* $\mathbb{K}(X)$ *at* $\eta \in \mathbb{K}(X)$ *to be*

$$T_\eta(\mathbb{K}) := T^{\mathrm{ext}}_\eta(\mathbb{K}) \oplus T^{\mathrm{int}}_\eta(\mathbb{K}) \tag{6.1.7}$$

and define the gradient *as*

$$\nabla^{\mathbb{K}} := (\nabla^{\mathbb{K}}_{ext}, \nabla^{\mathbb{K}}_{int})$$

whenever the objects exist.

Existence for cylinder functions

Fix $F = g_F(\langle \rho_1, \eta\rangle, \ldots, \langle \rho_N, \eta\rangle) \in \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X))$.

Proposition 6.1.16. *The gradient* $\nabla^{\mathbb{K}}$ *of* F *exists and equals*

$$\begin{aligned} \nabla^{\mathbb{K}} F(\eta) = \Bigg(&\sum_{i=1}^{N} \partial_i g_F(\langle \rho_1, \eta\rangle, \ldots, \langle \rho_N, \eta\rangle)\rho_i, \\ &\sum_{i=1}^{N} \partial_i g_F(\langle \rho_1, \eta\rangle, \ldots, \langle \rho_N, \eta\rangle)\nabla^X \rho_i \Bigg) \\ =: &\sum_{i=1}^{N} \partial_i g_F(\langle \rho_1, \eta\rangle, \ldots, \langle \rho_N, \eta\rangle)\left(\rho_i, \nabla^X \rho_i\right) \qquad \in T_\eta(\mathbb{K}) \end{aligned}$$

and for all $h \in C_0(X)$ and $v \in V_0(X)$ we have

$$\left\langle \nabla^{\mathbb{K}} F(\eta), (h, v) \right\rangle_{T_\eta(\mathbb{K})} = \nabla^{\mathbb{K}}_{h,v} F(\eta).$$

Proof. Using Definition 6.1.15, the result follows by Propositions 6.1.8 and 6.1.14. For the formula, we calculate:

$$\begin{aligned}
&\left\langle \nabla^{\mathbb{K}} F(\eta), (h, v) \right\rangle_{T_\eta(\mathbb{K})} \\
=&\sum_{i=1}^{N} \partial_i g_F(\langle \rho_1, \eta\rangle, \dots, \langle \rho_N, \eta\rangle) \left\langle \left(\rho_i, \nabla^X \rho_i\right), (h, v) \right\rangle_{T_\eta(\mathbb{K})} \\
=&\sum_{i=1}^{N} \partial_i g_F(\langle \rho_1, \eta\rangle, \dots, \langle \rho_N, \eta\rangle) \Big(\langle \rho_i, h\rangle_{T_\eta^{\mathrm{ext}}(\mathbb{K})} + \left\langle \nabla^X \rho_i, v \right\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})} \Big) \\
=&\nabla^{\mathbb{K}}_{h,v} F(\eta).
\end{aligned}$$

□

6.2 Dense subsets of $L^2(\mathbb{K}(X), \mathcal{G}_\theta)$

Let $\mathcal{G}_\lambda$ be a fixed Levy measure with Levy intensity measure λ on $\mathbb{R}_+$. In order to obtain a Dirichlet form, we will have to show that the corresponding bilinear form is densely defined in $L^2(\mathbb{K}(X), \mathcal{G}_\lambda)$. It is sufficient to prove

$$\mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X)) \subset L^2(\mathcal{G}_\lambda) := L^2(\mathbb{K}(X), \mathcal{B}(\mathbb{K}(X)), \mathcal{G}_\lambda) \text{ is dense,}$$

which is the task of this section.

Remark 6.2.1. *In the case of a configuration space $\Gamma(\mathbb{R}^d)$ over $\mathbb{R}^d$, the dense set one normally considers is $\mathcal{F}C_b^\infty(\Gamma(\mathbb{R}^d), C_0^\infty(\mathbb{R}^d))$. Since $\mathbb{K}(X)$ can be embedded as a topological subspace in $\Gamma(\hat{X})$ and $\mathcal{G}_\theta(\mathbb{K}(X)) = 1$, the set $\mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(\mathbb{R}_+ \times X))$ lies dense in $L^2(\mathbb{K}(X), \mathcal{G}_\theta)$.*

Although this set is large enough to construct a Dirichlet form on it, we have to prove that $\mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X)) \subset L^2(\mathbb{K}(X), \mathcal{G}_\theta)$ is dense because the gradient $\nabla^{\mathbb{K}(X)}$ only act on cylindrical functions of the latter class.

A general strategy for showing the denseness of cylindrical functions

As a motivation for our strategy, we first of all repeat some facts about the Borel σ-algebra of the underlying space X and cylindrical functions.

Remark 6.2.2.
As in [MR92, Section II.3a)], let us suppose for this motivation that S is a locally convex topological real vector space which is a Suslin space*, i.e. the continuous image of a complete separable metric space.*

1. *[Bad70, Exposé n^0 8, N^0 7 Corollaire], as well as [Sch73, Part II, Ch. I, Thm. 3, p.162], implies that the Borel σ-algebra of S, $\mathcal{B}(S)$, coincides with the one generated by the dual space of S, i.e.*
$$\mathcal{B}(S) = \sigma(S'), \tag{6.2.1}$$
where S' denotes the topological dual space of S.

2. *By [Sch73, Part II, Ch. I, Lemma 4, p.162] there exists a countable subset of S' separating the points in S. Let*
$$\mathcal{F}C_b^\infty(S, S') := \left\{ f(l_1, \ldots, l_N) \mid N \in \mathbb{N},\ f \in C_b^\infty(\mathbb{R}^N),\ l_1, \ldots, l_N \in S' \right\}$$
denote the set of cylindrical functions.

3. *By the Lindelöf property, one gets that $\mathcal{F}C_b^\infty(S, S')$ separates the points in S.*

4. *Using (6.2.1), one shows that (cf. [Hag08, Proof of Thm 4.1.15, (a) Claim, P.52], resp. cf. the technique below for the denseness argument)*
$$\mathcal{F}C_b^\infty(S, S') \subset L^2(S, \mathcal{B}(S), \mu) \text{ dense w.r.t. } L^2(S, \mu),$$
where μ is a probability measure on $(S, \mathcal{B}(S))$.

Unfortunately, we do not have these nice properties for the cone $\mathbb{K}(X)$. Thus, we need more arguments to get the point separating set and to deduce the denseness result.

6.2.1 A set of point separating functions

We show that there exists a countable subset $\mathbb{S}_\mathbb{K}$ of $\mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X))$ which separates the points of $\mathbb{K}(X)$. Since X is separable, there exists a countable set $\mathbb{D}_X$ of points which lies dense in X, i.e.,
$$\forall x \in X,\ r > 0,\ \exists q \in \mathbb{D}_X :\ d(x, q) \le r,$$
where $d := d_X : X \times X \to \mathbb{R}$ denotes the metric in X. We denote for each $r \in \mathbb{R}$ the ball with radius r around $q \in X$ by
$$B_r(q) := \{x \in X \mid d(x, q) \le r\} \qquad \text{and} \qquad B_r := B_r(0). \tag{6.2.2}$$

Definition 6.2.3. *We consider the countable family of functions*

$$\mathbb{S}_{\mathbb{K}} := \bigcup_{N\in\mathbb{N}} \bigcup_{q\in\mathbb{D}_{\mathbb{X}}} \bigcup_{r\in\mathbb{Q}} \{c_N(\langle\cdot,\varphi_{q,r}\rangle)\} \quad \subset \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X)).$$

Here, we choose $c_N \in C_b^\infty(\mathbb{R})$ monotone such that

$$\left\{\begin{array}{ll} c_N(r) \geq -1 & \text{if } r < 0 \\ c_N(r) = r & \text{if } 0 \leq r \leq N \\ c_N(r) \leq N+1 & \text{if } N \leq r \end{array}\right\} \tag{6.2.3}$$

and $\varphi_{q,r} \in C_b^\infty(X)$ with (cf. (6.2.2))

$$1\!\!1_{B_{r/10}(q)} \leq \varphi_{q,r} \leq 1\!\!1_{B_r(q)}. \tag{6.2.4}$$

Remark 6.2.4 (Existence of these "spiked" functions)**.**
The function $\varphi_{q,r}$ has a "spike" centered at q of heights 1.

1. *In the case of X being $\mathbb{R}^d$, $d \in \mathbb{N}$, we can obtain these functions in the following way: We define for $\varepsilon > 0$ $g_\varepsilon \in C_0^\infty(X)$ such that supp $g_\varepsilon \subset B_\varepsilon$ and obtain $\varphi_{q,r} \in C_0^\infty(X)$ by setting*

$$\tilde{\varphi}_{q,r}(\cdot) := \left\{\begin{array}{ll} 1 & \text{if } d(\cdot,q) \leq \frac{r}{2}, \\ 0 & \text{else,} \end{array}\right\} = 1\!\!1_{B_{r/2}(q)}$$

   and $\varphi_{N,q,r} := g_{r/10} \star \varphi_{N,q,r}$, where $\star$ denotes the convolution of two functions.

2. *In the general case of X being a connected Riemannian manifold, the existence of the functions $\varphi_{q,r}$, $q \in X$ and $r \in \mathbb{R}$, follows by the existence of smooth bump functions (cf. [Lee03, Proposition 2.26,P.55]): If $r \leq 0$, then we choose $\varphi_{q,r} = 0$. Otherwise, we apply the mentioned result for the closed set $B_{r/10}(q)$ and the open set $\{x \in X | d_X(x,q) < r/2\}$.*

   Here, we used that the ("distance") metric $d = d_X$, which corresponds to the Riemannian metric, exists for any connected Riemannian manifold and that its metric topology is the same as the original manifold topology (cf. [Lee03, Proposition 11.20, P.278]).

Proposition 6.2.5. *The countable set $\mathbb{S}_{\mathbb{K}}$ separates the points of $\mathbb{K}(X)$.*

Proof. First, we sketch the idea of the proof: Fix $\{\eta,\eta'\} \in \mathbb{K}(X)$. Initially we find a compact set $A_N \in \mathbb{R}_+ \times X$ on which the two elements $\eta,\eta' \in \mathbb{K}(X)$ differ. Then we consider the function $\varphi_{0,N}$ being supported by B_{N+1}. If the associated function in $\mathbb{S}$ does not separate the points, we "change the spike of $\varphi_{0,N}$" and choose an "appropriately spiked function $\varphi_{q,r}$" that separates η and η'.

It is important to identify a correct "spike". Therefore, we will order the points of η and η' lying in the strip $\mathbb{R}_+ \times B_{N+1}$ appropriately, consider the first point $\hat{x}_d$ which belongs only to one of them and does not lie on the border of the strip and "cut" the sum over the marks of the ordered points.

Fix $\eta, \eta' \in \mathbb{K}(X)$: $\eta \neq \eta'$ and consider the covering $(A_N)_{N\in\mathbb{N}}$ of $\mathbb{R}_+ \times X$, where $A_N := \big(\big[\frac{1}{N}, N\big] \times B_N\big)$. We find $N \in \mathbb{N}$ such that there exists $\hat{x}_\star \in \eta \cap A_N$ with $\hat{x}_\star \neq \hat{x}'$ for all $\hat{x}' \in \eta' \cap A_N$ or $\hat{x}'_\star \in \eta' \cap A_N$ with $\hat{x}'_\star \neq \hat{x}$ for all $\hat{x} \in \eta \cap A_N$. (From now on, $'$ indicates a property related to η'.) Set

$$\begin{aligned}&\tilde{M} := \langle \mathbb{1}_{B_{N+1}}, \eta\rangle < \infty, \quad M' := \langle \mathbb{1}_{B_{N+1}}, \eta'\rangle < \infty \quad \text{and}\\ &M := \max(M, M').\end{aligned}$$

Here, they are finite by the definition of $\mathbb{K}(X)$ (cf. (3.1.1)).

If $\langle \eta, \varphi_{0,N}\rangle \neq \langle \eta', \varphi_{0,N}\rangle$, then $\langle \cdot, \varphi_{0,N}\rangle$ separates these points and we choose $c_M(\langle \cdot, \varphi_{0,N}\rangle) \in \mathbb{S}_{\mathbb{K}(X)}$ to separate η and η'.

Otherwise, we have to identify the correct "spike": We count some points of η and η' lying in the strip $\mathbb{R}_+ \times B_{N+1}$: Let us define for the fixed parameter N and M the set

$$B := B_{N,M} :=]0, M] \times B_{N+1}.$$

It contains all the points of the restriction of η, resp. η', to the strip $\mathbb{R}_+ \times B_{N+1}$. We enumerate the points of η_B, resp. η'_B, lying in B by sorting them by the size of their marks, i.e.

$$\begin{aligned}\eta_B &= \bigcup_{n=1}^{|\eta_B|} \{\hat{x}_n\} : \quad (\tilde{M} \geq)\ s_1 \geq s_2 \geq \cdots \geq s_n \geq \ldots \text{ and}\\ \eta'_B &= \bigcup_{n=1}^{|\eta'_B|} \{\hat{x}'_n\} : \quad (M' \geq)\ s'_1 \geq s'_2 \geq \cdots \geq s'_n \geq \ldots .\end{aligned}$$

Furthermore, we assume that the points of η'_B are ordered such that the ordering of $\eta'_B \cap \eta_B$ in η'_B is the same as in η_B. (This is possible because for each mark there are at most finitely many points $\hat{x}$ having that mark.)

For the next step we need a formal fail safe for the case that one of the elements has just finitely many points in the strip and they are "contained" in the other one. If $|\eta_B| < |\eta'_B|$, we pick $x_1 \in X \backslash \{x_1, \ldots, x_{|\eta_B|}, x'_1, \ldots, x'_{|\eta_B|}, x'_{|\eta_B|+1}\}$

and set $s_{|\eta_B|+1} = 0$ and $x_{|\eta_B|+1} = x_1$. If $|\eta'_B| < |\eta_B|$, we do a similar definition, namely we choose $x'_1 \in X \backslash \{x_1, \ldots, x_{|\eta'_B|}, x_{|\eta'_B|+1}, x'_1, \ldots, x'_{|\eta'_B|}\}$ and set $s'_{|\eta'_B|+1} = 0$ and $x'_{|\eta'_B|+1} = x'_1$.

Fix $d \in \mathbb{N}$ to be the smallest index such that the corresponding "points" do not lie both in η and η' and on the border of the strip, i.e., such that

$$\Big(x_d \neq x'_d \text{ or } s_d \neq s'_d\Big) \qquad \text{and}$$
$$\Big(d(x_d, 0) < N+1 \text{ or } d(x'_d, 0) < N+1\Big).$$

(At the latest this will happen at $\hat{x}_d = \hat{x}_\star$ or $\hat{x}'_d = \hat{x}'_\star$.)

If $x_d = x'_d$, then w.l.o.g. $s'_d \leq s_d$. (Otherwise we change the role of η and η'.) If $x_d \neq x'_d$, then w.l.o.g. $d(x_d, 0) < N+1$ (It is not on the border of the strip).

We will identify the "correct" spike, i.e., the correct function $\varphi_{q,r}$, which separates the points. To this end, we choose $t' \in \mathbb{N}$ such that for all $n \geq t'$, $n \in \mathbb{N}$, the sum of the remaining marks is small, i.e.,

$$\sum_{k=n+1}^{|\eta'_B|} s'_k \leq \varepsilon_d := \frac{1}{2}(s_d - s'_{\tilde{d}}),$$

where

$$\tilde{d} := \min\{k \geq d | s'_k < s_d\}$$

is the smallest index after d such that the corresponding mark differs from s_d and set $s'_\infty := 0$. If $x_d = x'_d$, then $\tilde{d} = d$. If $|\eta'_B| < \infty$, we choose $t' = |\eta'_B|$.

The next idea is that there shall be no difference between the functions f_N and φ_N for the first t' points of η'_B except maybe for the point x_d. To this end, we calculate the minimal distance of x_d to these points and the border of the strip. In detail, we choose $r \in \mathbb{Q}$ such that

$$0 < r \leq \min\{d(x_d, x'_k) \big| 1 \leq k \leq t', x'_k \neq x_d\} \wedge \big(N + 1 - d(x_d, 0)\big).$$

Due to the definition of $\mathbb{K}(X)$, there is at most one x'_k, $k \leq |\eta'_B|$, such that $x'_k = x_d$. Thus, we set

$$d' := \min\{k | x'_k = x_d\} \text{ and } x'_\infty := 0.$$

As a last step we choose a $q \in \mathbb{D}_X$ such that $d(x_d, q) \leq \frac{r}{10}$ and choose

$$\varphi_{q,r} \leq 1.$$

This $\varphi_{q,r}$ separates η and η':

$$
\begin{aligned}
\langle \varphi_{q,r},\eta\rangle - \langle \varphi_{q,r},\eta'\rangle =& \sum_{\substack{k=1\\k\neq d}}^{|\eta_B|} \underbrace{s_k\varphi_{q,r}(x_k)}_{\geq 0} + s_d\varphi_{q,r}(x_d) \\
& - \sum_{\substack{k=1\\k\neq d'}}^{t'} \underbrace{s'_k\varphi_{q,r}(x'_k)}_{=0} - \underbrace{s'_{d'}\varphi_{q,r}(x'_{d'})}_{=s'_{d'}\varphi_{q,r}(x_d)}\big) - \sum_{\substack{k=t'+1\\k\neq d'}}^{|\eta'_B|} s'_k\underbrace{\varphi_{q,r}(x'_k)}_{\leq 1} \\
\geq& (s_d - s'_{d'})\varphi_{q,r}(x_d) - \varepsilon_d \\
=& (s_d - s'_{d'}) - \frac{1}{2}(s_d - s'_{\tilde{d}}) \\
\geq& (s_d - s'_{d'}) - \frac{1}{2}(s_d - s'_{d'}) > 0,
\end{aligned}
\tag{6.2.5}
$$

where, by the definition and ordering of the points, we have $s'_{d'} < s_d$ because $(s_d, x_d) \neq (s'_{d'}, x_d)$ and $d(x_d, 0) < N+1$. Since $\tilde{d}$ is the smallest index k such that $s'_k < s_d$, we get $s'_{d'} \leq s'_{\tilde{d}}$.

Furthermore, we choose $\tilde{N} \in \mathbb{N}$ such that

$$\tilde{N} \geq \max\left(\langle \varphi_{q,r},\eta\rangle, \langle \varphi_{q,r},\eta'\rangle\right) < \infty.$$

Hence (using also (6.2.5)), $c_{\tilde{N}}(\langle \varphi_{q,r},\cdot\rangle) \in \mathbb{S}_{\mathbb{K}}$ separates η and η'. Therefore, $\mathbb{S}_{\mathbb{K}}$ separates the points in $\mathbb{K}(X)$. □

6.2.2 Denseness criterium

We show that $\mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X)) \subset L^2(\mathbb{K}(X),\mu)$ is dense, where $\mu \in \mathcal{M}^1(\mathbb{K}(X))$.

Definition 6.2.6. *Let $(Z,\mathcal{B})$ be a Borel space and $\mathbb{F}$ a set of $\mathcal{B}$-measurable functions $f: Z \to \mathbb{R}$. Then we define for $k \in \mathbb{N}\cup\{0,\infty\}$ the set of* finitely based functions *$\mathcal{F}C_b^k(Z,\mathbb{F})$ by*

$$h \in \mathcal{F}C_b^k(Z,\mathbb{F}) \quad :\Leftrightarrow \quad h(\cdot) = g(f_1(\cdot),\ldots,f_N(\cdot)), \tag{6.2.6}$$

where $N \in \mathbb{N}$, $g \in C_b^k(\mathbb{R}^N)$ and $f_i \in \mathbb{F}$ for $1 \leq i \leq N$.

Theorem 6.2.7. *Let $(Z,\mathcal{B})$ be a standard Borel space*[3] *and μ be a finite measure on it. Assume that there exists a countable set and exists $l \in \mathbb{N}\cup\{0,\infty\}$ such that*

$$\mathbb{S} := \{f_n | n \in \mathbb{N}\} \subset \mathcal{F}C_b^l(Z,\mathbb{F}) =: \mathfrak{M}$$

[3] We recall its definition in Section A.1.

and that $\mathbb{S}$ separates the points of Z. Here, $\mathbb{F}$ is a set containing measurable functions $f : Z \to \mathbb{R}$.

Then for any $k \in \mathbb{N} \cup \{\infty\}$

$$\mathfrak{M} \subset L^k(\mu) := L^k(Z, \mathcal{B}, \mu) \text{ dense w.r.t. } L^k(\mu). \tag{6.2.7}$$

Before we prove this theorem, we note that it is sufficient for our purpose:

Corollary 6.2.8. *For any finite measure μ on $(\mathbb{K}(X), \mathcal{B}(\mathbb{K}(X)))$*

$$\mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X)) \subset L^2(\mu) := L^2(\mathbb{K}(X), \mathcal{B}(\mathbb{K}(X)), \mu)$$

is dense w.r.t. $L^2(\mu)$. In particular, this holds for $\mu = \mathcal{G}_\theta$.

Proof. Since $\mathbb{K}(X)$ is homeomorphic to $\Gamma_f(\hat{X}) \in \mathcal{B}(\Gamma(\hat{X}))$ (cf. Subsection 3.1.2 and Remark 2.2.8), we obtain by Theorem A.1.6 that $(\mathbb{K}(X), \mathcal{B}(\mathbb{K}(X)))$ is a standard Borel space.

Therefore, Proposition 6.2.5 and Theorem 6.2.7 yield the claim, where we consider the following set of measurable (cf. Lemma 2.1.1) functions

$$\mathbb{F} := \{\langle \phi, \cdot \rangle | \phi \in C_0^\infty(X)\}.$$

□

Proof of Theorem 6.2.7. We use a monotone class argument (see [Röc05, Definition 1.11.7, Satz 1.11.11, p.54f] or [Pro05, I Theorem 8]):

$$\mathfrak{H} := \overline{\mathfrak{M}}^{L^k(\mu)} \subset L^k(\mu)$$

is a monotone vector space.[4]

[4] Clearly, $1 \in \mathfrak{H}$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{H}$ such that $0 \le f_1 \le f_2 \le \cdots \le f_n \nearrow f$ and f bounded. We have to prove that $f \in \mathfrak{H}$. By the Lebesgue dominated convergence theorem and a diagonal argument there exists a sequence $g_n \in \mathfrak{M}$ such that $f = L^k(\mu) - \lim_{n\to\infty} g_n \in \mathfrak{H}$: Indeed, since the sequence $(f_n)_{n\in\mathbb{N}}(\in L^k(\mu))$ converges pointwisely monotone increasing to f and is bounded by $f \in L^\infty(\mu) \subset L^k(\mu)$, the Lebesgue dominated convergence theorem gives us that

$$L^k(\mu) - \lim_{n\to\infty} f_n = f$$

and thus w.l.o.g (eventually considering a subsequence) $\|f_n - f\|_{L^k(\mu)} \le \frac{1}{n}$. For each $f_n \in \mathfrak{H}$ there exists a sequence $g_{n,m} \in \mathfrak{M}$ such that $f_n = L^k(\mu) - \lim_{m\to\infty} g_{n,m}$. Furthermore w.l.o.g for all $m \in \mathbb{N} : m \ge n$ we have $\|f_n - g_{n,m}\|_{L^k(\mu)} \le \frac{1}{n}$.

Defining $g_n := g_{n,n}$ we obtain

$$\|g_n - f\|_{L^k(\mu)} \le \|g_{n,n} - f_n\|_{L^k(\mu)} + \|f_n - f\|_{L^k(\mu)} \le \frac{2}{n}$$

and hence $f \in \mathfrak{H}$.

Moreover, $\mathfrak{M}$ is a set of bounded functions, which is closed under multiplication. Then (by monotone classes, e.g. [Röc05, Satz 11.1.11] or [Pro05, I Theorem 8]) $\sigma(\mathfrak{M})_b \subset \mathfrak{H}$. Here, $\sigma(\mathfrak{M})_b$ denotes the set of all bounded, $\sigma(\mathfrak{M})$-measurable functions.

Claim:
$$\sigma(\mathfrak{M}) \stackrel{!}{=} \mathcal{B} \tag{6.2.8}$$

This yields

$$\mathfrak{M} \subset L^\infty(\mu) \stackrel{(6.2.8)}{=} \sigma(\mathfrak{M})_b \stackrel{\substack{\text{monotone}\\ \text{classes}}}{\subset} \mathfrak{H} = \overline{\mathfrak{M}}^{L^k(\mu)} \tag{6.2.9}$$

and hence we are done because the indicator functions are measurable.

Proof of the ***Claim****(cf (6.2.8)):*
Since we would like to apply Kuratowski's Theorem (cf. Theorem A.1.7), we set $\mathcal{D} := \sigma(\mathbb{S}) = \sigma(\{f_n \mid n \in \mathbb{N}\})$, which is countably generated, and consider

$$\begin{array}{cccc} \mathrm{id}\colon & (Z, \mathcal{B}) & \to & (Z, \mathcal{D}) \\ & z & \mapsto & z. \end{array}$$

The function id is one-to-one and measurable because the f_n are measurable. $(Z, \mathcal{D})$ is even a separable Borel space (cf. Definition A.1.4) because

$$\forall z \in Z\colon \{z\} \stackrel{!}{=} \bigcap_n \{f_n = f_n(z)\} \in \mathcal{D}.$$

The subset inclusion is obvious. z is the only element in the intersection because $\mathbb{S}$ is point separating; and thus for every $y \neq z$ there exists a $n' \in \mathbb{N}$ such that $f_{n'}(y) \neq f_{n'}(z)$.

Hence, by Kuratowski's Theorem (cf. Theorem A.1.7) id^{-1} is an isomorphism and $\mathcal{B} = \mathcal{D}$. Therefore, $\mathcal{B} = \mathcal{D} \subset \sigma\{\mathfrak{M}\} \subset \mathcal{B}$. □

Remark 6.2.9 (Further extensions)**.** *The above proof can be generalized:*

1. *The arguments work for any complete, separable metric space X, for which the countable set $\mathbb{S}_{\mathbb{K}(X)}$ (cf. Definition 6.2.3) exists and $m(A) < \infty$ for all $A \in \mathcal{B}(X)$ bounded.*

2. *The method to show the denseness also works*

   (a) for configuration spaces. Then we consider $s_x = 1$ and

   $$\Gamma(X) = \{\eta \in \mathbb{K}(X) | \forall \Lambda \in B_c(\mathbb{R}_+ \times X) \forall \hat{x} \in \eta_\Lambda s_x = 1\}.$$

   (b) $C_0^\infty(X)$ being replaced by $\{f : X \to \{0, 1, 2\}$ measurable$\}$.

6.3 Integration by parts and Dirichlet forms

Now we proceed to construct Dirichlet forms via the scheme that we outlined at the beginning of this chapter.

Let X still be a connected, orientated, separable C^∞-Riemannian manifold X with Riemannian metric d_X. It is equipped with its Borel σ-algebra $\mathcal{B}(X)$. Let

$$m(dx) = \rho\mathfrak{v}(dx) \tag{6.3.1}$$

be a non-atomic Radon measure on the state space X with Riemmanian volume element $\mathfrak{v}$, where $\rho > 0$ $\mathfrak{v}$-a.e. is a density such that $\rho^{\frac{1}{2}} \in H^{1,2}_{\mathrm{loc}}(X, \mathfrak{v})$, which denotes the local Sobolev space of $L^2_{\mathrm{loc}}(X, \mathfrak{v})$.

Further on, we denote the *logarithmic derivative* of m by

$$X \ni x \mapsto \beta^m(x) := \frac{\nabla^X \rho(x)}{\rho(x)} \in T_x(X), \tag{6.3.2}$$

where ∇^X denotes the *gradient* on X and $\beta^m := 0$ on $\{\rho = 0\}$.

6.3.1 Intrinsic motion for Levy measures

We do not have a quasi-invariance principle, thus we cannot go on with the construction scheme at the second point (as done, e.g., in [AKR98a]). This is due to the fact that the measure on $\mathbb{R}_+$ is infinite.

Let λ be a Levy measure on $\mathbb{R}_+$ with finite first moment, i.e., $m_1(\lambda) < \infty$, and $\mathcal{G}_\lambda$ the corresponding Levy process (cf. Theorem 3.1.7).

Definition 6.3.1. *For any $v \in V_0(X)$ we define the* intrinsic logarithmic derivative*of $\mathcal{G}_\lambda$ along v as*

$$\begin{aligned} \mathbb{K}(X) \ni \eta \quad \mapsto \quad & \langle \beta^{\mathcal{G}_\lambda}_{int}(\eta), v\rangle_{T^{\mathrm{int}}_\eta(\mathbb{K})} := \langle \eta, \beta^m_v\rangle \\ & := \int_X \langle \beta^m(x), v(x)\rangle_{T_x(X)} + div^X v(x)\eta(dx), \end{aligned}$$

where div^X *is the divergence on X with respect to m.*

Remark 6.3.2. *The* intrinsic logarithmic derivate *is independent of λ.*

Lemma 6.3.3. *Let $v \in V_0(X)$. If the first moment of λ exists, i.e.,*

$$m_1(\lambda) := \int_{\mathbb{R}_+} s\lambda(ds) < \infty, \tag{6.3.3}$$

then $\langle \beta_{int}^{\mathcal{G}_\lambda}(\cdot), v\rangle_{T^{\mathrm{int}}_{\cdot}(\mathbb{K})} \in L^1(K, \mathcal{G}_\lambda)$*. Moreover, if the second one is finite, i.e.,*

$$m_2(\lambda) := \int_{\mathbb{R}_+} s^2 \lambda(ds) < \infty, \tag{6.3.4}$$

then

$$\langle \beta_{int}^{\mathcal{G}_\lambda}(\cdot), v\rangle_{T^{\mathrm{int}}_{\cdot}(\mathbb{K})} \in L^2(K, \mathcal{G}_\lambda).$$

Proof. We define

$$f(x) := \langle v(x), \beta^m(x)\rangle_{T_x(X)} + \mathrm{div}^X v(x),\ x \in X.$$

We recall (compare Theorem 3.2.6) the formula for the first and second moments of the measure $\mathcal{G}_\lambda$, namely for all finitely supported Borel-measurable function $f \geq 0$ we have

$$\int_{\mathbb{K}(X)} \langle \eta, f\rangle \mathcal{G}_\lambda(d\eta) = m_1(\lambda)\langle f, m\rangle \qquad \text{resp.} \tag{6.3.5}$$

$$\int_{\mathbb{K}(X)} \langle \eta, f\rangle^2 \mathcal{G}_\lambda(d\eta) = m_2(\lambda)\langle f^2, m\rangle + m_1^2(\lambda)\langle f, m\rangle^2. \tag{6.3.6}$$

Therefore, it is sufficient to show that $f \in L^1(X, m) \cap L^2(X, m)$. To this end, we use the assumption that $\rho^{\frac{1}{2}} \in H^{1,2}_{loc}(X, \mathfrak{v})$ to show that $f \in L^1(X, m)$. Once obtained this property we deduce by the compactness of the support of v that each f is even in $L^1(X, m) \cap L^2(X, m)$.

First of all, we note that, due to $v \in V_0(X)$, X being finite-dimensional and (6.3.5), the integral over the divergence part is finite, i.e.,

$$\int_X \mathrm{div}^X v(x) m(dx) \leq C m(\Lambda) < \infty,$$

where we choose $C > 0$ and $\Lambda \in \mathcal{B}_c(X)$ appropriately. Moreover, by Cauchy-Schwartz

$$\begin{aligned}\int_X \langle v(x), \beta^m(x)\rangle_{T_x(X)} m(dx) &\leq \int_X |v(x)|^{1/2}_{T_x(X)} \cdot |v(x)|^{1/2}_{T_x(X)} |\beta^m(x)|_{T_x(X)} m(dx) \\ &\leq \Big(\int_X \underbrace{|v(x)|_{T_x(X)}}_{\leq C\mathbb{1}_\Lambda} m(dx)\Big)^{1/2} \cdot \Big(\int_X \underbrace{|v(x)|_{T_x(X)}}_{\leq C\mathbb{1}_\Lambda} |\beta^m(x)|^2_{T_x(X)} m(dx)\Big)^{1/2} \\ &\leq C \Big(\int_\Lambda \rho(x)\mathfrak{v}(dx)\Big)^{\frac{1}{2}} \Big(\int_\Lambda \Big|\frac{\nabla^X \rho}{\rho}(x)\Big|^2_{T_x(X)} \rho(x)\mathfrak{v}(dx)\Big)^{\frac{1}{2}}.\end{aligned}$$

Since $\rho^{\frac{1}{2}} \in H^{1,2}_{loc}(X, \mathfrak{v})$, i.e.,

$$\int_\Lambda \left| \mathbb{1}_{\rho>0} \frac{\nabla^X \rho}{\rho^{\frac{1}{2}}} \right|^2_{T_x X} + |\rho^{\frac{1}{2}}|^2 \mathfrak{v}(dx) < \infty \quad \Lambda \in \mathcal{B}_c(X),$$

the above product is finite. To show $f \in L^2(X, m)$, we remark

$$\begin{aligned} \int_X |f(x)|^2 m(dx) = & \int_X |v(x)|^2_{T_x(X)} |\beta^m(x)|^2_{T_x(X)} m(dx) \\ \leq & C^2 \int_\Lambda |\beta^m(x)|^2_{T_x(X)} m(dx) \end{aligned}$$

and conclude, as above, the assertion. □

Having a closer look at the last proof, we get

Corollary 6.3.4. *Let $p \in \mathbb{N}$ such that for all $k \in \{1, \dots, p\}$*

$$m_k(\lambda) := \int_{\mathbb{R}_+} s^k \lambda(ds) < \infty. \tag{6.3.7}$$

Then $\rho^{1/2} \in H^{1,p}_{loc}(X, \mathfrak{v})$ implies

$$\langle \beta^{\mathcal{G}_\lambda}_{int}(\cdot), v \rangle_{T^{\text{int}}_{\cdot}(\mathbb{K})} \in \bigcap_{1 \leq k \leq p} L^k(K, \mathcal{G}_\lambda).$$

For $\mathcal{G}_\lambda = \mathcal{G}_\theta$, $\theta > 0$, this holds for any $p \in \mathbb{N}$. In particular, in the basic model setting, we have

$$\langle \beta^{\mathcal{G}_\theta}_{int}(\cdot), v \rangle_{T^{\text{int}}_{\cdot}(\mathbb{K})} \in \bigcap_{1 \leq k < \infty} L^k(K, \mathcal{G}_\lambda).$$

Theorem 6.3.5. *Let $m_1(\lambda) < \infty$. Then for all F, $G \in \mathcal{F}C^1_b(\mathbb{K}(X), C^1_0(X))$ and any $v \in V_0(X)$ we have an* integration by parts formula

$$\begin{aligned} \int_{\mathbb{K}(X)} (\nabla^{\mathbb{K}}_{int,v} F)(\eta) G(\eta) \mathcal{G}_\lambda(d\eta) = & - \int_{\mathbb{K}(X)} F(\eta) (\nabla^{\mathbb{K}}_{int,v} G)(\eta) \mathcal{G}_\lambda(d\eta) \\ & - \int_{\mathbb{K}(X)} F(\eta) G(\eta) \langle \beta^{\mathcal{G}_\lambda}_{int}(\eta), v \rangle_{T^{\text{int}}_\eta(\mathbb{K})} \mathcal{G}_\lambda(d\eta). \end{aligned}$$

Proof. Let $\lambda_n(dt) := \mathbb{1}_{[\frac{1}{n}, \infty[}(t) \lambda(dt)$, which is finite. By Theorem 3.1.7, for each λ_n there exists a Poisson measure $\mathcal{P}_{\lambda_n}$. By definition, the gradient only "notes" changes in a finite set on the position space, it is continuous and

$\langle \eta - \eta_{\Lambda_N}, \mathbb{1}_\Delta \rangle$, where $\Delta \in \mathcal{B}_c(X)$: $\Lambda_N \in \mathbb{R}_+ \times \Delta$, becomes for $\Lambda_N \nearrow \hat{X}$ arbitrarily small. Hence, Lebesgue's dominated convergence theorem yields

$$\int_{\mathbb{K}(X)} (\nabla^{\mathbb{K}}_{\mathrm{int},v} F)(\eta) \mathcal{G}_\lambda(d\eta) = \lim_{\substack{\Lambda_N \in \mathcal{B}_c(\mathbb{R}_+ \times X) \\ \Lambda_N \nearrow \mathbb{R}_+ \times X}} \int_{\mathbb{K}(X)} (\nabla^{\mathbb{K}}_{\mathrm{int},v} F)(\eta_{\Lambda_N}) \mathcal{G}_\lambda(d\eta)$$

$$= \lim_{\substack{\Lambda_N \in \mathcal{B}_c(\mathbb{R}_+ \times X) \\ \Lambda_N \nearrow \mathbb{R}_+ \times X}} \lim_{n \to \infty} \int_{\mathbb{K}(X)} (\nabla^{\mathbb{K}}_{\mathrm{int},v} F)(\eta_{\Lambda_N}) \mathcal{G}_{\lambda_n}(d\eta)$$

Now we have a finite intensity measure λ_n on $\mathbb{R}_+$. Thus, we can use the assertion of [KdSS98, Theorem 3.5], which is valid in this setting.[5] Hence, the last line equals

$$- \lim_{\substack{\Lambda_N \in \mathcal{B}_c(\mathbb{R}_+ \times X) \\ \Lambda_N \nearrow \mathbb{R}_+ \times X}} \lim_{n \to \infty} \int_{\mathbb{K}(X)} F(\eta_{\Lambda_N}) \langle \beta^{\mathcal{G}_\lambda}_{\mathrm{int}}(\eta), v \rangle_{T^{\mathrm{int}}_\eta(\mathbb{K})} \mathcal{G}_{\lambda_n}(d\eta)$$

$$= - \int_{\mathbb{K}(X)} F(\eta) \langle \beta^{\mathcal{G}_\lambda}_{\mathrm{int}}(\eta), v \rangle_{T^{\mathrm{int}}_\eta(\mathbb{K})} \mathcal{G}_\lambda(d\eta),$$

where the finiteness follows by Lemma 6.3.3. □

Adjoint of the intrinsic gradient

Definition 6.3.6. *A function V is called an* intrinsic vector field *iff it is of the following form*

$$V(\eta) := \sum_{i=1}^{N} g_i(\eta) v_i$$

where for $i = 1, \ldots, N$ $g_i \in \mathcal{F}C^\infty_b(\mathbb{K}(X), C^\infty_0(X))$ and $v_i \in V_0(X)$. By $V_{cyl,X}(\mathbb{K}(X))$ we denote the set of all these intrinsic vector fields.

Lemma 6.3.7. *Let $V_1, V_2 \in V_{cyl,X}(\mathbb{K}(X))$. If $m_1(\lambda) < \infty$, then*

$$\int_{\mathbb{K}(X)} \langle V_1(\eta), V_2(\eta) \rangle_{T^{\mathrm{int}}_\eta(\mathbb{K})} \mathcal{G}_\lambda(d\eta) < \infty.$$

Proof. The intrinsic vector fields are bounded and finitely supported. Thus the integral is finite because the first moments of the Levy measure λ are

[5] The cutoff in $\mathbb{R}_+$ does not void the assertion because the quasi-invariance property of $\mathcal{G}_{\lambda_n}$ holds (cf. [KdSS98, Proposition 2.8]) and the integration by parts formula is obtained by using this property.

finite (cf. (6.3.5)): Indeed, there exist $\Lambda \in \mathcal{B}_c(X)$ and $C, \tilde{C} > 0$ such that

$$\begin{aligned}&\int_{\mathbb{K}(X)} \langle V_1(\eta), V_2(\eta)\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})}\, \mathcal{G}_\lambda(d\eta)\\ \le& C\sum_{i=1}^{N}\sum_{j=1}^{\tilde{N}}\int_{\mathbb{K}(X)}\int_X \langle v_i, \tilde{v}_j\rangle_{T_x(X)}\, 1\!\!1_\Lambda(x)\eta(dx)\mathcal{G}_\lambda(d\eta)\\ \le& \tilde{C}\int_{\mathbb{K}(X)} \langle 1\!\!1_\Lambda, \eta\rangle\, \mathcal{G}_\lambda(d\eta) = \tilde{C} m_1(\lambda) m(\Lambda) < \infty. \end{aligned} \tag{6.3.8}$$

□

Theorem 6.3.8. *Fix $V := \sum_{i=1}^N g_i v_i \in V_{cyl,X}(\mathbb{K}(X))$. Let λ be a Levy measure on $\mathbb{R}_+$ with $m_1(\lambda) + m_2(\lambda) < \infty$. Then for all $F \in \mathcal{F}C_b^1(\mathbb{K}(X), C_0^1(X))$*

$$\begin{aligned}&\int_{\mathbb{K}(X)} \left\langle \nabla_{int}^{\mathbb{K}} F(\eta), V(\eta)\right\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})} \mathcal{G}_\lambda(d\eta) =\\ &\quad -\int_{\mathbb{K}(X)} F(\eta)\sum_{i=1}^N \left\langle \nabla_{int}^{\mathbb{K}} g_i, v_i\right\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})} \mathcal{G}_\lambda(d\eta)\\ &\qquad -\int_{\mathbb{K}(X)} F(\eta)\langle \beta_{int}^{\mathcal{G}_\lambda}(\eta), V(\eta)\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})}\mathcal{G}_\lambda(d\eta),\end{aligned} \tag{6.3.9}$$

where all the integrals are finite. We can reformulate it as

$$\begin{aligned}\left(\left(\nabla_{int}^{\mathbb{K}}\right)^\star_{\mathcal{G}_\lambda} V\right)(\eta) =& -\sum_{i=1}^N \left\langle \nabla_{int}^{\mathbb{K}} g_i(\eta), v_i\right\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})}\\ &- \langle \beta_{int}^{\mathcal{G}_\lambda}(\eta), V(\eta)\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})}.\end{aligned} \tag{6.3.10}$$

Proof. The finiteness of the involved integrals follows by Lemma 6.3.7. The integration by parts formula (cf. Theorem 6.3.5) yields the result. Indeed,

$$\begin{aligned}&\int_{\mathbb{K}(X)} \left\langle \left(\nabla_{\mathrm{int}}^{\mathbb{K}} F\right)(\eta), V(\eta)\right\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})} \mathcal{G}_\lambda(d\eta)\\ =&\sum_{i=1}^N \int_{\mathbb{K}(X)} g_i(\eta)\left\langle \nabla_{\mathrm{int}}^{\mathbb{K}} F(\eta), v_i\right\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})} \mathcal{G}_\lambda(d\eta)\\ =&\sum_{i=1}^N \int_{\mathbb{K}(X)} F(\eta)\left(-\nabla_{\mathrm{int},v_i}^{\mathbb{K}} g_i(\eta) - \langle \beta_{\mathrm{int}}^{\mathcal{G}_\lambda}(\eta), g_i(\eta)\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})}\right)\mathcal{G}_\lambda(d\eta)\\ =&\int_{\mathbb{K}(X)} F(\eta)\Bigg(-\sum_{i=1}^N \left\langle \nabla_{\mathrm{int}}^{\mathbb{K}} g_i(\eta), v_i\right\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})} - \langle \beta_{\mathrm{int}}^{\mathcal{G}_\lambda}(\eta), V(\eta)\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})}\Bigg)\mathcal{G}_\lambda(d\eta)\\ =&\int_{\mathbb{K}(X)} F(\eta)\left(\left(\nabla_{\mathrm{int}}^{\mathbb{K}}\right)^\star_{\mathcal{G}_\lambda} V\right)(\eta)\mathcal{G}_\lambda(d\eta),\end{aligned}$$

where we used the definition of the adjoint in the last line. □

Intrinsic bilinear form

We define for $F,\ G \in \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X))$ a gradient bilinear form, namely

$$\mathcal{E}_{\mathrm{int}}^{\mathcal{G}_\lambda}(F,G) := \int_{\mathbb{K}(X)} \left\langle \nabla_{\mathrm{int}}^{\mathbb{K}} F(\eta), \nabla_{\mathrm{int}}^{\mathbb{K}} G(\eta) \right\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})} \mathcal{G}_\lambda(d\eta). \tag{6.3.11}$$

Remark 6.3.9. *By Lemma 6.3.7, $\mathcal{E}_{int}^{\mathcal{G}_\lambda}(F,G)$ is finite and by Theorem 6.2.7 densely defined.*

Definition 6.3.10. *For $F = g_F(\langle \rho_1, \eta\rangle, \dots, \langle \rho_N, \eta\rangle) \in \mathcal{F}C_b^2(\mathbb{K}(X), C_0^\infty(X))$ we define the* intrinsic generator *of $\mathcal{E}_{int}^{\mathcal{G}_\lambda}$*

$$\begin{aligned}(L_{int}^{\mathcal{G}_\lambda} F)(\eta) &:= -\left(S_{int}^{\mathbb{K}(X)} F\right)(\eta) - \langle \beta_{int}^{\mathcal{G}_\lambda}(\eta), \left(\nabla_{int}^{\mathbb{K}} F\right)(\eta)\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})} \\ &= -\left(S_{int}^{\mathbb{K}(X)} F\right)(\eta) - \sum_{i=1}^N \partial_i g_F(\langle \rho_1, \eta\rangle, \dots, \langle \rho_N, \eta\rangle) \int_X \Delta^X \rho_i(x) d\eta(x) \\ &\quad - \sum_{i=1}^N \partial_i g_F(\langle \rho_1, \eta\rangle, \dots, \langle \rho_N, \eta\rangle) \int_X \langle \beta^m(x), \nabla^X \rho_i(x)\rangle_{T_x(X)} d\eta(x), \end{aligned} \tag{6.3.12}$$

where Δ^X denotes the Laplace-Beltrami operator on X and

$$\left(S_{int}^{\mathbb{K}(X)} F\right)(\eta) := \sum_{l,k=1}^N \partial_l \partial_k g_F(\langle \rho_1, \eta\rangle, \dots, \langle \rho_N, \eta\rangle) \langle \nabla^X \rho_k, \nabla^X \rho_l\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})}.$$

Corollary 6.3.11. *Let $m_1(\lambda) + m_2(\lambda) < \infty$. We rewrite the bilinear form using the intrinsic generator: For all $F, G \in \mathcal{F}C_b^2(\mathbb{K}(X), C_0^\infty(X))$ we obtain*

$$\mathcal{E}_{int}^{\mathcal{G}_\lambda}(F,G) = \int_{\mathbb{K}(X)} (L_{int}^{\mathcal{G}_\lambda} F)(\eta) G(\eta) \mathcal{G}_\lambda(d\eta). \tag{6.3.13}$$

Proof. The result follows by Theorem 6.3.8. □

Proposition 6.3.12. *Let $m_1(\lambda) + m_2(\lambda) < \infty$. Then the bilinear form $(\mathcal{E}_{int}^{\mathcal{G}_\lambda}, \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X)))$ is well-defined, positive definite, symmetric and closable.*

Proof. The symmetry and positive definiteness of the form are clear:

$$\mathcal{E}_{\mathrm{int}}^{\mathcal{G}_\lambda}(F,G) = \int_{\mathbb{K}(X)} \langle \nabla_{\mathrm{int}}^{\mathbb{K}} F(\eta), \nabla_{\mathrm{int}}^{\mathbb{K}} G(\eta)\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})} \mathcal{G}_\lambda(d\eta) = \mathcal{E}_{\mathrm{int}}^{\mathcal{G}_\lambda}(G,F) \tag{6.3.14}$$

and

$$\mathcal{E}_{\text{int}}^{\mathcal{G}_\lambda}(F,F) = \int_{\mathbb{K}(X)} \langle \nabla_{\text{int}}^{\mathbb{K}} F(\eta), \nabla_{\text{int}}^{\mathbb{K}} F(\eta)\rangle_{T_\eta^{\text{int}}(\mathbb{K})} \mathcal{G}_\lambda(d\eta) \geq 0. \tag{6.3.15}$$

The closability now follows by [MR92, Proposition I.3.3]. □

Intrinsic Dirichlet form

We denote the closure of $(\mathcal{E}_{\text{int}}^{\mathcal{G}_\lambda}, \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X)))$ by $(\mathcal{E}_{\text{int}}^{\mathcal{G}_\lambda}, \mathcal{D}(\mathcal{E}_{\text{int}}^{\mathcal{G}_\lambda}))$.

Definition 6.3.13. *A Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is* conservative, *if*

$$1 \in \mathcal{D}(\mathcal{E}) \text{ and } \mathcal{E}(1,1) = 0.$$

Theorem 6.3.14. *Let $m_1(\lambda)+m_2(\lambda) < \infty$. The closure $(\mathcal{E}_{int}^{\mathcal{G}_\lambda}, \mathcal{D}(\mathcal{E}_{int}^{\mathcal{G}_\lambda}))$ of the intrinsic bilinear form $(\mathcal{E}_{int}^{\mathcal{G}_\lambda}, \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X)))$ is a conservative Dirichlet form.*

Proof. We give the details to get the contraction property. Let $\rho_\varepsilon \in C_b^2(\mathbb{R})$: 1.) $\rho_\varepsilon : \mathbb{R} \to [-\varepsilon, 1+\varepsilon]$ and $\rho_\varepsilon^{'} \leq 1$, 2.) $\rho_\varepsilon(t) = t \ \ \forall t \in [0,1]$ and 3.) $\forall t_1 \geq t_2 :$ $\rho_\varepsilon(t_1) \leq \rho_\varepsilon(t_2)$. Then $|\rho_\varepsilon(t)| \leq |\rho_\varepsilon'(t)||t| \leq |t|$ because $\rho_\varepsilon(0) = 0$ and $\rho_\varepsilon' \leq 1$. Hence,

$$\begin{aligned}
&\mathcal{E}_{\text{int}}^{\mathcal{G}_\lambda}(\rho_\varepsilon \circ F, \rho_\varepsilon \circ F) \\
= &\int_{\mathbb{K}(X)} \langle \nabla_{\text{int}}^{\mathbb{K}}(\rho_\varepsilon \circ F)(\eta), \nabla_{\text{int}}^{\mathbb{K}}(\rho_\varepsilon \circ F)(\eta)\rangle_{T_\eta \mathbb{K}(X)} \mathcal{G}_\lambda(d\eta) \\
= &\int_{\mathbb{K}(X)} \int_X \sum_{i,j=1}^N \partial_i(\rho_\varepsilon \circ g_F)(\langle \rho_1, \eta\rangle, \dots, \langle \rho_N, \eta\rangle) \\
&\qquad \partial_j(\rho_\varepsilon \circ g_F)(\langle \rho_1, \eta\rangle, \dots, \langle \rho_N, \eta\rangle) \langle \nabla^X \rho_i(x), \nabla^X \rho_j(x)\rangle_{T_x(X)} \eta(dx) \mathcal{G}_\lambda(d\eta), \\
= &\int_{\mathbb{K}(X)} (\rho_\varepsilon'(F(\eta)))^2 \int_X \sum_{i,j=1}^N \partial_i g_F(\langle \rho_1, \eta\rangle, \dots, \langle \rho_N, \eta\rangle) \\
&\qquad \partial_j g_F(\langle \rho_1, \eta\rangle, \dots, \langle \rho_N, \eta\rangle) \langle \nabla^X \rho_i(x), \nabla^X \rho_j(x)\rangle_{T_x(X)} \eta(dx) \mathcal{G}_\lambda(d\eta) \\
\leq &\int_{\mathbb{K}(X)} \int_X \sum_{i,j=1}^N \partial_i g_F(\langle \rho_1, \eta\rangle, \dots, \langle \rho_N, \eta\rangle) \\
&\qquad \partial_j g_F(\langle \rho_1, \eta\rangle, \dots, \langle \rho_N, \eta\rangle) \langle \nabla^X \rho_i(x), \nabla^X \rho_j(x)\rangle_{T_x(X)} \eta(dx) \mathcal{G}_\lambda(d\eta) \\
&= \mathcal{E}_{\text{int}}^{\mathcal{G}_\lambda}(F,F).
\end{aligned}$$

By [MR92, Propositions I.4.7 and I.4.10] the closure of the bilinear form is in fact a Dirichlet form. That it is conservative is obvious. □

Example 6.3.15. *Chosing $\lambda(dt) = \theta t^{-1} e^{-t} dt$, we obtain a Dirichlet form $\mathcal{E}_{int}^{\mathcal{G}_\theta}$ for the Gamma measure $\mathcal{G}_\theta$.*

6.3.2 Extrinsic motion related to Gibbs measures

In this subsection we construct a Dirichlet form describing the motion of marks. To this end, we enter the general scheme at the second point using the (extrinsic) quasi-invariance property of $\mathcal{G}_\theta$. By [LS01] this quasi-invariance property only holds for particular measures. Of those, the Gamma measures are prominent examples (for details cf. [LS01, Theorem 1]). Hence, we assume from now on that $\lambda = \lambda_\theta$, $\theta > 0$ fixed.

We deduce the Dirichlet form describing the movement of positions (and marks, cf. Subsections 6.3.3 and 6.3.4) w.r.t. a fixed Gibbs measure μ corresponding to the pair potential ϕ. From now on, we assume that $(\phi)_{\mathbb{K}(X)}$ holds and that

$$|\mathcal{K}'_\Delta| < \infty \qquad \forall \Delta \in \mathcal{B}_c(X). \tag{6.3.16}$$

Remark 6.3.16. *The property stated in (6.3.16) is equivalent to*

$$\left\{\Delta_{\mathcal{Q}} \middle| \Delta \in \mathcal{B}_c(X)\right\} \subset \mathcal{Q}_c(X),$$

where we use the notation introduced in (5.1.15). Hence, the assumption (6.3.16) allows us to use for the construction of the Dirichlet forms the usual cylinder functions

$$\mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X)).$$

One can avoid this condition. For $k \in \mathbb{N}_0^\infty$ let $C_{0,\mathcal{Q}}^k(X)$ denote the set of k-times continuously differentiable functions f that vanish outside of a set $\Delta_f \in \mathcal{Q}(X)$. If (6.3.16) does not hold, one can rewrite all the results of Chapter 6 using $C_{0,\mathcal{Q}}^k(X)$ instead of $C_0^k(X)$, e.g.,

$$\mathcal{F}C_b(\mathbb{K}(X), C_{0,\mathcal{Q}}(X)) \quad \textit{replaces} \quad \mathcal{F}C_b(\mathbb{K}(X), C_0(X)).$$

But for the simplicity and clarity of the outline, we assume it.

Adjoint of the extrinsic directional derivative

Before we obtain the integration by parts formula, we introduce

Definition 6.3.17. *For each $h \in C_0(X)$ and $\eta \in \mathbb{K}(X)$ we define the following* extrinsic logarithmic derivative

$$\langle \beta^\mu_{ext}(\eta), h\rangle_{T_\eta^{\mathrm{ext}}(\mathbb{K})} := \theta\langle h, m\rangle - \langle h, \eta\rangle - \beta \left\langle h, \phi\right\rangle_{T_{\eta\otimes\eta}}, \tag{6.3.17}$$

where

$$\langle h, \phi\rangle_{T_{\eta\otimes\eta}} := \int_{\mathbb{K}(X)} \int_{\mathbb{K}(X)} \phi(x,y)\,(h(x)+h(y))\,\eta(dx)\eta(dy). \tag{6.3.18}$$

Lemma 6.3.18. *Let $(\phi)_{\mathbb{K}(X)}$ (and (6.3.16)) hold. For each $h \in C_0(X)$, we have*

$$\langle \beta^{\mu}_{ext}(\eta), h\rangle_{T^{\mathrm{ext}}_{\eta}(\mathbb{K})} \in \bigcap_{p\in\mathbb{N}} L^p(\mathbb{K}(X), \mu). \tag{6.3.19}$$

Proof. For each $k \in \mathbb{N}_0$ there exists $\tilde{\Delta} \in \mathcal{Q}_c(X)$ and $C > 0$ such that

$$\begin{aligned}\int_{\mathbb{K}(X)} |\beta^{\mu}_{\mathrm{ext}}(h,\eta)|^k \,\mu(d\eta) &\le C^k \int_{\mathbb{K}(X)} \left(1 + \langle \mathbb{1}_{\tilde{\Delta}}, \eta\rangle + \left\langle \mathbb{1}_{\mathcal{U}_{\tilde{\Delta}}}, \eta\right\rangle^2\right)^k \mu(d\eta) \\ &\le C^k \sum_{j=0}^{2k} \binom{2k}{j} \int_{\mathbb{K}(X)} \left\langle \mathbb{1}_{\mathcal{U}_{\tilde{\Delta}}}, \eta\right\rangle^j \mu(d\eta) < \infty, \end{aligned} \tag{6.3.20}$$

where we deduce the finiteness by Theorem 5.3.10. □

Integration by parts for the extrinsic directional derivative

Theorem 6.3.19. *Let $(\phi)_{\mathbb{K}(X)}$ (and (6.3.16)) hold. Then we have for $F, G \in \mathcal{F}C_b^{\infty}(\mathbb{K}(X), C_0(X))$ and $h \in C_0(X)$ the following* extrinsic integration by parts formula

$$\begin{aligned}\int_{\mathbb{K}(X)} \nabla^{\mathbb{K}}_{ext,h} F(\eta) G(\eta)\mu(d\eta) = &-\int_{\mathbb{K}(X)} F(\eta)\nabla^{\mathbb{K}}_{ext,h} G(\eta)\mu(d\eta) \\ &- \int_{\mathbb{K}(X)} F(\eta)G(\eta)\langle \beta^{\mu}_{ext}(\eta), h\rangle_{T^{\mathrm{ext}}_{\eta}(\mathbb{K})}\mu(d\eta).\end{aligned}$$

It can be rewritten as

$$\begin{aligned}\left(\left(\nabla^{\mathbb{K}}_{ext,h}\right)^{\star,\mu} G\right)(\eta) = &-\langle \nabla^{\mathbb{K}}_{ext} G(\eta), h\rangle_{T^{\mathrm{ext}}_{\eta}(\mathbb{K})} \\ &- G(\eta)\langle \beta^{\mu}_{ext}(\eta), h\rangle_{T^{\mathrm{ext}}_{\eta}(\mathbb{K})}.\end{aligned} \tag{6.3.21}$$

Remark 6.3.20. *This integration by parts formula for the Gibbs measure μ is independent of the concrete structure of the Gibbs measure: It only depends on the potential ϕ (and, of course, on the Gamma measure $\mathcal{G}_{\theta}$ and the direction $h \in C_0(X)$ of the directional derivative).*

Proof of Theorem 6.3.19. To prove this result we use the (**DLR**) property of the measure μ. The integrability of the logarithmic derivative is given in Lemma 6.3.18.

By the product rule, it is enough to prove the integration by parts formula for $G = 1$. Fix $\tilde{\Delta} \in \mathcal{Q}_c(X)$ such that $F(\eta_{\tilde{\Delta}}) = F(\eta)$ and $h = h\mathbb{1}_{\tilde{\Delta}}$. Let

$\mathcal{U}_{\tilde{\Delta}} \in \mathcal{Q}_c(X)$ be chosen as in **(FR)**. In particular, $\phi(x,y) = 0$ if $x \in \tilde{\Delta}$ and $y \notin \mathcal{U}_{\tilde{\Delta}}$. Let $\Delta \in \mathcal{Q}_c(X)$ be such that $\mathcal{U}_{\tilde{\Delta}} \subset \Delta$.

Using the (**DLR**) equation (cf. (6.3.18) in Definition 5.3.5), we get

$$\begin{aligned}
&\int_{\mathbb{K}(X)} \nabla^{\mathbb{K}}_{\mathrm{ext},h} F(\eta)\mu(d\eta) = \int_{\mathbb{K}(X)} \nabla^{\mathbb{K}}_{\mathrm{ext},h} F(\eta_\Delta)\mu(d\eta) \\
&= \int_{\mathbb{K}(X)} \int_{K(\Delta)} \nabla^{\mathbb{K}}_{\mathrm{ext},h} F(\eta_\Delta \cup \xi_{\Delta^c})\pi_\Delta(d\eta_\Delta|\xi)\mu(d\xi) \\
&= \int_{\mathbb{K}(X)} \frac{1}{Z_\Delta(\xi)} \int_{K(\Delta)} \nabla^{\mathbb{K}}_{\mathrm{ext},h} F(\eta_\Delta) e^{-\beta H_\Delta(\eta_\Delta|\xi)} \mathcal{G}^\Delta_\theta(d\eta_\Delta)\mu(d\xi),
\end{aligned} \tag{6.3.22}$$

where $\mathcal{G}^\Delta_\theta$ denotes the Gamma measure on $\mathbb{K}(\Delta)$. We calculate the integration by parts formula for the inner integral. Since each factor in the integrals below is continuously differentiable and the logarithmic derivative is in $\bigcap_{p\in\mathbb{N}} L^p(\mathbb{K}(X),\mu)$ (cf. Lemma 6.3.18), we may interchange the differentiation and integration.

$$\begin{aligned}
&\int_{\mathbb{K}(\Delta)} \nabla^{\mathbb{K}}_{\mathrm{ext},h} F(\eta) e^{-\beta H_\Delta(\eta_\Delta|\xi)} \mathcal{G}_\theta(d\eta) \\
&= \frac{d}{dt} \int_{\mathbb{K}(\Delta)} F(e^{th}\eta) e^{-\beta H_\Delta(\eta_\Delta|\xi)} \mathcal{G}_\theta(d\eta) \Big|_{t=0} \\
&= \frac{d}{dt} \int_{\mathbb{K}(\Delta)} F(\eta) e^{-\beta H_\Delta(e^{-th}\eta_\Delta|\xi)} \frac{dM_{th}\mathcal{G}_\theta(\eta)}{d\mathcal{G}_\theta(\eta)} d\mathcal{G}_\theta(\eta) \Big|_{t=0} \\
&= \frac{d}{dt} \int_{\mathbb{K}(\Delta)} F(\eta) \exp\big(-\beta H_\Delta(e^{-th}\eta_\Delta|\xi) - \theta\langle th, m\rangle - \langle e^{-th} - 1, \eta\rangle\big) d\mathcal{G}_\theta(\eta) \Big|_{t=0}
\end{aligned} \tag{6.3.23}$$

We calculate the derivate of the relative energy:

$$\begin{aligned}
&\frac{d}{dt} H_\Delta(e^{-th}\eta|\xi)\Big|_{t=t_0} \\
&= \frac{d}{dt}\Bigg(\sum_{x,y\in\tau(\eta)\cap\Delta} e^{-th(x)} s_x e^{-th(y)} s_y \phi(x,y) + \sum_{\substack{x\in\tau(\eta)\cap\Delta \\ y\in\tau(\xi)\cap\Delta^c}} e^{-th(x)} s_x s_y \phi(x,y) \Bigg)\Bigg|_{t=t_0} \\
&= -\sum_{x,y\in\tau(\eta)\cap\Delta\cap\tilde{\Delta}} \big(h(x)+h(y)\big) e^{-t_0 h(x)} s_x e^{-t_0 h(y)} s_y \phi(x,y) \\
&\quad - \sum_{\substack{x\in\tau(\eta)\cap\Delta\cap\tilde{\Delta} \\ y\in\tau(\xi)\cap\Delta^c\cap\mathcal{U}_{\tilde{\Delta}}}} h(x) e^{-t_0 h(x)} s_x s_y \phi(x,y) \qquad (6.3.24) \\
&= -\langle h, \phi\rangle_{T_{\eta\otimes\eta}},
\end{aligned}$$

where the second sum in (6.3.24) vanishes because $\mathcal{U}_{\tilde{\Delta}} \subset \Delta$. Interchanging integration and differentiation in (6.3.22) and plugging (6.3.24) into (6.3.23), yields

$$\begin{aligned}
&\int_{\mathbb{K}(\Delta)} \nabla^{\mathbb{K}}_{\mathrm{ext},h} F(\eta) e^{-\beta H_\Delta(\eta_\Delta|\xi)} \mathcal{G}_\theta(d\eta) \\
&\qquad = \int_{\mathbb{K}(\Delta)} F(\eta) \langle \beta^\mu_{\mathrm{ext}}(\eta), h\rangle_{T^{\mathrm{ext}}_\eta(\mathbb{K})} e^{-\beta H_\Delta(\eta_\Delta|\xi)} \mathcal{G}_\theta(d\eta). \qquad (6.3.25)
\end{aligned}$$

Plugging (6.3.25) back into (6.3.22), we get using the (**DLR**) equation

$$\begin{aligned}
&\int_{\mathbb{K}(X)} \nabla^{\mathbb{K}}_{\mathrm{ext},h} F(\eta) \mu(d\eta) \\
&= \int_{\mathbb{K}(X)} \left(\frac{1}{Z_\Delta(\xi)} \int_{K(\Delta)} F(\eta) \langle \beta^\mu_{\mathrm{ext}}(\eta), h\rangle_{T^{\mathrm{ext}}_\eta(\mathbb{K})} e^{-\beta H_\Delta(\eta|\xi)} \mathcal{G}_\theta(d\eta) \right) \mu(d\xi) \\
&= \int_{\mathbb{K}(X)} F(\eta) \langle \beta^\mu_{\mathrm{ext}}(\eta), h\rangle_{T^{\mathrm{ext}}_\eta(\mathbb{K})} \mu(d\eta).
\end{aligned}$$

which implies the claim. □

Adjoint of the extrinsic gradient

Definition 6.3.21. *A function V is called a* extrinsic vector field *iff it is of the following form*

$$V(\eta) := \sum_{i=1}^N g_i(\eta)\phi_i$$

where $g_i \in \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0(X))$ and $\phi_i \in C_0(X)$ for $i = 1, \dots, N$. By $V_{cyl,\mathbb{R}_+}(\mathbb{K}(X))$ we denote the set of all these extrinsic vector fields.

Remark 6.3.22. *For $F \in \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0(X))$, we have $\nabla^{\mathbb{K}}_{ext} F \in V_{cyl}(\mathbb{K}(X))$.*

Lemma 6.3.23. *Let $V_1, V_2 \in V_{cyl,\mathbb{R}_+}(\mathbb{K}(X))$, then*

$$\int_{\mathbb{K}(X)} \langle V_1(\eta), V_2(\eta)\rangle_{T^{\mathrm{ext}}_\eta(\mathbb{K})} \mu(d\eta) < \infty.$$

Proof. The extrinsic vector fields are bounded and finitely supported. Thus the integral is finite because the first moment of μ is finite (cf. (5.3.13)). □

Theorem 6.3.24. *Fix $V := \sum_{i=1}^N g_i\phi_i \in V_{cyl,\mathbb{R}_+}(\mathbb{K}(X))$. Let $(\phi)_{\mathbb{K}(X)}$ (and (6.3.16)) hold. Then we have for all $F \in \mathcal{F}C_b^1(\mathbb{K}(X), C_0(X))$*

$$\int_{\mathbb{K}(X)} \left\langle \nabla^{\mathbb{K}}_{ext} F(\eta), V(\eta)\right\rangle_{T^{\mathrm{ext}}_\eta(\mathbb{K})} \mu(d\eta) = \\ - \int_{\mathbb{K}(X)} F(\eta) \sum_{i=1}^N \left\langle \nabla^{\mathbb{K}}_{ext} g_i(\eta), \phi_i\right\rangle_{T^{\mathrm{ext}}_\eta(\mathbb{K})} \mu(d\eta) \\ - \int_{\mathbb{K}(X)} F(\eta)\langle \beta^\mu_{ext}(\eta), V(\eta)\rangle_{T^{\mathrm{ext}}_\eta(\mathbb{K})}\mu(d\eta), \tag{6.3.26}$$

where all the integrals are finite. We can reformulate it as

$$\left(\left(\nabla^{\mathbb{K}}_{ext}\right)^\star_\mu V\right)(\eta) = -\sum_{i=1}^N \left\langle \nabla^{\mathbb{K}}_{ext} g_i(\eta), \phi_i\right\rangle_{T^{\mathrm{ext}}_\eta(\mathbb{K})} \\ - \langle \beta^\mu_{ext}(\eta), V(\eta)\rangle_{T^{\mathrm{ext}}_\eta(\mathbb{K})}. \tag{6.3.27}$$

Proof. The finiteness of the involved integrals follows by Lemma 6.3.23. Similar as in Theorem 6.3.8, we use the integration by parts formula (cf. Theorem 6.3.19) to derive the result. □

Extrinsic bilinear form

We define for $F, G \in \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0(\mathbb{K}(X)))$ the bilinear form

$$\mathcal{E}^\mu_{\mathrm{ext}}(F,G) := \int_{\mathbb{K}(X)} \langle \nabla^{\mathbb{K}}_{\mathrm{ext}} F(\eta), \nabla^{\mathbb{K}}_{\mathrm{ext}} G(\eta)\rangle_{T^{\mathrm{ext}}_\eta(\mathbb{K})}\mu(d\eta). \tag{6.3.28}$$

Remark 6.3.25. *It is finite (cf. Lemma 6.3.23) and densely defined (cf. Corollary 6.2.8).*

Definition 6.3.26. *For each $F \in \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0(X))$ we define the* extrinsic generator *for all $\eta \in \mathbb{K}(X)$ by*

$$(L^\mu_{ext} F)(\eta) := -\left(S^{\mathbb{K}(X)}_{ext} F\right)(\eta) - \langle \beta^\mu_{ext}(\eta), \nabla^{\mathbb{K}}_{ext} F(\eta)\rangle_{T^{\mathrm{ext}}_\eta(\mathbb{K})},$$

where for all $\eta \in \mathbb{K}(X)$

$$(S^{\mathbb{K}(X)}_{ext} F)(\eta) := \sum_{l,k=1}^N \partial_l\partial_k g_F(\langle \rho_1, \eta\rangle, \dots, \langle \rho_N, \eta\rangle)\langle \rho_k, \rho_l\rangle_{T^{\mathrm{ext}}_\eta(\mathbb{K})}.$$

Corollary 6.3.27. *Let $(\phi)_{\mathbb{K}(X)}$ (and (6.3.16)) hold. We have for $F,\ G \in \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0(X))$ that*

$$\mathcal{E}^\mu_{ext}(F,G) = \int_{\mathbb{K}(X)} \left\langle \nabla^{\mathbb{K}}_{ext}F(\eta), \nabla^{\mathbb{K}}_{ext}G(\eta)\right\rangle_{T^{\mathrm{ext}}_\eta(\mathbb{K})} \mu(d\eta)$$
$$= \int_{\mathbb{K}(X)} (L^\mu_{ext}F)(\eta)G(\eta)\mu(d\eta). \tag{6.3.29}$$

Proof. The claim follows by Theorem 6.3.24. Namely, for an arbitrary cylindrical function $F(\eta) = g_F(\langle\rho_1,\eta\rangle,\ldots,\langle\rho_N,\eta\rangle) \in \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0(X))$ we choose $\phi_i = \rho_i$ for $i = 1,\ldots,N$ and

$$g_i(\eta) := \partial_i g_F(\langle\rho_1,\eta\rangle,\ldots,\langle\rho_N,\eta\rangle).$$

□

Proposition 6.3.28. *Let $(\phi)_{\mathbb{K}(X)}$ (and (6.3.16)) hold. Then the bilinear form $(\mathcal{E}^\mu_{ext}, \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0(X)))$ is closable, symmetric and* positive definite.

Proof. This assertion follows similarly to Proposition 6.3.12 with the obvious adaptions. □

Extrinsic Dirichlet form

We denote the closure of $(\mathcal{E}^\mu_{\mathrm{ext}}, \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0(X)))$ by $(\mathcal{E}^\mu_{\mathrm{ext}}, \mathcal{D}(\mathcal{E}^\mu_{\mathrm{ext}}))$.

Theorem 6.3.29. *Let $(\phi)_{\mathbb{K}(X)}$ (and (6.3.16)) hold. Then $(\mathcal{E}^\mu_{ext}, \mathcal{D}(\mathcal{E}^\mu_{ext}))$ is a conservative Dirichlet form.*

Proof. The claim follows as in Theorem 6.3.14 with the obvious changes. □

6.3.3 Intrinsic motion related to Gibbs measures

From now on, let $(\phi)_{\mathbb{K}(X)}$ (and (6.3.16)) hold and, in addition, $\phi \in C^1(X\times X)$.

Integration by parts for the directional derivative

Before we calculate the integration by parts formula, we introduce

Definition 6.3.30. *Let $(\phi)_{\mathbb{K}(X)}$ (and (6.3.16)) hold and $\phi \in C^1(X \times X)$. For each $v \in V_0(X)$ and each $\eta \in \mathbb{K}(X)$ we define the* intrinsic directional logarithmic derivative

$$\eta \mapsto \langle\beta^\mu_{int}(\eta), v\rangle_{T^{\mathrm{int}}_\eta(\mathbb{K})} := \langle\beta^{\mathcal{G}_\theta}_{int}(\eta), v\rangle_{T^{\mathrm{int}}_\eta(\mathbb{K})} - \beta\left\langle\nabla^X\phi, v\right\rangle_{T_\eta\otimes T_\eta}. \tag{6.3.30}$$

Here,

$$\langle \beta^{\mathcal{G}_\theta}_{int}(\eta), v\rangle_{T^{\mathrm{int}}_\eta(\mathbb{K})} = \int_X \langle \beta^m(x), v(x)\rangle_{T^X_x} + div^X v(x)\eta(dx)$$

and

$$\langle \nabla\phi, v\rangle_{T_\eta\otimes T_\eta} := \int_X\int_X \left(\nabla^X\phi(x,y), (v(x), v(y))\right)\eta(dx)\eta(dy) \tag{6.3.31}$$

Remark 6.3.31. *Note that*

$$\begin{aligned}\langle \nabla\phi, v\rangle_{T_\eta\otimes T_\eta} &= \sum_{x,y\in\tau(\eta)} \left(\nabla^X\phi(x,y), (v(x), v(y))\right) s_x s_y\\ &= \sum_{x,y\in\tau(\eta)} \left(\partial_1\phi(x,y)(v(x)) + \partial_2\phi(x,y)(v(y))\right) s_x s_y.\end{aligned} \tag{6.3.32}$$

Lemma 6.3.32. *We have*

$$\eta \mapsto \langle \beta^\mu_{int}(\eta), v\rangle_{T^{\mathrm{int}}_\eta(\mathbb{K})} \in \bigcap_{k\in\mathbb{N}} L^k(\mathbb{K}(X), \mu). \tag{6.3.33}$$

Proof. As in (6.3.20), we get the integrability of $\langle \beta^\mu_{\mathrm{int}}(\cdot), v\rangle_{T^{\mathrm{int}}_\cdot(\mathbb{K})}$. □

Theorem 6.3.33. *Let $(\phi)_{\mathbb{K}(X)}$ (and (6.3.16)) hold and $\phi \in C^1(X\times X)$. Then for each $v \in V_0(X)$ and $F, G \in \mathcal{F}C^\infty_b(\mathbb{K}(X), C^\infty_0(X))$ we have an* intrinsic integration by parts formula *for the gradient $\nabla^{\mathbb{K}}_{int,v}$:*

$$\begin{aligned}\int_{\mathbb{K}(X)} \left(\nabla^{\mathbb{K}}_{int,v}F\right)(\eta)G(\eta)\mu(d\eta) &= -\int_{\mathbb{K}(X)} F(\eta)\left(\nabla^{\mathbb{K}}_{int,v}G\right)(\eta)\mu(d\eta)\\ &- \int_{\mathbb{K}(X)} F(\eta)G(\eta)\langle \beta^\mu_{int}(\eta), v\rangle_{T^{\mathrm{int}}_\eta(\mathbb{K})}\mu(d\eta).\end{aligned} \tag{6.3.34}$$

Proof. It is enough to deduce the claim for $G = 1$. There exists $\tilde\Delta \in \mathcal{Q}_c(X)$: $F(\eta) = F(\eta_{\tilde\Delta})$ and $v = 1\!\!1_{\tilde\Delta}v$. Using the **(DLR)** property, we have for an arbitrary $\Delta \in \mathcal{Q}_c(X)$ with $\Delta \supset \mathcal{U}_{\tilde\Delta}$

$$\begin{aligned}&\int_{\mathbb{K}(X)} \left(\nabla^{\mathbb{K}}_{\mathrm{int},v}F\right)(\eta)\mu(d\eta) = \int_{\mathbb{K}(X)} \left(\nabla^{\mathbb{K}}_{\mathrm{int},v}F\right)(\eta_\Delta)\mu(d\eta)\\ &= \int_{\mathbb{K}(X)}\int_{\mathbb{K}(X)} \left(\nabla^{\mathbb{K}}_{\mathrm{int},v}F\right)(\eta_\Delta)\pi_\Delta(d\eta|\xi)\mu(d\xi)\\ &= \int_{\mathbb{K}(X)} \frac{1}{Z_\Delta(\xi)}\int_{K(\Delta)} \left(\nabla^{\mathbb{K}}_{\mathrm{int},v}F\right)(\eta_\Delta\cup\xi_{\Delta^c})e^{-\beta H_\Delta(\eta_\Delta|\xi)}\mathcal{G}^\Delta_\theta(d\eta_\Delta)\mu(d\xi).\end{aligned} \tag{6.3.35}$$

Since both factors in the integrand are in $\bigcap_{p\in\mathbb{N}} L^p(\mathbb{K}(\Delta), \mathcal{G}_\theta^\Delta)$, we get using (the proof of) Theorem 6.3.5 that

$$\begin{aligned}
&\int_{\mathbb{K}(X)} \left(\nabla^{\mathbb{K}}_{\mathrm{int},v} F\right)(\eta)\mu(d\eta) \\
=&\int_{\mathbb{K}(X)} \frac{1}{Z_\Delta(\xi)} \int_{K(\Delta)} F(\eta_\Delta) e^{-\beta H_\Delta(\eta_\Delta|\xi)} \Bigg(- \langle \beta^{\mathcal{G}_\theta}_{\mathrm{int}}(\eta_\Delta), v\rangle_{T^{\mathrm{int}}_{\eta_\Delta}(\mathbb{K})} \\
&\qquad - \beta \frac{d}{dt} H_\Delta \left(\left((\phi_t^\nu)^{-1}\right)^* \eta|\xi\right)\Big|_{t=0} \Bigg) \mathcal{G}_\theta^\Delta(d\eta_\Delta)\mu(d\xi). \qquad (6.3.36)
\end{aligned}$$

Hence, it remains to calculate

$$\begin{aligned}
&\frac{d}{dt} H_\Delta \left(\left((\phi_t^\nu)^{-1}\right)^* \eta|\xi\right)\Big|_{t=0} \\
=&\frac{d}{dt} \sum_{x,y\in\tau(\eta)\cap\Delta} \phi\left((\phi_t^\nu)^{-1}(x), (\phi_t^\nu)^{-1}(y)\right) s_x s_y \\
&+ \sum_{\substack{x\in\tau(\eta)\cap\Delta \\ y\in\tau(\xi)\cap\Delta^C}} \phi\left((\phi_t^\nu)^{-1}(x), y\right) s_x s_y \Big|_{t=0} \\
=&\sum_{x,y\,\in\tau(\eta)\cap\Delta} s_x s_y \big(\partial_1 \phi\left((\phi_t^\nu)^{-1}(x), (\phi_t^\nu)^{-1}(y)\right) (\psi_t(x)) \\
&\qquad + \partial_2 \phi\left((\phi_t^\nu)^{-1}(x), (\phi_t^\nu)^{-1}(y)\right) (\psi_t(y)) \big)\Big|_{t=0} \\
=&\sum_{x,y\in\tau(\eta)\cap\tilde{\Delta}} s_x s_y \big(\partial_1\phi(x,y)(\psi_t(x)) + \partial_2\phi(x,y)(\psi_t(x))\big)\Big|_{t=0}. \qquad (6.3.37)
\end{aligned}$$

Here, we used that $\phi((\phi_t^\nu)^{-1}(x), (\phi_t^\nu)^{-1}(y)) = 0$ for all t, $x \in \tilde{\Delta}$ and $y \notin \mathcal{U}_{\tilde{\Delta}}$. Hence, it is sufficient to consider

$$\begin{aligned}
X \ni x \mapsto &\psi_t(x) := \left(\frac{d}{dt} (\phi_t^\nu)^{-1} \right)(x) \\
&= - \left(\left(\frac{d}{dx} \phi_t^\nu \right) \circ \left((\phi_t^\nu)^{-1}(x)\right) \right)^{-1} \left(\left(\frac{d}{dt} \phi_t^\nu \right) \circ \left((\phi_t^\nu)^{-1}(x)\right) \right) \\
&= - \left(\left(\frac{d}{dx} \phi_t^\nu \right) \circ \left((\phi_t^\nu)^{-1}(x)\right) \right)^{-1} (v(x)). \qquad (6.3.38)
\end{aligned}$$

Moreover, $v \in V_0(X)$ is a smooth, compactly supported vector field and (cf. e.g. [Hag08, Theorem 4.2.11]) the derivative in (6.3.38) is bounded; therefore,

ψ_t is uniformly bounded for small t. Setting $t = 0$ yields

$$\psi_0 = -\left(\left(\frac{d}{dx}\phi_0^\nu\right) \circ \left((\phi_0^\nu)^{-1}(x)\right)\right)^{-1} (v(x)) = -v(x).$$

We plug (6.3.38) back into (6.3.37) and see that the later equals

$$\begin{aligned}&\sum_{x,y\in\tau(\eta)\cap\tilde{\Delta}} s_x s_y \big(\partial_1\phi(x,y)v(x) + \partial_2\phi(x,y)v(y)\big)\\ &= -\langle\nabla\phi, \mathbb{1}_{\tilde{\Delta}} v\rangle_{T_\eta\otimes T_\eta} = -\langle\nabla\phi, v\rangle_{T_\eta\otimes T_\eta}.\end{aligned} \tag{6.3.39}$$

Using (6.3.39) in (6.3.36), yields the claim. □

Adjoint of the intrinsic gradient

Lemma 6.3.34. *Let $V_1, V_2 \in V_{cyl,X}(\mathbb{K}(X))$, then*

$$\int_{\mathbb{K}(X)} \langle V_1(\eta), V_2(\eta)\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})}\, \mu(d\eta) < \infty.$$

Proof. The claim follows similarly as in Lemma 6.3.23 because V_1 and V_2 are bounded and finitely supported (cf. also Definition 6.3.6). □

Theorem 6.3.35. *Let $(\phi)_{\mathbb{K}(X)}$ (and (6.3.16)) hold and $\phi \in C^1(X\times X)$. Fix*

$$V := \sum_{i=1}^N g_i v_i \in V_{cyl,X}(\mathbb{K}(X)).$$

Then for any $F \in \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X))$

$$\begin{aligned}&\int_{\mathbb{K}(X)} \langle\nabla_{int}^{\mathbb{K}} F(\eta), V(\eta)\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})}\mu(d\eta)\\ &= \int_{\mathbb{K}(X)} F(\eta)\left(\sum_{i=1}^N \left(-\nabla_{int,v_i}^{\mathbb{K}} g_i\right)(\eta) - \langle\beta_{int}^\mu(\eta), V(\eta)\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})}\right)\mu(d\eta).\end{aligned} \tag{6.3.40}$$

In other words,

$$\left(\left(\nabla_{int}^{\mathbb{K}}\right)^{\star,\mu} V\right)(\eta) = \sum_{i=1}^N \left(-\nabla_{int,v_i}^{\mathbb{K}} g_i\right)(\eta) - \langle\beta_{int}^\mu(\eta), V(\eta)\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})}. \tag{6.3.41}$$

Proof. The claim follows by the intrinsic integration by parts formula of the Gibbs measure μ (cf. Theorem 6.3.33):

$$\int_{\mathbb{K}(X)} \langle \nabla^{\mathbb{K}}_{\text{int}} F(\eta), V(\eta)\rangle \mu(d\eta) = \int_{\mathbb{K}(X)} \sum_{i=1}^{N} \langle \left(\nabla^{\mathbb{K}}_{\text{int}} F\right)(\eta), v_i\rangle g_i(\eta)\mu(d\eta)$$

$$= \sum_{i=1}^{N} \int_{\mathbb{K}(X)} F(\eta) \cdot \left(-\nabla^{\mathbb{K}}_{\text{int},v_i} g_i(\eta) - g_i(\eta)\langle \beta^{\mu}_{\text{int}}(\eta), v_i(\eta)\rangle_{T^{\text{int}}_{\eta}(\mathbb{K})}\right) \mu(d\eta),$$

which implies the result by the linearity of the logarithmic derivative. □

Intrinsic bilinear form

We define for $F, G \in \mathcal{F}C^{\infty}_{b}(\mathbb{K}(X), C^{\infty}_{0}(X))$ the *intrinsic bilinear form*

$$\mathcal{E}^{\mu}_{\text{int}}(F, G) := \int_{\mathbb{K}(X)} \langle \nabla^{\mathbb{K}}_{\text{int}} F(\eta), \nabla^{\mathbb{K}}_{\text{int}} G(\eta)\rangle_{T^{\text{int}}_{\eta}(\mathbb{K})} \mu(d\eta).$$

Proposition 6.3.36. *Let $(\phi)_{\mathbb{K}(X)}$ hold and $\phi \in C^1(X \times X)$. The bilinear form $\left(\mathcal{E}^{\mu}_{int}, \mathcal{F}C^{\infty}_{b}(\mathbb{K}(X), C^{\infty}_{0}(X))\right)$ is well-defined, closable, symmetric and positive definite.*

Proof. By Lemma 6.3.34, the integral is finite. Since $\mathcal{F}C^{\infty}_{b}(\mathbb{K}(X), C^{\infty}_{0}(X)) \subset L^2(\mathbb{K}(X), \mu)$ dense (cf. Corollary 6.2.8), the bilinear form is densely defined. Thus, the bilinear form is well-defined. The other properties follow as in the proof of Proposition 6.3.11. □

Corollary 6.3.37. *For $F,\ G \in \mathcal{F}C^{\infty}_{b}(\mathbb{K}(X), C^{\infty}_{0}(X))$*

$$\mathcal{E}^{\mu}_{int}(F, G) \;=\; \int_{\mathbb{K}(X)} F(\eta)(L^{\mu}_{int} G)(\eta)\mu(d\eta),$$

where we define the intrinsic generator L^{μ}_{int} *as*

$$\begin{aligned}
(L^{\mu}_{int} G)(\eta) &:= \left(\left(\nabla^{\mathbb{K}}_{int}\right)^{\star,\mu} \nabla^{\mathbb{K}}_{int} G\right)(\eta) \\
= &- \sum_{i,j=1}^{N} \frac{\partial^2}{\partial_i \partial_j} g_G\left(\langle \varphi_1, \eta\rangle, \ldots, \langle \varphi_N, \eta\rangle\right) \langle \nabla^X \varphi_i, \nabla^X \varphi_j\rangle_{T^{\text{int}}_{\eta}(\mathbb{K})} \\
&- \sum_{i=1}^{N} \partial_i g_F(\langle \rho_1, \eta\rangle, \ldots, \langle \rho_N, \eta\rangle)\Bigg(\int_X \Delta^X \rho_i(x) d\eta(x) \\
&+ \int_X \langle \beta^m(x), \nabla^X \rho_i(x)\rangle_{T_x(X)} d\eta(x) - \langle \nabla\phi, \nabla^X \rho_i(x)\rangle_{T_\eta \otimes T_\eta}\Bigg)
\end{aligned} \tag{6.3.42}$$

Proof. To calculate the generator, we use $V(\eta) = \nabla^{\mathbb{K}}_{\text{int}} G(\eta)$ in (6.3.40) and follow the arguments to prove Corollary 6.3.11 with the obvious changes. □

Intrinsic Dirichlet form

We denote the closure of $(\mathcal{E}^\mu_{\mathrm{int}}, \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X)))$ by $(\mathcal{E}^\mu_{\mathrm{int}}, \mathcal{D}(\mathcal{E}^\mu_{\mathrm{int}}))$.

Theorem 6.3.38. *Let $(\phi)_{\mathbb{K}(X)}$ hold and $\phi \in C^1(X \times X)$. Then the closure $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is a conservative Dirichlet form.*

Proof. As for the proof of Theorem 6.3.14, the claim follows by showing the contraction property. □

6.3.4 Joint motion related to Gibbs measures

In this subsection we combine the extrinsic and intrinsic considerations to get a gradient Dirichlet form that corresponds to $\nabla^{\mathbb{K}}$ and describes a movement in both components.

Adjoint of the directive derivative

Theorem 6.3.39. *Let $(\phi)_{\mathbb{K}(X)}$ (and (6.3.16)) hold and $\phi \in C^1(X \times X)$. For each $h \in C_0(X)$, $v \in V_0(X)$ and $\eta \in \mathbb{K}(X)$ we define the following* logarithmic derivative

$$\langle \beta^\mu(\eta), (h, v)\rangle_{T_\eta(\mathbb{K})} := \langle \beta^\mu_{ext}(\eta), h\rangle_{T_\eta^{\mathrm{ext}}(\mathbb{K})} + \langle \beta^\mu_{int}(\eta), v\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})}.$$

We obtain for all F, $G \in \mathcal{F}C_b^1(\mathbb{K}(X), C_0^1(X))$, all $h \in C_0(X)$ and all $v \in V_0(X)$ an integration by parts *formula , i.e.,*

$$\int_{\mathbb{K}(X)} \nabla^{\mathbb{K}}_{h,v} F(\eta) G(\eta) \mu(d\eta) = -\int_{\mathbb{K}(X)} F(\eta) \nabla^{\mathbb{K}}_{h,v} G(\eta) \mu(d\eta) - \int_{\mathbb{K}(X)} F(\eta) G(\eta) \langle \beta^\mu(\eta), (h, v)\rangle_{T_\eta(\mathbb{K})} \mu(d\eta).$$

Proof. The result follows by Theorems 6.3.19 and 6.3.33. □

Adjoint of the gradient

Definition 6.3.40. *A function $V : \mathbb{K}(X) \to \mathbb{R}$ is called a* joint vector field*, iff it is of the following form*

$$V(\eta) := \left(\sum_{i=1}^N g_i(\eta)\phi_i, \sum_{i=1}^N h_i(\eta) v_i \right)$$

where for $i = 1, \ldots, N$ g_i, $h_i \in \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X))$, $\phi_i \in C_0(X)$ and $v_j \in V_0(X)$. By $V_{cyl}(\mathbb{K}(X))$ we denote the set of all these joint vector fields.

Moreover, we denote for each such V its projection to $V_{cyl,\mathbb{R}_+}(\mathbb{K}(X))$, respectively to $V_{cyl,X}(\mathbb{K}(X))$, by

$$V_{\mathbb{R}_+} := \sum_{i=1}^{N} g_i(\eta)\phi_i, \quad \textit{respectively by} \quad V_X := \sum_{i=1}^{N} h_i(\eta)v_i.$$

Remark 6.3.41. *Let $F \in \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X))$ then $\nabla^{\mathbb{K}} F \in V_{cyl}(\mathbb{K}(X))$.*

Lemma 6.3.42. *Let $V_1, V_2 \in V_{cyl}(\mathbb{K}(X))$, then*

$$\int_{\mathbb{K}(X)} \langle V_1(\eta), V_2(\eta)\rangle_{T_\eta(\mathbb{K})}\, \mu(d\eta) < \infty. \tag{6.3.43}$$

Proof. Using equation (6.1.7), this follows by Lemmas 6.3.23 and 6.3.34. □

Theorem 6.3.43. *Let $(\phi)_{\mathbb{K}(X)}$ (and (6.3.16)) hold and $\phi \in C^1(X \times X)$.Fix*

$$V := \left(\sum_{i=1}^{N} g_i\phi_i, \sum_{i=1}^{N} h_i v_i\right) \in V_{cyl}(\mathbb{K}(X)).$$

Then we have for all $F \in \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X))$

$$\begin{aligned} &\int_{\mathbb{K}(X)} \left\langle \nabla^{\mathbb{K}} F(\eta), V(\eta)\right\rangle_{T_\eta(\mathbb{K})} \mu(d\eta) \\ &\qquad = -\int_{\mathbb{K}(X)} F(\eta) \Bigg[\sum_{i=1}^{N} \left\langle \nabla^{\mathbb{K}}_{ext} g_i(\eta), \phi_i\right\rangle_{T_\eta^{\mathrm{ext}}(\mathbb{K})} \\ &\qquad\qquad + \left\langle \nabla^{\mathbb{K}}_{int} h_i(\eta), v_i\right\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})} \Bigg] \mu(d\eta) \\ &\qquad\quad - \int_{\mathbb{K}(X)} F(\eta)\langle \beta^\mu(\eta), V(\eta)\rangle_{T_\eta(\mathbb{K})} \mu(d\eta), \end{aligned} \tag{6.3.44}$$

where all the integrals are finite. We can reformulate in terms of the adjoint

$$\begin{aligned} &\left(\left(\nabla^{\mathbb{K}}\right)^{\star,\mu} V\right)(\eta) \\ &= -\sum_{i=1}^{N} \left\langle \nabla^{\mathbb{K}}_{ext} g_i(\eta), \phi_i\right\rangle_{T_\eta^{\mathrm{ext}}(\mathbb{K})} - \langle \beta^\mu_{ext}(\eta), V_{\mathbb{R}_+}(\eta)\rangle_{T_\eta^{\mathrm{ext}}(\mathbb{K})} \\ &\quad -\sum_{i=1}^{N} \left\langle \nabla^{\mathbb{K}}_{int} h_i(\eta), v_i\right\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})} - \langle \beta^\mu_{int}(\eta), V_X(\eta)\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})}. \end{aligned} \tag{6.3.45}$$

Proof. This follows by the definition of $T_\eta(\mathbb{K})$ (cf. (6.1.7)) and by Theorems 6.3.24 and 6.3.35. The finiteness of the involved integrals follows by Lemma 6.3.42. □

Bilinear form

We define for $F, G \in \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X))$ the joint bilinear form

$$\mathcal{E}^\mu(F,G) := \int_{\mathbb{K}(X)} \langle \nabla^{\mathbb{K}} F(\eta), \nabla^{\mathbb{K}} G(\eta)\rangle_{T_\eta(\mathbb{K})} \mu(d\eta). \tag{6.3.46}$$

Remark 6.3.44. *It is finite by Lemma 6.3.42 and densely defined by Corollary 6.2.8.*

Definition 6.3.45. *We define for all $F \in \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X))$ the* intrinsic generator *as*

$$\begin{aligned}(L^\mu F)(\eta) :=& (L^\mu_{ext} F)(\eta) + (L^\mu_{int} F)(\eta)\\ =:& -\left(S^{\mathbb{K}(X)} F\right)(\eta) - \langle \beta^\mu(\eta), \nabla^{\mathbb{K}} F(\eta)\rangle_{T_\eta(\mathbb{K})},\end{aligned}$$

where $\eta \in \mathbb{K}(X)$,

$$\begin{aligned}\left(S^{\mathbb{K}(X)} F\right)(\eta) &:= \left(S^{\mathbb{K}(X)}_{ext} F\right)(\eta) + \left(S^{\mathbb{K}(X)}_{int} F\right)(\eta),\\ (L^\mu_{ext} F)(\eta) &:= -\left(S^{\mathbb{K}(X)}_{ext} F\right)(\eta) - \langle \beta^\mu_{ext}(\eta), \nabla^{\mathbb{K}}_{ext} F(\eta)\rangle_{T^{\mathrm{ext}}_\eta(\mathbb{K})} \text{ and}\\ (L^\mu_{int} F)(\eta) &:= -\left(S^{\mathbb{K}(X)}_{int} F\right)(\eta) - \langle \beta^\mu_{int}(\eta), \nabla^{\mathbb{K}}_{int} F(\eta)\rangle_{T^{\mathrm{int}}_\eta(\mathbb{K})}.\end{aligned}$$

Corollary 6.3.46. *For $F,\ G \in \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X))$ we get*

$$\begin{aligned}\mathcal{E}^\mu(F,G) &= \int_{\mathbb{K}(X)} \left\langle \nabla^{\mathbb{K}} F(\eta), \nabla^{\mathbb{K}} G(\eta)\right\rangle_{T_\eta(\mathbb{K})} \mu(d\eta)\\ &= \int_{\mathbb{K}(X)} (L^\mu F)(\eta) G(\eta) \mu(d\eta). \end{aligned} \tag{6.3.47}$$

Proof. This follows by Theorem 6.3.43 and by Corollaries 6.3.11 and 6.3.27. □

Proposition 6.3.47. *$(\mathcal{E}^\mu, \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X)))$ is a closable, symmetric positive definite bilinear form.*

Proof. This follows by (6.1.7), Proposition 6.3.28 and Corollary 6.3.37. □

Dirichlet form

We denote the closure of $(\mathcal{E}^\mu, \mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X)))$ by $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$.

Theorem 6.3.48. *Then the closure (of the bilinear form) $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is a conservative Dirichlet form.*

Proof. This follows by the arguments used to prove Theorems 6.3.29 and 6.3.38. □

Chapter 7

Equilibrium processes

In Chapter 6 we obtained extrinsic, intrinsic and joint gradient Dirichlet forms. Now, we want to actually construct corresponding diffusions on $\mathbb{K}(\mathbb{R}^d)$. We achieve this, e.g., for the Gamma measure $\mathcal{G}_\theta$ in the basic model framework. The obtained diffusions are measure-valued diffusions.

According to [Hoc91] and [Sko97] the theory of measure-valued stochastic processes was initiated by [Daw75]. "Measure-valued processes have been used to describe the dynamics of populations whose underlying distributions are continuously changing, and which are therefore described via a distribution or random measure at each fixed time. They also arise as the diffusion approximation to certain real-valued processes describing spatially-distributed systems. Applications that lead to measure-valued processes in the diffusion limit include models that describe the behavior of systems of branching and diffusing particle [...], models describing frequency distributions of alleles in neutral, non-neutral and interactive populations [...]; and the continuous limit of hierarchically-structured branching and branching diffusion systems [...]."[1] (For details, references and examples we refer to the survey paper [Hoc91].)

Prominent examples for genetic models are the Fleming-Viot processes, which are supported by spaces that consist of probability measures (cf. [EK93, RS95, EK95], where [EK93] is a survey). A (technical) advantage of treating processes on the space of probability measures is that this space can be equipped with the Wasserstein-metric making it Polish (cf. Theorem A.2.8). This is quite useful for showing the quasi-regularity property.

In [Dyn89] the Ornstein-Uhlenbeck, the Fleming-Viot and the Dawson-Watanabe process are considered in a unified view. More recent publications

[1] This is quoted from [Hoc91, P.212].

consider entropic measures and Wasserstein diffusion (cf. [vRS09, AvR10]).

As we pointed out in Chapter 6, to a Dirichlet form, there exists an associated diffusion if and only if the Dirichlet form is quasi-regular and local. Hence, our task is to find conditions for the Dirichlet forms to be quasi-regular (and local) in our setting.

In [BBR06, P.269] it is outlined "that for any semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on a measurable Lusin space E there exists a Lusin topology with the given σ-algebra as the Borel σ-algebra so that $(\mathcal{E}, D(\mathcal{E}))$ becomes quasi-regular. However one has to enlarge E by a zero set." But, this set might not be exceptional, and hence, the associated process might not be located on E. In [BBR08, Theorem 2.2] a condition is presented such that there exists a right process with state space E. Unfortunately, we cannot apply these results because we do not know whether $\mathbb{K}(X)$ is a Lusin space.[2]

To get the quasi-regularity, we use a technique presented in [RS95] (cf. also [MR00] and [KLR06, Section 6]). It uses that the underlying state space is Polish. Hence, in our situation, our first task is to find a proper underlying Polish space.

Having the above remarks in mind, an initial idea is to treat this issue in the spirit of Fleming-Viot processes. In Section 7.1, we embed $\mathbb{K}(X)$ into the well-known Polish space $\mathcal{M}(X)$ of all Radon measures over X (cf. Section A.2). Here, as before (cf. Chapters 6), we consider a connected, orientated C^∞-Riemannian manifold X and a measure $m(dx) = \rho(x)\mathfrak{v}(dx)$ on $(X, \mathcal{B}(X))$ with

$$\rho \in H^{1,2}_{\mathrm{loc}}(X, \mathfrak{v}). \tag{7.0.1}$$

Then we obtain a process describing an extrinsic motion (cf. Theorem 7.1.1), if X is compact.

This is a good start, but those diffusions that we want to construct in the following

- for *extrinsic*, *intrinsic* and *joint motion*
- are located in $\mathbb{K}(\mathbb{R}^d)$ and
- are related to Gibbs perturbations of $\mathcal{G}_\theta$ w.r.t. to a pair potential $0 \le \phi \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ as treated in Section 5.3 (cf. $(\phi)_{\mathbb{K}(\mathbb{R}^d)}$).

[2] A *Lusin space* is the continuous one-to-one image of a Polish space.

And the results in Section 7.1 do not provide this. Hence, we have to analyze our situation more carefully.

A main difficulty to obtain diffusions on $\mathbb{K}(\mathbb{R}^d)$ is to find the correct underlying Polish space for the quasi-regularity property.[3]

For simplicity, let $X = \mathbb{R}^d$, $d \in \mathbb{N}$, and m be the Lebesgue measure dx on $\mathbb{R}^d$. We define a functional $d_{\ddot{\Gamma}_f}$ (cf. Definition 7.2.7) on the *configuration space of multiple configurations* in $\hat{\mathbb{R}}^d$ (cf. (7.2.1)), which is a metric on (cf. Definition 7.2.9)

$$\ddot{\Gamma}_f(\hat{\mathbb{R}}^d) := \left\{\gamma \in \ddot{\Gamma}(\hat{\mathbb{R}}^d) \middle| d_{\ddot{\Gamma}_f}(\gamma, \emptyset) < \infty\right\}.$$

The space $(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), d_{\ddot{\Gamma}_f})$ is Polish (cf. Theorem 7.2.11); and it is the space on which we will work.

Consider the pre-Dirichlet form defined for all $F \in \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), C_0(\hat{\mathbb{R}}^d))$ (cf. (7.2.15)) via (cf. Definition 7.2.18)

$$\begin{aligned}\mathcal{E}^{\mathcal{P}_\theta,\ddot{\Gamma}_f}(F,F) := \int_{\ddot{\Gamma}_f(\hat{\mathbb{R}}^d)} \int_{\hat{\mathbb{R}}^d} &\left(\sqrt{s}\frac{d}{ds}F(\gamma)\right)^2 \\ &+ \left(\frac{1}{\sqrt{s}}\frac{d}{dx}F(\gamma)\right)^2 \gamma(ds,dx)\mathcal{P}_\theta(d\gamma).\end{aligned}$$

Its closure $(\mathcal{E}^{\mathcal{P}_\theta,\ddot{\Gamma}_f}, \mathcal{D}(\mathcal{E}^{\mathcal{P}_\theta,\ddot{\Gamma}_f}))$ is a conservative Dirichlet form (cf. Theorem 7.2.22). As an essential step to obtain an associated diffusion, we prove that this Dirichlet form is quasi-regular (cf. Theorem 7.2.39).

We get a conservative diffusion[4] $\mathbf{M}^{\ddot{\Gamma}_f}$ that is properly associated with $(\mathcal{E}^{\mathcal{P}_\theta,\ddot{\Gamma}_f}, \mathcal{D}(\mathcal{E}^{\mathcal{P}_\theta,\ddot{\Gamma}_f}))$ (cf. Theorem 7.3.7). One drawback is that the process is only constructed on $\ddot{\Gamma}(\hat{\mathbb{R}}^d)$. In Theorem 7.3.12 we prove that it is actually a diffusion on the set $\Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d)$ consisting of all *pinpointing configurations with finite local mass* $d_{\ddot{\Gamma}_f}(\emptyset, \cdot)$ (cf. Definition 7.3.8). For the proof we show that $\ddot{\Gamma}(\hat{\mathbb{R}}^d)\backslash\Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d)$ is $\mathcal{E}^{\mathcal{P}_\theta,\ddot{\Gamma}_f}$-exceptional by extending a technique presented in [RS98].

In Section 7.4 we transfer this result to the cone $\mathbb{K}(\mathbb{R}^d)$ and get a main result of Chapter 7 (cf. Theorem 7.4.4 and Corollary 7.4.5):

[3] Since we do not know whether $\mathbb{K}(X)$ is a Lusin space, we cannot apply the abstract results [BBR06, BBR08].

[4] A diffusion is a strong Markov process with continuous sample paths

- There exist extrinsic, intrinsic and joint diffusions on $\mathbb{K}(\mathbb{R}^d)$, $d \geq 2$, describing the motion of marks and positions.

- In particular, there exists a diffusion describing the motion of the dense set $\tau(\eta_t) \in \mathbb{R}^d$, where $\eta_t \in \mathbb{K}(\mathbb{R}^d)$ for all $t \geq 0$.[5]

The above results extend to more general situations (cf. Theorem 7.4.4):

1. We can equip the Lebesgue measure dx with a density $\rho \in H^{1,2}_{\mathrm{loc}}(\mathbb{R}^d, dx)$ such that $m(dx) = \rho(x)dx$ fulfills (cf. (7.2.12))

$$m\left(\left\{x \in \mathbb{R}^d \;\middle|\; |x| \leq k\right\}\right) \leq MC^k.$$

2. In addition to the first extension, $\mathcal{G}_\theta$ can be replaced by a Gibbs perturbation of $\mathcal{G}_\theta$ w.r.t. some non-negative potential $\phi \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$.

7.1 An extrinsic process moving finite measures

We show the quasi-regularity property of an extrinsic Dirichlet form. Then there exists a corresponding Markov process (cf. Theorem 7.1.1 for the exact properties and for their definitions [MR92, Definitions IV.1.5, IV.1.8 and IV.1.13]).

Let X be compact for this section.

7.1.1 Embedding of $\mathbb{K}(X)$

$\mathbb{K}(X)$ can be embedded into $\mathcal{M}(X)$, which denotes the space of Radon measures. And, more importantly, the gradient and bilinear forms are well-defined, if we replace

$$\mathcal{F}C_b^\infty(\mathbb{K}(X), C_0(X)) \text{ by } \mathcal{F}C_b^\infty(\mathcal{M}(X), C_0(X)), \text{ resp.}$$
$$\mathcal{F}C_b^\infty(\mathbb{K}(X), C_0^\infty(X)) \text{ by } \mathcal{F}C_b^\infty(\mathcal{M}(X), C_0^\infty(X)).$$

For the Dirichlet forms we use that any Gamma measure $\mathcal{G}_\theta$, as well as all other considered measures, has full mass on on the cone $\mathbb{K}(X)$. Hence, the results that we obtained in Chapter 6 are carried over.

[5] The extrinsic motion exists also for $d = 1$.

Extending of well-known objects For each $h \in C_0(X)$ and $\eta \in \mathcal{M}(X)$, we extend the definition of the *translation* given in Subsection 6.1.2: It acts on each Radon measure by equipping it with a density, i.e.,

$$M_h(\eta)(dx) := e^{h(x)}\eta(dx).$$

We get similar to Proposition 6.1.7 for each $F \in \mathcal{F}C_b^\infty(\mathcal{M}(X), C_0^\infty(X))$, $h \in C_0(X)$ and $\eta \in \mathcal{M}(X)$

$$(\nabla^{\mathbb{K}}_{\mathrm{ext},h}F)(\eta) = \sum_{i=1}^{N} \partial_i g_F(\langle \rho_1, \eta\rangle, \ldots, \langle \rho_n, \eta\rangle)\langle h, \rho_i\rangle_{T_\eta^{\mathrm{ext}}(\mathcal{M})},$$

where $T_\eta^{\mathrm{ext}}(\mathcal{M}) := L^2(X, \eta)$.

7.1.2 Quasi-regularity of extrinsic Dirichlet forms

Theorem 7.1.1 (compare also [RS95, Theorem 3.4 or p.31f]). *Let X be compact and μ be a probability measure on $(\mathcal{M}(X), \mathcal{B}(\mathcal{M}(X)))$ having full support on $\mathbb{K}(X)$ and finite first moment, i.e.,*

$$\mathbb{E}_\mu[\langle 1, \cdot\rangle] < \infty.$$

For all $F,\ G \in \mathcal{F}C_b^\infty(\mathcal{M}(X), C_b(X))$ we define the extrinsic bilinear form

$$\mathcal{E}_{ext}^{\mu,\mathcal{M}}(F, G) := \int_{\mathcal{M}(X)} \langle \nabla^{\mathbb{K}}_{ext}F(\eta), \nabla^{\mathbb{K}}_{ext}G(\eta)\rangle_{T_\eta^{\mathrm{ext}}(\mathcal{M})}\mu(d\eta). \tag{7.1.1}$$

We assume that it is closable and that its closure $(\mathcal{E}_{ext}^{\mu,\mathcal{M}}, \mathcal{D}(\mathcal{E}_{ext}^{\mu,\mathcal{M}}))$ is a Dirichlet form. Then this Dirichlet form is quasi-regular and there exists an associated μ-tight special standard process, which is properly associated with $(\mathcal{E}_{ext}^{\mu,\mathcal{M}}, \mathcal{D}(\mathcal{E}_{ext}^{\mu,\mathcal{M}}))$.

Theorem 7.1.2. *Let $m(X) < \infty$ and μ be a Gibbs measure as treated in Section 6.3. Then $(\mathcal{E}_{ext}^{\mu,\mathcal{M}}, \mathcal{D}(\mathcal{E}_{ext}^{\mu,\mathcal{M}}))$ is quasi-regular and there exists an associated μ-tight special standard process, which is properly associated with $(\mathcal{E}_{ext}^{\mu,\mathcal{M}}, \mathcal{D}(\mathcal{E}_{ext}^{\mu,\mathcal{M}}))$.*

Proof. As in Section 6.3, we obtain that the bilinear form is closable and that the closure $(\mathcal{E}_{\mathrm{ext}}^{\mu,\mathcal{M}}, \mathcal{D}(\mathcal{E}_{\mathrm{ext}}^{\mu,\mathcal{M}}))$ is a conservative Dirichlet form because μ has full support on $\mathbb{K}(X)$ (cf. also Theorem 6.3.29). Thus, Theorem 7.1.1 implies the assertion. □

This is our motivation to prove Theorem 7.1.1, for which we use:

Definition 7.1.3. *Let ρ and ρ_1 be two metrics on E. They are* uniformly equivalent *if*

$$\mathrm{id} : (E, \rho_1) \to (E, \rho)$$

and its inverse are uniformly continuous.

Lemma 7.1.4. *Let (E, ρ) be a metric space. If $\varphi \in C_b^\infty(\mathbb{R})$ is a strictly increasing function (i.e. $\varphi' > 0$) with decreasing derivative, $\varphi(0) = 0$ and $|\varphi'| \leq 1$, then*

$$\rho_1 := \varphi \circ \rho$$

is a bounded metric, which is uniformly equivalent with ρ.

Proof. The boundedness, symmetry, positivity and triangle inequality (φ is increasing!) for the metric ρ_1 are clear. It remains to show that $\mathrm{id} : (E, \rho_1) \to (E, \rho)$, as well as its inverse, is uniformly continuous. For the inverse this property is obvious ($\rho_1 \leq |\varphi'| \cdot \rho \leq \rho$). Since $\varphi' \circ \varphi^{-1}\big|_{\mathbb{R}_0^+} \in C(\mathbb{R}_0^+)$ and $\exists \varepsilon_2 > 0 : \varphi'\big(\varphi^{-1}(0)\big) \geq \varepsilon_2 > 0$, there exists $\delta_1 > 0 :$

$$\varphi'\big(\varphi^{-1}(r)\big) \geq \frac{\varepsilon_2}{2} \; \forall r > 0 : |r| < \delta_1.$$

For $\varepsilon > 0$ pick $\delta := \min(\delta_1, \frac{\varepsilon_2}{2}\varepsilon)$. Then for all $x, y \in E$ with $\rho_1(x, y) \leq \delta$

$$\begin{aligned} \rho(x, y) &= \varphi^{-1}(\rho_1(x, y)) \leq \max_{0 \leq r \leq \rho_1(x,y)} \frac{1}{\varphi'\big(\varphi^{-1}(r)\big)} \rho_1(x, y) \\ &\leq \max_{0 \leq r \leq \delta} \frac{1}{\varphi'\big(\varphi^{-1}(r)\big)} \delta \leq \frac{1}{\frac{\varepsilon_2}{2}} \cdot \frac{\varepsilon_2}{2} \cdot \varepsilon \leq \varepsilon. \end{aligned}$$

□

Proof of Theorem 7.1.1. Since X is a complete, separable metric space, by Corollary A.2.10 $\mathcal{M}(X)$ is again Polish. Moreover, by Theorem A.2.11, there exist uniformly continuous functions $(\phi_i)_{i \in \mathbb{N}}$ on X, such that $\|\phi_i\|_\infty \leq 1$, and they are sufficient to get the metric which generates the topology on $\mathcal{M}(X)$:

$$\rho_0(\eta, \nu) = \sup_i \langle \phi_i, \eta - \nu \rangle, \quad \eta, \nu \in \mathcal{M}(X).$$

Similar as in Subsection 6.1.2 (compare Definition 6.1.4 and Proposition 6.1.8) we have for $F(\cdot) = g_F(\langle \rho_1, \cdot \rangle, \dots, \langle \rho_N, \cdot \rangle) \in \mathcal{F}C_b^1(\mathcal{M}(X), C_0(X))$ and $\eta \in \mathcal{M}(X)$

$$(\nabla_{\mathrm{ext}}^{\mathbb{K}} F)(\eta) = \sum_{i=1}^N \partial_i g_F(\langle \rho_1, \eta \rangle, \dots, \langle \rho_N, \eta \rangle) \rho_i \in T_\eta^{\mathrm{ext}}(\mathcal{M}) = L^2(X, \eta).$$

The Dirichlet form $(\mathcal{E}^{\mu,\mathcal{M}}_{\mathrm{ext}}, \mathcal{D}(\mathcal{E}^{\mu,\mathcal{M}}_{\mathrm{ext}}))$ is of the type studied in [RS95, Section 3] with core $\mathcal{F}C_b^\infty(\mathcal{M}(X), C_b(X))$; and for $F, G \in \mathcal{F}C_b^\infty(\mathcal{M}(X), C_b(X))$ we set for all $\eta \in \mathcal{M}(X)$

$$\begin{aligned} S^{\mathcal{M}}_{\mathrm{ext}}(F,G)(\eta) &:= \langle (\nabla^{\mathbb{K}}_{\mathrm{ext}} F)(\eta), (\nabla^{\mathbb{K}}_{\mathrm{ext}} G)(\eta)\rangle_{T^{\mathrm{ext}}_\eta(\mathcal{M})} \\ &= \sum_{i=1}^N \partial_i g_F(\langle \rho_1, \eta\rangle, \dots, \langle \rho_N, \eta\rangle) \sum_{j=1}^{\tilde N} \partial_j g_G(\langle \tilde\rho_1, \eta\rangle, \dots, \langle \tilde\rho_{\tilde N}, \eta\rangle) \\ &\quad \times \langle \rho_i(x)\tilde\rho_j(x), \eta\rangle. \end{aligned}$$

By [RS95, Lemma 3.2], $S^{\mathcal{M}}_{\mathrm{ext}}$ satisfies [RS95, (3.6)]: For any smooth function ϕ on $\mathbb{R}$ with $\phi(0) = 0$ and $|\phi'| \le 1$ we obtain

$$\begin{aligned} |S^{\mathcal{M}}_{\mathrm{ext}}(F, \phi\circ G)(\eta)| &= \Bigg| \int_X \sum_{i=1}^N \partial_i g_F(\langle \rho_1, \eta\rangle, \dots, \langle \rho_N, \eta\rangle)\rho_i(x) \\ &\quad \cdot \sum_{j=1}^{\tilde N} \partial_j (\phi\circ g_G)(\langle \tilde\rho_1, \eta\rangle, \dots, \langle \tilde\rho_{\tilde N}, \eta\rangle)\tilde\rho_j(x)\eta(dx)\Bigg| \\ &= \left|\sum_{i=1}^N \partial_i g_F(\langle \rho_1, \eta\rangle, \dots, \langle \rho_N, \eta\rangle)\right| |\phi'(G(\eta))| \cdot \\ &\quad \cdot \left|\sum_{j=1}^{\tilde N} \partial_j g_G(\langle \tilde\rho_1, \eta\rangle, \dots, \langle \tilde\rho_{\tilde N}, \eta\rangle)\right| \left|\int_X \rho_i(x)\tilde\rho_j(x)\eta(dx)\right| \\ &\le \left|S^{\mathcal{M}}_{\mathrm{ext}}(F,G)(\eta)\right|. \end{aligned}$$

Let φ be as in Lemma 7.1.4, then, by the same lemma, $\rho_1 := \varphi\circ\rho$ is a bounded metric uniformly equivalent with ρ. Since $(\mathcal{M}(X), \rho_0)$ is separable, there exists a countable dense set $(\nu_i)_{i\in\mathbb{N}} \subset \mathcal{M}(X)$. We define

$$f_{ij}(\eta) := \varphi(\langle \phi_j, \eta - \nu_i\rangle)\ \forall \eta \in \mathcal{M}(X).$$

Then we obtain, using that φ is increasing,

$$\sup_j f_{ij}(\eta) \quad = \quad \varphi(\sup_j \langle \phi_j, \eta - \nu_i\rangle) = \varphi(\rho_0(\eta, \nu_i)) = \rho_1(\eta, \nu_i).$$

In order to apply [RS95, Theorem 3.4] it remains to show

$$\sup_{ij} S^{\mathcal{M}}_{\mathrm{ext}}(f_{ij}) \in L^1(\mathcal{M}(X), \mu).$$

We have

$$S_{\text{ext}}^{\mathcal{M}}(f_{ij})(\eta) \;=\; \int_X (\varphi'(\langle \phi_j, \eta - \nu_i\rangle))^2 \phi_j^2(x)\eta(dx) \le \langle 1, \eta\rangle,$$

which, by assumption, is integrable w.r.t. μ. Thus the Dirichlet form $(\mathcal{E}_{\text{ext}}^{\mu,\mathcal{M}}, \mathcal{D}(\mathcal{E}_{\text{ext}}^{\mu,\mathcal{M}}))$ is quasi-regular by [RS95, Theorem 3.4]. By [MR92, Theorem IV.3.5, p.103] we obtain the mentioned associated Markov process. □

7.2 Quasi-regularity on multiple configurations

In this section, we prove that $(\mathcal{E}_{\text{ext}}^{\mathcal{P}_\theta,\ddot{\Gamma}_f}, \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), C_0(\hat{\mathbb{R}}^d)))$ is closable and that its closure is a quasi-regular Dirichlet form (cf. Subsections 7.2.4 and 7.2.6). To that end, we show that $(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), d_{\ddot{\Gamma}_f})$ is a Polish space (cf. Subsection 7.2.2).

For our considerations we need the *configuration space of multiple* locally finite *configurations* in $\hat{X}$

$$\ddot{\Gamma}(\hat{X}) := \left\{ \gamma = \sum_{y\in\gamma} m_y \delta_y \,\middle|\, m_y \in \mathbb{N} \text{ and } \gamma(\Lambda) < \infty,\ \forall\, \Lambda \in \mathcal{B}_c(\hat{X}) \right\}. \quad (7.2.1)$$

An equivalent representation is

$$\ddot{\Gamma}(\hat{X}) = \left\{ \ddot{\gamma} = (n^\gamma, \gamma) \middle| \gamma \in \Gamma(\hat{X}),\ n^\gamma : \gamma \to \mathbb{N} \right\}. \quad (7.2.2)$$

We define $|\ddot{\gamma}_\Lambda| := \sum_{y\in\gamma_\Lambda} n^\gamma(y)$. Here, the function n^γ yields the number of points at each position $y \in \gamma$. $\ddot{\Gamma}(\hat{X}) \subset \mathcal{M}(\hat{X})$ is equipped with the vague topology $\mathcal{O}(\ddot{\Gamma}(\hat{X}))$ inherited from $\mathcal{M}(\hat{X})$. The corresponding Borel σ-algebra is denoted by $\mathcal{B}(\ddot{\Gamma}(\hat{X}))$.[6]

We fix μ to be a probability measure on $(\ddot{\Gamma}(\hat{X}), \mathcal{B}(\ddot{\Gamma}(\hat{X}))$ such that

$$\langle h, \cdot\rangle \in L^1(\ddot{\Gamma}(\hat{X}), \mu) \text{ for all } h \in C_0^\infty(\hat{X}). \quad (7.2.3)$$

7.2.1 The embedding space for compact X

Let X be compact. We define the set of *multiple configurations with finite mass* by

$$\ddot{\Gamma}_f(\hat{X}) := \left\{ \gamma \in \ddot{\Gamma}(\hat{X}) \middle| \langle \gamma, s \otimes \mathbb{1}(x)\rangle < \infty \right\}, \quad (7.2.4)$$

[6] For details, we refer to, e.g., [AKR98a].

where $\ddot{\Gamma}(\hat{X})$ is defined by (7.2.1). We equip $\ddot{\Gamma}_f(\hat{X})$ with the subspace topology $\mathcal{O}(\ddot{\Gamma}_f(\hat{X}))$ of $(\ddot{\Gamma}(\hat{X}), \mathcal{O}(\ddot{\Gamma}(\hat{X})))$ and denote by $\mathcal{B}(\ddot{\Gamma}_f(\hat{X}))$ the corresponding Borel σ-algebra.

Remark 7.2.1. *If we consider the subset of* pinpointing *configurations, then we get $\Gamma_f(X)$, which corresponds to the cone $\mathbb{K}(X)$ (cf. (2.2.20) and also the proof of Theorem 5.3.7).*

We define a metric which yields an alternative (and for the later considerations very useful) description of $\ddot{\Gamma}_f(\hat{X})$:

Definition 7.2.2. *Fix $q \in (0,1)$. Then we choose a sequence $(f_n)_{n\in\mathbb{Z}}$ such that $f_n \in C_0^\infty(\mathbb{R}_+)$,*

$$\mathbb{1}_{[q^n,q^{n-1}]} \leq f_n \leq \mathbb{1}_{[q^{n+1},q^{n-2}]} \quad \text{and} \quad |f_n'| \leq q^{-(n+3)}\mathbb{1}_{[q^{n+1},q^n]\cup[q^{n-1},q^{n-2}]}. \tag{7.2.5}$$

We define the metric

$$d_f(\gamma,\gamma') := \sum_{n=-\infty}^{\infty} |\langle \gamma - \gamma', f_n(s)s \otimes \mathbb{1}(x)\rangle| \qquad \forall \gamma,\ \gamma' \in \ddot{\Gamma}(\hat{X}). \tag{7.2.6}$$

Remark 7.2.3. *Note that $(\ddot{\Gamma}_f(\hat{X}), d_f)$ is a metric space: The triangle inequality is clear and the rest follows by the following equivalence*

$$d_f(\gamma,\varnothing) < \infty \quad \Leftrightarrow \quad \langle \gamma, \mathbb{1}(x)s\rangle < \infty,$$

which holds for each $\gamma \in \ddot{\Gamma}(\hat{X})$ because

$$\langle \gamma, \mathbb{1}(x)s\rangle \leq \sum_{n=-\infty}^{\infty} \langle \gamma, \mathbb{1}(x)f_n(s)s\rangle \leq \sum_{n=-\infty}^{\infty} \langle \gamma, \mathbb{1}(x)\mathbb{1}_{[q^{n+1},q^{n-2}]}s\rangle \leq \langle \gamma, \mathbb{1}(x)3s\rangle.$$

Definition 7.2.4. *Let d_{MR} denote the metric defined in [MR00, (3.12)]. On $\ddot{\Gamma}_f(\hat{X})$, we introduce the metric*

$$d_{\ddot{\Gamma}_f}(\gamma,\gamma') := d_{MR}(\gamma,\gamma') + d_f(\gamma,\gamma'), \qquad \forall \gamma,\ \gamma' \in \ddot{\Gamma}_f(\hat{X}).$$

Remark 7.2.5. *Let us summarize a few properties of d_{MR} (cf. [MR00, Subsection 3.2, esp. (3.12), and (3.3)]):*

$$d_{MR}(\gamma,\gamma') := \sup_{k\in\mathbb{N}} c_k \xi\left(\overline{d_{\hat{X}}}\left(\phi_k\cdot\gamma, \phi_k\cdot\gamma'\right)\right), \tag{7.2.7}$$

where $0 < c_k \searrow 0$, $\xi \in C_0^\infty(\mathbb{R})$ *with* $0 \le \xi \le 1$ *on* $[0,\infty]$, $\xi(t) = t$ *on* $[-1/2, 1/2]$, $\xi' > 0$ *and* $\xi'' \le 0$, $\phi_k := (1+\delta_k)/\delta_k \; g_{E_k,\delta_k}$,

$$\overline{d_{\hat{X}}}(\gamma,\gamma') := \sup\left\{ |\overline{g}_{A,\varepsilon}(\gamma) - \overline{g}_{A,\varepsilon}(\gamma')\big| A \in B(\hat{X}), \varepsilon > 0 \right\},$$
$$g_{A,\varepsilon}(\hat{x}) := \frac{1}{1+\varepsilon}\left(\varepsilon - d_{\hat{X}}(\hat{x}, A) \wedge \varepsilon\right) \qquad \text{and} \qquad \overline{g}_{A,\varepsilon}(\gamma) = \int_{\hat{X}} g_{A,\varepsilon}(\hat{x})\gamma(d\hat{x}).$$

Here, $d_{\hat{X}}$ *denotes the metric on* $\hat{X}$ *(cf. (2.2.6)) and* $(E_k)_{k\in\mathbb{N}}$*, consisting of bounded sets, is a* well-exhausting sequence *of* $\hat{X}$ *with* $(\delta_k)_{k\in\mathbb{N}}$ *being the corresponding sequence of strictly positive numbers.*

This means that

$$\bigcup_{k\in\mathbb{I}} E_k = \hat{X} \qquad \text{and} \qquad E_k^{\delta_k} \subset E_{k+1}, \quad \forall k \in \mathbb{I},$$

where we define $A^\varepsilon := \{\hat{x} \in \hat{X} \big| d_{\hat{X}}(x, A) < \varepsilon\}$ *for all* $A \in \mathcal{B}(\hat{X})$ *and* $\varepsilon > 0$ *and* $(\delta_k)_{k\in\mathbb{I}}$ *is a sequence of strictly positive numbers.*

Theorem 7.2.6. *If* (X, d_X) *is a compact and complete separable metric space, then* $(\ddot{\Gamma}_f(\hat{X}), d_{\ddot{\Gamma}_f})$ *is a complete and separable metric space.*

Proof. Let $(\gamma_k)_{k\in\mathbb{N}}$ be a $d_{\ddot{\Gamma}_f}$-Cauchy sequence. Then it is a d_{MR}- and a d_f-Cauchy sequence. We define for $n \in \mathbb{Z}$

$$\gamma^{(n)} := \langle \gamma, \mathbb{1}(x) f_n(s) s \rangle \qquad \forall \gamma \in \ddot{\Gamma}(\hat{X})$$

and get the following embedding:

$$\begin{array}{ccccc} \ddot{\Gamma}(\hat{X}) & \hookleftarrow & \ddot{\Gamma}_f(\hat{X}) & \hookrightarrow & l^1 \\ \gamma_k & \leftarrow\!\shortmid & \gamma_k & \mapsto & \left(\gamma_k^{(n)}\right)_{n=-\infty}^{\infty} \end{array}.$$

Since $(\gamma_k)_{k\in\mathbb{N}}$ is a d_{MR}-Cauchy sequence in the complete space $(\ddot{\Gamma}(\hat{X}), d_{\mathrm{MR}})$, we deduce by [MR00, Proposition 3.9] that it converges vaguely to a multiple configuration $\gamma_0 \in \ddot{\Gamma}(\hat{X})$. By the vague convergence, $\gamma^{(n)}$ converges pointwisely to $\gamma_0^{(n)}$. Since $\left(\gamma_k^{(n)}\right)_{n=-\infty}^{\infty}$, $k \to \infty$, is a d_f-Cauchy sequence in the Banach space l^1, it converges to an element $a = \left(a^{(n)}\right)_{n=-\infty}^{\infty} \in l^1$. The l^1 convergence implies pointwise convergence, thus

$$\gamma_0^{(n)} = a^{(n)} \quad \forall n \in \mathbb{Z}$$

because the pointwise limits have to coincide. Hence, $\gamma_0 \in \ddot{\Gamma}_f(\hat{X})$ because $\gamma_0 \in \ddot{\Gamma}(\hat{X})$ and $a \in l^1$, i.e.,

$$d_f(\gamma_0, \varnothing) = \sum_{n=-\infty}^{\infty} \langle \gamma_0, \mathbb{1}(x) f_n(s) s \rangle = \sum_{n=-\infty}^{\infty} a^{(n)} < \infty.$$

Therefore, $(\ddot{\Gamma}_f(\hat{X}), d_{\ddot{\Gamma}_f})$ is a complete metric space.

It remains to prove the separability: To that end, let $\mathbb{S}_{\ddot{\Gamma}(\hat{X})}$ be a countable dense set in $\big(\ddot{\Gamma}(\hat{X}), d_{\mathrm{MR}}\big)$. We define

$$\mathbb{S}_{\ddot{\Gamma}_f(\hat{X})} := \left\{ \gamma_K := \left\{ (s,x) \in \gamma \big| q^K \leq s \leq q^{-K},\ x \in X \right\} \Big| \gamma \in \mathbb{S}_{\ddot{\Gamma}(\hat{X})}, K \in \mathbb{N} \right\}.$$

This is a countable and dense subset of $\big(\ddot{\Gamma}_f(\hat{X}), d_{\ddot{\Gamma}_f}\big)$. Indeed, let $\varepsilon > 0$ and $\gamma \in \ddot{\Gamma}_f(\hat{X})$. Since there exists $K := K_{MR}$ such that $c_k < \varepsilon$ and $\xi \leq 1$ for all $k > K$, we get (cf. (7.2.7))

$$d_{MR}(\gamma, \gamma_k) < \varepsilon \quad \forall k > K_{MR}. \tag{7.2.8}$$

Moreover, since $d_f(\gamma, \varnothing) < \infty$,

$$\exists K \geq K_{MR} :\ \forall k \geq K \qquad \sum_{\substack{|n|>k-2\\ n\in\mathbb{Z}}} |\langle \gamma, f_n(s) s \mathbb{1}_X(x) \rangle| < \varepsilon. \tag{7.2.9}$$

Since $\mathbb{S}_{\ddot{\Gamma}(\hat{X})} \subset \ddot{\Gamma}(\hat{X}) \ni \gamma_K$ dense, there exists $\gamma' \in \mathbb{S}_{\ddot{\Gamma}(\hat{X})}$:

$$\begin{aligned} &d_{MR}(\gamma_K, \gamma') < \varepsilon \quad \text{and} \\ &|\langle \gamma_K - \gamma', \mathbb{1}_X f_n(s) s \rangle| < \frac{\varepsilon}{2^{|n|+2}} \quad \forall n \in \mathbb{Z} :\ |n| \leq K+4. \end{aligned} \tag{7.2.10}$$

We pick γ'_{K+2}, then

$$d_{MR}(\gamma_K, \gamma'_{K+2}) \leq d_{MR}(\gamma_k, \gamma') < \varepsilon \tag{7.2.11}$$

because we have less points outside of $[q^K, q^{-K}] \times X$, in which lie all the points of γ_K. Furthermore,

$$\big|\langle \gamma_K - \gamma'_{K+2}, \mathbb{1}_X f_n(s) s \rangle\big| < \frac{\varepsilon}{2^{|n|+2}} \quad \forall n \in \mathbb{Z}$$

because $\langle f_{K+1}(s)s, \gamma_K\rangle = 0$ and $\langle f_{K+m}(s)s, \gamma'_{K+2}\rangle = 0$ for all $m \geq 3$. Thus,

$$d_{\ddot{\Gamma}_f}\left(\gamma, \gamma'_{K+2}\right) \leq \underbrace{d_{MR}\left(\gamma, \gamma_K\right)}_{\leq \varepsilon \text{ by } (7.2.8)} + \underbrace{d_{MR}\left(\gamma_K, \gamma'_{K+2}\right)}_{\leq \varepsilon \text{ by } (7.2.11)} + d_f(\gamma, \gamma_K) + d_f(\gamma_K, \gamma'_{K+2})$$

$$\leq 2\varepsilon + \sum_{|n|\leq K-2} \underbrace{|\langle \gamma - \gamma_K, f_n(s)s\mathbb{1}_X(x)\rangle|}_{=0 \quad \forall n \leq K-2} + \sum_{|n|\geq K-2} \left|\langle \underbrace{\gamma_K}_{\subset\gamma}, f_n(s)s\mathbb{1}_X(x)\rangle\right|$$

$$+ \underbrace{\sum_{|n|\geq K-2} |\langle \gamma, f_n(s)s\mathbb{1}_X(x)\rangle|}_{\leq \varepsilon \text{ by } (7.2.9)} + \sum_{|n|\leq K+3} \underbrace{\left|\langle \gamma - \gamma'_{K+2}, f_n(s)s\mathbb{1}_X(x)\rangle\right|}_{\leq\ \varepsilon\ 2^{-|n|-2}}$$

$$+ \sum_{|n|\geq K+3} \underbrace{\left|\langle \gamma - \gamma'_{K+2}, f_n(s)s\mathbb{1}_X(x)\rangle\right|}_{=0} \leq 2\varepsilon + 2\varepsilon + \frac{\varepsilon}{4}(1 + 2^0 + 1) < 5\varepsilon,$$

which shows the separability. □

7.2.2 Identifying a proper Polish space for $X = \mathbb{R}^d$

Let $X = \mathbb{R}^d$, $d \in \mathbb{N}$. We restrict the assumptions on m to get the corresponding assertions of Theorem 7.2.23 (and 7.2.29 and 7.2.31).

Basically, we assume that the mass does not "grow too fast". Then, using a well-exhausting sequence $\{X_k\}_{k\in\mathbb{N}}$ of $\mathbb{R}^d$, we replace $\mathbb{1}_X$ by an infinite sum, as we did replace $\mathbb{1}_{\mathbb{R}_+}s$ by $\sum_n sf_n(s)$.

Definition 7.2.7. *If $m(\mathbb{R}^d) < \infty$, then we choose $\mathbb{I} = \{1\}$, $X_1 = \mathbb{R}^d$ and $\phi_1 = \mathbb{1}_X$. Otherwise, let $\delta = 1/2$ and define for $k \in \mathbb{I} := \mathbb{N}$*

$$X_k := \{x \in \mathbb{R}^d |\ |x| \leq k\}$$

We assume that

$$\exists 1 \leq M, C < \infty : \qquad m(X_k) \leq MC^k, \qquad \forall k \in \mathbb{I}. \tag{7.2.12}$$

Fix $(\phi_k)_{k\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ such that

$$\begin{array}{ll} \mathbb{1}_{X_k\setminus X_{k-1}} \leq \phi_k \leq \mathbb{1}_{X_k^{1/4}\setminus X_{k-1}^{-1/4}} & \text{and} \\ |\partial_i\phi_k| \leq 4\mathbb{1}_{X_k^{1/4}\setminus X_{k-1}^{-1/4}} & \forall 1 \leq i \leq d \end{array}, \tag{7.2.13}$$

where for all $A \in \mathcal{B}(\mathbb{R}^d)$ and $\varepsilon > 0$, we set

$$A^\varepsilon := \left\{x \in X \middle| d_X(x, A) < \varepsilon\right\} \text{ and } \qquad A^{-\varepsilon} := \left\{x \in X \middle| d_X(x, A^c) > \varepsilon\right\}.$$

We define for all $\gamma, \gamma' \in \ddot{\Gamma}(\hat{\mathbb{R}}^d)$

$$d_f(\gamma, \gamma') := 2\sum_{n=-\infty}^{\infty}\sum_{k\in\mathbb{I}} \frac{1}{M(2C)^k} |\langle \phi_k \cdot f_n(s)s, \gamma - \gamma'\rangle| \in [0, \infty].$$

Remark 7.2.8. *Definition 7.2.7 is consistent with Definition 7.2.2. If* $\mathbb{R}^d$ *is equipped with the Lebesgue measure, i.e.* $\rho \equiv 1$*, then (7.2.12) is fulfilled.*

Definition 7.2.9. *We define*

$$d_{\ddot{\Gamma}_f}(\gamma, \gamma') := d_f(\gamma, \gamma') + d_{\mathrm{MR}}(\gamma, \gamma'), \quad \forall \gamma, \gamma' \in \ddot{\Gamma}(\hat{\mathbb{R}}^d),$$

and the set of multiple configurations with finite mass $d_{\ddot{\Gamma}_f}$ *by*

$$\ddot{\Gamma}_f(\hat{\mathbb{R}}^d) := \left\{ \gamma \in \ddot{\Gamma}(\hat{\mathbb{R}}^d) \middle| d_{\ddot{\Gamma}_f}(\gamma, \emptyset) < \infty \right\}, \tag{7.2.14}$$

where $\ddot{\Gamma}(\hat{\mathbb{R}}^d)$ *is defined by (7.2.1). We equip* $\ddot{\Gamma}_f(\hat{X})$ *with the subspace topology* $\mathcal{O}(\ddot{\Gamma}_f(\hat{X}))$ *of* $(\ddot{\Gamma}(\hat{X}), \mathcal{O}(\ddot{\Gamma}(\hat{X})))$ *and denote by* $\mathcal{B}(\ddot{\Gamma}_f(\hat{X}))$ *the corresponding Borel* σ*-algebra.*

Remark 7.2.10. *Note that* $d_{\ddot{\Gamma}_f}$ *is a metric on* $\ddot{\Gamma}_f(\hat{\mathbb{R}}^d)$*.*

Theorem 7.2.11. *Let the conditions stated in Definition 7.2.7 hold. Then* $\big(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), d_{\ddot{\Gamma}_f}\big)$ *is a complete, separable metric space.*

Proof. Making the obvious appropriate adaption of the proof of Theorem 7.2.6 yields this result. □

7.2.3 Bilinear forms on $\ddot{\Gamma}(\hat{X})$

We outline the bilinear forms that correspond to the extrinsic and intrinsic ones defined in Chapters 6.[7]

Let μ be a probability measure on $(\ddot{\Gamma}(\hat{X}), \mathcal{B}(\ddot{\Gamma}(\hat{X}))$ such that $\mu(\ddot{\Gamma}_f(\hat{X})) = 1$ and (7.2.3) holds.

Smooth cylinder functions on $\ddot{\Gamma}_f(\hat{X})$ We will also use some sets of *smooth cylinder functions*. For all $k, l \in \mathbb{N}_0 \cup \{\infty\}$, we write

$$\mathcal{F}C_b^k(\ddot{\Gamma}_f(\hat{X}), C_0^l(Y)).$$

By definition, $\mathcal{F}C_b^k(\Gamma, C_0^l(Y))$ consists of all functions F which can be represented as

$$\Gamma \ni \gamma \mapsto F(\gamma) = g_F(\langle \varphi_1, \gamma \rangle, \dots, \langle \varphi_N, \gamma \rangle), \tag{7.2.15}$$

with some $N \in \mathbb{N}$, $g_F \in C_b^k(\mathbb{R}^N)$ and $\varphi_i \in C_0^l(Y)$, $1 \le i \le N$.

[7]For this subsection we do not assume anymore that X is compact.

Extrinsic bilinear forms

Definition 7.2.12. *For all* $F,\ G \in \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{X}), C_0^\infty(\hat{X}))$ *we define the* extrinsic bilinear form

$$\mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}(F) := \mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}(F,F) := \int_{\ddot{\Gamma}_f(\hat{X})} \int_{\hat{X}} \left(\sqrt{s}\frac{d}{ds}F(\gamma)\right)^2 \gamma(ds,dx)\mu(d\gamma).$$

Remark 7.2.13. *Fix* $F,\ G \in \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{X}), C_0^\infty(\hat{X}))$*. Note that* $\mathcal{E}_{int}^{\mu}(F,G) < \infty$*. We denote*

$$S_{ext}(\varphi_i, \tilde{\varphi}_j)(\hat{x}) := s\frac{d}{ds}\varphi_i(\hat{x})\frac{d}{ds}\tilde{\varphi}_j(\hat{x})$$

and the extrinsic bilinear square field operator

$$\begin{aligned} S_{ext}^{\ddot{\Gamma}}(F,G)(\gamma) &:= \sum_{i=1}^{N} \partial_i F \sum_{j=1}^{\tilde{N}} \partial_j G \langle \gamma, S_{ext}(\varphi_i, \tilde{\varphi}_j)(\hat{x})\rangle \\ &= \int_{\hat{X}} \sum_{i=1}^{N}\sum_{j=1}^{\tilde{N}} \partial_i F\ \partial_j G\ S_{ext}(\varphi_i, \tilde{\varphi}_j)(\hat{x})\gamma(d\hat{x}), \end{aligned}$$

where $\partial_i F := \partial_i g_F(\langle\gamma,\varphi_1\rangle, \ldots, \langle\gamma,\varphi_N\rangle)$, $\partial_j G := \partial_j g_G(\langle\gamma,\tilde{\varphi}_1\rangle, \ldots, \langle\gamma,\tilde{\varphi}_{\tilde{N}}\rangle)$*. Then we can rewrite the bilinear form*

$$\mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}(F,G) = \int_{\ddot{\Gamma}_f(\hat{X})} S_{ext}^{\ddot{\Gamma}}(F,G)(\gamma)\mu(d\gamma).$$

Remark 7.2.14. *Heuristically, we get for* $F \in \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{X}), \{\mathrm{id}_{\mathbb{R}_+} \otimes \tilde{\phi} \mid \tilde{\phi} \in C_0^\infty(X)\})$ *that*

$$S_{ext}(\mathrm{id}_{\mathbb{R}_+} \otimes \tilde{\phi}_i, \mathrm{id}_{\mathbb{R}_+} \otimes \tilde{\phi}_j)(\hat{x}) = s\tilde{\phi}_i(x)\tilde{\phi}_j(x).$$

Intrinsic bilinear form

Definition 7.2.15. *For all* $F \in \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{X}), C_0^\infty(\hat{X}))$ *we define the* intrinsic bilinear form

$$\mathcal{E}_{int}^{\mathcal{G}_\theta,\ddot{\Gamma}_f}(F) := \mathcal{E}_{int}^{\mathcal{G}_\theta,\ddot{\Gamma}_f}(F,F) := \int_{\ddot{\Gamma}_f(\hat{X})} \int_{\hat{X}} \left(\frac{1}{\sqrt{s}}\frac{d}{dx}F(\gamma)\right)^2 \gamma(ds,dx)\mu(d\gamma).$$

Remark 7.2.16. *Fix* $F,\ G \in \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{X}), C_0^\infty(\hat{X}))$*, we have* $\mathcal{E}_{int}^{\mu}(F,G) < \infty$ *and*

$$\mathcal{E}_{int}^{\mu,\ddot{\Gamma}_f}(F,G) = \int_{\ddot{\Gamma}_f(\hat{X})} S_{int}^{\ddot{\Gamma}}(F,G)(\gamma)\mu(d\gamma).$$

Here, the square field operator is

$$\begin{aligned} S_{int}^{\ddot{\Gamma}}(F,G)(\gamma) :=& \sum_{i=1}^{N} \partial_i F \sum_{j=1}^{\tilde{N}} \partial_j G \langle \gamma, S_{int}(\varphi_i, \tilde{\varphi}_j)(\hat{x})\rangle \\ =& \int_{\hat{X}} \sum_{i=1}^{N} \sum_{j=1}^{\tilde{N}} \partial_i F\ \partial_j G\ S_{int}(\varphi_i, \tilde{\varphi}_j)(\hat{x})\gamma(d\hat{x}), \end{aligned}$$

where $\partial_i F := \partial_i g_F(\langle \gamma, \varphi_1\rangle, \ldots, \langle \gamma, \varphi_N\rangle)$, $\partial_j G := \partial_j g_G(\langle \gamma, \tilde{\varphi}_1\rangle, \ldots, \langle \gamma, \tilde{\varphi}_{\tilde{N}}\rangle)$ *and*

$$S_{int}(\varphi_i, \tilde{\varphi}_j)(\hat{x}) := \frac{1}{s}\langle \nabla^X \varphi_i(\hat{x}) \nabla^X \tilde{\varphi}_j(\hat{x})\rangle_{T_x(X)}.$$

Remark 7.2.17. *Heuristically, we get for* $F \in \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{X}), \{\mathrm{id}_{\mathbb{R}_+} \otimes \tilde{\phi} \mid \tilde{\phi} \in C_0^\infty(X)\})$ *that*

$$S_{int}(\mathrm{id}_{\mathbb{R}_+} \otimes \tilde{\phi}_i, \mathrm{id}_{\mathbb{R}_+} \otimes \tilde{\phi}_j)(\hat{x}) = s\langle \nabla^X \tilde{\phi}_i(x), \nabla^X \tilde{\phi}_j(x)\rangle_{T_x(X)}.$$

Joint Bilinear forms

Definition 7.2.18. *For all* $F,\ G \in \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{X}), C_0^\infty(\hat{X}))$ *we define the* joint bilinear form

$$\mathcal{E}^{\mu,\ddot{\Gamma}_f}(F,G) := \mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}(F,G) + \mathcal{E}_{int}^{\mu,\ddot{\Gamma}_f}(F,G).$$

Remark 7.2.19. *Note that* $\mathcal{E}^{\mu,\ddot{\Gamma}_f}(F,G) < \infty$ *and*

$$\mathcal{E}^{\mu,\ddot{\Gamma}_f}(F,G) = \int_{\ddot{\Gamma}_f} S^{\ddot{\Gamma}}(F,G)(\gamma)\mu(d\gamma),$$

where the square field operator is

$$\begin{aligned} S^{\ddot{\Gamma}}(F,G)(\gamma) :=& S_{ext}^{\ddot{\Gamma}}(F,G)(\gamma) + S_{int}^{\ddot{\Gamma}}(F,G)(\gamma) \\ =& \int_{\hat{X}} \sum_{i=1}^{N} \sum_{j=1}^{\tilde{N}} \partial_i F\ \partial_j G\ S(\varphi_i, \tilde{\varphi}_j)(\hat{x})\gamma(d\hat{x}), \end{aligned}$$

and

$$\begin{aligned} S(\varphi_i, \tilde{\varphi}_j)(\hat{x}) :=& S_{ext}(\varphi_i, \tilde{\varphi}_j)(\hat{x}) + S_{int}(\varphi_i, \tilde{\varphi}_j)(\hat{x}) \\ =& s\varphi_i(\hat{x})\tilde{\varphi}_j(\hat{x}) + \frac{1}{s}\langle \nabla^X \varphi_i(\hat{x}), \nabla^X \tilde{\varphi}_j(\hat{x})\rangle_{T_x(X)}. \end{aligned}$$

7.2.4 Closability of the bilinear forms

We identify some Gibbs measures for which the corresponding bilinear forms are closable.

Let $X = \mathbb{R}^d$, $d \in \mathbb{N}$, m on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and $(\phi_k)_{k\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ such that (7.0.1), (7.2.12) and (7.2.13) hold. Assume that μ is a Gibbs perturbation of $\mathcal{P}_\theta$ w.r.t. a pair potential

$$V(\hat{x}, \hat{y}) = s_x \phi(x, y) s_y,$$

where $\phi \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ such that $(\phi)_{\mathbb{K}(\mathbb{R}^d)}$ holds (cf. Subsection 5.3.3).

Extrinsic bilinear form

Theorem 7.2.20. *The extrinsic bilinear form* $(\mathcal{E}_{ext}^{\ddot{\Gamma}_f,\mu}, \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), C_0(\hat{\mathbb{R}}^d)))$ *is closable and the* extrinsic generator *is for all* $\gamma \in \Gamma_f(\hat{\mathbb{R}}^d)$

$$(L_{ext}^{\ddot{\Gamma}_f,\mu} F)(\gamma) := \int_{\hat{\mathbb{R}}^d} s\frac{d^2}{ds^2}F(\gamma) - s\frac{d}{ds}F(\gamma)\left(1 + 2\sum_{\hat{y}\in\gamma} \phi(x,y)s_y\right)\gamma(d\hat{x}). \tag{7.2.16}$$

We have

$$L_{ext}^{\ddot{\Gamma}_f,\mu} F \in \bigcap_{p\in\mathbb{N}} L^p(\ddot{\Gamma}_f, \mu). \tag{7.2.17}$$

The closure of $(\mathcal{E}_{ext}^{\ddot{\Gamma}_f,\mu}, \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), C_0(\hat{\mathbb{R}}^d)))$ *is denoted by* $(\mathcal{E}_{ext}^{\ddot{\Gamma}_f,\mu}, \mathcal{D}(\mathcal{E}_{ext}^{\ddot{\Gamma}_f,\mu}))$. *It is a conservative Dirichlet form.*

Proof. The idea to use the integration by parts formula of the Lebesgue measure on $\mathbb{R}_+$ to prove the assertion is from [AKR98b, Theorem 4.3].

Fix $F = g_F(\langle\phi_1,\gamma\rangle, \ldots, \langle\phi_N,\gamma\rangle) \in \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), C_0(\hat{\mathbb{R}}^d))$ and $\Lambda' \in B_c(\hat{\mathbb{R}}^d)$ such that $\varphi_1, \ldots, \varphi_N$ are supported by Λ'. Let $\mathcal{U}_{\Lambda'} \in \mathcal{B}_c(\hat{\mathbb{R}}^d)$:

$$\phi(\hat{x}, \hat{y}) = 0, \quad \forall \hat{x} \in \Lambda, \hat{y} \in \mathcal{U}_{\Lambda'}^c.$$

The integrability (cf. (7.2.17)) follows because there exists $C > 0$ such that

$$(L_{\text{ext}}^{\ddot{\Gamma}_f,\mu} F)(\gamma) \le 2C\langle s\mathbb{1}_{\Lambda'}, \gamma\rangle\big(1 + C\langle \mathbb{1}_{\mathcal{U}_{\Lambda'}} s, \gamma\rangle\big) \quad \in \bigcap_{p\in\mathbb{N}} L^p(\ddot{\Gamma}_f, \mu),$$

where we used Theorem 5.3.10. Fix $\Lambda \in \mathcal{B}_c(\hat{X})$ with $\mathcal{U}_{\Lambda'} \subset \Lambda$. By (**DLR**),

$$\int_{\ddot{\Gamma}_f(\hat{X})} S_{\text{ext}}^{\ddot{\Gamma}_f}(F)(\gamma)\mu(d\gamma) = \int_\Gamma \int_{\Gamma(\Lambda)} S_{\text{ext}}^{\ddot{\Gamma}_f}(F)(\gamma)\pi_\Lambda(d\gamma|\xi)\mu(\xi) \tag{7.2.18}$$

Since $|\gamma_\Lambda| < \infty$ for all $\gamma \in \Gamma(\hat{\mathbb{R}}^d)$, the inner integral equals μ-a.e.

$$\frac{1}{Z_\Lambda(\xi)} \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\Lambda^N} S_{\text{ext}}^{\ddot{\Gamma}_f}(F)(\sum_{i=1}^{N} \delta_{\hat{x}_i})$$
$$\times \exp\Big(- \sum_{i,j=1}^{N} V(\hat{x}_i, \hat{x}_j) - 2 \sum_{\substack{1 \le i \le N \\ \hat{y} \in \xi_{\Lambda^c}}} V(\hat{x}_i, \hat{y})\Big) \prod_{i=1}^{N} \lambda_\theta \otimes m(d\hat{x}_i). \qquad (7.2.19)$$

By the choice of Λ, the second summand in the exponent vanishes. We write $F(\sum_{i=1}^{N} \delta_{\hat{x}_i})$ as $\tilde{F}(s_1, x_1, \ldots, s_N, x_N)$, define $h(s) := s$, $s \in \mathbb{R}_+$, and use

$$S_{\text{ext}}^{\ddot{\Gamma}_f}(\tilde{F})(s_1, x_1, \ldots, s_N, x_N) = \sum_{i=1}^{N} \Big(\partial_{s_i} \tilde{F}(s_1, x_1, \ldots, s_N, x_N)\Big)^2 h(s_i). \quad (7.2.20)$$

Then the last integral equals

$$\sum_{i=1}^{N} \int_{\Lambda^{N-1}} \int_{\Lambda_X} \int_{\Lambda_{\mathbb{R}_+}} \left(\partial_{s_i} F(s_1, x_1, \ldots, s_N, x_N)\right)^2 h(s_i)$$
$$\times \exp\Big(- \sum_{i,j=1}^{N} V(\hat{x}_i, \hat{x}_j)\Big) \lambda_\theta(ds_i) m(dx_i) \prod_{\substack{1 \le j \le N, \\ i \ne j}} \lambda_\theta \otimes m(d\hat{x}_i). \qquad (7.2.21)$$

Using the integration by parts formula for the Lebesgue measure and the symmetry of the potential V yields that the inner integral equals

$$- \int_{\Lambda_{\mathbb{R}_+}} \tilde{F}(s_1, x_1, \ldots, s_N, x_N) \Bigg(\frac{d^2}{d{s_i}^2} F(s_1, x_1, \ldots, s_n, x_n)$$
$$+ \frac{d}{ds_i} F(s_1, x_1, \ldots, s_n, x_n) \Bigg(\frac{h'(s_i)}{h(s_i)} - 2 \sum_{\substack{1 \le j \le N \\ j \ne i}} \frac{d}{ds_i} V(s_i, x_i, s_j, x_j)$$
$$- \frac{d}{ds_i} V(s_i, x_i, s_i, x_i) + \frac{{\lambda_\theta}'(s_i) \rho(x_i)}{\lambda_\theta(s_i) \rho(x_i)} \Bigg) \Bigg) h(s_i) \lambda_\theta(s_i) \rho(x_i)$$
$$\times \exp\Big(- \sum_{i,j=1}^{N} V(\hat{x}_i, \hat{x}_j)\Big) ds_i + \underbrace{\int_{\partial \Lambda_{\mathbb{R}_+}} \ldots ds_i}_{=0}$$
$$=: \int_{\Lambda_{\mathbb{R}_+}} \tilde{F}(s_1, x_1, \ldots, s_N, x_N) (L_{\text{ext}}^{\ddot{\Gamma}_f, \mu} F)(\gamma, \hat{x}_i) ds_i. \qquad (7.2.22)$$

Using the additivity of $(L^{\ddot{\Gamma}_f,\mu}_{\mathrm{ext}} F)(\gamma, \hat{x}_i)$ and

$$\frac{h'(s_i)}{h(s_i)} + \frac{\lambda_\theta{}'(s_i)\rho(x_i)}{\lambda_\theta(s_i)\rho(x_i)} = \frac{1}{s} - (1 + \frac{1}{s}) = 1,$$

we get, plugging (7.2.22), (7.2.21), resp. (7.2.19) back in,

$$\int_{\ddot{\Gamma}_f(\hat{X})} S^{\ddot{\Gamma}_f}_{\mathrm{ext}}(\tilde{F})(\gamma)\mu(d\gamma) = \int_{\ddot{\Gamma}_f(\hat{X})} F(\gamma)(L^{\ddot{\Gamma}_f,\mu}_{\mathrm{ext}} F)(\gamma)\mu(d\gamma). \tag{7.2.23}$$

Obviously, $(\mathcal{E}^{\ddot{\Gamma}_f,\mu}_{\mathrm{ext}}, \mathcal{F}C^\infty_b(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), C_0(\hat{\mathbb{R}}^d)))$ is positive definite and symmetric. Hence, by [MR92, Proposition I.3.3.], it is closable. By arguments similar to the proof of Theorem 6.3.14, we see that its closure is a conservative Dirichlet form. □

Intrinsic bilinear form

Theorem 7.2.21. *Let $\phi \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ such that $(\phi)_{\mathbb{K}(\mathbb{R}^d)}$ holds. Then the intrinsic bilinear form $(\mathcal{E}^{\ddot{\Gamma}_f,\mu}_{int}, \mathcal{F}C^\infty_b(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), C_0(\hat{\mathbb{R}}^d)))$ is closable and the* intrinsic generator *is for all $\gamma \in \Gamma_f(\hat{\mathbb{R}}^d)$*

$$(L^{\ddot{\Gamma}_f,\mu}_{int} F)(\gamma) := \sum_{k=1}^d \int_{\hat{\mathbb{R}}^d} \frac{1}{s}\left(\frac{d}{dx^{(k)}}\right)^2 F(\gamma) + \frac{d}{dx^{(k)}} F(\gamma)$$
$$\times \left(\frac{1}{s}\frac{1}{\rho(x)}\left(\frac{d}{dx^{(k)}}\rho(x)\right) - 2\sum_{\hat{y}\in\gamma} s_y \frac{d}{dx^{(k)}}\phi(x,y)\right)\gamma(d\hat{x}), \tag{7.2.24}$$

where $x = \left(x^{(k)}\right)_{k=1}^d$. We have

$$L^{\ddot{\Gamma}_f,\mu}_{int} F \in \bigcap_{p\in\mathbb{N}} L^p(\ddot{\Gamma}_f, \mu). \tag{7.2.25}$$

The closure of $(\mathcal{E}^{\ddot{\Gamma}_f,\mu}_{int}, \mathcal{F}C^\infty_b(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), C_0(\hat{\mathbb{R}}^d)))$ is denoted by $(\mathcal{E}^{\ddot{\Gamma}_f,\mu}_{int}, \mathcal{D}(\mathcal{E}^{\ddot{\Gamma}_f,\mu}_{int}))$. It is a conservative Dirichlet form.

Proof. Adapting the proof of Theorem 7.2.21 appropriately, we get this result. In detail, we choose $h(s) = \frac{1}{s}$ and the equation corresponding to (7.2.20) is

$$S^{\ddot{\Gamma}}_{\mathrm{ext}}(F)(s_1, x_1, \ldots, s_N, x_N) = \sum_{i=1}^N h(s_i) \sum_{k=1}^d \left(\frac{d}{dx_i^{(k)}} \tilde{F}(s_1, x_1, \ldots, s_N, x_N)\right)^2.$$

□

Joint motion of marks and positions

Theorem 7.2.22. *Let $\phi \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ such that $(\phi)_{\mathbb{K}(\mathbb{R}^d)}$ holds. Then the joint bilinear form $(\mathcal{E}^{\ddot{\Gamma}_f,\mu}, \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), C_0(\hat{\mathbb{R}}^d)))$ is closable and the* joint generator *is for all $\gamma \in \Gamma_f(\hat{\mathbb{R}}^d)$*

$$L^{\ddot{\Gamma}_f,\mu}F := L_{ext}^{\ddot{\Gamma}_f,\mu}F + L_{int}^{\ddot{\Gamma}_f,\mu}F \in \bigcap_{p\in\mathbb{N}} L^p(\ddot{\Gamma}_f,\mu). \tag{7.2.26}$$

The closure of $(\mathcal{E}^{\ddot{\Gamma}_f,\mu}, \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), C_0(\hat{\mathbb{R}}^d)))$ is denoted by $(\mathcal{E}^{\ddot{\Gamma}_f,\mu}, \mathcal{D}(\mathcal{E}^{\ddot{\Gamma}_f,\mu}))$. It is a conservative Dirichlet form.

Proof. This follows by Theorems 7.2.20 and 7.2.21. □

7.2.5 Quasi-regularity for compact X

Let X be a compact and complete, separable metric space and μ be a probability measure on $(\ddot{\Gamma}_f(\hat{X}), \mathcal{B}(\ddot{\Gamma}_f(\hat{X})))$ with

$$\langle \mathrm{id}_{\mathbb{R}_+} \otimes \mathbb{1}_X, \cdot\rangle \in L^1(\ddot{\Gamma}_f(\hat{X}),\mu). \tag{7.2.27}$$

Theorem 7.2.23. *We assume that $\big(\mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}, \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{X}), C_0^\infty(\hat{X}))\big)$ is closable and that its closure $(\mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}, \mathcal{D}(\mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}))$ is a Dirichlet form. Then it is quasi-regular.*

Proof. We will apply [MR00, Proposition 4.1] to get this quasi-regularity. By Remark 7.2.13, the conditions $(S.1)$, $(S.2)$, $(S.3)$ and $(\mathcal{D}.1)$ in [MR00, Sections 1.1 and 2.1, cf. p. 267 and 281] are fulfilled. Thus, we show that

$$\begin{aligned}&d_{\ddot{\Gamma}_f}(\cdot,\gamma') = d_{MR}(\cdot,\gamma') + d_f(\cdot,\gamma') \in \mathcal{D}(\mathcal{E}_{\mathrm{ext}}^{\mu,\ddot{\Gamma}_f(\hat{X})}) \text{ and}\\ &S_{\mathrm{ext}}^{\ddot{\Gamma}}(d_{\ddot{\Gamma}_f}(\cdot,\gamma')) \le \eta \in L^1(\ddot{\Gamma}_f(\hat{X}),\mu),\end{aligned}$$

where η is independent of γ'. We prove that $d_{\ddot{\Gamma}_f}(\cdot,\gamma')$ can be approximated for each $\gamma' \in \ddot{\Gamma}(\hat{X})$ in the $\big(\mathcal{E}_{\mathrm{ext},1}^{\mu,\ddot{\Gamma}_f}\big)$-norm, i.e., in $\mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{X}), C_0^\infty(X)\otimes C_0^\infty(\mathbb{R}_+))$ $\subset \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{X}), C_0^\infty(\hat{X}))$ there exist elements $\tilde{F}_{l,k}$ such that

$$\big(\mathcal{E}_{\mathrm{ext},1}^{\mu,\ddot{\Gamma}_f}\big)^{1/2} - \lim_{\substack{k\to\infty\\ l\to\infty}} \tilde{F}_{l,k} = d_{\ddot{\Gamma}_f}(\cdot,\gamma') \in \mathcal{D}(\mathcal{E}_{\mathrm{ext}}^{\mu,\ddot{\Gamma}_f(\hat{X})}).$$

It is sufficient to prove the claim for each component of the metric $d_{\ddot{\Gamma}_f}$ separately, i.e., we have to show that there exists $\eta \in L^1(\ddot{\Gamma}_f(\hat{X}), \mu)$:

$$\begin{aligned}
&\forall \gamma' \in \ddot{\Gamma}_f(\hat{X})\ \exists (F^{MR}_{l,k})_{l,k\in\mathbb{N}},\ (F_{l,k})_{l,k\in\mathbb{N}} \in \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{X}), C_0^\infty(\hat{X})) :\\
&\quad \tilde{F}_{l,k} = F^{MR}_{l,k} + F_{l,k} \quad \forall k,l \in \mathbb{N},\ \text{for } l,k \to \infty\\
&\quad F^{MR}_{l,k} \to d_{MR}(\cdot,\gamma') \text{ and } F_{l,k} \to d_f(\cdot,\gamma') \text{ in } \left(\mathcal{E}^{\mu,\ddot{\Gamma}_f}_{\text{ext},1}\right)^{1/2}\text{-norm,}\\
&\quad S^{\ddot{\Gamma}}_{\text{ext}}(d_{MR}(\cdot,\gamma')) \le \eta \text{ and } S^{\ddot{\Gamma}}_{\text{ext}}(d_f(\cdot,\gamma')) \le \eta
\end{aligned}$$

because $S^{\ddot{\Gamma}}_{\text{ext}}(\tilde{F}_{l,k}) \le 2S^{\ddot{\Gamma}}_{\text{ext}}(F^{MR}_{l,k}) + 2S^{\ddot{\Gamma}}_{\text{ext}}(F_{l,k})$. For the metric d_{MR} we get this result by the arguments of [MR00, Proposition 4.8]. Hence, to obtain the quasi-regularity of $\mathcal{E}^{\mu,\ddot{\Gamma}_f}_{\text{ext}}$, it is sufficient to show that there exist $F_{l,k}$, $l,k \in \mathbb{N}$, in the core $\mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{X}), C_0^\infty(\hat{X}))$ with

$$\begin{aligned}
&\lim_{\substack{k\to\infty\\ l\to\infty}} F_{l,k} = \sum_{n=1}^{\infty} |\langle \cdot - \gamma', \mathbb{1}(x) f_n(s) s\rangle| = d_f(\cdot,\gamma') \in \mathcal{D}(\mathcal{E}^{\mu,\ddot{\Gamma}_f(\hat{X})}_{\text{ext}}) \text{ and}\\
&S^{\ddot{\Gamma}}_{\text{ext}}(d_f(\cdot,\gamma')) \le \eta \in L^1(\ddot{\Gamma}_f(\hat{X}), \mu).
\end{aligned}$$

This we prove now: Fix $\gamma' \in \ddot{\Gamma}_f(\hat{X})$ and choose $g(x) = \frac{x}{1+x}$, $x \in [0,\infty)$, $g(\infty) := 1$. We define for all $l,k \in \mathbb{N}$ (cf. [MR00, Lemma 4.7, Proof of (i)])

$$\begin{aligned}
F_{l,k}:\quad \ddot{\Gamma}_f(\hat{X}) &\to [0,\infty)\\
\gamma &\mapsto g\Bigg(\sum_{n=-k}^{k} \varphi_l\left(\langle \gamma - \gamma', \mathbb{1}(x) f_n(s) s\rangle\right) \Bigg),
\end{aligned} \tag{7.2.28}$$

where $0 \le \varphi_l \in C_b^\infty(\mathbb{R})$ such that for all $t \in \mathbb{R}$

$$\begin{aligned}
&\varphi_l(t) \xrightarrow[l\to\infty]{} |t|,\ \varphi_l(0) = 0,\ |\varphi_l'| \le 1,\ \varphi_l'(0) = 0,\ \varphi_l'(t) \xrightarrow[l\to\infty]{} \operatorname{sign} t,\\
&\varphi_l(r) = |r| \quad \forall r : \frac{1}{l} \le |r| \le l - \frac{1}{l},\ \varphi_l(r) = l, \quad \forall r : |r| \ge l + \frac{1}{l} \text{ and}\\
&|\varphi_l''(r)| \le \begin{cases} 0, & |r| \in \left]\frac{1}{l}, l - \frac{1}{l}\right[\ \cup\ \left]l + \frac{1}{l}, \infty\right[,\\ 4l, & \text{otherwise.} \end{cases}
\end{aligned}$$

Then $F_{l,k} \in \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{X}), C_0^\infty(\hat{X}))$. We check that $(F_{l,k})_{l,k\in\mathbb{N}}$ is a $\left(\mathcal{E}^{\mu,\ddot{\Gamma}_f}_{\text{ext},1}\right)^{1/2}$-Cauchy sequence, i.e.,

$$\begin{aligned}
\lim_{\substack{l,l'\to\infty\\ k,k'\to\infty}} \Bigg(&\int_{\ddot{\Gamma}_f(\hat{X})} S^{\ddot{\Gamma}}_{\text{ext}}(F_{l,k} - F_{l',k'})(\gamma)\mu(d\gamma)\\
&+ \int_{\ddot{\Gamma}_f(\hat{X})} \left(F_{l,k} - F_{l',k'}\right)^2(\gamma)\mu(d\gamma) \Bigg) \overset{!}{=} 0.
\end{aligned} \tag{7.2.29}$$

We treat the $L^2(\ddot{\Gamma}_f(\hat{X}),\mu)$ part first. Since $0 \le g \le 1$, we have $|F_{l,k}| \le 1$. Thus, we may apply Lebesgue's dominated convergence theorem and take the limit pointwisely. Since g is continuous, by Lemma 7.2.24 below the L^2 part becomes arbitrarily small.

It remains to treat the second part. This we do in Lemma 7.2.25 below, which is applicable by (7.2.27).

All in all, we obtain that $(F_{k,l})_{k,l\in\mathbb{N}}$ is a $\big(\mathcal{E}^{\mu,\ddot{\Gamma}_f(\hat{X})}_{\mathrm{ext},1}\big)^{1/2}$-Cauchy sequence. Therefore (cf. the bound obtained in (7.2.35) below and also [MR00, Lemma 4.7(i)] for the second assertion),

$$\lim_{\substack{k\to\infty\\ l\to\infty}} F_{l,k} = g\left(\sum_{n=1}^{\infty} |\langle \cdot - \gamma', \mathbb{1}(x) f_n(s) s\rangle|\right) = d_f(\cdot,\gamma') \in \mathcal{D}(\mathcal{E}^{\mu,\ddot{\Gamma}_f(\hat{X})}_{\mathrm{ext}}) \text{ and}$$
$$S^{\ddot{\Gamma}}_{\mathrm{ext}}(d_f(\cdot,\gamma')) \le \lim_{\substack{k\to\infty\\ l\to\infty}} S^{\ddot{\Gamma}}_{\mathrm{ext}}(F_{l,k}) \in L^1(\ddot{\Gamma}_f(\hat{X}),\mu).$$

Thus, it remains to show Lemmas 7.2.24 and 7.2.25:

Lemma 7.2.24. *For $n \in \mathbb{Z}$ we abbreviate*

$$x_n := x_n(\gamma,\gamma') := \langle \gamma - \gamma', \mathbb{1}(x) f_n(s) s\rangle. \tag{7.2.30}$$

Then

$$\lim_{\substack{l,l'\to\infty\\ k,k'\to\infty}} \left| \sum_{n=-k}^{k} \varphi_l(x_n) - \sum_{n=-k'}^{k'} \varphi_{l'}(x_n) \right| = 0. \tag{7.2.31}$$

Proof. Since $\sum_{n=-\infty}^{\infty} |x_n| \le d_f(\gamma,\varnothing) + d_f(\gamma',\varnothing) < \infty$, we get

$$\forall \varepsilon > 0\ \exists k_0 \in \mathbb{N} : \quad \sum_{|n|>k_0} x_n \le \varepsilon \quad \text{for all } k \ge k_0.$$

Fix $\varepsilon > 0$ with corresponding k_0 and let

$$l_0 := \frac{1}{\min\left\{x_n \middle| x_n \ne 0 \wedge |n| \le k_0\right\} \wedge \varepsilon} \vee \left(d_f(\gamma,\varnothing) + d_f(\gamma',\varnothing)\right), \tag{7.2.32}$$

then we have for $k, k' \geq k_0$ and $l, l' \geq l_0$ that

$$\begin{aligned}
&\left|\sum_{n=-k}^{k} \varphi_l(x_n) - \sum_{n=-k'}^{k'} \varphi_{l'}(x_n)\right| \\
&= \Bigg| \sum_{-k_0 \leq n \leq -k_0} \underbrace{(\varphi_l(x_n) - \varphi_{l'}(x_n))}_{=\varphi_{l_o}(x_n)-\varphi_{l_o}(x_n)=0} + \sum_{\substack{n\in\mathbb{Z},\\ k_0<|n|\leq k}} \varphi_l(x_n) - \sum_{\substack{n\in\mathbb{Z},\\ k_0<|n|\leq k'}} \varphi_{l'}(x_n)\Bigg| \\
&\leq \sum_{\substack{n\in\mathbb{Z}\\ k_0<|n|\leq k}} ||\varphi_l'||_\infty |x_n| + \sum_{\substack{n\in\mathbb{Z}\\ k_0<|n|\leq k'}} ||\varphi_{l'}'||_\infty |x_n| \leq 2\varepsilon,
\end{aligned}$$

and (7.2.31) is shown. □

Lemma 7.2.25. *Let* $\langle \mathrm{id}_{\mathbb{R}_+} \otimes \mathbb{1}_X, \cdot\rangle \in L^1(\ddot{\Gamma}_f(\hat{X}), \mu)$, *then*

$$\lim_{\substack{l,l'\to\infty\\ k,k'\to\infty}} \int_{\ddot{\Gamma}_f(\hat{X})} S^{\ddot{\Gamma}}_{ext}(F_{l,k} - F_{l',k'})(\gamma)\mu(d\gamma) = 0. \tag{7.2.33}$$

Remark 7.2.26. *If μ is a Poisson measure, we can use Mecke's identity (cf. Remark 1.1.6). This yields for $F \in \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{X}), C_0^\infty(\hat{X}))$ and for μ-a.e. $\gamma \in \ddot{\Gamma}_f(\hat{X})$ that (cf. Remark 7.2.13)*

$$S^{\ddot{\Gamma}}_{ext}(F)(\gamma) = \int_{\hat{X}} \left(\sum_{i=1}^{N} \partial_i F(\gamma + \delta_{(s_x,x)}) \cdot \sqrt{s}\frac{d}{ds}\phi_i(\hat{x})\right)^2 \lambda \otimes m(d\hat{x}).$$

But, we can prove the lemma without Mecke's identity.

Proof of Lemma 7.2.25. We prove this lemma by arguments similar to those used to prove Lemma 7.2.24. We want to apply Lebesgue's dominated convergence theorem. Hence, we estimate

$$S^{\ddot{\Gamma}}_{\mathrm{ext}}(F_{l,k} - F_{l',k'})(\gamma) \leq 2S^{\ddot{\Gamma}}_{\mathrm{ext}}(F_{l,k})(\gamma) + 2S^{\ddot{\Gamma}}_{\mathrm{ext}}(F_{l',k'})(\gamma).$$

We set

$$g_n(s) = \frac{d}{ds}\big(f_n(s)s\big) = f_n(s) + sf_n'(s).$$

Then, using (7.2.5),

$$\begin{aligned}
|g_n(s)| \leq& \mathbb{1}_{[q^{n+1},q^{n-2}]}(s) + q^{-(n+3)}\mathbb{1}_{[q^{n+1},q^n]\cup[q^{n-1},q^{n-2}]}(s)q^{n-2} \\
\leq& (1+q^{-5})\mathbb{1}_{[q^{n+1},q^{n-2}]}(s).
\end{aligned}$$

Moreover, $g' \le 1$ and $\varphi_l' \le 1$ $\forall l \in \mathbb{N}$. Hence (cf. Remark 7.2.13), the integrand in $S_{\mathrm{ext}}^{\ddot{\Gamma}}(F_{l,k})(\gamma)$ is dominated by

$$\begin{aligned}
&\Big| \sum_{n=-k}^{k} \underbrace{g'\Big(\sum_{n=-k}^{k} \varphi_l\,(x_n) \Big)}_{\le 1} \Big(\underbrace{\varphi_l'\,(x_n)}_{\le 1} g_n(s_x) \mathbb{1}_X(x) \Big) \Big|^2 s_x \\
\le & s_x \Big(\sum_{n=-k}^{k} |g_n(s_x)\mathbb{1}_X(x)| \Big)^2 \le s_x(1+q^{-5}) \Big| \sum_{n=-\infty}^{\infty} \mathbb{1}_X(x)\mathbb{1}_{[q^{n+1},q^{n-2}]}(s_x) \Big|^2 \\
\le & s_x(1+q^{-5})\,|3\mathbb{1}_X(x)|^2 . \qquad (7.2.34)
\end{aligned}$$

To summarize, we have obtained a bound for the integrand in (7.2.33) that is independent of γ' and by assumption integrable. This means that there exists $\eta \in L^1(\ddot{\Gamma}_f(\hat{X}), \mu)$ such that

$$\limsup_{l,k \nearrow \infty} S_{\mathrm{ext}}^{\ddot{\Gamma}}(F_{l,k})(\gamma) \le \big(1+q^{-5}\big) \int_{\hat{X}} s_x\,|3\mathbb{1}_X(x)|^2\,\gamma(d\hat{x}) \le \eta(\gamma). \qquad (7.2.35)$$

So, by Lebesgue's dominated convergence theorem, we may take the limit in (7.2.33) pointwisely. Using x_n as set in (7.2.30), we get

$$\begin{aligned}
& S_{\mathrm{ext}}^{\ddot{\Gamma}}(F_{l,k} - F_{l',k'})(\gamma) \\
=& \int_{\hat{X}} \Bigg(g'\Big(\sum_{n=-k}^{k} \varphi_l(x_n) \Big) \sum_{n=-k}^{k} \varphi_l'(x_n)\sqrt{s_x} g_n(s_x) \\
& \quad - g'\Big(\sum_{n=-k'}^{k'} \varphi_{l'}(x_n) \Big) \sum_{n=-k'}^{k'} \varphi_{l'}'(x_n)\sqrt{s_x} g_n(s_x) \Bigg)^2 \gamma(d\hat{x}) \\
=& \int_{\hat{X}} \Bigg(\Big(\sum_{n=-k}^{k} \varphi_l'(x_n)\sqrt{s_x} g_n(s_x) - \sum_{n=-k'}^{k'} \varphi_{l'}'(x_n)\sqrt{s_x} g_n(s_x) \Big) g'\Big(\sum_{n=-k}^{k} \varphi_l(x_n) \Big) \\
& \quad + \sum_{n=-k'}^{k'} \varphi_{l'}'(x_n)\sqrt{s_x} g_n(s_x) \Big(g'\Big(\sum_{n=-k}^{k} \varphi_l(x_n) \Big) - g'\Big(\sum_{n=-k'}^{k'} \varphi_{l'}(x_n) \Big) \Big) \Bigg)^2 \gamma(d\hat{x})
\end{aligned} \qquad (7.2.36)$$

By the choice of φ_l, in particular $l > l_0$, i.e., $|x_n| < l$ (cf. (7.2.32)),

$$|\varphi_l'(x_n)| \le \left\{ \begin{array}{ll} \|\varphi_l''\|_\infty |x_n| \le 4l\frac{1}{l}, & \text{for } x_n \in [-l, l], \\ 0, & \text{for } |x_n| \in \big(\frac{1}{l}, l - \frac{1}{l}\big) \ \cup \ \big(l + \frac{1}{l}, \infty\big) \end{array} \right\} \le 4.$$

Hence, using also (7.2.34), the sum in the last summand in (7.2.36) is dominated by $4(1+q^{-5}) \cdot 3\mathbb{1}_X(x)$. Moreover, using in addition (7.2.31) and the continuity of g', the last summand in (7.2.36) becomes arbitrarily small for $l, k, l', k' \nearrow \infty$. It remains to prove that the first summand in (7.2.36) becomes arbitrarily small. Since g' is bounded, this follows by Lemma 7.2.27 below.

Lemma 7.2.27.

$$\lim_{\substack{l,l'\to\infty\\ k,k'\to\infty}} \left| \sum_{n=k}^{k} \varphi_l'(x_n)\sqrt{s_x} g_n(s_x) - \sum_{n=k'}^{k'} \varphi_{l'}'(x_n)\sqrt{s_x} g_n(s_x) \right| \stackrel{!}{=} 0, \qquad (7.2.37)$$

Proof. This we deduce similarly to the proof of Lemma 7.2.27. Namely,

$$\forall \varepsilon > 0\ \exists k_1 \in \mathbb{N} : \quad \sum_{\substack{n\in\mathbb{Z}\\ |n|>k_0}} 4\sqrt{s_x} g_n(s_x) \le \varepsilon \quad \text{for all } k \ge k_1.$$

Let $\varepsilon > 0$ with corresponding k_1 and

$$\frac{l_1}{2} := \frac{1}{\min\left\{x_n \middle| x_n \neq 0 \wedge |n| \le k_1\right\} \wedge \varepsilon} \vee \left(d_f(\gamma, \varnothing) + d_f(\gamma', \varnothing)\right),$$

then we get for $k, k' \ge k_1$ and $l, l' \ge l_1$ that

$$\begin{aligned}
&\left| \sum_{n=-k}^{k} \varphi_l'(x_n)\sqrt{s_x} g_n(s_x) - \sum_{n=-k'}^{k'} \varphi_{l'}'(x_n)\sqrt{s_x} g_n(s_x) \right| \\
&= \Bigg| \sum_{-k_1 \le n \le -k_1} \underbrace{(\varphi_l'(x_n) - \varphi_{l'}'(x_n))}_{=\varphi_{l_1}'(x_n) - \varphi_{l_1}'(x_n) = 0} \sqrt{s_x} g_n(s_x) \\
&\quad + \sum_{\substack{n\in\mathbb{Z},\\ k_1<|n|\le k}} \varphi_l'(x_n)\sqrt{s_x} g_n(s_x) - \sum_{\substack{n\in\mathbb{Z},\\ k_1<|n|\le k'}} \varphi_{l'}'(x_n)\sqrt{s_x} g_n(s_x) \Bigg| \\
&\le \sum_{\substack{n\in\mathbb{Z}\\ k_1<|n|\le k}} 4\sqrt{s_x} g_n(s_x) + \sum_{\substack{n\in\mathbb{Z}\\ k_1<|n|\le k'}} 4\sqrt{s_x} g_n(s_x) \le 2\varepsilon,
\end{aligned}$$

and (7.2.43) is shown. □

This concludes the proof of Lemma 7.2.25. □

And thus, also the proof of Theorem 7.2.23 is completed. □

Remark 7.2.28. *Actually, Theorem 7.2.23 holds also if we choose $C_0^\infty(\mathbb{R}_+) \otimes C_0^\infty(X)$ instead of $C_0^\infty(\hat{X})$ (cf. (7.2.28)).*

Intrinsic bilinear form

Theorem 7.2.29. *Let* $\big(\mathcal{E}_{int}^{\mu,\ddot{\Gamma}_f}, \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{X}), C_0^\infty(\hat{X}))\big)$ *be closable. Assume that its closure* $(\mathcal{E}_{int}^{\mu,\ddot{\Gamma}_f}, \mathcal{D}(\mathcal{E}_{int}^{\mu,\ddot{\Gamma}_f}))$ *is a Dirichlet form. Then it is quasi-regular.*

Proof. The arguments used for $\mathcal{E}_{\mathrm{ext}}^{\mu,\ddot{\Gamma}_f}$ in the proof of Theorem 7.2.23 hold for $\mathcal{E}_{\mathrm{int}}^{\mu,\ddot{\Gamma}_f}$, if we replace ext by int and use Lemma 7.2.30 instead of Lemma 7.2.25.

Lemma 7.2.30. *We have*

$$\lim_{\substack{l,l'\to\infty\\ k,k'\to\infty}} \int_{\ddot{\Gamma}_f(\hat{X})} S_{int}^{\ddot{\Gamma}}(F_{l,k}-F_{l',k'})(\gamma)\mu(d\gamma) = 0. \tag{7.2.38}$$

Proof. Because of Remark 7.2.16, the claim follows by

$$S_{\mathrm{int}}\left(\mathrm{id}_{\mathbb{R}_+} f_n \otimes \mathbb{1}_X\right)(\hat{x}) = \frac{1}{s}\underbrace{\left|\nabla^X\left(\mathrm{id}_{\mathbb{R}_+} f_n \otimes \mathbb{1}_X\right)(\hat{x})\right|^2_{T_x(X)}}_{=0} = 0.$$

□

Hence, Theorem 7.2.29 is proved. □

Joint bilinear form

Theorem 7.2.31. *Let* $\big(\mathcal{E}^{\mu,\ddot{\Gamma}_f}, \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{X}), C_0^\infty(\hat{X}))\big)$ *is closable and that its closure* $(\mathcal{E}^{\mu,\ddot{\Gamma}_f}, \mathcal{D}(\mathcal{E}^{\mu,\ddot{\Gamma}_f}))$ *is a Dirichlet form. Then it is quasi-regular.*

Proof. This follows by the arguments of the proof of Theorem 7.2.23, where we replace Lemma 7.2.25 by Lemma 7.2.32.

Lemma 7.2.32. *Let* $\langle \mathrm{id}_{\mathbb{R}_+} \otimes \mathbb{1}_X, \cdot\rangle \in L^1(\ddot{\Gamma}_f(\hat{X}), \mu)$*, then*

$$\lim_{\substack{l,l'\to\infty\\ k,k'\to\infty}} \int_{\ddot{\Gamma}_f(\hat{X})} S^{\ddot{\Gamma}}(F_{l,k}-F_{l',k'})(\gamma)\mu(d\gamma) = 0. \tag{7.2.39}$$

Proof. The claim follows by Lemmas 7.2.25 and 7.2.30 because (cf. Remark 7.2.19)

$$S^{\ddot{\Gamma}}(F_{l,k}-F_{l',k'})(\gamma) := S_{\mathrm{ext}}^{\ddot{\Gamma}}(F_{l,k}-F_{l',k'})(\gamma) + S_{\mathrm{int}}^{\ddot{\Gamma}}(F_{l,k}-F_{l',k'})(\gamma).$$

□

Therefore, Theorem 7.2.31 is shown. □

7.2.6 Quasi-regularity for $X = \mathbb{R}^d$

From now on, we fix $X = \mathbb{R}^d$, $d \in \mathbb{N}$, m on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $(\phi_k)_{k\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ and μ as in Subsection 7.2.4. This means that we fix $X = \mathbb{R}^d$, $d \in \mathbb{N}$, m on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and $(\phi_k)_{k\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ such that (7.0.1), (7.2.12) and (7.2.13) hold. And μ is a Gibbs perturbation of $\mathcal{P}_\theta$ w.r.t. a pair potential $V(\hat{x}, \hat{y}) = s_x\phi(x, y)s_y$ with $\phi \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ satisying $(\phi)_{\mathbb{K}(\mathbb{R}^d)}$.

Proposition 7.2.33. *Let μ be as described above. Then*

$$\int_{\ddot{\Gamma}(\mathbb{R}^d)} d_{\ddot{\Gamma}_f}(\gamma, \emptyset)\mu(d\gamma) < \infty.$$

In particular, $\ddot{\Gamma}_f(\mathbb{R}^d) \in \mathcal{B}(\ddot{\Gamma}(\mathbb{R}^d))$ *and* $\mu(\ddot{\Gamma}_f(\mathbb{R}^d)) = 1$.

Proof. We note that $d_{\ddot{\Gamma}_f}(\cdot, \emptyset)$ is a local mass map (w.r.t. $\lambda_\theta \otimes m$):

$$\begin{aligned}&\int_{\hat{X}} d_{\ddot{\Gamma}_f}(\{\hat{x}\}, \varnothing)\lambda_\theta \otimes m(d\hat{x})\\ \leq & 2\int_{\mathbb{R}_+} 3s_x\lambda_\theta(ds_x)\sum_{i\in\mathbb{I}} \frac{3}{M(2C)^i} m(X_{i+1}) \leq 6\int_{\mathbb{R}_+} s_x\lambda_\theta(ds_x)3C < \infty,\end{aligned}$$

where we applied (7.2.12) and (7.2.13). The measurability follows as in Proposition 7.2.1. Hence, by (the proof of) Theorem 4.3.34 (cf. Remark 5.2.12) the assertion follows. □

Extrinsic motion

Theorem 7.2.34. *Let μ be as above. Then* $(\mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}, \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), C_0(\hat{\mathbb{R}}^d)))$ *is closable and its closure* $(\mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}, \mathcal{D}(\mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}))$ *is a* quasi-regular *Dirichlet form.*

Proof. By Theorem 7.2.20, we get the closability. To show that the Dirichlet form is quasi-regular, we assume that w.l.o.g. $\mathbb{I} = \mathbb{N}$. Replacing $F_{l,k}$ in (7.2.28) by

$$\begin{aligned} F_{l,k}\colon\quad \ddot{\Gamma}_f(\mathbb{R}^d) &\to [0,\infty)\\ \gamma &\mapsto g\Bigg(\sum_{n=-k}^{k}\sum_{i\in\mathbb{I}} \frac{1}{M(2C)^i}\varphi_l\left(\langle\gamma-\gamma', \phi_i(x)f_n(s)s\rangle\right)\Bigg), \end{aligned} \tag{7.2.40}$$

we obtain the result adapting the arguments appropriately. In particular, we use Lemmas 7.2.35 and 7.2.36 instead of Lemmas 7.2.24 and 7.2.27 and follow the arguments of Lemma 7.2.25 in this new situation. In more detail, in (7.2.34) $\mathbb{1}_X$ is replaced by $2\sum_i \frac{1}{M(2C)^i}\phi_i$. By Proposition 7.2.33, the changed estimate is integrable.

Lemma 7.2.35. *For $n \in \mathbb{Z}$, $i \in \mathbb{I}$ we abbreviate*

$$x_{n,i} := x_{n,i}(\gamma,\gamma') := \langle \gamma - \gamma', \phi_i(x) f_n(s)s \rangle. \tag{7.2.41}$$

Then

$$\lim_{\substack{l,l'\to\infty \\ k,k'\to\infty}} \sum_{i\in\mathbb{I}} \frac{1}{M(2C)^i} \left| \sum_{n=-k}^{k} \varphi_l(x_{n,i}) - \sum_{n=-k'}^{k'} \varphi_{l'}(x_{n,i}) \right| = 0. \tag{7.2.42}$$

Proof. This follows by the same kind of arguments as Lemma 7.2.24. □

Lemma 7.2.36. *Let $\langle \mathrm{id}_{\mathbb{R}_+} \otimes \sum_{i\in\mathbb{I}} \frac{1}{M(2C)^i} \mathbb{1}_{X_i}, \cdot \rangle \in L^1(\ddot{\Gamma}_f(\mathbb{R}^d), \mu)$. Then*

$$\lim_{\substack{l,l'\to\infty \\ k,k'\to\infty}} \sum_{i\in\mathbb{I}} \frac{1}{M(2C)^i} |\phi_i(x)| \left| \sum_{n=k}^{k} \varphi'_l(x_n) f_n(s_x) s_x - \sum_{n=k'}^{k'} \varphi'_{l'}(x_n) f_n(s_x) s_x \right| = 0. \tag{7.2.43}$$

Proof. This follows by the same kind of arguments as Lemmas 7.2.27 and 7.2.24, and using (7.2.12) instead of (7.2.27). □

Therefore, Theorem 7.2.34 holds. □

Intrinsic motion

Theorem 7.2.37. *Let μ be as above. Then $(\mathcal{E}_{int}^{\mu,\ddot{\Gamma}_f}, \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), C_0(\hat{\mathbb{R}}^d)))$ is closable and its closure $(\mathcal{E}_{int}^{\mu,\ddot{\Gamma}_f}, \mathcal{D}(\mathcal{E}_{int}^{\mu,\ddot{\Gamma}_f}))$ is a* quasi-regular *Dirichlet form.*

Proof. The proof is similar to the changes used to prove Theorem 7.2.34, where we apply Lemma 7.2.38 instead of Lemma 7.2.30.

Lemma 7.2.38. *Let $\langle \mathrm{id}_{\mathbb{R}_+} \otimes \sum_{i\in\mathbb{I}} \frac{1}{M(2C)^i} \mathbb{1}_{X_i}, \cdot \rangle \in L^1(\ddot{\Gamma}_f(\mathbb{R}^d), \mu)$. Then*

$$\lim_{\substack{l,l'\to\infty \\ k,k'\to\infty}} \int_{\ddot{\Gamma}_f(\mathbb{R}^d)} S_{int}^{\ddot{\Gamma}}(F_{l,k} - F_{l',k'})(\gamma)\mu(d\gamma) = 0. \tag{7.2.44}$$

Proof. The integrand $S_{\mathrm{int}}^{\ddot{\Gamma}}(F_{l,k})(\gamma)$ is dominated by

$$\frac{1}{s_x} \sum_{\substack{i,j\in\mathbb{I} \\ -k\le m,n\le k}} \underbrace{g'\Big(\sum_{\substack{i\in\mathbb{I} \\ -k\le n\le k}} \varphi_l(x_{i,n})\Big)}_{\le 1} \underbrace{g'\Big(\sum_{\substack{j\in\mathbb{I} \\ -k\le m\le k}} \varphi_l(x_{j,m})\Big)}_{\le 1} \underbrace{\varphi'_l(x_{i,n})}_{\le 1} \underbrace{\varphi'_l(x_{j,m})}_{\le 1}$$

$$f_n(s_x)s_x \frac{1}{M(2C)^i} f_m(s_x) s_x \frac{1}{M(2C)^j} \left|\nabla^X \phi_i(x)\right|^2_{T_x(X)}.$$

Since

$$\left|\nabla^X \phi_i(x)\right|^2_{T_x(X)} \leq 4^2 d \mathbb{1}_{X_i^{1/4}\setminus X_{i-1}^{-1/4}}(x) \mathbb{1}_{X_j^{1/4}\setminus X_{j-1}^{-1/4}}(x)$$

the last line is dominated by

$$4^2 ds_x \underbrace{\Big(\sum_{n\in\mathbb{Z}} f_n(s_x)\Big)^2}_{\leq 3} \sum_{i,j\in\mathbb{I}} \frac{1}{M(2C)^i} \frac{1}{M(2C)^j} \mathbb{1}_{X_i^{1/4}\setminus X_{i-1}^{-1/4}}(x) \mathbb{1}_{X_j^{1/4}\setminus X_{j-1}^{-1/4}}(x)$$

$$\leq 12^2 ds_x \sum_{i\in\mathbb{I}} \frac{1}{M(2C)^i} \left(\frac{1}{M(2C)^{i+1}} + \frac{1}{M(2C)^i} + \frac{1}{M(2C)^{i-1}} \right) \mathbb{1}_{X_i^{1/4}\setminus X_{i-1}^{-1/4}}(x)$$

$$\leq 12^2 ds_x \cdot 3 \sum_{i\in\mathbb{I}} \frac{1}{M(2C)^i} \left(\mathbb{1}_{X_{i+1}}(x) + \mathbb{1}_{X_i}(x) + \mathbb{1}_{X_{i-1}}(x) \right),$$

where we used (7.2.13) and that each set $B_i := X_i^{1/4}\setminus X_{i-1}^{-1/4}$ overlaps at most with the three sets B_{i+1}, B_i, B_{i-1}, resp. (for the last step) with X_{i+1}, X_i, X_{i-1}. By assumption the last line is integrable.

Hence, following and adapting the arguments as in Lemma 7.2.25 (cf., e.g., (7.2.36)) combined with Lemma 7.2.36, yields the proof. □

Hence, Theorem 7.2.37 is proved. □

Joint motion

Theorem 7.2.39. *Let μ be as above. Then $(\mathcal{E}^{\mu,\ddot{\Gamma}_f}, \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), C_0(\hat{\mathbb{R}}^d)))$ is closable and its closure $(\mathcal{E}^{\mu,\ddot{\Gamma}_f}, \mathcal{D}(\mathcal{E}^{\mu,\ddot{\Gamma}_f}))$ is a* quasi-regular *Dirichlet form.*

Proof. With the same reasoning as given in the proof of Theorem 7.2.31, it is sufficient to combine Lemma 7.2.38 with the analogue version of Lemma 7.2.25 to prove this result. □

7.3 Diffusions on multiple configurations

We show that the associated processes for the above considered quasi-regular Dirichlet forms are diffusions (cf. Subsection 7.3.1). Then we give conditions to see that they actually sit on $\Gamma_{\mathbb{K}(X)} \subset \Gamma_f(\hat{X})$.

As in Subsection 7.2.6, we fix $X = \mathbb{R}^d$, $d \in \mathbb{N}$, m on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)$ and $(\phi_k)_{k\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ such that (7.0.1), (7.2.12) and (7.2.13) hold. And μ is a Gibbs perturbation of $\mathcal{P}_\theta$ w.r.t. a pair potential

$$V(\hat{x}, \hat{y}) = s_x \phi(\hat{x}, \hat{y}) s_y,$$

where $\phi \in C^1(\hat{\mathbb{R}}^d \times \hat{\mathbb{R}}^d)$ satisfies $(\phi)_{\mathbb{K}(\mathbb{R}^d)}$ (cf. Subsection 5.3.1).

7.3.1 Diffusions on multiple configurations

We briefly mention the locality of the considered quasi-regular Dirichlet forms, before we present the associated diffusions.

Locality

Definition 7.3.1 (cf. [MR92, Def. V.1.1, p.148])**.** *Let E be a Lusin space and μ be a measure on it. A quasi-regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E,\mu)$ is said to have the* local property *(or to be* local*) if for all $F, G \in \mathcal{D}(\mathcal{E})$ with supp$(|F|\mu) \cap$ supp$(|G|\mu) = \emptyset$, we have*

$$\mathcal{E}(u,v) = 0.$$

Proposition 7.3.2. *The Dirichlet form $(\mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}, \mathcal{D}(\mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}))$ is local.*

Proposition 7.3.3. *The Dirichlet form $(\mathcal{E}_{int}^{\mu,\ddot{\Gamma}_f}, \mathcal{D}(\mathcal{E}_{int}^{\mu,\ddot{\Gamma}_f}))$ is local.*

Proposition 7.3.4. *The Dirichlet form $(\mathcal{E}^{\mu,\ddot{\Gamma}_f}, \mathcal{D}(\mathcal{E}^{\mu,\ddot{\Gamma}_f}))$ is local.*

Proofs of Propositions 7.3.2, 7.3.3 and 7.3.4. This is evident using the proof of [MR00, Proposition 4.12] (cf. also [MR92, Ex. V.I.12(ii), p.154]. □

Associated diffusions

Theorem 7.3.5. *Let $X = \mathbb{R}^d$, m and μ be as described above. Then there exists a conservative diffusion process (i.e., a conservative strong Markov process with continuous sample paths)*

$$\mathbf{M}_{ext}^{\ddot{\Gamma}_f} = \Big(\mathbf{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t\geq 0}, (\mathbf{\Theta}_t)_{t\geq 0}, (\mathbf{X}(\mathbf{t}))_{t\geq 0}, (\mathbf{P}_\gamma)_{\gamma\in\ddot{\Gamma}_f(\hat{X})}\Big)$$

on $\ddot{\Gamma}_f(\hat{X})$ which is properly associated with $\big(\mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}, \mathcal{D}\big(\mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}\big)\big)$, i.e., for all ($\mu$-versions) of $F \in L^2(\ddot{\Gamma}_f(\hat{X}),\mu)$ and all $t > 0$ the function

$$\ddot{\Gamma}_f(\hat{X}) \ni \gamma \mapsto p_t F(\gamma) := \int_\Omega F(\mathbf{X}(\mathbf{t}))\mathbf{dP}_\gamma$$

is an $\mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}$-quasi-continuous version of $\exp(-tL_{ext}^{\mu,\ddot{\Gamma}_f})F$, where $L_{ext}^{\mu,\ddot{\Gamma}_f}$ is the generator of $\big(\mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}, \mathcal{D}\big(\mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}\big)\big)$ (cf. [MR92, Section I.2]). $\mathbf{M}$ is up to μ-equivalence unique (cf. [MR92, Theorem VI.6.4]). In particular, $\mathbf{M}$ is μ-symmetric (i.e., $\int Gp_tFd\mu = \int Fp_tGd\mu$ for all $F, G : \ddot{\Gamma}_f(\hat{X}) \to \mathbb{R}$, $\mathcal{B}(\ddot{\Gamma}_f(\hat{X}))$-measurable) and has μ as an invariant measure.

Proof. Since the Dirichlet form is quasi-regular (cf. Theorem 7.2.23 resp. 7.2.37) and local (cf. Proposition 7.3.2) this follows by [MR92, Theorem IV.3.5 resp.V.1.1]). □

Theorem 7.3.6. *The assertions of Theorem 7.3.5 hold with* $\mathcal{E}_{int}^{\mu,\ddot{\Gamma}_f}$ *and* $\mathbf{M}_{int}^{\ddot{\Gamma}_f}$ *replacing* $\mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}$*, resp.* $\mathbf{M}_{ext}^{\ddot{\Gamma}_f}$.

Proof. Since the Dirichlet form is quasi-regular (cf. Theorem 7.2.29 resp. 7.2.37) and local (cf. Proposition 7.3.3) this follows by [MR92, Theorem IV.3.5 resp.V.1.1]). □

Theorem 7.3.7. *The assertions of Theorem 7.3.5 hold with* $\mathcal{E}^{\mu,\ddot{\Gamma}_f}$ *and* $\mathbf{M}^{\ddot{\Gamma}_f}$ *replacing* $\mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}$*, resp.* $\mathbf{M}_{ext}^{\ddot{\Gamma}_f}$.

Proof. Since the Dirichlet form is quasi-regular (cf. Theorem 7.2.31 resp. 7.2.39) and local (cf. Proposition 7.3.4) this follows by [MR92, Theorem IV.3.5 resp.V.1.1]). □

7.3.2 Exceptional set

The next theorems yield that the constructed diffusions over $\ddot{\Gamma}(\hat{\mathbb{R}}^d)$ actually sit on $\Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d) \subset \Gamma_f(\mathbb{R}^d)$, if $d \geq 2$.

Definition 7.3.8. *We denote the set of* pinpointing configurations with local mass $d_{\ddot{\Gamma}_f}(\varnothing,\cdot)$ *by*

$$\Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d) := \left\{\gamma \in \ddot{\Gamma}_f(\hat{\mathbb{R}}^d) \middle| \gamma(\mathbb{R}_+ \times \{x\}) \leq 1 \qquad \forall x \in \mathbb{R}^d \right\}.$$

Remark 7.3.9. *In general* $\Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d) \subset \Gamma_f(\hat{\mathbb{R}}^d)$ *(cf. Definition 4.3.12).*

Theorem 7.3.10. *Let* $X = \mathbb{R}^d$*,* $d \geq 1$*, and* μ *be as described in the beginning of Section 7.3 with* $\phi \geq 0$*. Then the set* $\ddot{\Gamma}(\hat{\mathbb{R}}^d) \backslash \Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d)$ *is* $\mathcal{E}_{ext}^{\mu,\ddot{\Gamma}_f}$*-exceptional and the assertions of Theorem 7.3.5 hold with* $\ddot{\Gamma}_f(\hat{\mathbb{R}}^d)$ *being replaced by* $\Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d)$*. In particular, this holds for the set* $\ddot{\Gamma}(\hat{\mathbb{R}}^d) \backslash \Gamma_f(\mathbb{R}^d)$.

Proof. Compare the proof of Theorem 7.3.12. □

Theorem 7.3.11. *Let* $X = \mathbb{R}^d$*,* $d \geq 2$*, and* μ *be as above with* $\phi \geq 0$*. Then the set* $\ddot{\Gamma}(\hat{\mathbb{R}}^d) \backslash \Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d)$ *is* $\mathcal{E}_{int}^{\mu,\ddot{\Gamma}_f}$*-exceptional and the assertions of Theorem 7.3.6 hold with* $\ddot{\Gamma}_f(\hat{\mathbb{R}}^d)$ *being replaced by* $\Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d)$*. In particular, this holds for the set* $\ddot{\Gamma}(\hat{\mathbb{R}}^d) \backslash \Gamma_f(\mathbb{R}^d)$.

Proof. See the proof of Theorem 7.3.12. □

Theorem 7.3.12. *Let $X = \mathbb{R}^d$, $d \geq 2$, and μ be as above with $\phi \geq 0$. Then the set $\ddot{\Gamma}(\hat{\mathbb{R}}^d)\backslash\Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d)$ is $\mathcal{E}^{\mu,\ddot{\Gamma}_f}$-exceptional and the assertions of Theorem 7.3.7 hold with $\ddot{\Gamma}_f(\hat{\mathbb{R}}^d)$ being replaced by $\Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d)$. In particular, this holds for the set $\ddot{\Gamma}(\hat{\mathbb{R}}^d)\backslash\Gamma_f(\mathbb{R}^d)$.*

To prove these results, we use the following Proposition.

Proposition 7.3.13. *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Dirichlet form with Polish state space E. The union of an increasing sequence of $\mathcal{E}$-exceptional sets is $\mathcal{E}$-exceptional. The same holds for the intersection of a decreasing sequence of $\mathcal{E}$-exceptional sets.*

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be an increasing sequence of $\mathcal{E}$-exceptional sets. By [MR92, Theorem III.2.8], we have using the notation of [MR92, Section III.2]

$$\mathrm{Cap}_{h,g}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sup_{n\geq 1} \mathrm{Cap}_{h,g}\left(A_n\right) = 0,$$

where we used that the capacity $\mathrm{Cap}_{h,g}(A)$ of $A \subset E$ corresponding to the two excessive functions h and g (cf. [MR92, p.78]) is 0, i.e.,

$$\mathrm{Cap}_{h,g}(A) = 0, \text{ if and only if } A \subset E \text{ is } \mathcal{E}\text{-exceptional}$$

(cf. [MR92, Theorem III.2.11]). This yields the first assertion.

For the second one, let $(A_n)_{n\in\mathbb{N}}$ be a decreasing sequence of $\mathcal{E}$-exceptional sets. By [MR92, Theorem III.2.8],

$$\mathrm{Cap}_{h,g}\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \inf_{n\geq 1} \mathrm{Cap}_{h,g}\left(A_n\right) = 0.$$

□

Proof of Theorems 7.3.12, 7.3.10 and 7.3.11.
We extend the idea used to prove [RS98, Proposition 1]: Namely, by Proposition 7.3.13, it is sufficient to show for all $a \in \mathbb{N}$ that the set

$$N_a := \left\{\gamma \in \ddot{\Gamma}(\hat{\mathbb{R}}^d) \,\middle|\, \sup_{y\in[-a,-a]^d} \left|\left\{\hat{x} \in \gamma \,\middle|\, \hat{x} \in [e^{-a}, e^a] \times \{y\}\right\}\right| \geq 2\right\}$$

is exceptional because $\bigcup_{a\in\mathbb{N}} N_a = \ddot{\Gamma}(\hat{\mathbb{R}}^d)\backslash\Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d)$. Fix $a \in \mathbb{N}$. We show that the function $u := u_{(a)} = \mathbb{1}_{N_a}$ is quasi-continuous. This we will show by approximating u with continuous functions $u_n \in \mathcal{D}(\mathcal{E}^{\mu,\ddot{\Gamma}_f})$.

To this end, we start with a smooth partition of $\hat{\mathbb{R}}^d$. Let $\phi \in C_b^\infty(\mathbb{R})$ satisfy $\mathbb{1}_{[0,1]} \leq \phi \leq \mathbb{1}_{[-1/2,3/2]}$ and $|\phi'| \leq 3\mathbb{1}_{[-1/2,3/2]}$. For any $n \in \mathbb{N}$ and $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d$ we define a $C_0^\infty(\hat{\mathbb{R}}^d)$ function by (compare [RS98, (12)])

$$\phi_{i,n}(\hat{x}) := \phi\left(\frac{\frac{1}{a}\ln(s)+1}{2}\right)\prod_{k=1}^d \phi(nx_k - i_k).$$

Moreover, we note that $\phi_{i,n} \leq I_{i,n}$ where

$$\begin{aligned} I_{i,n}(\hat{x}) :=& \mathbb{1}_{[-2,2]}(\frac{1}{a}\ln(s))\prod_{k=1}^d \mathbb{1}_{[-1/2,3/2]}(nx_k - i_k) \\ =& \mathbb{1}_{[e^{-2a},e^{2a}]}(s)\prod_{k=1}^d \mathbb{1}_{[-1/2,3/2]}(nx_k - i_k). \end{aligned}$$

We calculate the partial derivatives

$$\begin{aligned} \partial_0\phi_{i,n}(\hat{x}) :=& \frac{d}{ds}\phi_{i,n}(\hat{x}) = \phi'\left(\frac{\frac{1}{a}\ln(s)+1}{2}\right)\frac{1}{2as}\prod_{k=1}^d \phi(nx_k - i_k) \\ \partial_j\phi_{i,n}(\hat{x}) =& \phi\left(\frac{\frac{1}{a}\ln(s)+1}{2}\right)\phi'(nx_k - i_k)n\prod_{k=1,k\neq j}^d \phi(nx_k - i_k) \quad 1 \leq j \leq d. \end{aligned}$$

Hence,

$$\partial_0\phi_{i,n}(\hat{x}) \leq \frac{3}{2as}I_{i,n}(\hat{x}) \qquad \text{and } \partial_j\phi_{i,n}(\hat{x}) \leq 3nI_{i,n}(\hat{x}) \text{ for } 1 \leq j \leq d. \quad (7.3.1)$$

Let ψ be a smooth function on $\mathbb{R}$ satisfying $\mathbb{1}_{[1,\infty)} \leq \psi \leq \mathbb{1}_{[2,\infty)}$ and $|\psi'| \leq 2\mathbb{1}_{(2,\infty)}$. We pick the lattice $A_n := [-na, na]^d \cap \mathbb{Z}^d$ and define a continuous element of $\mathcal{D}(\mathcal{E}^{\mu,\ddot{\Gamma}_f(\hat{\mathbb{R}}^d)})$ by:

$$u_n(\gamma) := \psi\left(\sup_{i\in A_n}\langle\phi_i,\gamma\rangle\right). \quad (7.3.2)$$

Then $u_n \to u$ pointwisely for $n \to \infty$. Namely, for a fixed γ we have $|\gamma(a)| := |\gamma\cap([e^{-a},e^a]\times[-a,a]^d)| < \infty$. Hence, there exists a fixed and finite number of points $x \in \tau(\gamma(a))$, which have a minimal distance ε_X to each other. Choose n large enough such that in each box $B_{i,n} := [-1/2n, 1/2n]^d + i/n$ there are only points having the same support. Hence for the chosen $n = n(\gamma)$:

$$u_n(\gamma) = \begin{cases} 1, & \text{if } \gamma \in N_a, \\ 0, & \text{if } \gamma \notin N_a. \end{cases}$$

It suffices to show that $\sup_n \mathcal{E}^{\mu,\ddot{\Gamma}_f}(u_n,u_n) < \infty$ to apply [RS98, Lemma 1]. We prepare to estimate $S^{\ddot{\Gamma}}(u_n)$:

$$\left(\psi'\left(\sup_{i\in A_n}\langle\phi_{i,n},\gamma\rangle\right)\right)^2 \leq 4\mathbb{1}_{\{\gamma|\sup_{i\in A_n}\langle\phi_{i,n},\gamma\rangle>1\}} \leq 4\mathbb{1}_{\{\gamma|\sup_{i\in A_n}\langle I_{i,n},\gamma\rangle\geq 2\}}, \tag{7.3.3}$$

where we use that $\langle I_{i,n},\gamma\rangle \in \mathbb{N}_0$ for the final inequality. Hence, using first $S^{\ddot{\Gamma}}(u\vee v) \leq S^{\ddot{\Gamma}}(u)\vee S^{\ddot{\Gamma}}(v)$, then (7.3.1) and (7.3.3), we get

$$\begin{aligned}
&S^{\ddot{\Gamma}}(u_n)(\gamma) = \left(\psi'\left(\sup_{i\in A_n}\langle\phi_{i,n},\gamma\rangle\right)\right)^2 S^{\ddot{\Gamma}}\left(\sup_{i\in A_n}\langle\phi_{i,n},\cdot\rangle\right)(\gamma)\\
\leq&\left(\psi'\left(\sup_{i\in A_n}\langle\phi_{i,n},\gamma\rangle\right)\right)^2 \sup_{i\in A_n} S^{\ddot{\Gamma}}\left(\langle\phi_{i,n},\cdot\rangle\right)(\gamma)\\
=&\left(\psi'\left(\sup_{i\in A_n}\langle\phi_{i,n},\gamma\rangle\right)\right)^2 \sup_{i\in A_n}\int_{\hat{\mathbb{R}}^d}\frac{1}{s_x}\left|\nabla^X\phi_{i,n}(\hat{x})\right|^2 + \left|\sqrt{s_x}\partial_0\phi_{i,n}(\hat{x})\right|^2\gamma(d\hat{x})\\
\leq& 4e^{2a}\mathbb{1}_{\{\gamma|\sup_{i\in A_n}\langle I_{i,n},\gamma\rangle\geq 2\}}9(n^2d+1)\sup_{i\in A_{i,n}}\langle I_{i,n},\gamma\rangle\\
\leq& 36e^{2a}(n^2d+1)\sum_{i\in A_{i,n}}\mathbb{1}_{\{\gamma|\langle I_{i,n},\gamma\rangle\geq 2\}}\langle I_{i,n},\gamma\rangle.
\end{aligned}\tag{7.3.4}$$

By the proof of Proposition 4.2.3, more detailed by (4.2.5), we see for μ being a Gibbs measure with non-negative potential as considered in Chapter 5 that

$$\begin{aligned}
&\int_{\{\gamma|\langle I_{i,n},\gamma\rangle\geq 2\}}\langle I_{i,n},\gamma\rangle\mu(d\gamma) \leq 2\left(\int_{\hat{\mathbb{R}}^d} I_{i,n}(\hat{x})\lambda\otimes m(d\hat{x})\right)^2\\
\leq&\underbrace{\int_{\mathbb{R}_+}\mathbb{1}_{[e^{-2a},e^{2a}]}(s_x)\lambda(ds_x)}_{\leq C<\infty}\prod_{k=1}^{d}\underbrace{\left(\int_{\mathbb{R}^d}\mathbb{1}_{[-1/2,3/2]}(nx_k-i_k)m(dx)\right)}_{\leq m\left(\left[\frac{-1/2}{n},\frac{3/2}{n}\right]+\frac{i_k}{n}\right)}
\end{aligned}\tag{7.3.5}$$

where we used that

$$-1/2 \leq nx_k - i_k \leq 3/2 \Leftrightarrow \frac{-1/2+i_k}{n} \leq x_k \leq \frac{3/2+i_k}{n}.$$

Since $m(dx) = \rho(x)dx$, by (7.3.4) and (7.3.5)

$$\begin{aligned}
\int_{\ddot{\Gamma}_f(\hat{\mathbb{R}}^d)} S^{\ddot{\Gamma}}(u_n)(\gamma)\mu(d\gamma) &\leq 36e^{2a}(n^2d+1)\sum_{i\in A_{i,n}} Cm\left(\left[\frac{-1/2}{n},\frac{3/2}{n}\right]^d+\frac{i}{n}\right)^2\\
&= 36e^{2a}(n^2d+1)C\sum_{i\in A_{i,n}}\left(\int_{\left[\frac{-1/2}{n},\frac{3/2}{n}\right]^d+i/n}\rho(x)dx\right)^2.
\end{aligned}$$

By Cauchy-Schwartz, $\rho \in L^2_{\text{loc}}(\mathbb{R}^d, dx)$ (cf. (7.0.1)) and the translation invariance of the Lebesgue measure, the later sum is dominated by

$$\underbrace{\sum_{i \in A_{i,n}} \left(\int_{\left[\frac{-1/2}{n}, \frac{3/2}{n}\right]^d + i/n} \rho^2(x) dx \right)}_{= \int_{[-a,a]} \rho^2(x) dx \le C'(a) < \infty} \cdot \underbrace{\left(\int_{\left[\frac{-1/2}{n}, \frac{3/2}{n}\right]^d + i/n} dx \right)}_{=(2/n)^d} \le n^{-d} 2^d C'(a) < \infty.$$

Summing up, we get that for $d \geq 2$ there exists $\tilde{C} < \infty$:

$$\sup_{n \in \mathbb{N}} \mathcal{E}^{\mu, \ddot{\Gamma}_f}(u_n) \le \tilde{C} \sup_{n \in \mathbb{N}} \left(n^{2-d} d + n^{-d} \right) < \infty. \tag{7.3.6}$$

This concludes the proof of Theorem 7.3.12. Moreover, we note that with the same choices and (similar) calculations, we get that

$$\sup_{n \in \mathbb{N}} \mathcal{E}^{\mu, \ddot{\Gamma}_f}_{\text{ext}}(u_n) \le \tilde{C} \sup_{n \in \mathbb{N}} \left(n^{-d} \right) < \infty \quad \text{for } d \geq 1 \text{ and}$$
$$\sup_{n \in \mathbb{N}} \mathcal{E}^{\mu, \ddot{\Gamma}_f}_{\text{int}}(u_n) \le \tilde{C} \sup_{n \in \mathbb{N}} \left(n^{2-d} d \right) < \infty \quad \text{for } d \geq 2.$$

Hence, Theorems 7.3.10 and 7.3.11 are also shown. □

We summarize the last three results:

Theorem 7.3.14. *Let $X = \mathbb{R}^d$, $d \geq 2$, and μ be as above. Then the set $\ddot{\Gamma}(\hat{\mathbb{R}}^d) \backslash \Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d)$ is $\mathcal{E}^\mu_{ext}$-, resp. $\mathcal{E}^\mu_{int}$- or $\mathcal{E}^\mu$,-exceptional. In particular, this holds for the set $\ddot{\Gamma}(\hat{\mathbb{R}}^d) \backslash \Gamma_f(\mathbb{R}^d)$.*

Proof. This is clear by the last three theorems. □

7.4 Diffusions on the cone

Our final goal is to obtain a diffusion on the cone $\mathbb{K}(\mathbb{R}^d)$. To that end, we have shown in Theorem 7.3.14 that the set of points in $\ddot{\Gamma}_f(\hat{\mathbb{R}}^d)$ that are not pinpointing is exceptional. Hence, the corresponding Markov process sits on the subset of pinpointing points in $\ddot{\Gamma}_f(\hat{\mathbb{R}}^d)$, which is denoted by $\Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d)$. This subset equals $\Gamma_f(\mathbb{R}^d)$ if $m(\mathbb{R}^d) < \infty$.

7.4.1 Diffusions on $\mathbb{K}(\mathbb{R}^d)$

By the next theorem, we will obtain diffusions on the cone $\mathbb{K}(\mathbb{R}^d)$ in the basic model framework (cf. also Corollary 7.4.5).

Theorem 7.4.1. *Let the conditions of Theorem 7.3.14 hold. Then the constructed diffusions describe motion of marks and positions on the cone* $\mathbb{K}(\mathbb{R}^d)$.

Proof. There exists an injective map that maps $\Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d)$ in $\mathbb{K}(\mathbb{R}^d)$. If $m(\mathbb{R}^d) < \infty$, then $\Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d) = \Gamma_f(\mathbb{R}^d)$ and the map is bijective. $\square$

Extrinsic motion on $\mathbb{K}(\mathbb{R}^d)$

Theorem 7.4.2. *Let $X = \mathbb{R}^d$, $d \geq 0$, and m be such that (7.0.1), (7.2.12) and (7.2.13) hold. (In particular, in the basic model framework this holds.) Let μ be a Gibbs perturbation of $\mathcal{G}_\theta$ w.r.t. a pair potential $0 \leq \phi \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ for which $(\phi)_{\mathbb{K}(\mathbf{X})}$ holds (cf. Subsection 5.3.1 and Definition 5.3.5). Then there exists a diffusion on $\mathbb{K}(\mathbb{R}^d)$ describing an extrinsic motion.*

More precisely (cf. Theorem 7.3.5), there exists a conservative diffusion process (i.e., a conservative strong Markov process with continuous sample paths)

$$\mathbf{M}_{ext}^{\mathbb{K}(\mathbb{R}^d)} = \Big(\mathbf{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t\geq 0}, (\mathbf{\Theta}_t)_{t\geq 0}, (\mathbf{X}(t))_{t\geq 0}, (\mathbf{P}_\gamma)_{\eta\in\mathbb{K}(\mathbb{R}^d)}\Big)$$

on $\mathbb{K}(\mathbb{R}^d)$ which is properly associated with $\big(\mathcal{E}_{ext}^{\mu,\mathbb{K}}, \mathcal{D}\big(\mathcal{E}_{ext}^{\mu,\mathbb{K}}\big)\big)$,[8] i.e., for all ($\mu$-versions) of $F \in L^2(\mathbb{K}(\mathbb{R}^d), \mu)$ and all $t > 0$ the function

$$\mathbb{K}(\mathbb{R}^d) \ni \eta \mapsto p_t F(\eta) := \int_\Omega F(\mathbf{X}(t)) d\mathbf{P}_\eta$$

is an $\mathcal{E}_{ext}^{\mu,\mathbb{K}}$-quasi-continuous version of $\exp(-tL_{ext}^{\mu,\mathbb{K}})F$, where $L_{ext}^{\mu,\mathbb{K}}$ is the generator of $\big(\mathcal{E}_{ext}^{\mu,\mathbb{K}}, \mathcal{D}\big(\mathcal{E}_{ext}^{\mu,\mathbb{K}}\big)\big)$ (cf. [MR92, Section I.2]). $\mathbf{M}$ is up to μ-equivalence unique (cf. [MR92, Theorem VI.6.4]). In particular, $\mathbf{M}$ is μ-symmetric (i.e., $\int G p_t F d\mu = \int F p_t G d\mu$ for all $F, G : \mathbb{K}(\mathbb{R}^d) \to \mathbb{R}$, $\mathcal{B}(\ddot{\Gamma}_f(\hat{X}))$-measurable) and has μ as an invariant measure.

Proof. By Theorem 7.2.20, the assumptions of Theorem 7.2.34 (and Proposition 7.2.33) are fulfilled.

To actually obtain continuous paths in $\mathbb{K}(\mathbb{R}^d)$, we show the following claim: If $\gamma_n \to \gamma$ in $(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), d_{\ddot{\Gamma}_f})$, then $\langle \gamma_n, sf(x)\rangle \to \langle \gamma, sf(x)\rangle$ for all

[8] This is the Dirichlet form on $\mathbb{K}(\mathbb{R}^d)$ that corresponds to $\big(\mathcal{E}_{\text{ext}}^{\mu,\ddot{\Gamma}_f}, \mathcal{D}\big(\mathcal{E}_{\text{ext}}^{\mu,\ddot{\Gamma}_f}\big)\big)$, i.e.,

$$\mathcal{E}_{\text{ext}}^{\mu,\mathbb{K}}(F)(\eta) := \mathcal{E}_{\text{ext}}^{\mu,\ddot{\Gamma}_f}\big(F(T^{-1}(\eta))\big),$$

where $T : \Gamma_f(\mathbb{R}^d) \to \mathbb{K}(\mathbb{R}^d)$ (cf. (3.1.2)). This holds because $\ddot{\Gamma}_f(\hat{\mathbb{R}}^d) \backslash \Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d)$ is exceptional.

$f \in C_0(\hat{\mathbb{R}}^d)$. We use the notation of Definition 7.2.7 to deduce this claim. There exists $k \in \mathbb{I}$ such that

$$f \leq \|f\|_\infty \mathbb{1}_{X_k}.$$

For all $\varepsilon > 0$, we find $I \in \mathcal{B}_c(\mathbb{R}_+)$ such that

$$d_f(\gamma \cap (I^c \times X_k), \varnothing) < \varepsilon.$$

Let $g_N \in C_0^\infty(\mathbb{R}_+)$ be such that $\mathbb{1}_{[\frac{1}{N},N]} \leq g_N$. By the vague convergence there exists N_0 such that for all $n \geq N_0$

$$|\langle f g_N, \gamma_n\rangle - \langle f g_N, \gamma\rangle| < \varepsilon.$$

By the d_f convergence, there exists $N_1 > N_0$ such that for all $n \geq N_1$

$$d_f(\gamma_n \cap (I \times X_k), \gamma \cap (I \times X_k)) < \varepsilon.$$

Hence, for all $n \geq N_1$, we get, using also Definition 7.2.7,

$$\begin{aligned}
&|\langle sf(x), \gamma_n\rangle - \langle sf(x), \gamma\rangle| \\
\leq &|\langle s g_N(s) f(x), \gamma_n\rangle - \langle s g_N(s) f(x), \gamma\rangle| + M(2C)^k \|f\|_\infty \\
&\quad \times (d_f(\gamma_n \cap (I^c \times X_k), \varnothing) + d_f(\gamma \cap (I^c \times X_k), \varnothing)) \\
< &\varepsilon + M(2C)^k \|f\|_\infty \Big(d_f(\gamma_n \cap (I^c \times X_k), \gamma \cap (I^c \times X_k)) \\
&\qquad + d_f(\gamma \cap (I^c \times X_k), \varnothing) + \varepsilon \Big) \\
< &\varepsilon + 3\varepsilon M(2C)^k \|f\|_\infty.
\end{aligned}$$

This shows the claim.

Therefore, the assertions follow by Theorem 7.4.1, which projects the diffusions sitting on $\Gamma_{\mathbb{K}}(\hat{\mathbb{R}}^d)$ (cf. Theorems 7.3.10 and 7.3.5) onto $\mathbb{K}(\mathbb{R}^d)$. □

Intrinsic Motion

We obtain a result for the intrinsic motion that is similar to Theorem 7.4.2.

Theorem 7.4.3. *Let $X = \mathbb{R}^d$, $d \geq 2$, and the assumptions of Theorem 7.4.2 hold. Then there exists an analoguos diffusion on $\mathbb{K}(\mathbb{R}^d)$ describing an intrinsic motion.*

Proof. The claim follows similarly to the proof of Theorem 7.4.2, where we replace the "extrinsic" arguments by their "intrinsic" counterparts. □

Joint motion of marks and positions

We obtain a result for the motion of marks and positions on $\mathbb{K}(\mathbb{R}^d)$ that is similar to Theorem 7.4.2.

Theorem 7.4.4. *Let $X = \mathbb{R}^d$, $d \geq 2$, and the assumptions of Theorem 7.4.2 hold. Then there exists a diffusion on $\mathbb{K}(\mathbb{R}^d)$ describing the joint motion of marks and positions.*

Proof. The result follows similarly to the proof of Theorem 7.4.2, where we use instead of the "extrinsic" arguments the corresponding "joint" ones. $\square$

Corollary 7.4.5. *Assume that we are in the basic model framework with $d \geq 2$. Then there exist extrinsic, intrinsic and joint diffusions on $\mathbb{K}(\mathbb{R}^d)$ describing the motion of marks and positions. In particular, there exists a diffusion describing the motion of the dense set $\tau(\eta_t) \in \mathbb{R}^d$, where $\eta_t \in \mathbb{K}(\mathbb{R}^d)$ for all $t \geq 0$.*

Proof. This follows from Theorems 7.4.2, 7.4.3 and 7.4.4, which hold in this case for $\mu = \mathcal{G}_\theta$. $\square$

7.4.2 Extension of Dirichlet forms on $\mathbb{K}(\mathbb{R}^d)$

We show a connection to Dirichlet forms considered in Section 6.3.

We fix $X = \mathbb{R}^d$, $d \geq 2$, and $m(dx) = dx$.

Theorem 7.4.6. *Under the assumptions of Theorem 7.4.2, we have*

$$\mathcal{F}C_b^\infty\left(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), \left\{\mathrm{id}_{\mathbb{R}_+} \otimes \varphi \middle| \varphi \in C_0^\infty(\mathbb{R}^d)\right\}\right) \subset \mathcal{D}\left(\mathcal{E}_{ext}^{\ddot{\Gamma}_f,\mu}\right).$$

In particular for the Lebesgue measure on $\mathbb{R}^d$, the Dirichlet form that corresponds to $\mathcal{E}_{ext}^{\mathcal{G}_\theta}$ over $\Gamma_f(\mathbb{R}^d)$ is extended by $\big(\mathcal{E}_{ext}^{\ddot{\Gamma}_f,\mathcal{P}_\theta}, \mathcal{D}(\mathcal{E}_{ext}^{\ddot{\Gamma}_f,\mathcal{P}_\theta})\big)$.

Proof. Similar as in the proof of Theorem 7.2.23, we approximate

$$F \in \mathcal{F}C_b^\infty\left(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), \left\{\mathrm{id}_{\mathbb{R}_+} \otimes \varphi \middle| \varphi \in C_0^\infty(\mathbb{R}^d)\right\}\right)$$

by elements $F_k \in \mathcal{F}C_b^\infty(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), C_0(\hat{\mathbb{R}}^d))$ in the $\big(\mathcal{E}_{\mathrm{ext},1}^{\ddot{\Gamma}_f,\mu}\big)$-norm.

Since $g_F \in C_b^\infty(\mathbb{R}^N)$ and because of the structure of $S_{\mathrm{ext}}^{\ddot{\Gamma}}$, it is sufficient to consider w.l.o.g. $F = g_F(\langle \cdot, \mathrm{id}_{\mathbb{R}_+} \otimes \varphi\rangle)$. We choose

$$F_k := g_F\left(\langle \cdot, f_k \cdot \mathrm{id}_{\mathbb{R}_+} \otimes \varphi\rangle\right), \quad \text{for } k \in \mathbb{N}$$

where the $f_k \in C_0^\infty(\mathbb{R}_+)$ such that

$$\begin{aligned}&\mathbb{1}_{[2^{-k},2^k]} \le f_k \le \mathbb{1}_{[2^{-k-1},2^{k+1}]} \quad \text{and}\\ &|f_k'(s)| \le 3\cdot 2^{k+1}\mathbb{1}_{[2^{-k-1},2^{-k}]}(s) + \mathbb{1}_{[2^k,2^{k+1}]}(s)\end{aligned}$$

Obviously, the F_k, $k \in \mathbb{N}$, converge pointwisely to F for $k \nearrow \infty$. We check that $(F_k)_{k\in\mathbb{N}}$ form a $\big(\mathcal{E}^{\ddot{\Gamma}_f,\mu}_{\mathrm{ext},1}\big)$ Cauchy sequence, i.e.,

$$\lim_{k,k'\to\infty}\left(\int_{\ddot{\Gamma}_f(\hat{\mathbb{R}}^d)} S^{\ddot{\Gamma}}_{\mathrm{ext}}(F_k - F_{k'})(\gamma)\mu(d\gamma) + \int_{\ddot{\Gamma}_f(\hat{\mathbb{R}}^d)} (F_k - F_{k'})^2\mu(d\gamma)\right) \stackrel{!}{=} 0. \tag{7.4.1}$$

We treat the $L^2(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d),\mu)$ part first. Let g_F be bounded by C. Then also $\sup_{l\in\mathbb{N}} ||F_l||_\infty < C$. Thus, we may apply Lebesgue's dominated convergence theorem and take the limit pointwisely. Since g is continuous, the convergence follows by

$$\langle\gamma, (\mathbb{1}_{\mathbb{R}_+} - f_k)\cdot \mathrm{id}_{\mathbb{R}_+}\otimes\varphi\rangle \le \langle\gamma, \mathbb{1}_{[2^{-k},2^k]^c}\,\mathrm{id}_{\mathbb{R}_+}\otimes\varphi\rangle \underset{k\to\infty}{\longrightarrow} 0. \tag{7.4.2}$$

Let us consider the first part in (7.4.1). Again we will use Lebesgue's dominated convergence theorem. Fix $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ and $M_\varphi < \infty$ such that $\varphi \le M_\varphi \mathbb{1}_\Delta$. We define

$$g_k(s) := f_k(s) + s f_k'(s) \quad \text{and note } |g_k'(s)| \le 7\mathbb{1}_{[2^{-k-1},2^{k+1}]}.$$

Since $S^{\ddot{\Gamma}}_{\mathrm{ext}}(F_k - F_{k'}) \le 2S^{\ddot{\Gamma}}_{\mathrm{ext}}(F_k) + 2S^{\ddot{\Gamma}}_{\mathrm{ext}}(F_{k'})$, it is sufficient to consider

$$\begin{aligned}S^{\ddot{\Gamma}}_{\mathrm{ext}}(F_k)(\gamma) =& g_F'\Big(\langle\gamma, f_k\cdot\mathrm{id}_{\mathbb{R}_+}\otimes\varphi\rangle\Big)^2 \cdot \int_{\hat{\mathbb{R}}^d} s_x\Big(g_k(s)\varphi(x)\Big)^2\gamma(d\hat{x})\\ \le& 49C^2M_\varphi^2\int_{\hat{\mathbb{R}}^d} s_x\mathbb{1}_\Delta(x)\gamma(d\hat{x}).\end{aligned}$$

Hence (cf. Theorem 5.2.10),

$$\int_{\ddot{\Gamma}_f(\hat{X})} S^{\ddot{\Gamma}}_{\mathrm{ext}}(F_k)\mu(d\gamma) \le 49C^2M_\varphi^2\int_{\hat{\mathbb{R}}^d} s\mathbb{1}_\Delta(x)\lambda_\theta\otimes m(d\hat{x}) < \infty.$$

Thus it is sufficient to prove the convergence pointwisely. Since g is continuous, the pointwise convergence follows by (7.4.2) because

$$\begin{aligned}&S^{\ddot{\Gamma}}_{\mathrm{ext}}(F_k - F_{k'})(\gamma)\\ &= \int_{\hat{\mathbb{R}}^d}\bigg(\Big(g_F'\big(\langle\gamma, f_k\otimes\varphi\rangle\big) - g_F'\big(\langle\gamma, f_{k'}\cdot\mathrm{id}_{\mathbb{R}_+}\otimes\varphi\rangle\big)\Big)g_k(s_x)\varphi(x)\\ &\qquad - g_F'\big(\langle\gamma, f_{k'}\cdot\mathrm{id}_{\mathbb{R}_+}\otimes\varphi\rangle\big)\Big(g_{k'}(s_x) - g_k(s_x)\Big)\varphi(x)\bigg)^2 s_x\gamma(d\hat{x}).\end{aligned}$$

Thus, the claim follows. □

Theorem 7.4.7. *Under the assumptions of Theorem 7.4.2, we have*

$$\mathcal{F}C_b^\infty\left(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), \left\{\mathrm{id}_{\mathbb{R}_+}\otimes\varphi \middle| \varphi \in C_0^\infty(\mathbb{R}^d)\right\}\right) \subset \mathcal{D}\left(\mathcal{E}_{int}^{\ddot{\Gamma}_f,\mu}\right).$$

In particular for the Lebesgue measure on $\mathbb{R}^d$*, the Dirichlet form that corresponds to* $\mathcal{E}_{int}^{\mathcal{G}_\theta}$ *on* $\Gamma_f(\mathbb{R}^d)$ *is extended by* $\big(\mathcal{E}_{int}^{\ddot{\Gamma}_f,\mathcal{P}_\theta}, \mathcal{D}(\mathcal{E}_{int}^{\ddot{\Gamma}_f,\mathcal{P}_\theta})\big)$.

Proof. This follows by adapting the arguments of the proof of Theorem 7.4.6 in an obvious way and using that there exists $M < \infty$ and $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ such that $\mathbb{R}^d \ni x \mapsto |\nabla^{\mathbb{R}^d}\varphi(x)|_{T_x(\mathbb{R}^d)} \le M\mathbb{1}_\Delta$. □

Theorem 7.4.8. *Under the assumptions of Theorem 7.4.2, we have*

$$\mathcal{F}C_b^\infty\left(\ddot{\Gamma}_f(\hat{\mathbb{R}}^d), \left\{\mathrm{id}_{\mathbb{R}_+}\otimes\varphi \middle| \varphi \in C_0^\infty(\mathbb{R}^d)\right\}\right) \subset \mathcal{D}\left(\mathcal{E}^{\ddot{\Gamma}_f,\mu}\right).$$

In particular for the Lebesgue measure on $\mathbb{R}^d$*, the Dirichlet form that corresponds to* $\mathcal{E}^{\mathcal{G}_\theta}$ *on* $\Gamma_f(\mathbb{R}^d)$ *is extended by* $\big(\mathcal{E}^{\ddot{\Gamma}_f,\mathcal{P}_\theta}, \mathcal{D}(\mathcal{E}^{\ddot{\Gamma}_f,\mathcal{P}_\theta})\big)$.

Proof. This follows by combining the arguments of Theorems 7.4.6 and 7.4.7. □

Appendix A

Spaces of measures

We briefly outline some general facts concerning measurable spaces and the space of finite Radon measures that we use in our considerations.

In Section A.1, we present some facts about standard Borel spaces. (Borel space in the notion of [Pat67] are measurable spaces.)

In Section A.2, we collect some results concerning the space $\mathcal{M}^+(X)$ of all non-negative *finite* Radon measures over X, and also the space of probability and the one of general finite Radon measures. Corollary A.2.10 and Theorem A.2.11 allow us to prove the quasi-regularity in Section 7.1.

A.1 Kuratowski's theorem

Definition A.1.1 (see [Pat67, p.1])**.** *Let Y be a topological space. The* Borel σ-algebra $\mathcal{B}_Y$ *of Y is defined to be the smallest σ-algebra of subsets of Y which contains all the open subsets of Y.*

Definition A.1.2 (see [Pat67, Definition VI.1.2, p.6])**.** *A* Borel space *(or a measurable space) $(Z, \mathcal{B})$ is a pair, where Z is an abstract set and $\mathcal{B}$ is a σ-algebra of subsets of Z.*

Definition A.1.3. *A set or a class of sets $\mathcal{D}$ is called* denumerable*, if there exists a surjective map $j : \mathbb{N} \to \mathcal{D}$.*

Definition A.1.4 (see [Pat67, Definition V.2.1, p.132])**.** *A Borel space $(Z, \mathcal{B})$ is said to be* countably generated *if there exists a denumerable class $\mathcal{D} \subset \mathcal{B}$ such that $\mathcal{D}$ generates $\mathcal{B}$. $(Z, \mathcal{B})$ is called* separable *if it is countably generated and for each $z \in Z$, the single point set $\{z\} \in \mathcal{B}$.*

Definition A.1.5 (see [Pat67, Definition V.2.2, p.133])**.** *A countably generated Borel space $(Z, \mathcal{B})$ is called* standard *if there exists a complete separable metric space Y such that the σ-algebras $\mathcal{B}$ and $\mathcal{B}_Y$ are σ-isomorphic.*

Theorem A.1.6 (see [Pat67, Theorem V.2.2,p.133]). *If the Borel space $(Z, \mathcal{B})$ is countably generated, then there exists a separable metric space Y such that $\mathcal{B}$ and $\mathcal{B}_Y$ are σ-isomorphic.*

If Z is a separable metric space and $\mathcal{B}_Z$ the class of Borel subsets of Z, then $(Z, \mathcal{B}_Z)$ is standard if and only if Z is a Borel set in some complete separable metric space Z in which Z can be embedded as a topological subspace.

In this case Z is a Borel set in every complete separable metric space in which it is a topological subspace.

Theorem A.1.7 (Kuratowski, see [Pat67, Theorem V.2.4, p.135]). *Let $(Z, \mathcal{B})$ be a standard Borel space, $(Y, \mathcal{C})$ be a countably generated one and ϕ a one-to-one map of Z into Y which is measurable.*

Then $Y' := \phi(Z) \in \mathcal{C}$ and ϕ is a Borel isomorphism between the Borel spaces $(Z, \mathcal{B})$ and $(Y', \mathcal{C}_{Y'})$, where $\mathcal{C}_{Y'} := \{A \cap Y' | A \in \mathcal{C}\}$ is the trace σ-algebra.

A.2 Properties of Radon measures

In this section we only consider *finite* measures. The main property that we prove in this section is that the space of all *finite* non-negative Radon measures over a complete, separable metric space X is Polish (cf. Corollary A.2.10) and that its metric can be described by a supremum of countably many functions (cf. Theorem A.2.11).

To that end we collect some results concerning spaces of (finite) measures over metric spaces. A good reference is [Bog07a, Bog07b], whose setting even includes general topological spaces. Note that measures in the sense of [Bog07b] can be negative and are finite, cf. [Bog07a, 1.3.2. Definition]. We call those (possibly negative) measures finite "signed measures".

Let X be a topological space.

Definition A.2.1 (see [Bog07b, Definitions 7.1.1., 7.1.5, 7.2.1]).

1. *A countably additive (finite) signed measure on the Borel σ-algebra $\mathcal{B}(X)$ is called a* Borel measure *on X. By $\mathcal{M}_{\mathcal{B}}(X)$ we denote the set of all Borel measures.*

2. *A Borel measure μ on X is called a finite* Radon measure *if for every $B \in \mathcal{B}(X)$ and $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset B$ such that*

$$|\mu|(B \backslash K_\varepsilon) < \varepsilon.$$

By $\mathcal{M}_r(X)$ we denote the set of all Radon measures.

3. *A Borel measure μ on a topological space X is called* τ-additive*if for every increasing net of open sets $(U_\lambda)_{\lambda\in\Lambda}$ in X, one has the equality*

$$|\mu|\left(\bigcup_{\lambda\in\Lambda} U_\lambda\right) = \lim_{\lambda} |\mu|\left(U_\lambda\right). \tag{A.2.1}$$

If (A.2.1) is fulfilled for all nets with $\bigcup_\lambda U_\lambda = X$, then μ is called τ_0-additive. *By $\mathcal{M}_\tau(X)$ we denote the set of all τ-additive measures.*

Moreover, let us denote by $\mathcal{M}^+_{\mathcal{B}}(X)$, $\mathcal{M}^+_r(X)$, $\mathcal{M}^+_\tau(X)$ the corresponding classes of non-negative measures.

Definition A.2.2. *A nonnegative set function μ defined on some system of $\mathcal{A}$ of subsets of a topological space is called* regular *if for every $A \in \mathcal{A}$ and every $\varepsilon > 0$, there exists a closed set F_ε such that $F_\varepsilon \subset A$, $A\backslash F_\varepsilon \in \mathcal{A}$ and $\mu\left(A\backslash F_\varepsilon\right) < \varepsilon$.*

An additive set function μ of bounded variation on an algebra is called regular *if its total variation $|\mu|$ is regular.*

Remark A.2.3. *We emphasize that in the setting that we treat we do not have to distinguish them (cf. Theorem A.2.5).*

Definition A.2.4 (see [Bog07b, 6.1.2 Definition])**.** 1. *X is called* Hausdorff *if every two distinct points in X possess disjoint neighborhoods.*

2. *A Hausdorff space X is called* regular *if, for every point $x \in X$ and every closed set $Z \subset X$ not containing x, there exist disjoint open sets U and V such that $x \in U$, $Z \subset V$.*

3. *A Hausdorff space X is called* completely regular *if, for every point $x \in X$ and every closed set $Z \in X\backslash\{x\}$, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ and $f(z) = 0$ for all $z \in Z$.*

Theorem A.2.5 (see [Bog07b, 7.1.17 Theorem and 7.2.2. Proposition])**.** *Let X be a metric space. Then every Borel measure μ on X is regular. If X is complete and separable, then the measure μ is Radon.*

Proposition A.2.6. 1. *Every Radon measure is τ-additive.*

2. *Every τ-additive measure on a regular space is regular. In particular, every τ-additive measure on a compact space is Radon.*

3. *Every Borel measure on a separable metric space X is τ-additive.*

From now on X shall be a metric space. Let us introduce a norm on $\mathcal{M}_\tau(X)$ that will turn out to be consistent with the weak topology.

Definition A.2.7 (see [Bog07b, Section 8.3, P.181]). *Let (X, ρ_X) be a metric space. We define the* Kantorovich-Rubinshstein norm *on $\mathcal{M}_\tau(X)$ via*

$$\|\mu\|_0 := \sup\left\{\int_X f d\mu \Big| f \in Lip_1(X), \sup_{x\in X}|f(x)| \le 1\right\}, \tag{A.2.2}$$

where the space of Lipschitz functions with Lipschitz constant 1 is defined by

$$Lip_1(X) := \big\{f : X \to \mathbb{R} \big| |f(x) - f(y)| \le \rho_X(x, y),\ \forall x, y \in X\big\}.$$

By ρ_o we denote the corresponding metric, i.e.

$$\rho_0(\eta, \nu) = \|\eta - \nu\|_0 \quad \text{for all } \eta, \nu \in \mathcal{M}_\tau(X).$$

Theorem A.2.8 (see [Bog07b, 8.3.2 Theorem]). *The topology generated by $\|\cdot\|_0$ coincides with the weak topology on the set $\mathcal{M}^+_\tau(X)$ of nonnegative τ-additive measures. In addition, on the set $\mathcal{P}_\tau$ of probability τ-additive measures the weak topology is generated by the* Lèvy-Prohorov metric*:*

$$d_P(\mu, \nu) = \inf\{\varepsilon > 0 : \nu(B) \le \mu(B^\varepsilon) + \varepsilon,\ \mu(B) \le \nu(B^\varepsilon) + \varepsilon,\ \forall B \in \mathcal{B}(X)\},$$

where $B^\varepsilon := \{x | dist(x, B) < \varepsilon\}$.

Theorem A.2.9 (see [Bog07b, 8.9.4 Theorem]). *Let X be completely regular.*

1. *The space $\mathcal{M}^+_\tau(X)$ equipped with the weak topology is metrizable iff X is metrizable. In that case, the metrizability of $\mathcal{M}^+_\tau$ by a complete metric is necessary and sufficient for the metrizability of X by a complete metric.*

2. *If X is separable, then $\mathcal{M}_\tau(X)$, as well as $\mathcal{M}^+_\tau(X)$, is separable in the weak topology.*

The same assertions hold, if we replace $\mathcal{M}^+_\tau(X)$ by $\mathcal{P}_\tau(X)$ which denotes the space of all τ-additive measures over X with total mass 1.

Corollary A.2.10 (see also [Sch73, Part II, Appendix, Thm. 7, p.385]). *Let X be a complete, separable metric space. Then the set $\mathcal{M}^+(X)$ of nonnegative (finite) Radon measures, which coincides with the one of nonnegative Borel and τ-additive ones in this case, together with the metric ρ_o is a complete separable metric space.*

Theorem A.2.11. *Let X be a complete, separable metric space. Then there exists a* countable *sequence $(\phi_i)_{i\in\mathbb{N}}$ of uniformly continuous functions on X with $\|\phi_i\|_\infty \le 1$ and*

$$\rho_0(\eta, \eta') = \sup_i \langle \phi_i, \eta - \eta'\rangle \quad \forall \eta, \eta' \in \mathcal{M}^+_\tau(X).$$

Proof. By the separability of X there exists a countable dense set $\mathbb{S}_X := \{x_i \in X | i \in \mathbb{N}\}$. For all $r \in \mathbb{Q}$, for all $\{q_i | i \in \mathbb{N}\} \subset \mathbb{Q}$ and for all $\{x_i | i \in \mathbb{N}\} \subset \mathbb{S}_X$ such that there exists $f \in \mathrm{Lip}_1(X)$ with $\|f\|_\infty \leq 1$ and $|f(x_i) - q_i| < r$ we choose $\phi = \phi_{r,(x_i,q_i)_{i\in\mathbb{N}}} = f \in \mathrm{Lip}_1(X)$ and denote the countable union of all these by

$$\Phi := \{\phi_i | i \in \mathbb{N}\}.$$

We have that $\forall \eta, \eta' \in \mathcal{M}^+(X)$, $\forall \delta > 0$ $\exists f_\delta \in \mathrm{Lip}_1(X) : \|f\|_\infty \leq 1$ and

$$\rho_0(\eta, \eta') - \delta \leq \int_X f_\delta(x)(\eta - \eta')(dx).$$

Moreover, there exists $\phi_\delta \in \Phi$:

$$|f_\delta(x_i) - \phi_\delta(x_i)| \leq \frac{\delta}{4}, \quad \forall x_i \in \mathbb{S}_X.$$

We define for all $j \in \mathbb{N}$

$$A_j := B_{\delta/4}(x_j) \cap \bigcup_{i=1,\dots,j-1} A_i$$

and obtain, using in addition the Lipschitz continuity,

$$\begin{aligned}
\rho_0(\eta, \eta') - \delta \leq &\sum_{j\in\mathbb{N}} \Bigg(\int_{A_j} \underbrace{|f_\delta(x) - f_\delta(x_j)|}_{\leq \delta/4} + \underbrace{|f_\delta(x_j) - \phi_\delta(x_j)|}_{\leq \delta/4} |\eta - \eta'|(dx) \\
&+ \int_{A_j} \underbrace{|\phi_\delta(x_j) - \phi_\delta(x)|}_{\leq \delta/4} |\eta - \eta'|(dx) + \int_{A_j} \phi_\delta(x)(\eta - \eta')(dx) \Bigg) \\
\leq &\frac{3}{4}\delta \|\eta - \eta'\|_{\mathrm{tv}} + \int_X \phi_\delta(x)(\eta - \eta')(dx) \\
\leq &\frac{3}{4}\delta \|\eta - \eta'\|_{\mathrm{tv}} + \sup_{\phi_i \in \Phi} \int_X \phi_i(x)(\eta - \eta')(dx) \\
\leq &\frac{3}{4}\delta \|\eta - \eta'\|_{\mathrm{tv}} + \rho_0(\eta, \eta'),
\end{aligned}$$

where we denote for each $\eta \in \mathcal{M}_\tau(X)$ by $\|\eta\|_{\mathrm{tv}}$ its total variation norm, which is finite because we treat finite non-negative measures η. Hence, we obtain taking the limit $\delta \searrow 0$ that

$$\begin{aligned}
\rho_0(\eta, \eta') \leq &\lim_{\delta \searrow 0} \frac{3}{4}\delta \|\eta - \eta'\| + \sup_{\phi_i \in \Phi} \int_X \phi_i(x)(\eta - \eta')(dx) \\
= &\sup_{\phi_i \in \Phi} \int_X \phi_i(x)(\eta - \eta')(dx) \leq \rho_0(\eta, \eta').
\end{aligned}$$

□

Remark A.2.12. *If (under stronger assumptions on the space X) the functions in Theorem A.2.11 could be chosen to be continuously differentiable, then the arguments of Sections 7.1 would even yield an associated diffusion for the Dirichlet form $(\mathcal{E}^{\mathcal{G}_\theta}_{int}, \mathcal{D}(\mathcal{E}^{\mathcal{G}_\theta}_{int}))$ and not only for $(\mathcal{E}^{\mathcal{G}_\theta}_{ext}, \mathcal{D}(\mathcal{E}^{\mathcal{G}_\theta}_{ext}))$.*[1]

[1] If one considers Fleming-Viot processes, then one only considers movement w.r.t. marks (cf. e.g. [RS95, Section 4(c), p.31]).

Part IV

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List of Symbols

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