Variational Convergence of Nonlinear Partial Differential Operators on Varying Banach Spaces

Dissertation zur Erlangung des Doktorgrades
der Fakultät für Mathematik
der Universität Bielefeld

vorgelegt von
Jonas M. Tölle
im Januar 2010

Gedruckt auf alterungsbeständigem Papier nach DIN-ISO 9706.

Variational Convergence of Nonlinear Partial Differential Operators on Varying Banach Spaces

Dissertation zur Erlangung des Doktorgrades
der Fakultät für Mathematik
der Universität Bielefeld

vorgelegt von
Dipl. Math. Jonas M. Tölle
aus Bielefeld
im Januar 2010

1. Gutachter: Prof. Dr. Michael Röckner
2. Gutachter: Prof. Dr. Yuri G. Kondratiev

Tag der mündlichen Prüfung: 5. Februar 2010

Abstract

In this doctoral thesis, a new approach towards variational convergence of quasi-linear monotone partial differential operators is elaborated. To this end, we analyze more explicitly the so-called Kuwae-Shioya-convergence of metric spaces in the case of Banach spaces. For the first time, weak Banach space topologies are included. We achieve the objective to be able to formulate reasonable (topological) statements about convergence of vectors, functionals or operators such that each element of a convergent sequence lives in/on another distinct Banach space. Banach space-convergence is considered a natural generalization of Gromov-Hausdorff-convergence of compact metric spaces. The associated theory is developed here completely and justified by several examples. Among other things, we are able to consider varying $L^{p_n}(\Omega_n, \mathscr{F}_n, \mu_n)$-spaces such that the measurable space $(\Omega_n, \mathscr{F}_n)$ as well as the measure μ_n as well as the degree of integrability p_n varies for positive integers n, and such that the limit $n \to \infty$ is given sense.

Inside the framework of varying spaces, we show that a number of classical results on variational convergence still hold. Explicit applications are given for the equivalence of so-called Mosco-convergence of convex functionals and the strong-graph-convergence of the associated subdifferential operators. In the case of abstract L^p-spaces, we prove an elaborate result yielding a general transfer-method that enables us to carry over classical results (for one fixed space) to the case of varying spaces. More precisely, we construct isometries that respect the asymptotic topology of the varying Banach spaces and allow us to transform back to one fixed Banach space.

We are considering four types of quasi-linear partial differential operators mapping a Banach space X to its dual space X^*. All of these operators are characterized completely via variational methods by lower semi-continuous convex functionals on a Banach space V embedded properly into X. As operators to be approximated, we present the weighted (non-homogeneous) Φ-Laplacian in $\mathbb{R}^d$, the weighted p-Laplacian in $\mathbb{R}^d$, the 1-Laplacian with vanishing trace in a bounded domain, and the generalized porous medium resp. fast diffusion operator in an abstract measure space. When taking the Mosco-approximation of their energies, we generally vary the weights (measures). In the second and third case, p is also varied. When dealing with approximations of such kind, varying spaces occur naturally. In the theory of homogenization, the special case of two-scale convergence has already been being employed for some time.

Furthermore, we develop an alternative approach towards weighted p-Sobolev spaces of first order, which enables us to consider weights in a class different from the Muckenhoupt class. We prove a new result on density of smooth functions in weighted p-Sobolev spaces, which is known and well-studied as "Markov uniqueness" for $p = 2$. This problem is also known as "$H = W$", that is, the coincidence of the strong and the weak Sobolev space. With the help of this result, we are able to identify the Mosco-limit of weighted p-Laplace operators.

Zusammenfassung

In dieser Dissertation wird ein neuer Zugang zur Variationskonvergenz von quasi-linearen monotonen partiellen Differenzialoperatoren erarbeitet. Zu diesem Zwecke wird die so genannte Kuwae-Shioya-Konvergenz von metrischen Räumen genauer im Banach-Raum-Fall untersucht. Erstmals werden schwache Banach-Raum-Topologien mit eingebunden. Wir erfüllen das Ziel, sinnvolle (topologische) Aussagen über Konvergenz von Vektoren, Funktionalen und Operatoren treffen zu können, so dass jedes Element einer konvergenten Folge in oder auf einem ausgezeichneten, von den anderen verschiedenen Banach-Raum definiert ist. Banach-Raum-Konvergenz wird als natürliche Verallgemeinerung der Gromov-Hausdorff-Konvergenz von kompakten metrischen Räumen angesehen. Die zugehörige Theorie wird von uns vollständig entwickelt und mit etlichen Beispielen gerechtfertigt. Wir sind unter anderem in der Lage, variierende $L^{p_n}(\Omega_n, \mathscr{F}_n, \mu_n)$-Räume zu betrachten, so dass sowohl der messbare Raum $(\Omega_n, \mathscr{F}_n)$, als auch das Maß μ_n sowie der Integrierbarkeitsgrad p_n für natürliche Zahlen n variieren, und so dass der Grenzübergang $n \to \infty$ sinnvoll ist.

Im Rahmen der variierenden Räume zeigen wir, dass einige klassische Resultate über Variationskonvergenz weiterhin gelten. Genaue Anwendung für konkrete Operatoren findet die Äquivalenz der so genannten Mosco-Konvergenz von konvexen Funktionalen und der Konvergenz der zugehörigen Subdifferenzialoperatoren im starken Graphen Sinne. Im Fall von abstrakten L^p-Räumen beweisen wir ein aufwändiges Resultat, in dem Isometrien konstruiert werden, welche die asymptotische Topologie der variierenden Banach-Räume respektieren und es erlauben, eine Folge von Banach-Räumen auf einen festen Banach-Raum zurückzutransformieren.

Wir betrachten vier verschiedene Typen von quasi-linearen partiellen Differentialoperatoren, welche einen Banach-Raum X in dessen Dualraum X^* abbilden. Sämtliche dieser Operatoren werden vollständig durch variationelle Methoden mittels unterhalbstetiger konvexer Funktionale auf einem in X echt eingebetteten Banach-Raum V beschrieben. Als zu approximierende Operatoren präsentieren wir den gewichteten (nicht-homogenen) Φ-Laplace Operator in $\mathbb{R}^d$, den gewichteten p-Laplace Operator in $\mathbb{R}^d$, den 1-Laplace Operator mit verschwindender Spur in einer beschränkten Domäne und den verallgemeinerten poröse Medien- bzw. schnelle Diffusions-Operator in einem abstrakten Maßraum. Bei der Mosco-Approximation derer Energien werden generell die Gewichte (Maße) und im zweiten und dritten Fall p variiert. Bei Approximationen dieser Art treten variierende Räume natürlicherweise auf. In der Theorie der Homogenisation findet dies bereits seit einiger Zeit in dem Spezialfall der Zwei-Skalen-Konvergenz Anwendung.

Weiterhin entwickeln wir einen alternativen Zugang zu gewichteten p-Sobolev-Räumen der ersten Ordnung, der uns eine Klasse von Gewichten einsetzen lässt, die sich von der Muckenhoupt-Klasse unterscheidet. Wir zeigen ein neues Resultat über Dichtheit glatter Funktionen in gewichteten p-Sobolev-Räumen, welches für $p = 2$ als „Markoff-Eindeutigkeit“ bekannt und wohlstudiert ist. Dieses Problem ist auch als „$H = W$“ bekannt; das Zusammenfallen des starken und schwachen Sobolev-Raumes. Mit Hilfe dieses Resultats sind wir in der Lage, den Mosco-Grenzwert von gewichteten p-Laplace Operatoren zu identifizieren.

Acknowledgements

First of all, I would like to express my gratitude to Prof. Dr. Michael Röckner, who is the supervisor of this doctoral thesis and to whom I am deeply indebted. Without his personal support and kind patience, motivating discussions, clear insight into the problems involved and formulation of precise questions, this work would not have been possible. I can hardly stress enough how grateful I am for the fantastic opportunity to be a part of the Beijing–Bielefeld Graduate College, to study and do research both in Germany and China and to have had the possibility to attend many interesting workshops, conferences and summer schools all over the world.

For advice, support and hospitality during my stays in Beijing and Wuhan, I would like to thank my "Chinese-side advisor" Prof. Dr. Ma Zhi-Ming. Furthermore, I would like to thank the second referee of this work, Prof. Dr. Yuri G. Kondratiev. For taking their time for my defense, I would like to thank Prof. Dr. Dr. h.c. Claus Michael Ringel and Prof. Dr. Wolf-Jürgen Beyn.

For fruitful discussions and proofreading the manuscript, a great thanks goes to Dr. Oleksandr Kutovyi. For offering a number of important remarks, I would like to thank the anonymous referee of my preprint from 2007. For proofreading my posters, I would like to thank Prof. Dr. Ludwig Streit.

For helpful assistance and enduring kindness, I would like to thank the faculty administrators Nicole Zimmermann, Gaby Windhorst, Nadja Epp, Hanne Litschewsky, Margret Meyer-Heidemann and Waltraud Bartsch. Also, a big thanks goes to Sven Wiesinger for his continuing and by all means incredible efforts. For guaranteeing a pleasant stay in Beijing, I would like to thank the staff member of the Chinese Academy of Sciences, Zhai Xiaoyun.

For love, unshaken support and constant encouragement, I am deeply grateful to my family, in particular, to my parents Soile and Norbert, to my brother Felix and to my cousin Benjamin.

For infinite helpfulness and hospitality in China, many inspiring discussions about Chinese culture and mathematics and their friendship, I am kindly asking Jin Peng, Liu Wei and Ouyang Shun-Xiang to accept my best thanks.

I am gratefully paying tribute to my friends and companions Angélica Yohana Pachón Pinzón, Felipe Torres Tapia, Łukasz Derdziuk, Nâzım Hikmet Tekmen, Eko Nugroho, Sven Struckmeier, who visited 武當山 with me.

Thanks to Christiane for sending me a motivating postcard from each conference she has visited, and thanks to Christin for her backup and invaluable advice. Also, I am grateful to Michael for lending an ear or two from time to time in our kitchen. Last, not least, thanks to Verena for her excellent company in a laborious year 2009.

Financial support by the DFG International Graduate College 1132 "Stochastics and Real World Models" Beijing–Bielefeld, the Graduate College of the Chinese Academy of Sciences, Beijing, and the Faculty of Mathematics, Universität Bielefeld, are gratefully acknowledged.

Jonas M. Tölle

Bielefeld, January 2010

Erratum

By the time of the defense of this doctoral thesis (February 5th, 2010), it was noticed by the first referee, Prof. Dr. Michael Röckner, that some parts of Section 8.3 contain wrong reasoning. It has also been asserted in the official report about this thesis. We agree that our proof of Theorems 8.24 and 8.25 relies on wrong conclusions. It is an open question whether the claimed result still holds true. We are currently working on a new corrected proof.

Jonas M. Tölle

Bielefeld, February 2010

Update

This version of the book contains a completely rewritten Section 8.3. Also, the proof of Lemma 3.19 and some typographical errors have been corrected.

Jonas M. Tölle

Bielefeld, August 2010

Contents

1 Introduction

The contents of this doctoral thesis touch different fields of functional analysis, PDE theory and topology. Our central interests are variational convergence (in the applications) and approximation (in the theoretical part). We also present a result on smooth approximation in first order weighted p-Sobolev spaces, which is connected to variational convergence.

Firstly, our aim was to study convergence of (nonlinear) second order quasi-linear partial differential operators on Banach spaces. Secondly, we were aimed at developing a general theory of variational convergence on varying Banach spaces, encompassing earlier approaches, but with strong orientation to new applications. In our analysis of partial differential operators, we restrict ourselves to the situation of maximal monotone graphs. In variational calculus, such kinds of operators are usually attached canonically to a Banach space, as e.g. the (Dirichlet) Laplacian $-\Delta$ is attached to the Sobolev space $H_0^{1,2}$. It seems quite plausible, that a reasonable concept of convergence for partial differential operators, which "encode" different types of partial differential equations, should provide the convergence of solutions. Within variational calculus, a monotone operator is usually representable by an energy functional or a variational kernel. In turn, the solutions of the corresponding (abstract) Cauchy equation are represented by the associated (nonlinear) semigroup (or flow) and the spectral properties can be encoded into the associated (nonlinear) resolvent. Of course, for sequences of continuous operators, pointwise (strong Banach space) convergence is sufficient for the convergence of the solutions to the associated (abstract) operator equations. Failing to be continuous or even to be single-valued, another concept of convergence is needed for monotone graphs. For several reasons, strong graph convergence (also called G-convergence) is the right choice. By definition, if an operator is represented by a graph $\Gamma \subset X \times X^*$ (and so for a sequence of graphs $\{\Gamma_n\}$), where (X, X^*) is a dual pair of Banach spaces, then $\Gamma_n \xrightarrow[n\to\infty]{G} \Gamma$ in the *strong graph sense* if for each pair $[x, y] \in \Gamma$ there is a sequence of pairs $\{[x_n, y_n]\}$ with $[x_n, y_n] \in \Gamma_n$ such that

$$\|x_n - x\|_X \to 0 \quad \text{and} \quad \|y_n - y\|_{X^*} \to 0.$$

The latter is equivalent to the so-called Painlevé-Kuratowski convergence of graphs whenever the graphs are maximal monotone, see Definition 7.32 and Lemma 7.36. Its use dates back to Tosio Kato's "generalized convergence" [Kat66], and probably back further. The celebrated Trotter-Neveu-Kato Theorem (in the linear case) and the Brézis-Attouch Theorem (in the maximal monotone case) prove that strong graph convergence is exactly the right notion, if one demands equivalent strong (pointwise) convergence of resolvents, semigroups and Yosida approximations. See e.g. [Tro58, Kat66, Bré73, Rei82, Att84].

As the main feature, the solutions of the associated Cauchy equations converge if and only if the partial differential operators converge in the strong graph sense. In this work, we are able to generalize to the case that the pairs of Banach spaces "vary along", i.e., $(X_n, X_n^*) \to (X, X^*)$ in the sense specified below. In other words, the operator equation is solved in another Banach space for each element of a sequence (in the "natural" Banach space to a particular equation), and still, the solutions converge in a varying-Banach-space-sense, see Theorems 7.23 and 7.24.

As one of the main results of this work (see Chapter 8, Corollaries 8.8, 8.14, 8.26, 8.39), we are able to prove the following strong graph convergence (G-convergence) of nonlinear monotone operators (where the representing formulae are heuristic):

1. Let w_n, w be weights, let φ be a gauge function. Then
$$\operatorname{div}\left[w_n\varphi(|\nabla u|)\operatorname{sign}(\nabla u)\right] \to \operatorname{div}\left[w\varphi(|\nabla u|)\operatorname{sign}(\nabla u)\right],$$
if $w_n \to w$ weakly in L^1_{loc}.

2. Let w_n, w be weights, let $\{p_n\} \subset (1,\infty)$, $1 < p < \infty$. Then
$$\operatorname{div}\left[w_n|\nabla u|^{p_n-2}\nabla u\right] \to \operatorname{div}\left[w|\nabla u|^{p-2}\nabla u\right],$$
if $p_n \to p$ in $\mathbb{R}$ and $w_n \to w$ weakly in L^1_{loc}.

3. Let $\{p_n\} \subset (1,\infty)$. Then
$$\operatorname{div}\left[|\nabla u|^{p_n-2}\nabla u\right] \to \operatorname{div}\left[\frac{\nabla u}{|\nabla u|}\right],$$
if $p_n \to 1$ in $\mathbb{R}$.

4. Let μ_n, μ be σ-finite measures. Let $L_n^{\mu_n}, L^\mu$ be linear dissipative (second order partial or pseudo differential) operators, which are self-adjoint in $L^2(\mu_n)$, $L^2(\mu)$ resp. Let $\mathscr{F}_n^*$, $\mathscr{F}^*$ resp. be their abstract Green spaces. Denote by $\overline{L}_n^{\mu_n} : \mathscr{F}_n \to \mathscr{F}_n^*$, $\overline{L}^\mu : \mathscr{F} \to \mathscr{F}^*$ resp. the canonical extensions of the operators $L_n^{\mu_n}, L^\mu$ resp. Let φ be a gauge function. Then
$$\overline{L}_n^{\mu_n}\left[\varphi(|u|)\operatorname{sign}(u)\right] \to \overline{L}^\mu\left[\varphi(|u|)\operatorname{sign}(u)\right],$$
if $\mu_n \to \mu$ weakly in the sense of measures and $\overline{L}_n^{\mu_n} \to \overline{L}^\mu$ strongly.

Other conditions, needed in order to ensure the above convergences, are also imposed, see Theorems 8.4, 8.12 and 8.38. Operator no. 4 is a generalization of the well-known porous medium and fast diffusion operators (indeed, to get the classical porous medium equation, set $L^\mu = -\Delta$ on a bounded domain Ω, $\mu = \mathrm{d}x$, $\varphi(t) = |t|^{p-2}t$, $p > 2$).

As can be seen from the formulae of our four main applications above, each operator in the approximating sequence is defined on another Banach space. This is one of the new aspects of our work. In particular, we have to give sense to convergence "along"

$$
\begin{gathered}
L^{\Phi}(\mathbb{R}^d; w_n\,\mathrm{d}x) \to L^{\Phi}(\mathbb{R}^d; w\,\mathrm{d}x),\\
L^{p_n}(\mathbb{R}^d; w_n\,\mathrm{d}x) \to L^{p}(\mathbb{R}^d; w\,\mathrm{d}x),\\
L^{p_n}(\Omega; \mathrm{d}x) \to L^{1}(\Omega; \mathrm{d}x),\\
\mathscr{F}_n^* \to \mathscr{F}^*,
\end{gathered}
$$

respectively, where $\Phi(x) := \int_0^x \varphi(t)\,\mathrm{d}t$ and L^{Φ} is an Orlicz space. Dual spaces have to be treated, too. For this reason, we had to study (and develop) the theory of varying Banach spaces. Note that we are not assuming any kind of monotonicity of the convergence. This is one of the advantages of a varying-space-framework.

In order to illustrate the abstract results formulated below, suppose we are given a nonlinear semigroup $(S_t^n)_{t\geq 0}$, for each p_n-Laplace operator $A_n := \operatorname{div}\left[w_n|\nabla u|^{p_n-2}\nabla u\right]$ as above (assuming that it is dissipative or its domain is embedded into the Hilbert space L^2_w), i.e., for $f \in D(A_n)$ the semigroup $S_t^n f$ is a solution to the initial value problem

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}u(t) + A_n u(t) &\ni 0,\ 0 < t < \infty,\\
u(0) &= f.
\end{aligned}
$$

Define $(S_t)_{t\geq 0}$ correspondingly for the p-Laplace operator $A := \operatorname{div}\left[w|\nabla u|^{p-2}\nabla u\right]$. Assume we can prove, that $A_n \xrightarrow[n\to\infty]{G} A$ in the strong graph sense in (or, so to say, along) the varying space $L^{p_n}_{w_n} \to L^p_w$, as we do, in fact, in Corollary 8.14. Then for all $t \geq 0$ and for all $\varphi, \psi \in C_0(\mathbb{R}^d)$,

$$
\lim_n \int_{\mathbb{R}^d} \psi S_t^n \varphi\, w_n\,\mathrm{d}x = \int_{\mathbb{R}^d} \psi S_t \varphi\, w\,\mathrm{d}x,
$$

and

$$
\lim_n \int_{\mathbb{R}^d} |S_t^n \varphi|^{p_n}\ w_n\,\mathrm{d}x = \int_{\mathbb{R}^d} |S_t \varphi|^{p}\ w\,\mathrm{d}x.
$$

As a matter of fact, this convergence also characterizes the varying-space-strong-graph-convergence $A_n \xrightarrow[n\to\infty]{G} A$, see Chapters 5–7, in particular, Lemma 7.9 and Theorem 7.24.

In this work, we shall refer to the well-known abstract results on solutions to monotone operator equations, and do not discuss explicit solutions. Anyhow, we do not assume coercivity for our operators, since we are solely treating convergence problems. We assume that the reader is familiar with the standard theory of maximal monotone operators in Banach space, see e.g. [Sho97].

Varying Banach spaces

In recent years, variational convergence on varying spaces has been getting greater attention. In the theory of homogenization of elliptic partial differential operators, around

1990, Gabriel Nguetseng [Ngu89] and Grégoire Allaire [All92] have started to develop a theory for convergence in varying L^2-spaces of periodic functions. Today, this approach is well-known as "two-scale convergence". As a matter of fact, it is covered by our theory, see Paragraph 6.6. With a slightly different focus, Vasiliĭ V. Zhikov considered convergence of of elliptic operators with varying weights [Zhi98]. Recently, Zhikov and Pastukhova have also studied so-called "variable spaces" [ZP07], and again, their definition and ours coincide, see Paragraph 5.2.1. Our work on varying Banach spaces is based on and inspired by the work of Kazuhiro Kuwae and Takashi Shioya [KS03, KS08]. In their first paper, they considered varying metric measure spaces and varying Hilbert spaces. In their second paper they extended their study to varying metric spaces. Our basic definitions and terminology are borrowed from them. Their framework is used e.g. in the works [Kol05, Kol06, Töl06, GKR07, Kol08, Hin09].

Generally, when considering an *asymptotic topology* of Banach spaces $\{E_i\}$ indexed by a topological space I (e.g. the real line), one wants to know which sequences are converging in the *asymptotic space*

$$\mathfrak{E} := \dot{\bigcup_{i\in I}} E_i,$$

(the union being forced to be disjoint by labeling). Let $\{u_n\}$ be a sequence in $\mathfrak{E}$, u a point in $\mathfrak{E}$. By disjointness there are unique $i_n \in I$ with $u_n \in E_{i_n}$ and $i_\infty \in I$ with $u \in E_{i_\infty}$. For a reasonable notion of (strong) convergence $u_n \to u$ it is natural to demand

$$i_n \to i_\infty \quad \text{in } I\text{-topology},$$

and, furthermore, that the following axioms hold.

(1) For any $u \in E_{i_\infty}$ there exists a sequence $\{u_n\}$ such that $u_n \in E_{i_n}$, $n \in \mathbb{N}$ and $u_n \to u$.

(2) For any sequence $\{u_n\}$, with $u_n \in E_{i_n}$, $n \in \mathbb{N}$ and any $u \in E_{i_\infty}$ the following statement holds:
If $u_n \to u$, then
$$\lim_n \|u_n\|_{E_{i_n}} = \|u\|_{E_{i_\infty}}.$$

(3) For any two sequences $\{u_n\}$, $\{v_n\}$ with $u_n, v_n \in E_{i_n}$, $n \in \mathbb{N}$ and any $u \in E_{i_\infty}$ the following statement holds:
If $u_n \to u$ and $\lim_n \|u_n - v_n\|_{E_{i_n}} = 0$, then $v_n \to u$.

(4) For any two sequences $\{u_n\}$, $\{v_n\}$ with $u_n, v_n \in E_{i_n}$, $n \in \mathbb{N}$ and any two $u, v \in E_{i_\infty}$ and any $\alpha, \beta \in \mathbb{R}$ the following statement holds:
If $u_n \to u$ and $v_n \to v$, then $\alpha u_n + \beta v_n \to \alpha u + \beta v$.

Kuwae and Shioya [KS08] have suggested the name *asymptotic relation* for a topology on $\mathfrak{E}$ satisfying the above properties. Zhikov and Pastukhova [ZP07] call it a *variable space*. We shall call it a *strong linear asymptotic relation* (distinguishing from the *weak* *asymptotic relation* defined in this work). In its metric space version, it is a generalization of the famous Gromov-Hausdorff convergence of compact metric spaces as treated by Mikhail L. Gromov in [Gro99].

In most of Chapter 5, the above general framework is simplified to a "sequential asymptotics" along a fixed sequence of Banach spaces $E_n \to E_\infty$. In particular, $I = \mathbb{N} \cup \{\infty\}$ and $i_n := n \to \infty =: i_\infty$, so that we are considering convergence in the space

$$\mathfrak{E} := \dot{\bigcup_{n\in\mathbb{N}}} E_n \dot{\cup} E_\infty,$$

where, for reasons of the specific topology of the space $\mathbb{N} \cup \{\infty\}$, sequences $\{u_n\}$ with $u_n \in E_n$, $n \in \mathbb{N}$, converging to points $u \in E_\infty$, remain the most interesting ones.

The starting point is always a sequence of linear maps $\{\Phi_n\}$, which is an approximation of the identity on E_∞. More precisely, $\Phi_n : C \subset E_\infty \to E_n$, where C is a dense linear subspace of E_∞, and

$$\lim_n \|\Phi_n(\varphi)\|_{E_n} = \|\varphi\|_{E_\infty} \quad \forall \varphi \in C.$$

No continuity of Φ_n is assumed. Such a sequence $\{\Phi_n\}$ is called *linear metric approximation.* Many concrete examples of metric approximations are given in Chapter 6. We implement the general functional analytic method suggested by Kuwae and Shioya [KS03] which defines a convergence $u_n \to u$ of elements $u_n \in E_n$, $n \in \mathbb{N}$, $u \in E_\infty$ as follows. $u_n \to u$ in the strong sense if and only if there is an approximating sequence $\{\varphi_m\}$ in C with $\lim_m \|\varphi_m - u\|_{E_\infty} = 0$ for which it holds that

$$\lim_m \overline{\lim_n} \|u_n - \Phi_n(\varphi_m)\|_{E_n} = 0.$$

Strong convergence defined in this way satisfies the convergence axioms of a linear strong asymptotic relation.

Let $\mathfrak{E}^* := \dot{\bigcup}_n E_n^* \dot{\cup} E_\infty^*$ be the asymptotic space of Banach space duals. For the first time, we are able to define a natural *linear weak convergence* on $\mathfrak{E}$. This was previously done by Kuwae and Shioya for Hilbert spaces or $CAT(0)$ spaces only. *Weak* convergence* on $\mathfrak{E}^*$ can also be defined naturally. Weak convergence does not depend on the metric approximations used to construct the strong asymptotic relations. Anyhow, when given metric approximations $\{\Phi_n : C \to E_n\}$ on $\mathfrak{E}$ and $\{\Phi_n^* : C^* \to E_n^*\}$ on $\mathfrak{E}^*$, the definition of weak convergence simplifies as follows. $u_n \rightharpoonup u$ in the weak sense if and only if $\sup_n \|u_n\|_{E_n} < \infty$ and

$$\lim_n {}_{E_n^*}\langle \Phi_n^*(\varphi^*), u_n \rangle_{E_n} = {}_{E_\infty^*}\langle \varphi^*, u \rangle_{E_\infty} \quad \forall \varphi^* \in C^*.$$

To make this possible, we demand the property of *asymptotic duality*, which is,

$$\lim_n {}_{E_n^*}\langle \Phi_n^*(\varphi^*), \Phi_n(\varphi) \rangle_{E_n} = {}_{E_\infty^*}\langle \varphi^*, \varphi \rangle_{E_\infty} \quad \forall \varphi \in C,\ \forall \varphi^* \in C^*.$$

It is always satisfied in Hilbert spaces but it needs to be verified in general Banach spaces.

The basic example of a varying Banach space is $\dot{\bigcup}_n L^p(\Omega;\mu_n)\dot{\cup}L^p(\Omega;\mu)$, where $\Omega \subset \mathbb{R}^d$, $\{\mu_n\}$ is a sequence of positive Radon measures such that $\mu_n \to \mu$ in the vague sense, where μ is a regular positive Radon measure. If $p > 1$, we can prove that a sequence $\{u_n\}$, with $u_n \in L^p(\Omega;\mu_n)$, converges strongly to $u \in L^p(\Omega;\mu)$ if and only if

$$\lim_n \int_\Omega \varphi u_n \,\mathrm{d}\mu_n = \int_\Omega \varphi u \,\mathrm{d}\mu \quad \forall \varphi \in C_0(\Omega),$$

and

$$\lim_n \int_\Omega |u_n|^p \,\mathrm{d}\mu_n = \int_\Omega |u|^p \,\mathrm{d}\mu.$$

New results in the theory of varying spaces

In this work (see Chapters 5–7), we are able to extend the knowledge on varying spaces in the following points.

- The theory is developed for general (real) Banach spaces with no restriction on the geometry.
- We implement weak and weak* Banach space topologies and duality naturally; this was done previously for Hilbert spaces and $CAT(0)$ spaces only, see Chapter 5.
- We give a natural (and purely functional analytic) foundation for two-scale convergence including topology and geometry (as e.g. the Kadeč-Klee property), see Section 6.6.
- We prove the Kolesnikov-Isometric-Theorem [Kol05] for abstract separable atomless L^p-spaces, $1 < p < \infty$, cf. Theorem 5.68 and Proposition 6.3. It was previously known only for separable Hilbert spaces. In Section 7.1, we present a general transfer-method (with examples), which enables us to carry over classical results, as e.g. the Trotter Theorem, automatically to the case of varying spaces.
- We topologically justify the use of sequences instead of nets. This was a pitfall of the previous works by Kuwae and Shioya. See Theorems 5.32, 5.37, 5.76.
- We are able to consider varying $L^p(\Omega, \mathscr{F}, \mu)$-spaces such that the measurable space $(\Omega, \mathscr{F})$ as well as the measure μ as well as p varies, see Section 6.2.
- We include a large class of Orlicz spaces with varying measures. This will be especially interesting in the future, because we are developing a method for homogenization and two-scale convergence for nonlinear equations involving Orlicz spaces, see Lemmas 6.17, 6.19.
- We prove the Trotter-Neveu-Kato and Brézis-Attouch Theorems mentioned above in varying Banach spaces, see Theorems 7.10, 7.23, 7.24.

- We prove that *Mosco convergence* (and *slice convergence* in the non-reflexive case) of convex functionals are equivalent to strong graph convergence of the associated subdifferential operators, see Theorems 7.43, 7.46.

- In Section 5.14, we sketch the more general, so-called *asymptotic topology*, which allows us to consider continua of metric spaces and gives us the natural notions of topology and convergence either of nets or of sequences. All previous parts of Chapter 5 then embed naturally. This might be particularly interesting in two- or more parameter convergences.

Variational methods

Variational methods emerge naturally in the study of partial differential equations. Solutions to many linear and nonlinear partial differential equations can be equivalently characterized by minimizing certain functionals (of functions). We mention the Dirichlet principle as a basic example with numerous generalizations both in PDE and operator theory. From it, we seize the idea that some operators, each understood as a graph in the product of two linear spaces, have a variational description by a bivariate functional which fully represents the given operator. All examples of partial differential operators in this work have such a representation. In fact, our examples reduce to a situation that is symmetric in some sense, that is to say, subdifferential operators of convex (univariate) functionals. One might ask how restrictive this class of operators is. At least it includes all self-adjoint dissipative linear operators on Hilbert spaces and all maximal cyclically monotone (multi-valued) operators from a Banach space into its dual.

Variational convergence is the counterpart to pointwise convergence in the analysis of (univariate or bivariate) functionals. It has been considered both purely in terms of sequential convergence or coming from (or inducing a) topology. The most famous notion of a variational convergence is Ennio De Giorgi's Γ-convergence [DG77]. Several generalizations and refinements have been developed since then. We would like to refer to two monographs on this subject. The first one by Gianni Dal Maso [DM93] spreads many concrete examples throughout the text, touching classical applications as approximation of elliptic PDEs and homogenization. From the second one by Hédy Attouch [Att84], we have borrowed several methods and techniques. Some of our generalized results are based on Attouch's proofs. The mode of convergence which deserves our main interest is Umberto Mosco's version of Γ-convergence, called simply Mosco convergence [Mos69, Mos94]. By definition, a convex functional $f : X \to \mathbb{R} \cup \{+\infty\}$ on a Banach space X is the *Mosco limit* of a sequence of convex functionals $\{f_n : X \to \mathbb{R} \cup \{+\infty\}\}$ if

$$\forall x \in X \; \forall x_n \in X,\; n \in \mathbb{N},\; x_n \rightharpoonup x \text{ weakly}: \quad \varliminf_n f_n(x_n) \geq f(x), \tag{M1}$$

$$\forall y \in X \; \exists y_n \in X,\; n \in \mathbb{N},\; y_n \to y \text{ strongly}: \quad \varlimsup_n f_n(y_n) \leq f(y). \tag{M2}$$

In the setting of general, not necessarily reflexive Banach spaces, we shall advance to its natural generalization, the so-called *slice convergence* [Bee92, AB93, CT98]. For the sake of completeness, we also mention Wijsman convergence. The connection was drawn in [BF89].

The main idea about variational convergence can be rephrased as follows. Given a variational representation $f : X \to \mathbb{R} \cup +\infty$ of a graph $\partial f \subset X \times X^*$ in a Banach space X with dual X^*, how can we establish strong graph convergence of a sequence of graphs

$$\partial f_n \to \partial f$$

in some sense, using only information on the variational forms $\{f_n\}$?

The benefit of Mosco's celebrated theorem [Att84, Theorem 3.66], reformulated in the language of partial differential operators, is that we can establish convergence of second-order objects merely with the analysis of first-order objects. For varying spaces, this generalized in Theorems 7.43 and 7.46. Following this philosophy, and for motivation, we shall now present four nonlinear operators fully described by proper, lower semi-continuous, convex functionals on Banach spaces, which are among our main applications; i.e. we can prove that they are Mosco and slice limits in varying spaces, see below.

Four nonlinear operators

Each of the following operators is an operator of the type $A : X \to X^*$, where X is a Banach space and X^* its dual. Only operator no. 3 is multi-valued, hence considered as a graph $A_3 \subset X_3 \times X_3^*$. All four operators A_i, $i = 1, \dots, 4$ are subdifferential operators, hence maximal cyclically monotone with an "energy functional" $f_i : X_i \to [0, +\infty]$, $i = 1, \dots, 4$, which is proved to be proper, lower semi-continuous and convex. In formulae, $A_i := \partial f_i$, $i = 1, \dots, 4$. We can specify even more, namely each f_i is of the type

$$f_i(x) = \begin{cases} g_i(x), & \text{if } x \in V_i, \\ +\infty, & \text{if } x \in X_i \setminus V_i, \end{cases}$$

where X_i, $i = 1, \dots, 4$ are separable Banach spaces, V_i, $i = 1, \dots, 4$ are separable Banach spaces embedded linearly, continuously and densely into X_i and g_i, $i = 1, \dots, 4$ are real-valued, lower semi-continuous and convex functionals on V_i. We, however, note that our variational theory can treat much more general situations.

1. The weighted Φ-Laplacian

Let $\Omega = \mathbb{R}^d$. Let $\Phi : [0, \infty) \to [0, \infty)$ be a nice Young-function satisfying Δ_2 and ∇_2 growth conditions as explained in Appendix C. Let φ be its left-derivative. Let Ψ be the Young conjugate of Φ. Let $d \geq 1$, $w \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $0 < w < +\infty$ almost everywhere, satisfying the Φ-Hamza condition (S1) from Chapter 3. Let $X_1 := L^\Phi_w(\mathbb{R}^d)$ be the Orlicz space with measure $w\,\mathrm{d}x$ and dual $L^\Psi_w(\mathbb{R}^d)$ and let $V_1 := W^{1,\Phi}_w(\mathbb{R}^d)$ be the weighted Orlicz-Sobolev space of first order, see [Vui87]. We consider

$$g_1(u) := \int_{\mathbb{R}^d} \Phi(|\nabla u|)\, w\, \mathrm{d}x, \quad u \in W^{1,\Phi}_w(\mathbb{R}^d).$$

On smooth functions, the associated operator $A_1 = \partial f_1$ is proved to have the representation

$$ {}_{L^\Psi_w}\langle A_1 u, v\rangle_{L^\Phi_w} = -\int_{\mathbb{R}^d} \operatorname{div}\left[w\varphi(|\nabla u|)\operatorname{sign}(\nabla u)\right] v \,\mathrm{d}x, \quad u \in C_0^\infty,\ v \in L^\Phi_w. $$

We call A_1 a *weighted* Φ*-Laplace* operator. Because of being non-homogeneous in general, the name "strongly nonlinear" has been suggested for operators of this type. Other variants are known, and can be e.g. defined by altering the placing of weights in the Sobolev space. Equations of Φ-Laplace-type have been studied by Jean-Pierre Gossez et. al. [Gos82, Gos86, GM87] in the unweighted case. The works [GHMS96, GHLMS99, GM02] study the eigenvalue problem. Generally, the lack of homogeneity in any Orlicz-space setting extending L^p generates the need of a more precise analysis of the modular functional $\int \Phi(\cdot)\,\mathrm{d}x$ and the so-called Orlicz and Luxemburg norms. We collect the facts needed in Appendix C. Weighted Orlicz-Sobolev spaces are discussed in Chapter 3.

The following special case of a weighted Φ-Laplacian is known better:

2. The weighted p-Laplacian

Let $\Omega = \mathbb{R}^d$. Restrict the preceding example to the case of $\Phi(t) := \frac{1}{p}|t|^p$, for any $1 < p < \infty$. Let $q := p/(p-1)$. Suppose that w has a derivative in the sense of Schwartz distributions $\mathrm{D}w$ which is locally integrable. Let $\varphi := w^{1/p}$. Suppose that

$$ \varphi \in H^{1,p}_{\mathrm{loc}}(\mathbb{R}^d) $$

and set

$$ \beta := p\frac{\nabla\varphi}{\varphi}. $$

Suppose that $\beta \in L^q_w(\mathbb{R}^d \to \mathbb{R}^d)$, in fact, only $\varphi^{p-2}\nabla\varphi \in L^q_{\mathrm{loc}}(\mathbb{R}^d \to \mathbb{R}^d)$ is needed.

Let $X_2 := L^p_w(\mathbb{R}^d)$ and $V_2 := W^{1,p}_w(\mathbb{R}^d)$. We consider

$$ g_2(u) := \frac{1}{p}\int_{\mathbb{R}^d} |\nabla u|^p \, w \,\mathrm{d}x, \quad u \in W^{1,p}_w(\mathbb{R}^d). $$

On smooth functions, the associated operator $A_2 = \partial f_2$ is proved to have the representation

$$ {}_{L^q_w}\langle A_2 u, v\rangle_{L^p_w} = -\int_{\mathbb{R}^d} \operatorname{div}\left[w|\nabla u|^{p-2}\nabla u\right] v \,\mathrm{d}x, \quad u \in C_0^\infty,\ v \in L^p_w. $$

But also for $u \in C_0^\infty(\mathbb{R}^d)$

$$ -A_2 u = \operatorname{div}\left[|\nabla u|^{p-2}\nabla u\right] + \left\langle |\nabla u|^{p-2}\nabla u, \beta\right\rangle \quad \text{in } L^q_w. $$

This indicates that the operator A_2 can be regarded as a first-order perturbation of the (non-weighted) p-Laplace operator. Therefore, we are suggesting the alternative name "*generalized nonlinear p-Schrödinger operator*". The p-Laplace operator has numerous applications in physics, e.g. nonlinear diffusions, non-Newtonian fluids, flows in porous media and plasma physics, compare with [Día85]. In the case $p = 2$ (with weight), including also infinite dimensional domains, the generalized Schrödinger operator (weighted Laplacian) has been studied extensively by Sergio Albeverio, Michael Röckner et. al. in e.g. [AR89, AR90a, AR90b, AKR90, MR92], including the associated probability theory.

3. The 1-Laplacian

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let $X_3 := L^1(\Omega, \mathrm{d}x)$, $X_3^* = L^\infty(\Omega, \mathrm{d}x)$. Let $V_3 := BV_0(\Omega)$, that is, the functions of bounded variation with vanishing trace on Ω. For $u \in BV_0(\Omega)$, we consider

$$g_3(u) := \|\mathrm{D}u\|(\Omega) := \sup\left\{\left|\int_\Omega u \operatorname{div}\eta \,\mathrm{d}x\right| \;\middle|\; \eta \in C_0^\infty(\Omega \to \mathbb{R}^d),\ \|\eta\|_\infty \le 1\right\},$$

the *total variation*. When $u \in H_0^{1,1}(\Omega) \subset BV_0(\Omega)$, then $\|\mathrm{D}u\|(\Omega) = \int_\Omega |\nabla u| \,\mathrm{d}x$.

When $u \in H_0^{1,1}(\Omega)$ and

$$A_3 u := -\operatorname{div}[\operatorname{sign}(\nabla u)] = -\operatorname{div}\left[\frac{\nabla u}{|\nabla u|}\right] \in L^\infty(\Omega),$$

then $u \in D(\partial f_3) = \{u \in BV_0(\Omega) \mid \partial f_3(u) \neq \varnothing\}$ and $A_3 u \in \partial f_3 u$. As mentioned above, it is multi-valued since a requirement for maximal monotonicity of the sole sign-function is that $\operatorname{sign}(0) = \{x \in \mathbb{R}^d \mid |x| \le 1\}$, $0 \in \mathbb{R}^d$. The space BV and the form f_3 have been well-studied e.g. in [Giu77, Giu84, Zie89, JLJ98]. The operator A_3 has been studied e.g. by Vladislav Fridman, Bernd Kawohl, Friedemann Schuricht and Enea Parini [Fri03, Sch06, KS07, Par09]. See [BDPR09] for the stochastic case. The 1-Laplace operator has applications in material science and in image processing, see [AFP00, AK06].

4. Generalized porous medium and fast diffusion operators

We follow a setting by Jiagang Ren, Michael Röckner and Feng-Yu Wang [RRW07, RW08]. Let $(L, D(L))$ be a linear partial (or pseudo) differential operator with order ≤ 2, which is associated to a transient (symmetric) Dirichlet form $(\mathscr{E}, \mathscr{F})$ on some $L^2(\mu)$-space. Let $\mathscr{F}_e$ be the extended Dirichlet space and $\mathscr{F}_e^*$ its dual. If e.g. in the *classical case*, $L = -\Delta$ on a bounded domain $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary conditions, then $\mathscr{F} = \mathscr{F}_e = H_0^{1,2}$ and $\mathscr{F}_e^* = H^{-1}$.

Let Φ be a nice Young function with Δ_2 and ∇_2 growth conditions. Let φ be its left-derivative. Set $X_4 := \mathscr{F}_e^*$ and $X_4^* = \mathscr{F}_e$. X_4 is a Hilbert space and the Riesz-isometry $-\overline{L} : X_4^* \to X_4$ is an extension of $-L$. Let $V_4 := L^\Phi(\mu) \cap \mathscr{F}_e$ ($L^p(\Omega, \mathrm{d}x) \cap H^{-1}$ in the classical case, $1 < p < \infty$). We consider

$$g_4(u) := \int_\Omega \Phi(|u|) \,\mathrm{d}\mu.$$

Since the Riesz map of $\mathscr{F}_e$ is the operator $\overline{L}$ that we are using to define the subgradient operator, we have to distinguish between the dualization and the inner product of $\mathscr{F}_e^*$. We write ${}_{\mathscr{F}_e}\langle\cdot,\cdot\rangle_{\mathscr{F}_e^*}$ for the dualization and $(\cdot,\cdot)_{\mathscr{F}_e^*}$ for the inner product. The associated operator $A_4 = \partial f_4$ is proved to have the representation

$$ {}_{\mathscr{F}_e}\langle A_4 u, v\rangle_{\mathscr{F}_e^*} = \int \varphi(|u|)\operatorname{sign}(u) v \,\mathrm{d}\mu, \quad u \in D(A_4) \subset V_4,\ v \in V_4. $$

Applying the Riesz isometry, we get

$$ (A_4 u, v)_{\mathscr{F}_e^*} = -\int \overline{L}\left[\varphi(|u|)\operatorname{sign}(u)\right] v \,\mathrm{d}\mu, \quad u \in D(A_4) \subset V_4,\ v \in V_4, $$

which is a subgradient operator on a Hilbert space. Reducing to the classical case, it takes the shape

$$ A_4 u = \Delta\left[|u|^{p-2}u\right] \quad \text{in } H^{-1} $$

which is the *porous medium operator* for $p > 2$ and the *fast diffusion operator* for $p < 2$. A famous lecture by Donald G. Aronson [Aro86] forms a highly readable introduction to the porous medium equation. Juan Luis Vázquez's book [Váz07] provides a systematic and comprehensive reference. In [Váz06], both porous medium and fast diffusion type equations are studied extensively. A number of physical applications for these equations are known, such as to describe processes involving a flow of fluid through a porous medium. As other applications, we mention nonlinear diffusion and heat transfer, plasma physics, lubrication, material science.

In all four cases, the approximants are of similar type, except in case no. 3. Now what is new about these Mosco-approximations, and why can they be treated only in the framework of varying Banach spaces? In case no. 1, we vary the weight, yielding spaces $L^{\Phi}_{w_n} \to L^{\Phi}_{w}$, see Section 8.1. In case no. 2, we vary both p and the weight at the same time, resulting in an approximation along $L^{p_n}_{w_n} \to L^{p}_{w}$, see Section 8.2. In case no. 3, we approximate the critical functional defined above (in the sense of slice-convergence) by functionals of the type

$$ \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x, \quad u \in H_0^{1,p}(\Omega), $$

yielding an approximation along $L^p \to L^1$, see Section 8.3. Our proof involves also the Legendre transforms of the functionals because in the non-reflexive L^1-case a dual statement for convergence is needed and cannot be derived theoretically from reflexivity alone. In case no. 4 we vary the operators L and presume that the associated Dirichlet forms converge pointwise on a core of L (which is weaker than Mosco convergence). The measures involved are also varied, see Section 8.4.

For linear monotone operators, Mosco convergence methods have been used for approximation for quite some time now. We mention the works [Kol05, Kol06, Kol08] by Alexander V. Kolesnikov, which have inspired us lastingly. An interesting approach using square field operators was created by Kazuhiro Kuwae and Toshihiro Uemura [KU96, KU97]. Other advances in convergence of linear operators, that have influenced our work, are given by [AKS86, Mer94, RZ97, Hin98, Mat99]. An interesting application of G-convergence of nonlinear operators to stochastic partial differential equations is given in Ioana Ciotir's works [Cio09, Cio10].

Weighted Sobolev spaces

Another main result of this thesis concerns weighted first order Sobolev spaces and their uniqueness. For the operators A_1 and A_2 above, such spaces are needed. The theory of unweighted Sobolev spaces is folklore [Ada75, GT77, Maz85, Zie89, EG92], less so, in the weighted case, where a number of different approaches exist. Usually, when one uses Schwartz distributions for the definition, as in the fundamental works of Alois Kufner and Bohumír Opic [Kuf80, KO84], one needs additional regularity on the weights and the boundary of the domain in order to verify basic properties. In this context, the so-called Muckenhoupt class is widely known. A weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is said to belong to A_p (the p-Muckenhoupt class; here $1 < p < \infty$) if there is a constant $K > 0$ such that

$$\left(\frac{1}{\operatorname{vol} B}\int_B w \,\mathrm{d}x\right) \cdot \left(\frac{1}{\operatorname{vol} B}\int_B w^{-1/(p-1)} \,\mathrm{d}x\right)^{p-1} \le K,$$

for all balls $B \subset \mathbb{R}^d$. We refer to the lecture notes by Bengt Ove Turesson [Tur00] for a detailed discussion of this class. Weighted Sobolev spaces with Muckenhoupt weights have many nice properties. A standard construction for weighted first order Sobolev spaces $H^{1,p}_w$ can be found in the book by Juha Heinonen, Tero Kilpeläinen and Olli Martio [HKM93]. In contrary to it, we shall follow the approach of Sergio Albeverio, Michael Röckner and Shigeo Kusuoka [AR90a, AKR90]. We shall only need *Hamza's condition*

$$w = 0 \ \mathrm{d}x\text{-a.e. on } \mathbb{R}^d \setminus R(w),$$

where $R(w)$ is the *p-regular set* of w defined as

$$R(w) := \left\{ x \in \mathbb{R}^d \;\middle|\; \int_{|x-y|<\varepsilon} w^{-1/(p-1)} \,\mathrm{d}y < \infty \text{ for some } \varepsilon > 0 \right\}.$$

The stronger condition $R(w) = \mathbb{R}^d$ is known better, since it is equivalent to

$$w^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^d).$$

Although we treat only the case that the domain is all of $\mathbb{R}^d$, our weaker condition includes cases where w is zero on a set of positive measure and where w has possible singularities at the boundary of $R(w)$. It is strong enough in order to verify that the weighted Sobolev space $H^{1,p}_w$ is a space of functions included in L^p_w, and that it is a lattice and functions therein possess locally absolutely continuous versions on the "conditional regular sets" on the lines parallel to the coordinate axes. These properties are proved in Chapter 3 (including Orlicz-Sobolev spaces). Another advantage of this approach is that one can consider possibly infinite dimensional spaces replacing $\mathbb{R}^d$. We shall also need another characterization for $f \in W_w^{1,p}$, based on the integration by parts formula

$$\int \partial_i f \eta \, w \, \mathrm{d}x = -\int f \partial_i \eta \, w \, \mathrm{d}x - \int f \eta \beta_i \, w \, \mathrm{d}x \quad \forall \eta \in C_0^\infty,$$

where $\beta_i = \partial_i \log(w) = \partial_i w / w$ is the so-called logarithmic derivative of w. Obviously, some local weak differentiability has to be assumed for the weight.

Another interesting topic to study is the uniqueness question for Sobolev spaces. It is classically called "strong equals weak" or "$H = W$" and credited to Norman G. Meyers and James Serrin [MS64]. In case with weights, everything becomes more complicated, see e.g. [Kil97, Zhi98]. If $p = 2$, it has been well studied under the name "Markov uniqueness", see e.g. [RZ92, ARZ93a, ARZ93b, RZ94, Ebe99, Sta99a]. Another suggestive name would be "problem of smooth approximation", because it is known that density of C_0^∞ in $W_w^{1,p}$ is equivalent to the problem. When studying solutions to degenerate elliptic equations, it is related to the so-called Lavrent'ev phenomenon, which was first observed in [Lav26].

Generally, it is known that the Muckenhoupt condition mentioned above is sufficient for $H_w^{1,p} = W_w^{1,p}$. Anyhow, it seems quite restrictive and hard to verify. Luckily, inspired by a proof of Patrick Cattiaux and Myriam Fradon [CF96], we are able to impose a new sufficient condition for $H_w^{1,p} = W_w^{1,p}$ (for the domain $\Omega = \mathbb{R}^d$) as follows

$$w^{1/p} \in H^{1,p}_{\mathrm{loc}}, \; \frac{\partial_i w}{w^{1/p}} \in L^q_{\mathrm{loc}} \quad \forall 1 \leq i \leq d,$$

see Theorem 4.4 and Section 4.2. We note that for $p = 2$, the condition above reduces to the well-known one

$$\sqrt{w} \in H^{1,2}_{\mathrm{loc}},$$

which was first proved to be sufficient for $H_w^{1,2} = W_w^{1,2}$ in [RZ94].

The proof of Theorem 4.4 uses the integration by parts formula presented above. In connection with the approach using Schwartz distributions, we are able to transfer the result to the classical definition of the weak Sobolev space $W_w^{1,p}$, see Section 4.3.

As was noticed by Alexander V. Kolesnikov [Kol05] and by V. V. Zhikov [Zhi98] in a different context, the problem $H = W$ is indeed related to condition (M2) for Mosco convergence. Intuitively, the idea behind is clear, since (M2) is a condition demanding that "a limit is attained", and, on the other hand, $H = W$ is a condition for smooth approximation. Thus our result on uniqueness has a direct application for the Mosco convergence of functional no. 2 above.

Overview

This work consists of an introduction, 7 subsequent chapters and 3 appendices. We shortly present the topics of the chapters as follows:

Chapter 1 contains the introduction, the guide for the reader below and the acknowledgements.

Chapter 2 collects well-known facts from the theory of (linear and nonlinear) operators, bilinear forms and convex functionals, with an emphasis on the calculus of variations. Chapters 7 and 8 heavily depend on these facts.

Chapter 3 introduces weighted first order Sobolev spaces on $\mathbb{R}^d$ that belong to an Orlicz-space integrability class. We distinguish between the strong weighted Φ-Sobolev space $H_w^{1,\Phi}$ and the weak weighted Φ-Sobolev space $W_w^{1,\Phi}$. Fundamental properties are proved.

Chapter 4 is devoted to the study of $H = W$. A new condition for equality of the strong and weak weighted Sobolev spaces in the L^p-integrability class, $1 < p < \infty$, is given.

Chapter 5 contains the abstract functional analytic and topological theory of varying Banach spaces. The theory is fully developed. Many useful topological and geometric properties of both strong and weak topologies are proved. In particular, we explain what we mean by (strong and weak) asymptotic relations and metric approximations and how these two concepts depend on each other, as sketched in the figures of Section 5.13.

Chapter 6 gives many concrete examples for metric approximations, and, as a consequence, for asymptotic relations and varying spaces. Among them, we present varying L^p-spaces, varying Orlicz-spaces, varying finite dimensional approximations and two-scale convergence.

Chapter 7 provides the (abstract) theory of variational convergence of operators, forms, convex functionals and "spectral objects", as semigroups and resolvents, everything done in the varying-space-framework. Γ, Mosco, slice and graph convergence are introduced.

Chapter 8 gives four applications for variational convergence of quasi-linear partial differential operators. In all four cases, the underlying Banach spaces are varied. We prove Mosco and slice convergence, and, as a result, strong graph convergence.

Appendix A recalls some facts from general topology, needed mainly in Chapter 5.

Appendix B collects facts from the general (geometric) theory of Banach spaces, needed in Chapters 5 and 6.

Appendix C gives a short introduction to Young functions and Orlicz spaces with general measures. Orlicz spaces are used particularly in Chapters 3, 6 and 8.

For a more detailed summary (with further comments on the novelty of the respective results), we refer to the following "guide for the reader".

A guide for the reader

In Chapter 2, we collect all preliminary results and notions that are needed thereafter (except for those collected in the appendix). In Section 2.1, we introduce the symbols and notations we are using in this work. Section 2.2 briefly summarizes the framework of linear monotone operators (resolvents, semigroups) and bilinear forms on Hilbert spaces. It is needed for our results on linear operators in Sections 7.2 and 7.3. Section 2.3 gives a short introduction to nonlinear multi-valued monotone operators on Banach spaces and their resolvents, Yosida approximations and semigroups. We need these notions in the results of Section 7.4. Section 2.4 explains the basics of the variational theory of convex functionals on normed spaces, including subgradients, differentiability, the Legendre transform, infimal convolution and Moreau-Yosida approximation. These concepts are needed for our results on variational convergence in Sections 7.5 and 7.6. All examples of concrete operators in Chapter 8 are subdifferentials of convex functionals on Banach spaces. In particular, for Chapter 8, we need a variational setting based on an embedding $V \hookrightarrow X$ of Banach spaces. It is developed in Section 2.5. The interplay of the two spaces V and X with subgradients is revealed in Proposition 2.50. Corollary 2.51 and Lemma 2.52 are highly useful in order to establish single-valuedness and lower semi-continuity of subdifferentials in Chapter 8.

In Chapter 3, we present weighted first order Orlicz-Sobolev spaces on $\mathbb{R}^d$ in a nutshell. The chapter starts with four conditions (S1)–(S4) on the weights, needed for a reasonable construction of weighted Sobolev spaces, see Definitions 3.1 and 3.2. Only condition (S2) is well-known in the literature as $w^{1/(1-p)} \in L^1_{\mathrm{loc}}$, which ensures that $H^{1,p}_w$ is a space of functions. In fact, we only need the weaker condition (S1) (involving the so-called regular set $R(w)$), which is implied by (S2). (S2) turns out to be particularly useful, when comparing weak derivatives with distributional derivatives, see Remark 3.9. Section 3.1 develops the construction of the strong weighted Orlicz-Sobolev space $H^{1,\Phi}_w$. Note that our (closability) condition (3.3) is directly implied by (S1). The construction is quite standard, namely, taking the completion of C^∞-functions with finite Orlicz-Sobolev-norm. In Lemma 3.10, we use a partition of unity in order to prove that smooth compactly supported functions are dense in $H^{1,\Phi}_w$. Section 3.2 presents the Albeverio-Kusuoka-Röckner-Zhang approach towards the weak weighted Orlicz-Sobolev space $W^{1,\Phi}_w$. Note that it was previously only known for $W^{1,2}_w$. Integration by parts formula (3.4) is used as a definition. It is both intuitive and significant, as central properties can be derived from it directly, see Lemmas 3.13 and 3.14 and also Lemma 4.5. In addition to (S1), we are demanding (S3) and (S4), related to the so-called logarithmic derivative β of w. Unfortunately, in order to prove that $W^{1,\Phi}_w$ is a lattice, we need another characterization of functions in $W^{1,\Phi}_w$, known as absolute continuity on lines. We prove in Proposition 3.15 that this property is both necessary and sufficient. Using this representation, we are able to explain the relationship with distributional derivatives in Corollary 3.16, prove the Leibniz rule in Corollary 3.17 and prove a chain rule with Lipschitz functions in Lemma 3.18. Finally, in Lemma 3.19, we prove that bounded and compactly supported functions are dense in $W^{1,\Phi}_w$. Note that in both cases, the (closed) gradient depends on the weight w.

Chapter 4 is devoted to the problem $H = W$. We impose the condition (HW) on the weight w (being closely related to (S3) and (S4)), in order to derive uniqueness $H = W$ from it. In Section 4.1, we shortly discuss the famous Muckenhoupt class, which is a rather strong condition for $H = W$ to hold. Our main result in Section 4.2 is Theorem 4.4, which states that $H_w^{1,p}(\mathbb{R}^d) = W_w^{1,p}(\mathbb{R}^d)$ whenever (HW) is assumed. The central steps in the proof, which is based on an approximation by smoothing mollifiers, depend heavily on the integration by parts-Lemma 4.5, on the inequalities (4.12) and (4.13) involving the Hardy-Littlewood maximal function and on the classical Lemma 4.6 about difference quotients for Sobolev functions. We shall first prove Lemma 4.5 with the help of another regularization of the weight w. Using several approximations and localization, we are able to prove that C_0^∞ is dense in $W_w^{1,p}$. Note that the results of Chapter 3, in particular Lemma 3.19, are crucial. In Section 4.3, we recover the classical definition of $W_w^{1,p}$ as in equation (4.17), which uses distributional derivatives as a definition (instead of the integration by parts formula (3.4)). Luckily, under conditions (S2) and (HW), both definitions coincide, which is a result of Lemma 4.7 and Proposition 4.10.

In Chapter 5, we develop the general theory of varying Banach spaces. We suggest for readers interested in the general (but convenient) approach to read Appendix A completely before starting with Chapter 5. Section 5.1 contains a motivating summary on classical results about convergence of compact metric spaces. In Section 5.2, we define axiomatically what we mean by a varying Banach space and call it a linear strong asymptotic relation. For the strong topology, there are four related definitions, namely 5.6, 5.8, 5.10 and 5.12. In our applications, we shall need only Definition 5.12. The others differ in topological details. That the definitions make sense, is proved later in Section 5.7. Lemma 5.9 embeds the framework of Section 5.1 into the larger picture. In Subsection 5.2.3, we shall discuss three important topological properties of strong asymptotic relations, namely, the Hausdorff property, the existence of a countable base, and the regularity, which combined together (under separability assumptions on the Banach spaces involved) yield the metrization property. Section 5.3 discusses weak and weak* topologies of varying Banach spaces, also axiomatically, see Definition 5.18. In Subsection 5.3.1, we prove a Banach-Alaoglu-type compactness theorem (Theorem 5.22) for weak* topologies of varying Banach space duals. The proof is far more complicated than the classical one, since we are not working in a linear space anymore. Nevertheless, the result is quite useful, when no reflexivity of the spaces is assumed. Section 5.4 introduces metric approximations, see Definition 5.24. Note that concrete examples of metric approximations are given in Chapter 6. As mentioned above, they are always the starting point, when constructing strong and weak (linear) asymptotic relations. In Section 5.5, we define strong convergence (coming from a metric approximation), see Definition 5.28. We prove that it generates a sequential topology, see Theorem 5.32. A more abstract, but in some sense more natural way of dealing with the topology can be found in the proof of Theorem 5.76. Anyhow, we note that, to the best of our knowledge, this proof is new and necessary when talking about topologies coming from a convergence. This has not been addressed in previous works on asymptotic relations since nets were used instead of sequences. Now we justify the use of sequences in the asymptotic relation topologies. We highly recommend the reader to take notice of

Lemma 5.27, which shall be applied many times in this work. It provides an extremely elegant way to deal with diagonal sequences, due to Attouch and Wets. In Section 5.6, weak and weak* convergence are defined, see Definition 5.33. Two highly useful properties of weak convergence are proved in Lemmas 5.35 and 5.36. We shall use them many times. Finally, in Theorem 5.37 we prove that weak and weak* convergence generate a sequential topology. In Section 5.7, we prove that the topologies defined in Sections 5.5 and 5.6 are indeed strong and weak asymptotic relation topologies, see Theorem 5.38 and 5.45. Uniqueness is also proved, given the compatibility condition (C). More on condition (C) related to metric approximations can be found in Subsection 5.7.1. In Subsection 5.7.2, we discuss a property called asymptotic duality (Definitions 5.42 and 5.43), which enables us to talk about weak topology and dual pairs of varying Banach spaces. It turns out to be very natural, see Proposition 5.44. It can be verified easily in applications, see Chapter 6. In Lemmas 5.46 and 5.47, we prove an asymptotic lower semi-continuity of the norms w.r.t. the weak and weak* topologies. In Lemma 5.48, we verify convergence of normalized duality maps, using the compactness result in Theorem 5.22. Lemma 5.49 recovers the usual definition of weak convergence in varying spaces found in the literature. Section 5.8 briefly discusses the so-called asymptotic continuity of metric approximations, which is generic in some sense, see Lemma 5.51. In Section 5.9, we prove the sequential relative compactness of bounded sets for the weak topology, see Lemma 5.53. It works for reflexive and separable varying Banach spaces. In Section 5.10, we introduce the Kadeč-Klee property (which is known to hold in varying Hilbert spaces), and prove that it is implied by the so-called asymptotic uniform convexity, see Definition 5.56 and Theorem 5.57. Recall that the Kadeč-Klee property states that $u_n \to u$ strongly if and only if $u_n \rightharpoonup u$ weakly and $\|u_n\| \to \|u\|$, see Definition 5.54. We shall need it in Chapter 8. Proposition 5.58 gives conditions when the normalized duality maps are strongly converging. Lemma 5.59 and 5.60 together give a new characterization of strong convergence using weak convergence, supposing that the Kadeč-Klee property holds. In Section 5.11, we introduce so-called asymptotic (strong and weak) embeddings and discuss their basic properties. Lemma 5.66 will prove to be useful in applications, for it gives a method for verifying strong asymptotic embeddings. Note that asymptotic embeddings naturally occur in our variational framework based on Banach space embeddings $V \hookrightarrow X$. In Section 5.12, we present a major result of this work, namely an abstract isometric theorem, which is Theorem 5.68, stating that an asymptotic relation is metrically isometrically isomorphic (both strongly and weakly) to $E \times (\mathbb{N} \cup \{\infty\})$ and that there exist compatible isometric metric approximations. The proof of the result involves linear algebra, matrix manipulations, geometry of Banach spaces (as e.g. Schauder bases and orthogonality), Banach limits and, most importantly, the abstract condition (I), as defined in 5.67. (I) is easily verified for Hilbert spaces. In Chapter 6, we shall verify it for abstract atomless separable L^p-spaces. Lemmas 5.69, 5.70 are crucial for the proof of Theorem 5.68. Lemma 5.71 might be of its own interest in the isometric theory of Banach spaces. To the best of our knowledge, it is new. Corollary 5.72 is needed in Section 7.1. Proposition 5.73 yields the isometric metric on $E \times (\mathbb{N} \cup \{\infty\})$, implying that, under some conditions, linear strong asymptotic relations are Polish. As a feature, we give an overview on the relations of strong linear asymptotic relations and different types of linear metric approximations in five figures in Section 5.13. Section 5.14, an addendum, is devoted to the so-called asymptotic topology, an observation by us, providing a convenient way to introduce varying (and continua of) metric spaces. We have already used it in the presentation above. In Lemma 5.75, it can be seen that topological properties of asymptotic relations occur quite naturally. Theorem 5.76 is a generalization of Theorems 5.32 and 5.38 with a shorter proof due to the more general situation. In Subsection 5.14.1, an example of a continuum of Hilbert spaces is given. It is kept in the spirit of the examples in Chapter 6.

Chapter 6 consists of examples of metric approximations of varying Banach spaces. Some of them are used in Chapter 8 for our applications to operator convergence. In Section 6.1, the Hilbert space case is treated, which is, however, already well-known. In particular, separable infinite dimensional Hilbert spaces always satisfy condition (I) as a result of Lemma 6.2. In Section 6.2, we prove an elaborate result stating that separable atomless L^p-spaces, with $1 < p < \infty$ fixed, always satisfy condition (I). We quote many results from the isometric theory of Banach spaces and from the classification theory, basis theory and geometry for L^p-spaces. To the best of our knowledge, only Lemma 6.12 is new, and might be of its own interest; it is explaining the rôle of duality for isometries between L^p-spaces on different measure spaces. All preliminary results are needed to verify condition (I). Finally, concrete metric approximations are given, starting from Subsection 6.2.1. We treat the cases $\dot{\bigcup}_n L^p(\mu_n)\dot{\cup}L^p(\mu)$, i.e. varying measure. Subsection 6.2.2 treats the case $\dot{\bigcup}_n L^p(\Omega_n;\mu_n)\dot{\cup}L^p(\Omega;\mu)$ of varying domain and measure (where the domain can be infinite dimensional). In Subsection 6.2.3, we consider varying p (and varying measure), excluding the critical (non-reflexive) cases $p = 1$ and $p = \infty$, that is, $\dot{\bigcup}_n L^{p_n}(\mu_n)\dot{\cup}L^p(\mu)$. In Subsection 6.2.4, we present metric approximations for $\dot{\bigcup}_n L^{p_n}(\Omega)\dot{\cup}L^1(\Omega)$ and for $\dot{\bigcup}_n L^{q_n}(\Omega)\dot{\cup}L^\infty(\Omega)$, $\Omega \subset \mathbb{R}^d$ bounded. In Section 6.3, we give a metric approximation for Orlicz spaces with varying measure. The main difficulty is the non-homogeneity of the Young function Φ involved. Nevertheless, we are able to prove the convergence of norms, see Lemmas 6.17 and 6.19. This is a major step towards two-scale convergence of Orlicz spaces and homogenization of non-homogeneous degenrate elliptic equations. In Section 6.4, we relate so-called scales of Banach spaces to asymptotic relations. In Section 6.5, we prove that finite dimensional approximations in Banach spaces are always linear asymptotic relations. As an important feature of our work, in Section 6.6, we are able to prove that two-scale convergence, as used in the theory of homogenization, can always be embedded into our framework. This has topological and analytical implications.

Chapter 7 provides the (abstract) functional analytic foundation of variational convergence of operators and forms in varying spaces. In Section 7.1, a general method, suggested by a referee in 2007, is presented, which enables us to transfer abstract results on a fixed space to varying spaces. It is applied in the proofs of Theorems 7.10, 7.15 and 7.24. It is explained in Proposition 7.2 and Corollary 7.3. It works in separable Hilbert spaces and separable non-atomic L^p-spaces, $1 < p < \infty$, as a consequence of Theorem 5.68, Lemma 6.2 and Proposition 6.3. Section 7.2 discusses modes of convergence for sequences of bounded linear operators on varying Banach spaces. In Sections 7.2.1 and 7.3, we prove results of variational convergence in Theorems 7.10, 7.12, 7.15, Proposition 7.18 within the linear case. Note that for varying spaces, only Theorem 7.12 (the symmetric case) was known before. Theorem 7.10 is the celebrated Trotter-Neveu-Kato Theorem, Theorem 7.12 is the Mosco theorem for symmetric forms. Section 7.3 uses the framework of generalized bilinear forms as introduced in Section 2.2. We go over to the non-symmetric situation in Section 7.4. We explicitly prove (without the methods from Section 7.1) the nonlinear Trotter-type Theorem 7.23 due to Attouch. The nonlinear-semigroup-convergence-Theorem 7.24 is due to Brézis, where we shall use the methods from Section 7.1. In Section 7.5, we finally introduce Γ, Mosco and slice convergence (for varying spaces) in Definitions 7.25, 7.26 and 7.27. Some asymptotic properties of Γ-limits are discussed in Proposition 7.29. Curiously, the classical compactness of the Γ-topology extends to our case, as proved in Theorem 7.31. For the proof of Theorem 7.38, we need the notion of epilimits, as introduced in Definition 7.32. Some properties are discussed thereafter. In Subsection 7.5.2, we reprove a famous result by Attouch (Theorem 7.38), stating that the Mosco topology is exactly the one making the Legendre transform bicontinuous. This result is rather technical, but needed in the proofs of Theorems 7.43 and 7.46. Finally, in Section 7.6, we prove that in reflexive (varying) spaces, Mosco convergence of convex functionals is equivalent to G-convergence of their subgradients, see Theorem 7.43. Moreover, in non-reflexive spaces, we lift a result by Combari and Thibault to the case varying spaces, Theorem 7.46. It states that slice convergence of convex functionals is equivalent to the G-convergence of their subgradients. In fact, Theorem 7.43 is included therein. We shall apply both Theorems 7.43 and 7.46 in Corollaries 8.8, 8.14, 8.26, 8.39, which are, as a matter of fact, condensing our main results.

Chapter 8 is divided into four sections corresponding to the four operators presented above. Each of it exhibits a similar structure. First the convex functional for the operator is defined, as there are, respectively, the weighted Φ-energy, the weighted p-energy, the 1-homogeneous total variation and the Φ-modular for the porous medium and fast diffusion operator. We prove respective Gâteaux differentiability in Lemmas 8.1, 8.9 and 8.36. Regularity properties of the respective functionals are discussed in Lemmas 8.2, 8.10 and 8.37. The associated subgradient operators appear thereafter as, respectively, the weighted Φ-Laplacian, the weighted p-Laplacian, the 1-Laplacian and the porous medium operator and the fast diffusion operator, see Remarks 8.3, 8.11, 8.22 and the paragraph below Lemma 8.37, respectively. Then we state the conditions and prove the approximations for our respective convex functionals. This is done in Theorem 8.4 (Mosco convergence), Theorem 8.12 (Mosco convergence), Theorems 8.24, 8.25 (together: slice convergence) and in Theorem 8.38 (Mosco convergence). Below each of the theorems, we give a corollary with our main results, the strong graph convergence of the four operators as described above, see Corollaries 8.8, 8.14, 8.26, 8.39. Additionally, in Section 8.1, there are some results specific to Orlicz spaces, in particular, the method how we can avoid the pitfalls coming from non-homogeneity. In Section 8.3, we recall the definition of the space BV, which is needed for the 1-Laplacian. In Section 8.4, we quote some results on intersections and sums of Banach spaces, needed for the Ren-Röckner-Wang framework.

Appendix A collects the facts from general topology that are needed for our topological arguments in Chapter 5. Some of them might be not so well-known, as e.g. Fréchet and sequential spaces, as well as convergence operators.

Appendix B collects the necessary facts from the geometry of Banach spaces that are used in this work. It starts with convexity and smoothness properties of the norm (Section B.1) and then discusses duality maps (Section B.2), the Kadeč-Klee property (Section B.3) and all necessary facts (only a few) from the theory of bases in Banach spaces (Section B.4). At last, we give a short review on orthogonality in Banach spaces (Section B.5), which is needed in some proofs.

At the end of the book, there is an introduction to Orlicz spaces, Appendix C. Everything necessary is collected; on Young functions (Section C.1) and on Orlicz spaces with general measures (Section C.2). As usually, it is kept short and is by no means complete. Anyway, it contains everything we need.

2 Preliminaries

In this chapter, we collect some well-known facts about (linear and non-linear) operators and related energy-forms. We shall restrict ourselves to those facts needed in our later results and proofs. There are many books on linear dissipative operators in Hilbert space as well as on nonlinear monotone operators in Banach space. We refrain from the impossible task of presenting the theory comprehensively and refer to the literature as e.g. [DS57, Bré73, Tan79, Paz83, BP86, Zei90a, Zei90b, Sho97].

2.1 Basic notations

For our convenience (in indexing), zero is not a natural number for us; hence $\mathbb{N} = \{1, 2, \dots\}$ *strictly positive integers*. For a *metric space* X with *distance* $d_X(\cdot,\cdot)$ we write $B(x,\rho) = B_X(x,\rho) := \{y \in X \mid d_X(y,x) < \rho\}$ for the *open ball* with *center* $x \in X$ and *radius* $\rho \in [0,\infty)$. Similarly, $\overline{B}(x,\rho) = \overline{B}_X(x,\rho) := \{y \in X \mid d_X(y,x) \leq \rho\}$ the *closed ball*. For *real Banach spaces* E, F with *norms* $\|\cdot\|_E, \|\cdot\|_F$ we denote the set of all *bounded linear operators* from E to F by $\mathscr{L}(E,F)$ with *operator norm* $\|\cdot\|_{\mathscr{L}(E,F)}$. For convenience we set $\mathscr{L}(E) := \mathscr{L}(E,E)$. For linear operators T denote the *(algebraic) kernel* by $\ker(T)$ and the *range* by $\operatorname{ran}(T)$. For a real Banach space E we denote by $E^* := \mathscr{L}(E,\mathbb{R})$ its *(topological) dual* with norm $\|\cdot\|_{E^*} := \|\cdot\|_{\mathscr{L}(E,\mathbb{R})}$. Let ${}_{E^*}\langle\cdot,\cdot\rangle_E$ denote the *dualization* between E^* and E, i.e., ${}_{E^*}\langle f,x\rangle_E := f(x) \in \mathbb{R}$ given $f \in E^*, x \in E$. We shall also refer to this notation as the *duality bracket*. Denote by $E^{**} := \mathscr{L}(E^*,\mathbb{R})$ the *bidual* of E. For a *real Hilbert space* H we will denote the *inner product* by $(\cdot,\cdot)_H$ and define the *norm* by $\|\cdot\|_H := (\cdot,\cdot)_H^{1/2}$. From now on we shall frequently omit the attribute "real" since we will deal with real vector spaces only. We abbreviate $\alpha := \alpha\,\mathrm{Id}$ for any $\alpha \in \mathbb{R}$. Let $\overline{\mathbb{R}} := [-\infty,+\infty]$ be the so-called *extended real numbers*. As usually, $a+(\pm\infty) = (\pm\infty)+a = \pm\infty$ for $a \in \mathbb{R}$. Also $(+\infty)+(+\infty) = +\infty$ and $(-\infty)+(-\infty) = -\infty$. $+\infty+(-\infty)$ and $-\infty+(+\infty)$ are not defined. Moreover, $a\cdot(\pm\infty) = (\pm\infty)\cdot a = \pm\infty$ for $a \in (0,\infty]$ and $a\cdot(\pm\infty) = (\pm\infty)\cdot a = \mp\infty$ for $a \in [-\infty,0)$. Also $\mathbb{R}_+ := [0,+\infty)$, $\overline{\mathbb{R}}_+ := [0,+\infty]$ and $\mathbb{R}_\infty := (-\infty,+\infty]$. Also $\overline{\mathbb{N}} := \mathbb{N}\cup\{\infty\}$. Every of these sets carries a topology well-known as the one-point (two-point) compactification. For $d = 1, 2, \dots$ let $\mathbb{R}^d = \times_{i=1}^d \mathbb{R}$ which we shall usually (unless stated differently) equip with the *Euclidean distance*, namely the *Hilbertian metric* of $\ell^2(d)$. When it is clear from the context, we will write $|\cdot| := |\cdot|_{\mathbb{R}^d} := \|\cdot\|_{\ell^2(d)}$ and $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{\mathbb{R}^d} := (\cdot,\cdot)_{\ell^2(d)}$. Here $\ell^2(d)$, and more generally, $\ell^p(d)$, $\ell^p := \ell^p(\infty)$ resp. for $1 \leq p \leq \infty$ are equal to $\mathbb{R}^d$ (equipped with the *p-norm*), resp. equal to the *sequence spaces* of *p-summable real sequences* (equipped with the *p-norm*). For $v \in \mathbb{R}^d$ we define the *vector-valued sign function*

$$\operatorname{sign}(v) := \begin{cases} \frac{v}{|v|}, & \text{if } v \neq 0, \\ 0, & \text{if } v = 0. \end{cases}$$

Clearly, $|\operatorname{sign}(v)| \in \{0,1\}$. For a subset A of $\mathbb{R}^d$, we write $\operatorname{diam} A$ for the *diameter* of A and $\operatorname{vol} A$ for the *volume* of A (when A is a measurable set). A *domain* $\Omega \subset \mathbb{R}^d$ is a non-empty, connected, open subset. For a domain Ω we say that a set D is *compactly contained* in Ω and write $D \Subset \Omega$, if D is *relatively compact* and $\overline{D} \subset \Omega$, where $\overline{D}$ is the *closure* in $\mathbb{R}^d$. For a *topological space* T with *topology* $\mathscr{T}$ we write $\mathscr{T}$-lim for a *limit* w.r.t. to this topology, and interchangeably T-lim, when it is clear which topology is meant. We shall sometimes write $\|\cdot\|_E$-lim to indicate the limit in the strong topology of a normed space E. For a subset $A \subset T$, we write $\overline{A}$ for the *closure*, $\operatorname{int} A$ for the *interior* and ∂A for the *boundary*. For extended real-valued *sequences* or *nets* we denote the *limit superior* by $\overline{\lim}$ and the *limit inferior* by $\underline{\lim}$. For a real linear space V and a subset $U \subset V$ we set

$$\operatorname{lin} U := \left\{ v = \sum_{i=1}^{m} \alpha_i u_i \;\middle|\; m \in \mathbb{N},\ \alpha_i \in \mathbb{R},\ u_i \in U \right\}$$

and

$$\operatorname{co} U := \left\{ v = \sum_{i=1}^{m} \alpha_i u_i \;\middle|\; m \in \mathbb{N},\ \alpha_i \in [0,1],\ \sum_{i=1}^{m} \alpha_i = 1,\ u_i \in U \right\}.$$

A subset S of a linear space V is *linear* iff $\operatorname{lin} S = S$ and is *convex* iff $\operatorname{co} S = S$. If V is a real topological vector space and $U \subset V$ we write $\overline{\operatorname{lin}} U$ and $\overline{\operatorname{co}} U$ for the closure of $\operatorname{lin} U$ and $\operatorname{co} U$ resp. For extended real-valued functions $u, v : S \to \overline{\mathbb{R}}$ (S is any set) we write

$$u \vee v := \sup(u,v),\ u \wedge v := \inf(u,v),\ u^+ := u \vee 0,\ u^- := -(u \wedge 0).$$

For two sets A, B we write $A \subset B$ (or $B \supset A$) if for each $x \in A$ it holds that $x \in B$. We shall not use the notation $\subseteq$ or $\supseteq$. If $A \subset B$ and we would like to point out that there is an element $x \in B$ with $x \notin A$ we will write $A \subsetneqq B$ (or $B \supsetneqq A$). If $A \cap B = \varnothing$, we write $A \dot{\cup} B$ instead of $A \cup B$, the *disjoint union*. For a set A included in a set B we define the *indicator function* 1_A as a function from B to $\{0,1\}$ such that $1_A(x) = 1$ iff $x \in A$ and $1_A(x) = 0$ iff $x \in B \setminus A$. The *Kronecker delta* $\delta_{i,j}$ is a special indicator function function on $\mathbb{N} \times \mathbb{N}$ with $\delta_{i,j} := 1_\Delta(i,j)$, where $\Delta := \{(i,i) \mid i \in \mathbb{N}\}$ is the *diagonal*. We say that a function $f : V \to \mathbb{R}_\infty$ on a real vector space V is *positively homogeneous of degree* $p \in \mathbb{R}$ if $f(tx) = t^p f(x)$ for all $t > 0$. We say that f is *even*, if $f(x) = f(-x)$ for every $x \in V$ and we say that f is *odd*, if $f(-x) = -f(x)$ for every $x \in V$. We use the notation $D(\cdot)$ instead of $\operatorname{dom}(\cdot)$ to indicate the *domain* or the *effective domain* of a function or an operator. The arg min and the arg max are defined to be the sets of arguments such that an extended-real-valued expression attains its infimum or supremum. For an extended-real valued function $f : X \to \overline{\mathbb{R}}$ on a topological space X, denote by $\operatorname{supp} f := \overline{\{x \in X \mid f(x) \neq 0\}}$ its *support*. For two measures μ, ν on a measurable space $(\Omega, \mathscr{A})$ we write $\mu \ll \nu$ if $\mu(A) = 0$ whenever $\nu(A) = 0$ for any $A \in \mathscr{A}$, i.e. μ is *absolutely continuous* w.r.t. ν. On a measure space $(\Omega, \mathscr{A}, \mu)$, if $f : \Omega \to \overline{\mathbb{R}}$ is a measurable function, we occasionally write $\widetilde{f}$ or $\widetilde{f}^{\mu}$ for the equivalence class of all functions that differ from f on a μ-null set. We use the notation $*$ for the *convolution*.

2.2 Bilinear forms and linear operators

For all of this section, fix a separable Hilbert space H.

Definition 2.1. *A family of bounded linear operators $T_t : H \to H$, $t \geq 0$ is called* C_0-contraction semigroup *if $\|T_t\|_{\mathscr{L}(H)} \leq 1$ for all $t \geq 0$,*

$$\lim_{t \to 0} \|T_t x - x\|_H = 0 \quad \forall x \in H, \tag{2.1}$$

and

$$T_t T_s = T_{t+s} \quad \forall t, s \geq 0. \tag{2.2}$$

A family of bounded linear operators $G_\alpha : H \to H$, $\alpha > 0$ is called C_0-contraction resolvent *if $\|\alpha G_\alpha\|_{\mathscr{L}(H)} \leq 1$ for all $\alpha > 0$,*

$$\lim_{\alpha \to +\infty} \|\alpha G_\alpha x - x\|_H = 0 \quad \forall x \in H, \tag{2.3}$$

and

$$G_\alpha - G_\beta = (\beta - \alpha) G_\alpha G_\beta \quad \forall \alpha, \beta > 0. \tag{2.4}$$

A linear operator $A : D(A) \subset H \to H$ with linear domain $D(A)$ is called infinitesimal generator of a C_0-contraction semigroup *if $D(A) \subset H$ is dense, $(A, D(A))$ is closed, $(-Ax, x)_H \geq 0$ for every $x \in D(A)$, $(\alpha - A)(D(A)) = H$ for some $\alpha > 0$.*

Compare [MR92, Ch. I.1] for details. See also [Kat66], [RS75, Ch. X.8] and [Paz83]. There is a one-to-one correspondence between the above classes.

2.2.1 Symmetric forms

Definition 2.2. *We say that a functional*

$$F : H \to [0, +\infty]$$

is a (non-negative) quadratic form *(with extended real values) if there exists a linear subspace $D(F) \subset H$ and a (non-negative) symmetric bilinear form $\mathscr{E} : D(F) \times D(F) \to \mathbb{R}$ (i.e., $D(F) \ni u \mapsto \mathscr{E}(u, v)$ is linear for every $v \in D(F)$, $D(F) \ni v \mapsto \mathscr{E}(u, v)$ is linear for every $u \in D(F)$, $\mathscr{E}(u, v) = \mathscr{E}(v, u)$ for all $u, v \in D(F)$ and $\mathscr{E}(u, u) \geq 0$ for all $u \in D(F)$) such that*

$$F(x) = \begin{cases} \mathscr{E}(x, x), & \text{if } x \in D(F), \\ +\infty, & \text{if } x \in X \setminus D(F). \end{cases} \tag{2.5}$$

The notation $D(F)$ is justified because it is the effective domain of F. We write $D(\mathscr{E})$ for $D(F)$.

Remark 2.3. *Every non-negative quadratic form is convex.*

The following proposition is purely algebraic and well-known. It draws the essential connection between non-negative quadratic forms and the parallelogram identity.

Proposition 2.4. *Let $F : H \to [0, +\infty]$ be an arbitrary functional. If*

(i) $F(0) = 0$,

(ii) $F(tx) \le t^2 F(x)$ *for every* $x \in H$ *and for every* $t > 0$,

(iii) $F(x+y) + F(x-y) \le 2F(x) + 2F(y)$ *for every* $x, y \in H$,

then F is a quadratic form. Conversely, if F is a quadratic form, then (i)–(iii) are satisfied, and, in addition,

(iv) $F(tx) = t^2 F(x)$ *for every* $x \in H$ *and for every* $t \in \mathbb{R}$, $t \neq 0$,

(v) $F(x+y) + F(x-y) = 2F(x) + 2F(y)$ *for every* $x, y \in H$.

In this case, $D(F) := \{x \in H \mid F(x) < +\infty\}$ *is a linear space and the bilinear form* $\mathscr{E}$ *in* (2.5) *is given by*

$$\mathscr{E}(x,y) = \frac{1}{4}\left[F(x+y) - F(x-y)\right].$$

Proof. See [DM93, Proposition 11.9]. □

2.2.2 Coercive forms

Let $\mathscr{A}$ be a bilinear form on H with domain $\mathscr{V} \subset H$. The *symmetric part* $\widetilde{\mathscr{A}}$ of $\mathscr{A}$ is defined by

$$\widetilde{\mathscr{A}}(u,v) := \frac{1}{2}\left[\mathscr{A}(u,v) + \mathscr{A}(v,u)\right], \quad u, v \in \mathscr{V}.$$

The *antisymmetric part* $\check{\mathscr{A}}$ of $\mathscr{A}$ is defined by

$$\check{\mathscr{A}}(u,v) := \frac{1}{2}\left[\mathscr{A}(u,v) - \mathscr{A}(v,u)\right], \quad u, v \in \mathscr{V}.$$

It is clear that $\mathscr{A} = \widetilde{\mathscr{A}} + \check{\mathscr{A}}$. For $\alpha > 0$, set

$$\mathscr{A}_\alpha(u,v) := \mathscr{A}(u,v) + \alpha(u,v)_H, \quad u, v \in \mathscr{V}.$$

$\widetilde{\mathscr{A}_\alpha}$ is defined similarly. We suppose that $(\mathscr{A}, \mathscr{V})$ is a *coercive closed form* with *sector constant* $K \ge 1$, that is,

(1) $(\widetilde{\mathscr{A}}, \mathscr{V})$ is a non-negative definite, symmetric, closed form,

(2) $(\mathscr{A}, \mathscr{V})$ satisfies the *weak sector condition*, i.e., there exists a (sector) constant $K \ge 1$ such that

$$|\mathscr{A}_1(u,v)| \le K \mathscr{A}_1(u,u)^{1/2} \mathscr{A}_1(v,v)^{1/2} \quad \text{for all } u, v \in \mathscr{V}.$$

By *closed* we mean that $\mathscr{V}$, equipped with the norm $\|\cdot\|_{\mathscr{V}} := \widetilde{\mathscr{A}_1}(\cdot)^{1/2}$, is a Hilbert space. Identifying H with its dual H^* we obtain a dense and continuous embedding $\mathscr{V} \subset H \equiv H^* \subset \mathscr{V}^*$. The pairing between $\mathscr{V}$ and $\mathscr{V}^*$ is expressed by ${}_{\mathscr{V}^*}\langle\ ,\ \rangle_{\mathscr{V}}$.

2.2.3 Generalized forms

We continue with an introduction to so-called generalized (Dirichlet) forms, following the framework of [Sta99b, Section I]. We will present bilinear forms $\mathscr{E}$ which are associated with some coercive closed form $(\mathscr{A},\mathscr{V})$ and some properly chosen linear operator $(\Lambda, D(\Lambda, H))$.

Now suppose that $(\mathscr{A},\mathscr{V})$ is a *coercive closed form* with sector constant $K \geq 1$

Let Λ be a linear operator on H with a linear domain $D(\Lambda, H)$. We assume the following:

(1) Λ generates a C_0-semigroup of contractions $(U_t)_{t\geq 0}$ on H.

(2) $(U_t)_{t\geq 0}$ can be restricted to a C_0-semigroup of contractions on $\mathscr{V}$.

Denote the infinitesimal generator of the restricted semigroup by $(\Lambda, D(\Lambda,\mathscr{V}))$. Note that the adjoint operator $(\widehat{\Lambda}, D(\widehat{\Lambda},\mathscr{V}^*))$ of $(\Lambda, D(\Lambda,\mathscr{V}))$ also satisfies the conditions above. In particular, $D(\Lambda, H)\cap\mathscr{V}$ is dense in $\mathscr{V}$. It follows from [Sta99b, Lemma I.2.3] that $\Lambda : D(\Lambda, H)\cap\mathscr{V} \to \mathscr{V}^*$ is closable. Let us denote its closure by $(\Lambda,\mathscr{F})$. Then $\mathscr{F}$ is a Hilbert space with (graphical) inner product

$$(\cdot,\cdot)_{\mathscr{F}} := (\cdot,\cdot)_{\mathscr{V}} + (\Lambda\cdot,\Lambda\cdot)_{\mathscr{V}^*}.$$

Furthermore, define $\widehat{\mathscr{F}} := D(\widehat{\Lambda},\mathscr{V}^*)\cap\mathscr{V}$ with (graphical) inner product

$$(\cdot,\cdot)_{\widehat{\mathscr{F}}} := (\cdot,\cdot)_{\mathscr{V}} + (\widehat{\Lambda}\cdot,\widehat{\Lambda}\cdot)_{\mathscr{V}^*}.$$

$\mathscr{F}$ and $\widehat{\mathscr{F}}$ are dense in $\mathscr{V}$, ${}_{\mathscr{V}^*}\langle\Lambda u,u\rangle_{\mathscr{V}} \leq 0$ for $u\in\mathscr{F}$, ${}_{\mathscr{V}^*}\left\langle\widehat{\Lambda}u,u\right\rangle_{\mathscr{V}} \leq 0$ for $u\in\widehat{\mathscr{F}}$ and $D(\Lambda,\mathscr{V})$ is dense in $\mathscr{F}$, cf. [Sta99b, Lemma I.2.5] and [Sta99b, Lemma I.2.6].

Now for given $\mathscr{A}$ and Λ, define the *bilinear form $\mathscr{E}$ associated with $(\mathscr{A},\mathscr{V})$ and* $(\Lambda, D(\Lambda, H))$ on H by

$$\mathscr{E}(u,v) := \begin{cases} \mathscr{A}(u,v) - {}_{\mathscr{V}^*}\langle\Lambda u,v\rangle_{\mathscr{V}}, & \text{if } u\in\mathscr{F}, v\in\mathscr{V}, \\ \mathscr{A}(u,v) - {}_{\mathscr{V}^*}\left\langle\widehat{\Lambda}v,u\right\rangle_{\mathscr{V}}, & \text{if } u\in\mathscr{V}, v\in\widehat{\mathscr{F}}. \end{cases}$$

We extend $\mathscr{E}$ to a form defined on H and taking values in $\overline{\mathbb{R}}$ by setting $\mathscr{E}(u,v) = +\infty$ for every other case, even if $u\in H\setminus\mathscr{V}$ and $v=0$.

We also define the *co-form* $\widehat{\mathscr{E}}$ by

$$\widehat{\mathscr{E}}(u,v) := \begin{cases} \mathscr{A}(v,u) - {}_{\mathscr{V}^*}\left\langle\widehat{\Lambda}u,v\right\rangle_{\mathscr{V}}, & \text{if } u\in\widehat{\mathscr{F}}, v\in\mathscr{V}, \\ \mathscr{A}(v,u) - {}_{\mathscr{V}^*}\langle\Lambda v,u\rangle_{\mathscr{V}}, & \text{if } u\in\mathscr{V}, v\in\mathscr{F}. \end{cases}$$

Remark 2.5. *Let $(\mathscr{A},\mathscr{V})$ be a coercive closed form and $\Lambda = 0$. Clearly $\mathscr{F} = \mathscr{V} = \widehat{\mathscr{F}}$ and $\mathscr{E} = \mathscr{A}$ is a generalized form by* [Sta99b, Example I.4.9 (i)]. *This is the case of (Dirichlet) forms as described in* [MR92].

Let us recall some useful facts. As usually, we define for $\alpha > 0$

$$\mathscr{E}_\alpha(u,v) := \mathscr{E}(u,v) + \alpha(u,v)_H.$$

Proposition 2.6. *For all $\alpha > 0$ there exist continuous, linear bijections $W_\alpha : \mathscr{V}^* \to \mathscr{F}$ and $\widehat{W}_\alpha : \mathscr{V}^* \to \widehat{\mathscr{F}}$ such that*

$$\mathscr{E}_\alpha(W_\alpha f, v) = {}_{\mathscr{V}^*}\langle f, v\rangle_{\mathscr{V}} = \mathscr{E}_\alpha(v, \widehat{W}_\alpha f)$$

for all $f \in \mathscr{V}^*$, $v \in \mathscr{V}$. $(W_\alpha)_{\alpha>0}$ and $(\widehat{W}_\alpha)_{\alpha>0}$ satisfy the resolvent equation (cf. [Sta99b, Proposition I.3.4]*).*

Furthermore, there exists a unique C_0-resolvent $(G_\alpha)_{\alpha>0}$ and a unique C_0-coresolvent $(\widehat{G}_\alpha)_{\alpha>0}$ on H (being the restrictions of W_α, $\widehat{W}_\alpha$ resp. to H). such that for all $\alpha > 0$, $f \in H$ and $u \in \mathscr{V}$

$$\begin{aligned} &G_\alpha(H) \subset \mathscr{F}, \quad \widehat{G}_\alpha(H) \subset \widehat{\mathscr{F}}, \\ &\mathscr{E}_\alpha(G_\alpha f, u) = \mathscr{E}_\alpha(u, \widehat{G}_\alpha f) = (f,u)_H. \end{aligned} \tag{2.6}$$

$\widehat{G}_\alpha$ is the adjoint of G_α and αG_α, $\alpha\widehat{G}_\alpha$ are contraction operators. Also, we have for $u \in \mathscr{V}$ that

$$\lim_{\alpha\to\infty} \alpha G_\alpha u = u$$

strongly in $\mathscr{V}$ and thus in H.

Proof. See [Sta99b, Section I.3]. □

Note that the second line of (2.6) is equivalent with

$${}_{\mathscr{V}^*}\langle (M_\alpha - \Lambda)G_\alpha f, g\rangle_{\mathscr{V}} = (f,g)_H = {}_{\mathscr{V}^*}\left\langle (\widehat{M}_\alpha - \widehat{\Lambda})\widehat{G}_\alpha f, g\right\rangle_{\mathscr{V}}, \quad f \in H,\ g \in \mathscr{V},$$

where for $\alpha > 0$ we set $M_\alpha : \mathscr{V} \to \mathscr{V}^*$, ${}_{\mathscr{V}^*}\langle M_\alpha u, \cdot\rangle_{\mathscr{V}} := \mathscr{A}_\alpha(u,\cdot)$ and $\widehat{M}_\alpha : \mathscr{V} \to \mathscr{V}^*$, ${}_{\mathscr{V}^*}\left\langle \widehat{M}_\alpha u, \cdot\right\rangle_{\mathscr{V}} := \mathscr{A}_\alpha(\cdot, u)$.

Define approximate forms $\mathscr{E}^{(\beta)}$, $\beta > 0$ of $\mathscr{E}$ by

$$\mathscr{E}^{(\beta)}(u,v) = \beta(u - \beta G_\beta u, v)_H, \quad u,v \in H$$

and set $\mathscr{E}^{(\beta)}_\alpha(u,v) = \mathscr{E}^{(\beta)}(u,v) + \alpha(u,v)_H$.

Proposition 2.7. *(i) $\mathscr{E}^{(\beta)}(u,v) = \mathscr{E}(\beta G_\beta u, v)$ for $u \in H$, $v \in \mathscr{V}$.*

(ii) $\mathscr{E}^{(\beta)}(u,u) = \mathscr{E}(\beta G_\beta u, \beta G_\beta u) + \beta \|u - \beta G_\beta u\|_H^2$ for $u \in H$.

(iii) $\lim_{\beta\to\infty} \mathscr{E}^{(\beta)}(u,v) = \mathscr{E}(u,v)$ for $u \in \mathscr{F}$, $v \in \mathscr{V}$.

(iv) If $\sup_{\beta>0} \mathscr{E}_1^{(\beta)}(u,u) < \infty$, then $u \in \mathscr{V}$.

Proof. For (i)–(iii), see [MR92, Lemma I.2.11] and [Sta98, Proposition 2.7 (iii)]. For (iv) see [Hin98, Proposition 2.3 (iv)]. □

Let $(T_t)_{t\geq 0}$, $(\widehat{T}_t)_{t\geq 0}$ resp. be the C_0-semigroup of contractions, the C_0-cosemigroup of contractions resp. associated with $(G_\alpha)_{\alpha>0}$, $(\widehat{G}_\alpha)_{\alpha>0}$ resp.

2.3 Monotone operators

Let X, Y be two general sets. A *multi-valued map* F is a binary relation $\sim_F$ (or correspondence) between elements in X and Y. Define $F(x) := \{y \in Y \mid x \sim_F y\} \subset 2^Y$, where 2^Y is the power set of Y. Hence F becomes a map $F : X \to 2^Y$ which includes the empty set as a possible target. The *graph* $\Gamma(F) \subset X \times Y$ is defined as $[x, y] \in \Gamma(F)$ iff $x \sim_F y$. We consider F as a map and as a graph interchangeably ($F = \Gamma(F)$) and shall write $[x, y] \in F$ iff $y \in F(x)$ iff $x \sim_F y$. $D(F) := \{x \in X \mid F(x) \neq \varnothing\}$ is called the *effective domain* of F. $\operatorname{ran} F := \bigcup_{x \in X} F(x)$ is called the *range*. When each $F(x)$, $x \in D(F)$ contains exactly one element of Y, we identify F with the function $f : D(F) \to Y$, $f(x) := y$ whenever $[x, y] \in F$. A multi-valued map G is said to *extend* F if $F \subset G$ in the sense of graphs. A function $f : D(F) \to Y$ is called *selection* of F if $f(x) \in F(x)$ for all $x \in D(F)$.

For this section, fix a Banach space X with (topological) dual X^*. Let $J : X \to 2^{X^*}$ be the normalized duality map of X, see Appendix B.2.

Definition 2.8. *A graph $A \subset X \times X^*$ is called* monotone operator *or* monotone graph *if*

$$ {}_{X^*}\langle y_1 - y_2, x_1 - x_2 \rangle_X \geq 0 \quad \forall [x_1, y_1], [x_2, y_2] \in A. $$

A monotone operator A is called maximal monotone *if for any extension B of A it must hold: When B is monotone, then $B = A$.*

By an application of Zorn's Lemma, we see that any monotone graph has a maximal monotone extension. The following lemma is called "monotonicity trick". Compare with Lemmas 5.59, 7.22 below. Many authors take it as a definition of maximal monotonicity.

Lemma 2.9. *Let A be a monotone operator. Let $x \in D(A)$ and $y \in X^*$. Consider the following conditions:*

(i) For all pairs $[u, v] \in A$ it holds that

$$ {}_{X^*}\langle y - v, x - u \rangle_X \geq 0. $$

(ii) $[x, y] \in A$.

(ii) implies (i). (i) implies (ii) if and only if A is maximal monotone.

Proof. See [AC84, Ch. 3, §1, Proposition 1]. □

Lemma 2.10. *Let $A \subset X \times X^*$ be maximal monotone. Then $A(x)$ is convex and weak*-closed for all $x \in D(A)$. In particular, when X is reflexive, $A(x)$ is convex and strongly closed for all $x \in D(A)$.*

Proof. See [Dei85, Theorem 23.3] and the fact that by Mazur's Theorem in reflexive Banach spaces weak* closed convex sets are strongly closed, cf. [HHZ96, Theorem 56]. □

Lemma 2.11. *Suppose that X is reflexive. Let $A \subset X \times X^*$ be maximal monotone. Then both $\overline{D(A)}$ and $\overline{\mathrm{ran}(A)}$ are convex.*

Proof. See [Bar76, Ch. II, Corollary 1.2]. □

Definition 2.12. *Denote by A^0 the* principal section *of a maximal monotone operator A, i.e. the selection obtained by $A^0x := \mathrm{proj}_{Ax}(0)$, $x \in D(A)$, where "proj" denotes the metric projection, see e.g.* [Sin70b].

For $x \in D(A)$, A^0x are the elements of Ax with minimal norm. A^0 is single-valued if and only if X is reflexive and strictly convex (Definition B.1), see Lemma 2.10, [Sin70b]. For properties of A^0 in the Hilbert space case, we refer to [Bré73].

2.3.1 Resolvents and Yosida-approximation

The following result is known as Minty's Theorem in the Hilbert space case.

Theorem 2.13 (Browder-Rockafellar)**.** *Suppose that X is reflexive and has the Kadeč-Klee property (see Appendix B.3). Let $A : X \to 2^{X^*}$ be a monotone operator. Then A is maximal monotone if and only if*

$$\mathrm{ran}(A + J) = X^*.$$

In that case, $A + \varepsilon J$ is maximal monotone, one-to-one and onto for each $\varepsilon > 0$.

Proof. See [Bro76, Chapter 7]. □

Suppose that X is reflexive and has the Kadeč-Klee property (see Appendix B.3). Let $A : X \to 2^{X^*}$ be maximal monotone. For every $\lambda > 0$, for every $x \in X$, there exists a unique element $R^A_\lambda x$ belonging to $D(A)$ such that

$$J(R^A_\lambda x - x) + \lambda A(R^A_\lambda x) \ni 0.$$

We introduce the notation

$$A_\lambda x := \frac{1}{\lambda} J(x - R^A_\lambda x),$$

thus

$$A_\lambda x \in A(R^A_\lambda x).$$

Definition 2.14. *The family of operators $(A_\lambda)_{\lambda>0}$ is called the* Yosida-approximation *of A and the family of operators $(R^A_\lambda)_{\lambda>0}$ is called the* resolvent *of A.*

Remark 2.15. *When A is linear and maximal monotone on a Hilbert space H, then $R^A_\lambda = (1/\lambda)G^{-A}_{1/\lambda}$ for each $\lambda > 0$, where $(G^A_\alpha)_{\alpha>0}$ is the resolvent as defined in 2.1.*

Proposition 2.16. *Suppose that X is reflexive and has the Kadeč-Klee property (see Appendix B.3). Let $A : X \to 2^{X^*}$ be maximal monotone. Then:*

(i) *The operators $R^A_\lambda : X \to X$ and $A_\lambda : X \to X^*$ are continuous (both X and X^* equipped with the strong (normed) topologies). When X is a Hilbert space, the resolvents R^A_λ are contraction maps while each A_λ is a Lipschitz map with continuity constant $1/\lambda$.*

(ii) *For $\lambda > 0$, A_λ is a maximal monotone operator and the family $(A_\lambda)_{\lambda>0}$ satisfies the* resolvent equation

$$(A_\lambda)_\mu = A_{\lambda+\mu} \quad \forall \lambda, \mu > 0.$$

(iii) *The net $\{A_\lambda\}$ converges in the graph sense to A as $\lambda \searrow 0$, that is,*

$$\forall [x,y] \in A,\ \exists x_\lambda \in X,\ \lambda > 0 :\ \lim_{\lambda \searrow 0} \|x_\lambda - x\|_X = 0,\ \lim_{\lambda \searrow 0} \|A_\lambda x_\lambda - y\|_{X^*} = 0.$$

Moreover,

$$\forall x \in D(A) :\ \lim_{\lambda \searrow 0} \left\|A_\lambda x - A^0 x\right\|_{X^*} = 0.$$

Here A^0 denotes the principal section *of A, i.e. the selection obtained by $A^0 x := \operatorname{proj}_{Ax}(0)$, $x \in D(A)$. Also*

$$\forall x \in X :\ \lim_{\lambda \searrow 0} \left\|R^A_\lambda x - \operatorname{proj}_{\overline{D(A)}} x\right\|_X = 0.$$

*In both terms "*proj*" denotes the metric projection, see e.g.* [Sin70b].

The proof can be found in [Att84, Proposition 3.56]. We recall an important inequality from this proof, used later in the proof of Theorem 7.23. For any $[x_0, y_0] \in A$, $\lambda > 0$, $x \in X$:

$$\left\|x - R^A_\lambda x\right\|_X^2 \leq \left\|x - R^A_\lambda x\right\|_X (\|x - x_0\|_X + \lambda \|y_0\|_{X^*}) + \lambda \|y_0\|_{X^*} \|x - x_0\|_X \qquad (2.7)$$

and thus

$$\left\|x - R^A_\lambda x\right\|_X \leq 2 (\|x - x_0\|_X + \lambda \|y_0\|_{X^*}) \qquad (2.8)$$

and R^A_λ is a bounded operator.

2.3.2 Semigroups

For this paragraph, fix a Hilbert space H and identify it with its dual.

Definition 2.17. *Let $C \subset H$. A family of (possibly nonlinear) maps $S_t : C \subset H \to H$, $t \geq 0$ is called* (nonlinear) nonexpansive semigroup *on C (or plainly* semigroup*) if it satisfies the following conditions:*

(i) *$S_0 = \mathrm{Id}_C$, $S_t(C) \subset C$ and $S_t \circ S_s = S_{t+s}$ for all $t, s \geq 0$.*

(ii) *$\lim_{t \searrow 0} \|S_t x - x\|_H = 0$ for all $x \in C$.*

(iii) *$\|S_t x - S_t y\|_H \leq \|x - y\|_H$ for all $x, y \in C$ and for all $t \geq 0$.*

Theorem 2.18 (Komura-Kato)**.** *Let $A \subset H \times H$ be maximal monotone. Let $A^0(x) := \mathrm{proj}_{Ax}(0)$, $x \in D(A)$ be its principal section. Then the initial value problem*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} u(t) + Au(t) \ni 0,\ 0 < t < \infty, \\ u(0) = f \end{aligned} \tag{2.9}$$

has a unique solution $t \mapsto S_t f$ for all $f \in D(A)$, where the time-derivative is to be understood in the sense of weak convergence in H. Moreover,

(i) $S_t f \in D(A)$ for all $f \in D(A)$, $t \geq 0$,

(ii) $t \mapsto S_t f$ is Lipschitz continuous on $[0, \infty)$ for all $f \in D(A)$,

(iii) For $\mathrm{d}t$-a.a. $t \in (0, \infty)$ the derivative $\frac{\mathrm{d}}{\mathrm{d}t} S_t f$ exists in the strong sense and solves (2.9). Furthermore, $\left\|\frac{\mathrm{d}}{\mathrm{d}t} S_t f\right\|_H \leq \left\|A^0 f\right\|_H$,

and $(S_t)_{t \geq 0}$ is a nonexpansive semigroup on $D(A)$ which can be uniquely extended to a nonexpansive semigroup on $\overline{D(A)}$.

Proof. See [Kom67], [Bré73, Théorème 3.1] and [Zei90b, Theorem 31.A]. □

Conversely, a Hille-Yosida-Phillips-type theorem holds:

Theorem 2.19 (Brézis)**.** *Let $(S_t)_{t \geq 0}$ be a semigroup on a (strongly) closed convex subset $C \subset H$. Then there exists a unique maximal monotone operator $A \subset H \times H$ such that $\overline{D(A)} = C$ and $(S_t)_{t \geq 0}$ coincides with solution of (2.9) for A. In particular, we have*

$$\|\cdot\|_H - \lim_{t \searrow 0} \frac{S_t x - x}{t} = -A^0 x \quad \forall x \in D(A). \tag{2.10}$$

Proof. See [Bré73, Théorème 4.1]. □

In the situation of Theorems 2.18, 2.19, $-A$ is called the *generator* of the semigroup $(S_t)_{t \geq 0}$.

2.4 Convex functionals

Definition 2.20. *A function $f : X \to \overline{\mathbb{R}}$ on a topological space X is called* lower semi-continuous at a point $x \in X$ *if for every $t \in \mathbb{R}$ with $t < f(x)$, there is an open neighborhood U of x such that $t < f(y)$ for all $y \in U$. We say that f is* lower semi-continuous *if it is lower semi-continuous at every point $x \in X$.*

Remark 2.21. *In this work, we shall use lower semi-continuity exclusively w.r.t. the strong Banach space topology. Also, in this work, we usually abbreviate "lower semi-continuous" to "*l.s.c.*".*

Lemma 2.22. *Suppose that X satisfies the first axiom of countability. Let $f : X \to \overline{\mathbb{R}}$ be a function and let $x \in X$. The following statements are equivalent:*

(i) f *is l.s.c. at* x.

(ii) $f(x) \leq \underline{\lim}_n f(x_n)$ *for every sequence* $\{x_n\}$ *which converges to* x *in* X.

(iii) $f(x) \leq \lim_n f(x_n)$ *for every sequence* $\{x_n\}$ *which converges to* x *in* X *such that* $\lim_n f(x_n)$ *exists and is less than* $+\infty$.

Proof. See [DM93, Proposition 1.3]. □

For the sequel, recall that $\mathbb{R}_\infty = (-\infty, +\infty]$.

Definition 2.23. *Let* X *be a Banach space and* $f : X \to \mathbb{R}_\infty$ *a functional.* f *is called* convex *if*

$$f(tu + (1-t)v) \leq tf(u) + (1-t)f(v) \tag{2.11}$$

for every $u, v \in X$, $t \in [0,1]$. *If* (2.11) *holds with "*$\leq$*" replaced by "*$<$*" whenever* $t \in (0,1)$ *and* $u \neq v$, f *is called* strictly convex. f *is called* (strictly) concave *if* $-f$ *is (strictly) convex.* f *is called* proper *if* $f(u) < +\infty$ *for some* $u \in X$. *We write* $D(f) := \{u \in X \mid f(u) < +\infty\}$ *for the* effective domain. *For any set* $S \subset X$ *we define the* indicator function $I_S : X \to \{0, +\infty\}$ *by*

$$I_S(u) := \begin{cases} 0, & \text{if } u \in S, \\ +\infty, & \text{if } u \in X \setminus S. \end{cases}$$

I_S *is not to be confused with the indicator function* 1_S. *The* epigraph *of* f *is given by*

$$\operatorname{epi}(f) := \{[u,a] \in X \times \mathbb{R} \mid f(u) \leq a\}.$$

Remark 2.24. I_S *is proper if and only if* S *is non-void.* I_S *is convex if and only if* S *is convex.* $u \in D(f)$ *if and only if* $[u,a] \in \operatorname{epi}(f)$ *for some* $a \in \mathbb{R}$.

Lemma 2.25. *Let* X *be a Banach space and* $f : X \to \mathbb{R}_\infty$. *If* f *is convex and* $\lambda \geq 0$, *then* λf *is convex. If* f_1, f_2 *are convex, then* $f_1 + f_2$ *is convex. If each* f_i, $i \in I$ *(any index set) is convex, then* $\sup_{i\in I} f_i$ *is convex and* $\operatorname{epi}(\sup_{i\in I} f_i) = \bigcap_{i\in I} \operatorname{epi}(f_i)$. f *is proper, l.s.c. and convex if and only if* $\operatorname{epi}(f)$ *is, respectively, non-void, (strongly) closed and convex in* $X \times \mathbb{R}$.

Proof. See [Sho97, Proposition II.7.1]. □

We shall not need the next Proposition but include it on account of completeness.

Proposition 2.26. *A proper, convex, l.s.c.* $f : X \to \mathbb{R}_\infty$ *on a Banach space* X *is strongly continuous on* $\operatorname{int} D(f)$.

Proof. See [Sho97, Proposition II.7.3]. □

2.4.1 The subgradient

Definition 2.27. *Let $f : X \to \mathbb{R}_\infty$ be proper and convex on a Banach space X. The* subdifferential *or* subgradient *of f is a possibly multi-valued operator $\partial f : X \to 2^{X^*}$ defined by $[x, x^*] \in \partial f$ if*

$$f(x) - f(u) \leq {}_{X^*}\langle x^*, x - u\rangle_X \qquad \forall u \in X, \tag{2.12}$$

which makes sense if $x \in D(f)$. Set $D(\partial f) := \{x \in X \mid \partial f(x) \neq \varnothing\}$, which is called the effective domain *of ∂f. If $x \in D(\partial f)$, we say that f is* subdifferentiable *at x.*

It is obvious that $D(\partial f) \subset D(f)$. Conversely, if f is l.s.c., proper and convex, $\operatorname{int} D(f) \subset D(\partial f)$ and $D(\partial f)$ is dense in $D(f)$, cf. [BP86, Ch. 2, §2, Corollaries 2.1, 2.2].

Lemma 2.28. *If a convex function $f : X \to \mathbb{R}_\infty$ is finite and continuous at some point $x_0 \in X$, then $x_0 \in D(\partial f)$, i.e., f is subdifferentiable at x_0.*

Proof. See [BP86, Ch. 2, §2, Proposition 2.2]. □

Definition 2.29. *Let $f : X \to \mathbb{R}_\infty$ be proper and convex on a Banach space X. For any $\varepsilon > 0$ define the* ε-subgradient $\partial_\varepsilon f$ of f *by*

$$\partial_\varepsilon f(x) := \{x^* \in X^* \mid f(u) - f(x) + \varepsilon \geq {}_{X^*}\langle x^*, u - x\rangle_X \ \forall u \in X\}, \quad x \in X.$$

Remark 2.30. *In the above definition, for any $\varepsilon > 0$ and any $x \in X$, it holds that $\partial_\varepsilon f(x) = \{x^* \in X^* \mid f(x) + f^*(x^*) - {}_{X^*}\langle x^*, x\rangle_X \leq \varepsilon\}$, where f^* denotes the Legendre transform of f, see Paragraph 2.4.3 below.*

The next lemma is known as the "Brøndsted–Rockafellar Lemma" in the literature. Its proof involves Zorn's Lemma.

Lemma 2.31 (Brøndsted–Rockafellar)**.** *Let $f : X \to \mathbb{R}_\infty$ be proper, l.s.c. and convex on a Banach space X. Let $\varepsilon > 0$. If $[x, x^*] \in \partial_\varepsilon f$, then there exists a pair $[y, y^*] \in \partial f$ such that*

$$\|x - y\|_X \leq \sqrt{\varepsilon} \quad \text{and} \quad \|x^* - y^*\|_{X^*} \leq \sqrt{\varepsilon}.$$

Proof. See [BR65]. □

Definition 2.32. *An operator $A : X \to 2^{X^*}$ is called* cyclically monotone, *if*

$$\forall n \in \mathbb{N} : \quad \sum_{i=1}^{n} {}_{X^*}\langle y_i, x_i - x_{i+1}\rangle_X \geq 0 \quad \forall [x_i, y_i] \in A,\ 1 \leq i \leq n,\ x_{n+1} := x_1.$$

A is called maximal cyclically monotone *if its graph is not properly contained in any other cyclically monotone graph.*

Theorem 2.33 (Rockafellar)**.** *If $f : X \to \mathbb{R}_\infty$ is a proper, l.s.c. and convex functional on a Banach space X, then $\partial f : X \to 2^{X^*}$ is a maximal cyclically monotone operator. Let $A : X \to 2^{X^*}$. In order that there exists a proper, l.s.c., convex functional $f : X \to \mathbb{R}_\infty$ such that $A = \partial f$, it is necessary and sufficient that A be a maximal cyclically monotone operator. Moreover, in this case A determines f up to an additive constant. Indeed,* (2.13) *in Theorem 2.34 below holds with $\partial f = A$ (where one can start with any $x_0 \in D(A)$).*

Proof. See [Roc70, Theorems A and B]. See also [BP86, Ch. 2, §2, Theorem 2.2]. □

Theorem 2.34 (Rockafellar)**.** *Let X be a Banach space and $f : X \to \mathbb{R}_\infty$ be proper, l.s.c. and convex. For each $x \in X$ and $x_0 \in D(\partial f)$*

$$f(x) = \sup_{x_0 \curvearrowright x} \left\{ f(x_0) + \sum_{i=1}^{n-1} {}_{X^*}\langle x_i^*, x_{i+1} - x_i \rangle_X \right\}, \tag{2.13}$$

where $x_0 \curvearrowright x$ denotes all finite chains $x_0 = x_1, x_2, \ldots, x_n = x$ contained in $D(\partial f)$, and where $x_i^* \in \partial f(x_i)$.

Proof. See [Roc66, Roc70]. □

Equation (2.13) is called *integration formula* for subgradients.

2.4.2 Gâteaux and Fréchet differentiability

Definition 2.35. *Let X be a normed linear space. A function $f : X \to \mathbb{R}$ is said to be* Gâteaux differentiable *at a point $x \in X$ if the* directional derivative in direction $h \in X$

$$f'(x,h) := \lim_{t \searrow 0} \frac{f(x+th) - f(x)}{t}$$

exists for each $h \in X$ and if $h \mapsto f'(x,h)$ is a linear and strongly continuous map. In this case we denote this map by $\nabla_G f$ or $\nabla_G f(x)$ which is an element in X^* that satisfies

$${}_{X^*}\langle \nabla_G f(x), h \rangle_X = f'(x,h) \quad \forall h \in X.$$

f is called Fréchet differentiable *at a point $x \in X$ if it is Gâteaux differentiable at x and for any $\varepsilon > 0$ there is $\delta > 0$ such that*

$$|f(x+h) - f(x) - {}_{X^*}\langle \nabla_G f(x), h \rangle_X| \le \varepsilon \|h\|_X \quad \textit{whenever} \quad \|h\|_X < \delta.$$

In that case we write $\nabla_F = \nabla_G$.

Proposition 2.36. *If a convex function f is Gâteaux differentiable at x_0, then $\partial f(x_0)$ consists of a single element $x_0^* = \nabla_G f(x_0)$. Conversely, if f is continuous at x_0 and if $\partial f(x_0)$ contains a single element, then f is Gâteaux differentiable at x_0 and $\nabla_G f(x_0) = \partial f(x_0)$.*

Proof. See [BP86, Ch. 2, §2, Proposition 2.4]. □

Remark 2.37. *In connection with Proposition 2.36, we would like to mention that generic differentiability of continuous convex functions on Banach spaces has been an active interest of research for quite many years now. The class of Banach spaces known best in this context are the so-called* Asplund spaces*. Skipping the details, let us only recall the result that if X is a separable Banach space, then any continuous convex function defined on a non-empty open convex subset D of X is generically Gâteaux differentiable at each point of some dense G_δ subset of D. If X^* is separable, the corresponding statement holds with Gâteaux replaced by Fréchet. For details, we refer to* [Phe89].

2.4.3 The Legendre-Fenchel transform

Definition 2.38. *Let $f : X \to \mathbb{R}_\infty$. Define the* Legendre-Fenchel transform *(or plainly the* Legendre transform*) $f^* : X^* \to \mathbb{R}_\infty$ by*

$$f^*(x^*) := \sup_{x \in X} \left[{}_{X^*}\langle x^*, x \rangle_X - f(x) \right], \quad x^* \in X^*. \tag{2.14}$$

f^* is also called the conjugate function*. The* biconjugate function *$f^{**} : X \to \mathbb{R}_\infty$ is defined via*

$$f^{**}(x) := \sup_{x^* \in X^*} \left[{}_{X^*}\langle x^*, x \rangle_X - f^*(x^*) \right], \quad x \in X. \tag{2.15}$$

Conjugation reverses the order, that is, $f \geq g$ implies $g^* \geq f^*$. Also $f^{**} \leq f$. f^* and f^{**} are always convex, cf. [BP86, Ch. 2, §1, Proposition 1.8].

Theorem 2.39 (Fenchel–Moreau)**.** *Let $f : X \to \mathbb{R}_\infty$ be proper. Then $f^{**} = f$ if and only if f is convex and l.s.c.*

Proof. See [BP86, Ch. 2, §1, Theorem 1.4]. □

Clearly, for every $x \in X$ and every $x^* \in X^*$

$$f(x) + f^*(x^*) \geq {}_{X^*}\langle x^*, x \rangle_X \tag{2.16}$$

and

$$f^*(x^*) + f^{**}(x) \geq {}_{X^*}\langle x^*, x \rangle_X \,. \tag{2.17}$$

We also define the conjugate $(f^*)^* : X^{**} \to \mathbb{R}_\infty$ of the conjugate. It coincides with f^{**} if X is reflexive. In general, the restriction of $(f^*)^*$ to X coincides with f^{**}.

Proposition 2.40. *Let $f : X \to \mathbb{R}_\infty$ be a proper convex function. Then the following three conditions are equivalent:*

(i) $[x, x^*] \in \partial f$.

(ii) $f(x) + f^*(x^*) \leq {}_{X^*}\langle x^*, x \rangle_X$.

(iii) $f(x) + f^*(x^*) = {}_{X^*}\langle x^*, x \rangle_X$.

If, in addition, f is l.s.c., then (i)–(iii) are equivalent to:

(iv) $[x^*, x] \in \partial f^*$.

Proof. See [BP86, Proposition 2.2.1]. □

If one defines for a proper convex functional $g : X^* \to \mathbb{R}_\infty$ the *dual subgradient* $\partial^* f : X^* \to 2^X$ by $[x^*, x] \in \partial^* g$ if

$$g(x^*) - g(u^*) \leq {}_{X^*}\langle x^* - u^*, x \rangle_X \qquad \forall u^* \in X^*, \tag{2.18}$$

then by [BP86, Ch. 2, §3, Remark 2.1]

$$[x, x^*] \in \partial f \quad \Leftrightarrow \quad [x^*, x] \in \partial^* f^*. \tag{2.19}$$

If X is reflexive, $\partial^* g = \partial g$, for $g : X^* \to \mathbb{R}_\infty$. If one instead considers $\partial f^* \subset X^* \times X^{**}$ in the non-reflexive case, the situation is more complicated, see [Roc70]. We note also that, in Hilbert spaces, two possible definitions for subgradients exist, namely, one involving the inner product and one involving the dualization. They are interchangeable by application of the Riesz map.

2.4.4 Infimal convolution and Moreau-Yosida approximation

Definition 2.41. *Let X be a normed space. Let $C \subset X$ be a set. A function $f : X \to \mathbb{R}_\infty$ is called* weakly coercive over C *if*

$$\lim_{n\to\infty} f(x_n) = +\infty, \quad \textit{whenever } x_n \in C,\ n \in \mathbb{N},\ \lim_n \|x_n\|_X = +\infty.$$

f is plainly called weakly coercive *if it is weakly coercive over X.*
A function $f : X \to \mathbb{R}_\infty$ is called coercive over C *if*

$$\lim_{n\to\infty} \frac{f(x_n)}{\|x_n\|_X} = +\infty, \quad \textit{whenever } x_n \in C,\ n \in \mathbb{N},\ \lim_n \|x_n\|_X = +\infty.$$

f is plainly called coercive *if it is coercive over X.*

Lemma 2.42. *Let X be a reflexive Banach space. If $f : X \to \mathbb{R}_\infty$ is proper, convex, l.s.c. and coercive,* $\operatorname{ran} \partial f = X^*$.

Proof. See [BP86, Ch. 2, §2, Proposition 2.5]. □

Proposition 2.43. *Let X be a reflexive Banach space. Let $C \subset X$ be a non-empty closed convex subset. Let $f : X \to \mathbb{R}_\infty$ be convex, l.s.c. and proper. Suppose either that C is bounded or that f is weakly coercive over C. Then*

$$\arg\min_{u\in C} f(u)$$

is non-empty. Moreover, if f is strictly convex, it contains exactly one element.

Proof. See [ET76, Ch. II, Proposition 1.2]. □

Definition 2.44. *Let X be a normed linear space. Given two functions $\varphi, \psi : X \to \mathbb{R}_\infty$ their* infimal convolution *$\varphi \diamond \psi$ is the well-defined function from X to $\mathbb{R}_\infty$ equal to*

$$(\varphi \diamond \psi)(x) := \inf_{u \in X} [\varphi(u) + \psi(x-u)], \quad x \in X.$$

If φ, ψ are convex, so is $\varphi \diamond \psi$, cf. [Mor70].

Lemma 2.45. *Let X be a normed linear space. Suppose that we are given two functions $\varphi, \psi : X \to \mathbb{R}_\infty$. Then*

$$(\varphi \diamond \psi)^* = \varphi^* + \psi^*.$$

Suppose that X is a reflexive Banach space and both φ, ψ are l.s.c, convex, proper and $D(\varphi) - D(\psi)$ is a neighborhood of the origin, then

$$(\varphi + \psi)^* = \varphi^* \diamond \psi^*.$$

Proof. See [Att84, p. 267–268, Proposition 3.4]. □

Proposition 2.46. *Let X be a normed linear space and $f : X \to \mathbb{R}_\infty$ proper, convex and l.s.c. Then, for every $\lambda > 0$ the so-called* Moreau-Yosida approximation

$$(f)_\lambda := f \diamond \frac{1}{2\lambda} \|\cdot\|_X^2 \tag{2.20}$$

is a continuous convex function. Moreover,

$$\lim_{\lambda \searrow 0} (f)_\lambda = \sup_{\lambda > 0} (f)_\lambda = f$$

pointwise.

If X is a reflexive Banach space, the function to be minorized in (2.20) is weakly coercive by an argument similar to [ET76, Ch. II, p. 39].

By Lemma 2.45 for proper, convex, l.s.c. f, g,

$$(f_\lambda)^* = f^* + \frac{\lambda}{2} \|\cdot\|_{X^*}^2 \tag{2.21}$$

and if X is a reflexive Banach space,

$$(g^*)_\lambda = \left[g + \frac{\lambda}{2} \|\cdot\|_X^2 \right]^* \tag{2.22}$$

Theorem 2.47. *Let $f : X \to \mathbb{R}_\infty$ be proper, convex and l.s.c. Let $A := \partial f$. Then $(f)_\lambda$ is Gâteaux differentiable on X and $A_\lambda = \partial f_\lambda$ (the Yosida approximation) for every $\lambda > 0$. In addition,*

$$(f)_\lambda(x) = \frac{\lambda}{2} \|A_\lambda x\|^2 + f(R_\lambda x) \quad \forall \lambda > 0 \, \forall x \in X$$

and

$$f(R_\lambda x) \le (f)_\lambda(x) \le f(x) \quad \forall \lambda > 0 \, \forall x \in X.$$

Also, if X is a reflexive Banach space, the infimum in (2.20) *is attained (uniquely) and is equal to the resolvent R_λ of A, compare with Paragraph 2.3.1.*

Proof. See [BP86, Ch. 2, §2.3, Theorem 2.3] and [Att84, Theorem 3.24]. □

2.5 A variational setting

We present a non-standard variational setting for convex functionals based on an embedding $V \hookrightarrow X$, where V and X are reflexive Banach spaces (usually X is a Hilbert space). Nevertheless, in the case of linear symmetric operators this framework is well-known (see e.g. [Tan79] and the preceding Paragraph 2.2.1 about quadratic forms).

2.5.1 Embeddings of Banach spaces

Let X be a separable reflexive Banach space with dual X^*. Let V be a Banach space. Suppose that V can be embedded linearly, densely and continuously (w.r.t. the strong topologies) into X, i.e., there is a linear one-to-one map $i : V \to X$ with dense range such that for a positive constant M_0 it holds that

$$\|i(v)\|_X \leq M_0 \|v\|_V . \tag{2.23}$$

We shall write $V \overset{i}{\hookrightarrow} X$ and occasionally identify $i(V)$ with V such that $V \subset X$ is a dense linear subspace equipped with a stronger topology.

Let $i^* : X^* \to V^*$ be the *linear adjoint* of i, i.e., ${}_{V^*}\langle i^*(x^*), v\rangle_V := {}_{X^*}\langle x^*, i(v)\rangle_X$. If $x^*\restriction_V$ denotes the restriction of $x^* \in X^*$ to $i(V)$, then for $v \in V$

$$|{}_{X^*}\langle x^*\restriction_V, i(v)\rangle_X| = |{}_{X^*}\langle x^*, i(v)\rangle_X| \leq \|x^*\|_{X^*} \|i(v)\|_X \leq M_0 \|x^*\|_{X^*} \|v\|_V . \tag{2.24}$$

Hence i^* is continuous. Since $i(V)$ is dense in X, the correspondence $x^* \mapsto x^*\restriction_V$ and hence the map i^* is one-to-one, so that we can identify X^* with $i^*(X^*)$ and may write $X^* \subset V^*$. Since $\|x^*\restriction_V\|_{V^*} \leq M_0 \|x^*\|_{X^*}$ by (2.24), X^* has a stronger topology than V^*.

Lemma 2.48. *Denote by J_X the normalized duality map of X, see Appendix B.2. $i^* J_X i(V)$ is total in V^* (see Definition B.19). In particular, $i^*(X^*)$ is dense in V^*, whenever V is reflexive by Lemma B.20.*

Proof. Let $v \in V \setminus \{0\}$, $w \in J_X(i(v))$. Then

$${}_{V^*}\langle i^*(w), v\rangle_V = {}_{X^*}\langle w, i(v)\rangle_X = \|i(v)\|_X^2 \neq 0,$$

since $v \neq 0$ and i is one-to-one. □

Lemma 2.49. *If V is reflexive, it is separable.*

Proof. Since X is assumed to be separable so is X^* by reflexivity. But X^* is densely and continuously embedded into V^* by the lemma proved before. Hence V^* is separable by an elementary $\varepsilon/2$ argument. Reflexivity again proves that V is separable. □

If V and X are reflexive, we shall refer to the correspondence

$$V \overset{i}{\hookrightarrow} X \overset{J}{\to} X^* \overset{i^*}{\hookrightarrow} V^* \tag{G}$$

as an *evolution quadruple* or a *Gelfand quadruple*. In the literature, however, X is usually assumed to be a separable Hilbert space. Note that J might be multi-valued but it is onto if and only if X is reflexive by [Dei85, Theorem 12.3]. It is single valued e.g. when X is smooth, see Lemma B.3. We shall call $V \overset{i}{\hookrightarrow} X$ plainly a *strong embedding*.

2.5.2 Embeddings and convex functionals

Let V, X be reflexive Banach spaces. Suppose that $V \overset{i}{\hookrightarrow} X$ is a strong embedding. Let $X^* \overset{i^*}{\hookrightarrow} V^*$ be the dual embedding. Suppose that we are given a finite-valued, convex functional $g : V \to \mathbb{R}$ with $D(g) = V$. Define $f : X \to \mathbb{R}_\infty$ by

$$f(x) := \begin{cases} g(i^{-1}(x)), & if\ x \in i(V), \\ +\infty, & if\ x \in X \setminus i(V). \end{cases}$$

Then f is proper and convex. Furthermore,

$$g(v) = f(i(v)), \quad \forall v \in V$$

and

$$D(f) = i(D(g)) = i(V)$$

and

$$D(\partial g) \subset V, \quad D(\partial f) \subset i(V).$$

The following elementary proposition was proved by Ottavio Caligaris in [Cal76, Dimostrazione della Proposizione 4.9] (when X is a Hilbert space).

Proposition 2.50. *In the above situation,*

$$i^*(\partial f(i(v))) = \partial g(v) \cap i^*(X^*) \quad \forall v \in V.$$

Proof. Let $v \in V$. We prove $i^*(\partial f(i(v))) \supset \partial g(v) \cap i^*(X^*)$ first. Let $[v, w] \in \partial g \cap (V \times i^*(X^*))$. If no such pair exists, the claim is trivial. Now,

$$g(z) - g(v) \geq {}_{V^*}\langle w, z - v\rangle_V \quad \forall z \in V.$$

Also,

$$f(i(z)) - f(i(v)) \geq {}_{X^*}\big\langle (i^*)^{-1}(w), i(z - v)\big\rangle_X \quad \forall z \in V$$

and hence

$$f(y) - f(i(v)) \geq {}_{X^*}\big\langle (i^*)^{-1}(w), y - i(v)\big\rangle_X \quad \forall y \in i(V),$$

which yields by definition of f

$$f(y) - f(i(v)) \geq {}_{X^*}\big\langle (i^*)^{-1}(w), y - i(v)\big\rangle_X \quad \forall y \in X.$$

We have proved that $[i(v), (i^*)^{-1}(w)] \in \partial f$ and hence $w \in i^*(\partial f(i(v)))$.

We prove $i^*(\partial f(i(v))) \subset \partial g(v) \cap i^*(X^*)$. Let $[v, w] \in i^*(\partial f(i(v)))$. If no such pair exists, we are done. Now, $w \in i^*(X^*)$ and $(i^*)^{-1}(w) \in \partial f(i(v))$, hence

$$f(y) - f(i(v)) \geq {}_{X^*}\big\langle (i^*)^{-1}(w), y - i(v)\big\rangle_X \quad \forall y \in X,$$

and, in particular,

$$f(y) - f(i(v)) \geq {}_{X^*}\big\langle (i^*)^{-1}(w), y - i(v)\big\rangle_X \quad \forall y \in i(V) \subset X,$$

and hence

$$f(i(z)) - f(i(v)) \geq {}_{X^*}\big\langle (i^*)^{-1}(w), i(z - v)\big\rangle_X \quad \forall z \in V.$$

We get that

$$g(z) - g(v) \geq {}_{V^*}\langle w, z - v\rangle_V \quad \forall z \in V.$$

We have proved that $[v, w] \in \partial g$. Noting that $w \in i^*(X^*)$ completes the proof. $\square$

If we identify $i(V)$ with V and $i^*(X^*)$ with X^*, and if g is l.s.c., we are able to summarize

$$\begin{aligned}
& D(f) = D(g) = V, \\
\forall v \in V: \quad & f(v) = g(v), \\
\forall v \in V: \quad & \partial f(v) = \partial g(v) \cap X^*, \\
& D(\partial g) \subset V, \\
& D(\partial f) \subset V,
\end{aligned}$$

if, additionally, g is coercive in V, and f is l.s.c. in X, by Lemma 2.42 and Proposition 2.50, we have that

$$\begin{aligned}
\operatorname{ran} \partial g &= V^*, \\
\operatorname{ran} \partial f &= X^*,
\end{aligned}$$

and when g is strongly continuous in V, by Lemma 2.28,

$$D(\partial g) = V.$$

Corollary 2.51. *When $g : V \to \mathbb{R}$ is Gâteaux differentiable in V, then $\partial f : D(\partial f) \subset V \subset X \to X^*$ is single-valued.*

Proof. Immediate from Propositions 2.36, 2.50 and the fact that the embedding $X^* \hookrightarrow V^*$ is one-to-one. $\square$

In some cases, one can verify lower semi-continuity of f easily:

Lemma 2.52. *Suppose that V and X are reflexive with a strong embedding $V \hookrightarrow X$. Let f, g be as above and suppose that g is l.s.c in V. Let S be a set in X. Suppose that $\sup_{x \in S} \|x\|_X < +\infty$ and that $\sup_{x \in S} f(x) < +\infty$ always imply that $\sup_{x \in S} \|x\|_V < +\infty$. Then f is l.s.c. in X.*

Proof. Clearly, $\sup_{x\in S}\|x\|_X < +\infty$ and $\sup_{x\in S} f(x) < +\infty$ imply that $S \subset V$ and $\sup_{x\in S} g(x) < +\infty$. Let $u_n, u \in X$, $n \in \mathbb{N}$ with $u_n \to u$ strongly in X. Suppose that $\underline{\lim}_n f(u_n) < +\infty$. Extract a subsequence of $\{u_n\}$ (also denoted by $\{u_n\}$) such that $\lim_n f(u_n) = C \in \mathbb{R}$. Hence $\{\|u_n\|_V\}$ is bounded by the assertion. But by reflexivity of V, we can extract a subsequence (also denoted by $\{u_n\}$) such that $u_n \rightharpoonup v$ weakly in V for some $v \in V$, see e.g. [Yos78, Ch. V, §2, Theorem 1]. By Mazur's Lemma (cf. Lemma B.7) there exists a sequence of convex combinations

$$v_n := \sum_{k=n}^{N_n} \lambda_k^{(n)} u_k, \quad \text{with} \ \sum_{k=n}^{N_n} \lambda_k^{(n)} = 1, \ \ \lambda_k^{(n)} \geq 0, \ \ n \leq k \leq N_n,$$

such that $v_n \to v$ strongly in V. But also by Lemma B.7, $v_n \to u$ strongly in X since $u_n \to u$ strongly in X. Therefore $v = u$. Since we can do so for every subsequence, $v_n \to u$ strongly in V. By applying (B.2), convexity and lower semi-continuity of g successively, we get that

$$\lim_n f(u_n) = \lim_n g(u_n) \geq \lim_n \sum_{k=n}^{N_n} \lambda_k^{(n)} g(u_k) \geq \lim_n g\left(\sum_{k=n}^{N_n} \lambda_k^{(n)} u_k\right) \geq g(u) = f(u),$$

which is the desired lower semi-continuity of f. $\square$

3 Weighted Orlicz-Sobolev spaces

In this chapter, we introduce first order Sobolev spaces such that functions and their weak gradients belong to an Orlicz space integrability class w.r.t. a weight as e.g. $L^\Phi_w := L^\Phi(\mathbb{R}^d; w\,\mathrm{d}x)$. We are restricting to the case that the domain is all of $\mathbb{R}^d$. Nevertheless, the weight w is allowed to be zero on a set of positive Lebesgue measure. The weighted case for $\Phi(x) = (1/2)|x|^2$ with finite and infinite dimensional domains has been studied well by Sergio Albeverio, Shigeo Kusuoka, Michael Röckner and Tu-Sheng Zhang in [AR89, AR90a, AR90b, AKR90, RZ92, ARZ93a, ARZ93b, RZ94]. See also [MR92, FOT94] for the related functional analytic and probabilistic theory. The spaces $H^{1,2}_w$ and $W^{1,2}_w$ are of particular interest in (mathematical) quantum field theory, where the weight w is related to the potential of a generalized Schrödinger operator. For $1 < p < \infty$, weighted Sobolev spaces have been studied e.g. in [Kuf80, KO84, HKM93]. For weighted Orlicz-Sobolev spaces, we refer to [Vui87].

Our approach generalizes that in [AR90a, AKR90, ARZ93b] analogously. The main advantage, compared to the standard definition (see Paragraph 4.3 below), is that we have an integration by parts formula (3.4), which is non-distributional and rather involves the logarithmic derivative of the weight w. We remind the reader that our definition of the *strong Sobolev space* $H^{1,\Phi}_w$ is classical and all later results could be restricted to this case. Anyhow, we are also interested in the *weak Sobolev space* $W^{1,\Phi}_w$, where our approach might seem unfamiliar. For convenience, we give conditions on w such that the classical case is recovered, see Paragraph 4.3. Also, in Chapter 4, we give a new condition for the equality $H = W$.

For all of this chapter, fix a dimension $d \in \mathbb{N}$ and a *complementary pair* of *nice Young functions* (Φ, Ψ) such that $\Phi, \Psi \in \Delta_2 \cap \nabla_2$. Denote by (φ, ψ) the pair of corresponding *gauges*, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}\Phi = \varphi$, $\frac{\mathrm{d}}{\mathrm{d}t}\Psi = \psi$ and $\varphi = \psi^{-1}$. We use the notation $\|\cdot\|_{(\Phi,w)}$ for the *Luxemburg norm* of L^Φ_w and $\|\cdot\|_{\Phi,w}$ for the *Orlicz norm* of L^Φ_w. We use the same notation for the norms of $L^\Phi_w(\Omega \to \mathbb{R}^d)$ ($\Omega \subset \mathbb{R}^d$), that is, the $\mathbb{R}^d$-valued Orlicz functions. They are defined either via Bochner integrals or component-wise. $\mathbb{R}^d$ carries the *Euclidean metric* with norm $|\cdot|$ and scalar product $\langle\cdot,\cdot\rangle$. See Appendix C for details. We shall use the terms (weighted) Φ-Sobolev space and (weighted) Orlicz-Sobolev space interchangeably. We use the notation ∂_i for the *partial derivative in direction* i (either in the *classical sense* or in the *weak sense*) and ∇ for the *(classical or weak) gradient*. In order to avoid confusion, we use the notation D_i for the *i-th partial derivative in the sense of Schwartz distributions* and D for the *gradient in the sense of Schwartz distributions*, see [RS80, Ch. V.4] or [Alt06, Ch. 3].

Definition 3.1. *For a fixed pair of nice Young functions* (Φ, Ψ) *with the pair of gauges*

(φ, ψ) and a weight $w \in L^1_{\text{loc}}(\mathbb{R}^d)$, $w \geq 0$ define the Φ-regular set

$$R(w) := R^\Phi(w) := \left\{ x \in \mathbb{R}^d \;\middle|\; \int_{B(x,\varepsilon)} \psi\left(\frac{1}{w(y)}\right) \mathrm{d}y < \infty \;\; \text{for some } \varepsilon > 0 \right\},$$

where we set $\psi(1/0) := +\infty$ and $\psi(1/+\infty) := 0$.

$R(w)$ is the largest open set such that $\psi(1/w)$ is locally integrable. Clearly, $w > 0$ $\mathrm{d}x$-a.e. on $R(w)$. When $\Phi(x) := (1/2)|x|^2$, $\varphi = \psi = \mathrm{Id}$, $R(\cdot)$ is the regular set as in [AR90a, Proposition 5.1].

Definition 3.2. *Fix a pair of nice Young functions (Φ, Ψ) and gauges (φ, ψ) and a weight $w \in L^1_{\text{loc}}(\mathbb{R}^d)$, $w \geq 0$. Consider the following conditions (“*S*” is for “*Sobolev*”):*

(S1) *$w = 0$ $\mathrm{d}x$-a.e. on $\mathbb{R}^d \setminus R(w)$.*

(S2) *$R(w) = \mathbb{R}^d$.*

(S3) *$\mathrm{D}_i w \in L^1_{\text{loc}}(\mathbb{R}^d)$, $\forall 1 \leq i \leq d$.*

(S4) *$\beta_i := \mathrm{D}_i w / w \in L^\Psi_w(\mathbb{R}^d)$, $\forall 1 \leq i \leq d$.*

Indeed, L^Ψ_w in (S4) can be replaced by $L^\Psi_{w,\text{loc}}$.

Remark 3.3. *Conditions* (S2) *just states that*

$$\psi\left(\frac{1}{w}\right) \in L^1_{\text{loc}}(\mathbb{R}^d; \mathrm{d}x),$$

which is a well-known condition in the literature. It is obvious that (S2) *implies* (S1)*. Condition* (S1) *is also known as* Hamza's condition, *due to* [Ham75].

β_i is also called logarithmic derivative *of w because of the heuristic formula $\beta_i = \partial_i \log(w)$.*

Lemma 3.4. *When for $\mathrm{d}x$-a.a. $x \in \{w > 0\}$,*

$$\operatorname*{ess\,inf}_{y \in B(x,\delta)} w(y) > 0 \tag{3.1}$$

for some $\delta = \delta(x) > 0$, then w satisfies (S1)*. In particular, each lower semi-continuous function $w \geq 0$ satisfies* (S1)*.*

Proof. Let $x \in \{w > 0\}$ be such that (3.1) holds with some $\delta = \delta(x) > 0$. Obviously, for some constant $C > 0$,

$$\operatorname*{ess\,sup}_{y \in B(x,\delta)} \frac{1}{w(y)} \leq C.$$

It is enough to prove that $x \in R(w)$. But by monotonicity of ψ,

$$\begin{aligned} \int_{B(x,\delta)} \psi\left(\frac{1}{w(y)}\right) \mathrm{d}y &\leq \operatorname{vol} B(x,\delta) \operatorname*{ess\,sup}_{y \in B(x,\delta)} \psi\left(\frac{1}{w(y)}\right) \\ &\leq \operatorname{vol} B(x,\delta) \psi\left(\operatorname*{ess\,sup}_{y \in B(x,\delta)} \frac{1}{w(y)}\right) \leq \operatorname{vol} B(x,\delta) \psi(C) < \infty. \end{aligned}$$

□

Lemma 3.5. *Suppose that* (S1) *holds for* $w \geq 0$, $w \in L^1_{\mathrm{loc}}$. *Then*

$$L^\Phi_w(R(w)) \subset L^1_{\mathrm{loc}}(R(w)).$$

Proof. Let $u \in L^\Phi_w(R(w))$ and $O \Subset R(w)$, open and bounded. By the Hölder inequality (C.1) for Orlicz spaces

$$\int_{\overline{O}} |u| \,\mathrm{d}x \leq \left\| u 1_{R(w)} \right\|_{\Phi,w} \left\| \frac{1}{w} 1_{\overline{O}} \right\|_{(\Psi,w)}. \tag{3.2}$$

By Lemma C.12, $\left\|(1/w)1_{\overline{O}}\right\|_{(\Psi,w)}$ is finite iff $\int_{\overline{O}} \Psi(1/w)w \,\mathrm{d}x$ is. By Lemma C.4, the latter is finite iff

$$\int_{\overline{O}} \psi\left(\frac{1}{w}\right)\frac{1}{w} w \,\mathrm{d}x = \int_{\overline{O}} \psi\left(\frac{1}{w}\right) \mathrm{d}x$$

is. But this term is in turn finite by (S1) and compactness of $\overline{O}$. □

3.1 The strong Sobolev space H

Let $C^\infty := C^\infty(\mathbb{R}^d)$ be the set of infinitely often continuously differentiable real-valued functions on $\mathbb{R}^d$. For $u \in C^\infty$, we define the *weighted Luxemburg* Φ*-Sobolev norm* by

$$\|u\|_{(1,\Phi,w)} := \|u\|_{(\Phi,w)} + \|\nabla u\|_{(\Phi,w)}$$

and the *weighted Orlicz* Φ*-Sobolev norm*

$$\|u\|_{1,\Phi,w} := \|u\|_{\Phi,w} + \|\nabla u\|_{\Phi,w}.$$

We impose another condition, which is in fact implied by (S1), and is referred to as *closability.* For any sequence $\{u_n\} \in C^\infty$,

$$\lim_n \|u_n\|_{\Phi,w} = 0 \text{ and } \{u_n\} \text{ is } \|\nabla\cdot\|_{\Phi,w}\text{-Cauchy, always imply that } \lim_n \|\nabla u_n\|_{\Phi,w} = 0. \tag{3.3}$$

Lemma 3.6. *Condition* (S1) *implies* (3.3).

Proof. Let $\{u_n\} \in C^\infty$ such that $\|u_n\|_{\Phi,w} \to 0$ and that $\{u_n\}$ is $\{\|\nabla\cdot\|_{\Phi,w}\}$-Cauchy. By Lemma 3.5, we conclude that $\{\nabla u_n\}$ is a Cauchy sequence in the Fréchet space $L^1_{\mathrm{loc}}(R(w) \to \mathbb{R}^d)$ and hence converges to some $v \in L^1_{\mathrm{loc}}(R(w) \to \mathbb{R}^d)$. Let $\eta \in C^\infty_0(R(w))$, $1 \leq i \leq d$,

$$0 = \lim_n \int_{R(w)} u_n \partial_i \eta \,\mathrm{d}x = -\lim_n \int_{\operatorname{supp}\eta \cap R(w)} \partial_i u_n \eta \,\mathrm{d}x = \int_{\operatorname{supp}\eta\cap R(w)} v_i \eta \,\mathrm{d}x,$$

which yields $v = 0 \in \mathbb{R}^d$ $\mathrm{d}x$-a.s on $R(w)$. By a standard diagonal procedure there is a subsequence $\{\nabla u_{n_k}\}$ with $\nabla u_{n_k}(x) \to 0$ for $\mathrm{d}x$-a.a. $x \in R(w)$. Recalling that Φ is continuous, by Fatou's lemma

$$\int_{\mathbb{R}^d} \Phi(|\nabla u_n|)\, w \,\mathrm{d}x \leq \varliminf_k \int_{R(w)} \Phi(|\nabla(u_n - u_{n_k})|)\, w \,\mathrm{d}x,$$

which is arbitrarily small for large n by assumption. Hence $\|\nabla u_n\|_{\Phi,w} \to 0$. □

Definition 3.7. *Let $H^{1,\Phi}_w$ be the abstract completion $\widetilde{X}$ of the set*

$$X := \left\{ u \in C^\infty(\mathbb{R}^d) \;\middle|\; \|u\|_{1,\Phi,w} < \infty \right\}$$

w.r.t. the norm $\|\cdot\|_{1,\Phi,w}$.

Lemma 3.8. *Suppose that* (S1) *holds. Then $H^{1,\Phi}_w$ can be identified with a linear subspace of L^Φ_w which is a Banach space with either norm. In particular, an element in $H^{1,\Phi}_w$ is uniquely characterized by its L^Φ_w-class.*

Proof. If $[u_n] \in \widetilde{X}$ is a class, then by completeness of L^Φ_w there are $\widetilde{u} \in L^\Phi_w$, $\widetilde{v}_i \in L^\Phi_w$, $1 \le i \le d$ such that $\lim_n \|u_n - \widetilde{u}\|_{\Phi,w} = \lim_n \|u_n - \widetilde{u}\|_{(\Phi,w)} = 0$ and for each $1 \le i \le d$, $\lim_n \|\partial_i u_n - \widetilde{v}_i\|_{\Phi,w} = \lim_n \|\partial_i u_n - \widetilde{v}_i\|_{(\Phi,w)} = 0$. Set $\nabla\widetilde{u} := (\widetilde{v}_1, \ldots, \widetilde{v}_d)$ which is a unique (class of a) function in $L^\Phi_w(\Omega \to \mathbb{R}^d)$ by (3.3). The representation does not depend on the choice of the sequence in $[u_n]$ by definition of the abstract completion. Therefore $\widetilde{X}$ is linearly isometrically isomorphic to a linear subspace Y of $L^\Phi_w(\Omega \to \mathbb{R} \times \mathbb{R}^d)$. By completeness of $\widetilde{X}$, Y is closed and hence reflexive (cf. [HHZ96, Proposition 67], [Bea85, Part 3, Ch. II, §1, Proposition 4]). We have proved that $H^{1,\Phi}_w$ is a reflexive Banach space with the Sobolev norm and a subspace of L^Φ_w. □

Remark 3.9. *Under condition* (S2)*, each $u \in H^{1,\Phi}_{w,\mathrm{loc}}$ is a Schwartz distribution and $\mathrm{D}u = \nabla u$ $\mathrm{d}x$-a.s. To see this fix $1 \le i \le d$ and let $\{u_n\} \subset C^\infty$ with $u_n \to u$ and $\partial_i u_n \to \partial_i u$ in L^Φ_w. By the proof of Lemma 3.6, $u_n \to u$, $\partial_i u_n \to \partial_i u$ in $L^1_{\mathrm{loc}}(R(w)) = L^1_{\mathrm{loc}}(\mathbb{R}^d)$. In particular, u and $\partial_i u$ are Schwartz distributions. Now for any $\zeta \in C_0^\infty(\mathbb{R}^d)$,*

$$\left| \int u\partial_i\zeta + \zeta\partial_i u \,\mathrm{d}x \right| = \left| \int_{\operatorname{supp}\zeta} (u - u_n)\partial_i\zeta + (\partial_i u - \partial_i u_n)\zeta \,\mathrm{d}x \right| \to 0.$$

Hence $\nabla u = \mathrm{D}u$ $\mathrm{d}x$-a.s. on $\mathbb{R}^d$.

Lemma 3.10. *The $w\,\mathrm{d}x$ classes of $C_0^\infty(\mathbb{R}^d)$ are dense in $H^{1,\Phi}_w$.*

Proof. Compare with [HKM93, Theorem 1.27]. Let $u \in H^{1,\Phi}_w$. For $j \in \mathbb{N}$ let $A_j := B(0, j+1) \setminus \overline{B}(0, j-1)$ and choose functions $\eta_j \in C_0^\infty(A_j)$, $0 \le \eta_j \le 1$, such that

$$\sum_{j=1}^\infty \eta_j(x) = 1$$

for each $x \in \mathbb{R}^d$; i.e. η_j is a partition of unity subordinate to the covering $\{A_j\}_{j\in\mathbb{N}}$, see [Alt06, §2.19]. Then $u\eta_j \in H^{1,\Phi}_w(A_j)$ has compact support in A_j. Let $\varepsilon > 0$ and pick $\zeta_j \in C_0^\infty(A_j)$ with

$$\|\zeta_j - u\eta_j\|_{1,\Phi,w} < 2^{-(j+1)}\varepsilon.$$

Then $\zeta = \sum_{j=1}^\infty \zeta_j \in C^\infty(\mathbb{R}^d)$ and

$$\|\zeta - u\|_{1,\Phi,w} \le \left\| \sum_{j=1}^\infty \zeta_j - u \sum_{j=1}^\infty \eta_j \right\|_{1,\Phi,w} < \varepsilon/2.$$

Now choose $j_0 \in \mathbb{N}$ such that

$$\left\|\zeta 1_{\mathbb{R}^d \setminus B(0,j_0)}\right\|_{1,\Phi,w} < \varepsilon/2.$$

Then $\vartheta := \sum_{j=1}^{j_0} \zeta_j \in C_0^\infty(\mathbb{R}^d)$ and

$$\|\vartheta - \zeta\|_{1,\Phi,w} \le \left\|\zeta 1_{\mathbb{R}^d \setminus B(0,j_0)}\right\|_{1,\Phi,w} < \varepsilon/2.$$

Hence

$$\|\vartheta - u\|_{1,\Phi,w} < \varepsilon,$$

which proves the desired density. □

3.2 The weak Sobolev space W

Lemma 3.11. *Fix $1 \le i \le d$. Let $e_i \in \mathbb{R}^d$ be the unit vector in direction i. Suppose that* (S3) *holds. Then there is a version $\widetilde{w}$ of w differing on a set of Lebesgue measure zero in $\{e_i\}^\perp$, such that for $y \in \{e_i\}^\perp$ and each compact interval $I \subset \mathbb{R}$ the map $I \ni t \mapsto \widetilde{w}(y + te_i)$ is absolutely continuous for almost all $y \in \{e_i\}^\perp$. Furthermore, for almost all $y \in \{e_i\}^\perp$,*

$$\mathbb{R} \setminus R(w(y + \cdot e_i)) \subset \{t \,|\, \widetilde{w}(y + te_i) = 0\}.$$

Recall that the almost sure inclusion "$\supset$" holds automatically.

Proof. The first part follows from a well-known theorem due to Nikodým, cf. [Miz73, Theorem 2.7]. The second part follows from absolute continuity and Lemma 3.4. □

Definition 3.12. *Suppose that* (S1), (S3) *and* (S4) *hold. Let $W_w^{1,\Phi}(\mathbb{R}^d)$ be the set of $w\,\mathrm{d}x$-equivalence classes of functions $f \in L_w^\Phi(\mathbb{R}^d)$ such that for each $1 \le i \le d$ there exists a function $\overline{\partial}_i f \in L_w^\Phi(\mathbb{R}^d)$ which satisfies*

$$\int \overline{\partial}_i f \eta w \,\mathrm{d}x = -\int f \partial_i \eta w \,\mathrm{d}x - \int f \eta \beta_i w \,\mathrm{d}x \tag{3.4}$$

for all $\eta \in C_0^\infty(\mathbb{R}^d)$. For $f \in W_w^{1,\Phi}$, define $\overline{\nabla} f := (\overline{\partial}_1 f, \dots, \overline{\partial}_d f) \in L_w^\Phi(\mathbb{R}^d \to \mathbb{R}^d)$.

Lemma 3.13. *Suppose that* (S1), (S3) *and* (S4) *hold. $W_w^{1,\Phi}$ is a Banach space with either* Luxemburg-Sobolev norm

$$\|f\|_{(1,\Phi,w)} := \|f\|_{(\Phi,w)} + \left\|\overline{\nabla} f\right\|_{(\Phi,w)}$$

or Orlicz-Sobolev norm

$$\|f\|_{1,\Phi,w} := \|f\|_{\Phi,w} + \left\|\overline{\nabla} f\right\|_{\Phi,w}.$$

In particular, an element in $W_w^{1,\Phi}$ is uniquely characterized by its L_w^Φ-class.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $W_w^{1,\Phi}$ which is the same in either norm by Δ_2-regularity. By completeness of L_w^Φ there is a function $f \in L_w^\Phi$ with $\|f_n - f\|_{\Phi,w} \to 0$. Also, for $1 \le i \le d$ there are $g_i \in L_w^\Phi$ such that $\left\|\overline{\partial}_i f_n - g_i\right\|_{\Phi,w} \to 0$. By (3.4) for all $\eta \in C_0^\infty$

$$\int \overline{\partial}_i f_n \eta w \,\mathrm{d}x = -\int f_n \partial_i \eta w \,\mathrm{d}x - \int f_n \eta \beta_i w \,\mathrm{d}x.$$

By Hölder's inequality (C.1), using (S4), we can pass on to the limit

$$\int g_i \eta w \,\mathrm{d}x = -\int f \partial_i \eta w \,\mathrm{d}x - \int f \eta \beta_i w \,\mathrm{d}x.$$

Hence $\overline{\nabla} f = (g_1, \dots, g_d)$ and $\|f_n - f\|_{1,\Phi,w} \to 0$. The limits for the Luxemburg-Sobolev norm follow by the Δ_2-condition. □

Lemma 3.14. *Suppose that* (S1), (S3) *and* (S4) *hold. Then* $H_w^{1,\Phi} \subset W_w^{1,\Phi}$.

Proof. By the Leibniz formula and the definition of β_i, each $f \in C_0^\infty$ satisfies (3.4) with $\overline{\nabla} f = \nabla f$. The claim follows now by Lemma 3.10 and the completeness of $W_w^{1,\Phi}$. □

We point out that in general $H_w^{1,\Phi} \subsetneqq W_w^{1,\Phi}$. In [Zhi98], V. V. Zhikov constructs a counterexample in the weighted $W_w^{1,2}$-case (assuming (S2)). In the subsequent chapter, we shall give two types of sufficient conditions on the weight for $H = W$ to hold.

We need another representation of functions in $W_w^{1,\Phi}$, known as *absolute continuity on lines.* It is needed to verify important properties of Sobolev functions as the Leibniz formula and a chain rule for Lipschitz functions.

Proposition 3.15. *Suppose that* (S1), (S3) *and* (S4) *hold. Fix* $1 \le i \le d$. *Let* $e_i \in \mathbb{R}^d$ *be the unit vector in direction* i. *Then* $f \in W_w^{1,\Phi}$ *has a representative* $\widetilde{f}^i$ *such that* $t \mapsto \widetilde{f}^i(y + te_i)$ *is absolutely continuous for* ($d-1$*-dim.) Lebesgue almost all* $y \in \{e_i\}^\perp$ *on any compact subinterval of* $R(w(y + \cdot e_i))$. *In that case, for* $\mathrm{d}y$*-a.a.* $y \in \{e_i\}^\perp$, $\mathrm{d}t$*-a.a.* $t \in R(w(y + \cdot e_i))$, *setting* $x := y + te_i$, $\overline{\partial}_i f(x) = \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{f}^i(y + te_i)$.

Conversely, if for each $1 \le i \le d$ *a function* $f \in L_w^\Phi$ *has a* $\mathrm{d}y$*-version* $\widetilde{f}^i$, *which is locally absolutely continuous on almost all of* $R(w(y + \cdot e_i))$, *equal to zero on* $\mathbb{R}^d \setminus R(w)$ *and satisfies* $\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{f}^i(\cdot + \cdot e_i) \in L_w^\Phi$, *then* $f \in W_w^{1,\Phi}$.

Proof. For the $W_w^{1,2}$-case, the first part was proved in [ARZ93b, Lemma 2.2], and the second part was proved in [AKR90, Proposition 2.4 (ii)].

Fix $1 \le i \le d$. By Lemma 3.11, fix a version of w (denoted also by w) such that the map $t \mapsto w(y + te_i)$ is locally absolutely continuous on $\mathbb{R}$ for $\mathrm{d}y$-a.a. $y \in \{e_i\}^\perp$. By (3.4), for any $\eta \in C_0^\infty(\mathbb{R}^d)$,

$$\int \overline{\partial}_i f \eta w \,\mathrm{d}x = -\int f \partial_i \eta w \,\mathrm{d}x - \int f \eta \beta_i w \,\mathrm{d}x.$$

By Fubini's theorem (and the fundamental lemma of variational calculus [Alt06, §2.21]) for $\mathrm{d}y$-a.a. $y \in \{e_i\}^\perp$ and for all $\eta \in C_0^\infty(\mathbb{R})$

$$\begin{aligned}
&-\int \left[\overline{\partial}_i f(y+te_i) + f(y+te_i)\beta_i(y+te_i)\right] w(y+te_i)\eta(t)\,\mathrm{d}t \\
&= \int \frac{\mathrm{d}}{\mathrm{d}t}\eta(t) f(y+te_i) w(y+te_i)\,\mathrm{d}t,
\end{aligned} \tag{3.5}$$

and hence for $\mathrm{d}y$-a.a. $y \in \{e_i\}^\perp$ the map

$$t \mapsto f(y+te_i)w(y+te_i)$$

has a distributional derivative which lies in $L^1_{\mathrm{loc}}(\mathbb{R})$. Hence by a well-known theorem of Nikodým [Miz73, Theorem 2.7] it has an absolutely continuous $\mathrm{d}t$-version on any compact interval in $\mathbb{R}$. By Lemma 3.11, $R(w(y+\cdot e_i)) \supset \{w(y+\cdot e_i) > 0\}$ $\mathrm{d}y$-a.s. and hence $R(w(y+\cdot e_i)) = \{w(y+\cdot e_i) > 0\}$ $\mathrm{d}y$-a.s. We conclude that $t \mapsto f(y+te_i)$ has a version $\widetilde{f}^i$ which is absolutely continuous on any compact subinterval of $R(w(y+\cdot e_i))$ for almost all $y \in \{e_i\}^\perp$. By the Leibniz formula for absolutely continuous functions and integration by parts, (3.5) proves that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{f}^i(y+te_i) = \overline{\partial}_i f(y+te_i)$$

where the equality holds in the sense of $w\,\mathrm{d}x$-classes.

Let us prove the converse. Fix $1 \le i \le d$. By Lemma 3.11, fix an absolutely continuous version of $t \mapsto w(y+te_i)$ locally on $\mathbb{R}$ for $\mathrm{d}y$-a.a. $y \in \{e_i\}^\perp$ (denoted also by w). Let $f \in L^\Phi_w$ with a locally absolutely continuous version $\widetilde{f}^i$ with derivative in L^Φ_w and such that f is equal to zero on $\mathbb{R}^d \setminus R(w)$. Denote the maps $t \mapsto \widetilde{f}^i(y+te_i)$, $t \mapsto w(y+te_i)$ by $\widetilde{f}^{i,y}$, $w^{i,y}$ resp. Since $R(w^{i,y})$ as an open subset of $\mathbb{R}$ is a countable disjoint union of open intervals it is easy to see that the set of isolated points of $\mathbb{R} \setminus R(w^{i,y})$ is countable. Since $\mathbb{R} \setminus R(w^{i,y}) = \{w^{i,y} > 0\}$ for $\mathrm{d}y$-a.a. $y \in \{e_i\}^\perp$, it follows that for $\mathrm{d}y$-a.a. $y \in \{e_i\}^\perp$, $\frac{\mathrm{d}}{\mathrm{d}t}w^{i,y} = 0$ on $\mathbb{R} \setminus R(w^{i,y})$ $\mathrm{d}t$-a.s. Let $\eta \in C_0^\infty(\mathbb{R})$. Hence by the Leibniz formula, for $\mathrm{d}y$-a.a. $y \in \{e_i\}^\perp$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{f}^{i,y}\eta w^{i,y}) = \frac{\mathrm{d}\widetilde{f}^{i,y}}{\mathrm{d}t}\eta w^{i,y} + \widetilde{f}^{i,y}\frac{\mathrm{d}\eta}{\mathrm{d}t}w^{i,y} + \widetilde{f}^{i,y}\eta\frac{\mathrm{d}w^{i,y}}{\mathrm{d}t}\cdot 1_{R(w^{i,y})}, \tag{3.6}$$

$\mathrm{d}t$-a.e. on $\mathbb{R}$. Since, clearly, for $\mathrm{d}y$-a.a. y

$$\frac{\mathrm{d}w^{i,y}}{\mathrm{d}t}\cdot 1_{R(w^{i,y})} = \frac{\mathrm{D}_i w(y+te_i)}{w}w$$

$\mathrm{d}t$-a.s. on $\mathbb{R}$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{f}^{i,y}\eta w^{i,y}) \in L^1(\mathbb{R};\mathrm{d}t)$$

and hence by absolute continuity (see e.g. [Alt06, U1.6 (2), p. 68])

$$\int \frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{f}^{i,y}\eta w^{i,y})\,\mathrm{d}t = 0.$$

To see now that $\widetilde{f}^{i,y}$ satisfies (3.4), one needs only to multiply (3.6) with a function $\zeta \in C_0^\infty(\{e_i\}^\perp)$ and note that the linear hull of functions of the type $\eta \otimes \zeta$ ($\eta \in C_0^\infty(\mathbb{R})$) is dense in $C_0^\infty(\mathbb{R}^d)$ w.r.t. uniform convergence by a locally compact version of the Stone-Weierstraß Theorem [Cho69, p. 28 (iii)]. However, the issue of product measurability needs some caution (since we would like to apply Fubini's theorem). It can be treated by the abstract disintegration formula in [Rip76, Proposition 1] (or [DM78, Ch. III, 70]), see [AR90a, Section 5 (a)] for details. □

Corollary 3.16. *Suppose that* (S2)*,* (S3) *and* (S4) *hold. Let $f \in W_w^{1,\Phi}$. Then for each $1 \le i \le d$ there is a* $\mathrm{d}x$*-version $\widetilde{f}$ of f such that $\overline{\partial}_i \widetilde{f} = \mathrm{D}_i \widetilde{f}$* $\mathrm{d}x$*-a.s.*

Proof. Obvious since $R(w) = \mathbb{R}^d$. □

Picking appropriate locally absolutely continuous versions, one immediately obtains the following Leibniz formula:

Corollary 3.17. *Suppose that* (S1)*,* (S3) *and* (S4) *hold. If $f, g \in W_w^{1,\Phi}$ and if fg, $f\overline{\partial}_i g$ and $g\overline{\partial}_i f$ are in L_w^Φ for all $1 \le i \le d$, then $fg \in W_w^{1,\Phi}$ and $\overline{\partial}_i(fg) = f\overline{\partial}_i g + g\overline{\partial}_i f$ for all $1 \le i \le d$. Then also, $\overline{\nabla}(fg) = f\overline{\nabla}g + g\overline{\nabla}f$.*

The following lemma implies that we can truncate Sobolev functions. This is known as the "sub-Markov property" or as the "lattice property" of the Sobolev space.

Lemma 3.18. *Suppose that* (S1)*,* (S3) *and* (S4) *hold. Suppose that $f \in W_w^{1,\Phi}$ and that $F : \mathbb{R} \to \mathbb{R}$ is Lipschitz. Then $F \circ f \in W_w^{1,\Phi}$ with*

$$\overline{\nabla}(F \circ f) = (F' \circ f) \cdot \overline{\nabla} f \quad w\,\mathrm{d}x\text{-a.s.}$$

In particular, when $F(t) := N \wedge t \vee -N$, $N \in \mathbb{N}$ is a cut-off function,

$$|\overline{\nabla}(F \circ f)| \le |\overline{\nabla} f| \quad w\,\mathrm{d}x\text{-a.s.} \tag{3.7}$$

Proof. By Proposition 3.15, pick a locally absolutely continuous version $\widetilde{f}^i$ of f for $1 \le i \le d$. The chain rule follows by a well-known chain rule for Lipschitz functions versus absolutely continuous functions [BH84, Démonstration de la Proposition 5]. Repeating the procedure for all $1 \le i \le d$, the claim follows. □

We remark that, indeed, we are able to prove the lattice property now. (For the notion of (Banach) lattices, we refer to [MN91]). The procedure is standard and can be excellently seen in [HKM93, Theorem 1.18 et sqq.]. As another consequence, bounded and compactly supported functions are dense, which is crucial for the results in the next chapter.

Lemma 3.19. *Suppose that* (S1)*,* (S3) *and* (S4) *hold. The set of bounded and compactly supported functions in $W_w^{1,\Phi}$ is dense in $W_w^{1,\Phi}$.*

Proof. For $f \in W_w^{1,\Phi}$, let $f_n := (n \wedge f \vee -n)\eta_n$, where $\{\eta_n\} \subset C_0^\infty(\mathbb{R}^d)$, $0 \le \eta_n \le 1$, $\eta_n \uparrow 1$, $\eta_n \equiv 1$ on $B(0,n)$, $|\nabla \eta_n| \le C$. Then, by Corollary 3.17 and Lemma 3.18, $f_n \in W_w^{1,\Phi}$ and

$$\overline{\nabla} f_n = (n \wedge f \vee -n)\nabla \eta_n + \eta_n \overline{\nabla}(n \wedge f \vee -n). \tag{3.8}$$

Furthermore,

$$\Phi(|f_n|)\, w \le \Phi(|f|)\, w \in L^1(\mathrm{d}x),$$

and, by (3.7), (3.8) and convexity,

$$\Phi(|\overline{\nabla} f_n|)\, w \le \frac{1}{2}\Phi(2C|f|)\, w + \frac{1}{2}\Phi(2|\nabla f|)\, w \in L^1(\mathrm{d}x).$$

Note that by continuity of Φ, by (3.8) and Lemma 3.18

$$\Phi(|f_n - f|)\, w \to 0 \quad \text{and} \quad \Phi(|\overline{\nabla} f_n - \overline{\nabla} f|)\, w \to 0 \quad \mathrm{d}x\text{-a.s.}$$

as $n \to \infty$. Hence an application of Lebesgue's dominated convergence theorem and Lemma C.16 yields the assertion. □

Note that the last two statements also hold for $H_w^{1,\Phi}$. Anyhow, the proof of Lemma 3.18 for $H_w^{1,\Phi}$ needs some caution, because the Lipschitz function has to be approximated by smooth functions. The method is well-known, we refer to [MR92, Proposition I.4.7, Example II.2.c)] and [FOT94, Equation (1.1.5), Theorem 1.4.1].

Lemma 3.20. *$H_w^{1,\Phi}$ as well as $W_w^{1,\Phi}$ is separable and reflexive.*

Proof. We can identify $H_w^{1,\Phi}(\mathbb{R}^d)$ with a linear subspace of the reflexive and separable space $L_w^\Phi(\mathbb{R}^d \to \mathbb{R}^{1+d})$ (where, for our convenience, $\mathbb{R}^{1+d}$ is equipped with the norm $|x_1| + \sqrt{\sum_{i=2}^{1+d} |x_i|^2}$, $(x_1, x_2, \dots, x_{1+d}) \in \mathbb{R}^{1+d}$) via the map $Pu := (u, \partial_1 u, \dots, \partial_d u)$. Since (by our choice of the norms) $\|u\|_{1,\Phi,w} = \|Pu\|_{\Phi,w}$, we see that P is an isometry. $\operatorname{ran} P$ is linear and closed as it is isometrically isomorphic to a complete space. The claim follows now by well-known inheritances for closed linear subspaces (cf. [HHZ96, Proposition 67], [Bea85, Part 3, Ch. II, §1, Proposition 4]). The case of $W_w^{1,\Phi}(\mathbb{R}^d)$ works similarly. □

4 $H = W$ for weighted p-Sobolev spaces

The problem of "$H = W$" for Sobolev spaces has many names. Classically, it was called "strong equals weak" and solved by Norman G. Meyers and James Serrin in [MS64] for the unweighted $\mathbb{R}^d$-case. In fact, it already appeared in the earlier work of Jacques Deny and Jacques-Louis Lions [DL54]. For $p = 2$ with weights, the problem is known as "Markov uniqueness". Another suggestive name would be "problem of smooth approximation". For the linear case (resp. Hilbert space case, resp. $p = 2$ case), Markov uniqueness is related to the core problem for linear symmetric operators. We refer to Andreas Eberle's lecture notes [Ebe99] for a detailed discussion. See also [RS09] for a recent work on arbitrary sub-domains of $\mathbb{R}^d$. For $1 < p < \infty$, the problem $H = W$ is also called "attainability problem" and related to the so-called Lavrent'ev phenomenon (due to his work [Lav26]), we refer to [Zhi98, Pas07, ZP08, Zhi09]. A widely known condition on the weight for $H = W$ to hold (in arbitrary sub-domains of $\mathbb{R}^d$) is the so-called Muckenhoupt condition. We shall discuss it shortly in section 4.1.

For the spaces $H^{1,2}_w(\mathbb{R}^d)$ and $W^{1,2}_w(\mathbb{R}^d)$, Michael Röckner and Tu-Sheng Zhang have proved in [RZ92, RZ94] that

$$\sqrt{w} \in H^{1,2}_{\mathrm{loc}}(\mathrm{d}x) \tag{4.1}$$

is necessary and sufficient for Markov uniqueness. See also [ARZ93b]. Motivated by this, we are suggesting a similar condition for p-Sobolev spaces, $1 < p < \infty$, namely

$$w^{1/p} \in H^{1,p}_{\mathrm{loc}},\ \frac{\partial_i w}{w^{1/p}} = p w^{(p-2)/p} \partial_i (w^{1/p}) \in L^q_{\mathrm{loc}} \quad \forall 1 \le i \le d. \tag{HW}$$

Or equivalently, upon setting $w = \varphi^p$,

$$\varphi \in H^{1,p}_{\mathrm{loc}},\ \frac{\partial_i \varphi^p}{\varphi} = p\varphi^{p-2}\partial_i\varphi \in L^q_{\mathrm{loc}} \quad \forall 1 \le i \le d. \tag{HW}$$

Note that in this part, φ refers to $w^{1/p}$ and not to the gauge of an N-function! Clearly for $p = 2$, (4.1) and (HW) are equivalent. As a matter of fact, the proofs of Röckner and Zhang involve deep probability theory known to work only for $p = 2$. We do not know whether it is possible to extend their probabilistic methods to the case $1 < p < \infty$, $p \neq 2$. Nevertheless, there is a completely analytic proof by Patrick Cattiaux and Myriam Fradon [CF96]. Luckily, we were able to transfer its arguments to the case $1 < p < \infty$, as is done in the proof of Theorem 4.4 below. The classical case is recovered in Paragraph 4.3.

We remark that $H = W$ is crucial for the problem of identifying the Mosco limit of p-energies (condition (M2) in Definition 7.26). The results of this chapter can be directly applied to Mosco convergence problems in Chapter 8, see Theorem 8.12. In this context,

$H = W$ becomes an "attainability condition". We conjecture that the proof of Theorem 4.4 also works for weighted Orlicz-Sobolev spaces $W^{1,\Phi}_w$ such that $\Phi \in \Delta'$, see Definition C.6. As above, we remark that we are working on all of $\mathbb{R}^d$ as a domain.

4.1 Muckenhoupt weights

We start with a well-known sufficient condition on w such that $H = W$ holds; the so-called Muckenhoupt condition.

Definition 4.1. *Let Φ, Ψ be two complementary N-functions with gauges φ, ψ. A weight $w \in L^1_{\text{loc}}(\mathbb{R}^d)$ is called an A_Φ-*weight *(is said to belong to the* Φ-Muckenhoupt class*), written $w \in A_\Phi$, if there is a global $K = K_{\Phi,w} > 0$ with*

$$\left(\frac{1}{\operatorname{vol} B}\int_B \varepsilon w \,\mathrm{d}x\right) \cdot \varphi\left(\frac{1}{\operatorname{vol} B}\int_B \psi\left(\frac{1}{\varepsilon w}\right)\mathrm{d}x\right) \le K \tag{4.2}$$

for every ball $B \subset \mathbb{R}^d$ and every $\varepsilon > 0$.

It is straightforward that an A_Φ-weight satisfies (S2).

For a measurable function $f : \mathbb{R}^d \to \mathbb{R}$, let

$$Mf(x) := \sup\left\{\frac{1}{\operatorname{vol} B}\int_B |f(y)| \,\mathrm{d}y \;\middle|\; B \subset \mathbb{R}^d \text{ is a ball with } x \in B\right\} \tag{4.3}$$

be the *Hardy-Littlewood maximal operator.*

Lemma 4.2. *Suppose that $\Phi \in \Delta_2 \cap \nabla_2$. If $w \in A_\Phi \cup A_p$, where p^{-1} is the upper index of Φ (cf. Definition C.10), then*

$$\int_{\mathbb{R}^d} \Phi(Mf)\, w \,\mathrm{d}x \le K \int_{\mathbb{R}^d} \Phi(|f|)\, w \,\mathrm{d}x, \tag{4.4}$$

where $K > 0$ is as in (4.2).

Proof. See [KT82, Theorem 1]. □

Proposition 4.3. *Let $\Phi \in \Delta_2 \cap \nabla_2$. Suppose that $w \in A_\Phi \cup A_p$, where p^{-1} is the upper index of Φ (cf. Definition C.10). Then $H^{1,\Phi}_w(\mathbb{R}^d) = W^{1,\Phi}_w(\mathbb{R}^d)$.*

Proof. Let $\eta \in C_0^\infty(\mathbb{R}^d)$, $\eta \ge 0$, $\int \eta \,\mathrm{d}x = 1$, $\operatorname{supp}\eta \subset B(0,1)$, $\eta_\varepsilon(x) := \varepsilon^{-d}(x/\varepsilon)$, $\varepsilon > 0$, be a standard mollifier. By Lemma 3.19, it is enough to approximate functions in $W^{1,\Phi}_w$ which are bounded and compactly supported. Let $u \in W^{1,\Phi}_w(\mathbb{R}^d)$ be such a function, $u_\varepsilon := u * \eta_\varepsilon$, $\varepsilon > 0$. $u_\varepsilon \in L^1(\mathrm{d}x)$ for all $\varepsilon > 0$. Then $u \in L^1(\mathrm{d}x)$ and $u_\varepsilon \to u$ in $L^1(\mathrm{d}x)$. A subsequence converges $\mathrm{d}x$-almost surely.

By [Ste93, Ch. II, §2, p. 57, Ch. V, §2, p. 198]

$$|u_\varepsilon| \le M(u) \quad \text{and} \quad |\nabla u_\varepsilon| \le M(|\mathrm{D}u|) \quad \text{for all } \varepsilon > 0 \text{ pointwise.}$$

Combined with (4.4) and Lebesgue's dominated convergence theorem, a subsequence of $\{u_\varepsilon\}$ converges in $W^{1,\Phi}_w$ to u (taking into account that $\overline{\nabla} u = \mathrm{D}u$ $\mathrm{d}x$-a.e. by (S2) and Corollary 3.16). □

4.2 Weakly differentiable weights: A new condition for uniqueness

We arrive at the main result of this chapter. Fix $1 < p < \infty$, $d \in \mathbb{N}$. Our proof is inspired by that of Patrick Cattiaux and Myriam Fradon in [CF96]. See also [Fra97]. Recall that we denote the gradient in $W^{1,p}_w$ by $\overline{\nabla}$ or $\overline{\partial}_i$ which is generally not equal to the distributional gradient. Recall that $w = \varphi^p$.

Theorem 4.4. *Let $1 < p < \infty$, $q := p/(p-1)$, $d \in \mathbb{N}$. Suppose that for a weight $w \geq 0$, $w \in L^1_{\text{loc}}(\mathrm{d}x)$ conditions* (S1) *and* (HW) *hold.*

Then $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{1,p}_w(\mathbb{R}^d)$, or equivalently, $H^{1,p}_w(\mathbb{R}^d) = W^{1,p}_w(\mathbb{R}^d)$.

Recall that if $p = 2$, (HW) reduces to $\sqrt{w} \in H^{1,2}_{\text{loc}}$. Clearly, (HW) implies that $\nabla w = \mathrm{D}w \in L^1_{\text{loc}}(\mathbb{R}^d \to \mathbb{R}^d)$ and that

$$\beta_i := p\frac{\partial_i \varphi}{\varphi} \in L^q_{\text{loc}}(w\,\mathrm{d}x) \quad \forall 1 \leq i \leq d. \tag{4.5}$$

Hence $w = \varphi^p$ satisfies (S3) and (S4). Note that (3.4) for $f \in W^{1,p}_w$ then becomes

$$\int \overline{\partial}_i f \zeta \varphi^p \,\mathrm{d}x = -\int f \partial_i \zeta \varphi^p \,\mathrm{d}x - p \int f \zeta \frac{\partial_i \varphi}{\varphi} \varphi^p \,\mathrm{d}x, \quad \zeta \in C_0^\infty. \tag{4.6}$$

The proof of Theorem 4.4 depends on Lemma 3.19 in an essential way. For the proof of Lemma 3.19, the property of $W^{1,p}_w$ being a lattice is crucial. We remind the reader that this property had to be derived from the alternative representation of classes in $W^{1,p}_w$ provided by Proposition 3.15.

For the approximation, an integration by parts-lemma is needed. Compare with [CF96, Lemma 2.9].

Lemma 4.5. *Suppose that* (S1) *and* (HW) *hold for w. Set $\varphi := w^{1/p}$. Let $f \in W^{1,p}_w(\mathbb{R}^d)$ such that f is bounded and compactly supported. Then for every $\zeta \in C_0^\infty(\mathbb{R}^d)$ and every $1 \leq i \leq d$*

$$\int \overline{\partial}_i f \zeta \varphi \,\mathrm{d}x + \int f \partial_i \zeta \varphi \,\mathrm{d}x + \int f \zeta \partial_i \varphi \,\mathrm{d}x = 0. \tag{4.7}$$

Proof. For all of the proof fix $1 \leq i \leq d$. Let us first assure ourselves that all three integrals in (4.7) are well-defined. Clearly,

$$|\overline{\partial}_i f \zeta \varphi|^p \leq \|\zeta\|^p_\infty \, |\overline{\partial}_i f|^p \varphi^p 1_{\operatorname{supp}\zeta} \in L^1(\mathrm{d}x),$$

and hence, by compact support,

$$|\overline{\partial}_i f \zeta \varphi| \in L^1(\mathrm{d}x).$$

A similar argument works for the second integral. The third integral is well-defined because by $\varphi \in H^{1,p}_{\text{loc}}$ we have that

$$|f \zeta \partial_i \varphi|^p \leq \|f\zeta\|^p_\infty \, |\partial_i \varphi|^p 1_{\operatorname{supp}\zeta} \in L^1(\mathrm{d}x)$$

and hence, by compact support,

$$|f\zeta\partial_i\varphi| \in L^1(\mathrm{d}x).$$

Let $M \in \mathbb{N}$ and $\vartheta_M \in C_0^\infty(\mathbb{R})$ with

$$\vartheta_M(t) = t \text{ for } t \in [-M, M],\ |\vartheta_M| \le M + 1,\ |\vartheta_M'| \le 1$$

and

$$\operatorname{supp}(\vartheta_M) \subset [-3M, 3M].$$

Define

$$\varphi_M := \vartheta_M\left(\frac{1}{\varphi^{p-1}}\right) 1_{\{\varphi>0\}} + 0 \cdot 1_{\{\varphi=0\}}.$$

By boundedness, $\varphi_M \in L^p_{\mathrm{loc}}$. Furthermore, define

$$\Phi_M := (1-p)\vartheta_M'\left(\frac{1}{\varphi^{p-1}}\right)\frac{\partial_i\varphi}{\varphi^p} 1_{\{\varphi>0\}} + 0 \cdot 1_{\{\varphi=0\}}.$$

Since $\vartheta_M'(1/\varphi^{p-1}) = 0$ on $\{\varphi^{p-1} \le 1/(3M)\}$ and

$$|\Phi_M| \le (p-1)\frac{|\partial_i\varphi|}{\varphi^p} 1_{\{\varphi^{p-1}>1/(3M)\}} = (p-1)\frac{|\partial_i\varphi|}{\varphi^p} 1_{\{\varphi^p>(1/(3M))^q\}},$$

hence $\Phi_M \in L^p_{\mathrm{loc}}$. Using an approximation by smoothing mollifiers and the chain rule, we see that $\Phi_M = \partial_i\varphi_M$ (where ∂_i denotes the usual weak derivative) and that $\varphi_M \in H^{1,p}_{\mathrm{loc}}$.

Since $\varphi \in H^{1,p}_{\mathrm{loc}}$ and since φ_M is bounded, we have that $\varphi_M\partial_i\varphi \in L^p_{\mathrm{loc}}$. Also, $\varphi\partial_i\varphi_M \in L^p_{\mathrm{loc}}$, since

$$|\varphi\partial_i\varphi_M| \le (p-1)\frac{|\partial_i\varphi|}{\varphi^{p-1}} 1_{\{\varphi^{p-1}>1/(3M)\}} \le (p-1)3M|\partial_i\varphi|. \tag{4.8}$$

Now by the usual Leibniz rule for weak derivatives (see e.g. [Alt06, §2.24])

$$\varphi\varphi_M \in H^{1,p}_{\mathrm{loc}} \quad \text{and} \quad \partial_i(\varphi\varphi_M) = \varphi_M\partial_i\varphi + (1-p)\vartheta_M'\left(\frac{1}{\varphi^{p-1}}\right)\frac{\partial_i\varphi}{\varphi^{p-1}}$$

where by convention $\partial_i\varphi/\varphi^{p-1} = 0$ on $\{\varphi = 0\}$. Consider the term $\varphi_M\varphi^p$. As readily seen, $\varphi\varphi_M \in H^{1,p}_{\mathrm{loc}}$. By (HW), $\varphi^{p-1} \in L^q_{\mathrm{loc}}$ and $\partial_i(\varphi^{p-1}) = (p-1)\varphi^{p-2}\partial_i\varphi \in L^q_{\mathrm{loc}}$. Therefore $\varphi^{p-1} \in H^{1,q}_{\mathrm{loc}}$. Hence $\varphi\varphi_M(\partial_i\varphi^{p-1}) \in L^1_{\mathrm{loc}}$ and $\partial_i(\varphi\varphi_M)\varphi^{p-1} \in L^1_{\mathrm{loc}}$. It follows that $\varphi_M\varphi^p \in H^{1,1}_{\mathrm{loc}}$ and by the Leibniz rule for weak derivatives

$$\partial_i(\varphi_M\varphi^p) = p\varphi_M\varphi^{p-1}\partial_i\varphi + (1-p)\vartheta_M'\left(\frac{1}{\varphi^{p-1}}\right)\partial_i\varphi \in L^1_{\mathrm{loc}}.$$

Let $\zeta \in C_0^\infty(\mathbb{R}^d)$. Applying integration by parts, we see that

$$\int \partial_i\zeta\varphi_M\varphi^p \,\mathrm{d}x = -p\int \zeta\varphi_M\frac{\partial_i\varphi}{\varphi}\varphi^p \,\mathrm{d}x + (p-1)\int \zeta\frac{\partial_i\varphi}{\varphi^p}\vartheta_M'\left(\frac{1}{\varphi^{p-1}}\right)\varphi^p \,\mathrm{d}x. \tag{4.9}$$

Moreover, by (4.8) $\partial_i\varphi_M \in L^p_{\text{loc}}(\varphi^p\,\mathrm{d}x)$. $\varphi_M \in L^p_{\text{loc}}(\varphi^p\,\mathrm{d}x)$ was also verified above. Therefore $\varphi_M \in W^{1,p}_{w,\text{loc}}$ (the weighted local Sobolev space) and

$$\overline{\partial}_i\varphi_M = (1-p)\frac{\partial_i\varphi}{\varphi^p}\vartheta'_M\left(\frac{1}{\varphi^{p-1}}\right).$$

Though we have not defined $W^{1,p}_{w,\text{loc}}$ explicitly, it should be clear what we mean by it. For instance, the Leibniz rule also holds in $W^{1,p}_{w,\text{loc}}$, and so we would like to give sense to the expression $\overline{\partial}_i(f\varphi_M) = \varphi_M\overline{\partial}_i f + f\overline{\partial}_i\varphi_M$. But $\varphi_M \in W^{1,p}_{w,\text{loc}}$, $f \in W^{1,p}_w$ and f is bounded and compactly supported, $f\overline{\partial}_i\varphi_M \in L^p_{w,\text{loc}}$ since f is bounded and compactly supported and finally $\varphi_M\overline{\partial}_i f \in L^p_{w,\text{loc}}$ since φ_M is bounded. Hence $f\varphi_M \in W^{1,p}_{w,\text{loc}}$ and the Leibniz rule holds (locally). By definition of $\overline{\partial}_i$ for $\zeta \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned}\int \overline{\partial}_i f\zeta\varphi_M\varphi^p\,\mathrm{d}x =&(p-1)\int f\zeta\frac{\partial_i\varphi}{\varphi^p}\vartheta'_M\left(\frac{1}{\varphi^{p-1}}\right)\varphi^p\,\mathrm{d}x \\ &-\int f\partial_i\zeta\varphi_M\varphi^p\,\mathrm{d}x - p\int f\zeta\varphi_M\frac{\partial_i\varphi}{\varphi}\varphi^p\,\mathrm{d}x\end{aligned}\tag{4.10}$$

Now let $M\to\infty$ in (4.10). Note that $\varphi_M \to (1/\varphi^{p-1})1_{\{\varphi>0\}}$ $\mathrm{d}x$-a.s. and $\vartheta'_M(1/\varphi^{p-1}) \to 1$ $\mathrm{d}x$-a.s. In order to apply Lebesgue's dominated convergence theorem, we verify

$$|\overline{\partial}_i f\zeta\varphi_M\varphi^p| \le 2|\overline{\partial}_i f\varphi|\,\|\zeta\|_\infty\,1_{\operatorname{supp}\zeta} \in L^1(\mathrm{d}x),$$

where we have used that

$$|\varphi_M\varphi^{p-1}| \le (M+1)/M \le 2.$$

Furthermore,

$$\begin{aligned}|f\zeta\partial_i\varphi\vartheta'_M\left(1/\varphi^{p-1}\right)| &\le |f\partial_i\varphi|\,\|\zeta\|_\infty\,1_{\operatorname{supp}\zeta} \in L^1(\mathrm{d}x),\\ |f\partial_i\zeta\varphi_M\varphi^p| &\le 2|f\varphi|\,\|\partial_i\zeta\|_\infty\,1_{\operatorname{supp}\zeta} \in L^1(\mathrm{d}x),\\ &\text{and}\\ |f\zeta\varphi_M\partial_i\varphi\varphi^{p-1}| &\le 2|f\partial_i\varphi|\,\|\zeta\|_\infty\,1_{\operatorname{supp}\zeta} \in L^1(\mathrm{d}x).\end{aligned}$$

The formula obtained, when passing on to the limit $M\to\infty$ in (4.10), is exactly the desired statement. □

Proof of Theorem 4.4. Let $f \in W^{1,p}_w$ be (a class of) a function which is bounded and compactly supported. By Lemma 3.19, we are done if we can approximate f by C_0^∞-functions. Let $\{\eta_\varepsilon\}_{\varepsilon>0}$ be a (radially symmetric) standard mollifier, i.e.,

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^d}\eta\left(\frac{x}{\varepsilon}\right),$$

where $\eta \in C_0^\infty(\mathbb{R}^d)$ with $\eta \ge 0$, $\eta(x) = \eta(|x|)$, $\operatorname{supp}\eta \subset \overline{B}(0,1)$ and $\int\eta\,\mathrm{d}x = 1$. Since f is bounded and compactly supported, $\eta_\varepsilon * f \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(\eta_\varepsilon * f) \subset$

$\operatorname{supp} f + \varepsilon B(0,1)$ and $|\eta_\varepsilon * f| \le \|f\|_\infty$. We claim that there exists a sequence $\varepsilon_n \searrow 0$ such that $\eta_{\varepsilon_n} * f$ converges to f in $W^{1,p}_w$. The L^p_w-part is easy. Since $\eta_\varepsilon * f, f \in L^1(\mathrm{d}x)$, $\lim_{\varepsilon\searrow 0} \|\eta_\varepsilon * f - f\|_{L^1(\mathrm{d}x)} = 0$. Therefore we can extract a subsequence $\{\varepsilon_n\}$ such that $\eta_{\varepsilon_n} * f \to f$ $\mathrm{d}x$-a.s. Set $\varphi := w^{1/p}$. For $\varepsilon_n \le 1$

$$|(\eta_{\varepsilon_n} * f)\varphi - f\varphi|^p \le 2^p \|f\|_\infty^p |\varphi|^p 1_{\operatorname{supp} f + B(0,1)} \in L^1(\mathrm{d}x).$$

By Lebesgue's dominated convergence theorem, $\lim_n \|\eta_{\varepsilon_n} * f - f\|_{L^p_w} = 0$.

Fix $1 \le i \le d$. We are left to prove $\partial_i(\eta_{\varepsilon_n} * f) \to \overline{\partial}_i f$ in L^p_w for some sequence $\varepsilon_n \searrow 0$. Or equivalently,

$$\varphi\partial_i(\eta_{\varepsilon_n} * f) \to \varphi\overline{\partial}_i f \quad \text{in } L^p(\mathrm{d}x).$$

We shall use another approximation, as in

$$\begin{aligned}&\int |\varphi\partial_i(\eta_\varepsilon * f) - \varphi\overline{\partial}_i f|^p \,\mathrm{d}x\\ \le& 2^{p-1}\left[\int |\varphi\overline{\partial}_i f - (\eta_\varepsilon * (\varphi\overline{\partial}_i f))|^p \,\mathrm{d}x + \int |(\eta_\varepsilon * (\varphi\overline{\partial}_i f)) - \varphi\partial_i(\eta_\varepsilon * f)|^p \,\mathrm{d}x\right].\end{aligned}$$

The first term tends to zero as $\varepsilon \searrow 0$ by a well-known fact [Alt06, Satz 2.14]. We continue with studying the second term. Recall that $\eta_\varepsilon(x) = \eta_\varepsilon(|x|)$.

$$\begin{aligned}&\int |\varphi\partial_i(\eta_\varepsilon * f) - (\eta_\varepsilon * (\varphi\overline{\partial}_i f))|^p \,\mathrm{d}x\\ =&\int \left|\varphi(x)\int \partial_i\eta_\varepsilon(x-y)f(y)\,\mathrm{d}y - \int \eta_\varepsilon(x-y)\varphi(y)\overline{\partial}_i f(y)\,\mathrm{d}y\right|^p \mathrm{d}x\\ =&\int \Bigg|\int \partial_i\eta_\varepsilon(x-y)f(y)[\varphi(x)-\varphi(y)]\,\mathrm{d}y\\ &+\int \partial_i\eta_\varepsilon(x-y)f(y)\varphi(y) - \eta_\varepsilon(x-y)\varphi(y)\overline{\partial}_i f(y)\,\mathrm{d}y\Bigg|^p \mathrm{d}x\\ &\text{apply Lemma 4.5 with } \zeta(y) := \eta_\varepsilon(y-x) \text{ (noting that } \partial_i\eta_\varepsilon(x-y) = -\partial_i\eta_\varepsilon(y-x))\\ =&\int \left|\int \partial_i\eta_\varepsilon(x-y)f(y)[\varphi(x)-\varphi(y)]\,\mathrm{d}y + \int \eta_\varepsilon(x-y)f(y)\partial_i\varphi(y)\,\mathrm{d}y\right|^p \mathrm{d}x\\ \le& 2^{p-1}\left[\int \left|\int \partial_i\eta_\varepsilon(x-y)f(y)[\varphi(x)-\varphi(y)]\,\mathrm{d}y\right|^p \mathrm{d}x + \int |\eta_\varepsilon * (f\partial_i\varphi)|^p \,\mathrm{d}x\right]\\ \le& 2^{p-1}\int \left|\int \partial_i\eta_\varepsilon(x-y)f(y)[\varphi(x)-\varphi(y)]\,\mathrm{d}y\right|^p \mathrm{d}x + 2^{p-1}\|f\partial_i\varphi\|^p_{L^p(\mathrm{d}x)}.\end{aligned}$$

We would like to control the first term. Replace φ by $\widehat{\varphi} \in H^{1,p}$ defined by:

$$\widehat{\varphi} = \varphi\xi \;\text{ with }\; \xi \in C_0^\infty(\mathbb{R}^d) \;\text{ and }\; 1_{\operatorname{supp} f + B(0,2)} \le \xi \le 1_{\operatorname{supp} f + B(0,3)}.$$

Let $h_\varepsilon : \mathbb{R}^d \to \mathbb{R}^d$, $h_\varepsilon(x) := -\varepsilon x$. Then upon substituting $y = x + \varepsilon z$ (which leads to

$\mathrm{d}y = \varepsilon^d \,\mathrm{d}z$)

$$
\begin{aligned}
&\int\left|\int \partial_i\eta_\varepsilon(x-y)f(y)[\varphi(x)-\varphi(y)]\,\mathrm{d}y\right|^p\,\mathrm{d}x\\
=&\int\left|\int_{B(0,1)} \partial_i\eta_\varepsilon(-\varepsilon z)f(x+\varepsilon z)[\widehat{\varphi}(x)-\widehat{\varphi}(x+\varepsilon z)]\varepsilon^d\,\mathrm{d}z\right|^p\,\mathrm{d}x\\
&\quad\text{since by the chain rule } -\varepsilon(\partial_i\eta_\varepsilon)(-\varepsilon z)=\partial_i(\eta_\varepsilon\circ h_\varepsilon)(z)=(1/\varepsilon^d)\partial_i(\eta)(z)\\
=&\int\left|\int_{B(0,1)} \partial_i\eta(z)f(x+\varepsilon z)\frac{\widehat{\varphi}(x)-\widehat{\varphi}(x+\varepsilon z)}{\varepsilon}\,\mathrm{d}z\right|^p\,\mathrm{d}x\\
\le&2^{p-1}\int\left|\int_{B(0,1)} \partial_i\eta(z)f(x+\varepsilon z)\left\langle-\nabla\widehat{\varphi}(x+\varepsilon z),z\right\rangle\,\mathrm{d}z\right|^p\,\mathrm{d}x\\
&+2^{p-1}\int\left|\int_{B(0,1)} \partial_i\eta(z)f(x+\varepsilon z)\left[\frac{\widehat{\varphi}(x)-\widehat{\varphi}(x+\varepsilon z)}{\varepsilon}+\left\langle\nabla\widehat{\varphi}(x+\varepsilon z),z\right\rangle\right]\,\mathrm{d}z\right|^p\,\mathrm{d}x
\end{aligned}
$$

By Jensen's inequality, the first term is bounded by

$$C(p,d)\,\|\partial_i\eta\|_\infty^p\sum_{j=1}^d\|f\partial_j\varphi\|^p_{L^p(\mathrm{d}x)}\,,$$

where $C(p,d)$ is a positive constant depending only on p and d.

Concerning the second term, we use Jensen's inequality and Fubini's theorem to see that it is bounded by

$$C'(p,d)\,\|\partial_i\eta\|_\infty^p\,\|f\|_\infty^p\int_{B(0,1)}\int\left|\frac{\widehat{\varphi}(x)-\widehat{\varphi}(x+\varepsilon z)}{\varepsilon}+\left\langle\nabla\widehat{\varphi}(x+\varepsilon z),z\right\rangle\right|^p\,\mathrm{d}x\,\mathrm{d}z,$$

where $C'(p,d)$ is a positive constant depending only on p and d. Let us investigate the inner integral. We need a lemma on difference quotients. Compare with [GT77, Proof of Lemma 7.23].

Lemma 4.6. *Let $z\in B(0,1)\subset\mathbb{R}^d$ and $u\in H^{1,p}(\mathrm{d}x)$. Set for $\varepsilon>0$*

$$\Delta_\varepsilon u(x):=\frac{u(x-\varepsilon z)-u(x)}{\varepsilon}$$

for some representative of u. Then

$$\|\Delta_\varepsilon u-\langle\nabla u,z\rangle\|_{L^p(\mathrm{d}x)}\to 0$$

as $\varepsilon\searrow 0$.

Proof. Start with $u\in C^1\cap H^{1,p}$. By the fundamental theorem of calculus

$$\Delta_\varepsilon u(x)=\frac{1}{\varepsilon}\int_0^\varepsilon\langle\nabla u,z\rangle\,(x-sz)\,\mathrm{d}s.$$

Use Fubini's Theorem to get

$$\int |\Delta_\varepsilon u(x) - \langle \nabla u(x), z\rangle|^p \,\mathrm{d}x = \frac{1}{\varepsilon}\int_0^\varepsilon \int |\langle \nabla u(x-sz), z\rangle - \langle \nabla u(x), z\rangle|^p \,\mathrm{d}x\,\mathrm{d}s. \quad (4.11)$$

By a well-known property of L^p-norms [Alt06, Satz 2.14] the map

$$s \mapsto \int |\langle \nabla u(x-sz), z\rangle - \langle \nabla u(x), z\rangle|^p \,\mathrm{d}x$$

is continuous in zero. Hence $s = 0$ is a Lebesgue point of this map. Therefore the right hand side of (4.11) tends to zero as $\varepsilon \searrow 0$. The claim can be extended to functions in $H^{1,p}$ by an approximation by smooth functions as e.g. in [Zie89, Theorem 2.3.2]. □

By variable substitution, we get

$$\int \left| \frac{\widehat{\varphi}(x-\varepsilon z) - \widehat{\varphi}(x)}{\varepsilon} + \langle \nabla \widehat{\varphi}(x), z\rangle \right|^p \,\mathrm{d}x.$$

By the preceding lemma, the term converges to zero pointwise as $\varepsilon \searrow 0$ for each fixed $z \in B(0,1)$. Let for $g \in L^1_{\mathrm{loc}}$,

$$Mg(x) := \sup_{\rho>0} \frac{1}{\operatorname{vol} B(x,\rho)} \int_{B(x,\rho)} |g(y)| \,\mathrm{d}y,$$

be the *centered Hardy–Littlewood maximal function.* We shall need the useful inequality

$$|u(x) - u(y)| \le c|x-y| \left[M|\nabla u|(x) + M|\nabla u|(y) \right] \quad (4.12)$$

for any $u \in H^{1,p}$, for all $x, y \in \mathbb{R}^d \setminus N$, where N is a set of Lebesgue measure zero and c is a positive constant depending only on d and p. For a proof see e.g. [AK09, Corollary 4.3]. The inequality is credited to L. I. Hedberg [Hed72].

Also for all $u \in L^p$

$$\|Mu\|_{L^p} \le c' \|u\|_{L^p} \quad (4.13)$$

by the maximal function theorem [Ste70, Theorem I.1] and $c' > 0$ depends only on d and p.

Hence for $\mathrm{d}z$-a.a. $z \in B(0,1)$

$$\int \left| \frac{\widehat{\varphi}(x-\varepsilon z) - \widehat{\varphi}(x)}{\varepsilon} + \langle \nabla \widehat{\varphi}(x), z\rangle \right|^p \,\mathrm{d}x \le C(p,d) \|\nabla \widehat{\varphi}\|^p_{L^p(\mathrm{d}x)} |z|^p 1_{B(0,1)} \in L^1(\mathrm{d}z).$$

The desired convergence to zero as $\varepsilon \searrow 0$ follows now by the preceding discussion and Lebesgue's dominated convergence theorem.

We have proved that

$$\begin{aligned} &\int |\varphi \partial_i(\eta_\varepsilon * f) - (\eta_\varepsilon * (\varphi \overline{\partial}_i f))|^p \,\mathrm{d}x \\ \le & C(d,p,\operatorname{supp} f) \left[\sum_{j=1}^d \|f \partial_j \varphi\|^p_{L^p(\mathrm{d}x)} + \|f\|^p_\infty \theta(\varepsilon) \right] \end{aligned} \quad (4.14)$$

with $\theta(\varepsilon) \to 0$ as $\varepsilon \searrow 0$, and θ depends only on $\operatorname{supp} f$.

To conclude the proof, we shall need another approximation. Let $f_\delta := \eta_\delta * f$ for $\delta > 0$. By Lebesgue's dominated convergence theorem again, we can prove that there is a subnet (also denoted by $\{f_\delta\}$), such that

$$\sum_{j=1}^{d} \|(f - f_\delta)\partial_j \varphi\|_{L^p(\mathrm{d}x)}^p \to 0 \tag{4.15}$$

as $\delta \searrow 0$. We shall estimate with the help of (4.14) (f replaced by $f - f_\delta$ therein)

$$\begin{aligned}
&\left\|\varphi\partial_i(\eta_\varepsilon * f) - (\eta_\varepsilon * (\varphi\overline{\partial}_i f))\right\|_{L^p(\mathrm{d}x)}^p \\
\le & 2^{p-1}\left\|\varphi\partial_i(\eta_\varepsilon * (f - f_\delta)) - (\eta_\varepsilon * (\varphi\overline{\partial}_i (f - f_\delta)))\right\|_{L^p(\mathrm{d}x)}^p \\
&+ 2^{p-1}\left\|\varphi\partial_i(\eta_\varepsilon * f_\delta) - (\eta_\varepsilon * (\varphi\overline{\partial}_i f_\delta))\right\|_{L^p(\mathrm{d}x)}^p \\
\le & C(d, p, \operatorname{supp} f)\left[\sum_{j=1}^{d} \|f - f_\delta \partial_j \varphi\|_{L^p(\mathrm{d}x)}^p + \|f - f_\delta\|_\infty^p \,\theta(\varepsilon)\right] \\
&+ 2^{p-1}\left\|\varphi\partial_i(\eta_\varepsilon * f_\delta) - (\eta_\varepsilon * (\varphi\overline{\partial}_i f_\delta))\right\|_{L^p(\mathrm{d}x)}^p .
\end{aligned}$$

The use of (4.14) is justified, since $\widehat{\varphi} = \varphi$ on $\operatorname{supp} f + B(0, 2)$, thus on $\operatorname{supp}(f - f_\delta) + B(0, 1)$. Taking (4.15) into account, by choosing first δ and then ε, the first term above can be controlled (since $\|f - f_\delta\|_\infty \le 2\|f\|_\infty$). If we can prove for any $\zeta \in C_0^\infty$

$$\left\|\varphi\partial_i(\eta_\varepsilon * \zeta) - (\eta_\varepsilon * (\varphi\overline{\partial}_i \zeta))\right\|_{L^p(\mathrm{d}x)}^p \to 0 \tag{4.16}$$

as $\varepsilon \searrow 0$, we can control the second term above and hence are done. But

$$\begin{aligned}
&\|\varphi\partial_i(\eta_\varepsilon * \zeta) - (\eta_\varepsilon * (\varphi\partial_i \zeta))\|_{L^p(\mathrm{d}x)}^p \\
\le & \int \left| \int \eta_\varepsilon(x - y)\partial_i \zeta(y)\,[\varphi(x) - \varphi(y)]\,\mathrm{d}y \right|^p \mathrm{d}x \\
& \text{upon substituting } y = x + \varepsilon z \ (\mathrm{d}y = \varepsilon^d\,\mathrm{d}z) \text{ and using “Jensen” and “Fubini”} \\
\le & C(d, p)\,\|\eta\|_\infty^p\,\|\partial_i\zeta\|_\infty^p \int_{B(0,1)} \|(\varphi\xi_\zeta)(\cdot) - (\varphi\xi_\zeta)(\cdot + \varepsilon z)\|_{L^p(\mathrm{d}x)}^p\,\mathrm{d}z,
\end{aligned}$$

where $\xi_\zeta \in C_0^\infty(\mathbb{R}^d)$ with $\xi_\zeta = 1$ on $\operatorname{supp}\zeta + B(0, 1)$.

$$\|(\varphi\xi_\zeta)(\cdot) - (\varphi\xi_\zeta)(\cdot + \varepsilon z)\|_{L^p(\mathrm{d}x)}^p$$

tends to zero as $\varepsilon \searrow 0$ again by [Alt06, Satz 2.14]. By inequalities (4.12) and (4.13) for $\mathrm{d}z$-a.a. $z \in B(0, 1)$

$$\|(\varphi\xi_\zeta)(\cdot) - (\varphi\xi_\zeta)(\cdot + \varepsilon z)\|_{L^p(\mathrm{d}x)}^p \le c(d, p)\,\|\nabla(\varphi\xi_\zeta)\|_{L^p(\mathrm{d}x)}^p\,|\varepsilon z|^p 1_{B(0,1)} \in L^1(\mathrm{d}z),$$

thus we can apply Lebesgue's dominated convergence theorem.

The proof is complete. $\square$

4.3 The classical case recovered

The weak weighted p-Sobolev space $W^{1,p}_w$ is usually defined differently in the literature, see [Kuf80, KO84]. Let $1 < p < \infty$ and

$$\widetilde{W^{1,p}_w}(\mathbb{R}^d) := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathrm{d}x) \;\middle|\; u \in L^p_w(\mathbb{R}^d),\ \mathrm{D}u \in L^p_w(\mathbb{R}^d \to \mathbb{R}^d) \right\}. \tag{4.17}$$

We also refer to [Kil97] for a discussion on $H = W$ using Definition (4.17). It is related to the so-called weighted Poincaré inequality. We shall not go into details. For the case $p = 2$, the results of this section are known by [AR90b].

Lemma 4.7. *Under* (S2), (HW), $W^{1,p}_w(\mathbb{R}^d) \subset \widetilde{W^{1,p}_w}(\mathbb{R}^d)$. *In particular,* $\overline{\nabla} u = \mathrm{D}u$ $\mathrm{d}x$*-a.e.*

Proof. Let $u \in W^{1,p}_w$. By (S2), u is a distribution in L^1_{loc}, see Lemma 3.5. By Theorem 4.4, there is a sequence $\{\eta_k\} \subset C_0^\infty$ with $\eta_k \to u$ in $L^p_w(\mathbb{R}^d)$ and $\nabla\eta_k \to \overline{\nabla} u$ in $L^p_w(\mathbb{R}^d \to \mathbb{R}^d)$. By (S2), the sequences converge in L^1_{loc}, too. Let $\zeta \in C_0^\infty$. Then

$$\int \eta_k \partial_i \zeta \,\mathrm{d}x = -\int \partial_i \eta_k \zeta \,\mathrm{d}x$$

and in the limit,

$$\int u \partial_i \zeta \,\mathrm{d}x = \int \overline{\partial}_i u \zeta \,\mathrm{d}x,$$

hence $\mathrm{D}u = \overline{\nabla} u$ $\mathrm{d}x$-a.e. □

Basically, the claim could also have been deduced form the fact that our definition of the strong Sobolev space $H^{1,p}_w$ is classical. For the converse, we need the lattice property of $\widetilde{W^{1,p}_w}(\mathbb{R}^d)$.

Lemma 4.8. *Under* (S2), (HW), *for* $f \in \widetilde{W^{1,p}_w}(\mathbb{R}^d)$, *bounded and compactly supported, it holds that*

$$\mathrm{D}_i(f\varphi) = f\partial_i\varphi + \varphi\mathrm{D}_i f$$

in the sense of Schwartz distributions for any $1 \le i \le d$.

Proof. Let $M \in \mathbb{N}$ and ϑ_M as in the proof of Lemma 4.5, that is, $\vartheta_M \in C_0^\infty(\mathbb{R})$ with

$$\vartheta_M(t) = t \text{ for } t \in [-M, M],\ |\vartheta_M| \le M + 1,\ |\vartheta_M'| \le 1$$

and

$$\mathrm{supp}(\vartheta_M) \subset [-3M, 3M].$$

Set $\varphi_M := \vartheta_M(\varphi)$. φ_M is bounded and $|\varphi_M| \le |\varphi|$. By the chain rule for Sobolev functions, $\mathrm{D}_i\varphi_M = \partial_i\varphi_M = \vartheta_M'(\varphi)\partial_i\varphi$ a.e. By an approximation by smoothing mollifiers one easily proves that $\mathrm{D}_i(f\varphi_M) = f\mathrm{D}_i\varphi_M + \varphi_M\mathrm{D}_i f$ in the sense of distributions (just note that by (S2) f is locally integrable and $\mathrm{D}_i f$ is locally integrable, also f and φ_M are bounded, $f\varphi_M$ is locally integrable). Let $\eta \in C_0^\infty$. Then for every $1 \le i \le d$

$$\int \partial_i \eta f \varphi_M \,\mathrm{d}x = -\int \eta \mathrm{D}_i f \varphi_M \,\mathrm{d}x - \int \eta f \vartheta'_M(\varphi)\partial_i \varphi \,\mathrm{d}x. \tag{4.18}$$

As $M \to \infty$, $\varphi_M \to \varphi$ $\mathrm{d}x$-a.e., $\vartheta'_M(\varphi) \to 1$ $\mathrm{d}x$-a.e. But we are allowed to apply Lebesgue's dominated convergence theorem by

$$|\partial_i \eta f \varphi_M| \le |f\varphi| \, \|\partial_i \eta\|_\infty 1_{\operatorname{supp}\eta} \in L^1(\mathrm{d}x),$$
$$|\eta \mathrm{D}_i f \varphi_M| \le |\mathrm{D}_i f \varphi| \, \|\eta\|_\infty 1_{\operatorname{supp}\eta} \in L^1(\mathrm{d}x),$$
and
$$|\eta f \vartheta'_M(\varphi)\partial_i\varphi| \le |f\partial_i\varphi| \, \|\eta\|_\infty 1_{\operatorname{supp}\eta} \in L^1(\mathrm{d}x).$$

Here, we have used that $f\varphi, \mathrm{D}_i f\varphi \in L^p$ and that f is bounded and $\partial_i\varphi$ is locally integrable. The formula obtained in the limiting procedure as $M \to \infty$ is exactly the assertion. □

Lemma 4.9. *Under* (S2), *each* $f \in \widetilde{W_w^{1,p}(\mathbb{R}^d)}$ *has a version* $\widetilde{f}^i$ *for* $1 \le i \le d$ *which is locally absolutely continuous on almost all lines parallel to the coordinate axes.*

Proof. Assuming (S2), by the proof of Lemma 3.5, each $\mathrm{D}_i f$ is locally integrable. Therefore the desired locally absolutely continuous version exists by [Miz73, Theorem 2.7]. □

Proposition 4.10. *Under* (S2), (HW), $\widetilde{W_w^{1,p}(\mathbb{R}^d)} \subset W_w^{1,p}(\mathbb{R}^d)$ *and* $\overline{\nabla} u = \mathrm{D}u$ $\mathrm{d}x$*-a.e.*

Proof. By (S2), we conclude from Lemma 4.9 and the proof of Lemma 3.18, that $\widetilde{W_w^{1,p}(\mathbb{R}^d)}$ is a lattice. Now by an adaption of Lemma 3.19, bounded and compactly supported functions are dense in $\widetilde{W_w^{1,p}(\mathbb{R}^d)}$. Let f be such a function. Fix $1 \le i \le d$. Let $\{\eta_k\} \subset C_0^\infty$ be a sequence such that

$$\eta_k \to \varphi^{p-1} \quad \text{in } L^q_{\mathrm{loc}}.$$

Since $\varphi \in H^{1,p}_{\mathrm{loc}}$, such a sequence can be obtained by a mollifier. Also

$$\partial_i \eta_k \to (p-1)\varphi^{p-2}\partial_i\varphi \quad \text{in } L^q_{\mathrm{loc}}$$

since $\varphi^{p-2}\partial_i\varphi \in L^q_{\mathrm{loc}}$ by (HW). Fix $\zeta \in C_0^\infty$. Now by Lemma 4.8

$$\begin{aligned}\int \partial_i \zeta \eta_k f \varphi \,\mathrm{d}x &= -\int \zeta \partial_i \eta_k f\varphi \,\mathrm{d}x - \int \zeta \eta_k \mathrm{D}_i(f\varphi)\,\mathrm{d}x \\ &= -\int \zeta \partial_i \eta_k f\varphi \,\mathrm{d}x - \int \zeta\eta_k \mathrm{D}_i f\varphi\,\mathrm{d}x - \int \zeta\eta_k f\partial_i\varphi\,\mathrm{d}x.\end{aligned} \tag{4.19}$$

The limit $k \to \infty$ exists by Hölder's inequality, taking into account that $f\varphi \in L^p$, $\mathrm{D}_i f\varphi \in L^p$, $\partial_i\varphi \in L^p_{\mathrm{loc}}$. Taking the limit yields the formula

$$\int \partial_i \zeta f \varphi^p \,\mathrm{d}x = (1-p)\int \zeta f \frac{\partial_i\varphi}{\varphi}\varphi^p \,\mathrm{d}x - \int \zeta \mathrm{D}_i f \varphi^p \,\mathrm{d}x - \int \zeta f \frac{\partial_i \varphi}{\varphi}\varphi^p\,\mathrm{d}x,$$

which proves that $f \in W_w^{1,p}$ and $\overline{\nabla} f = \mathrm{D}f\ \varphi^p \mathrm{d}x$-a.s. and hence, since by (S2) $L_w^p(\Omega) \subset L^1(\Omega)$ for each bounded open set $\Omega \subset \mathbb{R}^d$, $\overline{\nabla} f = \mathrm{D}f\ \mathrm{d}x$-a.s. Together with Lemma 4.7, we have proved that $W_w^{1,p} \cap L^\infty = \widetilde{W_w^{1,p}} \cap L^\infty$. The claim follows now by an approximation in $\|\cdot\|_{1,p,w}$-norm and the fact that $W_w^{1,p}$ is a Banach space in this norm. $\square$

5 The general theory of varying Banach spaces

5.1 Gromov-Hausdorff convergence

This section is to be read as a prequel to the theory of varying spaces. That it is embedded therein, we can see in Lemma 5.9. For the next definition see [KS08, Section 2]. See also [Gro99].

Definition 5.1. *A metric space X is called* boundedly compact*, if its bounded and closed subsets are compact.*

A pair (X, o) is called pointed metric space*, if X is a metric space and $o \in X$ is a distinguished point, called the* basepoint*.*

Let $X, Y \subset Z$, where Z is a metric space.

$$d_Z^H(X,Y) := \inf\{\rho > 0 \mid X \subset B_Z(Y,\rho),\ Y \subset B_Z(X,\rho)\}$$

is called the Hausdorff distance*.*

Let X, Y be compact metric spaces.

$$d^{GH}(X,Y) := \inf\left\{d_Z^H(X,Y) \mid Z \text{ a metric space},\ X, Y \hookrightarrow Z \text{ isometrically}\right\}$$

is called the Gromov-Hausdorff distance*.*

Let X, Y be compact metric spaces. Let $\Phi : X \to Y$ be any map.

$$\operatorname{dis}\Phi := \sup_{x,y\in X} |d_Y(\Phi(x),\Phi(y)) - d_X(x,y)|$$

is called the distorsion*.*

Let X, Y be compact metric spaces. If a map $\Phi : X \to Y$ satisfies $\operatorname{dis}\Phi < \varepsilon$ *and $B_Y(\Phi(X),\varepsilon) = Y$, then it is called an* ε-approximation*.*

The Hausdorff distance turns the collection $\mathscr{C}(Z)$ of all non-empty compact subsets of Z into a metric space. The Gromov-Hausdorff distance turns the collection of all isometry classes of compact metric spaces into a metric space.

Lemma 5.2. *Let X, Y be compact metric spaces. If $d^{GH}(X,Y) < \varepsilon$, then there exists a 2ε-approximation from X to Y.*

Conversely, if there exists an ε-approximation from X to Y, then $d^{GH}(X,Y) < 2\varepsilon$.

Proof. See [BBI01, Corollary 7.3.28]. □

Definition 5.3. *Let $\mathscr{N}$ be a directed set and $\{(X_\nu, o_\nu)\}_{\nu\in\mathscr{N}}$ a net of pointed boundedly compact metric spaces. We say that $\{(X_\nu, o_\nu)\}_{\nu\in\mathscr{N}}$* Gromov-Hausdorff converges *to a pointed boundedly compact metric space (X, o) if for any $\rho > 0$ there exist nets $\{\rho_\nu\}_{\nu\in\mathscr{N}}$, $\{\varepsilon_\nu^\rho\}_{\nu\in\mathscr{N}}$ of positive real numbers such that $\rho_\nu \searrow \rho$ and $\varepsilon_\nu^\rho \searrow 0$ and there exist ε_ν^ρ-approximations $\varphi_\nu^\rho : B_{X_\nu}(o_\nu, \rho_\nu) \to B_X(o, \rho)$ such that $\varphi_\nu^\rho(o_\nu) = o$.*

In the compact case, we have the following.

Lemma 5.4. *Let $\mathscr{N}$ be a directed set and let $\{X_\nu\}_{\nu\in\mathscr{N}}$ be a net of compact metric spaces and let X be a compact metric space. If for a net of points $\{o_\nu\}$, $o_\nu \in X_\nu$, $\nu \in \mathscr{N}$, $o \in X$ it holds that*

$$(X_\nu, o_\nu) \to (X, o) \quad \text{in the Gromov-Hausdorff sense,} \tag{5.1}$$

then

$$\lim_{\nu\in\mathscr{N}} d^{GH}(X_\nu, X) = 0. \tag{5.2}$$

Conversely, if (5.2) *holds and $o \in X$, then one can find $o_\nu \in X_\nu$, $\nu \in \mathscr{N}$ such that* (5.1) *holds.*

Proof. See [BBI01, Exercise 8.1.2]. □

The following proposition is proved in [KS08, Proposition 2.3]. See also [Gro99, Proof of 3.5(b)].

Proposition 5.5. *Let $\mathscr{N}$ be a directed set and $\{(X_\nu, o_\nu)\}_{\nu\in\mathscr{N}}$ a net of pointed boundedly compact metric spaces, such that $\{(X_\nu, o_\nu)\}_{\nu\in\mathscr{N}}$* Gromov-Hausdorff converges *to a pointed boundedly compact metric space (X, o). Then there exists a boundedly compact metric $d_{\mathfrak{X}}$ on the disjoint union*

$$\mathfrak{X} := \dot{\bigcup_{\nu\in\mathscr{N}}} X_\nu \dot{\cup} X$$

such that:

(i) The restrictions of $d_{\mathfrak{X}}$ on X_ν, $\nu \in \mathscr{N}$, X coincide with the original metrics d_{X_ν}, $\nu \in \mathscr{N}$, d_X respectively.

(ii) $\{X_\nu\}_{\nu\in\mathscr{N}}$ compactly Hausdorff-converges *to X in $\mathfrak{X}$, that is, for any $\rho > 0$ there exists a net $\{\rho_\nu\}_{\nu\in\mathscr{N}}$ with $\rho_\nu \searrow \rho$ such that*

$$\mathbb{R}\text{-}\lim_{\nu\in\mathscr{N}} d_{\mathfrak{X}}^H(B_{\mathfrak{X}}(o, \rho_\nu) \cap X_\nu, B_{\mathfrak{X}}(o, \rho) \cap X) = 0.$$

(iii) $\mathbb{R}\text{-}\lim_{\nu\in\mathscr{N}} d_{\mathfrak{X}}(o_\nu, o) = 0$.

According to the references, the following metric matches our purposes. Let $\{p_1 = o, p_2, \ldots\} = P \subset X$ be a countable dense set (which exists since X is boundedly compact). By Gromov-Hausdorff convergence, there is a net $\{N_\nu\}$ of natural numbers increasing to $+\infty$ and points $p_n^\nu \in X_\nu$, $p_1^\nu = o_\nu$, $\nu \in \mathscr{N}$, $n \in \mathbb{N}$ such that for all $\nu \in \mathscr{N}$

$$|d_{X_\nu}(p_n^\nu, p_m^\nu) - d_X(p_n, p_m)| \le \frac{1}{N_\nu} \quad \forall 1 \le n, m \le N_\nu.$$

The proposed metric is

$$d_{\mathfrak{X}}(x,y) := \inf_{1\le n\le N_\nu} \left[d_{X_\nu}(x,p_n^\nu) + d_X(p_n,y) + \frac{1}{N_\nu} \right],$$

whenever $x \in X_\nu$, $y \in X$, $d_{\mathfrak{X}} := d_{X_\nu}$ on $X_\nu \times X_\nu$, $\nu \in \mathscr{N}$, $d_{\mathfrak{X}} := d_X$ on $X \times X$,

$$d_{\mathfrak{X}}(x,y) := \inf_{z\in X} \left[d_{\mathfrak{X}}(x,z) + d_{\mathfrak{X}}(z,y) \right],$$

whenever $x \in X_{\nu_1}$, $y \in X_{\nu_2}$, $\nu_1,\nu_2 \in \mathscr{N}$, $\nu_1 \neq \nu_2$. Such a metric satisfies

$$d_{\mathfrak{X}}(p_n^\nu, p_n) = \frac{1}{N_\nu} \quad \forall 1 \le n \le N_\nu. \tag{5.3}$$

5.2 Strong asymptotic relation

We work within a framework originating from K. Kuwae and T. Shioya. The standard references are [KS03, KS08].

Let $\mathscr{N}$ be a directed set. Suppose that for each $\nu \in \mathscr{N}$ there is a Banach space E_ν and suppose that there is a Banach space E. No index or the index "∞" (which is to be understood as a symbol $\notin \mathscr{N}$) will refer to E in the sequel.

Using set-theoretic labeling, we define the disjoint unions

$$\mathfrak{E} := \dot{\bigcup_{\nu\in\mathscr{N}}} E_\nu \dot{\cup} E, \quad \mathfrak{E}^* := \dot{\bigcup_{\nu\in\mathscr{N}}} E_\nu^* \dot{\cup} E^*, \quad \mathfrak{E}^{**} := \dot{\bigcup_{\nu\in\mathscr{N}}} E_\nu^{**} \dot{\cup} E^{**},$$

where " *", " **" resp. denote the Banach space dual, Banach space bidual resp. Note that by labeling, the union is forced to be pairwise disjoint. Therefore, $0 \in E_{\nu_1}$ is not equal to $0 \in E_{\nu_2}$ when $\nu_1 \neq \nu_2$. In the sequel, when writing 0, it should be clear in which space it is.

Definition 5.6 (Asymptotic Relation, Net Version)**.** *We call a topology τ on $\mathfrak{E}$ a* linear strong asymptotic relation *between $\{E_\nu\}_{\nu\in\mathscr{N}}$ and E (or on $\mathfrak{E}$) if τ satisfies the following conditions:*

(A1) *The relative topologies of E_ν, $\nu \in \mathscr{N}$, E in $(\mathfrak{E},\tau)$ coincide with the original strong topologies.*

(A2) *For any $u \in E$ there exists a net $\{u_\nu\}_{\nu\in\mathscr{N}}$ such that $u_\nu \in E_\nu$, $\nu \in \mathscr{N}$ and $\tau\text{-}\lim_{\nu\in\mathscr{N}} u_\nu = u$.*

(A3) *For any subnet $\{\nu_\mu\}_{\mu\in\mathscr{M}}$ of $\{\nu\}_{\nu\in\mathscr{N}}$, for any net $\{u_\mu\}_{\mu\in\mathscr{M}}$, with $u_\mu \in E_{\nu_\mu}$, $\mu \in \mathscr{M}$ and any $u \in E$ the following statement holds:*
If $\tau\text{-}\lim_{\mu\in\mathscr{M}} u_\mu = u$, then

$$\mathbb{R}\text{-}\lim_{\mu\in\mathscr{M}} \|u_\mu\|_{E_{\nu_\mu}} = \|u\|_E.$$

(A4) *For any subnet* $\{\nu_\mu\}_{\mu\in\mathscr{M}}$ *of* $\{\nu\}_{\nu\in\mathscr{N}}$, *for any two nets* $\{u_\mu\}_{\mu\in\mathscr{M}}$, $\{v_\mu\}_{\mu\in\mathscr{M}}$ *with* $u_\mu, v_\mu \in E_{\nu_\mu}$, $\mu \in \mathscr{M}$ *and any* $u \in E$ *the following statement holds:*
If $\tau\text{-}\lim_{\mu\in\mathscr{M}} u_\mu = u$ *and* $\mathbb{R}\text{-}\lim_{\mu\in\mathscr{M}} \|u_\mu - v_\mu\|_{E_{\nu_\mu}} = 0$, *then* $\tau\text{-}\lim_{\mu\in\mathscr{M}} v_\mu = u$.

(AL) *For any subnet* $\{\nu_\mu\}_{\mu\in\mathscr{M}}$ *of* $\{\nu\}_{\nu\in\mathscr{N}}$, *for any two nets* $\{u_\mu\}_{\mu\in\mathscr{M}}$, $\{v_\mu\}_{\mu\in\mathscr{M}}$ *with* $u_\mu, v_\mu \in E_{\nu_\mu}$, $\mu \in \mathscr{M}$ *and any two* $u, v \in E$ *and any* $\alpha, \beta \in \mathbb{R}$ *the following statement holds:*
If $\tau\text{-}\lim_{\mu\in\mathscr{M}} u_\mu = u$ *and* $\tau\text{-}\lim_{\mu\in\mathscr{M}} v_\mu = v$, *then* $\tau\text{-}\lim_{\mu\in\mathscr{M}} \left[\alpha u_\mu + \beta v_\mu\right] = \alpha u + \beta v$.

Our definition using subnets $\{\nu_\mu\}$ of $\{\nu\}$ might seem unnecessary at first glance. The reason is that the directed set $\{\nu\}$ is used already to index the spaces $\{E_\nu\}$. Hence, when considering any net in $\dot{\bigcup}_{\nu\in\mathscr{N}} E_\nu$, that converges to some point in E, we use this trick to keep track of the space in which an element u_μ of a net $\{u_\mu\}$ lies. Therefore, one could think of picking $\{\nu_\mu\}$ exactly such that $u_\mu \in E_{\nu_\mu}$. See also Section 5.14 for a more elegant definition (when $\mathscr{N}$ carries a topology).

It is important to realize that $O \subset E$, $\|\cdot\|_E$-open, is not a τ-neighborhood of its points! This would violate (A2). $\mathfrak{E}$ is the τ-closure of $\dot{\bigcup}_{\nu\in\mathscr{N}} E_\nu$ in $\mathfrak{E}$.

A strong asymptotic relation between $\{E_\nu\}_{\nu\in\mathscr{N}}$ and E is not a (topological) convergence in general. This was noted by K. Kuwae and T. Shioya in [KS08, Remark 3.2]. To see this, suppose that there is an asymptotic relation τ on $\dot{\bigcup}_{\nu\in\mathscr{N}} E_\nu \dot{\cup} E$. Then for any subspace $U \subset E$ the restricted topology on $\dot{\bigcup}_{\nu\in\mathscr{N}} E_\nu \dot{\cup} U$ is also an asymptotic relation, only (AL) fails to hold in general if U is not linear. Note also that an asymptotic relation depends on the norms of the spaces involved (and not only on the topologies). An equivalent renorming leads to different limit points in E.

Remark 5.7. *We also impose the following condition,*

(A3') *For any subnet* $\{\nu_\mu\}_{\mu\in\mathscr{M}}$ *of* $\{\nu\}_{\nu\in\mathscr{N}}$, *for any two nets* $\{u_\mu\}_{\mu\in\mathscr{M}}$, $\{v_\mu\}_{\mu\in\mathscr{M}}$, *with* $u_\mu, v_\mu \in E_{\nu_\mu}$, $\mu \in \mathscr{M}$ *and any two* $u, v \in E$ *the following statement holds:*
If $\tau\text{-}\lim_{\mu_{\mathscr{M}}} u_\mu = u$ *and* $\tau\text{-}\lim_{\mu\in\mathscr{M}} v_\mu = v$, *then*

$$\mathbb{R}\text{-}\lim_{\mu\in\mathscr{M}} \|u_\mu - v_\mu\|_{E_{\nu_\mu}} = \|u - v\|_E\,,$$

which is strictly stronger than (A3) *if* (AL) *is lacking. Clearly* (A3) *and* (AL) *imply* (A3'). *In the sequel we will always demand* (AL).

Given (A3), (A4), *we can replace* (AL) *equivalently by*

(AL') *For any subnet* $\{\nu_\mu\}_{\mu\in\mathscr{M}}$ *of* $\{\nu\}_{\nu\in\mathscr{N}}$, *for any two nets* $\{u_\mu\}_{\mu\in\mathscr{M}}$, $\{v_\mu\}_{\mu\in\mathscr{M}}$ *with* $u_\mu, v_\mu \in E_{\nu_\mu}$, $\mu \in \mathscr{M}$ *and any two* $u, v \in E$ *and any two nets* $\{\alpha_\mu\}_{\mu\in\mathscr{M}}, \{\beta_\mu\}_{\mu\in\mathscr{M}} \subset \mathbb{R}$ *and any two* $\alpha, \beta \in \mathbb{R}$ *such that* $\mathbb{R}\text{-}\lim_{\mu\in\mathscr{M}} \alpha_\mu = \alpha$, $\mathbb{R}\text{-}\lim_{\mu\in\mathscr{M}} \beta_\mu = \beta$ *the following statement holds:*
If $\tau\text{-}\lim_{\mu\in\mathscr{M}} u_\mu = u$ *and* $\tau\text{-}\lim_{\mu\in\mathscr{M}} v_\mu = v$, *then* $\tau\text{-}\lim_{\mu\in\mathscr{M}} \left[\alpha_\mu u_\mu + \beta_\mu v_\mu\right] = \alpha u + \beta v$.

which follows from (A3), (A4), (AL) *and the triangle inequality.*

A (not necessarily linear) strong asymptotic relation was introduced by K. Kuwae and T. Shioya in [KS08] for metric spaces, therein plainly called asymptotic relation, defined as follows.

Definition 5.8 (Asymptotic Relation, Metric Version)**.** *If each X_ν, $\nu \in \mathscr{N}$, X is a metric space (with distances d_ν, $\nu \in \mathscr{N}$, d respectively) we will still call a topology τ on $\mathfrak{X}$* asymptotic relation *if the following conditions are satisfied:*

(A1) *The relative topologies of X_ν, $\nu \in \mathscr{N}$, X in $(\mathfrak{X}, \tau)$ coincide with the original strong topologies.*

(A2) *For any $u \in X$ there exists a net $\{u_\nu\}_{\nu\in\mathscr{N}}$ such that $u_\nu \in X_\nu$, $\nu \in \mathscr{N}$ and $\tau\text{-}\lim_{\nu\in\mathscr{N}} u_\nu = u$.*

(A3) *For any subnet $\{\nu_\mu\}_{\mu\in\mathscr{M}}$ of $\{\nu\}_{\nu\in\mathscr{N}}$, for any two nets $\{u_\mu\}_{\mu\in\mathscr{M}}$, $\{v_\mu\}_{\mu\in\mathscr{M}}$ with $u_\mu, v_\mu \in X_{\nu_\mu}$, $\mu \in \mathscr{M}$ and any two $u, v \in X$ the following statement holds:*
If $\tau\text{-}\lim_{\mu\in\mathscr{M}} u_\mu = u$ and $\tau\text{-}\lim_{\mu\in\mathscr{M}} v_\mu = v$, then

$$\mathbb{R}\text{-}\lim_{\mu\in\mathscr{M}} d_{\nu_\mu}(u_\mu, v_\mu) = d(u, v).$$

(A4) *For any subnet $\{\nu_\mu\}_{\mu\in\mathscr{M}}$ of $\{\nu\}_{\nu\in\mathscr{N}}$, for any two nets $\{u_\mu\}_{\mu\in\mathscr{M}}$, $\{v_\mu\}_{\mu\in\mathscr{M}}$ with $u_\mu, v_\mu \in X_{\nu_\mu}$, $\mu \in \mathscr{M}$ and any $u \in X$ the following statement holds:*
If $\tau\text{-}\lim_{\mu\in\mathscr{M}} u_\mu = u$ and $\mathbb{R}\text{-}\lim_{\mu\in\mathscr{M}} d_{\nu_\mu}(u_\mu, v_\mu) = 0$, then $\tau\text{-}\lim_{\mu\in\mathscr{M}} v_\mu = u$.

Additionally, if each X_ν, $\nu \in \mathscr{N}$, X is a metrizable topological vector space, we call an asymptotic relation on $\mathfrak{X}$ linear *whenever:*

(AL) *For any subnet $\{\nu_\mu\}_{\mu\in\mathscr{M}}$ of $\{\nu\}_{\nu\in\mathscr{N}}$, for any two nets $\{u_\mu\}_{\mu\in\mathscr{M}}$, $\{v_\mu\}_{\mu\in\mathscr{M}}$ with $u_\mu, v_\mu \in X_{\nu_\mu}$, $\mu \in \mathscr{M}$ and any two $u, v \in X$ and any $\alpha, \beta \in \mathbb{R}$ the following statement holds:*
If $\tau\text{-}\lim_{\mu\in\mathscr{M}} u_\mu = u$ and $\tau\text{-}\lim_{\mu\in\mathscr{M}} v_\mu = v$, then $\tau\text{-}\lim_{\mu\in\mathscr{M}} \big[\alpha u_\mu + \beta v_\mu\big] = \alpha u + \beta v$.

The next lemma was stated in [KS08, Remark 3.2].

Lemma 5.9. *If a net of pointed boundedly compact metric spaces $\{(X_\nu, o_\nu)\}_{\nu\in\mathscr{N}}$ converges in the Gromov-Hausdorff sense to a pointed boundedly compact metric space (X, o), then $d_{\mathfrak{X}}$ as in Proposition 5.5 constitutes an asymptotic relation in the sense of Definition 5.8 on $\mathfrak{X}$.*

Proof. (A1) follows from Proposition 5.5 (i).

For (A2) let $x \in X$. Let $P \subset X$ be a countable dense set as in (5.3). Let $p_k \in P$, $k \in \mathbb{N}$ be a sequence such that

$$\lim_k d_X(p_k, x) = 0. \tag{5.4}$$

By (5.3) there are $p_k^\nu \in X_\nu$, $\nu \in \mathscr{N}$ such that for all $k \in \mathbb{N}$

$$\lim_{\nu\in\mathscr{N}} d_{\mathfrak{X}}(p_k^\nu, p_k) = 0,$$

and hence by Proposition 5.5 (i) and continuity of "$d_{\mathfrak{X}}(\cdot,\cdot)$"

$$\lim_{\nu\in\mathscr{N}} d_{\mathfrak{X}}(p_k^\nu, x) = d_{\mathfrak{X}}(p_k, x) = d_X(p_k, x). \tag{5.5}$$

Combining (5.4) and (5.5) by a diagonal procedure (see e.g. "diagonal lemma" 5.27 below) we find a net of natural numbers k_ν increasing to $+\infty$ such that

$$\lim_{\nu\in\mathscr{N}} d_{\mathfrak{X}}(p_{k_\nu}^\nu, x) = 0.$$

(A3) follows from

$$\begin{aligned}\left|d_{\nu_\mu}(u_\mu, v_\mu) - d(u,v)\right| &= |d_{\mathfrak{X}}(u_\mu, v_\mu) - d_{\mathfrak{X}}(u,v)| \\ &\leq |d_{\mathfrak{X}}(u_\mu, u) + d_{\mathfrak{X}}(v_\mu, v) + d_{\mathfrak{X}}(u,v) - d_{\mathfrak{X}}(u,v)|\,,\end{aligned}$$

where we have used Proposition 5.5 (i).

(A4) follows with from

$$d_{\mathfrak{X}}(v_\mu, u) \leq d_{\mathfrak{X}}(v_\mu, u_\mu) + d_{\mathfrak{X}}(u_\mu, u) = d_{\nu_\mu}(v_\mu, u_\mu) + d_{\mathfrak{X}}(u_\mu, u),$$

where we have used Proposition 5.5 (i). □

5.2.1 "Variable space" by Zhikov and Pastukhova

Quite recently, by V. V. Zhikov and S. E. Pastukhova in [ZP07], the following definition for a "variable Banach space" was proposed:

Definition 5.10 (Zhikov-Pastukhova)**.** *Let X_ε, $\varepsilon > 0$ X be Banach spaces. A* strong convergence *$x_\varepsilon \to x$ of a net of elements $\{x_\varepsilon\}$, $x_\varepsilon \in X_\varepsilon$, $\varepsilon > 0$ to an element $x \in X$ is defined by the following properties:*

1. *If $x_\varepsilon \to x$, then $\lim_{\varepsilon\to 0} \|x_\varepsilon\|_{X_\varepsilon} = \|x\|_X$.*
2. *If $x_\varepsilon \to x$ and $y_\varepsilon \to y$, then $x_\varepsilon - y_\varepsilon \to x - y$.*
3. *If, for an arbitrary $\delta > 0$, there exist $y_\varepsilon \in X_\varepsilon$ and $y \in X$ such that*
$$y_\varepsilon \to y, \quad \overline{\lim_{\varepsilon\to 0}}\, \|x_\varepsilon - y_\varepsilon\| \leq \delta, \quad \|x - y\| \leq \delta,$$
then $x_\varepsilon \to x$.
4. *Every element $x \in X$ is the strong limit of some net $\{x_\varepsilon\}$, $x_\varepsilon \in X_\varepsilon$, $\varepsilon > 0$.*

This is how the original definition in [ZP07] reads. The reader readily verifies that asymptotic relations and variable spaces are one and the same thing.

5.2.2 $\mathscr{N} = \mathbb{N}$

Suppose that our framework reduces to the case of

$$(\mathscr{N}, \succeq) = (\mathbb{N}, \geq)$$

and

$$\mathfrak{E} = \dot{\bigcup_{n \in \mathbb{N}}} E_n \dot{\cup} E.$$

Suppose that $\{u_k\}$ is a sequence in $\dot{\bigcup}_n E_n$. Picking $n_k \in \mathbb{N}$ such that $u_k \in E_{n_k}$ for all $k \in \mathbb{N}$, we get a sequence $\{n_k\}$ of natural numbers, which is unique by disjointness. Moreover, if the sequence $\{u_k\}$ is eventually in $\dot{\bigcup}_{n \geq N} E_n$ for each $N \in \mathbb{N}$, then $\lim_n n_k = +\infty$. For this reason, when we are saying below that $\{n_k\}$ is a countable subnet of $\{n\}$, we mean by saying so that $\{n_k\}$ is a sequence of natural numbers with $\lim_n n_k = +\infty$. For our purposes, saying that $\{n_k\}$ is a subsequence of $\{n\}$, is not satisfactory, since many cases of sequences $\{u_k\}$ in $\dot{\bigcup}_n E_n$, converging (eventually) to a point in E, would be excluded. See also Lemma A.1 (v) on subnets.

Remark 5.11. *It is a well-known topological fact that a net converges if and only if all of its subnets converge. Therefore it follows that $\tau\text{-}\lim_k u_k = u$, whenever $u_k := v_{n_k}$, $\{n_k\}$ is a countable subnet of $\{n\}$ and $\tau\text{-}\lim_n v_n = u$, $v_n \in E_n$ (as in* (A2) *below) for any $u \in E$.*

Note that condition (A1) in the definition below is sharpened. The deeper reason for this can be seen in Lemma 5.75 at the end of this chapter.

Definition 5.12 (Asymptotic Relation, Sequence Version)**.** *We call a topology τ on $\mathfrak{E}$ a* linear strong asymptotic relation *between $\{E_n\}_{n \in \mathbb{N}}$ and E if τ satisfies the following conditions:*

(A1) *E_n, $n \in \mathbb{N}$, E are closed in $(\mathfrak{E}, \tau)$, and the relative topologies of E_n, $n \in \mathbb{N}$, E in $(\mathfrak{E}, \tau)$ coincide with the original strong topologies. Also, E_n, $n \in \mathbb{N}$ are open in $(\mathfrak{E}, \tau)$.*

(A2) *For any $u \in E$ there exists a sequence $\{u_n\}_{n \in \mathbb{N}}$ such that $u_n \in E_n$, $n \in \mathbb{N}$ and $\tau\text{-}\lim_{n \to \infty} u_n = u$.*

(A3) *For any countable subnet $\{n_k\}_{k \in \mathbb{N}}$ of $\{n\}_{n \in \mathbb{N}}$, for any sequence $\{u_k\}_{k \in \mathbb{N}}$, with $u_k \in E_{n_k}$, $k \in \mathbb{N}$ and any $u \in E$ the following statement holds:*
If $\tau\text{-}\lim_{k \to \infty} u_k = u$, then

$$\mathbb{R}\text{-}\lim_{k \to \infty} \|u_k\|_{E_{n_k}} = \|u\|_E .$$

(A4) *For any countable subnet $\{n_k\}_{k \in \mathbb{N}}$ of $\{n\}_{n \in \mathbb{N}}$, for any two sequences $\{u_k\}_{k \in \mathbb{N}}$, $\{v_k\}_{k \in \mathbb{N}}$ with $u_k, v_k \in E_{n_k}$, $k \in \mathbb{N}$ and any $u \in E$ the following statement holds:*
If $\tau\text{-}\lim_{k \to \infty} u_k = u$ and $\mathbb{R}\text{-}\lim_{k \to \infty} \|u_k - v_k\|_{E_{n_k}} = 0$, then $\tau\text{-}\lim_{k \to \infty} v_k = u$.

(AL) *For any countable subnet* $\{n_k\}_{k\in\mathbb{N}}$ *of* $\{n\}_{n\in\mathbb{N}}$*, for any two sequences* $\{u_k\}_{k\in\mathbb{N}}$*,* $\{v_k\}_{k\in\mathbb{N}}$ *with* $u_k, v_k \in E_{n_k}$*,* $k \in \mathbb{N}$ *and any two* $u, v \in E$ *and any* $\alpha, \beta \in \mathbb{R}$ *the following statement holds:*
If $\tau\text{-}\lim_{k\to\infty} u_k = u$ *and* $\tau\text{-}\lim_{k\to\infty} v_k = v$*, then* $\tau\text{-}\lim_{k\to\infty} \left[\alpha u_k + \beta v_k\right] = \alpha u + \beta v$.

Define also:

(A3') *For any countable subnet* $\{n_k\}_{k\in\mathbb{N}}$ *of* $\{n\}_{n\in\mathbb{N}}$*, for any two sequences* $\{u_k\}_{k\in\mathbb{N}}$*,* $\{v_k\}_{k\in\mathbb{N}}$*, with* $u_k, v_k \in E_{n_k}$*,* $k \in \mathbb{N}$ *and any two* $u, v \in E$ *the following statement holds:*
If $\tau\text{-}\lim_{k_{\mathscr{M}}} u_k = u$ *and* $\tau\text{-}\lim_{k\to\infty} v_k = v$*, then*

$$\mathbb{R}\text{-}\lim_{k\to\infty} \|u_k - v_k\|_{E_{n_k}} = \|u - v\|_E\,,$$

As well as:

(AL') *For any countable subnet* $\{n_k\}_{k\in\mathbb{N}}$ *of* $\{n\}_{n\in\mathbb{N}}$*, for any two sequences* $\{u_k\}_{k\in\mathbb{N}}$*,* $\{v_k\}_{k\in\mathbb{N}}$ *with* $u_k, v_k \in E_{n_k}$*,* $k \in \mathbb{N}$ *and any two* $u, v \in E$ *and any two sequences* $\{\alpha_k\}_{k\in\mathbb{N}}, \{\beta_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ *and any two* $\alpha, \beta \in \mathbb{R}$ *such that* $\mathbb{R}\text{-}\lim_{k\to\infty} \alpha_k = \alpha$*,* $\mathbb{R}\text{-}\lim_{k\to\infty} \beta_k = \beta$ *the following statement holds:*
If $\tau\text{-}\lim_{k\to\infty} u_k = u$ *and* $\tau\text{-}\lim_{k\to\infty} v_k = v$*, then* $\tau\text{-}\lim_{k\to\infty} \left[\alpha_k u_k + \beta_k v_k\right] = \alpha u + \beta v$.

(A3') and (AL') follow from (A3), (A4) and (AL).

Note the special rôle of E and the similarities between τ on $\mathfrak{E}$ and the topology of the Alexandroff one-point compactification on $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. $\overline{\mathbb{N}}$ is a compact metric space with the metric

$$d_{\overline{\mathbb{N}}}(n, m) := \left|\frac{1}{n} - \frac{1}{m}\right|$$

(setting $\frac{1}{\infty} := 0$) and hence isometrically isomorphic to the compact set

$$\overline{\mathbb{N}}^{-1} := \dot{\bigcup_{n\in\mathbb{N}}} \left\{\frac{1}{n}\right\} \dot{\cup} \{0\}$$

with the usual metric. This is not a coincidence, see Section 5.14 at the end of this chapter.

5.2.3 Topological bases and regularity

We continue with further topological properties of linear asymptotic relations for Banach spaces. Assume always $\mathscr{N} = \mathbb{N}$.

Lemma 5.13. *Let* $\mathfrak{E}$*,* $\mathscr{N} = \mathbb{N}$ *be as above and suppose that* $(\mathfrak{E}, \tau)$ *is a linear strong asymptotic relation. Let* $\{n_k\}_{k\in\mathbb{N}}$ *be a countable subnet of* $\{n\}_{n\in\mathbb{N}}$*. Let* $\{u_k\}_{k\in\mathbb{N}}$ *with* $u_k \in E_{n_k}$*,* $k \in \mathbb{N}$*. Then* $\tau\text{-}\lim_k u_k = 0 \in E$ *if and only if* $\mathbb{R}\text{-}\lim_k \|u_k\|_{E_{n_k}} = 0$.

Proof. The "only if"-part is a consequence of (A3). Note that by (A2) and (AL) $E_{n_k} \ni 0 \to 0 \in E$ in τ-topology. Then by (A4) we obtain the "if"-part. $\square$

Compare the following lemma with Lemma 5.75 (iii) below.

Lemma 5.14. *Suppose that $\mathscr{N} = \mathbb{N}$. Suppose either that* (A3') *holds for sequences replaced by nets or that τ is first-countable. Then $(\mathfrak{E}, \tau)$ is Hausdorff.*

Proof. Let $\{u_i\}_{i\in I}$ be a net in $\mathfrak{E}$ with two limits $u, v \in \mathfrak{E}$. Suppose that $u \in E_n$ for some $n \in \mathbb{N}$. Since E_n is open by (A1), $\{u_i\}$ is eventually in E_n. But since E_n is closed by (A1), all limits of $\{u_i\}$ must lie in E_n. Hence $v \in E_n$. But then $\lim_{i\in I} u_n \ni u, v$ in the original topology of E_n which is of course Hausdorff; hence $u = v$. Suppose now that $u \in E$. If $v \in E_n$ for some $n \in \mathbb{N}$, we get a contradiction by the argument above. Hence $v \in E$. For all $N \in \mathbb{N}$, $\dot{\bigcup}_{n\geq N} E_n \dot{\cup} E$ is an open neighborhood of u. Hence there is a net of natural numbers $\{n_i\}_{i\in I}$ such that $u_i \in E_{n_i}$ for $i \in I$ and $\lim_{i\in I} n_i = +\infty$. Then by (A3')

$$0 = \lim_{i\in I} 0 = \lim_{i\in I} \|u_i - u_i\|_{E_{n_i}} = \|u - v\|_E .$$

Hence $u = v$. If, alternatively, τ is assumed to be first-countable, by [Eng89, Proposition 1.6.17], the "sequence-version" of condition (A3') is sufficient for the above proof. $\square$

For the following lemma, recall that separable metric spaces are second countable, cf. [Kel75, Ch. 4, Theorem 11, p. 120].

Lemma 5.15. *Let $\mathscr{N} = \mathbb{N}$, let $\mathfrak{E} = \dot{\bigcup}_{n\in\mathbb{N}} E_n \dot{\cup} E$ have a linear asymptotic relation τ and suppose that each E_n, $n \in \mathbb{N}$, E is separable. Then $\mathfrak{E}$ is second countable (as a topological space) with a base of open sets given by*

$$\bigcup_{n\in\mathbb{N}} \mathscr{O}_n \cup \bigcup_{i,j,k\in\mathbb{N}} \{O_{ijk}\}$$

where $\mathscr{O}_n$ is a countable base of open sets for E_n and O_{ijk} is given by

$$O_{ijk} := B_E(u^i, 1/k) \cup \bigcup_{j'\geq j} B_{E_{j'}}(u^i_{j'}, 1/k)$$

where $\{u^1, u^2, \ldots\} \subset E$ is a countable dense set and $u^i_j \in E_j$ with $\tau\text{-}\lim_{j\to\infty} u^i_j = u^i$.

Proof. The proof goes exactly as in [KS03, Lemma 2.13]. We repeat it for convenience. We would like to apply [Kel75, p. 47, Theorem 11].

O_{ijk} is a open neighborhood of u^i in $\mathfrak{E}$, since every net converging to a point in O_{ijk} is eventually in O_{ijk} by (A1) and (A3') (using that $\tau\text{-}\lim_j u^i_j = u^i$ and hence every subnet of it). Assume that $u \in O_{i_1j_1k_1} \cap O_{i_2j_2k_2} \cap E$. By [Kel75, p. 47, Theorem 11] it suffices to prove that there are $i_3, j_3, k_3 \in \mathbb{N}$ such that

$$u \in O_{i_3j_3k_3} \subset O_{i_1j_1k_1} \cap O_{i_2j_2k_2}. \tag{5.6}$$

Since $\left\|u-u^{i_p}\right\|_E<1/k_p$ for $p=1,2$, we find a number $k_3\in\mathbb{N}$ such that

$$\left\|u-u^{i_p}\right\|_E+2/k_3<1/k_p,\quad p=1,2, \tag{5.7}$$

and then find a number $i_3\in\mathbb{N}$ such that

$$\left\|u-u^{i_3}\right\|_E<1/k_3. \tag{5.8}$$

By (5.7), the triangle inequality and (5.8)

$$\begin{aligned}\left\|u^{i_3}-u^{i_p}\right\|_E+2/k_3&<\left\|u^{i_3}-u^{i_p}\right\|_E+1/k_p-\left\|u-u^{i_p}\right\|_E\\&\le 1/k_p+\left\|u-u^{i_3}\right\|_E<1/k_p+1/k_3,\quad p=1,2.\end{aligned} \tag{5.9}$$

By (A3') there exists $j_3\in\mathbb{N}$ such that for any $j\ge j_3$ and $p=1,2$,

$$\left\|u_j^{i_3}-u_j^{i_p}\right\|_{E_j}+1/k_3<1/k_p. \tag{5.10}$$

It follows from the triangle inequality that

$$B(x,r-\|x-y\|)\subset B(y,r)\quad\text{if}\quad x\in B(y,r). \tag{5.11}$$

Therefore the inequalities (5.8), (5.9), (5.10) together imply (5.6).

Suppose now that $u\in O^1\cap O^2$, where $O^1\in\mathscr{O}_{n_1}$, $O^2\in\mathscr{O}_{n_2}$, $n_1,n_2\in\mathbb{N}$. Then we must have $n_1=n_2$. By the property of $\mathscr{O}_{n_1}$ being a basis, we find $O^3\in\mathscr{O}_{n_1}$ with $u\in O^3\subset O^1\cap O^2$.

Suppose that $u\in O^1\cap O_{i,j,k}$, for some $i,j,k\in\mathbb{N}$ and $O^1\in\mathscr{O}_{n_0}$, $n_0\in\mathbb{N}$. We must have $n_0\ge j$. Also $u\in B_{E_{n_0}}(u^i_{n_0},1/k)=:B$. But since $O^1\cap B$ is an open neighborhood of u in E_{n_0}, we find an open set $O^2\in\mathscr{O}_{n_0}$ with $u\in O^2\subset O^1\cap B=O^1\cap O_{i,j,k}$.

Suppose that $u\in O_{i_1,j_1,k_1}\cap O_{i_2,j_2,k_2}$ for some $i_1,j_1,k_1,i_2,j_2,k_2\in\mathbb{N}$ and that $u\in E_{n_0}$ for some $n_0\in\mathbb{N}$. We must have $n_0\ge j_1$ and $n_0\ge j_2$. Hence $u\in B_{E_{n_0}}(u^{i_1}_{n_0},1/k_1)\cap B_{E_{n_0}}(u^{i_2}_{n_0},1/k_2)=:B_1\cap B_2$. But $B_1\cap B_2$ is an open neighborhood of u in E_{n_0} and hence by the property of $\mathscr{O}_{n_0}$ being a basis we find an open set $O\in\mathscr{O}_{n_0}$ with $u\in O\subset B_1\cap B_2\subset O_{i_1,j_1,k_1}\cap O_{i_2,j_2,k_2}$.

The proof is complete by an application of [Kel75, p. 47, Theorem 11].

□

The preceding result shows that in the separable case we know the topology of a strong asymptotic relation whenever we know what sequences converge. This is not clear a priori. Therefore we will later assume from time to time that a strong asymptotic relation is Fréchet (see Appendix A).

Lemma 5.16. *In the situation of Lemma 5.15 $(\mathfrak{E},\tau)$ is a regular topological space and hence separably metrizable.*

Proof. Each Hausdorff second countable regular topological space is separably metrizable by [Kel75, p. 125, Theorem 17]. We are left to show that $\mathfrak{E}$ is regular. It is enough to show that for each $x \in \mathfrak{E}$ and each element O in the base described in Lemma 5.15 such that $x \in O$ there exists an closed neighborhood A of x such that $A \subset O$ (cf. [Eng89, Proposition 1.5.5]).

Suppose that $x \in E_{n_0}$ for some $n_0 \in \mathbb{N}$. W.l.o.g. $x \in O \in \mathscr{O}_{n_0}$. E_{n_0} is regular w.r.t. its original topology hence there is a closed neighborhood $A \subset O$ containing x. But A is also a τ-neighborhood and τ-closed by (A1).

Suppose that $x \in E$ and $x \in O_{i_1 j_1 k_1}$ for some $i_1, j_1, k_1 \in \mathbb{N}$. Select k_2 such that $2/k_2 < 1/k_1 - \|x - u_{i_1}\|_E$ and u_{i_2} such that $\|x - u_{i_2}\|_E < 1/k_2$. Then $1/k_2 < 1/k_1 - \|u_{i_1} - u_{i_2}\|_E$ and hence by (5.11) $x \in \overline{B}_E(u_{i_2}, 1/k_2) \subset B_E(u_{i_1}, 1/k_1)$. By (A3') there is j_2 such that

$$O_{i_2,j_2,k_2} \subset A := \overline{B}_E(u_{i_2}, 1/k_2) \cup \bigcup_{j' \geq j_2} \overline{B}_{E_{j'}}(u_{j'}^{i_2}, 1/k_2) \subset O_{i_1,j_1,k_1}.$$

Hence A is a τ-neighborhood. It remains to prove that it is τ-closed. But by (A1), (A3') there is no sequence (net) in A which converges to a point in $\mathfrak{E} \setminus A$. $\square$

5.3 Weak and weak* asymptotic relation

Definition 5.17. *A pair of real linear spaces (X, Y) is called* dual pair *if there exists a bilinear map ${}_X\langle \cdot, \cdot \rangle_Y : X \times Y \to \mathbb{R}$ which* separates the points, *i.e.:*

$$\text{for every } x \in X \setminus \{0\} \text{ there is a } y \in Y \text{ with } {}_X\langle x, y \rangle_Y \neq 0,$$
$$\text{for every } y \in Y \setminus \{0\} \text{ there is a } x \in X \text{ with } {}_X\langle x, y \rangle_Y \neq 0.$$

If (X, Y) is a dual pair, we define the locally convex Hausdorff topology $\sigma(X, Y)$ on X via the family of semi-norms $\{p_y(x) := |{}_X\langle x, y \rangle_Y| \mid y \in Y\}$ (we also define $\sigma(Y, X)$ on Y in the obvious way).

If $\mathfrak{E}^{(i)} := \dot{\bigcup}_{\nu \in \mathscr{N}} E_\nu^{(i)} \dot{\cup} E^{(i)}$ for a directed set $\mathscr{N}$ and real linear spaces $E_\nu^{(i)}$, $\nu \in \mathscr{N}$, $E^{(i)}$, $i = 1, 2$, we say that $(\mathfrak{E}^{(1)}, \mathfrak{E}^{(2)})$ is a dual pair *if $(E_\nu^{(1)}, E_\nu^{(2)})$ is a dual pair for each $\nu \in \mathscr{N}$ and $(E^{(1)}, E^{(2)})$ is a dual pair.*

If E is a Banach space and E^* its dual with pairing ${}_{E^*}\langle \cdot, \cdot \rangle_E$, $\sigma(E, E^*)$ is the *weak* topology on E and $\sigma(E^*, E)$ is the *weak** topology on E^*. If E is reflexive, $\sigma(E^*, E)$ coincides with $\sigma(E^*, E^{**})$. When speaking of dual pairs of Banach spaces we shall always use the standard dualization which is continuous w.r.t to the strong and weak topologies.

Definition 5.18. *Let $(\mathfrak{E}, \mathfrak{F})$ be a dual pair consisting of Banach spaces such that $\tau(\mathfrak{E})$ and $\tau(\mathfrak{F})$ are strong linear asymptotic relations on $\mathfrak{E}$ and $\mathfrak{F}$ respectively. We call a topology $\sigma = \sigma(\mathfrak{E}, \mathfrak{F})$ on $\mathfrak{E}$ a* linear σ-asymptotic relation *with respect to $\mathfrak{F}$ between $\{E_\nu\}_{\nu \in \mathscr{N}}$ and E if σ satisfies the following conditions:*

(W1) *The relative topologies of E_ν, $\nu \in \mathscr{N}$, E in $(\mathfrak{E}, \sigma)$ coincide with the original $\sigma(E_\nu, F_\nu)$ (resp. $\sigma(E, F)$) topologies.*

(W2) *For any* $u \in E$ *there exists a net* $\{u_\nu\}_{\nu\in\mathscr{N}}$ *such that* $u_\nu \in E_\nu$, $\nu \in \mathscr{N}$ *and* $\sigma\text{-}\lim_{\nu\in\mathscr{N}} u_\nu = u$.

(W3) $\sigma(\mathfrak{E},\mathfrak{F})$ *is weaker than* $\tau(\mathfrak{E})$ *(or equal to).*

(W4) *For any subnet* $\{\nu_\mu\}_{\mu\in\mathscr{M}}$ *of* $\{\nu\}_{\nu\in\mathscr{N}}$*, for any net* $\{u_\mu\}_{\mu\in\mathscr{M}} \subset \mathfrak{E}$*, with* $u_\mu \in E_{\nu_\mu}$, $\mu \in \mathscr{M}$ *and any* $u \in E$ *the following statement holds:*
$\sigma(\mathfrak{E},\mathfrak{F})\text{-}\lim_{\mu\in\mathscr{M}} u_\mu = u$ *if and only if* $\sup_{\mu\in\mathscr{M}} \|u_\mu\|_{E_{\nu_\mu}} < +\infty$ *and*

$$\mathbb{R}\text{-}\lim_{\mu\in\mathscr{M}} {}_{E_{\nu_\mu}}\langle u_\mu, v_\mu\rangle_{F_{\nu_\mu}} = {}_E\langle u, v\rangle_F$$

for every $v \in F$ *and every net* $\{v_\mu\}_{\mu\in\mathscr{M}} \subset \mathfrak{F}$ *with* $v_\mu \in F_{\nu_\mu}$, $\mu \in \mathscr{M}$ *and* $\tau(\mathfrak{F})\text{-}\lim_{\mu\in\mathscr{M}} v_\mu = v$.

(WL) *For any subnet* $\{\nu_\mu\}_{\mu\in\mathscr{M}}$ *of* $\{\nu\}_{\nu\in\mathscr{N}}$*, for any two nets* $\{u_\mu\}_{\mu\in\mathscr{M}}$, $\{v_\mu\}_{\mu\in\mathscr{M}}$ *with* $u_\mu, v_\mu \in E_{\nu_\mu}$, $\mu \in \mathscr{M}$ *for any two* $u, v \in E$ *and for any* $\alpha, \beta \in \mathbb{R}$ *the following statement holds:*
If $\sigma\text{-}\lim_{\mu\in\mathscr{M}} u_\mu = u$ *and* $\sigma\text{-}\lim_{\mu\in\mathscr{M}} v_\mu = v$*, then* $\sigma\text{-}\lim_{\mu\in\mathscr{M}} \big[\alpha u_\mu + \beta v_\mu\big] = \alpha u + \beta v$.

If in the above definition $\mathfrak{F} = \mathfrak{E}^*$*, then we call a linear* σ*-asymptotic relation* $\sigma(\mathfrak{E},\mathfrak{E}^*)$ *a* linear weak asymptotic relation *on* $\mathfrak{E}$.

If in the above definition $\mathfrak{E} = \mathfrak{F}^*$*, then we call a linear* σ*-asymptotic relation* $\sigma(\mathfrak{F}^*,\mathfrak{F})$ *a* linear weak* asymptotic relation *on* $\mathfrak{F}^*$.

As in the case of strong asymptotic relations, a $\sigma(E,F)$-open subset of E is not a $\sigma(\mathfrak{E},\mathfrak{F})$-neighborhood of its points. This would violate (W2).

Remark 5.19. *We can replace* (WL) *by*

(WL') *For any subnet* $\{\nu_\mu\}_{\mu\in\mathscr{M}}$ *of* $\{\nu\}_{\nu\in\mathscr{N}}$*, for any two nets* $\{u_\mu\}_{\mu\in\mathscr{M}}$, $\{v_\mu\}_{\mu\in\mathscr{M}}$ *with* $u_\mu, v_\mu \in E_{\nu_\mu}$, $\mu \in \mathscr{M}$ *for any two* $u, v \in E$ *and any two nets* $\{\alpha_\mu\}_{\mu\in\mathscr{M}}, \{\beta_\mu\}_{\mu\in\mathscr{M}} \subset \mathbb{R}$ *and any two* $\alpha, \beta \in \mathbb{R}$ *such that* $\mathbb{R}\text{-}\lim_{\mu\in\mathscr{M}} \alpha_\mu = \alpha$, $\mathbb{R}\text{-}\lim_{\mu\in\mathscr{M}} \beta_\mu = \beta$ *the following statement holds:*
If $\sigma\text{-}\lim_{\mu\in\mathscr{M}} u_\mu = u$ *and* $\sigma\text{-}\lim_{\mu\in\mathscr{M}} v_\mu = v$*, then* $\sigma\text{-}\lim_{\mu\in\mathscr{M}} \big[\alpha_\mu u_\mu + \beta_\mu v_\mu\big] = \alpha u + \beta v$.

This follows from (W4), (WL)*, the linearity and strong continuity of the dualization and the boundedness of norms of the nets involved (in particular,* (A3)*).*

Remark 5.20. *Let* $(\mathfrak{E},\mathfrak{F})$ *be a dual pair consisting of Banach spaces such that* $\tau(\mathfrak{E})$ *and* $\tau(\mathfrak{F})$ *are strong linear asymptotic relations on* $\mathfrak{E}$ *and* $\mathfrak{F}$ *respectively. If there is a linear* σ*-asymptotic relation* $\sigma(\mathfrak{E},\mathfrak{F})$ *on* $\mathfrak{E}$ *then for any subnet* $\{\nu_\mu\}_{\mu\in\mathscr{M}}$ *of* $\{\nu\}_{\nu\in\mathscr{N}}$*, for any two nets* $\{u_\mu\}_{\mu\in\mathscr{M}} \subset \mathfrak{E}$, $\{v_\mu\}_{\mu\in\mathscr{M}} \subset \mathfrak{F}$ *and any two points* $u \in E$, $v \in F$ *such that* $u_\mu \in E_{\nu_\mu}$, $v_\mu \in F_{\nu_\mu}$, $\mu \in \mathscr{M}$ *and* $\tau(\mathfrak{E})\text{-}\lim_{\mu\in\mathscr{M}} u_\mu = u$, $\tau(\mathfrak{F})\text{-}\lim_{\mu\in\mathscr{M}} v_\mu = v$ *we have that*

$$\mathbb{R}\text{-}\lim_{\mu\in\mathscr{M}} {}_{E_{\nu_\mu}}\langle u_\mu, v_\mu\rangle_{F_{\nu_\mu}} = {}_E\langle u, v\rangle_F\,. \tag{5.12}$$

This is a consequence of (W3) *and* (W4)*. Similarly,* (5.12) *holds if there is a linear σ-asymptotic relation $\sigma(\mathfrak{F}, \mathfrak{E})$ on $\mathfrak{F}$. We shall see later (cf. Theorem 5.45) that* (5.12) *is sufficient as well for a reasonable construction of linear σ-asymptotic relations on $\mathfrak{E}$ or $\mathfrak{F}$.*

Definition 5.21. *If $\mathscr{N} = \mathbb{N}$ we sharpen* (W1) *as follows:*

(W1) *E_n, $n \in \mathbb{N}$, E are closed in $(\mathfrak{E}, \sigma)$, and the relative topologies of E_n, $n \in \mathbb{N}$, E in $(\mathfrak{E}, \sigma)$ coincide with the original $\sigma(E_n, F_n)$ (resp. $\sigma(E, F)$) topologies. Also, E_n, $n \in \mathbb{N}$ are open in $(\mathfrak{E}, \sigma)$.*

5.3.1 A compactness result

The next result could be considered an asymptotic version of the Banach-Alaoglu Theorem.

Theorem 5.22. *Let $\mathscr{N} = \mathbb{N}$. Suppose that $\mathfrak{E}$ has a linear strong asymptotic relation τ which is Fréchet and Hausdorff. Suppose that $\mathfrak{E}^*$ has a linear weak* asymptotic relation $\sigma(\mathfrak{E}^*, \mathfrak{E})$ corresponding to τ which is Fréchet. Then*

$$\mathfrak{B}_1^* := \dot{\bigcup_{n \in \mathbb{N}}} \overline{B}_{E_n^*}(0,1) \dot{\cup} \overline{B}_{E^*}(0,1)$$

equipped with the weak topology induced by $(\mathfrak{E}^*, \sigma(\mathfrak{E}^*, \mathfrak{E}))$ is compact. (For the terminology see Appendix A).*

Proof. The set $c(\mathfrak{E}) := \big\{\{u_k\}_{k\in\mathbb{N}} \subset \mathfrak{E} \;\big|\; \{u_k\}_{k\in\mathbb{N}} \text{ converges in } \tau\big\}$ is not empty and contains for any $u \in \mathfrak{E}$ a convergent sequence converging to u (since constant sequences converge in a Fréchet space). Also by (A2) for any $u \in E$ there is a sequence $\{u_k\}$, $u_k \in E_k$ for $k \in \mathbb{N}$ such that $u_k \to u$ in τ.

We decompose $c(\mathfrak{E})$ into disjoint non-empty subsets by saying

$$\left.\begin{array}{l} \{u_k\} \in (1)_{n_0} \\ \{u_k\} \in (2) \\ \{u_k\} \in (3) \end{array}\right\} \quad \text{if} \quad \left\{\begin{array}{l} \lim\limits_k u_k \in E_{n_0} \text{ for some } n_0 \in \mathbb{N}, \\ \lim\limits_k u_k \in E \text{ and } \{u_k\} \text{ frequently (or eventually) in } E, \\ \lim\limits_k u_k \in E \text{ and } \{u_k\} \text{ eventually in } \mathfrak{E} \setminus E. \end{array}\right.$$

Note that (2) includes the case that $\{u_k\}$ is eventually in E. Therefore, by (A1),

$$c(\mathfrak{E}) = \dot{\bigcup_{n\in\mathbb{N}}} (1)_n \dot{\cup} (2) \dot{\cup} (3).$$

Let $\mathfrak{B}_1^*$ be as in the assertion. Let $\Delta \notin \mathbb{R}$ be an isolated point. (A1) and (A3) imply that the norms of each $\{u_k\} \in c(\mathfrak{E})$ are convergent. For $a, b \in \mathbb{R}$ we endow $[a,b] \cup \{\Delta\}$ with the topology induced by $\mathbb{R} \cup \{\Delta\}$ (i.e., the Alexandroff one point compactification of $\mathbb{R}$). Clearly $[a,b] \cup \{\Delta\}$ is compact.

Define

$$\mathfrak{X} := \bigtimes_{\{u_k\} \in c(\mathfrak{E})} \left(\left[-\lim_k \|u_k\|, \lim_k \|u_k\| \right] \cup \{\Delta\} \right)$$

and equip $\mathfrak{X}$ with the product topology. By Tychonoff's Theorem (cf. [Kel75, p. 143, Theorem 13]) $\mathfrak{X}$ is compact. A net $\{x_i\}_{i \in I}$ in $\mathfrak{X}$ converges to some x in $\mathfrak{X}$ if and only if

$$(\mathbb{R} \cup \{\Delta\})\text{-}\lim_{i \in I} x_i(\{u_k\}) = x(\{u_k\}) \tag{5.13}$$

for all $\{u_k\} \subset \mathfrak{E}$, $\{u_k\}$ τ-convergent. This is known as pointwise or coordinatewise convergence. The notation might be irritating, since we write the "coordinate" $\{u_k\}$ in the parentheses.

Define a map $\mathscr{I} : \mathfrak{B}_1^* \to \mathfrak{X}$ as follows

$$\big(\mathscr{I}(f)\big)(\{u_k\}) :=$$
$$\begin{cases} {}_{E_{n_0}^*}\left\langle f, \lim_k u_k \right\rangle_{E_{n_0}} & \text{if} \quad f \in E_{n_0}^*,\ \{u_k\} \in (1)_{n_0}, \\ \Delta & \text{if} \quad f \in E_{n_0}^*,\ \{u_k\} \in (1)_{n_1},\ n_0 \neq n_1, \\ \Delta & \text{if} \quad f \in E_{n_0}^*,\ \{u_k\} \in (2), \\ {}_{E_{n_0}^*}\left\langle f, u_{\inf\{k \mid u_k \in E_{n_0}\}} \right\rangle_{E_{n_0}} & \text{if} \quad f \in E_{n_0}^*,\ \{u_k\} \in (3),\ u_{k_0} \in E_{n_0} \text{ for some } k_0, \\ \Delta & \text{if} \quad f \in E_{n_0}^*,\ \{u_k\} \in (3),\ u_k \notin E_{n_0} \text{ for all } k, \\ \Delta & \text{if} \quad f \in E^*,\ \{u_k\} \in (1)_n \text{ for any } n, \\ \Delta & \text{if} \quad f \in E^*,\ \{u_k\} \in (2), \\ {}_{E^*}\left\langle f, \lim_k u_k \right\rangle_E & \text{if} \quad f \in E^*,\ \{u_k\} \in (3). \end{cases}$$

Claim. $\mathscr{I}$ is injective and a homeomorphism between $\mathfrak{B}_1^*$ and $\mathfrak{K} := \mathscr{I}(\mathfrak{B}_1^*)$. $\mathfrak{K}$ is closed (and hence compact as a closed subset of a compact space).

If we can prove the Claim, we are done, since then $\mathfrak{B}_1^*$ is compact as the continuous image of a compact set.

Injectivity of $\mathscr{I}$.

In what follows let $f, g \in \mathfrak{B}_1^*$, $f \neq g$. Suppose that $f, g \in E_{n_0}^*$ for some $n_0 \in \mathbb{N}$. Then there is $u_0 \in E_{n_0}$ with ${}_{E_{n_0}^*}\langle f, u_0 \rangle_{E_{n_0}} \neq {}_{E_{n_0}^*}\langle g, u_0 \rangle_{E_{n_0}}$. Pick $\{u_k\} \in (1)_{n_0}$ with $\lim_k u_k = u_0$. Hence $\mathscr{I}(f)(\{u_k\}) = {}_{E_{n_0}^*}\langle f, u_0 \rangle_{E_{n_0}} \neq {}_{E_{n_0}^*}\langle g, u_0 \rangle_{E_{n_0}} = \mathscr{I}(g)(\{u_k\})$.
Suppose that $f \in E_{n_0}^*$, $g \in E_{n_1}^*$, for some $n_0, n_1 \in \mathbb{N}$, $n_0 \neq n_1$. Denote $0 \in E_{n_0}$ by $0^{(n_0)}$. Then $\{0^{(n_0)}\}_{k \in \mathbb{N}} \in (1)_{n_0}$ and $\mathscr{I}(f)(\{0^{(n_0)}\}) = 0 \neq \Delta = \mathscr{I}(g)(\{0^{(n_0)}\})$.
Suppose that $f \in E_{n_0}^*$ for some $n_0 \in \mathbb{N}$ and $g \in E^*$. Denote $0 \in E$ by $0^{(\infty)}$. Define $u_k := 0^{(k)}$ for $k < n_0$ and $u_k := 0^{(k+1)}$ for $k \geq n_0$. Then $\{u_k\} \in (3)$ and $\lim_k u_k = 0^{(\infty)}$. Hence $\mathscr{I}(f)(\{u_k\}) = \Delta \neq 0 = \mathscr{I}(g)(\{u_k\})$.
Suppose that $f, g \in E^*$. Then there is $u_0 \in E$ with ${}_{E^*}\langle f, u_0 \rangle_E \neq {}_{E^*}\langle g, u_0 \rangle_E$. Pick by (A2) $\{u_k\} \in (3)$ with $\lim_k u_k = u_0$. Then $\mathscr{I}(f)(\{u_k\}) = {}_{E^*}\langle f, u_0 \rangle_E \neq {}_{E^*}\langle g, u_0 \rangle_E = \mathscr{I}(g)(\{u_k\})$ and the injectivity is proved.

Continuity of $\mathscr{I}$.

By Lemma A.4 (iv) we only need to consider sequences. Let $\{f_l\}_{l\in\mathbb{N}} \subset \mathfrak{B}_1^*$ be a convergent sequence such that $\lim_l f_l \subset \mathfrak{B}_1^*$. Let $f \in \lim_l f_l$. Suppose that $f \in E_{n_0}^*$ for some $n_0 \in \mathbb{N}$. Then $\{f_l\}$ is eventually in $E_{n_0}^*$ and converges pointwisely for points from E_{n_0} by (W1). $\lim_l \mathscr{I}(f_l)(\{u_k\}) = \mathscr{I}(f)(\{u_k\})$ for every $\{u_k\} \in c(\mathfrak{E})$ follows.
Suppose that $f \in E^*$ and $\{f_l\}$ is eventually in E^*. By (W1) $\{f_l\}$ converges pointwisely for points from E. $\lim_l \mathscr{I}(f_l)(\{u_k\}) = \mathscr{I}(f)(\{u_k\})$ for every $\{u_k\} \in c(\mathfrak{E})$ follows.
Suppose that $f \in E^*$ and $\{f_l\}$ is eventually in $\mathfrak{E}^* \setminus E^*$. Only finitely many f_l are in any fixed $E_{n_0}^*$, $n_0 \in \mathbb{N}$ since otherwise there would be infinitely many members of the sequence in the open set $E_{n_0}^*$ (by (W1)) which contradicts the convergence. Hence there is a subsequence $\{n_l\}$ of $\{l\}$ and an index $L \in \mathbb{N}$ for which $f_l \in E_{n_l}^*$ for $l \geq L$. By (W4) $\lim_{l\geq L}\, {}_{E_{n_l}^*}\langle f_l, u_l\rangle_{E_{n_l}} = {}_{E^*}\langle f, u\rangle_E$ for any $\{u_l\} \in (3)$ with $u_l \in E_{n_l}$ for every $l \geq L$. Clearly $\lim_l \mathscr{I}(f_l)(\{u_k\}) = \mathscr{I}(f)(\{u_k\})$ for every $\{u_k\} \in c(\mathfrak{E})$. Suppose that $\{f_l\}$ is frequently in E^* and frequently in $\mathfrak{E} \setminus E^*$. Then the above arguments apply for two distinct subsequences of $\{f_l\}$.

Continuity of $\mathscr{I}^{-1}$.

We shall write "$\rightharpoonup^*$" for weak* convergence of nets in $\mathfrak{E}^*$. Let $\{x_i\}_{i\in I}$ be a convergent net in $\mathfrak{K}$ such that $\lim_{i\in I} x_i \subset \mathfrak{K}$. Let $x \in \lim_{i\in I} x_i$ and $f := \mathscr{I}^{-1}(x)$; also $f_i := \mathscr{I}^{-1}(x_i)$. Suppose that $f \in E_{n_0}^*$ for some $n_0 \in \mathbb{N}$. Let $\{u_k\} \in (1)_{n_0}$. Then f_i is eventually in $E_{n_0}^*$ since $x(\{u_k\}) \in \mathbb{R}$ and we cannot jump from Δ to $[-\lim_k u_k, \lim_k u_k]$. Therefore $\lim_{i\in I}\, {}_{E_{n_0}^*}\langle f_i, \lim_k u_k\rangle_{E_{n_0}} = {}_{E_{n_0}^*}\langle f, \lim_k u_k\rangle_{E_{n_0}}$. But since we can attain any limit in E_{n_0} with a sequence $\{u_k\} \in (1)_{n_0}$, $f_i \underset{i\in I}{\rightharpoonup^*} f$ by (W1).
Suppose that $f \in E^*$. Suppose that $\{f_i\}$ is frequently in E^*. Then there is a cofinal set $J \subset I$ with $f_j \in E^*$ for $j \in J$ and $\lim_{j\in J}\, {}_{E^*}\langle f_j, \lim_k u_k\rangle_E = {}_{E^*}\langle f, \lim_k u_k\rangle_E$ for every $\{u_k\} \in (3)$. Since we can attain any limit in E with a sequence $\{u_k\} \in (3)$ by (A2) and since the restricted weak* topology of E^* coincides with the one induced by $\mathfrak{E}^*$ we have that $f_j \underset{j\in J}{\rightharpoonup^*} f$. There are two cases now. Firstly, it is possible that $\{f_i\}$ is not frequently in $\mathfrak{E}^* \setminus E^*$. Then $\{f_i\}$ is eventually in E^* and $J = I$. Secondly, it is possible that $\{f_i\}$ is also frequently in $\mathfrak{E}^* \setminus E^*$. Then there is a cofinal set $J' \subset I$ with $f_{j'} \in \mathfrak{E}^* \setminus E^*$ for $j' \in J'$. Define $\{n_{j'}\}$ by $f_{j'} \in E_{n_{j'}}^*$. Then $\{n_{j'}\}$ converges to ∞ since otherwise it would have an accumulation point at some $n_0 \in \mathbb{N}$ which would mean that we had to jump from a compact subset of $\mathbb{R}$ to Δ (for sequences in $(1)_{n_0}$). Let $\{u_k\} \in (3)$ such that $u_k \in E_k$ for every k (which exists by (A2)). Also $\lim_{j'\in J'} u_{n_{j'}} = \lim_k u_k$. Then $\lim_{j'\in J'}\, {}_{E_{n_{j'}}^*}\left\langle f_{j'}, u_{n_{j'}}\right\rangle_{E_{n_{j'}}} = {}_{E^*}\langle f, \lim_k u_k\rangle_E$ which yields the weak* convergence $f_{j'} \underset{j'\in J'}{\rightharpoonup^*} f$ by (W4). Since we can choose J and J' such that their union is all I convergence of the whole net is proved.
Suppose that $\{f_i\}$ is eventually in $\mathfrak{E}^* \setminus E^*$. Then J' above is all I and the above argument applies to the whole net. The continuity is proved.

Closedness of $\mathfrak{K}$.

By Lemma A.4 (ii) and the above proof that $\mathscr{I}$ is a homeomorphism, $\mathfrak{K}$ is a Fréchet space. Therefore it is enough to prove that for any sequence $\{x_l\}_{l\in\mathbb{N}} \subset \mathfrak{K}$, which is

convergent, all limit points are contained in $\mathfrak{K}$.
Let $\{x_l\}$ be such a sequence and $x \in \lim_l x_l \subset \mathfrak{X}$. Suppose that $x(\{u_k\}) = \Delta$ for all $\{u_k\} \in c(\mathfrak{E})$. Then there exists an index $L \in \mathbb{N}$ with $x_l(\{u_k\}) = \Delta$ for $l \geq L$ and all $\{u_k\} \in c(\mathfrak{E})$ which is a contradiction to $x_l \in \mathfrak{K}$ for every l. Hence there is a sequence $\{u_k\} \in c(\mathfrak{E})$ with $x(\{u_k\}) \in \mathbb{R}$ and an index $L \in \mathbb{N}$ such that

$$x_l(\{u_k\}) \in \mathbb{R} \text{ for } l \geq L \text{ and } \mathbb{R}\text{-}\lim_{l \geq L} x_l(\{u_k\}) = x(\{u_k\}). \tag{5.14}$$

Suppose that $\{u_k\} \in (1)_{n_0}$ for some $n_0 \in \mathbb{N}$. It follows that $\mathscr{I}^{-1}(x_l) \in E^*_{n_0}$, $l \geq L$. Furthermore, $\mathbb{R}\text{-}\lim_{l \geq L} x_l(\{v_k\}) = x(\{v_k\}) \in \mathbb{R}$ for any $\{v_k\} \in (1)_{n_0}$. We define a functional f^x on E_{n_0} by

$$f^x(v) := \lim_{l \geq L} x_l(\{v_k\}), \quad \text{for some } \{v_k\} \in (1)_{n_0},\ \lim_k v_k = v,$$

which does not depend on the choice of $\{v_k\}$ by the definition of $\mathscr{I}$. Let $\alpha, \beta \in \mathbb{R}$ and $v, w \in E_{n_0}$ and $\{v_k\}, \{w_k\} \in (1)_{n_0}$ with $\lim_k v_k = v$, $\lim_k w_k = w$ and v_k is in the same space as w_k for each k. We prove linearity by

$$\begin{aligned}
&f^x(\alpha v + \beta w) \\
=&\lim_{l \geq L} x_l(\{\alpha v_k + \beta w_k\}) \\
=&\lim_{l \geq L} {}_{E^*_{n_0}}\left\langle \mathscr{I}^{-1}(x_l), \alpha v + \beta w \right\rangle_{E_{n_0}} \\
=&\lim_{l \geq L} \left[\alpha\, {}_{E^*_{n_0}}\left\langle \mathscr{I}^{-1}(x_l), v \right\rangle_{E_{n_0}} + \beta\, {}_{E^*_{n_0}}\left\langle \mathscr{I}^{-1}(x_l), w \right\rangle_{E_{n_0}} \right] \\
=&\alpha \lim_{l \geq L} {}_{E^*_{n_0}}\left\langle \mathscr{I}^{-1}(x_l), v \right\rangle_{E_{n_0}} + \beta \lim_{l \geq L} {}_{E^*_{n_0}}\left\langle \mathscr{I}^{-1}(x_l), w \right\rangle_{E_{n_0}} \\
=&\alpha \lim_{l \geq L} x_l(\{v_k\}) + \beta \lim_{l \geq L} x_l(\{w_k\}) \\
=&\alpha f^x(v) + \beta f^x(v).
\end{aligned}$$

Moreover, $|x_l(\{v_k\})| \leq \lim_k \|v_k\|_{E_{n_0}} = \|v\|_{E_{n_0}}$ for all $l \geq L$. Hence $|f^x(v)| \leq \|v\|_{E_{n_0}}$. Therefore $f^x \in \mathfrak{B}^*_1 \cap E^*_{n_0}$. It is left to prove that $\mathscr{I}(f^x) = x$. $(\mathscr{I} f^x)(\{v_k\}) = \lim_{l \geq L} x_l(\{v_k\}) = x(\{v_k\})$ if $\{v_k\} \in (1)_{n_0}$. If $\{v_k\} \in (1)_{n_1}$, $n_1 \neq n_0$ or $\{v_k\} \in (2)$ or $\{v_k\} \in (3)$ such that $v_k \notin E_{n_0}$ for all k then $(\mathscr{I} f^x)(\{v_k\}) = \Delta = \lim_{l \geq L} x_l(\{v_k\}) = x(\{v_k\})$ since $x_l(\{v_k\}) = \Delta$ for each $l \geq L$. Suppose at last that $\{v_k\} \in (3)$ such that some $v_{k_0} \in E_{n_0}$ and denote the smallest such index also by k_0. Let $\{v_k^{(k_0)}\} \in (1)_{n_0}$ with $\lim_k v_k^{(k_0)} = v_{k_0}$. Then $(\mathscr{I} f^x)(\{v_k\}) = {}_{E^*_{n_0}}\langle f^x, v_{k_0} \rangle_{E_{n_0}} = \lim_{l \geq L} x_l(\{v_k^{(k_0)}\}) = \lim_{l \geq L} x_l(\{v_k\}) = x(\{v_k\})$.

Suppose now that (5.14) holds for some $\{u_k\} \in (3)$. We can assume that (5.14) does not hold for any $\{u_k\} \in (1)_n$ and any $n \in \mathbb{N}$ since otherwise we could proceed as above. Let $\mathscr{I}^{-1}(x_l) \in E^*_{n_l}$, $n_l \in \mathbb{N} \cup \{\infty\}$. Equation (5.14) holds also for any $\{v_k\} \in (3)$ such that v_k is in the same space as u_k for each k. We define a functional f^x on E by

$$f^x(v) := \lim_{l \geq L} x_l(\{v_k\}), \quad \text{for some } \{v_k\} \in (3),$$
$$\lim_k v_k = v, \ v_k \text{ is in the same space as } u_k \text{ for each } k.$$

f^x does not depend on the choice of $\{v_k\}$ by (AL) nor on the definition of $\mathscr{I}$. Let $\alpha, \beta \in \mathbb{R}$ and $v, w \in E$ and $\{v_k\}, \{w_k\} \in (3)$ with $\lim_k v_k = v$, $\lim_k w_k = w$ and v_k is in the same space as w_k for each k. We prove linearity as follows (supposing for a while that $\mathscr{I}^{-1}(x_l) \notin E^*$ for all $l \geq L$)

$$\begin{aligned}
&f^x(\alpha v + \beta w)\\
=&\lim_{l \geq L} x_l(\{\alpha v_k + \beta w_k\})\\
=&\lim_{l \geq L} {}_{E^*_{n_l}}\left\langle \mathscr{I}^{-1}(x_l), \alpha v_{\inf\{k \mid v_k \in E_{n_l}\}} + \beta w_{\inf\{k \mid w_k \in E_{n_l}\}} \right\rangle_{E_{n_l}}\\
=&\lim_{l \geq L} \left[\alpha \; {}_{E^*_{n_l}}\left\langle \mathscr{I}^{-1}(x_l), v_{\inf\{k \mid v_k \in E_{n_l}\}} \right\rangle_{E_{n_l}} + \beta \; {}_{E^*_{n_l}}\left\langle \mathscr{I}^{-1}(x_l), w_{\inf\{k \mid w_k \in E_{n_l}\}} \right\rangle_{E_{n_l}} \right]\\
=&\alpha \lim_{l \geq L} {}_{E^*_{n_l}}\left\langle \mathscr{I}^{-1}(x_l), v_{\inf\{k \mid v_k \in E_{n_l}\}} \right\rangle_{E_{n_l}} + \beta \lim_{l \geq L} {}_{E^*_{n_l}}\left\langle \mathscr{I}^{-1}(x_l), w_{\inf\{k \mid w_k \in E_{n_l}\}} \right\rangle_{E_{n_l}}\\
=&\alpha \lim_{l \geq L} x_l(\{v_k\}) + \beta \lim_{l \geq L} x_l(\{w_k\})\\
=&\alpha f^x(v) + \beta f^x(v),
\end{aligned}$$

and, similarly, $v_{\inf\{k \mid v_k \in E_{n_l}\}}$ and $w_{\inf\{k \mid w_k \in E_{n_l}\}}$ replaced by v and w for the remaining case $\mathscr{I}^{-1}(x_l) \in E^*$ for some l. For each $l \geq L$ we have $|x_l(\{v_k\})| \leq \lim_k \|v_k\| = \|v\|_E$. Hence $|f^x(v)| \leq \|v\|_E$ and $f^x \in \mathfrak{B}^*_1 \cap E^*$. It is left to prove that $\mathscr{I}(f_x) = x$. If $\{v_k\} \in (1)_n$ for some n, then $\mathscr{I}(f^x)(\{v_k\}) = \Delta = \lim_l x_l(\{v_k\}) = x(\{v_k\})$ by the above assumption that (5.14) does not hold for such $\{v_k\}$. If $\{v_k\} \in (2)$ the assertion is clear. Suppose that $\{v_k\} \in (3)$. Then $\mathscr{I}(f^x)(\{v_k\}) = \lim_l x_l(\{v_k\}) = x(\{v_k\})$.

The proof is complete. □

Corollary 5.23. *In the situation of Theorem 5.22 for a sequence* $\{f_k\} \subset \mathfrak{E}^*$ *with* $f_k \in E^*_{n_k}$, $\{n_k\} \subset \mathbb{N} \cup \{\infty\}$ *and* $\sup_k \|f_k\|_{E^*_{n_k}} < \infty$ *we can find a subsequence* $\{f_{k_l}\}$ *of* $\{f_k\}$ *and a point* $f \in \mathfrak{E}^*$ *such that* $\sigma(\mathfrak{E}^*, \mathfrak{E})$-$\lim_l f_{k_l} = f$.

Proof. Let $\{f_k\}$ be a sequence with $f_k \in E^*_{n_k}$, $\{n_k\} \subset \mathbb{N} \cup \{\infty\}$ and $\sup_k \|f_k\|_{E^*_{n_k}} =: K < \infty$. Set $\widetilde{f}_k := K^{-1} f_k$, $k \in \mathbb{N}$. Then $\{\widetilde{f}_k\} \subset \mathfrak{B}^*_1$. By Theorem 5.22 a subnet of $\{\widetilde{f}_k\}$ converges weak* to some $\widetilde{f} \in \mathfrak{B}^*_1$. But because $\mathfrak{B}^*_1$ is a compact Fréchet-space (as a subset of the Fréchet-space $(\mathfrak{E}^*, \sigma(\mathfrak{E}^*, \mathfrak{E}))$) it is also sequentially compact by Lemma A.4 (v) and hence the subnet is a subsequence which we denote by $\{\widetilde{f}_{k_l}\}$. Now by (WL) or (W1) $f_{k_l} = K \widetilde{f}_{k_l} \underset{l \to \infty}{\rightharpoonup}{}^{*} K \widetilde{f} =: f$. □

5.4 Metric approximation

We shall now be concerned with the question whether there exists a linear asymptotic relation for a net of Banach spaces $\{E_\nu\}$ and a limit space E. We will introduce the notion of a *linear metric approximation* and prove (cf. Theorem 5.38 below) that it guarantees the existence of an asymptotic relation and contains enough information to uniquely characterize its strong topology.

Let $\mathscr{N}$ be a directed set. For $\nu \in \mathscr{N}$ let E_ν be a Banach space. Let E be a Banach space, too. Let $\mathfrak{E} := \dot{\bigcup}_{\nu\in\mathscr{N}} E_\nu \dot{\cup} E$.

Definition 5.24 (Metric Approximation)**.** *We call a net*

$$\{\Phi_\nu : D(\Phi_\nu) \subset E \to E_\nu\}_{\nu\in\mathscr{N}}$$

of maps Φ_ν with domains $D(\Phi_\nu) \subset E$ a linear metric approximation *between $\{E_\nu\}_{\nu\in\mathscr{N}}$ and E if the following properties are fulfilled:*

(B1) *$\{D(\Phi_\nu)\}_{\nu\in\mathscr{N}}$ is monotone non-decreasing in $\nu \in \mathscr{N}$ ordered by the inclusion of sets and $\bigcup_{\nu\in\mathscr{N}} D(\Phi_\nu) =: C$ is dense in E with respect to its strong topology.*

(B2) *For any $u \in C$*

$$\overline{\mathbb{R}}\text{-}\lim_{\nu\in\mathscr{N}} \begin{Bmatrix} \|\Phi_\nu u\|_{E_\nu} & , \text{ if } u \in D(\Phi_\nu) \\ +\infty & , \text{ otherwise} \end{Bmatrix} = \|u\|_E .$$

(BL) *Each $D(\Phi_\nu)$ is a linear space (when non-empty) and each Φ_ν is a linear operator.*

No continuity of the Φ_ν's is assumed. Note also that (B2) holds for any subnet of a metric approximation.

Remark 5.25. *Adopting the notation of the above definition we conclude from the* polarization identity *for Hilbert spaces: if each E_ν, $\nu \in \mathscr{N}$, E are Hilbert spaces,* (B2) *is equivalent to*

$$\overline{\mathbb{R}}\text{-}\lim_{\nu\in\mathscr{N}} \begin{Bmatrix} (\Phi_\nu u, \Phi_\nu v)_{E_\nu} & , \text{ if } u, v \in D(\Phi_\nu) \\ +\infty & , \text{ otherwise} \end{Bmatrix} = (u, v)_E.$$

for every $u, v \in C$.

For the sake of completeness, we recall the original definition of Kuwae and Shioya in [KS08]. Let (E_ν, d_ν), $\nu \in \mathscr{N}$, (E, d) be metric spaces.

Definition 5.26 (Metric Approximation, Metric Version)**.** *We call a net*

$$\{\Phi_\nu : D(\Phi_\nu) \subset E \to E_\nu\}_{\nu\in\mathscr{N}}$$

of maps Φ_ν with domains $D(\Phi_\nu) \subset E$ a metric approximation *between $\{E_\nu\}_{\nu\in\mathscr{N}}$ and E if the following properties are fulfilled:*

(B1) *$\{D(\Phi_\nu)\}_{\nu\in\mathscr{N}}$ is monotone non-decreasing in $\nu \in \mathscr{N}$ ordered by the inclusion of sets and $\bigcup_{\nu\in\mathscr{N}} D(\Phi_\nu) =: C$ is dense in E with respect to its metric topology.*

(B2) *For any $u \in C$*

$$\overline{\mathbb{R}}\text{-}\lim_{\nu\in\mathscr{N}} \left\{ \begin{array}{ll} d_\nu(\Phi_\nu u, \Phi_\nu v) & , \text{ if } u, v \in D(\Phi_\nu) \\ +\infty & , \text{ otherwise} \end{array} \right\} = d(u, v).$$

A metric approximation is called linear*, if, additionally:*

(BL) *Each $D(\Phi_\nu)$ is a linear space and each Φ_ν is a linear map.*

Examples of concrete metric approximations are given in Chapter 6.

5.5 Strong convergence

We start with a non-standard diagonalization lemma which will turn out to be highly useful. Its proof is due to H. Attouch and R. J.-B. Wets, originally in [AW83b, Appendix], and can also be found in [Att84, Lemma 1.15 et seq.].

Lemma 5.27 (Attouch-Wets)**.** *Let $\{a_{n,m}\}_{n,m\in\mathbb{N}} \subset \overline{\mathbb{R}}$ be a doubly indexed sequence of extended real numbers. Then there exists a map $n \mapsto m(n)$ with $m(n) \uparrow +\infty$ as $n \to \infty$ such that*

$$\varliminf_{n\to\infty} a_{n,m(n)} \geq \varliminf_{m\to\infty} \left[\varliminf_{n\to\infty} a_{n,m} \right], \tag{5.15}$$

or, equivalently

$$\varlimsup_{n\to\infty} a_{n,m(n)} \leq \varlimsup_{m\to\infty} \left[\varlimsup_{n\to\infty} a_{n,m} \right]. \tag{5.16}$$

Moreover,

$$\varlimsup_{m\to\infty} \left[\varliminf_{n\to\infty} a_{n,m} \right] \geq \varliminf_{n\to\infty} a_{n,m(n)}. \tag{5.17}$$

We note that $m(n)$ in (5.15) *might be different from $m(n)$ in* (5.17)*.*

Given a metric approximation, we define a "convergence relation" on $\mathfrak{E}$ as follows. Later it turns out to be exactly the convergence coming from a strong asymptotic relation on $\mathfrak{E}$. Compare Appendix A for the topological procedure. Definition 5.28 makes sense in a larger context, too, see Remark 5.77 at the end of this chapter.

Definition 5.28. *Let $\mathscr{N} = \mathbb{N}$ and let $\mathfrak{E}$ have a linear metric approximation $\{\Phi_n : D(\Phi_n) \to E_n\}_{n\in\mathbb{N}}$. The following relation between sequences $\{u_k\} \subset \mathfrak{E}$ and points $u \in \mathfrak{E}$ is called* strong convergence *and denoted by $u_k \xrightarrow[k\to\infty]{} u$:*

Case 1*: $\{u_k\} \subset \mathfrak{E}$ converges strongly to $u \in E_{n_0}$, $n_0 \in \mathbb{N}$ if there exists $K \in \mathbb{N}$ such that $u_k \in E_{n_0}$ for every $k \geq K$ and $\lim_{k\geq K} \|u_k - u\|_{E_{n_0}} = 0$.*

Case 2: $\{u_k\} \subset \mathfrak{E}$ *converges strongly to* $u \in E$ *if there exists* $K \in \mathbb{N}$ *such that* $u_k \in E$ *for* $k \geq K$ *and* $\lim_{k\geq K} \|u_k - u\|_E = 0$.

Case 3: $\{u_k\} \subset \mathfrak{E}$ *converges strongly to* $u \in E$ *if there is a countable subnet* $\{n_k\}$ *of* $\{n\}$ *(i.e.,* $\lim_k n_k = \infty$*) such that* $u_k \in E_{n_k}$, $k \in \mathbb{N}$ *and there exists a sequence* $\{\widetilde{u}_m\} \subset \bigcup_{k\in\mathbb{N}} D(\Phi_{n_k}) \subset C$ *such that* $\lim_m \|\widetilde{u}_m - u\|_E = 0$ *and*

$$\lim_m \overline{\lim_k} \begin{Bmatrix} \|\Phi_{n_k}(\widetilde{u}_m) - u_k\|_{E_{n_k}} & , \text{ if } \widetilde{u}_m \in D(\Phi_{n_k}) \\ +\infty & , \text{ otherwise} \end{Bmatrix} = 0. \tag{5.18}$$

Case 4: $\{u_k\} \subset \mathfrak{E}$ *converges strongly to* $u \in E$ *if there are two disjoint subsequences* $\{k_l\}$ *and* $\{k_l'\}$ *of* $\{k\}$ *such that their union is all* $\mathbb{N}$ *and* Case 2 *holds for* $\{u_{k_l}\}$ *and* u *and* Case 3 *holds for* $\{u_{k_l'}\}$ *and* u.

Remark 5.29. *(i) Let* $\{u_k\} \subset \mathfrak{E}$ *be a sequence such that* $u_k \in E_{n_k}$, $n_k \in \overline{\mathbb{N}}$. *In order to converge strongly in the sense of Definition 5.28, it is necessary that* $\{n_k\}$ *converges in* $\overline{\mathbb{N}}$. *A deeper reason for this is explained in Remark 5.77.*

(ii) If Case 3 *in Definition 5.28 holds for some* $\{u_k\}$, u, $\{\widetilde{u}_m\}$, $\{n_k\}$ *then there exists a subsequence* $\{m_k\}$ *of* $\{m\}$ *and* $K \in \mathbb{N}$ *such that* $\widetilde{u}_{m_k} \in D(\Phi_{n_k})$ *for* $k \geq K$ *and*

$$\lim_{k\geq K} \|\Phi_{n_k}(\widetilde{u}_{m_k}) - u_k\|_{E_{n_k}} = 0. \tag{5.19}$$

This follows with Lemma 5.27.

Lemma 5.30. *In the situation of Definition 5.28 for a sequence* $\{u_k\} \subset \mathfrak{E}$ *and* $u \in E$ Case 3 *is equivalent to:*
There is a countable subnet $\{n_k\}$ *of* $\{n\}$ *(i.e.,* $\lim_k n_k = \infty$*) such that* $u_k \in E_{n_k}$, $k \in \mathbb{N}$ *and for any sequence* $\{\widetilde{u}_m\} \subset \bigcup_{k\in\mathbb{N}} D(\Phi_{n_k}) \subset C$ *such that* $\lim_m \|\widetilde{u}_m - u\|_E = 0$ *(if it exists) we have that*

$$\lim_m \overline{\lim_k} \begin{Bmatrix} \|\Phi_{n_k}(\widetilde{u}_m) - u_k\|_{E_{n_k}} & , \text{ if } \widetilde{u}_m \in D(\Phi_{n_k}) \\ +\infty & , \text{ otherwise} \end{Bmatrix} = 0. \tag{5.20}$$

Proof. It is trivial that (5.18) follows from (5.20). Let us prove the converse. Let $\{\widetilde{u}_m\} \subset C$ be such that (5.18) holds. Let $\{\widetilde{\widetilde{u}}_m\}$ be any sequence in C with $\lim_m \left\|\widetilde{\widetilde{u}}_m - u\right\|_E = 0$.

For each $\widetilde{\widetilde{u}}_m \in C$ by (B1) we have an index $\widetilde{\widetilde{n}}_m \in \mathbb{N}$ such that

$$\widetilde{\widetilde{u}}_m \in \bigcup_{n\geq\widetilde{\widetilde{n}}_m} D(\Phi_n).$$

By a diagonal procedure, we find $\widetilde{\widetilde{n}}_1 \leq \widetilde{\widetilde{n}}_2 \leq \widetilde{\widetilde{n}}_3 \leq \ldots$ (denoting the new $\widetilde{\widetilde{n}}_m$ by the same symbol) such that

$$\widetilde{\widetilde{u}}_m \in \bigcup_{n\geq\widetilde{\widetilde{n}}_m} D(\Phi_n) \quad \text{for all } m \in \mathbb{N}.$$

Do so for $\{\widetilde{\widetilde{u}}_m\}$, too, and obtain a sequence $\{\widetilde{\widetilde{n}}_m\}$ such that

$$\widetilde{\widetilde{u}}_m \in \bigcup_{n \geq \widetilde{\widetilde{n}}_m} D(\Phi_n) \quad \text{for all } m \in \mathbb{N}.$$

Now for $n \geq \widetilde{n}_m \vee \widetilde{\widetilde{n}}_m$ it holds that

$$\widetilde{u}_m, \widetilde{\widetilde{u}}_m \in D(\Phi_n).$$

We see that by the triangle inequality and (BL)

$$\begin{aligned}
&\lim_m \overline{\lim_k} \begin{Bmatrix} \left\| \Phi_{n_k}(\widetilde{\widetilde{u}}_m) - u_k \right\|_{E_{n_k}} & \text{, if } \widetilde{\widetilde{u}}_m \in D(\Phi_{n_k}) \\ +\infty & \text{, otherwise} \end{Bmatrix} \\
\leq &\lim_m \overline{\lim_k} \begin{Bmatrix} \left\| \Phi_{n_k}(\widetilde{u}_m) - u_k \right\|_{E_{n_k}} & \text{, if } \widetilde{u}_m, \widetilde{\widetilde{u}}_m \in D(\Phi_{n_k}) \\ +\infty & \text{, otherwise} \end{Bmatrix} \\
+ &\lim_m \overline{\lim_k} \begin{Bmatrix} \left\| \Phi_{n_k}(\widetilde{u}_m - \widetilde{\widetilde{u}}_m) \right\|_{E_{n_k}} & \text{, if } \widetilde{u}_m, \widetilde{\widetilde{u}}_m \in D(\Phi_{n_k}) \\ +\infty & \text{, otherwise} \end{Bmatrix}.
\end{aligned}$$

The first term tends to zero by strong convergence, (B1) and the reasoning above. The second term tends to zero by (BL), (B2) and the fact that both sequences approximate u. □

Lemma 5.31. *(i) Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence of gauges that converges uniformly on bounded subsets of $\mathbb{R}_+$ to a gauge φ. (Where we call a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a gauge if it is continuous, strictly increasing, $\varphi(0) = 0$ and $\varphi(t) \to +\infty$ as $t \to +\infty$.)*

Let $\{u_k\} \subset \mathfrak{E}$ such that there exists a countable subnet $\{n_k\}$ of $\{n\}$ such that $u_k \in E_{n_k}$ for every k. Then $u_k \xrightarrow[k\to\infty]{} u$ if and only if there exists a sequence $\{\widetilde{u}_m\} \subset C$ such that $\lim_m \|u_m - u\|_E = 0$ and

$$\lim_m \overline{\lim_k} \begin{Bmatrix} \varphi_{n_k}\left(\left\| \Phi_{n_k}(\widetilde{u}_m) - u_k \right\|_{E_{n_k}} \right) & \text{, if } \widetilde{u}_m \in D(\Phi_{n_k}) \\ +\infty & \text{, otherwise} \end{Bmatrix} = 0.$$

(ii) If $u \in C$, then $\{u_n\}$ defined as follows

$$u_n := \begin{cases} 0 \in E_n & \text{if } u \notin D(\Phi_n), \\ \Phi_n(u) \in E_n & \text{if } u \in D(\Phi_n) \end{cases}$$

for $n \in \mathbb{N}$ converges strongly to u.

Proof. (i): The "only if"-part follows from the uniform convergence of $\{\varphi_n\}$ and $\varphi(0) = 0$. For the "if"-part note that if $\varphi_n \to \varphi$ uniformly on bounded sets also $\varphi_n^{-1} \to \varphi^{-1}$ uniformly on bounded sets by e.g. [BDF91].

(ii): Note that there exists $n_0 \in \mathbb{N}$ such that $u \in D(\Phi_n)$ for all $n \geq n_0$. The assertion is clear from the linearity of the Φ_n's and (B2).

□

The next theorem constitutes the basis for a construction of an asymptotic relation (topology) from a given metric approximation. See further Theorem 5.38. The condition $\mathscr{N} = \mathbb{N}$ is not as restrictive as it might seem, see Lemma 5.75 (iv) and Theorem 5.76 at the end of this chapter.

Theorem 5.32. *Let $\mathscr{N} = \mathbb{N}$ and $\{\Phi_n : D(\Phi_n) \to E_n\}$ be a linear metric approximation. Then $\mathfrak{E}$ with the strong convergence from Definition 5.28 is an $\mathscr{S}^*$-space and thus has a Fréchet topology. For the terminology we refer to Appendix A.*

Proof. First notice that Case 1, Case 2, Case 3 and Case 4 are mutually exclusive. We have to verify (L1)–(L4) in Definition A.5 in the appendix.

(L1): Suppose that $\{u_k\} \subset \mathfrak{E}$ with $u_k = u$ for all $k \in \mathbb{N}$ and some $u \in \mathfrak{E}$. If $u \in E_{n_0}$ for some $n_0 \in \mathbb{N}$ we are in Case 1 and (L1) follows. If $u \in E$ we are in Case 2 and (L1) follows. Case 3 and Case 4 do not occur.

(L2): Suppose that $u \in \mathfrak{E}$ and $\{u_k\} \subset \mathfrak{E}$ with $u_k \xrightarrow[k\to\infty]{} u$. Let $\{u_{k_l}\}$ be any subsequence of $\{u_k\}$. If we are in Case 1 or Case 2 (L2) follows. Suppose that we are in Case 3, i.e., $u \in E$ and there is a countable subnet $\{n_k\}$ of $\{n\}$ with $u_k \in E_{n_k}$ for $k \in \mathbb{N}$ and there is $\{\widetilde{u}_m\} \subset C$ with $\lim_m \|\widetilde{u}_m - u\|_E = 0$ such that

$$\lim_m \varlimsup_k \begin{Bmatrix} \|\Phi_{n_k}(\widetilde{u}_m) - u_k\|_{E_{n_k}} & , \text{ if } \widetilde{u}_m \in D(\Phi_{n_k}) \\ +\infty & , \text{ otherwise} \end{Bmatrix} = 0.$$

Let $\{k_l\}$ be any subsequence of $\{k\}$. Then $\{n_{k_l}\}$ is a countable subnet of $\{n\}$ (cf. (A.3) in the appendix). Now $u_{k_l} \xrightarrow[l\to\infty]{} u$ follows from the inequality

$$\begin{aligned} \varlimsup_l \begin{Bmatrix} \left\|\Phi_{n_{k_l}}(\widetilde{u}_m) - u_{k_l}\right\|_{E_{n_{k_l}}} & , \text{ if } \widetilde{u}_m \in D(\Phi_{n_{k_l}}) \\ +\infty & , \text{ otherwise} \end{Bmatrix} & \\ \leq \varlimsup_k \begin{Bmatrix} \|\Phi_{n_k}(\widetilde{u}_m) - u_k\|_{E_{n_k}} & , \text{ if } \widetilde{u}_m \in D(\Phi_{n_k}) \\ +\infty & , \text{ otherwise} \end{Bmatrix} & \end{aligned}$$

which gives us (L2). By the part already proved a subsequence of a Case 4 convergent sequence either converges Case 2 or Case 3 or Case 4.

(L3): Let $u \in \mathfrak{E}$ and $\{u_k\} \subset \mathfrak{E}$ with $u_k \underset{k\to\infty}{\not\longrightarrow} u$. Suppose that $u \in E_{n_0}$ for some $n_0 \in \mathbb{N}$. Then no subsequence of $\{u_k\}$ satisfies Case 2 or Case 3 or Case 4. Suppose that Case 1 is also not satisfied for $\{u_k\}$. If there is no $K \in \mathbb{N}$ such that $u_k \in E_{n_0}$ for $k \geq K$ also no subsequence of $\{u_k\}$ has this property. If there is such an

index K, but $\overline{\lim}_{k\geq K}\left\|u_k-u\right\|_E>0$, there clearly is a subsequence $\{u_{k_l}\}$ and $L:=\inf\{l\mid k_l\geq K\}$ with $\lim_{l\geq L}\left\|u_{k_l}-u\right\|_E=\alpha\in(0,+\infty]$, which in turn means that no subsequence of $\{u_{k_l}\}$ converges to u in the sense of Case 1.
Suppose that $u\in E$. Suppose that $\{u_k\}$ violates Case 2 and Case 3 and hence Case 4. If we can prove that no subsequence of $\{u_k\}$ satisfies Case 2 and Case 3 then no subsequence satisfies Case 4. Clearly Case 1 holds for no subsequence of $\{u_k\}$. If there is an index $K'\in\mathbb{N}$ such that $u_k\in E$ for every $k\geq K'$ and $\overline{\lim}_{k\geq K'}\left\|u_k-u\right\|_E>0$, then there is a subsequence of $\{u_{k_l}\}$ of $\{u_k\}$ from which no subsequence satisfies Case 2 (as above). No subsequence of $\{u_k\}$ satisfies Case 3. Hence no subsequence satisfies Case 4.
Suppose that there is no such $K'\in\mathbb{N}$. Then no subsequence of $\{u_k\}$ satisfies Case 2 or Case 4. Case 3 is still assumed to be wrong for $\{u_k\}$. Suppose that there is no sequence of natural numbers $\{n_k\}$ with $\lim_k n_k=\infty$ such that $u_k\in E_{n_k}$ for $k\in\mathbb{N}$. Then no subsequence of $\{u_k\}$ has this property.
Therefore suppose that there is a sequence of natural numbers $\{n_k\}$ with $\lim_k n_k=\infty$ such that $u_k\in E_{n_k}$ for $k\in\mathbb{N}$.
Suppose that no sequences $\{\widetilde{u}_m\}\subset C$ with $\lim_m\left\|\widetilde{u}_m-u\right\|_E=0$ satisfies (5.18). Then

$$\overline{\lim_m}\,\overline{\lim_k}\begin{Bmatrix}\left\|\Phi_{n_k}(\widetilde{u}_m)-u_k\right\|_{E_{n_k}} & ,\ \text{if } \widetilde{u}_m\in D(\Phi_{n_k})\\ +\infty & ,\ \text{otherwise}\end{Bmatrix}>0$$

for all such sequences $\{\widetilde{u}_m\}$. Pick any of those $\{\widetilde{u}_m\}$ and a subsequence $\{u_{k_l}\}$ of $\{u_k\}$ and a subsequence $\{\widetilde{u}_{m_s}\}$ of $\{\widetilde{u}_m\}$ such that

$$\lim_s\lim_l\begin{Bmatrix}\left\|\Phi_{n_{k_l}}(\widetilde{u}_{m_s})-u_{k_l}\right\|_{E_{n_{k_l}}} & ,\ \text{if } \widetilde{u}_{m_s}\in D(\Phi_{n_{k_l}})\\ +\infty & ,\ \text{otherwise}\end{Bmatrix}=\alpha'\in(0,+\infty].$$

Since still $\lim_s\left\|\widetilde{u}_{m_s}-u\right\|_E=0$ we see that (5.20) is violated for any subsequence of $\{u_{k_l}\}$ and hence by Lemma 5.30 no subsequence of it satisfies Case 3 or Case 4. By selecting common subsequences for all cases above, if necessary, we obtain (L3).

(L4): Let $u\in E_{n_0}$ for some $n_0\in\mathbb{N}$ and $\{u_k\}\subset\mathfrak{E}$ such that $u_k\xrightarrow[k\to\infty]{}u$ in the sense of Case 1. Let $K\in\mathbb{N}$ be such that $u_k\in E_{n_0}$ for $k\geq K$. For each u_k let there be a sequence $\{u_l^{(k)}\}\subset\mathfrak{E}$ with $u_l^{(k)}\xrightarrow[l\to\infty]{}u_k$. For each $k\geq K$, this is a Case 1 convergence. In particular, for each $k\geq K$, there is an index $L(k)\in\mathbb{N}$ such that

$u_l^{(k)} \in E_{n_0}$ for all $l \geq L(k)$. Define

$$a_{l,k} := \begin{cases} \left\| u_l^{(k)} - u \right\|_{E_{n_0}}, & \text{if } l \geq L(k), \\ +\infty, & \text{otherwise,} \end{cases}$$
$$b_{l,k} := \begin{cases} \left\| u_l^{(k)} - u_k \right\|_{E_{n_0}}, & \text{if } l \geq L(k),\ k \geq K, \\ +\infty, & \text{otherwise,} \end{cases}$$
$$c_{l,k} := \begin{cases} \left\| u_k - u \right\|_{E_{n_0}}, & \text{if } l \geq L(k),\ k \geq K, \\ +\infty, & \text{otherwise.} \end{cases}$$

By Lemma 5.27, there exists a subsequence $\{k_l\}$ of $\{k\}$ such that

$$\overline{\lim_l}\, a_{l,k_l} \leq \overline{\lim_k}\,\overline{\lim_l}\, a_{l,k} \leq \overline{\lim_k}\,\overline{\lim_l}[b_{l,k} + c_{l,k}] \leq \overline{\lim_k}\,\overline{\lim_l}\, b_{l,k} + \overline{\lim_k}\,\overline{\lim_l}\, c_{l,k} = 0,$$

where we have used the triangle inequality. For each $s \in \mathbb{N}$ set

$$l_s := \max\{L(k_l) \mid l = 1, \ldots, s\}$$

. Then by the above

$$\lim_s \left\| u_{l_s}^{k_{l_s}} - u \right\|_{E_{n_0}} = 0,$$

which is the Case 1 convergence of $\{u_{l_s}^{k_{l_s}}\}_{s\in\mathbb{N}}$ to u and gives us (L4).

Let $u \in E$ and $\{u_k\} \subset \mathfrak{E}$ such that $u_k \xrightarrow[k\to\infty]{} u$ in the sense of Case 2. Let $K \in \mathbb{N}$ be such that $u_k \in E$ for $k \geq K$. For each u_k let there be a sequence $\{u_l^{(k)}\} \subset \mathfrak{E}$ with $u_l^{(k)} \xrightarrow[l\to\infty]{} u_k$. We say that $k \in \mathscr{K}$ if $k \geq K$ and $u_l^{(k)} \xrightarrow[l\to\infty]{} u_k$ Case 2 and associate $L(k)$ as above. We say that $k \in \mathscr{K}'$ if $k \geq K$ and $u_l^{(k)} \xrightarrow[l\to\infty]{} u_k$ Case 3 and associate $\{n_l^{(k)}\}, \{\widetilde{u}_m^{(k)}\}$ as in the definition of Case 3. If $u_l^{(k)} \xrightarrow[l\to\infty]{} u_k$ Case 4 we replace it by a subsequence which converges either Case 2 or Case 3.

At least one of $\mathscr{K}, \mathscr{K}'$ is unbounded. Let $\{\widetilde{u}_m\} \subset C$ such that $\lim_m \|\widetilde{u}_m - u\|_E =$

0. Set

$$a_{l,k}^m := \begin{cases} \|u_l^{(k)} - u\|_E, & \text{if } k \in \mathscr{K},\ l \geq L(k), \\ \left\|u_l^{(k)} - \Phi_{n_l^{(k)}}(\widetilde{u}_m)\right\|_{E_{n_l^{(k)}}}, & \text{if } k \in \mathscr{K}',\ \widetilde{u}_m \in D(\Phi_{n_l^{(k)}}), \\ +\infty, & \text{otherwise,} \end{cases}$$

$$b_{l,k}^m := \begin{cases} \|u_l^{(k)} - u_k\|_E, & \text{if } k \in \mathscr{K},\ l \geq L(k), \\ \left\|u_l^{(k)} - \Phi_{n_l^{(k)}}(\widetilde{u}_m^{(k)})\right\|_{E_{n_l^{(k)}}}, & \text{if } k \in \mathscr{K}',\ \widetilde{u}_m, \widetilde{u}_m^{(k)} \in D(\Phi_{n_l^{(k)}}), \\ +\infty, & \text{otherwise,} \end{cases}$$

$$c_{l,k}^m := \begin{cases} \|u_k - u\|_E\,, & \text{if } k \in \mathscr{K},\ l \geq L(k), \\ \left\|\Phi_{n_l^{(k)}}(\widetilde{u}_m^{(k)} - \widetilde{u}_m)\right\|_{E_{n_l^{(k)}}}, & \text{if } k \in \mathscr{K}',\ \widetilde{u}_m, \widetilde{u}_m^{(k)} \in D(\Phi_{n_l^{(k)}}), \\ +\infty, & \text{otherwise.} \end{cases}$$

By applying Lemma 5.27 twice, there exists a subsequence $\{k_m\}$ of $\{k\}$ and a subsequence $\{m_l\}$ of $\{m\}$ such that

$$\begin{aligned} \overline{\lim_l}\, a_{l,k_{m_l}}^{m_l} &\leq \overline{\lim_m}\,\overline{\lim_l}\, a_{l,k_m}^m \leq \overline{\lim_k}\,\overline{\lim_m}\,\overline{\lim_l}\, a_{l,k}^m \\ &\leq \overline{\lim_k}\,\overline{\lim_m}\,\overline{\lim_l}[b_{l,k}^m + c_{l,k}^m] \leq \overline{\lim_k}\,\overline{\lim_m}\,\overline{\lim_l}\, b_{l,k}^m + \overline{\lim_k}\,\overline{\lim_m}\,\overline{\lim_l}\, c_{l,k}^m = 0, \end{aligned}$$

where we have used the triangle inequality and (B2). The convergence of $b_{l,k}^m$ to zero can be seen by picking a diagonal sequence of increasing domains for all $\widetilde{u}_m^{(k)}$ exactly as in the proof of Lemma 5.30. By a similar argument as above, we conclude that there is a sequence $\{u_{l_s}^{(k_{m_{l_s}})}\}_{s\in\mathbb{N}}$ which strongly converges to u. If $\mathscr{K}$ is bounded this is a Case 3 convergence, and if $\mathscr{K}'$ is bounded, it is a Case 2 convergence. If both $\mathscr{K}, \mathscr{K}'$ are unbounded it is a Case 4 convergence.

Let $u \in E$ and $\{u_k\} \subset \mathfrak{E}$ such that $u_k \xrightarrow[k\to\infty]{} u$ in the sense of Case 3. Let $\{n_k\}$, $\{\widetilde{u}_m\}$ be as in the definition of Case 3, $u_k \in E_{n_k}$ for $k \in \mathbb{N}$ etc. For each u_k let there be a sequence $\{u_l^{(k)}\} \subset \mathfrak{E}$ with $u_l^{(k)} \xrightarrow[l\to\infty]{} u_k$ Case 1. To $k \in \mathbb{N}$ associate $L(k) \in \mathbb{N}$ such that $u_l^{(k)} \in E_{n_k}$ for $l \geq L(k)$. Set

$$a_{l,k}^m := \begin{cases} \left\|u_l^{(k)} - \Phi_{n_k}(\widetilde{u}_m)\right\|_{E_{n_k}}, & \text{if } l \geq L(k),\ \widetilde{u}_m \in D(\Phi_{n_k}), \\ +\infty, & \text{otherwise,} \end{cases}$$

$$b_{l,k}^m := \begin{cases} \left\|u_l^{(k)} - u_k\right\|_{E_{n_k}}, & \text{if } l \geq L(k),\ \widetilde{u}_m \in D(\Phi_{n_k}), \\ +\infty, & \text{otherwise,} \end{cases}$$

$$c_{l,k}^m := \begin{cases} \|u_k - \Phi_{n_k}(\widetilde{u}_m)\|_{E_{n_k}}, & \text{if } l \geq L(k),\ \widetilde{u}_m \in D(\Phi_{n_k}), \\ +\infty, & \text{otherwise.} \end{cases}$$

Applying Lemma 5.27 twice, there is a subsequence $\{k_l\}$ of $\{k\}$ such that

$$\begin{aligned}\overline{\lim_m}\,\overline{\lim_l}\, a^m_{l,k_l} \le \overline{\lim_m}\,\overline{\lim_k}\,\overline{\lim_l}\, a^m_{l,k} &\le \overline{\lim_m}\,\overline{\lim_k}\,\overline{\lim_l}[b^m_{l,k} + c^m_{l,k}] \\ &\le \overline{\lim_m}\,\overline{\lim_k}\,\overline{\lim_l}\, b^m_{l,k} + \overline{\lim_m}\,\overline{\lim_k}\,\overline{\lim_l}\, c^m_{l,k} = 0,\end{aligned}$$

where we have used the triangle inequality. By similar arguments as above there is a subsequence $\{l_s\}$ of $\{l\}$ such that $\{u^{(k_{l_s})}_{l_s}\}$ converges Case 3 to u.

If $u \in E$ and $\{u_k\} \subset \mathfrak{E}$ such that $u_k \xrightarrow[k\to\infty]{} u$ in the sense of Case 4, we can extract a subsequence $\{u_{k_s}\}$ such that we are in either one of the above cases. (L4) follows.

□

Justified by Theorems 5.32, A.6, we will start to use the notation "lim" for strong convergence. We also occasionally write s-lim or $\to$. If we consider strong convergence on $\mathfrak{E}^*$, we also write $\to^*$. This still has to be done with caution, since we have no a priori Hausdorff property and limits might not be unique. We will obtain the Hausdorff property a posteriori, see Theorem 5.38.

5.6 Weak and weak* convergence

The procedure for σ-convergence is similar. We need to assume the existence of linear strong asymptotic relations in duality. "Consistency" in some sense is proved in Subsection 5.7.2. It is related to the notion of so-called *asymptotic duality* (cf. Definitions 5.42 and 5.43 below), which is a stronger condition than mere duality.

Definition 5.33. *Let $\mathscr{N} = \mathbb{N}$. Suppose that $(\mathfrak{E}, \mathfrak{F})$ is a dual pair consisting of Banach spaces (E_n, F_n), $n \in \mathbb{N}$, (E, F) in duality. Suppose that $\mathfrak{E}$, $\mathfrak{F}$ have linear strong asymptotic relations $\tau(\mathfrak{E})$, $\tau(\mathfrak{F})$ respectively. The following relation between sequences $\{u_k\} \subset \mathfrak{E}$ and points $u \in \mathfrak{E}$ is called* σ-convergence *(also* $\sigma(\mathfrak{E}, \mathfrak{F})$-convergence*) and denoted by $u_k \xrightarrow[k\to\infty]{\sigma} u$:*

Case 1*: $\{u_k\} \subset \mathfrak{E}$ σ-converges to $u \in E_{n_0}$, $n_0 \in \mathbb{N}$ if there exists an index $K \in \mathbb{N}$ such that $u_k \in E_{n_0}$ for every $k \ge K$ and $\lim_{k\ge K}\, {}_{E_{n_0}}\langle u_k - u, x\rangle_{F_{n_0}} = 0$ for every $x \in F_{n_0}$.*

Case 2*: $\{u_k\} \subset \mathfrak{E}$ σ-converges to $u \in E$ if there exists an index $K \in \mathbb{N}$ such that $u_k \in E$ for $k \ge K$ and $\lim_{k\ge K}\, {}_E\langle u_k - u, x\rangle_F = 0$ for every $x \in F$.*

Case 3*: $\{u_k\} \subset \mathfrak{E}$ σ-converges to $u \in E$ if there is a countable subnet $\{n_k\}_{k\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ such that $u_k \in E_{n_k}$ for $k \in \mathbb{N}$, $\sup_k \|u_k\|_{E_{n_k}} < +\infty$ and*

$$\lim_k\, {}_{E_{n_k}}\langle u_k, x_k\rangle_{F_{n_k}} = {}_E\langle u, x\rangle_F$$

for all $x \in F$ and all sequences $x_k \in F_{n_k}$, $k \in \mathbb{N}$ such that $\tau(\mathfrak{F})$-$\lim_k x_k = x$.

Case 4: *$\{u_k\} \subset \mathfrak{E}$ σ-converges to $u \in E$ if there are two disjoint subsequences $\{k_l\}$ and $\{k'_l\}$ of $\{k\}$ such that their union is all* $\mathbb{N}$ *and* Case 2 *holds for $\{u_{k_l}\}$ and u and* Case 3 *holds for $\{u_{k'_l}\}$ and u.*

If $\mathfrak{F} = \mathfrak{E}^*$ σ-convergence is called weak convergence *and denoted by "$\rightharpoonup$" or "$\xrightharpoonup{w}$". If $\mathfrak{E} = \mathfrak{G}^*$ and $\mathfrak{F} = \mathfrak{G}^{**}$ for some $\mathfrak{G}$, we shall also write "$\rightharpoonup^*$" or "$\xrightharpoonup{w^*}$", but still call it a weak convergence on $\mathfrak{G}^*$. If $\mathfrak{E} = \mathfrak{F}^*$ σ-convergence is called* weak* convergence *and denoted by "$\rightharpoonup^*$" or "$\xrightharpoonup{w^*}$".*

Remark 5.34. *If $\mathfrak{E} = \mathfrak{F}^*$ it follows from the uniform boundedness principle (cf.* [Yos78, p. 69, Corollary 1]*) that in* Case 1 *and* Case 2 $\sup_k \|u_k\| < +\infty$.

If $\mathfrak{F} = \mathfrak{E}^*$ it follows from the uniform boundedness principle and the fact that $\|u\|_E = \|\iota_E(u)\|_{E^{**}}$ that in Case 1 *and* Case 2 $\sup_k \|u_k\| < +\infty$, *where $\iota_E : E \to E^{**}$ is the canonical isometric embedding.*

The following lemma is highly useful.

Lemma 5.35. *Suppose that $D \subset F$ is a strongly dense subset. Let $\{n_k\}$ be a countable subnet of $\{n\}$. Let $\{u_k\} \subset \mathfrak{E}$ be such that $u_k \in E_{n_k}$, $k \in \mathbb{N}$ and $\sup_k \|u_k\|_{E_{n_k}} < \infty$.*

For each $d \in D$ pick an arbitrary $\{d_k\} \subset \mathfrak{F}$ with $d_k \in F_{n_k}$, $k \in \mathbb{N}$ and $\tau(\mathfrak{F})\text{-}\lim_k d_k = d$ strongly.

If

$$\lim_k {}_{E_{n_k}}\langle u_k, d_k\rangle_{F_{n_k}} = {}_E\langle u, d\rangle_F \tag{5.21}$$

for all $d \in D$, then $u_k \xrightarrow[k]{\sigma} u$ Case 3.

A similar statement holds for Case 1 *and* Case 2*, too.*

Proof. Let $M := \sup_k \|u_k\|_{E_{n_k}} < \infty$. Let $f_k \in F_{n_k}$, $k \in \mathbb{N}$, $f \in F$ with $\tau(\mathfrak{F})\text{-}\lim_k f_k = f$, which exists by (A2) and a subnet argument. Let $d_m \in D$ with $\lim_m \|d_m - f\|_F = 0$. For each d_m, $m \in \mathbb{N}$ pick $d_k^m \in F_{n_k}$, $k \in \mathbb{N}$ with $\tau(\mathfrak{F})\text{-}\lim_k d_k^m = d_m$ such that (5.21) holds. We have that

$$\begin{aligned}
&\left| {}_{E_{n_k}}\langle u_k, f_k\rangle_{F_{n_k}} - {}_E\langle u, f\rangle_F \right| \\
\leq &\left| {}_{E_{n_k}}\langle u_k, f_k - d_k^m\rangle_{F_{n_k}} \right| + \left| {}_{E_{n_k}}\langle u_k, d_k^m\rangle_{F_{n_k}} - {}_E\langle u, d_m\rangle_F \right| + |\, {}_E\langle u, d_m - f\rangle_F| \\
\leq &M \, \|f_k - d_k^m\|_{F_{n_k}} + \left| {}_{E_{n_k}}\langle u_k, d_k^m\rangle_{F_{n_k}} - {}_E\langle u, d_m\rangle_F \right| + \|u\|_E \, \|d_m - f\|_F \\
\xrightarrow[k\to\infty]{} &(M + \|u\|_F) \, \|f - d_m\|_F \\
\xrightarrow[m\to\infty]{} &0,
\end{aligned}$$

where we have used (A3') and (5.21). Case 1 and Case 2 can be verified as in [Yos78, Ch. V.1, Theorem 3, p. 121, Theorem 10, p. 125]. □

Lemma 5.36. *The assumption of* $\sup_k \|u_k\|_{E_{n_k}} < +\infty$ *in* Case 3 *is necessary.*

Proof. If $\mathfrak{F} = \mathfrak{E}^*$ then by the Hahn-Banach Theorem (cf. [Yos78, p. 108, Corollary 2]) for each $u \in \mathfrak{E}$, such that $u \in E_{n_0}$, $n_0 \in \mathbb{N} \cup \{\infty\}$ there is a point $f \in F_{n_0} \subset \mathfrak{F}$ such that ${}_{F_{n_0}}\langle f, u\rangle_{E_{n_0}} = \|f\|_{F_{n_0}} \|u\|_{E_{n_0}}$.

If $\mathfrak{E} = \mathfrak{F}^*$ then by the Bishop-Phelps Theorem B.16 the set of points f in each $E_{n_0} \subset \mathfrak{E}$ for some $n_0 \in \mathbb{N} \cup \{\infty\}$ such that there is $u \in F_{n_0} \subset \mathfrak{F}$ with ${}_{E_{n_0}}\langle f, u\rangle_{F_{n_0}} = \|f\|_{E_{n_0}} \|u\|_{F_{n_0}}$ is strongly dense in E_{n_0}.

Suppose that $\sup_k \|u_k\|_{E_{n_k}} = +\infty$. Then there exists a subsequence $\{k_l\}$ of $\{k\}$ such that $\|u_{k_l}\|_{E_{n_{k_l}}} \geq l$ for each $l \in \mathbb{N}$.

Let $x_l \in F_{n_{k_l}}$, $l \in \mathbb{N}$ be such that ${}_{E_{n_{k_l}}}\langle \widetilde{u}_{k_l}, x_l\rangle_{F_{n_{k_l}}} = \|\widetilde{u}_{k_l}\|_{E_{n_{k_l}}} \|x_l\|_{F_{n_{k_l}}}$ for some norm-attaining $\widetilde{u}_{k_l} \in E_{n_{k_l}}$ with

$$\|u_{k_l} - \widetilde{u}_{k_l}\|_{E_{n_{k_l}}} \leq 1/l \tag{5.22}$$

(if $\mathfrak{F} = \mathfrak{E}^*$ it is enough to consider $\widetilde{u}_{k_l} = u_{k_l}$).

Let $f_{k_l} \in F_{n_{k_l}}$, $l \in \mathbb{N}$, $f \in F$ such that $\tau(\mathfrak{F})\text{-}\lim_l f_{k_l} = f$, which exists by (A2) and a subnet argument. We have that

$$\begin{aligned}
&\left| {}_{E_{n_{k_l}}}\langle \widetilde{u}_{k_l}, f_{k_l}\rangle_{F_{n_{k_l}}} - {}_E\langle u, f\rangle_F \right| \\
&\qquad \leq \left| {}_{E_{n_{k_l}}}\langle \widetilde{u}_{k_l} - u_{k_l}, f_{k_l}\rangle_{F_{n_{k_l}}} \right| + \left| {}_{E_{n_{k_l}}}\langle u_{k_l}, f_{k_l}\rangle_{F_{n_{k_l}}} - {}_E\langle u, f\rangle_F \right| \\
&\qquad\qquad \leq \|\widetilde{u}_{k_l} - u_{k_l}\|_{E_{n_{k_l}}} \|f_{k_l}\|_{F_{n_{k_l}}} + \left| {}_{E_{n_{k_l}}}\langle u_{k_l}, f_{k_l}\rangle_{F_{n_{k_l}}} - {}_E\langle u, f\rangle_F \right|,
\end{aligned}$$

which tends to zero by (5.22), (A3) and the Case 3 σ-convergence of $\{u_{k_l}\}$. Therefore $\widetilde{u}_{k_l} \xrightarrow[l\to\infty]{\sigma} u$ also Case 3 (except for the uniform boundedness).

Moreover,

$$\frac{1}{l} \geq \|u_{k_l} - \widetilde{u}_{k_l}\|_{E_{n_{k_l}}} \geq \left| \|u_{k_l}\|_{E_{n_{k_l}}} - \|\widetilde{u}_{k_l}\|_{E_{n_{k_l}}} \right| \geq \|u_{k_l}\|_{E_{n_{k_l}}} - \|\widetilde{u}_{k_l}\|_{E_{n_{k_l}}}$$

and hence

$$\|\widetilde{u}_{k_l}\|_{E_{n_{k_l}}} \geq \|u_{k_l}\|_{E_{n_{k_l}}} - \frac{1}{l} \geq l - \frac{1}{l}.$$

Define

$$\widetilde{x}_l := \frac{1}{l} \cdot \frac{x_l}{\|x_l\|_{F_{n_{k_l}}}}.$$

By Corollary 5.13 $\widetilde{x}_l \to 0 \in E$ strongly. Hence

$${}_{E_{n_{k_l}}}\langle \widetilde{u}_{k_l}, \widetilde{x}_l\rangle_{F_{n_{k_l}}} \to {}_E\langle u, 0\rangle_F = 0,$$

but

$${}_{E_{n_{k_l}}}\langle \widetilde{u}_{k_l}, \widetilde{x}_l\rangle_{F_{n_{k_l}}} = \frac{1}{l} \, {}_{E_{n_{k_l}}}\left\langle \widetilde{u}_{k_l}, \frac{x_l}{\|x_l\|_{E_{n_{k_l}}}} \right\rangle_{F_{n_{k_l}}} = \frac{1}{l} \cdot \|\widetilde{u}_{k_l}\|_{E_{n_{k_l}}} \geq 1 - \frac{1}{l^2}.$$

This is a contradiction and hence $\sup_k \|u_k\|_{E_{n_k}} < +\infty$. For Cases 1 and 2 see Remark 5.34. □

Theorem 5.37. *Let $\mathscr{N} = \mathbb{N}$. Suppose that $(\mathfrak{E}, \mathfrak{F})$ is a dual pair consisting of Banach spaces (E_n, F_n), $n \in \mathbb{N}$, (E, F) in duality. Suppose that $\mathfrak{E}$, $\mathfrak{F}$ have linear strong asymptotic relations $\tau(\mathfrak{E})$, $\tau(\mathfrak{F})$ respectively which are assumed to be Fréchet. Then $\mathfrak{E}$ with the $\sigma(\mathfrak{E}, \mathfrak{F})$-convergence from Definition 5.33 is an $\mathscr{L}^*$-space and thus has a sequential topology. If each of F_n, $n \in \mathbb{N}$ and F are separable, then $\mathfrak{E}$ with the σ-convergence is an $\mathscr{S}^*$-space and thus has a Fréchet topology. For the terminology we refer to Appendix A.*

Proof. Note that Cases 1, 2, 3 and 4 are mutually exclusive. Let us first verify (L1)–(L3) in Definition A.5 in the appendix. For verifying (L4) therein assume that each of F_n, $n \in \mathbb{N}$, F are separable.

(L1): Clear; only Cases 1 and 2 occur.

(L2): Let $\{u_k\} \subset \mathfrak{E}$, $u \in \mathfrak{E}$ with $u_k \xrightarrow[k\to\infty]{\sigma} u$. Case 1 and Case 2 are clear. Case 3 follows since (L2) holds for any strongly convergent sequence $\tau(\mathfrak{F})\text{-}\lim_k x_k = x$. Case 4 follows.

(L3): Let $\{u_k\} \subset \mathfrak{E}$, $u \in \mathfrak{E}$ with $u_k \not\xrightarrow[k\to\infty]{\sigma} u$.
First suppose that $u \in E_{n_0}$ for some $n_0 \in \mathbb{N}$. If there is no $K \in \mathbb{N}$ with the property that $u_k \in E_{n_0}$ for $k \geq K$ then no subsequence of $\{u_k\}$ has this property. Also Cases 2 and 3 and hence 4 do not occur for any subsequence of $\{u_k\}$. If there is an index $K \in \mathbb{N}$ with $u_k \in E_{n_0}$ for $k \geq K$ then there exists a subsequence $\{k_l\}$ of $\{k\}$ and a point $x_0^{(n_0)} \in F_{n_0}$ with

$$\left| {}_{F_{n_0}}\left\langle x_0^{(n_0)}, u_{k_l} - u \right\rangle_{E_{n_0}} \right| \to \alpha \in (0, +\infty], \quad \text{as } l \to \infty;$$

no subsequence of which fulfills σ-convergence Case 1. Cases 2 and 3 and hence 4 do not occur for any subsequence.
Suppose that $u \in E$. Suppose that there is an index $K' \in \mathbb{N}$ such that $u_k \in E$ for $k \geq K'$. Then there is a point $x_0 \in F$ and a subsequence $\{k_l\}$ of $\{k\}$ such that

$$\left| {}_F\langle x_0, u_{k_l} - u \rangle_E \right| \to \alpha' \in (0, +\infty], \quad \text{as } l \to \infty;$$

no subsequence of which satisfies Case 2. Case 1 and Case 3 and hence Case 4 do not hold for any subsequence.
Now suppose that there is no $K' \in \mathbb{N}$ with $u_k \in E$ for $k \geq K'$. Suppose that Case 3 and Case 4 are violated. If there is no countable subnet of $\{n_k\}$ of $\{n\}$ such that $u_k \in E_{n_k}$ for $k \in \mathbb{N}$ then Cases 1, 2, 3, 4 do not hold for any subsequence of $\{u_k\}$ and u.
Then suppose that there is such a countable subnet $\{n_k\}$ with $u_k \in E_{n_k}$, $k \in \mathbb{N}$. Then (as Case 3 and 4 are assumed to be violated) there exists a $\tau(\mathfrak{F})$-convergent

sequence $\{x_k\}$, $x_k \in F_{n_k}$, $k \in \mathbb{N}$, $x \in F$ with $\tau(\mathfrak{F})\text{-}\lim_k x_k = x$ (whose existence follows by (A2)) such that

$$\overline{\lim_k} \left| {}_{F_{n_k}}\langle x_k, u_k \rangle_{E_{n_k}} - {}_F\langle x, u \rangle_E \right| = \alpha'' \in (0, +\infty].$$

Now by $\tau(\mathfrak{F})$-(L2) for $\{x_k\}$ there is a subsequence $\{u_{k_l}\}$ no subsequence of which σ-converges Case 3 (and hence 4) to u. Case 1 and 2 do not occur for any subsequence. Select a common subsequence if necessary and (L3) is proved.

(L4): Let $u \in E_{n_0}$ for some $n_0 \in \mathbb{N}$ and $\{u_k\} \subset \mathfrak{E}$ such that $u_k \xrightarrow[k\to\infty]{\sigma} u$ in the sense of Case 1. Let $\{x_m\}$ be a countable dense subset of F_{n_0}. Let $K \in \mathbb{N}$ be such that $u_k \in E_{n_0}$ for $k \geq K$. For each u_k let there be a sequence $\{u_l^{(k)}\} \subset \mathfrak{E}$ with $u_l^{(k)} \xrightarrow[l\to\infty]{\sigma} u_k$. For each $k \geq K$, this is a Case 1 convergence. In particular, there is an index $L(k) \in \mathbb{N}$ with $u_l^{(k)} \in E_{n_0}$ for each $l \geq L(k)$. Define

$$a_{l,k}^{(m)} := \begin{cases} {}_{F_{n_0}}\left\langle x_m, u_l^{(k)} - u \right\rangle_{E_{n_0}}, & \text{if } l \geq L(k),\ k \geq K, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$b_{l,k}^{(m)} := \begin{cases} {}_{F_{n_0}}\left\langle x_m, u_l^{(k)} - u_k \right\rangle_{E_{n_0}}, & \text{if } l \geq L(k),\ k \geq K, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$c_{l,k}^{(m)} := \begin{cases} {}_{F_{n_0}}\langle x_m, u_k - u \rangle_{E_{n_0}}, & \text{if } l \geq L(k),\ k \geq K, \\ +\infty, & \text{otherwise.} \end{cases}$$

By Lemma 5.27 there exists a subsequence $\{k_l\}$ of $\{k\}$ such that

$$\begin{aligned} \overline{\lim_l} |a_{l,k_l}^{(m)}| \leq \overline{\lim_k}\, \overline{\lim_l} |a_{l,k}^{(m)}| &\leq \overline{\lim_k}\, \overline{\lim_l} |b_{l,k}^{(m)} + c_{l,k}^{(m)}| \\ &\leq \overline{\lim_k}\, \overline{\lim_l} |b_{l,k}^{(m)}| + \overline{\lim_k}\, \overline{\lim_l} |c_{l,k}^{(m)}| = 0, \end{aligned}$$

where we have used the linearity of the duality bracket. For each $s \in \mathbb{N}$ set $l_s := \max\{L(k_l) \mid l = 1, \dots, s\}$. Then by the above

$$\lim_s \left| {}_{F_{n_0}}\left\langle x_m, u_{l_s}^{k_{l_s}} - u \right\rangle_{E_{n_0}} \right| = 0, \quad \text{for each fixed } m \in \mathbb{N}. \tag{5.23}$$

Therefore we get maps $s \mapsto l_s^m$ and $s \mapsto k_{l_s}^m$ for each m such that (5.23) holds. By a standard diagonal argument we can extract subsequences such that (5.23) holds for all x_m's at the same time.

Note that $\{u_{l_s}^{k_{l_s}}\}_{s\in\mathbb{N}}$ is uniformly norm bounded and hence by an $\varepsilon/3$-argument exactly as in the proof of Lemma 5.35 we get Case 1 σ-convergence of $\{u_{l_s}^{k_{l_s}}\}_{s\in\mathbb{N}}$ to u. This is the first part of (L4).

Let $u \in E$ and $\{u_k\} \subset \mathfrak{E}$ such that $u_k \xrightarrow[k\to\infty]{\sigma} u$ in the sense of Case 2. Let $K \in \mathbb{N}$ be such that $u_k \in E$ for $k \geq K$. For each u_k let there be a sequence $\{u_l^{(k)}\} \subset \mathfrak{E}$ with $u_l^{(k)} \xrightarrow[l\to\infty]{\sigma} u_k$. We say that $k \in \mathscr{K}$ if $k \geq K$ and $u_l^{(k)} \xrightarrow[l\to\infty]{\sigma} u_k$ Case 2 and associate $L(k)$ as above. We say that $k \in \mathscr{K}'$ if $k \geq K$ and $u_l^{(k)} \xrightarrow[l\to\infty]{\sigma} u_k$ Case 3 and associate countable subnets $\{n_l^{(k)}\}$ as in the definition of Case 3. If $u_l^{(k)} \xrightarrow[l\to\infty]{\sigma} u_k$ Case 4 we pick a subsequence $\{l_s\}$ of $\{l\}$ such that $u_{l_s}^{(k)} \xrightarrow[l\to\infty]{\sigma} u_k$ either Case 2 or Case 3 and replace the original sequence with it. At least one of $\mathscr{K}, \mathscr{K}'$ is unbounded. Let $\{x_m\} \subset F$ be a countable dense subset. By (A2) for each x_m, $m \in \mathbb{N}$ pick a sequence $\{x_l^{(m),k}\}$ with $x_l^{(m),k} \in F_{n_l^{(k)}}$ for $l \in \mathbb{N}$ and $\tau(\mathfrak{F})\text{-}\lim_l x_l^{(m),k} = x_m$ strongly; do so for all $k \in \mathscr{K}'$. Set

$$a_{l,k}^{(m)} := \begin{cases} {}_F\left\langle x_m, u_l^{(k)} - u \right\rangle_E, & \text{if } k \in \mathscr{K},\ l \geq L(k), \\ {}_{F_{n_l^{(k)}}}\left\langle x_l^{(m),k}, u_l^{(k)} \right\rangle_{E_{n_l^{(k)}}} - {}_F\langle x_m, u\rangle_E, & \text{if } k \in \mathscr{K}', \\ +\infty, & \text{otherwise,} \end{cases}$$

$$b_{l,k}^{(m)} := \begin{cases} {}_F\left\langle x_m, u_l^{(k)} - u_k \right\rangle_E, & \text{if } k \in \mathscr{K},\ l \geq L(k), \\ {}_{F_{n_l^{(k)}}}\left\langle x_l^{(m),k}, u_l^{(k)} \right\rangle_{E_{n_l^{(k)}}} - {}_F\langle x_m, u_k\rangle_E, & \text{if } k \in \mathscr{K}', \\ +\infty, & \text{otherwise,} \end{cases}$$

$$c_{l,k}^{(m)} := \begin{cases} {}_F\langle x_m, u_k - u\rangle_E, & \text{if } k \in \mathscr{K},\ l \geq L(k), \\ {}_F\langle x_m, u_k - u\rangle_E, & \text{if } k \in \mathscr{K}', \\ +\infty, & \text{otherwise.} \end{cases}$$

By applying Lemma 5.27 there exists a subsequence $\{k_l\}$ of $\{k\}$ such that

$$\begin{aligned} \overline{\lim_l} |a_{l,k_l}^{(m)}| &\leq \overline{\lim_k}\,\overline{\lim_l} |a_{l,k}^{(m)}| \\ &\leq \overline{\lim_k}\,\overline{\lim_l} |b_{l,k}^{(m)} + c_{l,k}^{(m)}| \leq \overline{\lim_k}\,\overline{\lim_l} |b_{l,k}^{(m)}| + \overline{\lim_k}\,\overline{\lim_l} |c_{l,k}^{(m)}| = 0, \end{aligned}$$

where we have used the linearity of the duality bracket. By Lemma 5.35 and similar arguments as above (extracting a sequence working for all x_m at the same time), we conclude that there is a sequence $\{u_{l_s}^{k_{l_s}}\}_{s\in\mathbb{N}}$ which σ-converges to u. If $\mathscr{K}$ is bounded, this is a Case 3 convergence, and if $\mathscr{K}'$ is bounded, it is a Case 2 convergence. If both $\mathscr{K}, \mathscr{K}'$ are unbounded, it is a Case 4 convergence.

Let $u \in E$ and $\{u_k\} \subset \mathfrak{E}$ such that $u_k \xrightarrow[k\to\infty]{\sigma} u$ in the sense of Case 3. Let $\{n_k\}$ be a countable subnet as in the definition of Case 3, $u_k \in E_{n_k}$ for $k \in \mathbb{N}$. For each u_k let there be a sequence $\{u_l^{(k)}\} \subset \mathfrak{E}$ with $u_l^{(k)} \xrightarrow[l\to\infty]{\sigma} u_k$ Case 1. To $k \in \mathbb{N}$

associate $L(k) \in \mathbb{N}$ such that $u_l^{(k)} \in E_{n_k}$ for $l \geq L(k)$. Pick a countable dense set $\{x_m\} \subset F$ and by (A2) for each m pick a sequence $\{x_k^{(m)}\}$ with $x_k^{(m)} \in F_{n_k}$, $k \in \mathbb{N}$ and $\tau(\mathfrak{F})\text{-}\lim_k x_k^{(m)} = x_m$ strongly. Set

$$a_{l,k}^{(m)} := \begin{cases} {}_{F_{n_k}}\left\langle x_k^{(m)}, u_l^{(k)} \right\rangle_{E_{n_k}} - {}_F\langle x_m, u\rangle_E, & \text{if } l \geq L(k), \\ +\infty, & \text{otherwise,} \end{cases}$$
$$b_{l,k}^{(m)} := \begin{cases} {}_{F_{n_k}}\left\langle x_k^{(m)}, u_l^{(k)} - u_k \right\rangle_{E_{n_k}}, & \text{if } l \geq L(k), \\ +\infty, & \text{otherwise,} \end{cases}$$
$$c_{l,k}^{(m)} := \begin{cases} {}_{F_{n_k}}\left\langle x_k^{(m)}, u_k \right\rangle_{E_{n_k}} - {}_F\langle x_m, u\rangle_E, & \text{if } l \geq L(k), \\ +\infty, & \text{otherwise.} \end{cases}$$

Applying Lemma 5.27 there is a subsequence $\{k_l\}$ of $\{k\}$ such that

$$\begin{aligned} \overline{\lim_l} |a_{l,k_l}^{(m)}| \leq \overline{\lim_k}\, \overline{\lim_l} |a_{l,k}^{(m)}| \leq \overline{\lim_k}\, \overline{\lim_l} |b_{l,k}^{(m)} + c_{l,k}^{(m)}| \\ \leq \overline{\lim_k}\, \overline{\lim_l} |b_{l,k}^{(m)}| + \overline{\lim_k}\, \overline{\lim_l} |c_{l,k}^{(m)}| = 0, \end{aligned}$$

where we have used the linearity of the duality bracket. By similar arguments as above (with e.g. an application of Lemma 5.35 and picking a common subsequence that the convergence works for all m at the same time) we conclude: there is a subsequence $\{l_s\}$ of $\{l\}$ such that $\{u_{l_s}^{(k_{l_s})}\}$ converges Case 3 to u.

If $\{u_k\} \subset \mathfrak{E}$, $u \in E$ are such that $u_k \xrightarrow[k\to\infty]{\sigma} u$ Case 4, we can extract a subsequence such that we are in either one of the two cases above.

The proof of (L4) is complete.

□

The above Theorem justifies the use of lim for σ-, weak- and weak*-convergence. We shall write σ- lim, $\sigma(\mathfrak{E},\mathfrak{F})$- lim, w- lim and w^*- lim in the sequel. We shall still write $\rightharpoonup$ for weak convergence. We also write $\rightharpoonup^*$ for weak* convergence and $\sigma(\mathfrak{E}^*, \mathfrak{E}^{**})$-convergence interchangeably, when doing so does not lead to confusion.

5.7 Correspondence

By the following theorem the existence of linear strong asymptotic relations is guaranteed given a linear metric approximation. This procedure is particularly applied to all of the examples in Chapter 6, where explicit metric approximations are presented. The converse is proved in Theorem 5.40.

Theorem 5.38. *Let $\mathscr{N} = \mathbb{N}$ and $\mathfrak{E}$ as above. Suppose that we have a linear metric approximation $\{\Phi_n : D(\Phi_n) \to E_n\}$. Then the Fréchet-topology τ of $\mathfrak{E}$ defined in 5.28 and verified in Theorem 5.32 is a linear strong asymptotic relation on $\mathfrak{E}$. Moreover, if τ' is another Fréchet linear strong asymptotic relation on $\mathfrak{E}$ such that the following compatibility condition holds*

$$\forall u \in C,\ \exists N \in \mathbb{N} :\ u \in D(\Phi_n)\ \forall n \geq N : \quad \tau'\text{-}\lim_{n \geq N} \Phi_n(u) = u, \tag{C}$$

then $\tau' = \tau$.

Such a topology τ is Hausdorff a posteriori by Lemma 5.14 since the proofs of (A3), (A4) and (AL) below work also for nets.

Proof of Theorem 5.38. We will verify (A1)–(A4), (AL) in Definition 5.12 and then prove the uniqueness given condition (C). Recall Cases 1–4 in Definition 5.28.

(A1): E_n, $n \in \mathbb{N}$, E are closed by Case 1 and Case 2 convergence. Since Case 1 convergence is the only one valid for E_n, $n \in \mathbb{N}$, from its definition we deduce that E_n, $n \in \mathbb{N}$ are open in $(\mathfrak{E}, \tau)$. By Case 3 convergence and (A2) below this will not hold for E.

We would like to prove that the relative topologies of E_n, $n \in \mathbb{N}$, E coincide with the original strong topologies (cf. Lemma A.1 (iv)).

Therefore let $O \in \tau$, $n_0 \in \mathbb{N} \cup \{\infty\}$. We want to prove that $O \cap E_{n_0}$ is open in E_{n_0} endowed with the original strong topology. Suppose that $O \cap E_{n_0}$ is non-empty and let $\{x_k\} \subset E_{n_0}$ be a sequence which $\|\cdot\|_{E_{n_0}}$-converges to $x \in O \cap E_{n_0}$. But this means that $\{x_k\}$ converges to x Case 1 or Case 2. Therefore the sequence is eventually in O, but since it is a Case 1 or Case 2 convergence, it is also eventually in E_{n_0}.

Conversely, let $n_0 \in \mathbb{N}$, $O \subset E_{n_0}$ open in the original strong topology. Any Case 1 convergent (other cases do not occur) sequence $\{x_k\}$ converging to a point $x \in O$ is eventually in O by the definition of the convergence. Therefore $O \in \tau$ (and of course $O \cap E_{n_0} = O$).

Let $O \subset E$ open in the original strong topology. Then $O \cup \bigcup_n E_n \in \tau$ by the definition of the convergence (especially Case 2 and Case 4). But clearly $[O \cup \bigcup_n E_n] \cap E = O$.

(A1) is proved.

(A2): Let $u \in E$. We would like to construct a Case 3 convergent sequence $\{u_n\}$ with $u_n \in E_n$, $n \in \mathbb{N}$ and $\tau\text{-}\lim_n u_n = u$. Let $\{\widetilde{u}_m\}, \{\widetilde{\widetilde{u}}_{m'}\} \subset C$ be sequences such that $\lim_m \|\widetilde{u}_m - u\|_E = 0$ and $\lim_{m'} \left\|\widetilde{\widetilde{u}}_{m'} - u\right\|_E = 0$. Set

$$a_n^{m,m'} := \begin{cases} \left\|\Phi_n(\widetilde{u}_m) - \Phi_n(\widetilde{\widetilde{u}}_{m'})\right\|_{E_n}, & \text{if } \widetilde{u}_m, \widetilde{\widetilde{u}}_{m'} \in D(\Phi_n), \\ +\infty, & \text{otherwise.} \end{cases}$$

Then by Lemma 5.27 there exists a subsequence $\{m'_n\}$ of $\{m'\}$ such that

$$\overline{\lim_m}\,\overline{\lim_n}\, a_n^{m,m'_n} \le \overline{\lim_m}\,\overline{\lim_{m'}}\,\overline{\lim_n}\, a_n^{m,m'} = \overline{\lim_m}\,\overline{\lim_{m'}} \left\| \widetilde{u}_m - \widetilde{\widetilde{u}}_{m'} \right\|_E = 0,$$

where we have used (BL) and Lemma 5.31 (ii). Define

$$u_n := \begin{cases} \Phi_n(\widetilde{\widetilde{u}}_{m'_n}) \in E_n, & \text{if } \widetilde{\widetilde{u}}_{m'_n} \in D(\Phi_n), \\ 0 \in E_n, & \text{otherwise.} \end{cases}$$

Then $u_n \xrightarrow[n\to\infty]{} u$ strongly, which is a Case 3 convergence. (A2) is proved.

(A3): Let $\{n_k\}$ be a countable subnet of $\{n\}$. Let $\{u_k\}, \{v_k\} \subset \mathfrak{E}$ with $u_k, v_k \in E_{n_k}$, $k \in \mathbb{N}$. Let $u, v \in E$ and suppose that $\tau\text{-}\lim_k u_k = u$, $\tau\text{-}\lim_k v_k = v$ in the sense of Case 3. Let $\{\widetilde{u}_m\}, \{\widetilde{v}_m\} \subset C$ such that $\lim_m \|\widetilde{u}_m - u\|_E = \lim_m \|\widetilde{v}_m - v\|_E = 0$.

We shall verify that

$$\lim_k \|u_k\|_{E_{n_k}} = \|u\|_E\,,$$

which particularly implies that

$$\sup_k \|u_k\|_{E_{n_k}} < +\infty.$$

Evidently,

$$\Big| \|u_k\|_{E_{n_k}} - \|u\|_E \Big| \le \left\{ \begin{array}{ll} \|u_k - \Phi_{n_k}(\widetilde{u}_m)\|_{E_{n_k}} & , \text{ if } \widetilde{u}_m \in D(\Phi_{n_k}), \\ +\infty & , \text{ otherwise,} \end{array} \right\}$$
$$+ \left\{ \begin{array}{ll} \Big| \|\Phi_{n_k}(\widetilde{u}_m)\|_{E_{n_k}} - \|u\|_E \Big| & , \text{ if } \widetilde{u}_m \in D(\Phi_{n_k}), \\ +\infty & , \text{ otherwise} \end{array} \right\}.$$

The first term tends to zero as $m, k \to \infty$ by Case 3 convergence. Taking $k \to \infty$ in the second term together with (B2) we obtain

$$|\|\widetilde{u}_m\|_E - \|u\|_E| \le \|\widetilde{u}_m - u\|_E\,,$$

which tends to zero as $m \to \infty$; (A3) is proved.

(A4): Let $u_k \in E_{n_k}$ for a countable subnet $\{n_k\}$ of $\{n\}$. We shall prove that $u_k \to 0 \in E$ if and only if $\|u_k\|_{E_{n_k}} \to 0$. Notice that Case 1 and 2 do not occur.

Assume that $\|u_k\|_{E_{n_k}} \to 0$. Set $\widetilde{u}_m = 0 \in \bigcap_n D(\Phi_n) \subset C$ for every $m \in \mathbb{N}$, since zero is in each domain by (BL). Then

$$\lim_m \overline{\lim_k} \|u_k - \Phi_{n_k}(0)\|_{E_{n_k}} = \lim_m \overline{\lim_k} \|u_k\|_{E_k} = 0,$$

hence $u_k \to 0$ strongly Case 3.

Assume now that $u_k \to 0 \in E$ strongly. Then there exists a sequence $\{\widetilde{u}_m\} \subset C$ with $\lim_m \|\widetilde{u}_m\|_E = 0$ and

$$\lim_m \overline{\lim_k} \begin{Bmatrix} \|u_k - \Phi_{n_k}(\widetilde{u}_m)\|_{E_{n_k}} & , \text{ if } \widetilde{u}_m \in D(\Phi_{n_k}), \\ +\infty & , \text{ otherwise} \end{Bmatrix} = 0.$$

Clearly, using (B2),

$$\begin{aligned}\lim_m \overline{\lim_k} \|u_k\|_{E_{n_k}} &\le \lim_m \overline{\lim_k} \begin{Bmatrix} \|u_k - \Phi_{n_k}(\widetilde{u}_m)\|_{E_{n_k}} & , \text{ if } \widetilde{u}_m \in D(\Phi_{n_k}), \\ +\infty & , \text{ otherwise} \end{Bmatrix} \\ &\quad + \lim_m \overline{\lim_k} \begin{Bmatrix} \|\Phi_{n_k}(\widetilde{u}_m)\|_{E_{n_k}} & , \text{ if } \widetilde{u}_m \in D(\Phi_{n_k}), \\ +\infty & , \text{ otherwise} \end{Bmatrix} \\ &= \lim_m \|\widetilde{u}_m\|_E = 0.\end{aligned}$$

Combined with (AL) below (A4) now follows. Note that the proofs of (A3), (A4) and (AL) also work for nets, this is needed for Lemma 5.14.

(AL): Obvious from Lemma 5.30 and (BL).

In order to complete the proof, let τ' be another Fréchet linear asymptotic relation on $\mathfrak{E}$ such that (C) holds. Recall that both τ and τ' are Hausdorff spaces a posteriori and Fréchet by Theorem 5.32 and the assertion respectively. Therefore, if we can prove that each τ-convergent sequence is τ'-convergent to the same point and vice versa, we are done. τ-convergence is divided into Cases 1–4.

Let us prove the first implication. Case 1 and Case 2 convergent sequences also converge in τ' by (A1).

Let $u \in E$, $u_k \to u$ Case 3. Then there is a countable subnet $\{n_k\}$ of $\{n\}$ such that $u_k \in E_{n_k}$ for each $k \in \mathbb{N}$ and a sequence $\{\widetilde{u}_m\} \subset C$ with $\lim_m \|\widetilde{u}_m - u\|_E = 0$ and

$$\lim_m \overline{\lim_k} \begin{Bmatrix} \|\Phi_{n_k}(\widetilde{u}_m) - u_k\|_{E_{n_k}} & , \text{ if } \widetilde{u}_m \in D(\Phi_{n_k}) \\ +\infty & , \text{ otherwise} \end{Bmatrix} = 0. \tag{5.24}$$

By (A2) there is a sequence $\{v_n\}$, $v_n \in E_n$ such that $\tau'\text{-}\lim_n v_n = u$. Also $\{w_k\}$, where $w_k := v_{n_k}$, τ'-converges to u (see Remark 5.11). By (C) for each $\widetilde{u}_m$ there is a number $N_m \in \mathbb{N}$ with $\widetilde{u}_m \in D(\Phi_n)$ for all $n \ge N_m$ and $\tau'\text{-}\lim_{n \ge N_m} \Phi_n(\widetilde{u}_m) = \widetilde{u}_m$. Since $\{n_k\}$ is a countable subnet of $\{n\}$, for each N_m there is an index K with $w_k \in E_n$ for all $n \ge N_m$ and $k \ge K$. Then by (A3')

$$\lim_{k \ge K} \|w_k - \Phi_{n_k}(\widetilde{u}_m)\|_{E_{n_k}} = \|u - \widetilde{u}_m\|_E .$$

By an easy application of Lemma 5.27 and (A3') there is a subsequence $\{m_k\}$ of $\{m\}$ with $\widetilde{u}_{m_k} \in D(\Phi_{n_k})$ and

$$\lim_k \|w_k - \Phi_{n_k}(\widetilde{u}_{m_k})\|_{E_{n_k}} = 0.$$

By (A4) τ'-$\lim_k \Phi_{n_k}(\widetilde{u}_{m_k}) = u$. But by extracting another subsequence, if necessary, (also denoted by $\{m_k\}$) combined with (5.24) we get

$$\lim_k \|u_k - \Phi_{n_k}(\widetilde{u}_{m_k})\|_{E_{n_k}} = 0.$$

Which again by (A4) yields that τ'-$\lim_k u_k = u$.

Case 4 convergence can be managed by dividing into two disjoint subsequences and applying the arguments above.

Let us prove the converse implication. Suppose that $\{u_k\} \subset \mathfrak{E}$ with τ'-$\lim_k u_k = u$ for some $u \in \mathfrak{E}$. Suppose that $\{u_k\}$ is eventually in any of E_n, $n \in \mathbb{N}$, E. Then by (A1) we have Case 1 or Case 2 convergence, which is similar for τ. If $u \in E_{n_0}$, $n_0 \in \mathbb{N}$ this is the case since E_{n_0} is open in τ' by (A1).

Suppose that $u \in E$ and $\{u_k\}$ is eventually in $\mathfrak{E} \setminus E$. Then there is an index $K \in \mathbb{N}$ with $u_k \notin E$ for $k \geq K$ and there is a map $k \mapsto n_k$ with $u_k \in E_{n_k}$ for $k \geq K$. Let $\{\widetilde{u}_m\} \subset C$ with $\lim_m \|\widetilde{u}_m - u\|_E = 0$. By (C) for each m there is a number $N_m \in \mathbb{N}$ such that $\widetilde{u}_m \in D(\Phi_n)$ for all $n \geq N_m$ and τ'-$\lim_n \Phi_n(\widetilde{u}_m) = \widetilde{u}_m$. For each $n' \in \mathbb{N}$, $\bigcup_{n \geq n'} E_n \cup E$ is an open neighborhood of any point in E. Therefore for fixed m by convergence $u_k \in \bigcup_{n \geq N_m} E_n$ for $k \geq K'$, where $K' \geq K$ depends on N_m. The same argument also shows that $\{n_k\}$ is indeed a countable subnet (in particular, (A.1) in the appendix holds; not to be confused with (A1)). Hence by (A3')

$$\overline{\lim_{k \geq K'}} \|\Phi_{n_k}(\widetilde{u}_m) - u_k\|_{E_{n_k}} = \|\widetilde{u}_m - u\|_E ,$$

which by taking the limit $m \to \infty$ proves that τ-$\lim_k u_k = u$ Case 3.

Suppose now that $u \in E$ and $\{u_k\}$ is frequently in E as well as frequently in $\mathfrak{E} \setminus E$. There exists a subsequence $\{u_{k_s}\}$ and a map $s \mapsto n_{k_s}$ (which proves to be a countable subnet of $\{n\}$ by similar arguments as above) of $\{n\}$ such that $u_{k_s} \in E_{n_{k_s}}$ and a disjoint subsequence $\{u_{k'_s}\}$ with $\{u_{k'_s}\} \subset E$. We have divided the proof in two sub-cases which both have been proved above and yield that the τ' convergence is a Case 4 convergence.

The proof is complete. $\square$

Note that τ' in (C) above is always Fréchet if it is a linear asymptotic relation and each E_n, $n \in \mathbb{N}$, E is separable. This follows from Lemma 5.15 and Lemma A.4 (i).

5.7.1 Compatibility of metric approximations

Let $\{\Phi_n^{(i)} : D(\Phi_n^{(i)}) \to E_n\}$, $i = 1, 2$ be two linear metric approximations on $\mathfrak{E}$, let $C_i := \bigcup_n D(\Phi_n^{(i)})$.

Definition 5.39. *We say that* $\{\Phi^{(i)}\}$, $i = 1, 2$ *are* compatible *if they generate the same asymptotic relation on* $\mathfrak{E}$.

If $C_1 \cap C_2$ is dense in E, we can test compatibility via

$$\overline{\mathbb{R}}\text{-}\lim_n \left\{ \begin{array}{ll} \left\|(\Phi_n^{(1)} - \Phi_n^{(2)})(u)\right\|_{E_n} & , \text{ if } u \in D(\Phi_n^{(1)}) \cap D(\Phi_n^{(2)}) \\ +\infty & , \text{ otherwise} \end{array} \right\} = 0 \tag{5.25}$$

for all $u \in C_1 \cap C_2$. This is an easy consequence of Theorem 5.38.

Theorem 5.40. *Let $\mathscr{N} = \mathbb{N}$. Suppose that $\mathfrak{E}$ has a linear strong asymptotic relation τ. If E is separable, then there exists a linear metric approximation $\{\Phi_n\}$ such that* (C) *in Theorem 5.38 holds with τ' replaced by τ. In other words, the topology of the strong convergence generated by $\{\Phi_n\}$ coincides with τ.*

Proof. By [OP75] from the separability of E it follows that there exists a fundamental total biorthogonal sequence $\{(e_i, e_i^*)\}_{i\in\mathbb{N}} \subset E \times E^*$ such that $\sup_{i\in\mathbb{N}} \|e_i\|_E \|e_i^*\|_{E^*} \le M < +\infty$.

By (A2) for each e_i pick a sequence $\{e_n^{(i)}\}$ with $e_n^{(i)} \in E_n$ and $\tau\text{-}\lim_n e_n^{(i)} = e_i$. Set $\Phi_n(e_i) := e_n^{(i)}$ and extend linearly to $\operatorname{lin}\{e_i\} =: D(\Phi_n) =: C$ for any n. (B1) follows since $\{e_i\}$ is total. (B2) follows from (A3) and (AL). $\{\Phi_n : D(\Phi_n) \to E_n\}$ is compatible (i.e., satisfies (C)) by definition. □

Employing Hamel bases, more can be said:

Corollary 5.41. *Let $\mathscr{N} = \mathbb{N}$. Suppose that $\mathfrak{E}$ has a linear strong asymptotic relation τ. Then there exists a linear metric approximation $\{\Phi_n\}$ with $D(\Phi_n) = E$ such that* (C) *in Theorem 5.38 holds with τ' replaced by τ. In other words, the topology of the strong convergence generated by $\{\Phi_n\}$ coincides with τ.*

Proof. Choose a Hamel basis $\mathscr{B}$ for E (which always exists by Zorn's Lemma). Then $\operatorname{lin}\mathscr{B} = E$. The rest of the proof is similar to the proof of Theorem 5.40. Note that we use the Axiom of (uncountable) choice. □

5.7.2 Asymptotic duality

Definition 5.42. *Let $\mathscr{N} = \mathbb{N}$. Suppose that $(\mathfrak{E}, \mathfrak{F})$ is a dual pair consisting of Banach spaces (E_n, F_n), $n \in \mathbb{N}$, (E, F) in duality. Suppose that $\{\Phi_n : D(\Phi_n) \to E_n\}$, $\{\Psi_n : D(\Psi_n) \to F_n\}$ are linear metric approximations on $\mathfrak{E}$, $\mathfrak{F}$ respectively. Set $C := \bigcup_n D(\Phi_n)$ and $D := \bigcup_n D(\Psi_n)$. We say that $\{\Phi_n\}$ and $\{\Psi_n\}$ are* asymptotically dual (asymptotically in duality) *if*

$$\overline{\mathbb{R}}\text{-}\lim_n \left\{ \begin{array}{ll} {}_{E_n}\langle \Phi_n(u), \Psi_n(v)\rangle_{F_n} & , \text{ if } u \in D(\Phi_n), v \in D(\Psi_n) \\ +\infty & , \text{ otherwise} \end{array} \right\} = {}_E\langle u, v\rangle_F$$

for each $u \in C$ and $v \in D$.

Definition 5.43. *Let $\mathscr{N} = \mathbb{N}$. Suppose that $(\mathfrak{E}, \mathfrak{F})$ is a dual pair consisting of Banach spaces (E_n, F_n), $n \in \mathbb{N}$, (E, F) in duality. Suppose that $\mathfrak{E}$, $\mathfrak{F}$ have linear strong asymptotic relations $\tau(\mathfrak{E})$, $\tau(\mathfrak{F})$ respectively. We say that $\mathfrak{E}$ and $\mathfrak{F}$ are* asymptotically dual (asymptotically in duality) *if for any countable subnet $\{n_k\}$ of $\{n\}$, for any two sequences $\{u_k\} \subset \mathfrak{E}$, $\{v_k\} \subset \mathfrak{F}$ and any two points $u \in E$, $v \in F$ such that $u_k \in E_{n_k}$, $v_k \in F_{n_k}$ and $\tau(\mathfrak{E})\text{-}\lim_k u_k = u$, $\tau(\mathfrak{F})\text{-}\lim_k v_k = v$ we have that*

$$\mathbb{R}\text{-}\lim_k {}_{E_{n_k}}\langle u_k, v_k\rangle_{F_{n_k}} = {}_E\langle u, v\rangle_F \,. \tag{5.26}$$

(Compare Remark 5.20).

Note that $(\mathfrak{E}, \mathfrak{F})$ are asymptotically dual if and only if $(\mathfrak{F}, \mathfrak{E})$ are.

Proposition 5.44. *Let $\mathscr{N} = \mathbb{N}$. Suppose that $(\mathfrak{E}, \mathfrak{F})$ is a dual pair consisting of Banach spaces (E_n, F_n), $n \in \mathbb{N}$, (E, F) in duality. Suppose that $\{\Phi_n : D(\Phi_n) \to E_n\}$, $\{\Psi_n : D(\Psi_n) \to F_n\}$ are linear metric approximations on $\mathfrak{E}$, $\mathfrak{F}$ respectively. Set $C := \bigcup_n D(\Phi_n)$ and $D := \bigcup_n D(\Psi_n)$. If $\{\Phi_n\}$ and $\{\Psi_n\}$ are in duality, then the strong linear asymptotic relations generated by them on $\mathfrak{E}$ and $\mathfrak{F}$, respectively, (cf. Theorems 5.32, 5.38) are asymptotically dual. Conversely, if $\mathfrak{E}$ and $\mathfrak{F}$ are asymptotically dual, then any two linear metric approximations compatible to the linear strong asymptotic relations on $\mathfrak{E}$ and $\mathfrak{F}$ resp. are asymptotically in duality.*

Proof. Fix a countable subnet $\{n_k\}$ of $\{n\}$, two sequences $\{u_k\} \subset \mathfrak{E}$, $\{v_k\} \subset \mathfrak{F}$ and two points $u \in E$, $v \in F$ such that $u_k \in E_{n_k}$, $v_k \in F_{n_k}$, $k \in \mathbb{N}$ and such that $\tau(\mathfrak{E})\text{-}\lim_k u_k = u$, $\tau(\mathfrak{F})\text{-}\lim_k v_k = v$. Let $\{\widetilde{u}_m\} \subset C$, $\{\widetilde{v}_m\} \subset D$ with $\lim_m \|\widetilde{u}_m - u\|_E = 0$, $\lim_m \|\widetilde{v}_m - v\|_F = 0$. By Case 3 convergence

$$\lim_m \overline{\lim_k} \begin{Bmatrix} \|\Phi_{n_k}(\widetilde{u}_m) - u_k\|_{E_{n_k}} & , \text{ if } \widetilde{u}_m \in D(\Phi_{n_k}) \\ +\infty & , \text{ otherwise} \end{Bmatrix} = 0,$$

and

$$\lim_m \overline{\lim_k} \begin{Bmatrix} \|\Psi_{n_k}(\widetilde{v}_m) - v_k\|_{F_{n_k}} & , \text{ if } \widetilde{v}_m \in D(\Psi_{n_k}) \\ +\infty & , \text{ otherwise} \end{Bmatrix} = 0.$$

Fix $m \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $\widetilde{u}_m \in D(\Phi_{n_k})$ and $\widetilde{v}_m \in D(\Psi_{n_k})$. We have that:

$$\begin{aligned}
&\left| {}_{E_{n_k}}\langle u_k, v_k\rangle_{F_{n_k}} - {}_E\langle u, v\rangle_F \right| \\
\le &\left| {}_{E_{n_k}}\langle u_k, v_k\rangle_{F_{n_k}} - {}_{E_{n_k}}\langle \Phi_{n_k}(\widetilde{u}_m), \Psi_{n_k}(\widetilde{v}_m)\rangle_{F_{n_k}} \right| \\
&+ \left| {}_{E_{n_k}}\langle \Phi_{n_k}(\widetilde{u}_m), \Psi_{n_k}(\widetilde{v}_m)\rangle_{F_{n_k}} - {}_E\langle \widetilde{u}_m, \widetilde{v}_m\rangle_F \right| \\
&+ \left| {}_E\langle \widetilde{u}_m, \widetilde{v}_m\rangle_F - {}_E\langle u, v\rangle_F \right| \\
\le &\left| {}_{E_{n_k}}\langle u_k - \Phi_{n_k}(\widetilde{u}_m), v_k\rangle_{F_{n_k}} \right| + \left| {}_{E_{n_k}}\langle \Phi_{n_k}(\widetilde{u}_m), v_k - \Psi_{n_k}(\widetilde{v}_m)\rangle_{F_{n_k}} \right| \\
&+ \left| {}_{E_{n_k}}\langle \Phi_{n_k}(\widetilde{u}_m), \Psi_{n_k}(\widetilde{v}_m)\rangle_{F_{n_k}} - {}_E\langle \widetilde{u}_m, \widetilde{v}_m\rangle_F \right| \\
&+ \left| {}_E\langle \widetilde{u}_m - u, \widetilde{v}_m\rangle_F \right| + \left| {}_E\langle u, \widetilde{v}_m - v\rangle_F \right| \\
\le &\|u_k - \Phi_{n_k}(\widetilde{u}_m)\|_{E_{n_k}} \|v_k\|_{F_{n_k}} + \|\Phi_{n_k}(\widetilde{u}_m)\|_{E_{n_k}} \|v_k - \Psi_{n_k}(\widetilde{v}_m)\|_{F_{n_k}} \\
&+ \left| {}_{E_{n_k}}\langle \Phi_{n_k}(\widetilde{u}_m), \Psi_{n_k}(\widetilde{v}_m)\rangle_{F_{n_k}} - {}_E\langle \widetilde{u}_m, \widetilde{v}_m\rangle_F \right| \\
&+ \|\widetilde{u}_m - u\|_E \|\widetilde{v}_m\|_F + \|u\|_E \|\widetilde{v}_m - v\|_F .
\end{aligned}$$

Note that strongly convergent sequences are norm-bounded by (A3). We finish the argument by using the assertion and taking the limit first $k \to \infty$ and then $m \to \infty$.

The converse is clear. $\square$

Theorem 5.45. *Let $\mathscr{N} = \mathbb{N}$. Suppose that $(\mathfrak{E}, \mathfrak{F})$ is a dual pair consisting of Banach spaces (E_n, F_n), $n \in \mathbb{N}$, (E, F) in duality. Suppose that $\mathfrak{E}$, $\mathfrak{F}$ have linear strong asymptotic relations $\tau(\mathfrak{E})$, $\tau(\mathfrak{F})$ respectively. Suppose that $\mathfrak{E}$ and $\mathfrak{F}$ are asymptotically dual. Then the sequential topology defined by the σ-convergence in Definition 5.33 and verified in Theorem 5.37 is a linear $\sigma(\mathfrak{E}, \mathfrak{F})$ asymptotic relation on $\mathfrak{E}$. $\sigma(\mathfrak{E}, \mathfrak{F})$ is a Fréchet topology if each F_n, $n \in \mathbb{N}$, F is separable. The corresponding statement with $\mathfrak{E}$ and $\mathfrak{F}$ interchanged holds. Conversely, the existence of a linear σ-asymptotic relation either on $\mathfrak{E}$ or on $\mathfrak{F}$ implies asymptotic duality whenever $(\mathfrak{E}, \mathfrak{F})$ is merely a dual pair.*

Proof. Denote the topology coming from the σ-convergence in Definition 5.33 by σ. It is sequential by Theorem 5.37 and Fréchet if F_n, $n \in \mathbb{N}$, F are separable again by Theorem 5.37. We will verify (W1)–(W4) and (WL) in Definition 5.18. Recall Cases 1–4 in Definition 5.33.

(W1): E_n, $n \in \mathbb{N}$, E are closed by Case 1 and Case 2 convergence. Since Case 1 convergence is the only one valid for E_n, $n \in \mathbb{N}$, by its definition we deduce that E_n, $n \in \mathbb{N}$ are open in $(\mathfrak{E}, \sigma)$. By Case 3 convergence and (W2) below this will not hold for E.

We would like to prove that the relative topologies of E_n, $n \in \mathbb{N}$, E coincide with the original $\sigma(E_n, F_n)$-, $n \in \mathbb{N}$, $\sigma(E, F)$-topologies (cf. Lemma A.1 (iv)).

Therefore let $O \in \sigma$, $n_0 \in \mathbb{N} \cup \{\infty\}$. We want to prove that $O \cap E_{n_0}$ is open in E_{n_0} endowed with the original $\sigma(E_{n_0}, F_{n_0})$-topology. Suppose that $O \cap E_{n_0}$ is non-empty and let $\{x_k\} \subset E_{n_0}$ be a sequence which $\sigma(E_{n_0}, F_{n_0})$-converges to $x \in O \cap E_{n_0}$. But this means that $\{x_k\}$ converges to x Case 1 or Case 2. Therefore the sequence is eventually in O, but since it is a Case 1 or Case 2 convergence, it is also eventually in E_{n_0}.

Conversely, let $n_0 \in \mathbb{N}$, $O \subset E_{n_0}$ open in the original $\sigma(E_{n_0}, F_{n_0})$-topology. Any Case 1 convergent (other cases do not occur) sequence $\{x_k\}$ converging to a point $x \in O$ is eventually in O by the definition of the convergence. Therefore $O \in \sigma$ (and of course $O \cap E_{n_0} = O$).

Let $O \subset E$ open in the original $\sigma(E, F)$-topology. Then $O \cup \bigcup_n E_n \in \sigma$ by the definition of the convergence (especially Case 2 and Case 4). But clearly $[O \cup \bigcup_n E_n] \cap E = O$.

(W1) is proved.

(W2): Follows from (A2) and (W3) below.

(W3): Follows from asymptotic duality and (W4) below.

(W4): Follows from the definition of Case 3 convergence.

(WL): Obvious from the definition and the bilinearity of the dualizations.

Interchanging the rôles of $\mathfrak{E}$ and $\mathfrak{F}$ completes the proof.

For the converse, consider Remark 5.20. $\square$

We conclude the section with four useful lemmas.

Lemma 5.46. *Let $\mathscr{N} = \mathbb{N}$, let $(\mathfrak{E}, \mathfrak{E}^*)$ be a dual pair of two linear strong asymptotic relations which are asymptotically dual. Suppose that $u_n \in E_n$, $n \in \mathbb{N}$, $u \in E$. If $u_n \rightharpoonup u$ weakly as $n \to \infty$, then $\sup_n \|u_n\|_{E_n} < +\infty$ and*

$$\varliminf_n \|u_n\|_{E_n} \geq \|u\|_E .$$

Proof. The first part was proved in Lemma 5.36. From this it follows that $\varliminf_n \|u_n\|_{E_n} < +\infty$. Extract a subsequence $\{u_{n_k}\}$ with $\lim_k \|u_{n_k}\|_{E_{n_k}} = \varliminf_n \|u_n\|_{E_n}$. Since weak convergence is a sequential convergence, $u_{n_k} \rightharpoonup u$ as $k \to \infty$. By the Hahn-Banach Theorem pick a point $f \in E^*$ with $\|f\|_{E^*} = 1$ and ${}_{E^*}\langle f, u\rangle_E = \|u\|_E$. By (A2) pick $f_n \to^* f$ strongly in $\mathfrak{E}^*$. By (A3) and weak convergence

$$\varliminf_n \|u_n\|_{E_n} = \lim_k \|f_{n_k}\|_{E^*_{n_k}} \lim_k \|u_{n_k}\|_{E_{n_k}} \geq \lim_k {}_{E^*_{n_k}}\langle f_{n_k}, u_{n_k}\rangle_{E_{n_k}} = {}_{E^*}\langle f, u\rangle_E = \|u\|_E ,$$

which proves the assertion. □

Lemma 5.47. *Let $\mathscr{N} = \mathbb{N}$, let $(\mathfrak{E}^*, \mathfrak{E})$ be a dual pair of two linear strong asymptotic relations which are asymptotically dual. Suppose that $f_n \in E^*_n$, $n \in \mathbb{N}$, $f \in E^*$. If $f_n \rightharpoonup^* f$ weakly* as $n \to \infty$, then $\sup_n \|f_n\|_{E^*_n} < +\infty$ and*

$$\varliminf_n \|f_n\|_{E^*_n} \geq \|f\|_{E^*} .$$

Proof. The first part was proved in Lemma 5.36.

Let $\varepsilon > 0$. There is a point $x \in E$ with $\|x\|_E = 1$ such that $|{}_{E^*}\langle f, x\rangle_E| \geq \|f\|_{E^*} - \varepsilon$. By (A2), (AL') pick $x_n \in E_n$, $n \in \mathbb{N}$ such that $x_n \to x$ strongly and $\|x_n\|_{E_n} = 1$ for all $n \in \mathbb{N}$. We have that

$$\lim_n \left| {}_{E^*_n}\langle f_n, x_n\rangle_{E_n} \right| = |{}_{E^*}\langle f, x\rangle_E| \geq \|f\|_{E^*} - \varepsilon.$$

But since there is a natural number $n(\varepsilon) \in \mathbb{N}$ such that

$$\|f_n\|_{E^*_n} \geq \left| {}_{E^*_n}\langle f_n, x_n\rangle_{E_n} \right| \geq \|f\|_{E^*} - 2\varepsilon \quad \forall n \geq n(\varepsilon),$$

it follows that

$$\varliminf_n \|f_n\|_{E^*_n} \geq \|f\|_{E^*} .$$

□

Lemma 5.48. *Let $\mathscr{N} = \mathbb{N}$ and J_n, $n \in \mathbb{N}$, J the normalized duality maps of E_n, $n \in \mathbb{N}$, E respectively. Suppose that $\mathfrak{E}$ has a linear strong asymptotic relation τ and that it is separable. Suppose that $\mathfrak{E}^*$ has a linear weak* asymptotic relation σ^* corresponding to τ. Then for each sequence $\{v_n\} \subset \mathfrak{E}$ with $v_n \in E_n$, $n \in \mathbb{N}$ and each $v \in E$ such that $\tau\text{-}\lim_n v_n = v$ we have that for each $v^*_n \in J_n v_n$, $n \in \mathbb{N}$, $v^* \in Jv$*

$$\lim_n {}_{E^*_n}\langle v^*_n, u_n\rangle_{E_n} = {}_{E^*}\langle v^*, u\rangle_E \tag{5.27}$$

for all $\{u_n\} \subset \mathfrak{E}$, $u_n \in E_n$, $n \in \mathbb{N}$, $u \in E$ with $\tau\text{-}\lim_n u_n = u$.

Proof. Compare [Cio90, Ch. I, Theorem 4.12]. Suppose that (5.27) does not hold. Then by (W4) there exist $v \in E$, an index $N \in \mathbb{N}$, a weak* open set $V^* \subset \mathfrak{E}^*$ with $Jv \subset V^*$ and a sequence $\{v_n\} \subset \mathfrak{E}$ with $v_n \in E_n$ and $\tau\text{-}\lim_n v_n = v$ such that for some $v_n^* \in J_n v_n$ it holds that $v_n^* \notin V^*$ for all $n \geq N$.

Let F_n be the weak* closure of the set $\{v_n^*, v_{n+1}^*, \ldots\}$. Then $F_N \supset F_{N+1} \supset \ldots$, but by $\|v_n^*\|_{E_n^*} = \|J_n v_n\|_{E_n^*} = \|v_n\|_{E_n}$ the sets are bounded and hence compact by Theorem 5.22. By compactness there exists $v^* \in \bigcap_{n=N}^{\infty} F_n$. $v^* \notin V^*$ and, in particular, $v^* \notin Jv$.

From the definition of F_n and v^* if follows that for each $n \geq N$ there is an index $m \geq n$ such that

$$\left| {}_{E^*}\langle v^*, v\rangle_E - {}_{E_k^*}\langle v_k^*, v_k\rangle_{E_k} \right| \leq \frac{1}{n} \quad \text{for all } k \geq m,$$

in other words, there is a subsequence v_{n_m} such that

$$\lim_m \left| {}_{E^*}\langle v^*, v\rangle_E - \|v_{n_m}\|_{E_{n_m}}^2 \right| = 0$$

hence by (A3) and an $\varepsilon/2$-argument

$${}_{E^*}\langle v^*, v\rangle_E = \|v\|^2 . \tag{5.28}$$

But

$$v^* \in F_n \subset \left\{ x^* \in \mathfrak{E}^* \;\middle|\; \|x^*\| \leq \sup_{m \geq n} \|v_m^*\|_{E_m^*} = \sup_{m \geq n} \|v_m\|_{E_m} \right\} \quad \text{for every } n \geq N$$

which leads to $\|v^*\|_{E^*} \leq \overline{\lim}_n \|v_n\|_{E_n} = \|v\|_E$ which combined with (5.28) gives $v^* \in Jv$; the desired contradiction. □

Lemma 5.49. *Let $\mathfrak{E}$ and $\mathfrak{E}^*$ have linear strong asymptotic relations. Suppose that $\mathfrak{E}$ and $\mathfrak{E}^*$ are asymptotically dual. Let $\{\Phi_n^* : D(\Phi_n^*) \to E_n^*\}$ be any linear metric approximation compatible with the linear strong asymptotic relation on $\mathfrak{E}^*$. Then $u_n \rightharpoonup u$ weakly if and only if $\sup_n \|u_n\|_{E_n} < +\infty$ and*

$$\overline{\mathbb{R}}\text{-}\lim_n \begin{Bmatrix} {}_{E_n^*}\langle \Phi_n^*(v), u_n\rangle_{E_n}, & \text{if } v \in D(\Phi_n^*) \\ +\infty, & \text{otherwise,} \end{Bmatrix} = {}_{E^*}\langle v, u\rangle_E$$

for any $v \in C^* = \bigcup_n D(\Phi_n^*)$.

A similar statement holds for the weak convergence.*

Proof. Clear from Lemma 5.35 and compatibility. □

5.8 Asymptotic continuity

Definition 5.50. *Let $\mathscr{N} = \mathbb{N}$. A linear strong duality approximation $\{\Phi_n : D(\Phi_n) \to E_n\}$ is called* asymptotically continuous *if for every $n \in \mathbb{N}$ we have that $D(\Phi_n) = E$, Φ_n is continuous and*

$$\Phi_n(u_n) \xrightarrow[n\to\infty]{} u \quad \textit{strongly}$$

for every sequence $\{u_n\} \subset E$, $u \in E$ with $\lim_n \|u_n - u\|_E = 0$.

Lemma 5.51. *Let $\mathscr{N} = \mathbb{N}$. Let $\{\Phi_n : D(\Phi_n) \to E_n\}$ be a linear metric approximation. Suppose that E is separable and possesses a Schauder basis. Then there exists an asymptotically continuous linear metric approximation $\{\overline{\Phi}_n : E \to E_n\}$ which is compatible with $\{\Phi_n\}$.*

Proof. The idea is similar to that in the proof of [KS08, Lemma 3.7]. Fix a Schauder basis $\mathscr{E} = (e_1, e_2, \dots)$ of E with coefficient functionals $\mathscr{F} = (f_1, f_2, \dots)$ in E^*. Let $P_j(u) := \sum_{i=1}^{j} {}_{E^*}\langle f_i, u\rangle_E \, e_i$, $j \in \mathbb{N}$ be the canonical projections. By Lemma 5.40 we can find a metric approximation $\{\widetilde{\Phi}_n : \operatorname{lin}\mathscr{E} \to E_n\}$ which is compatible with $\{\Phi_n\}$. Define

$$\varepsilon_{j,n} := \sup\left\{ \left| \left\| \widetilde{\Phi}_m(u) \right\|_{E_m} - \|u\|_E \right| \;\middle|\; m \geq n,\ u \in P_j(E) \right\}.$$

For fixed $j \in \mathbb{N}$ we have $\lim_n \varepsilon_{j,n} = 0$. By Lemma 5.27 there is a sequence $j_n \uparrow \infty$ with $\lim_n \varepsilon_{j_n,n} = 0$. Define $\overline{\Phi}_n : E \to E_n$ via $\overline{\Phi}_n := \widetilde{\Phi}_n \circ P_{j_n}$ which are continuous linear operators each as a composition of a continuous linear projection with finite dimensional range and a finite dimensional linear operator. For $u \in E$ we have

$$\begin{aligned}
&\left| \left\| \overline{\Phi}_n(u) \right\|_{E_n} - \|u\|_E \right| \\
&\qquad\qquad \leq \left| \left\| \overline{\Phi}_n(u) \right\|_{E_n} - \|P_{j_n}(u)\|_E \right| + \left| \|P_{j_n}(u)\|_E - \|u\|_E \right| \\
&\qquad\qquad\qquad\qquad \leq \varepsilon_{j_n,n} + \|P_{j_n}(u) - u\|_E \xrightarrow[n\to\infty]{} 0,
\end{aligned}$$

hence $\{\overline{\Phi}_n\}$ is a metric approximation. We would like to verify the asymptotic continuity. Take $\{u_n\} \subset E$, $u \in E$ with $\lim_n \|u_n - u\|_E = 0$. We have that $\lim_n \|P_{j_n}(u) - u\|_E = 0$. By a Theorem due to Banach (cf. [HHZ96, Theorem 237]) $\sup_j \|P_j\|_{\mathscr{L}(E)} < \infty$ and hence also $\lim_n \|P_{j_n}(u_n) - u\|_E = 0$. Therefore $\lim_n \|P_{j_n}(u_n) - P_{j_n}(u)\|_E = 0$. Since $\lim_n \varepsilon_{j_n,n} = 0$ we get by linearity that $\lim_n \left\| \overline{\Phi}_n(u_n) - \overline{\Phi}_n(u) \right\|_{E_n} = 0$. Let τ' be the strong topology on $\mathfrak{E}$ generated by $\{\overline{\Phi}_n\}$. Since by definition τ'-$\lim_n \overline{\Phi}_n(u) = u$ by (A4) and the above we get that τ'-$\lim_n \overline{\Phi}_n(u_n) = u$ which proves asymptotic continuity in τ'. We are left to verify that $\{\overline{\Phi}_n\}$ is compatible with $\{\Phi_n\}$. Let τ be the topology on $\mathfrak{E}$ generated by $\{\Phi_n\}$. We already know that $\{\widetilde{\Phi}_n\}$ generates τ. Let $u \in \operatorname{lin}\mathscr{E}$. We have that $\lim_n \|P_{j_n}(u) - u\|_E = 0$. Together with $\lim_n \varepsilon_{j_n,n} = 0$ we get that $\lim_n \left\| \overline{\Phi}_n(u) - \widetilde{\Phi}_n(u) \right\|_{E_n} = \lim_n \left\| \widetilde{\Phi}_n P_{j_n}(u) - \widetilde{\Phi}_n(u) \right\|_{E_n} = 0$. By Theorem 5.38 $\tau = \tau'$ and compatibility is proved, cf. equation (5.25). □

5.9 Asymptotic reflexivity and weak compactness

Definition 5.52. *Let $\mathscr{N} = \mathbb{N}$. We call a strong linear asymptotic relation on $\mathfrak{E}$* asymptotically reflexive *if $\mathfrak{E}^*$ has a strong linear asymptotic relation, $\mathfrak{E}$ and $\mathfrak{E}^*$ are asymptotically dual and each E_n, $n \in \mathbb{N}$, E is reflexive. We say that $\mathfrak{E}$ is* separable *if each E_n, $n \in \mathbb{N}$, E is separable.*

$\mathfrak{E}$ is asymptotically reflexive if and only if $\mathfrak{E}^*$ is (cf. [HHZ96, Proposition 66]). Then $\mathfrak{E}^*$ is asymptotically dual both to $\mathfrak{E}$ and $\mathfrak{E}^{**}$ and the weak and weak* topologies on $\mathfrak{E}^*$ coincide. If $\mathfrak{E}$ is asymptotically reflexive and separable, $\mathfrak{E}^*$ is separable as well by [Yos78, Ch. V.2, p. 126, Lemma].

Lemma 5.53. *Let $\mathscr{N} = \mathbb{N}$. Suppose that we are given a linear asymptotic relation on $\mathfrak{E}$ which is asymptotically reflexive and separable. Then the weak topology restricted to*

$$\mathfrak{B}_1 := \dot{\bigcup_{n\in\mathbb{N}}} \overline{B}_{E_n}(0,1) \dot{\cup} \overline{B}_E(0,1)$$

is sequentially compact and countably compact. In particular, norm-bounded sequences in $\mathfrak{E}$ have weakly convergent subsequences; the limit point lies in the Banach space determined by the "Convergence Cases" in Definition 5.33.

Proof. The second claim is equivalent to the first one by (WL) (cf. Corollary 5.23); we shall prove the second one.

For Case 1 and 2 convergence the result is classical (cf. [Yos78, p. 126, Chapter V.2, Theorem 1]). Case 4 follows from Case 2 since no uniqueness of the limit is asserted.

Let $\{n_k\}$ be a countable subnet of $\{n\}$ and $u_k \in E_{n_k}$ for $k \in \mathbb{N}$.

Set $M := \sup_k \|u_k\|_{E_{n_k}}$ which is supposed to be finite. Since E and hence E^* is separable (by reflexivity and [Yos78, p. 126, Chapter V.2 Lemma]) we can pick a countable dense subset $E^* \supset S = \{s_1, s_2, \ldots\}$. By (A2) for each $i \in \mathbb{N}$ pick a sequence $\{s_k^i\}$ with $s_k^i \in E^*_{n_k}$ for each $k \in \mathbb{N}$ and $\lim_k s_k^i = s_i$ strongly. Clearly by uniform boundedness of $\{u_k\}$

$$\overline{\lim_k} \; {}_{E^*_{n_k}}\langle s_k^i, u_k\rangle_{E_{n_k}}$$

is finite for every $i \in \mathbb{N}$. By a standard diagonal argument we can select a common subsequence of $\{u_k\}$ (still denoted by $\{u_k\}$) such that

$$\lim_k \; {}_{E^*_{n_k}}\langle s_k^i, u_k\rangle_{E_{n_k}} \tag{5.29}$$

exists and is finite for all $i \in \mathbb{N}$.

Now let $\varepsilon > 0$, $x \in E^*$. Pick an index $i_0 \in \mathbb{N}$ with

$$\|s_{i_0} - x\|_{E^*} < \varepsilon.$$

Pick $x_k \in E_{n_k}$, $k \in \mathbb{N}$ with $\lim_k x_k = x$ strongly in $\mathfrak{E}^*$. Now by (A3') and (5.29)

$$\begin{aligned}
&\left| {}_{E^*_{n_k}}\langle x_k, u_k\rangle_{E_{n_k}} - {}_{E^*_{n_l}}\langle x_l, u_l\rangle_{E_{n_l}} \right| \\
\le &\left| {}_{E^*_{n_k}}\left\langle x_k - s_k^{i_0}, u_k\right\rangle_{E_{n_k}} \right| + \left| {}_{E^*_{n_l}}\left\langle s_l^{i_0} - x_l, u_l\right\rangle_{E_{n_l}} \right| + \left| {}_{E^*_{n_k}}\left\langle s_k^{i_0}, u_k\right\rangle_{E_{n_k}} - {}_{E^*_{n_l}}\left\langle s_l^{i_0}, u_l\right\rangle_{E_{n_l}} \right| \\
\le &\left\| x_k - s_k^{i_0} \right\|_{E^*_{n_k}} \|u_k\|_{E_{n_k}} + \left\| s_l^{i_0} - x_l \right\|_{E^*_{n_l}} \|u_l\|_{E_{n_l}} + \left| {}_{E^*_{n_k}}\left\langle s_k^{i_0}, u_k\right\rangle_{E_{n_k}} - {}_{E^*_{n_l}}\left\langle s_l^{i_0}, u_l\right\rangle_{E_{n_l}} \right| \\
\le &(2M+1)\varepsilon \qquad \text{for large } k, l.
\end{aligned}$$

Hence $\left\{ {}_{E^*_{n_k}}\langle x_k, u_k\rangle_{E_{n_k}} \right\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and thus convergent.

Define $\ell : E^* \to \mathbb{R}$ via

$$\ell : x \mapsto \lim_k \ {}_{E^*_{n_k}}\langle x_k, u_k\rangle_{E_{n_k}} .$$

We claim that ℓ is well-defined and that $\ell \in E^{**}$.
ℓ does not depend on the choice of x_k above by

$$\lim_k \left| {}_{E^*_{n_k}}\langle x_k, u_k\rangle_{E_{n_k}} - {}_{E^*_{n_k}}\langle \widetilde{x}_k, u_k\rangle_{E_{n_k}} \right| \le M \cdot \lim_k \|x_k - \widetilde{x}_k\|_{E^*_{n_k}} = 0,$$

where $\{\widetilde{x}_k\}$ is another sequence with $\widetilde{x}_k \in E_{n_k}$, $k \in \mathbb{N}$ with $\lim_k \widetilde{x}_k = x$ strongly in $\mathfrak{E}^*$. Linearity follows from (AL), the linearity of limits and the bilinearity of the duality bracket. The continuity can be seen as

$$\left| {}_{E^{**}}\langle \ell, x\rangle_{E^*} \right| = \left| \lim_k \ {}_{E^*_{n_k}}\langle x_k, u_k\rangle_{E_{n_k}} \right| = \lim_k \left| {}_{E^*_{n_k}}\langle x_k, u_k\rangle_{E_{n_k}} \right| \le M \|x\|_{E^*} .$$

Hence $\ell \in E^{**}$. Therefore by reflexivity of E there exists $u \in E$ with $\ell = \iota_E(u)$, where $\iota_E : E \to E^{**}$ is the canonical isometry. We have proved now that

$${}_{E^*}\langle x, u\rangle_E = {}_{E^{**}}\langle \ell, x\rangle_{E^*} = \lim_k \ {}_{E^*_{n_k}}\langle x_k, u_k\rangle_{E_{n_k}}$$

for any $\mathfrak{E}^*$-strongly convergent sequence $\{x_k\}$. Since $x \in E^*$ was arbitrary, $u_k \rightharpoonup u$ weakly in $\mathfrak{E}$ by (W4).

$\mathfrak{B}_1$ is a weakly closed subset of the sequential space weak-$\mathfrak{E}$ by (W1), properties of weak convergence in fixed spaces and Lemma 5.46. Therefore $\mathfrak{B}_1$ with the weak topology is a sequential space by Lemma A.4 (iii). Hence sequential compactness implies countable compactness by Lemma A.4 (v). □

5.10 The asymptotic Kadeč-Klee property

Definition 5.54. *Let $\mathscr{N} = \mathbb{N}$. Suppose that both $\mathfrak{E}$ and $\mathfrak{E}^*$ have a linear strong asymptotic relation and that they are asymptotically dual. We say that $\mathfrak{E}$ possesses the* asymptotic Kadeč-Klee property, *if each E_n, $n \in \mathbb{N}$, E has the Kadeč-Klee property (see Appendix B.3) and a sequence $\{u_n\}$, $u_n \in E_n$, $n \in \mathbb{N}$ converges strongly to $u \in E$ if and only if $\{u_n\}$ converges weakly to u and $\lim_n \|u_n\|_{E_n} = \|u\|_E$.*

Lemma 5.55. *Suppose that $\mathfrak{E}$ has the asymptotic Kadeč-Klee property. Let $u_n \in E_n$, $n \in \mathbb{N}$, $u \in E$. Let $\{\Phi^*_n : D(\Phi^*_n) \to E^*_n\}$ be any linear metric approximation compatible with the linear strong asymptotic relation on $\mathfrak{E}^*$. Then $u_n \to u$ strongly if and only if $\lim_n \|u_n\|_{E_n} = \|u\|_E$ and*

$$\overline{\mathbb{R}}\text{-}\lim_n \left\{ \begin{array}{r} {}_{E^*_n}\langle \Phi^*_n(v), u_n\rangle_{E_n}, \ \text{if } v \in D(\Phi^*_n) \\ +\infty, \ \text{otherwise,} \end{array} \right\} = {}_{E^*}\langle v, u\rangle_E$$

for any $v \in C^* = \bigcup_n D(\Phi^*_n)$.

Proof. Clear from Lemma 5.49. □

Definition 5.56. *Let $\mathscr{N} = \mathbb{N}$. We say that a linear asymptotic relation $\mathfrak{E}$ is* asymptotically uniformly convex, *if each E_n, $n \in \mathbb{N}$, E is uniformly convex with respective moduli δ_n, $n \in \mathbb{N}$, δ and*

$$\delta_0(\varepsilon) := \inf_n \delta_n(\varepsilon) > 0 \text{ for each } \varepsilon > 0. \tag{5.30}$$

We say that a linear asymptotic relation $\mathfrak{E}$ is asymptotically uniformly smooth, *if each E_n, $n \in \mathbb{N}$, E is uniformly smooth with respective moduli η_n, $n \in \mathbb{N}$, η and*

$$\eta_0(\varepsilon) := \inf_n \eta_n(\varepsilon) > 0 \text{ for each } \varepsilon > 0. \tag{5.31}$$

It follows from [Bea85, Part 3, Ch. II, §1, Proposition 6, §2, Proposition 2], that either asymptotic uniform convexity or asymptotic uniform smoothness of $\mathfrak{E}$ implies that both $\mathfrak{E}$ and $\mathfrak{E}^*$ are asymptotically reflexive.

Theorem 5.57. *If $\mathfrak{E}$ is asymptotically uniformly convex, it possesses the asymptotic Kadeč-Klee property.*

Proof. The "only if"-part follows from (A3) and (W3).

Suppose that the "if"-part is false. Let $\{u_n\}$, $u_n \in E_n$, $\|u_n\|_{E_n} = 1$, $u \in E$, $\|u\|_E = 1$ such that $u_n \rightharpoonup u$ and $u_n \not\to u$. Let $\{w_n\}$, $w_n \in E_n$, $w_n \to u$ by (A2). Set $v_n := (1/\|w_n\|_{E_n})w_n$ for every $n \in \mathbb{N}$. Then $v_n \to u$ by (A3) and (AL'). Since $u_n \not\to u$ by (A4) there are $\varepsilon > 0$ and $N \in \mathbb{N}$ such that

$$\|u_n - v_n\|_{E_n} \geq \varepsilon \quad \text{for every } n \geq N.$$

By uniform convexity of each E_n

$$\frac{1}{2}\|u_n + v_n\|_{E_n} \leq 1 - \delta_n(\varepsilon).$$

But $\inf_n \delta_n(\varepsilon) \geq \delta_0(\varepsilon) > 0$. Therefore

$$\sup_{n \geq N} \frac{1}{2}\|u_n + v_n\|_{E_n} \leq 1 - \delta_0(\varepsilon), \quad n \geq N. \tag{5.32}$$

Pick $v_n^* \in E_n^*$, $\|v_n^*\|_{E_n^*} = 1$, $n \in \mathbb{N}$ such that $v_n^* \to^* v^*$ strongly in $\mathfrak{E}^*$ where $v^* \in J(u)$; we can do so by (A2), (A3) and (AL). Now by weak convergence,

$$\frac{1}{2}\|u_n + v_n\|_{E_n} \geq \frac{1}{2}\,{}_{E_n^*}\langle v_n^*, u_n + v_n\rangle_{E_n} \to \|u\|_E^2 = 1.$$

Pick $n \geq N$ such that

$$\frac{1}{2}\|u_n + v_n\|_{E_n} \geq 1 - \delta_0(\varepsilon)/2,$$

which contradicts (5.32). The "if"-part is proved. □

Proposition 5.58. *Let $\mathfrak{E}^*$ possess the asymptotic Kadeč-Klee property and let $\mathfrak{E}$ be asymptotically reflexive and separable. Then $\mathfrak{E}$ possesses* asymptotically strongly continuous duality mappings*, that is,*

$$v_n \to^* v \quad \textit{whenever} \quad u_n \in E_n,\ n \in \mathbb{N},\ u \in E,\ u_n \to u,\ v_n \in J_n(u_n),\ n \in \mathbb{N},\ v \in J(u).$$

Proof. Let $u_n \in E_n$, $n \in \mathbb{N}$, $u \in E$, $u_n \to u$ strongly. Let $v_n \in J_n(u_n)$, $n \in \mathbb{N}$, $v \in J(u)$. By Lemma 5.48 $v_n \rightharpoonup^* v$ weakly* and hence weakly in $\mathfrak{E}^*$ by asymptotic reflexivity. But $\|v_n\|_{E_n^*} = \|u_n\|_{E_n} \to \|u\|_E = \|v\|_{E^*}$. □

Lemma 5.59. *Let $\mathfrak{E}$ as well as $\mathfrak{E}^*$ possess the asymptotic Kadeč-Klee property and be asymptotically reflexive and separable. Let $x_n \in E_n$, $y_n \in J_n(x_n)$, $n \in \mathbb{N}$, $x \in E$, $y \in E^*$ such that*

$$x_n \rightharpoonup x \quad \textit{weakly in } \mathfrak{E} \textit{ as } n \to \infty, \tag{5.33}$$

$$y_n \rightharpoonup^* y \quad \textit{weakly in } \mathfrak{E}^* \textit{ as } n \to \infty, \tag{5.34}$$

$$\lim_n {}_{E_n^*}\langle y_n, x_n\rangle_{E_n} = {}_{E^*}\langle y, x\rangle_E\,. \tag{5.35}$$

Then $y \in J(x)$, $x_n \to x$ strongly in $\mathfrak{E}$ and $y_n \to^* y$ strongly in $\mathfrak{E}^*$ as $n \to \infty$.

Proof. Compare with the proof of Lemma 7.22 below.

Let $\{x_n\}$, $\{y_n\}$, x, y be as in the assertion. Recall that each J_n, $n \in \mathbb{N}$, J is a maximal monotone operator by Lemma B.14.

Let $u \in E$, $v \in J(u)$. Let $u_n \in E_n$, $n \in \mathbb{N}$ with $u_n \to u$ strongly in $\mathfrak{E}$. Let $v_n \in J_n(u_n)$, $n \in \mathbb{N}$, $v \in J(u)$. Then by Proposition 5.58 $v_n \to v$ strongly in $\mathfrak{E}^*$. Now, since each J_n is monotone,

$${}_{E_n^*}\langle y_n, x_n\rangle_{E_n} - {}_{E_n^*}\langle v_n, x_n\rangle_{E_n} - {}_{E_n^*}\langle y_n - v_n, u_n\rangle_{E_n} = {}_{E_n^*}\langle y_n - v_n, x_n - u_n\rangle_{E_n} \ge 0.$$

Passing on to the limit,

$${}_{E^*}\langle y, x\rangle_E - {}_{E^*}\langle v, x\rangle_E - {}_{E^*}\langle y - v, u\rangle_E \ge 0,$$

so

$${}_{E^*}\langle y - v, x - u\rangle_E \ge 0,$$

which is true for all $[u, v] \in J$. Hence by maximal monotonicity $[x, y] \in J$.

Clearly,

$$\|y_n\|^2_{E_n^*} = \|x_n\|^2_{E_n} = {}_{E_n^*}\langle y_n, x_n\rangle_{E_n} \to {}_{E^*}\langle y, x\rangle_E = \|y\|^2_{E^*} = \|x\|^2_E$$

as $n \to \infty$. The claim follows from the asymptotic Kadeč-Klee property of $\mathfrak{E}$ and $\mathfrak{E}^*$ respectively. □

Lemma 5.60. *Suppose that $\mathfrak{E}$, as well as $\mathfrak{E}^*$, has the asymptotic Kadeč-Klee property and that it is asymptotically reflexive and separable. Then $u_n \to u$ strongly, if and only if*

$$\lim_n {}_{E_n^*}\langle v_n, u_n\rangle_{E_n} = {}_{E^*}\langle v, u\rangle_E$$

for all $v_n \rightharpoonup v$ weakly.

Proof. The "only if"-part is trivial. For the "if"-part, recognize that $u_n \rightharpoonup u$ weakly. Therefore $\sup_n \|u_n\|_{E_n} < +\infty$ by Lemma 5.36. By Lemma 5.53 a subsequence of $\{J_n(u_n)\}$ converges weakly to some $\widetilde{v} \in E^*$. But by Lemma 5.59 $\widetilde{v} = Ju$. Since we can do so with any subsequence, we have for the whole sequence $\lim_n \|u_n\|_{E_n} = \|u\|_E$. The claim is proved. □

5.11 Asymptotic embeddings

Let $\mathscr{N} = \mathbb{N}$. Let X_n, V_n, $n \in \mathbb{N}$, X, V be a separable reflexive Banach spaces such that

$$V_n \overset{i_n}{\hookrightarrow} X_n, \ \ n \in \mathbb{N}, \ \ V \overset{i}{\hookrightarrow} X$$

are embedded densely, linearly and continuously with continuous linear embeddings i_n, $n \in \mathbb{N}$, i respectively. Set $M_n := \|i_n\|_{\mathscr{L}(V_n,X_n)} = \|i_n^*\|_{\mathscr{L}(X_n^*,V_n^*)}$, $n \in \mathbb{N}$, $M := \|i\|_{\mathscr{L}(V,X)} = \|i^*\|_{\mathscr{L}(X^*,V^*)}$, the continuity constants of the embeddings.

Suppose that $\mathfrak{V} := \dot{\bigcup}_{n\in\mathbb{N}} V_n \dot{\cup} V$ and $\mathfrak{X} := \dot{\bigcup}_{n\in\mathbb{N}} X_n \dot{\cup} X$ both have linear strong asymptotic relations.

Definition 5.61. *Under the assumptions above, we say that the embedding $\mathfrak{V} \hookrightarrow \mathfrak{X}$ is* asymptotically strong *if*

$$\begin{aligned} &i_n(v_n) \to i(v) \ \ \textit{strongly in}\, \mathfrak{X} \\ &\textit{for all} \ \ v_n \in V_n, \ n \in \mathbb{N}, \ v \in V, \ v_n \to v, \ \textit{strongly in}\, \mathfrak{V}. \end{aligned}$$

We say that the embedding $\mathfrak{V} \hookrightarrow \mathfrak{X}$ is asymptotically weak *if each i_n, $n \in \mathbb{N}$, i is a weakly continuous linear map and*

$$\begin{aligned} &i_n(v_n) \to i(v) \ \ \textit{weakly in}\, \mathfrak{X} \\ &\textit{for all} \ \ v_n \in V_n, \ n \in \mathbb{N}, \ v \in V, \ v_n \to v, \ \textit{weakly in}\, \mathfrak{V}. \end{aligned}$$

We say that the embedding $\mathfrak{V} \hookrightarrow \mathfrak{X}$ is asymptotically compact *if each i_n, $n \in \mathbb{N}$, i is a compact linear map and*

$$\begin{aligned} &i_n(v_n) \to i(v) \ \ \textit{strongly in}\, \mathfrak{X} \\ &\textit{for all} \ \ v_n \in V_n, \ n \in \mathbb{N}, \ v \in V, \ v_n \to v, \ \textit{weakly in}\, \mathfrak{V}. \end{aligned}$$

By the discussion in Paragraph 2.5.1 the linear adjoints i_n^*, $n \in \mathbb{N}$, i^* of i_n, $n \in \mathbb{N}$, i, respectively, form an embedding $\mathfrak{X}^* \hookrightarrow \mathfrak{V}^*$, on the topological dual spaces. The continuity constants are equal to M_n, $n \in \mathbb{N}$, M respectively. For simplicity we shall assume that $\mathfrak{X}^*$ as well as $\mathfrak{V}^*$ has a linear strong asymptotic relation which is asymptotically dual to that of $\mathfrak{X}$, $\mathfrak{V}$ respectively.

Lemma 5.62. *If the embedding $\mathfrak{V} \hookrightarrow \mathfrak{X}$ is asymptotically strong, then $\underline{\lim}_n M_n \geq M$, moreover, if it is asymptotically compact, then $\lim_n M_n = M$.*

Proof. See Lemma 7.6 in Chapter 7. □

We prove an easy lemma:

Lemma 5.63. *$\mathfrak{V} \hookrightarrow \mathfrak{X}$ is asymptotically weak if $\mathfrak{X}^* \hookrightarrow \mathfrak{V}^*$ is asymptotically strong. If, in addition, $\mathfrak{V}$ and $\mathfrak{V}^*$ have the asymptotic Kadeč-Klee property, $\mathfrak{V} \hookrightarrow \mathfrak{X}$ is asymptotically weak only if $\mathfrak{X}^* \hookrightarrow \mathfrak{V}^*$ is asymptotically strong.*

Proof. Let $v_n \in V_n$, $n \in \mathbb{N}$, $v \in V$, $v_n \rightharpoonup v$ weakly. Let $x_n^* \in X_n^*$, $n \in \mathbb{N}$, $x^* \in X^*$, $x_n^* \to^* x^*$ strongly. By assumption, $\lim_n {}_{V_n^*}\langle i_n^*(x_n^*), v_n\rangle_{V_n} = {}_{V^*}\langle i^*(x^*), v\rangle_V$. Consequently, $\lim_n {}_{X_n^*}\langle x_n^*, i_n(v_n)\rangle_{X_n} = {}_{X^*}\langle x^*, i(v)\rangle_X$, which is the first part.

For the second part, let again $v_n \in V_n$, $n \in \mathbb{N}$, $v \in V$, $v_n \rightharpoonup v$ weakly and let $x_n^* \in X_n^*$, $n \in \mathbb{N}$, $x^* \in X^*$, $x_n^* \to^* x^*$ strongly. By assumption, $\lim_n {}_{X_n^*}\langle x_n^*, i_n(v_n)\rangle_{X_n} = {}_{X^*}\langle x^*, i(v)\rangle_X$ and hence $\lim_n {}_{V_n^*}\langle i_n^*(x_n^*), v_n\rangle_{V_n} = {}_{V^*}\langle i^*(x^*), v\rangle_V$. The claim follows with Lemma 5.60. ☐

And its converse:

Lemma 5.64. *$\mathfrak{V} \hookrightarrow \mathfrak{X}$ is asymptotically strong only if $\mathfrak{X}^* \hookrightarrow \mathfrak{V}^*$ is asymptotically weak. If, in addition, $\mathfrak{X}$ and $\mathfrak{X}^*$ have the asymptotic Kadeč-Klee property, $\mathfrak{V} \hookrightarrow \mathfrak{X}$ is asymptotically strong if $\mathfrak{X}^* \hookrightarrow \mathfrak{V}^*$ is asymptotically weak.*

Proof. Let $v_n \in V_n$, $n \in \mathbb{N}$, $v \in V$, $v_n \to v$ strongly and let $x_n^* \in X_n^*$, $n \in \mathbb{N}$, $x^* \in X^*$, $x_n^* \rightharpoonup^* x^*$ weakly. By assumption, $\lim_n {}_{X_n^*}\langle x_n^*, i_n(v_n)\rangle_{X_n} = {}_{X^*}\langle x^*, i(v)\rangle_X$ and hence $\lim_n {}_{V_n^*}\langle i_n^*(x_n^*), v_n\rangle_{V_n} = {}_{V^*}\langle i^*(x^*), v\rangle_V$, which is the first part.

For the second part, let again $v_n \in V_n$, $n \in \mathbb{N}$, $v \in V$, $v_n \rightharpoonup v$ strongly and let $x_n^* \in X_n^*$, $n \in \mathbb{N}$, $x^* \in X^*$, $x_n \rightharpoonup^* x$ weakly. By assumption, $\lim_n {}_{V_n^*}\langle i_n^*(x_n^*), v_n\rangle_{V_n} = {}_{V^*}\langle i^*(x^*), v\rangle_V$ and hence $\lim_n {}_{X_n^*}\langle x_n^*, i_n(v_n)\rangle_{X_n} = {}_{X^*}\langle x^*, i(v)\rangle_X$. The claim follows with Lemma 5.60. ☐

By a diagonal argument, we have an approximation result for asymptotic embeddings:

Lemma 5.65. *Suppose that $\mathfrak{V} \hookrightarrow \mathfrak{X}$ is a strong (weak) asymptotic embedding. Then for each $x \in X$ there exists a sequence $\{x_n\}$ such that $x_n \in i_n(V_n) \subset X_n$ for $n \in \mathbb{N}$ and $x_n \to x$ ($x_n \rightharpoonup x$) strongly (weakly) in $\mathfrak{X}$.*

Proof. Suppose that $\mathfrak{V} \hookrightarrow \mathfrak{X}$ is asymptotically strong. Let $x \in X$, let $\{v_m\}$ be a sequence in V such that $\|i(v_m) - x\|_X \to 0$. Pick for each v_m, $v_n^m \in V_n$, $n \in \mathbb{N}$ such that $v_n^m \to v_m$ in $\mathfrak{V}$ for each $m \in \mathbb{N}$. Then $i_n(v_n^m) \to i(v_m)$ in $\mathfrak{X}$. Let $x_n \in X_n$, $n \in \mathbb{N}$ with $x_n \to x$ strongly in $\mathfrak{X}$ (by (A2)). Then by (A3')

$$\lim_n \|x_n - i_n(v_n^m)\|_{X_n} = \|x - i(v_m)\|_X \,.$$

Hence by Lemma 5.27 there is a sequence of natural numbers $m_n \uparrow \infty$ such that

$$\overline{\lim_n} \|x_n - i_n(v_n^{m_n})\|_{X_n} = 0$$

and therefore $i_n(v_n^{m_n}) \to x$ strongly in $\mathfrak{X}$.

For the weak part fix a countable dense set $\{s_1, s_2, \ldots\} \subset X^*$. By (A2) pick for each $s_k \in X^*$, $k \in \mathbb{N}$, $s_n^k \in X_n^*$, $n \in \mathbb{N}$, $s_n^k \to^* s_k$ strongly. Let $x \in X$. Let $\{v_m\}$ be a sequence in V such that $i(v_m) \rightharpoonup x$ weakly in X (since i is weakly continuous). For each v_m, $m \in \mathbb{N}$, by (W2) pick $v_n^m \in V_n$, $n \in \mathbb{N}$ with $v_n^m \rightharpoonup v_m$ weakly in $\mathfrak{V}$ for each $m \in \mathbb{N}$. Hence

$$\overline{\lim_n} \left| {}_{X_n^*}\left\langle s_n^k, i_n(v_n^m) \right\rangle_{X_n} - {}_{X^*}\langle s_k, i(v_m)\rangle_X \right| = 0 \quad \forall k$$

and

$$\overline{\lim_m} \left| {}_{X^*}\langle s_k, i(v_m) - x\rangle_X \right| = 0 \quad \forall k.$$

By Lemma 5.27 for each $k \in \mathbb{N}$ there is a sequence of natural numbers $m_n^k \uparrow \infty$ (as $n \to \infty$) such that

$$\overline{\lim_n} \left| {}_{X_n^*}\left\langle s_n^k, i_n(v_n^{m_n^k}) \right\rangle_{X_n} - {}_{X^*}\langle s_k, x\rangle_X \right| = 0 \quad \forall k.$$

By a standard diagonal procedure we can find a sequence $m_n' \uparrow \infty$ such that

$$\overline{\lim_n} \left| {}_{X_n^*}\left\langle s_n^k, i_n(v_n^{m_n'}) \right\rangle_{X_n} - {}_{X^*}\langle s_k, x\rangle_X \right| = 0$$

for all k at the same time. Hence by Lemma 5.35 $i_n(v_n^{m_n'}) \rightharpoonup x$ weakly in $\mathfrak{X}$. $\square$

5.11.1 Embeddings via metric approximations

Lemma 5.66. *Let $V_n \hookrightarrow X_n$, $n \in \mathbb{N}$, $V \hookrightarrow X$ and suppose that $\mathfrak{V}$ and $\mathfrak{X}$ have metric approximations $\{(\Phi_n^{\mathfrak{V}}, D(\Phi_n^{\mathfrak{V}}))\}$, $\{(\Phi_n^{\mathfrak{X}}, D(\Phi_n^{\mathfrak{X}}))\}$ respectively, and set $C^{\mathfrak{V}} := \bigcup_n D(\Phi_n^{\mathfrak{V}})$ and $C^{\mathfrak{X}} := \bigcup_n D(\Phi_n^{\mathfrak{X}})$. Suppose now that $\widetilde{C} := i(C^{\mathfrak{V}}) \cap C^{\mathfrak{X}}$ is dense in X and that $i^{-1}(\widetilde{C})$ is dense in V. Suppose that for any $z \in i^{-1}(\widetilde{C})$*

$$\lim_n \left\{ \begin{array}{ll} \left\| i_n \Phi_n^{\mathfrak{V}}(z) - \Phi_n^{\mathfrak{X}}(i(z)) \right\|_{X_n} & , \text{ if } z \in D(\Phi_n^{\mathfrak{V}}),\ i(z) \in D(\Phi_n^{\mathfrak{X}}) \\ +\infty & , \text{ otherwise} \end{array} \right\} = 0. \tag{5.36}$$

Suppose also that the embedding constants M_n are uniformly bounded. Then the embedding $\mathfrak{V} \hookrightarrow \mathfrak{X}$ is asymptotically strong.

Proof. Let $v_n \in V_n$, $n \in \mathbb{N}$, $v \in V$, $v_n \to v$ strongly in $\mathfrak{V}$. Let $\widetilde{v}_m \in i^{-1}(\widetilde{C})$, $m \in \mathbb{N}$ such that $\lim_m \|\widetilde{v}_m - v\|_V = 0$ and $\lim_m \|i(\widetilde{v}_m) - i(v)\|_X = 0$. Then,

$$
\begin{aligned}
&\left\{ \begin{array}{ll} \left\|\Phi_n^{\mathfrak{X}}(i(\widetilde{v}_m)) - i_n(v_n)\right\|_{X_n} & , \text{ if } i(\widetilde{v}_m) \in D(\Phi_n^{\mathfrak{X}}) \\ +\infty & , \text{ otherwise} \end{array} \right\} \\
\leq &\left\{ \begin{array}{ll} \left\|\Phi_n^{\mathfrak{X}}(i(\widetilde{v}_m)) - i_n\Phi_n^{\mathfrak{V}}(\widetilde{v}_m)\right\|_{X_n} & , \text{ if } \widetilde{v}_m \in D(\Phi_n^{\mathfrak{V}}),\ i(\widetilde{v}_m) \in D(\Phi_n^{\mathfrak{X}}) \\ +\infty & , \text{ otherwise} \end{array} \right\} \\
+ &\left\{ \begin{array}{ll} \left\|i_n\Phi_n^{\mathfrak{V}}(\widetilde{v}_m) - i_n(v_n)\right\|_{X_n} & , \text{ if } \widetilde{v}_m \in D(\Phi_n^{\mathfrak{V}}),\ i(\widetilde{v}_m) \in D(\Phi_n^{\mathfrak{X}}) \\ +\infty & , \text{ otherwise} \end{array} \right\} \\
\leq &\left\{ \begin{array}{ll} \left\|\Phi_n^{\mathfrak{X}}(i(\widetilde{v}_m)) - i_n\Phi_n^{\mathfrak{V}}(\widetilde{v}_m)\right\|_{X_n} & , \text{ if } \widetilde{v}_m \in D(\Phi_n^{\mathfrak{V}}),\ i(\widetilde{v}_m) \in D(\Phi_n^{\mathfrak{X}}), \\ +\infty & , \text{ otherwise} \end{array} \right\} \\
+ &\left(\sup_n M_n\right) \cdot \left\{ \begin{array}{ll} \left\|\Phi_n^{\mathfrak{V}}(\widetilde{v}_m) - v_n\right\|_{V_n} & , \text{ if } \widetilde{v}_m \in D(\Phi_n^{\mathfrak{V}}),\ i(\widetilde{v}_m) \in D(\Phi_n^{\mathfrak{X}}) \\ +\infty & , \text{ otherwise} \end{array} \right\}
\end{aligned}
$$

which tends to zero by strong convergence and (5.36). □

5.12 A useful isometric result

In this section, we would like to discuss an abstract existence result for metric approximations (given asymptotic relations) which consist of surjective isometries between the limit space E and the approximating spaces E_n. We introduce the highly technical condition (I) (see below) which guarantees, among the assumption of existence of Schauder bases and isometric comparability, that such a metric approximation always exists. Condition (I) has been verified for separable Hilbert spaces and atomless separable L^p-spaces ($1 < p < \infty$), cf. Lemma 6.2 and Proposition 6.3 below. There is some indication that it might hold for a subclass of Orlicz spaces as well, but this is still an open question.

The use of the central result 5.68 is revealed in Section 7.1, where a general lifting method is presented. It allows us to transfer classical results on convergence of particular classes of functionals or (possibly nonlinear) operators, as e.g. the Trotter Theorem, to the situation of linear asymptotic relations. The general procedure was noticed by a referee in 2007, when the isometry-result was known only for Hilbert spaces. Another consequence is the complete metrization of $\mathfrak{E}$, which then turns out to be isometric to $\overline{\mathbb{N}} \times E$, cf. Proposition 5.73.

The proof of Theorem 5.68 is inspired by A. V. Kolesnikov's proof in [Kol05, Proposition 7.2] (see also [Töl06, Theorem 2.10]), but requires a somewhat finer reasoning in parts of it. We would like to point out that, different from Kolesnikov's proof, no existence of an injective metric approximation or a basis consisting of smooth vectors is assumed. As another feature, our proof works in general reflexive Banach spaces, not only Hilbert spaces. We collect the necessary facts from the theory of bases in Banach spaces in Appendix B.4.

Definition 5.67. *Let $\mathscr{N} = \mathbb{N}$. Suppose that $\mathfrak{E}$ has a linear strong asymptotic relation.*

We say that $\mathfrak{E}$ is asymptotically isometric *if there exists a compatible asymptotically continuous linear metric approximation $\{\Psi_n\}$ such that for each $n \in \mathbb{N}$, $\Psi_n : E \to E_n$ is surjective linear isometry.*

We say that $\mathfrak{E}$ possesses property (I) *if there exist surjective isometries $\{\mathscr{I}_n : E \to E_n\}$ and E admits a monotone normal Schauder basis $(e_i)_{i=1}^{\infty}$ such that for each $n, k, d \in \mathbb{N}$, $k \leq d$, and each subsequence $\{i_j\}_{j=1}^{l}$ of $\{i\}_{i=1}^{k}$, $l \leq k \leq d$ such that $\{i'_j\}_{j=1}^{d-l}$ is the complementary subsequence, the following statement holds:*
There exists a another linear basis $(f_i)_{i=1}^{k}$ of the finite dimensional space $\operatorname{lin}(e_i)_{i=1}^{k}$ such that:

Whenever $M := \operatorname{lin}(f_{i_j})_{j=1}^{l}$ is isometrically isomorphic (with isometry $\Phi_{n,k,d,l}$) to a closed linear subspace X of $Z := \operatorname{lin}(\mathscr{I}_n e_i)_{i=1}^{d}$, then X is complemented in Z with $X \oplus Y = Z$ and Y is isometrically isomorphic to the complement N, where $M \oplus N = \operatorname{lin}(e_i)_{i=1}^{d}$. Furthermore for n, k, d there exist surjective isometries $\Psi_{n,k,d,l} : E \to E_n$ such that $\Psi_{n,k,d,l}\restriction_M = \Phi_{n,k,d,l}$.

As already mentioned, we shall later see that separable Hilbert spaces and separable atomless L^p-spaces $(1 < p < \infty)$ always satisfy property (I), cf. Lemma 6.2 and Proposition 6.3 below.

Theorem 5.68. *Let $\mathscr{N} = \mathbb{N}$. Suppose that $\mathfrak{E}$, $\mathfrak{E}^*$ have linear strong asymptotic relations in duality. Suppose that $\mathfrak{E}$, as well as $\mathfrak{E}^*$, is asymptotically reflexive, separable and possesses property* (I).

Then $\mathfrak{E}$ and $\mathfrak{E}^*$ are asymptotically isometric.

We shall need a couple of lemmas first. They are all new.

Lemma 5.69. *Let $\mathscr{N} = \mathbb{N}$. Suppose that $\mathfrak{E}$ has a linear strong asymptotic relation. Let E be separable and reflexive. Suppose that E possesses a normal monotone Schauder basis $\mathscr{E} = (e_i)_{i=1}^{\infty}$. Suppose that for each $n \in \mathbb{N}$ there is a surjective isometry $\mathscr{I}_n : E \to E_n$. Then there exists a compatible linear metric approximation $\Phi_n : \operatorname{lin} \mathscr{E} \to E_n$ and a sequence of natural numbers $\{d_n\}$ increasing to $+\infty$ such that*

$$\operatorname{ran}(\Phi_n) \subset \operatorname{ran}\left(\mathscr{I}_n\restriction_{\operatorname{lin}(e_i)_{i=1}^{d_n}}\right).$$

Proof. Let $\{P_d\}_{d=1}^{\infty}$ be the associated projections to $\mathscr{E} = (e_i)_{i=1}^{\infty}$, cf. Definition B.21. By (A2) for each $i \in \mathbb{N}$ pick $e_n^{(i)} \in E_n$, $n \in \mathbb{N}$, such that $e_n^{(i)} \to e_i$. For each $n \in \mathbb{N}$, $(\mathscr{I}_n(e_i))_{i=1}^{\infty}$ is a monotone basis for E_n. Denote by $\{P_d^{(n)}\}_{d=1}^{\infty}$ the associated projections. Since for every $i \in \mathbb{N}$

$$\lim_{d\to\infty} \left\| P_d^{(n)} e_n^{(i)} - e_n^{(i)} \right\|_{E_n} = 0,$$

we can pick for each $n \in \mathbb{N}$ a number $d'_n \in \mathbb{N}$ such that

$$\left\| P_{d'_n}^{(n)} e_n^{(i)} - e_n^{(i)} \right\|_{E_n} \leq \frac{1}{n} \tag{5.37}$$

for all $1 \le i \le n$. Set $d_n := \max\{d'_1, \dots, d'_n\}$ for each $n \in \mathbb{N}$.

For $e_i \in \mathscr{E}$ define $\Phi_n(e_i) := P^{(n)}_{d_n}(e^{(i)}_n)$ and extend linearly to $\operatorname{lin}\mathscr{E}$. It is a linear metric approximation by (5.37) and (A3). Compatibility follows from (5.37) and (A4). □

Lemma 5.70. *Suppose that the conditions of Lemma 5.69 hold. Let (e_i), $\{\Phi_n\}$, $\{d_n\}$ be as in Lemma 5.69. Fix $k \in \mathbb{N}$ and set $\Phi^k_n := \Phi_n\restriction_{\operatorname{lin}(e_i)^k_{i=1}}$. Then $\lim_n \dim \operatorname{ran} \Phi^k_n = k$.*

Proof. We prove the lemma by contradiction. Suppose that $l^k_n := \dim \operatorname{ran} \Phi^k_n \not\to k$. Then there is a subsequence (denoted also by $\{l^k_n\}$) and an index $N_0 \in \mathbb{N}$ such that $l^k_n = k_0$ for $n \ge N_0$ and some $0 \le k_0 < k$. By linear dependence there exist $\alpha^n_i \in \mathbb{R}$ with some $\alpha^n_i \neq 0$ for each $n \ge N_0$ such that $\sum_{i=1}^k \alpha^n_i \Phi^k_n(e_i) = 0$.

If $k_0 = 0$, $k = 1$ we can pick $\alpha^n_1 = 1$ for all $n \ge N_0$. By (B2) and (BL), $0 = \lim_n \alpha^n_1 \Phi^1_n(e_1) = e_1$ which is a contradiction.

Therefore suppose that $k \ge 2$. We prove by induction that $(\alpha^n_1, \dots, \alpha^n_k)$ is contained in a compact subset of $\mathbb{R}^k$ away from zero uniformly in n.

Let $k = 2$:
We must have $\alpha^n_1 \Phi^2_n(e_1) = -\alpha^n_2 \Phi^2_n(e_2)$ for all n. W.l.o.g. $\alpha^n_1 = 1$ for all n. Fix $\varepsilon > 0$. There is $N_1 \ge N_0$ such that $1 - \varepsilon \le \left\|\Phi^2_n(e_i)\right\|_{E_n} \le 1 + \varepsilon$ for $i = 1, 2$, for all $n \ge N_1$. Therefore $\frac{1-\varepsilon}{1+\varepsilon} \le |\alpha^n_2| \le \frac{1+\varepsilon}{1-\varepsilon}$ for all $n \ge N_1$.

$k \mapsto k + 1$:
Assume that $0 < A \le \min(\alpha^n_1, \dots, \alpha^n_k) \le \max(\alpha^n_1, \dots, \alpha^n_k) \le B < \infty$ for all $n \ge N_1 \ge N_0$ and constants A, B independent of n. Fix $\varepsilon > 0$. Since the basis (e_i) is normalized, there is $N_2 \ge N_1$ such that $1 - \varepsilon \le \left\|\Phi^{k+1}_n(e_i)\right\|_{E_n} \le 1 + \varepsilon$ for all $1 \le i \le k + 1$ and $n \ge N_2$.

$$
\begin{aligned}
&|\alpha^n_{k+1}| \le |\alpha^n_{k+1}| \left[\left\|\Phi^{k+1}_n(e_{k+1})\right\|_{E_n} + \varepsilon\right] \\
&= \left\|\sum_{i=1}^k \alpha^n_i \Phi^{k+1}_n(e_i)\right\|_{E_n} + \varepsilon|\alpha^n_{k+1}| \le Bk(1+\varepsilon) + \varepsilon|\alpha^n_{k+1}|.
\end{aligned}
$$

Hence $|\alpha^n_{k+1}| \le Bk\frac{1+\varepsilon}{1-\varepsilon}$ for $n \ge N_2$.

Furthermore, note that since by monotonicity $\{e_1\} \perp \operatorname{lin}(e_i)^k_{i=2}$ (by Lemma B.31, see Appendix B.5 for the terminology)

$$
\begin{aligned}
&\left\|\Phi^{k+1}_n(e_1) + \sum_{i=2}^k \beta_i \Phi^{k+1}_n(e_i)\right\|_{E_n} \ge \left\|e_1 + \sum_{i=2}^k \beta_i e_i\right\|_E - \varepsilon \\
&\ge \|e_1\|_E - \varepsilon \ge \left\|\Phi^{k+1}_n(e_1)\right\|_{E_n} - 2\varepsilon,
\end{aligned}
$$

for any scalars $\beta_i \in \mathbb{R}$, $2 \le i \le k$. Then

$$
\begin{aligned}
&|\alpha_{k+1}^n| \ge |\alpha_{k+1}^n| \left[\left\| \Phi_n^{k+1}(e_{k+1}) \right\|_{E_n} - \varepsilon \right] \\
= &\left\| \sum_{i=1}^k \alpha_i^n \Phi_n^{k+1}(e_i) \right\|_{E_n} - \varepsilon |\alpha_{k+1}^n| \ge |\alpha_1^n| \left\| \Phi_n^{k+1}(e_1) + \sum_{i=2}^k \frac{\alpha_i^n}{\alpha_1^n} \Phi_n^{k+1}(e_i) \right\|_{E_n} - \varepsilon |\alpha_{k+1}^n| \\
&\ge A(\left\| \Phi_n^{k+1}(e_1) \right\|_{E_n} - 2\varepsilon) - \varepsilon |\alpha_{k+1}^n| \ge A(1-3\varepsilon) - \varepsilon |\alpha_{k+1}^n|.
\end{aligned}
$$

Hence $|\alpha_{k+1}^n| \ge A\frac{1-3\varepsilon}{1+\varepsilon}$ for $n \ge N_2$. By induction and finiteness of k we can find small ε such that $(\alpha_1^n, \ldots, \alpha_k^n)$ is contained in a compact subset of $\mathbb{R}^k \setminus \{0\}$ for large n away from zero. Therefore we can find a subsequence of $\{n\}$ (also denoted by $\{n\}$) such that $(\alpha_1^n, \ldots, \alpha_k^n)$ converges in $\mathbb{R}^k$ to some $(\alpha_1^\infty, \ldots, \alpha_k^\infty) \in \mathbb{R}^k \setminus \{0\}$. But by (B2) and (BL) ((AL) resp.), taking into account that $\left\| \sum_{i=1}^k \alpha_i^n \Phi_n^k(e_i) \right\|_{E_n} = 0$ for all large n we have that $\sum_{i=1}^k \alpha_i^\infty e_i = 0$. But at least one of the α_i^∞'s is not equal to zero which contradicts the linear independence of the e_i's. The proof is complete. □

Lemma 5.71. *Suppose that X, Y are separable reflexive Banach spaces with duals X^*, Y^*. Suppose that we are given linear isomorphisms $U : X \to Y$, $V : X^* \to Y^*$ with $\|U\|_{\mathscr{L}(X,Y)} = \|V\|_{\mathscr{L}(X^*,Y^*)} = 1$. Then U, V are isometries if and only if*

$$
{}_{Y^*}\langle Vv, Uu \rangle_Y = {}_{X^*}\langle v, u \rangle_X \quad \forall u \in X,\ \forall v \in X^*. \tag{5.38}
$$

Proof. Let us prove the "if"-part.

Let $u \in X$. Clearly $\|Uu\|_Y \le \|u\|_X$. But indeed by (5.38),

$$
\begin{aligned}
\|Uu\|_Y = \sup &\left\{ |{}_{Y^*}\langle v^*, Uu \rangle_Y| \;\middle|\; v^* \in Y^*,\ \|v^*\|_{Y^*} \le 1 \right\} \\
&\ge \sup \left\{ |{}_{Y^*}\langle Vv, Uu \rangle_Y| \;\middle|\; u \in X,\ v \in X^*,\ \|Vv\|_{Y^*} \le 1 \right\} \\
&\qquad \ge \sup \{ |{}_{X^*}\langle v, u \rangle_X| \mid u \in X,\ v \in X^*,\ \|v\|_{X^*} \le 1 \} = \|u\|_X,
\end{aligned}
$$

where we have used that $\|Vv\|_{Y^*} \le \|v\|_{X^*}$ for all $v \in X^*$. By reflexivity we can proceed similarly for V.

We prove the "only if"-part. The idea of the proof is due to D. Koehler and P. Rosenthal (cf. [KR70] and also [FJ03, Theorem 1.4.5]), who considered a semi-inner product context.

Suppose that U, V are surjective isometries. Let U^* be the linear adjoint of U. Set $W : X^* \to X^*$, $W := U^* V$, which is a surjective isometry.

Note that for any $u \in X$, $v \in X^*$,

$$
|{}_{X^*}\langle W^n v, u \rangle_X| \le \|W^n v\|_{X^*} \|u\|_X = \|v\|_{X^*} \|u\|_X
$$

so that $\{ {}_{X^*}\langle W^n v, u \rangle_X \}_{n \in \mathbb{N}}$ is a bounded sequence of scalars. Let F be a linear functional of norm 1 on ℓ^∞ such that F is a *Banach limit*, i.e., $F(\{1\}_{n\in\mathbb{N}}) = 1$, $F(\{x_n\}_{n\in\mathbb{N}}) =$

$F(\{x_{n+1}\}_{n\in\mathbb{N}})$ for all $\{x_n\}_{n\in\mathbb{N}} \in \ell^\infty$, and $F(\{x_n\}_{n\in\mathbb{N}}) \leq F(\{y_n\}_{n\in\mathbb{N}})$ whenever $x_n \leq y_n$ for all $n \in \mathbb{N}$, whose existence is given by Banach [Ban32, p. 34]. Then define

$$\langle\langle v, u\rangle\rangle := F(\{\,{}_{X^*}\langle W^n v, u\rangle_X\}_{n\in\mathbb{N}}).$$

The properties of F and W guarantee that $\langle\langle v, u\rangle\rangle$ defines a bilinear map that separates points and satisfies the Cauchy-Schwarz inequality. Therefore $\langle\langle v, u\rangle\rangle = {}_{X^*}\langle v, u\rangle_X$. Since F is a Banach limit, ${}_{X^*}\langle v, u\rangle_X = {}_{X^*}\langle Wv, u\rangle_X$ and hence

$${}_{X^*}\langle v, u\rangle_X = {}_{Y^*}\langle Vv, Uu\rangle_Y\,.$$

□

Before we prove Theorem 5.68, we still need some preparations. Let $A = (a_{i,j})_{1\leq i,j\leq m}$ be a *quadratic matrix*. Denote by ${}^{\mathrm{t}}A := (a'_{i,j})_{1\leq i,j\leq m}$, $a'_{i,j} := a_{j,i}$, $1 \leq i, j \leq m$ the *transposed matrix*. Denote by A^{-1} the *inverse matrix* of A whenever A is *invertible*. An invertible A is called *unitary* if $A^{-1} = {}^{\mathrm{t}}A$. Let $\delta_{i,j}$ be the *Kronecker delta*. Denote by $I_m := (\delta_{i,j})_{1\leq i,j\leq m}$ the *$m \times m$-unit matrix*. We write $|A| := \det A$ for the *determinant*. We consider real quadratic matrices with fixed finite number of rows m as elements of the Hilbert space $\ell^2(m)\otimes\ell^2(m)$ and denote the corresponding submultiplicative matrix-norm by $\|\cdot\|_{\ell^2(m)\otimes\ell^2(m)}$ which is equal to the canonical operator norm on $\mathscr{L}(\ell^2(m), \ell^2(m))$ as well as equivalent for each fixed m to

$$\|(a_{i,j})_{1\leq i,j,\leq m}\|_{\ell^1(m)\otimes\ell^\infty(m)} := \max_{1\leq j\leq m} \sum_{i=1}^{m} |a_{i,j}|.$$

As is standard, for two m-dimensional vector spaces X_m, Y_m with ordered bases $\mathscr{X}_m = (x_1, \ldots, x_m)$, $\mathscr{Y}_m = (y_1, \ldots, y_m)$ respectively and a real $m\times m$-matrix $A = (a_{i,j})$ we define the associated linear endomorphism $\mathscr{A} : X_m \to Y_m$ via

$$\mathscr{A}\left(\sum_{i=1}^{m} \alpha_i x_i\right) = \sum_{i=1}^{m} \alpha_i \sum_{j=1}^{m} a_{j,i} y_j \tag{5.39}$$

and write

$$\mathscr{A} = [A]^{\mathscr{X}_m}_{\mathscr{Y}_m} \tag{5.40}$$

and shall refer to this notation below, when we “make an operator from a matrix”. Compare [Lor88] for details.

Proof of Theorem 5.68. For all of the proof fix a normal monotone Schauder basis $\mathscr{E} = (e_1, e_2, \ldots)$ of E with a biorthogonal normal monotone Schauder basis $\mathscr{E}^* = (e_1^*, e_2^*, \ldots)$ in E^*. Let $\{\Phi_n\}$, $\{d_n\}$ be metric approximations as in the proof of Lemma 5.69 w.r.t. $\mathscr{E}$ and $\mathscr{E}^*$. Let $\{\Phi_n^*\}$, $\{d_n^*\}$ be the dual metric approximations. W.l.o.g. $d_n^* = d_n$ (see the proof of Lemma 5.69).

Fix $n, k \in \mathbb{N}$. Let $(f_{k,i})_{i=1}^k$ be the alternative basis for $\mathrm{lin}(e_i)_{i=1}^k$ coming from property (I), where $d = d_n$. Let $(f_{k,i}^*)_{i=1}^k$ be the dual basis, i.e., ${}_{E^*}\left\langle f_{k,j}^*, f_{k,i}\right\rangle_E = \delta_{i,j}$. Set $\Phi_n^k := \Phi_n\restriction_{\mathrm{lin}(f_{k,i})_{i=1}^k}$. Then $\mathrm{ran}(\Phi_n^k) = \mathrm{lin}(\Phi_n^k(f_{k,i}))_{i=1}^k$.

Extract a basis $\mathscr{B}_{n,k} = (b_{n,k,j} := \Phi_n^k(f_{k,i_j}))_{j=1}^{l_n^k}$ where $l_n^k := \dim \operatorname{ran}(\Phi_n^k) \leq k \wedge d_n$. Fix the sequence $\{i_j\}$ for later (it depends on n and k). Let $\mathscr{B}_{n,k}^* = (b_{n,k,j}^*)_{j=1}^{l_n^k} \subset \operatorname{lin}(\Phi_n^{k,*}(f_{i,k}^*))_{i=1}^k$ be any basis for the dual space of $\operatorname{lin}\mathscr{B}_{n,k}$. Let $\mathscr{F}_{n,k} := (f_{k,i_j}))_{j=1}^{l_n^k}$ and $\mathscr{F}_{n,k}^* := (f_{k,i_j}^*))_{j=1}^{l_n^k}$.

In the matrix notation (5.40) $\Phi_n^k = [I_{l_n^k}]_{\mathscr{F}_{n,k}}^{\mathscr{B}_{n,k}}$.

Define for $1 \leq j \leq l_n^k$

$$M_{n,k}^j := \Big({}_{E_n^*}\big\langle b_{n,k,i'}^*, b_{n,k,i} \big\rangle_{E_n} \Big)_{1 \leq i,i' \leq j}$$

which is a real $j \times j$-matrix. By [Mil70, Lemma 1.6] there exists a renumbering of $\mathscr{B}_{n,k}^*$ such that $\Delta_{n,k}^j := \det M_{n,k}^j \neq 0$ for every $1 \leq j \leq l_n^k$. We assume $\mathscr{B}_{n,k}^*$ to be renumbered in this way and shall show later that this does not restrict generality. Also denote the new matrix by $M_{n,k}^j$. Set $M_{n,k} := M_{n,k}^{l_n^k}$. Note that nothing about symmetry of $M_{n,k}^j$ can be said.

Define for $1 \leq j \leq l_n^k$ (with $\Delta^j = \Delta_{n,k}^j$, $b_i = b_{n,k,i}$, $b_i^* = b_{n,k,i}^*$)

$$x_j := \frac{\begin{vmatrix} \multicolumn{3}{c}{\Big({}_{E_n^*}\langle b_{i'}^*, b_i \rangle_{E_n} \Big)_{1 \leq i,i' \leq j-1}} & \begin{matrix} {}_{E_n^*}\langle b_1^*, b_j \rangle_{E_n} \\ \vdots \\ {}_{E_n^*}\big\langle b_{j-1}^*, b_j \big\rangle_{E_n} \end{matrix} \\ b_1 & \cdots & b_{j-1} & b_j \end{vmatrix}}{\Delta^{j-1}} \tag{5.41}$$

($\Delta^0 := 1$) and

$$x_j^* := \frac{\begin{vmatrix} \multicolumn{3}{c}{\Big({}_{E_n^*}\langle b_{i'}^*, b_i \rangle_{E_n} \Big)_{1 \leq i,i' \leq j-1}} & \begin{matrix} b_1^* \\ \vdots \\ b_{j-1}^* \end{matrix} \\ {}_{E_n^*}\big\langle b_j^*, b_1 \big\rangle_{E_n} & \cdots & {}_{E_n^*}\big\langle b_j^*, b_{j-1} \big\rangle_{E_n} & b_j^* \end{vmatrix}}{\Delta^j}. \tag{5.42}$$

By a result of Mil'man $\mathscr{X}_{n,k} = (x_j = x_{n,k,j} \mid 1 \leq j \leq l_n^k)$ and $\mathscr{X}_{n,k}^* = (x_j^* = x_{n,k,j}^* \mid 1 \leq j \leq l_n^k)$ are mutually biorthogonal bases for $\operatorname{lin}\mathscr{B}_{n,k}$, $\operatorname{lin}\mathscr{B}_{n,k}^*$ resp. (cf. [MM64], [Mil70, Lemma 1.7]). W.l.o.g. we can assume that the bases are normalized.

Let $S_{n,k} = (s_{i,j}^{n,k})$ and $T_{n,k} = (t_{i,j}^{n,k})$ be the invertible matrices defined by the basis transformations $\mathscr{B}_{n,k} \rightsquigarrow \mathscr{X}_{n,k}$ and $\mathscr{B}_{n,k}^* \rightsquigarrow \mathscr{X}_{n,k}^*$, in other words $S_{n,k}$ and $T_{n,k}$ have the unique scalar entries such that

$$x_{n,k,i} = \sum_{j=1}^{l_n^k} s_{j,i}^{n,k} b_{n,k,j} \text{ and } x_{n,k,i}^* = \sum_{j=1}^{l_n^k} t_{j,i}^{n,k} b_{n,k,j}^*, \text{ for } 1 \leq i \leq l_n^k,$$

cf. [Lor88, Kapitel III, §3]. Reformulating in the notation of (5.40) we say that $S_{n,k}$ and $T_{n,k}$ are such that $\mathrm{Id} = [S_{n,k}]^{\mathscr{X}_{n,k}}_{\mathscr{B}_{n,k}}$ and $\mathrm{Id} = [T_{n,k}]^{\mathscr{X}^*_{n,k}}_{\mathscr{B}^*_{n,k}}$.

Define $V_{n,k} = (v^{n,k}_{i,j})$, $W_{n,k} = (w^{n,k}_{i,j})$ to be $V_{n,k} := S^{-1}_{n,k}$, $W_{n,k} := T^{-1}_{n,k}$. By the transformation rule for bilinear forms (or 2-tensors, cf. [Lor89, Kapitel VI, §3, Satz 3])

$$I_{l^k_n} = {}^{\mathrm{t}}S_{n,k}M_{n,k}T_{n,k} \text{ and hence } {}^{\mathrm{t}}V_{n,k}W_{n,k} = M_{n,k}.$$

Also $\Phi_{n,k} = [S_{n,k}]^{\mathscr{X}_{n,k}}_{\mathscr{F}_{n,k}}$.

Define operators $\Psi_{n,k} : \operatorname{lin}\mathscr{F}_{n,k} \to \operatorname{lin}\mathscr{X}_{n,k}$ and $\Psi^*_{n,k} : \operatorname{lin}\mathscr{F}^*_{n,k} \to \operatorname{lin}\mathscr{X}^*_{n,k}$ by $\Psi_{n,k} := [W_{n,k}]^{\mathscr{X}_{n,k}}_{\mathscr{F}_{n,k}}$ and $\Psi^*_{n,k} := [{}^{\mathrm{t}}T_{n,k}]^{\mathscr{X}^*_{n,k}}_{\mathscr{F}^*_{n,k}}$.

Claim 1: ${}_{E^*_n}\left\langle \Psi^*_{n,k}(f^*_{k,i_{j'}}), \Psi_{n,k}(f_{k,i_j})\right\rangle_{E_n} = \delta_{j,j'}$, $1 \le j, j' \le l^k_n$, where $\{i_j\}$ is as in the basis extraction above.

The claim is verified as follows (where ${}^{\mathrm{t}}T_{n,k} = ({}^{\mathrm{t}}t^{n,k}_{j,j'})$)

$$\begin{aligned} {}_{E^*_n}\left\langle \Psi^*_{n,k}(f^*_{k,i_{j'}}), \Psi_{n,k}(f_{k,i_j})\right\rangle_{E_n} &= \sum_{s=1}^{l^k_n}\sum_{s'=1}^{l^k_n} w^{n,k}_{s,j}{}^{\mathrm{t}}t^{n,k}_{s',j'}\ {}_{E^*}\left\langle f^*_{k,i_{s'}}, f_{k,i_s}\right\rangle_E \\ &= \sum_{s=1}^{l^k_n}\sum_{s'=1}^{l^k_n} w^{n,k}_{s,j}t^{n,k}_{j',s'}\delta_{s,s'} = \delta_{j,j'}. \end{aligned}$$

Claim 2: $\Psi^{\mathrm{iso}}_{n,k} := \Psi_{n,k}/\left\|\Psi_{n,k}\right\|_{\mathscr{L}(E,E_n)}$ as well as $\Psi^{*,\mathrm{iso}}_{n,k} := \Psi^*_{n,k}/\left\|\Psi^*_{n,k}\right\|_{\mathscr{L}(E,E_n)}$ is a linear Banach-space isometry.

Claim 2 follows from Lemma 5.71 noting that $\mathscr{F}_{n,k}$ and $\mathscr{X}_{n,k}$ are reflexive as closed linear subspaces the reflexive spaces E, E_n resp. (cf. [HHZ96, Proposition 67]).

Claim 3: There exist isometries $\{\Psi_n\}$ which are (asymptotically) compatible with $\{\Phi_n\}$.

Let Ψ_{n,k,d_n,l^k_n} be as in property (I), coming from $\Psi^{\mathrm{iso}}_{n,k}$.

Let Φ^k_n as above. We would like to prove that for all $k \in \mathbb{N}$

$$\lim_n \left\|(\Psi_{n,k,d_n,l^k_n} - \Phi^k_n)(e_i)\right\|_{E_n} = 0 \quad \text{for all } 1 \le i \le k. \tag{5.43}$$

If we succeed in doing so, by Lemma 5.27 we can find $k_n \in \mathbb{N}$, $k_n \uparrow \infty$ such that

$$\lim_n \left\|(\Psi_{n,k_n,d_n,l^{k_n}_n} - \Phi^{k_n}_n)(e_i)\right\|_{E_n} = 0 \quad \text{for all } i \in \mathbb{N}. \tag{5.44}$$

Let $E \ni x = \sum_{i=1}^{\infty} {}_{E^*}\langle e_i^*, x\rangle_E \, e_i$. Then, upon setting $\Psi_n := \Psi_{n,k_n,d_n,l_n^{k_n}}$,

$$\begin{aligned}
&\overline{\lim_m}\,\overline{\lim_n} \left\| \Phi_n^{k_n}(x) - \Psi_n(x) \right\|_{E_n} \\
=&\overline{\lim_m}\,\overline{\lim_n} \left\| \left(\Phi_n^{k_n} - \Psi_n\right) \left(\sum_{i=1}^{m} {}_{E^*}\langle e_i^*, x\rangle_E \, e_i \right) - \Psi_n \left(\sum_{i=m+1}^{\infty} {}_{E^*}\langle e_i^*, x\rangle_E \, e_i \right) \right\|_{E_n} \\
\leq&\overline{\lim_m}\,\overline{\lim_n} \left\| \left(\Phi_n^{k_n} - \Psi_n\right) \left(\sum_{i=1}^{m} {}_{E^*}\langle e_i^*, x\rangle_E \, e_i \right) \right\|_{E_n} \\
&+ \overline{\lim_m} \left\| \left(\sum_{i=m+1}^{\infty} {}_{E^*}\langle e_i^*, x\rangle_E \, e_i \right) \right\|_{E} \\
=&0.
\end{aligned}$$

Now, since $\Phi_n^{k_n}$ is the restriction of a metric approximation, we find that $\Psi_n(x)$ converges strongly to x in $\mathfrak{E}$, i.e. is compatible.

Claim 4: (5.43) holds.

Recall that $l_n^k = \dim \operatorname{ran} \Phi_n^k$. As $(f_{k,i})_{i=1}^k$ is an alternative basis for $\operatorname{lin}(e_i)_{i=1}^k$ used in property (I), we infer that

$$l_n^k \to k \quad \text{as} \quad n \to \infty$$

as a consequence of Lemma 5.70. Also $d_n \geq k$ for large n.

Next we would like to prove that for large n, $\Delta_{n,k}^j = \det M_{n,k}^j \neq 0$ for all $1 \leq j \leq l_n$. Take N_0 such that $l_n^k = k$ for all $n \geq N_0$. We can assume again by Lemma 5.70 that $l_n^{k,*} = k$ for $n \geq N_0$, where $l_n^{k,*} := \dim \operatorname{ran}(\Phi_n^{k,*})$. For such n, $\mathscr{B}_{n,k}^* := (\Phi_n^{k,*}(f_{k,i}^*))_{i=1}^k$ is a basis for $\operatorname{ran}(\Phi_n^{k,*})$ (which was picked arbitrarily above). This yields by asymptotic duality of $\{\Phi_n^k\}$ and $\{\Phi_n^{k,*}\}$

$$\left\| M_{n,k}^j - I_j \right\|_{\ell^2(j)\otimes\ell^2(j)} \to 0 \quad \text{as } n \to \infty,\ n \geq N_0, \tag{5.45}$$

and

$$\left\| {}^{\mathrm{t}}M_{n,k}^j - I_j \right\|_{\ell^2(j)\otimes\ell^2(j)} \to 0 \quad \text{as } n \to \infty,\ n \geq N_0. \tag{5.46}$$

Therefore

$$\left\| I_j - M_{n,k}^j \right\|_{\ell^2(j)\otimes\ell^2(j)} < \frac{1}{2}$$

for all $n \geq N_1 \geq N_0$, some N_1, all $1 \leq j \leq k$ at the same time. An application of the Neumann-series (cf. [Yos78, Ch. II.1, Theorem 2]) shows that $M_{n,k}^j$ is invertible for $n \geq N_1$ and hence $\Delta_{n,k}^j \neq 0$, for all $1 \leq j \leq k$. Therefore, for large n, there is no need to rearrange the basis in (5.41).

Note that by definition for fixed k, $\left\| V_{n,k}^{-1} \right\| = \|S_{n,k}\|$ is bounded in n whenever each $\Delta_{n,k}^j$, $1 \leq j \leq l_n^k$ is bounded away from zero and from above uniformly in n (compare with (5.41)). But by the Leibniz formula $\det : \ell^2(k) \otimes \ell^2(k) \to \mathbb{R}$ is a continuous map and $\det I_j = 1$. Therefore $\lim_{n\geq N_1} \Delta^j_{n,k} \to 1$ for each j and $(\Delta^1_{n,k}, \ldots, \Delta^k_{n,k})$ contained in a compact subset of $\mathbb{R}^k$ separated from zero for large n.

Fix $f_{k,i}$ and make n large. Then for a constant $C = C(k) \geq 1$ depending only on k,

$$\begin{aligned}
&\left\|(\Psi_{n,k,d_n,l^k_n} - \Phi^k_n)(f_{k,i})\right\|_{E_n} \\
=&\left\|\sum_{j=1}^{k}({}^{\mathrm{t}}w^{n,k}_{j,i} - s^{n,k}_{j,i})x_{n,k,j}\right\|_{E_n} \\
\leq&\left\|{}^{\mathrm{t}}W_{n,k} - S_{n,k}\right\|_{\ell^1(k)\otimes\ell^\infty(k)} \max_{1\leq j\leq k} \|x_{n,k,j}\|_{E_n} \\
\leq&C(k)\left\|{}^{\mathrm{t}}M_{n,k} - I_k\right\|_{\ell^2(k)\otimes\ell^2(k)} \left\|V^{-1}_{n,k}\right\|_{\ell^2(k)\otimes\ell^2(k)} \\
\to&0
\end{aligned}$$

as $n \to \infty$ because of asymptotic duality and the fact that $V^{-1}_{n,k} = S_{n,k}$ stays bounded. By linearity Claim 4 is proved.

Claim 5: $\{\Psi_n\}$ is asymptotically continuous.

Let $u_n \in E$, $n \in \mathbb{N}$, $u \in E$. $\lim_n \|u_n - u\|_E = 0$. By compatibility $\Psi_n(u) \to u$ strongly in $\mathfrak{E}$. Isometry gives

$$\|\Psi_n(u_n) - \Psi_n(u)\|_{E_n} = \|u_n - u\|_E \to 0$$

as $n \to \infty$, which yields by (A4) that $\Psi_n(u_n) \to u$ strongly in $\mathfrak{E}$.

Repeat all the steps for $\mathfrak{E}^*$ and obtain an isometric asymptotically continuous metric approximation $\{\Psi^*_n\}$ which is in duality by Proposition 5.44. We note that then by Lemma 5.71

$$_{E^*_n}\langle \Psi^*_n(v), \Psi_n(u)\rangle_{E_n} = {}_{E^*}\langle v, u\rangle_E \quad \forall n \in \mathbb{N},\ \forall u \in E,\ \forall v \in E^*. \tag{5.47}$$

The proof is complete. □

Corollary 5.72. *In the situation of Theorem 5.68, for $u_n \in E_n$, $n \in \mathbb{N}$, $u \in E$ it holds that:*

$$\begin{gathered}
u_n \to (\rightharpoonup) u \ \textit{strongly (weakly) in } \mathfrak{E} \\
\textit{if and only if} \\
\Psi^{-1}_n u_n \to (\rightharpoonup) u \ \textit{strongly (weakly) in } E.
\end{gathered} \tag{5.48}$$

A similar statement holds for $\mathfrak{E}^*$ and $\{\Psi^*_n\}$.

Proof. Let $u_n \in E_n$, $n \in \mathbb{N}$, $u \in E$. Suppose that $u_n \to u$ strongly in $\mathfrak{E}$.

$$\left\|\Psi^{-1}_n(u_n) - u\right\|_E = \|u_n - \Psi_n(u)\|_{E_n} \to 0$$

as $n \to \infty$ by strong convergence, ($\Psi_n(u) \to u$ strongly in $\mathfrak{E}$ as $\{\Psi_n\}$ is a compatible metric approximation) and by (A3'). Suppose that $u_n \rightharpoonup u$ weakly in $\mathfrak{E}$. Let $v \in E^*$. By (5.47)

$$ {}_{E^*}\left\langle v, \Psi_n^{-1}(u_n) - u \right\rangle_E = {}_{E_n^*}\langle \Psi_n^*(v), u_n - \Psi_n(u) \rangle_{E_n} \to 0 $$

as $n \to \infty$.

Suppose now that $u_n \in E$, $n \in \mathbb{N}$, $u \in E$. Suppose that $u_n \to u$ strongly in E. Then $\Psi_n(u_n) \to u$ strongly in $\mathfrak{E}$ by asymptotic continuity. Suppose that $u_n \rightharpoonup u$ weakly in E. Then $\sup_n \|u_n\|_E < +\infty$ by the uniform boundedness principle. By isometry also $\sup_n \|\Psi_n(u_n)\|_{E_n} < +\infty$. Let $v \in E^*$. Again by (5.47)

$$ {}_{E_n^*}\langle \Psi_n^*(v), \Psi_n(u_n) \rangle_{E_n} = {}_{E^*}\langle v, u_n \rangle_E \to {}_{E^*}\langle v, u \rangle_E $$

as $n \to \infty$. Hence by Lemma 5.49 $\Psi_n(u_n) \rightharpoonup u$ weakly in $\mathfrak{E}$.

We can proceed similarly for $\mathfrak{E}^*$ and $\{\Psi_n^*\}$. □

Proposition 5.73. *If $\mathfrak{E}$ is separable and asymptotically isometric, then it admits a complete metric $d_{\mathfrak{E}}$ compatible with the strong asymptotic relation (making it a* Polish space*) such that $(\mathfrak{E}, d_{\mathfrak{E}})$ is isometrically isomorphic (in the sense of metric spaces) to $\overline{\mathbb{N}} \times E$ with the 1-product metric.*

Proof. Let $\{\Psi_n\}$ be the isometric metric approximation. Define for $u \in E_n$, $v \in E_m$, $n, m \in \overline{\mathbb{N}}$, (where we set $\frac{1}{\infty} := 0$, $\Psi_\infty := \mathrm{Id}_E$)

$$ d_{\mathfrak{E}}(u, v) := \left| \frac{1}{n} - \frac{1}{m} \right| + \left\| \Psi_n^{-1}(u) - \Psi_m^{-1}(v) \right\|_E . \tag{5.49} $$

Clearly $0 \le d_{\mathfrak{E}} < +\infty$. Symmetry follows from the definition. Also $d_{\mathfrak{E}}(u, u) = 0$ for all $u \in \mathfrak{E}$. Let $d_{\mathfrak{E}}(u, v) = 0$ for some $u, v \in \mathfrak{E}$, $u \in E_n$, $v \in E_m$. Then $|1/n - 1/m| = 0$, hence $n = m$. Therefore $\left\| \Psi_n^{-1}(u) - \Psi_n^{-1}(v) \right\|_E = \|u - v\|_{E_n} = 0$, $u = v$ follows. The triangle inequality follows from the triangle inequalities of the norms.

Let $u \in E_{n_0}$, $n_0 \in \mathbb{N}$ and $u_k \in \mathfrak{E}$, $k \in \mathbb{N}$ such that $d_{\mathfrak{E}}(u_k, u) \to 0$. Then for some $K \in \mathbb{N}$, $u_k \in E_{n_0}$ for $k \ge K$. And $\lim_{k \ge K} \left\| \Psi_{n_0}^{-1}(u_k) - \Psi_{n_0}^{-1}(u) \right\|_E = \lim_{k \ge K} \|u_k - u\|_{E_{n_0}} = 0$. By (A1) also $\tau\text{-}\lim_n u_k = u$.

Let $u \in E$ and $u_k \in \mathfrak{E}$, $k \in \mathbb{N}$ such that $d_{\mathfrak{E}}(u_k, u) \to 0$. Pick $\{u_{k_s}\}$, $\{u_{k_l}\}$ such that their union is all of $\{u_k\}$ and $u_{k_s} \in E_{n_s}$ for a countable subnet $\{n_s\}$ of $\{n\}$, $u_{k_l} \in E$ for all l. Then $\lim_l \|u_{k_l} - u\|_E = 0$ and $\lim_s \left\| \Psi_{n_s}^{-1}(u_{k_s}) - u \right\|_E = 0$. $\tau\text{-}\lim_k u_k = u$ follows now from Corollary 5.72.

The converse follows (similarly) by (A1) and Corollary 5.72.

Since τ is second-countable by Lemma 5.15, we can characterize the topology completely by convergent sequences.

We see that the map $\Psi : \mathfrak{E} \to \overline{\mathbb{N}} \times E$, $\Psi(u) := (n, \Psi_n^{-1}(u))$, whenever $u \in E_n$, $n \in \overline{\mathbb{N}}$, is a surjective isometry of metric spaces. We finally remark that $d_{\mathfrak{E}}$ is complete by completeness of E and $\overline{\mathbb{N}}$ recalling that $\overline{\mathbb{N}}$ is complete as a compact metric space.

The proof is complete. □

5.13 Overview

We conclude the chapter with a couple of figures summarizing the relations between strong asymptotic relations and metric approximations.

$\mathfrak{E} = \dot{\bigcup}_{\nu \in \mathscr{N}} E_\nu \dot{\cup} E$ Banach spaces, $\mathscr{N}$ directed set:

$(\mathfrak{E}, \tau)$ linear strong asymptotic relation.	$\xleftarrow{\text{strong convergence of nets, [KS08, Lemma 3.5]}}$ $\xrightarrow[\text{[KS08, Lemma 3.5]}]{}$	$\{(\Phi_\nu, D(\Phi_\nu))\}_{\nu \in \mathscr{N}}$ linear metric approximation.
τ Hausdorff.		

$\mathfrak{E} = \dot{\bigcup}_{n \in \mathbb{N}} E_n \dot{\cup} E$ Banach spaces:

$(\mathfrak{E}, \tau)$ linear strong asymptotic relation.	$\xleftarrow{\text{strong convergence of sequences, Theorems 5.32, 5.38}}$ $\xrightarrow[\text{Corollary 5.41, } \tau \text{ assumed to be Fréchet}]{}$	$\{(\Phi_n, D(\Phi_n))\}_{n \in \mathbb{N}}$ linear metric approximation.
τ Hausdorff, Fréchet.		

$\mathfrak{E} = \dot{\bigcup}_{n \in \mathbb{N}} E_n \dot{\cup} E$ separable Banach spaces:

$(\mathfrak{E}, \tau)$ linear strong asymptotic relation.	$\xleftarrow{\text{strong convergence of sequences, Theorems 5.32, 5.38, Lemmas 5.15, 5.16}}$ $\xrightarrow[\text{Theorem 5.40}]{}$	$\{(\Phi_n, D(\Phi_n))\}_{n \in \mathbb{N}}$ linear metric approximation.
τ Hausdorff, second countable, regular.		

$\mathfrak{E} = \dot{\bigcup}_{n \in \mathbb{N}} E_n \dot{\cup} E$ separable Banach spaces, E possesses a Schauder basis:

$(\mathfrak{E}, \tau)$ linear strong asymptotic relation.	$\xleftarrow{\text{strong convergence of sequences, Theorems 5.32, 5.38, Lemmas 5.15, 5.16}}$ $\xrightarrow[\text{Theorem 5.40, Lemma 5.51}]{}$	$\{(\Phi_n, D(\Phi_n))\}_{n \in \mathbb{N}}$ linear metric approximation.
τ Hausdorff, second countable, regular.		$\{(\overline{\Phi}_n, E)\}_{n \in \mathbb{N}}$ asymptotically continuous linear metric approximation.

$\mathfrak{E} = \dot{\bigcup}_{n\in\mathbb{N}} E_n \dot{\cup} E$ separable reflexive Banach spaces, $\mathfrak{E}$ possesses property (I):

$(\mathfrak{E}, \tau)$ linear strong asymptotic relation.	strong convergence of sequences, Theorems 5.32, 5.38, Lemmas 5.15, 5.16 $\longleftarrow$ $\longrightarrow$ Theorem 5.40, Lemma 5.51, Theorem 5.68	$\{(\Phi_n, D(\Phi_n))\}_{n\in\mathbb{N}}$ linear metric approximation.
τ Hausdorff, second countable, regular.		$\{(\overline{\Phi}_n, E)\}_{n\in\mathbb{N}}$ asymptotically continuous linear metric approximation.
$\mathfrak{E}$ isometrically isomorphic to $\overline{\mathbb{N}} \times E$.		$\{(\Psi_n, E)\}_{n\in\mathbb{N}}$ isometrically isomorphic asymptotically continuous linear metric approximation.

5.14 Addendum: Asymptotic topology

For the sake of completeness, we present a more general approach towards the Kuwae-Shioya theory, which includes both asymptotic relations, as in this work and [KS08], and the approach presented in [KS03, Section 2.2] as special cases. The advanced reader will realize the unifying character of this approach.

Definition 5.74. *Let I be a topological space. Let $\{E_i \mid i \in I\}$ be a family of metric spaces indexed by I. A topology τ on the disjoint union*

$$\mathfrak{E} := \dot{\bigcup_{i\in I}} E_i$$

is called an asymptotic topology *relative to I if for each point $i_0 \in I$ and each net $\{i_\nu\}_{\nu\in\mathscr{N}}$ (where $\mathscr{N}$ is a directed set) such that $i_0 \in (I\text{-}\lim_{\nu\in\mathscr{N}} i_\nu)$ the topology τ restricted to $\dot{\bigcup}_{\nu\in\mathscr{N}} E_{i_\nu} \dot{\cup} E_{i_0}$ is an asymptotic relation of metric spaces (in the sense of Definition 5.8).*

In particular, a net $\{u_\nu\}_{\nu\in\mathscr{N}}$ of elements included in $\mathfrak{E}$ converges to some $u \in \mathfrak{E}$ if and only if there is a net $\{i_\nu\}_{\nu\in\mathscr{N}}$ of elements included in I such that there exists an element $i_0 \in I$ such that $i_0 \in (I\text{-}\lim_{\nu\in\mathscr{N}} i_\nu)$ and $u_\nu \in E_{i_\nu}$ for all $\nu \in \mathscr{N}$ and such that there exists $u \in E_{i_0}$ with $u \in (\tau\text{-}\lim_{\nu\in\mathscr{N}} u_\nu)$, where τ is restricted to the asymptotic relation on $\dot{\bigcup}_{\nu\in\mathscr{N}} E_{i_\nu} \dot{\cup} E_{i_0}$.

We remark that there might be many different asymptotic topologies for $\dot{\bigcup}_{i\in I} E_i$.

Lemma 5.75. *(i) If $J \subset I$ is closed, then $\dot{\bigcup}_{j\in J} E_j$ is closed in τ. In particular, if I is a T_1-space, then each E_i, $i \in I$ is closed in τ.*

(ii) *If $J \subset I$ is open, then $\dot{\bigcup}_{j\in J} E_j$ is open in τ. In particular, if I is discrete, then each E_i, $i \in I$ is open in τ.*

(iii) *If I is Hausdorff, then τ is Hausdorff.*

(iv) *If I is Fréchet, and if each possible (sequential) asymptotic relation $\dot{\bigcup}_{n\in\mathbb{N}} E_{i_n} \dot{\cup} E_{i_0}$ is Fréchet, so is τ.*

(v) *If I is countable (as a set) and each E_i, $i \in I$ is separable, then τ is second countable and regular.*

Proof. (i): Let $J \subset I$ be a closed subset. Let $\{x_\nu\}$ be any convergent net included in $\dot{\bigcup}_{j\in J} E_j$. Let x be any of its limit points. Then there is a net $\{i_\nu\}$ converging to some $i_0 \in I$ such that $x \in E_{i_0}$ and $x_\nu \in E_{i_\nu}$. Clearly $\{i_\nu\}$ is included in J. But J is closed and hence $i_0 \in J$. Therefore $x \in \dot{\bigcup}_{j\in J} E_j$ which is proved to be closed.

(ii): Let $J \subset I$ be an open subset. Let $x \in \dot{\bigcup}_{j\in J} E_j$. Let $\{x_\nu\}$ be a net in $\mathfrak{E}$ which converges to x. Then there is a net $\{i_\nu\}$ converging to j_0 such that $x \in E_{j_0}$, $j_0 \in J$ and $x_\nu \in E_{i_\nu}$. But since J is open, $\{i_\nu\}$ is eventually in J. Therefore $\{x_\nu\}$ is eventually in $\dot{\bigcup}_{j\in J} E_j$ which is thereby open.

(iii): Suppose that I is Hausdorff. Let $\mathscr{N}$ be a directed set. Let $\{u_\nu\}$ be a convergent net included in $\mathfrak{E}$. Suppose that $u, v \in (\tau\text{-}\lim_{\nu\in\mathscr{N}} u_\nu)$, $u \in E_{i_0}$, $v \in E_{i_1}$. Let $\{i_\nu\}$ be a convergent net included in I such that $u_\nu \in E_{i_\nu}$. We must have $i_0, i_1 \in (I\text{-}\lim_{\nu\in\mathscr{N}} i_\nu)$. By the Hausdorff property, $i_0 = i_1$ and hence $u, v \in E_{i_0}$. Now by (A3)

$$d_{i_0}(u, v) \le \overline{\lim_{\nu\in\mathscr{N}}}\, d_{i_\nu}(u_\nu, u_\nu) = 0.$$

(iv): Let $A \subset \mathfrak{E}$. Let $x \in \overline{A}$ (the τ-closure). Let $J := \{j \in I \mid A \cap E_j \ne \varnothing\}$ and $J' := \{j \in I \mid \overline{A} \cap E_j \ne \varnothing\}$. By definition of τ-convergence (of nets), $J' = \overline{J}$. Since I is Fréchet, there is a sequence $\{j_n\} \subset J$ converging to j_0, where $j_0 \in \overline{J}$ such that $x \in E_{j_0}$. Then $\mathscr{E} := \dot{\bigcup}_{n\in\mathbb{N}} E_{j_n} \dot{\cup} E_{j_0}$ is an asymptotic relation which is Fréchet. Therefore for each $y \in \overline{A \cap \mathscr{E}}$ (the closure w.r.t the asymptotic relation topology), there exists a sequence $\{y_n\}$ included in $A \cap \mathscr{E}$ converging to y. By definition the relative topology on $\mathscr{E}$ coming from τ coincides with the asymptotic relation topology and hence $\overline{A \cap \mathscr{E}} = \overline{A} \cap \mathscr{E}$. But x is contained in this set. Therefore, upon setting $y := x$, the assertion is proved.

(v): Repeat the steps in [KS03, Lemma 2.13] and Lemmas 5.15, 5.16 for metric spaces. □

Theorem 5.76. *Suppose that for each $i, j \in I$, we are given maps $\Phi^i_j : D(\Phi^i_j) \subset E_i \to E_j$ such that $D(\Phi^i_i) = E_i$ and $\Phi^i_i = \mathrm{Id}_{E_i}$ and, moreover, for each $i_0 \in I$ and each net $\{i_\nu\}_{\nu\in\mathscr{N}}$ with $I\text{-}\lim_{\nu\in\mathscr{N}} i_\nu = i_0$ the net $\{\Phi^{i_0}_{i_\nu} : D(\Phi^{i_0}_{i_\nu}) \subset E_{i_0} \to E_{i_\nu}\}_{\nu\in\mathscr{N}}$ is a metric approximation (in the sense of Definition 5.26 with sequences therein replaced by nets).*

Assume additionally that for all $i, j, k \in I$ and each net $\{k_\nu\}$ such that $I\text{-}\lim_{\nu\in\mathscr{N}} k_\nu = k$ it holds that for all $u \in D(\Phi^i_k)$, $v \in D(\Phi^j_k)$

$$\lim_{\nu\in\mathscr{N}} \left\{\begin{array}{ll} d_{k_\nu}(\Phi^i_{k_\nu}(u), \Phi^j_{k_\nu}(v)) & , \textit{ if } u \in D(\Phi^i_{k_\nu}),\ v \in D(\Phi^j_{k_\nu}), \\ +\infty & , \textit{ otherwise} \end{array}\right\} = d_k(\Phi^i_k(u), \Phi^j_k(v)). \tag{5.50}$$

If we define for $u_\nu \in E_{i_\nu}$, $\nu \in \mathscr{N}$, $u \in E_{i_0}$ that $u_\nu \to u$ converges if (definition) for one sequence $\{\widetilde{u}_m\}_{m\in\mathbb{N}} \subset \bigcup_{\nu\in\mathscr{N}} D(\Phi^{i_0}_{i_\nu})$ with $\lim_{m\to\infty} d_{i_0}(\widetilde{u}_m, u) = 0$ we have that

$$\lim_{m\to\infty} \overline{\lim_{\nu\in\mathscr{N}}} \left\{\begin{array}{ll} d_{i_\nu}(\Phi^{i_0}_{i_\nu}(\widetilde{u}_m), u_\nu) & , \textit{ if } \widetilde{u}_m \in D(\Phi^{i_0}_{i_\nu}) \\ +\infty & , \textit{ otherwise} \end{array}\right\} = 0, \tag{5.51}$$

then this convergence is a convergence class (of nets) in the sense Definition A.7 and the topology generated on $\dot{\bigcup}_{i\in I} E_i$ by this convergence class is an asymptotic topology in the sense of Definition 5.74 (cf. Theorem A.8).

Proof. Let us check (i)–(iv) in Definition A.7 in the appendix.

(i): Let $\nu \in \mathscr{N}$ be a directed set. Let $u_\nu = u \in \mathfrak{E}$ for every $\nu \in \mathscr{N}$. Let $i_0 \in I$ such that $u_\nu, u \in E_{i_0}$. Clearly $\{i_0\}_{\nu\in\mathscr{N}}$ converges to i_0 in I. Also, $\lim_{\nu\in\mathscr{N}} d_{i_0}(u_\nu, \Phi^{i_0}_{i_0}(u)) = d_{i_0}(u, u) = 0$, since $D(\Phi^{i_0}_{i_0}) = E_{i_0}$ and $\Phi^{i_0}_{i_0} = \mathrm{Id}_{E_{i_0}}$. (i) is proved.

(ii): Let $\nu \in \mathscr{N}$ be a directed set. Let $\{u_\nu\} \subset \mathfrak{E}$, $u \in \mathfrak{E}$ such that $u_\nu \to u$. Let $i_\nu \in I$, $\nu \in \mathscr{N}$, $i_0 \in I$ be such that $u_\nu \in E_{i_\nu}$, $n \in \mathscr{N}$, $i_0 \in E_{i_0}$. It is necessary that $I\text{-}\lim_{\nu\in\mathscr{N}} i_\nu = i_0$. Let $\{\widetilde{u}_m\}_{m\in\mathbb{N}} \subset \bigcup_{\nu\in\mathscr{N}} D(\Phi^{i_0}_{i_\nu})$ with $\lim_{m\to\infty} d_{i_0}(\widetilde{u}_m, u) = 0$ such that $\lim_{m\to\infty} \overline{\lim}_{\nu\in\mathscr{N}} a_{m,\nu} = 0$, where

$$a_{m,\nu} := \left\{\begin{array}{ll} d_{i_\nu}(\Phi^{i_0}_{i_\nu}(\widetilde{u}_m), u_\nu) & , \text{ if } \widetilde{u}_m \in D(\Phi^{i_0}_{i_\nu}) \\ +\infty & , \text{ otherwise} \end{array}\right\}, \quad m \in \mathbb{N},\ \nu \in \mathscr{N}.$$

Let $\{u_\mu\}_{\mu\in\mathscr{M}}$ be a subnet of $\{u_\nu\}_{\nu\in\mathscr{N}}$. Clearly, $I\text{-}\lim_{\mu\in\mathscr{M}} i_\mu = i_0$. For each fixed $m \in \mathbb{N}$, $\{a_{m,\mu}\}_{\mu\in\mathscr{M}}$ is a subnet of $\{a_{m,\nu}\}_{\nu\in\mathscr{N}}$ included in $\overline{\mathbb{R}}_+$. Therefore

$$\overline{\lim_{\mu\in\mathscr{M}}}\, a_{m,\mu} \leq \overline{\lim_{\nu\in\mathscr{N}}}\, a_{m,\nu},$$

(ii) follows.

(iii): Let $\nu \in \mathscr{N}$ be a directed set. Let $\{u_\nu\} \subset \mathfrak{E}$, $u \in \mathfrak{E}$ such that $u_\nu \not\to u$. Let $i_\nu \in I$, $\nu \in \mathscr{N}$, $i_0 \in I$ be such that $u_\nu \in E_{i_\nu}$, $n \in \mathscr{N}$, $i_0 \in E_{i_0}$. Suppose that $i_\nu \not\to i_0$ in I. Then by [Kel75, p. 74] there is a subnet $\{i_\mu\}$ of $\{i_\nu\}$ no subnet of which converges to i_0. So we can suppose that $I\text{-}\lim_{\nu\in\mathscr{N}} i_\nu = i_0$. By an adaption of the argument in the proof of Lemma 5.30, (5.51) is equivalent to holding for all sequences $\{\widetilde{u}_m\} \in E_{i_0}$ with $\lim_{m\to\infty} d_{i_0}(\widetilde{u}_m, u) = 0$. Therefore, if (5.51) does not hold for $\{u_\nu\}$ and u, there is a sequence $\{\widetilde{u}_m\} \in E_{i_0}$ with $\lim_{m\to\infty} d_{i_0}(\widetilde{u}_m, u) = 0$ such that

$$\lim_{m\to\infty} \overline{\lim_{\nu\in\mathscr{N}}} \begin{Bmatrix} d_{i_\nu}(\Phi^{i_0}_{i_\nu}(\widetilde{u}_m), u_\nu) & , \text{ if } \widetilde{u}_m \in D(\Phi^{i_0}_{i_\nu}) \\ +\infty & , \text{ otherwise} \end{Bmatrix} > 0.$$

Since $\overline{\mathbb{R}}_+$ is compact, we can pick a subnet $\{u_\mu\}_{\mu\in\mathscr{M}}$ of $\{u_\nu\}_{\nu\in\mathscr{N}}$ and a subsequence $\{\widetilde{u}_{m_l}\}$ of $\{\widetilde{u}_m\}$ such that

$$\lim_{l\to\infty} \overline{\lim_{\mu\in\mathscr{M}}} \begin{Bmatrix} d_{i_\mu}(\Phi^{i_0}_{i_\mu}(\widetilde{u}_{m_l}), u_\mu) & , \text{ if } \widetilde{u}_{m_l} \in D(\Phi^{i_0}_{i_\mu}) \\ +\infty & , \text{ otherwise} \end{Bmatrix} = \alpha \in (0,\infty].$$

But $I\text{-}\lim_{\mu\in\mathscr{M}} i_\mu = i_0$ and $\lim_{l\to\infty} \overline{\lim}_{\mu\in\mathscr{M}} d_{i_\mu}(\widetilde{u}_{m_l}, u) = 0$. Since it is necessary for the convergence that (5.51) holds for all approximating sequences $\{\widetilde{\widetilde{u}}_m\}$ of u (see Lemma 5.30 again), we conclude that no subnet of $\{u_\mu\}$ converges to u and (iii) follows.

(iv): Let $\mathscr{N}$ be a directed set, let $\mathscr{M}_\nu$ be a directed set for each $\nu \in \mathscr{N}$, let $\mathscr{K} := \mathscr{N} \times \bigtimes_{\nu\in\mathscr{N}} \mathscr{M}_\nu$ be the product directed set (with the product order $(\nu, f) \succeq (\nu', f')$ iff $\nu \succeq \nu'$ and $f(\nu) \succeq f'(\nu)$ for all $\nu \in \mathscr{N}$). Suppose that $u_{\nu,\mu} \to u_\nu$ for each fixed $\nu \in \mathscr{N}$ and $u_\nu \to u$, where $\{u_{\nu,\mu}\}, \{u_\nu\} \subset \mathfrak{E}$, $u \in \mathfrak{E}$. Let $i_{\nu,\mu}$, i_ν, i_0, $\mu \in \mathscr{M}_\nu$, $\nu \in \mathscr{N}$ be in I such that $u_{\nu,\mu} \in E_{i_{\nu,\mu}}$, $u_\nu \in E_{i_\nu}$, $u \in E_{i_0}$, $\mu \in \mathscr{M}_\nu$, $\nu \in \mathscr{N}$. It is necessary that $I\text{-}\lim_{\mu\in\mathscr{M}_\nu} i_{\nu,\mu} = i_\nu$ for each fixed $\nu \in \mathscr{N}$ and that $I\text{-}\lim_{\nu\in\mathscr{N}} i_\nu = i_0$. By the diagonal lemma [Kel75, Ch. 2, Theorem 4, p. 69] it holds that $I\text{-}\lim_{(\nu,f)\in\mathscr{K}} i_{\nu,f(\nu)} = i_0$. For each $\nu \in \mathscr{N}$ let $\{\widetilde{u}^\nu_m\} \subset \bigcup_{\mu\in\mathscr{M}_\nu} D(\Phi^{i_\nu}_{i_{\nu,\mu}})$ be an approximating sequence for u_ν. Let $\{\widetilde{u}_k\} \subset \bigcup_{\nu\in\mathscr{N}} D(\Phi^{i_0}_{i_\nu})$ be an approximating sequence for u. Set

$$a_{k,\nu,m,\mu} := \begin{cases} d_{i_{\nu,\mu}}(u_{\nu,\mu}, \Phi^{i_0}_{i_{\nu,\mu}}(\widetilde{u}_k)), & \text{if } \widetilde{u}_k \in D(\Phi^{i_0}_{i_{\nu,\mu}}), \\ +\infty, & \text{otherwise}, \end{cases}$$

$$b_{k,\nu,m,\mu} := \begin{cases} d_{i_{\nu,\mu}}(u_{\nu,\mu}, \Phi^{i_\nu}_{i_{\nu,\mu}}(\widetilde{u}^\nu_m)), & \text{if } \widetilde{u}^\nu_m \in D(\Phi^{i_\nu}_{i_{\nu,\mu}}),\ \widetilde{u}_k \in D(\Phi^{i_0}_{i_{\nu,\mu}}), \\ +\infty, & \text{otherwise}, \end{cases}$$

$$c_{k,\nu,m,\mu} := \begin{cases} d_{i_{\nu,\mu}}(\Phi^{i_0}_{i_{\nu,\mu}}(\widetilde{u}_k), \Phi^{i_\nu}_{i_{\nu,\mu}}(\widetilde{u}^\nu_m)), & \text{if } \widetilde{u}^\nu_m \in D(\Phi^{i_\nu}_{i_{\nu,\mu}}),\ \widetilde{u}_k \in D(\Phi^{i_0}_{i_{\nu,\mu}}), \\ +\infty, & \text{otherwise}. \end{cases}$$

By convergence, (5.50) and the triangle inequality

$$\begin{aligned} &\overline{\lim_{k\to\infty}}\, \overline{\lim_{\nu\in\mathscr{N}}}\, \overline{\lim_{m\to\infty}}\, \overline{\lim_{\mu\in\mathscr{M}_\nu}}\, a_{k,\nu,m,\mu} \\ &\qquad \le \overline{\lim_{k\to\infty}}\, \overline{\lim_{\nu\in\mathscr{N}}}\, \overline{\lim_{m\to\infty}}\, \overline{\lim_{\mu\in\mathscr{M}_\nu}}\, b_{k,\nu,m,\mu} + \overline{\lim_{k\to\infty}}\, \overline{\lim_{\nu\in\mathscr{N}}}\, \overline{\lim_{m\to\infty}}\, \overline{\lim_{\mu\in\mathscr{M}_\nu}}\, c_{k,\nu,m,\mu} = 0. \end{aligned}$$

By a diagonal argument (see e.g. Lemma 5.27), for each $k \in \mathbb{N}$, $\nu \in \mathscr{N}$, we find a net of natural numbers $\{m^{k,\nu}_\mu\}$ with $\lim_{\mu\in\mathscr{M}_\nu} m^{k,\nu}_\mu = \infty$ such that for each $k \in \mathbb{N}$,

$\nu \in \mathscr{N}$,

$$\overline{\lim_{\mu\in\mathscr{M}_\nu}}\, a_{k,\nu,m_\mu^{k,\nu},\mu} \le \overline{\lim_{m\to\infty}}\,\overline{\lim_{\mu\in\mathscr{M}_\nu}}\, a_{k,\nu,m,\mu}.$$

When we apply diagonal lemma [Kel75, Ch. 2, Theorem 4, p. 69] again (in $\overline{\mathbb{R}}_+$) we get that for each $k \in \mathbb{N}$

$$\overline{\lim_{(\nu,f)\in\mathscr{K}}}\, a_{k,\nu,m_{f(\nu)}^{k,\nu},f(\nu)} = \overline{\lim_{\nu\in\mathscr{N}}}\,\overline{\lim_{\mu\in\mathscr{M}_\nu}}\, a_{k,\nu,m_\mu^{k,\nu},\mu},$$

and hence

$$\overline{\lim_{k\to\infty}}\,\overline{\lim_{(\nu,f)\in\mathscr{K}}}\, a_{k,\nu,m_{f(\nu)}^{k,\nu},f(\nu)} = 0,$$

which is the desired convergence $u_{\nu,f(\nu)} \to u$. (iv) is proved.

For the asymptotic relation part it is enough to notice that we can reprove Theorem 5.38 with sequences therein replaced by nets, norms therein replaced by distances. □

Remark 5.77. *Strong convergence, as defined in 5.28 is an example of such a convergence. To see this, suppose that $I := \overline{\mathbb{N}}$, with Banach spaces E_n, $n \in \mathbb{N}$, $E = E_\infty$. Fix a metric approximation $\{\Phi_n : D(\Phi_n) \subset E_\infty \to E_n\}$ and set $\Phi_n^\infty := \Phi_n$ for each $n \in \mathbb{N}$. All we have to do is to find maps $\Phi_n^m : D(\Phi_n^m) \subset E_m \to E_n$ for any $n, m \in \mathbb{N}$, $n \neq m$ and maps $\Phi_\infty^n : D(\Phi_\infty^n) \subset E_n \to E_\infty$. Setting $D(\Phi_n^m) = D(\Phi_\infty^n) = \varnothing$ for $n, m \in \mathbb{N}$ does the job.* (5.50) *is satisfied trivially. The metric approximations are indeed recovering convergence* Cases 1–4.

5.14.1 Example

Let $I = [0,1]$ with the usual topology. Let $E_\varepsilon := L^2([0,1];\mu_\varepsilon)$, $\varepsilon \in I$, where each μ_ε is a fully supported regular Borel measure on $\mathscr{B}([0,1])$ and the map $I \ni \varepsilon \mapsto \mu_\varepsilon$ is vaguely continuous. Let $\varepsilon, \delta \in I$. Set $D(\Phi_\delta^\varepsilon) := C_0([0,1])$ when $\delta \neq \varepsilon$ and $\Phi_\delta^\varepsilon(f) := \widetilde{f}$ (the μ_δ-equivalence class). Let $\varepsilon_0 \in I$, $\varepsilon_n \in I$, $n \in \mathbb{N}$ with $\lim_n |\varepsilon_n - \varepsilon_0| = 0$. $C_0([0,1])$ is dense in $L^2([0,1];\mu_{\varepsilon_0})$. Also for $f \in C_0([0,1])$ by vague continuity

$$\lim_n \left\|\Phi_{\varepsilon_n}^{\varepsilon_0}(f)\right\|_{L^2([0,1];\mu_{\varepsilon_n})} = \lim_n \sqrt{\int_0^1 f^2 \,\mathrm{d}\mu_{\varepsilon_n}} = \sqrt{\int_0^1 f^2 \,\mathrm{d}\mu_{\varepsilon_0}} = \left\|\widetilde{f}\right\|_{L^2([0,1];\mu_{\varepsilon_0})}.$$

Hence $\{\Phi_{\varepsilon_n}^{\varepsilon_0}\}$ is a linear metric approximation. Also for $\varepsilon_1, \varepsilon_2 \in I$, $f, g \in C_0([0,1])$ and $\{\varepsilon_n\}$, ε_0 as before,

$$\begin{aligned}\lim_n \left\|\Phi_{\varepsilon_n}^{\varepsilon_1}(f) - \Phi_{\varepsilon_n}^{\varepsilon_2}(g)\right\|_{L^2([0,1];\mu_{\varepsilon_n})} &= \lim_n \sqrt{\int_0^1 |f-g|^2 \,\mathrm{d}\mu_{\varepsilon_n}} \\ &= \sqrt{\int_0^1 |f-g|^2 \,\mathrm{d}\mu_{\varepsilon_0}} = \left\|\Phi_{\varepsilon_0}^{\varepsilon_1}(f) - \Phi_{\varepsilon_0}^{\varepsilon_2}(g)\right\|_{L^2([0,1];\mu_{\varepsilon_0})},\end{aligned}$$

hence (5.50) is satisfied as well.

More general examples can be found in the next chapter.

6 Examples of varying Banach spaces

6.1 Hilbert spaces

Let H_n, $n \in \mathbb{N}$, H be separable Hilbert spaces (with Riesz-maps $\mathscr{R}_n$, $n \in \mathbb{N}$, $\mathscr{R}$ resp.) such that $\mathfrak{H} = \dot\bigcup_n H_n \dot\cup H$ has a strong linear metric approximation. We will give several examples below. The case of L^2-spaces is embedded into the case of L^p-spaces below.

$\mathfrak{H}^*$ carries a canonical strong linear asymptotic relation via the convention $\{y_n\} \subset \mathfrak{H}^*$ converges to $y \in \mathfrak{H}^*$ strongly (weakly) in $\mathfrak{H}^*$ if and only if $\mathscr{R}_{n_k}^{-1}(y_k)$ converges strongly (weakly) to $\mathscr{R}_{n_\infty}^{-1}(y)$ in $\mathfrak{H}$, where $y_k \in H_{n_k}^*$, $k \in \mathbb{N}$, $y \in H_{n_\infty}$ for some $n_k \in \overline{\mathbb{N}}$, $k \in \mathbb{N}$, $n_\infty \in \overline{\mathbb{N}}$, $\overline{\mathbb{N}}\text{-}\lim_k n_k = n_\infty$. Equivalently, one can transfer a linear metric approximation $\{\Phi_n\}$ on $\mathfrak{H}$ (which always exists by Theorem 5.40) to $\mathfrak{H}^*$ by setting $\Phi_n^* := \mathscr{R}_n \Phi_n \mathscr{R}^{-1}$ and $D(\Phi_n^*) := \{y \in H^* \mid \mathscr{R}^{-1}(y) \in D(\Phi_n)\}$. By the parallelogram identity and (A3') it is straightforward that $(\mathfrak{H}, \mathfrak{H}^*)$ are asymptotically dual.

Remark 6.1. *Let H be a Hilbert space. By the* parallelogram identity

$$\|x+y\|_H^2 + \|x-y\|_H^2 = 2\,\|x\|_H^2 + 2\,\|y\|_H^2$$

for every $x, y \in H$, which is a necessary and sufficient characterization of a norm being a Hilbert space norm, we can conclude that H is uniformly convex with modulus $\delta_H(\varepsilon) = 1 - \sqrt{1 - \varepsilon^2/4} = \varepsilon^2/8 + o(\varepsilon^4)$. Since H is isometrically isomorphic to its dual, it follows that H is uniformly smooth with asymptotically equivalent modulus $\eta_H \simeq \delta_H$. (Compare [Bea85, Part 3, Ch. II, §1]*).*

By Remark 6.1 applied to Theorem 5.57, $\mathfrak{H}$ and $\mathfrak{H}^*$ possess the asymptotic Kadeč-Klee property. See [KS03] for a direct proof (in the Hilbert space case) of the resulting properties as proved in Section 5.10.

The lemma below recovers [Kol05, Proposition 7.2] as a consequence of Theorem 5.68.

Lemma 6.2. *Suppose that $\mathfrak{H}$ has a strong linear asymptotic relation consisting of separable infinite dimensional Hilbert spaces H_n, $n \in \mathbb{N}$, H. Then $\mathfrak{H}$ possesses property* (I)*.*

Proof. By a well-known fact each of H_n, $n \in \mathbb{N}$, H is isometrically isomorphic to ℓ^2 (cf. [HHZ96, Theorem 30]). For each $n \in \mathbb{N}$, let $\mathscr{I}_n : H \to H_n$ be a surjective isometry. Let $(e_i)_{i=1}^\infty$ be a fundamental total orthonormal basis of H. Fix $n, k, d, l \in \mathbb{N}$, $l \le k \le d$.

Let $\{i_j\}_{j=1}^l$, $\{i'_j\}_{j=1}^{d-l}$ subsequences such that their union is $\{i\}_{i=1}^d$. Suppose that $M := \mathrm{lin}(e_{i_j})_{j=1}^l$ is isometrically isomorphic to some $X \subset Z := \mathrm{lin}(\mathscr{I}_n(e_i))_{i=1}^d$. X is closed as a finite dimensional linear subspace. By the Hilbert space projection theorem (cf. [Con90, Ch. I, §2]) there is a closed Hilbert subspace Y with $X \perp Y$, $X \oplus_2 Y = Z$, and since X is isometric to $\ell^2(l)$, Y is isometric to $\ell^2(d-l)$ which in turn is isometric to $N := \mathrm{lin}(e_{i'_j})_{j=1}^{d-l}$. Let $\Phi_{n,k,d,l} : M \to X$, $\Phi^{\perp}_{n,k,d,l} : N \to Y$ be the isometries found. Set $\Psi_{n,k,d,l} := \Phi_{n,k,d,l} \oplus_2 \Phi^{\perp}_{n,k,d,l} \oplus_2 \mathscr{I}_n$ on $M \oplus_2 N \oplus_2 (M \oplus_2 N)^{\perp} = H$, which are mutually orthogonal linear subspaces. It is clear that $\Psi_{n,k,d,l} : H \to H_n$ is a surjective isometry (by e.g. taking into account that in a Hilbert space $\|x+y\|_H = \|x-y\|_H$ if and only if $x \perp y$ and using the parallelogram identity to get $\|x+y\|_H^2 = \|x\|_H^2 + \|y\|_H^2$ for $x \in M$, $y \in N$, etc.) and that $\Psi_{n,k,d,l}\restriction_M = \Phi_{n,k,d,l}$. □

6.2 L^p-spaces

For $1 \le p < \infty$ and normed spaces X, Y, let $X \oplus_p Y$ be the algebraic direct sum normed via $\|(x,y)\|_{X \oplus_p Y} := \left(\|x\|_X^p + \|y\|_Y^p\right)^{1/p}$, $(x,y) \in X \oplus_p Y$. By [JL01, Section 4]

$$\ell^p(n),\ \ell^p,\ L^p([0,1];\mathrm{d}x),\ \ell^p(n) \oplus_p L^p([0,1];\mathrm{d}x),\ \ell^p \oplus_p L^p([0,1];\mathrm{d}x), \quad n \in \mathbb{N}, \tag{6.1}$$

is a complete listing, up to isometry, of the separable $L^p(\mu)$ spaces when $1 \le p < \infty$, $p \ne 2$, which are all mutually non-isometric. If $p = 2$, $\ell^2(n)$, $n \in \mathbb{N}$, ℓ^2 is a listing of isometric classes of separable Hilbert spaces. If μ is purely non-atomic, then $L^p(\mu)$ is isometric to $L^p([0,1];\mathrm{d}x)$ (see also [Lac74, Ch. 5, §14, Theorem 9, Corollary]). A general representation for these kinds of isometries is known, see Lemma 6.11 below.

Proposition 6.3. *Let $1 < p < \infty$, $p \ne 2$, $q := p/(p-1)$. Suppose that $\mathfrak{E}$ has a linear strong asymptotic relation such that each E_n, $n \in \mathbb{N}$, E is a separable (infinite dimensional) non-atomic L^p-space, say, $E_n = L^p(\mu_n)$, $n \in \mathbb{N}$, $E = L^p(\mu)$. Set $\mathfrak{L}^p := \mathfrak{E}$. Suppose that $\mathfrak{L}^q := \mathfrak{E}^*$ also has a linear strong asymptotic relation which is asymptotically dual. Then $\mathfrak{L}^p$ and $\mathfrak{L}^q$ have property* (I).

Proof. First note that by (6.1) non-atomic L^p spaces are always infinite dimensional. each $L^p(\mu_n)$, $n \in \mathbb{N}$, $L^p(\mu)$ is isometrically isomorphic to $L^p([0,1],\mathrm{d}x) =: L^p[0,1]$. Recall that the sequence of equivalence classes $\mathscr{H} = (\widetilde{h}_i \mid i \in \mathbb{N})$ such that (h_i) are the *Haar functions*, i.e. the functions defined on $[0,1]$ by

$$\begin{aligned}
&h_1(t) \equiv 1,\\
&h_{2^k+l}(t) = \begin{cases} \sqrt{2^k} & \text{for} \quad t \in \left[\dfrac{2l-2}{2^{k+1}}, \dfrac{2l-1}{2^{k+1}}\right), \\ -\sqrt{2^k} & \text{for} \quad t \in \left[\dfrac{2l-1}{2^{k+1}}, \dfrac{2l}{2^{k+1}}\right), \\ 0 & \text{for the other } \ t, \end{cases}\\
&l = 1, 2, \ldots, 2^k,\\
&k = 0, 1, 2, \ldots
\end{aligned}$$

constitutes a monotone Schauder basis for $L^p[0,1]$ (for a proof see [Sin70a, Chapter I, Example 2.3] and especially equation (2.17) therein for the monotonicity). Set

$g_i^{(p)} := \left(1/\left\|\widetilde{h}_i\right\|_{L^p[0,1]}\right)\widetilde{h}_i$, in particular, $g_i^{(p)} = \widetilde{(h_i)^{2/p}}$ obtaining a normalized monotone Schauder basis for $L^p[0,1]$. Do so similarly for $L^q[0,1]$. As proved in [Sin70a, Ch. II, §2], the associated biorthogonal functionals to $(g_i^{(p)})$ are $(g_i^{(q)})$ (and vice versa by reflexivity). We obtain a pair of normal bases.

By isometry and surjectivity we can find a normalized monotone Schauder basis $(e_i)_{i=1}^\infty$ on $L^p(\mu)$. Denote by $\mathscr{I}_n : L^p(\mu) \to L^p(\mu_n)$ the surjective isometries, that exist by (6.1). Clearly $(\mathscr{I}_n e_i)_{i=1}^\infty$ is a normalized monotone Schauder basis for $L^p(\mu_n)$. We shall need a couple of results from the classification theory of Banach spaces. Lemma 6.12 is new. We also refer to the survey paper [Ran01].

Lemma 6.4. *Let $1 \le p < \infty$, $p \ne 2$, and $M \subset \ell^p$ be a closed linear subspace of ℓ^p. Then the following conditions are equivalent:*

(i) M is 1-complemented in ℓ^p.

(ii) M is isometrically isomorphic to $\ell^p(\dim M)$.

(iii) There exist vectors $(e_i)_{i=1}^{\dim M}$ of norm one and of the form

$$e_i = \sum_{k \in S_i} \lambda_k u_k,$$

with $S_i \subset \mathbb{N}$, $S_i \cap S_j = \varnothing$ for $i \ne j$ such that $M = \overline{\mathrm{lin}}(e_i)_{i=1}^{\dim M}$, where $(u_k)_{k=1}^\infty$ is the unit vector basis and λ_k are scalars.

Moreover, if these conditions are satisfied, the norm one projection $P : \ell^p \to \ell^p$ with range M is given by

$$Px = \sum_{i=1}^{\dim M} {}_{\ell^q}\langle e_i^*, x\rangle_{\ell^p}\, e_i$$

with $e_i^* := J_{\ell^p} e_i$ (where J is the normalized duality map).

Proof. See [LT77, Ch. 2]. □

Lemma 6.5. *Let $1 \le p < \infty$, $p \ne 2$ and let M be a linear subspace of $L^p(X, \mathscr{F}, \mu)$. The following conditions on M are equivalent.*

(i) M is the range of a contractive projection on $L^p(X, \mathscr{F}, \mu)$.

(ii) There is a measure space $(Y, \mathscr{G}, \nu)$ such that M is isometrically isomorphic to $L^p(Y, \mathscr{G}, \nu)$.

Furthermore, in (ii) we can always choose $Y = X$, $\mathscr{G}$ a sub-σ-algebra of $\mathscr{F}$ and $\nu \ll \mu$.

Proof. See [And66] and [BL74, Theorem 4.1]. See also [Lac74, Ch. 6, §17, Lemma 8, Theorem 3] for a more detailed discussion. □

For the notion of orthogonality in Banach spaces (needed in the next lemma) we refer to Appendix B.5.

Lemma 6.6. *Let X be a real Banach space and M a complemented subspace of X such that $M \oplus N = X$. Let P be a linear projection on X with range M. Then $M \perp N$ if and only if P is contractive. Also,* $\mathrm{Id} - P$ *is a contractive projection from X onto $N = \ker P$ if and only if $N \perp M$.*

Proof. See [Kin84]. □

Lemma 6.7. *Let M be a finite dimensional linear subspace of an L^p-space. Then there exists N complementary to M with $N \perp M$ if and only if there exist k disjoint subsets of atoms $B_1, \ldots, B_k$ such that $M = \bigoplus_{i=1}^k M_i$, where M_i is either $L^p(B_i)$ or a hyperplane of $L^p(B_i)$.*

Proof. See [Lin85, Corollary 3]. □

We also need a characterization of L^p-isometries. Consider a definition first.

Definition 6.8. *Let $(X, \mathscr{F}, \mu)$, $(Y, \mathscr{G}, \nu)$ be measure spaces. A set map $T : \mathscr{F} \to \mathscr{G}$ is called a* regular set isomorphism *if*

(i) $T(X \setminus F) = T(X) \setminus T(F)$ *for all* $F \in \mathscr{F}$.

(ii) $T(\bigcup_{i=1}^\infty F_i) = \bigcup_{i=1}^\infty T(F_i)$ *for all* $F_i \in \mathscr{F}$, $i \in \mathbb{N}$, *pairwise disjoint.*

(iii) $\nu(T(F)) = 0$ *if and only if* $\mu(F) = 0$.

T is called measure preserving *if* $\mu(F) = \nu(T(F))$ *for all* $F \in \mathscr{F}$.

The following two lemmas can be found in [FJ03, Ch. 3.2]. The first one follows from the definitions, the second one can be proved with the help of the Radon-Nikodým Theorem. See also [Doo53, Ch. X, §1].

Lemma 6.9. *Let $(X, \mathscr{F}, \mu)$, $(Y, \mathscr{G}, \nu)$ be measure spaces. Let $T : \mathscr{F} \to \mathscr{G}$ be a regular set isomorphism. Then the following holds true.*

(i) *If* $F_1 \subset F_2$, *then* $T(F_1) \subset T(F_2)$.

(ii) $T(\bigcup_{i=1}^\infty F_i) = \bigcup_{i=1}^\infty T(F_i)$ *for all* $F_i \in \mathscr{F}$, $i \in \mathbb{N}$.

(iii) $T(\bigcap_{i=1}^\infty F_i) = \bigcap_{i=1}^\infty T(F_i)$ *for all* $F_i \in \mathscr{F}$, $i \in \mathbb{N}$.

(iv) $T(F_1) \cap T(F_2) = \varnothing$ *if and only if* $F_1 \cap F_2 = \varnothing$.

Lemma 6.10. *Let $(X, \mathscr{F}, \mu)$, $(Y, \mathscr{G}, \nu)$ be measure spaces. Let $T : \mathscr{F} \to \mathscr{G}$ be a regular set isomorphism. T induces a unique linear transformation (also denoted by T) from $M(X, \mathscr{F}, \mu)$ to $M(Y, \mathscr{G}, \nu)$ (where M denotes the set of equivalence classes of a.e. finite measurable functions) such that:*

(i) $T1_F = 1_{T(F)}$ *for all* $F \in \mathscr{F}$*.*

(ii) If $f_n \to f$ μ*-a.e. for some* f_n*,* $n \in \mathbb{N}$*,* f $\mathscr{F}$*-measurable, then* $Tf_n \to Tf$ ν*-a.e.*

(iii) $(Tf)^{-1}(B) = T(f^{-1}(B))$ *for every Borel set* $B \subset \mathbb{R}$ *and every* $\mathscr{F}$*-measurable* f*.*

(iv) $T(f \cdot g) = (Tf) \cdot (Tg)$ *for all* $\mathscr{F}$*-measurable* f, g*.*

(v) $T\left(\frac{1}{f}\right) = \frac{1}{Tf}$ *for all* $\mathscr{F}$*-measurable* f *with* $f \neq 0$ μ*-a.e.*

(vi) $T(|f|) = |T(f)|$ *for all* $\mathscr{F}$*-measurable* f*.*

(vii) $T(f^\alpha) = (Tf)^\alpha$ *for any* $\alpha > 0$ *and all* $\mathscr{F}$*-measurable* f *with* $f \geq 0$ μ*-a.e.*

The characterization of L^p-isometries follows.

Lemma 6.11 (Lamperti)**.** *Let* $(X, \mathscr{F}, \mu)$*,* $(Y, \mathscr{G}, \nu)$ *be* σ*-finite measure spaces. Suppose that* U *is a linear isometry from* $L^p(X, \mathscr{F}, \mu)$ *to* $L^p(Y, \mathscr{G}, \nu)$*, where* $1 \leq p < \infty$*,* $p \neq 2$*. Then there exists a regular set isomorphism* T *from* $\mathscr{F}$ *into* $\mathscr{G}$ *and a function* h *defined on* Y *so that*

$$(Uf)(y) = h(y) \cdot (Tf)(y) \tag{6.2}$$

and h *satisfies*

$$\int_{T(F)} |h|^p \mathrm{d}\nu = \int_{T(F)} \frac{\mathrm{d}(\mu \circ T^{-1})}{\mathrm{d}\nu} \mathrm{d}\nu = \mu(F) \quad \forall F \in \mathscr{F}. \tag{6.3}$$

Conversely, for any h *and* T *as above, the operator* U *satisfying* (6.2) *is an isometry.*

Proof. See [Lam58] or [FJ03, Theorem 3.2.5]. □

Lemma 6.12. *Let* $1 < p < \infty$*,* $p \neq 2$*,* $q := p/(p-1)$*. Fix two* σ*-finite measure spaces* μ, ν *on* $(X, \mathscr{F}, \mu)$ *and* $(Y, \mathscr{G}, \nu)$*. Let* $U : L^p(\mu) \to L^p(\nu)$ *be a linear isometry. Let* $J_1 : L^p(\mu) \to L^q(\mu)$*,* $J_2 : L^p(\nu) \to L^q(\nu)$ *be the normalized duality maps. Then the map*

$$U^* := J_2 U J_1^{-1} : L^q(\mu) \to L^q(\nu)$$

is a linear isometry as well.

Proof. Recall that J_1^{-1} is the normalized duality map from $L^q(\mu)$ to $L^p(\mu)$. Therefore, $g = J_1^{-1}(f)$ if and only if

$$g(x) = \frac{|f(x)|^{q-1} \operatorname{sign}(f(x))}{\|f\|_{L^q(\mu)}^{q-2}} \quad \text{for } \mu\text{-a.e. } x \in X$$

and $g = J_2(f)$ if and only if

$$g(y) = \frac{|f(y)|^{p-1} \operatorname{sign}(f(y))}{\|f\|_{L^p(\nu)}^{p-2}} \quad \text{for } \nu\text{-a.e. } y \in Y.$$

Compare with Paragraph B.2.1 in the appendix. Let $f \in L^q(\mu)$. Justified by Lemma 6.11, suppose that U is of the form (6.2) with an regular set isomorphism T and a function h.

Now, using Lemma 6.10

$$
\begin{aligned}
&U^* f = J_2 U J_1^{-1} f \\
=&J_2 U \left[|f|^{q-1} \operatorname{sign}(f) \, \|f\|_{L^q(\mu)}^{2-q} \right] \\
=&J_2 U \left[|f|^{1/(p-1)} \operatorname{sign}(f) \, \|f\|_{L^q(\mu)}^{(p-2)/(p-1)} \right] \\
=&J_2 \left[hT \left[|f|^{1/(p-1)} \operatorname{sign}(f) \, \|f\|_{L^q(\mu)}^{(p-2)/(p-1)} \right] \right] \\
&\text{use Lemma 6.10 (vi) and (vii)} \\
=&J_2 \left[\left[|h^{p-1} T(f)|^{1/(p-1)} \operatorname{sign}(hT(f)) \right] \|f\|_{L^q(\mu)}^{(p-2)/(p-1)} \right] \\
=&\left| \left[|h^{p-1} T(f)|^{1/(p-1)} \operatorname{sign}(hT(f)) \right] \|f\|_{L^q(\mu)}^{(p-2)/(p-1)} \right|^{p-1} \\
&\cdot \operatorname{sign} \left(\left[|h^{p-1} T(f)|^{1/(p-1)} \operatorname{sign}(hT(f)) \right] \|f\|_{L^q(\mu)}^{(p-2)/(p-1)} \right) \\
&\cdot \left\| \left[|h^{p-1} T(f)|^{1/(p-1)} \operatorname{sign}(hT(f)) \right] \|f\|_{L^q(\mu)}^{(p-2)/(p-1)} \right\|_{L^p(\nu)}^{2-p} \\
=&\left| h^{p-1} T(f) \right| \|f\|_{L^q(\mu)}^{p-2} \operatorname{sign}\left(hT(f) \right) \\
&\cdot \left\| |h^{p-1} T(f)|^{1/(p-1)} \operatorname{sign}(hT(f)) \right\|_{L^p(\nu)}^{2-p} \|f\|_{L^q(\mu)}^{(p-2)(2-p)/(p-1)} \\
=&h^{p-1} T(f) \, \|f\|_{L^q(\mu)}^{p-2} \left\| |h^{p-1} T(f)|^{1/(p-1)} \operatorname{sign}(hT(f)) \right\|_{L^p(\nu)}^{2-p} \|f\|_{L^q(\mu)}^{(p-2)(2-p)/(p-1)} .
\end{aligned}
$$

Note that $f \mapsto h^{p-1} T(f)$ is a linear isometry from $L^q(\mu) \to L^q(\nu)$. To see this, consider $F \in \mathscr{F}$ and

$$
\int_{T(F)} (|h|^{p-1})^q \, \mathrm{d}\nu = \int_{T(F)} |h|^p \, \mathrm{d}\nu = \mu(F)
$$

and apply Lemma 6.11. Also the map $f \mapsto |f|^{1/(p-1)} \operatorname{sign} f$ is a duality map with gauge $t \mapsto t^{1/(p-1)}$ for L^q. Therefore,

$$
\left\| \left[|h^{p-1} T(f)|^{1/(p-1)} \operatorname{sign}(hT(f)) \right] \right\|_{L^p(\nu)}^{2-p} = \left\| h^{p-1} T(f) \right\|_{L^q(\nu)}^{(2-p)/(p-1)} = \|f\|_{L^q(\mu)}^{(2-p)/(p-1)}
$$

Finally,

$$
p - 2 + \frac{2-p}{p-1} + \frac{(p-2)(2-p)}{p-1} = 0,
$$

and hence

$$
U^* f = h^{p-1} T(f)
$$

which we have already verified to be a linear isometry. $\square$

Proof of Proposition 6.3 continued. Fix $k, n, d \in \mathbb{N}$, $d \geq k$. Considering monotonicity of $(e_i)_{i=1}^{\infty}$, by Lemma 6.5 and the classification of L^p-spaces, $\mathrm{lin}(e_i)_{i=1}^{k}$ and $\mathrm{lin}(e_i)_{i=1}^{d}$ are isometric to $\ell^p(k)$, $\ell^p(d)$ resp. Set $v_i^k := \mathscr{J}_k(u_i)$ for each $1 \leq i \leq k$, where $\mathscr{J}_k : \ell^p(k) \to \mathrm{lin}(e_i)_{i=1}^{k}$ is an isometry and (u_i) is the unit vector basis.

Suppose that $l \leq k$ and that there is a subsequence $(v_{i_j}^k)_{j=1}^{l}$ of $(v_i^k)_{i=1}^{k}$ such that $M := \mathrm{lin}(v_{i_j}^k)_{j=1}^{l}$ is isometric to $X \subset Z := \mathrm{lin}(\mathscr{I}_n(e_i))_{i=1}^{d}$. $(\mathscr{J}_k^{-1}(v_{i_j}^k))_{j=1}^{l}$ are disjointly supported in $\ell^p(k)$, therefore by Lemma 6.4 M is isometric to $\ell^p(l)$ and so is X. But X is a subspace of Z and Z is isometric to $\ell^p(d)$. Clearly by Lemma 6.5 X is in the range of a contractive projection. Let Y be the complement of X in Z, i.e., $X \oplus Y = Z$. Lemma 6.6 yields $X \perp Y$. But by Lemma 6.4 there are l atoms B_j in Z such that $X = \bigoplus_{j=1}^{l} B_j$. By Lemma 6.7 $Y \perp X$. This gives that Y is isometric to $\ell^p(d-l)$ by Lemma 6.6 and Lemma 6.4. On the other hand, the complement N of M in $\mathrm{lin}(e_i)_{i=1}^{d}$ is isometric to $\ell^p(d-l)$ by exactly the same orthogonality arguments.

We would like to extend the partial isometries to an isometry $\Psi : L^p(\mu) \to L^p(\mu_n)$ such that if we restrict Ψ we obtain the original isometries.

Firstly, $M \perp N$ and $N \perp M$. Since both are finite dimensional, by Lemma 6.7 we can find mutually disjoint atoms spanning both M and N. A similar statement holds for X and Y. Let $\Psi_1 : M \to X$ and $\Psi_2 : N \to Y$ be the isometries found above. Let $(x, y) \in M \oplus_p N$. Set $\Psi : (x, y) \mapsto (\Psi_1(x), \Psi_2(y))$. Clearly

$$\|\Psi(x,y)\|_{X \oplus_p Y} = \left(\|\Psi_1(x)\|_{E_n}^p + \|\Psi_2(y)\|_{E_n}^p\right)^{1/p} = \left(\|x\|_E^p + \|y\|_E^p\right)^{1/p} = \|(x,y)\|_{M \oplus_p N}.$$

But by disjointness and a well-known property of L^p-norms (cf. [LT79, Ch. 1.b], [MN91]) $\|x\|_E^p + \|y\|_E^p = \|x+y\|_E^p$ and $\|\Psi_1(x)\|_{E_n}^p + \|\Psi_2(y)\|_{E_n}^p = \|\Psi_1(x) + \Psi_2(y)\|_{E_n}^p$. Therefore $\Psi(z) := \Psi_1(x) + \Psi_2(y)$ (whenever $z = x + y$, $x \in M$, $y \in N$) is an isometry in the original norms of E and E_n.

Let J be the normalized duality map of E. Set $e_i^* := Je_i$ for $i \in \mathbb{N}$, which are the unique Schauder biorthogonal coefficient functionals. Let $E \ni x = \sum_{i=1}^{\infty} {}_{E^*}\langle e_i^*, x\rangle_E \, e_i$ and $E^* \ni y = \sum_{i=1}^{\infty} {}_{E^*}\langle y, e_i\rangle_E \, e_i^*$. Define Ψ^* and $\mathscr{I}_n^*$ as in Lemma 6.12 which therein are proved to be isometries. Define

$$\Theta(x) := \sum_{i=1}^{d} {}_{E_n^*}\langle \Psi^*(e_i^*), \Psi(x)\rangle_{E_n} \, \Psi(e_i) + \sum_{i=d+1}^{\infty} {}_{E_n^*}\langle \mathscr{I}_n^*(e_i^*), \mathscr{I}_n(x)\rangle_{E_n} \, \mathscr{I}_n(e_i).$$

and

$$\Theta^*(y) := \sum_{i=1}^{d} {}_{E_n^*}\langle \Psi^*(y), \Psi(e_i)\rangle_{E_n} \, \Psi^*(e_i^*) + \sum_{i=d+1}^{\infty} {}_{E_n^*}\langle \mathscr{I}_n^*(y), \mathscr{I}_n(e_i)\rangle_{E_n} \, \mathscr{I}_n^*(e_i^*).$$

Θ and Θ^* are clearly linear and continuous. Take into account that

$$\begin{aligned}
{}_{E_n^*}\langle \Psi^*(e_i^*), \Psi(x)\rangle_{E_n} &= {}_{E^*}\langle e_i^*, x\rangle_E, \quad \forall i \in \mathbb{N},\\
{}_{E_n^*}\langle \Psi^*(y), \Psi(e_i)\rangle_{E_n} &= {}_{E^*}\langle y, e_i\rangle_E, \quad \forall i \in \mathbb{N},\\
{}_{E_n^*}\langle \mathscr{I}_n^*(e_i^*), \mathscr{I}_n(x)\rangle_{E_n} &= {}_{E^*}\langle e_i^*, x\rangle_E, \quad \forall i \in \mathbb{N},\\
{}_{E_n^*}\langle \mathscr{I}_n^*(y), \mathscr{I}_n(e_i)\rangle_{E_n} &= {}_{E^*}\langle y, e_i\rangle_E, \quad \forall i \in \mathbb{N},
\end{aligned} \tag{6.4}$$

by Lemma 5.71. Since Ψ and $\mathscr{I}_n$ are linear and continuous, they interchange with the series. We also see that $\|\Theta(x)\|_{E_n} \leq \|x\|_E$ for every $x \in E$. Also

$$\begin{aligned}
\|\Theta\|_{\mathscr{L}(E,E_n)} &= \sup\left\{\|\Theta(x)\|_{E_n} \mid x \in E,\ \|x\|_E \leq 1\right\} \\
&\geq \sup\left\{\|\Theta(x)\|_{E_n} \mid x \in \operatorname{lin}(e_i)_{i=1}^d,\ \|x\|_E \leq 1\right\} \\
&= \sup\left\{\|\Psi(x)\|_{E_n} \mid x \in \operatorname{lin}(e_i)_{i=1}^d,\ \|x\|_E \leq 1\right\} \\
&= 1.
\end{aligned}$$

We have proved that $\|\Theta\|_{\mathscr{L}(E,E_n)} = 1$. Replace Θ^* by its normalized self.

By orthogonality, (6.4), continuity and linearity we get that

$$_{E_n^*}\langle \Theta^*(y), \Theta(x)\rangle_{E_n} = {}_{E^*}\langle y, x\rangle_E .$$

Hence both are isometries by Lemma 5.71. Surjectivity follows from surjectivity of Ψ, $\mathscr{I}_n$ and the invariance of (complemented) subspaces. $\Theta\restriction_M = \Psi_1$ follows from $M \subset \operatorname{lin}(e_i)_{i=1}^d$.

(I) is proved. The case of $\mathfrak{L}^q$ works similarly. $\square$

We can also say something about the asymptotic Kadeč-Klee property for asymptotic relations of L^p spaces with fixed p. Consider the following properties of convexity and smoothness:

If $(\Omega, \mathcal{F}, \mu)$ is a σ-finite measure space and $1 < p < \infty$. Then $L^p(\Omega, \mathcal{F}, \mu)$ is both uniformly convex and uniformly smooth with moduli depending on p but not on μ:

$$\delta_{L^p}(\varepsilon) := \begin{cases} \frac{p-1}{8}\varepsilon^2 + o(\varepsilon^2), & 1 < p \leq 2, \\ \frac{2^p}{p}\varepsilon^p + o(\varepsilon^p), & 2 \leq p < \infty; \end{cases} \quad \eta_{L^p}(\varepsilon) := \begin{cases} \frac{1}{p}\varepsilon^p + o(\varepsilon^p), & 1 < p \leq 2, \\ \frac{p-1}{2}\varepsilon^2 + o(\varepsilon^2), & 2 \leq p < \infty. \end{cases} \tag{6.5}$$

Cf. [Han56] and [Bea85, Part 3, Ch. II, §1, Proposition 8]. See [LT79, Ch. 1.e, p. 63] for the constants.

Finally we will present concrete metric approximations for L^p-spaces in order to obtain asymptotic relations. We divide the presentation into a couple of subsections. In all of what follows, let X be a Hausdorff topological space.

6.2.1 Varying measure

Let $(X, \mathscr{B}(X))$ be a measurable space such that X is a Hausdorff topological space and $\mathscr{B}(X)$ is the *Borel σ-algebra*, i.e., $\mathscr{B}(X)$ is generated by the open sets in X. We assume for convenience that $(X, \mathscr{B}(X))$ is a separable measurable space, which is for example the case when X is second countable.

Suppose that μ is a σ-finite measure on $\mathscr{B}(X)$ such that (cf. [Bau74, Hal74, Bog07a, Bog07b]):

(i) $\mu(B) \in [0, +\infty]$ for each $B \in \mathscr{B}(X)$ (*positivity*),

(ii) for each $x \in X$ there exists an open neighborhood $U \subset X$ of x such that $\mu(U) < +\infty$ (*local finiteness*),

(iii) for every Borel set $B \in \mathscr{B}(X)$ we have

$$\mu(B) = \sup\left\{\mu(C) \;\middle|\; C \subset B,\ C \text{ compact}\right\}$$

(*inner regularity*),

(iv) for every Borel set $B \in \mathscr{B}(X)$ we have

$$\mu(B) = \inf\left\{\mu(O) \;\middle|\; O \supset B,\ O \text{ open}\right\}$$

(*outer regularity*),

(v) for every compact set $C \in \mathscr{B}(X)$

$$\mu(C) < +\infty$$

(*Borel*),

(vi) $\mu(U) > 0$ for every $\varnothing \neq U \subset X$, U open (*full support*).

A measure satisfying (ii) and (iii) is called a *Radon measure*. A measure satisfying (ii), (iii), (v) is called *Borel measure*. If, in addition, a Borel measure satisfies (iv), it is called a *regular Borel measure*. In general, one defines the *support* $\operatorname{supp}\mu$ of a positive measure μ by

$$\operatorname{supp}\mu := \{x \in X \mid \exists O_x \ni x \text{ open},\ \mu(O_x) > 0\}\,.$$

Let μ_n, $n \in \mathbb{N}$, μ be fully supported regular positive Borel measures on X, i.e., all of the above conditions are satisfied. Assume that we are in one of the two cases:

$$\begin{aligned} &X \text{ is Souslinean},\ \mu_n,\ n \in \mathbb{N},\ \mu \text{ are finite},\ \mu_n \xrightharpoonup{w} \mu \text{ weakly},\\ &\text{i.e., } \lim_n \int_X f\,\mathrm{d}\mu_n = \int_X f\,\mathrm{d}\mu \quad \forall f \in C_b(X), \end{aligned} \tag{6.6}$$

where $C_b(X)$ denotes the continuous real-valued functions on X that are bounded.

$$\begin{aligned} &X \text{ is locally compact},\ \mu_n \xrightharpoonup{v} \mu \text{ vaguely},\\ &\text{i.e., } \lim_n \int_X f\,\mathrm{d}\mu_n = \int_X f\,\mathrm{d}\mu \quad \forall f \in C_0(X), \end{aligned} \tag{6.7}$$

where $C_0(X)$ denotes the continuous real-valued functions on X that have compact support.

Lemma 6.13. *If X is Souslinean and μ is finite and fully supported, then the μ-classes of $C_b(X)$ are dense in $L^p(X;\mu)$ for any $1 \leq p < \infty$.*

Proof. Let $\widetilde{C_b(X)}$ be the μ-classes of $C_b(X)$. By full support, $\widetilde{C_b(X)}$ separates the points of X. By [Sch73, Lemma 18, p. 108]

$$\widetilde{C_b(X)} \quad \text{generates} \quad \mathscr{B}(X). \tag{6.8}$$

Define

$$\mathscr{M} := \left\{ u \in \mathscr{B}_b(X) \mid \exists u_n \in C_b(X),\ n \in \mathbb{N} \text{ such that } \|u - u_n\|_{L^p(X;\mu)} \to 0 \right\}.$$

By definition, $C_b(X) \subset \mathscr{M}$ and $1_X \in \mathscr{M}$. Let us prove that $\mathscr{M}$ is a monotone vector space. Let

$$0 \le u^1 \le u^2 \le u^3 \le \cdots \le f,$$

such that $u^i \in \mathscr{M}$, $i \in \mathbb{N}$, f bounded and $u^i \uparrow f$ as $i \to \infty$. We have to prove that $f \in \mathscr{M}$. Note that by finiteness of the measure, it holds that $\mathscr{B}_b(X) \subset L^p(X;\mu)$. Let $u^i_n \in C_b(X)$, $i, n \in \mathbb{N}$ with $\left\|u^i_n - u^i\right\|_{L^p(X;\mu)} \to 0$ as $n \to \infty$ (for all i). By Lemma 5.27 and the triangle inequality, there is a sequence of natural numbers $\{i_n\}$ with $i_n \uparrow \infty$ as $n \to \infty$ such that

$$\begin{aligned} \overline{\lim_n} \left\|u^{i_n}_n - f\right\|_{L^p(X;\mu)} &\le \overline{\lim_i}\,\overline{\lim_n} \left[\left\|u^i_n - u^i\right\|_{L^p(X;\mu)} + \left\|u^i - f\right\|_{L^p(X;\mu)} \right] \\ &\le \overline{\lim_i}\,\overline{\lim_n} \left\|u^i_n - u^i\right\|_{L^p(X;\mu)} + \overline{\lim_i}\,\overline{\lim_n} \left\|u^i - f\right\|_{L^p(X;\mu)} = \overline{\lim_i} \left\|u^i - f\right\|_{L^p(X;\mu)}. \end{aligned}$$

But

$$|u^i - f|^p \le 2^p \|f\|^p_\infty \in L^1(X;\mu).$$

Hence an application of Lebesgue's dominated convergence theorem gives that

$$\lim_i \left\|u^i - f\right\|_{L^p(X;\mu)} = 0.$$

Now by (6.8) and the monotone class theorem, cf. [Sha88, Appendix A.0], $\mathscr{B}_b(X) \subset \mathscr{M}$, hence $\mathscr{B}_b(X) = \mathscr{M}$. But $\mathscr{B}_b(X)$ is dense in $L^p(X;\mu)$. Hence we have proved that $C_b(X)$ is dense in $L^p(X;\mu)$. □

Lemma 6.14. *If X is locally compact and μ is fully supported, then the μ-classes of $C_0(X)$ are dense in $L^p(X;\mu)$ for any $1 \le p < \infty$.*

Proof. The proof is contained in the proof of Lemma 6.16 below. □

Let $1 < p < \infty$, $q := p/(p-1)$. Set $C := \widetilde{C_b(X)}$ if we are in case one or $C := \widetilde{C_0(X)}$ if we are in case two. Define

$$\Phi_n : C \subset L^p(X;\mu) \to L^p(X;\mu_n),\ \ \Phi_n(x) := x,\ \ n \in \mathbb{N},$$

and

$$\Psi_n : C \subset L^q(X;\mu) \to L^q(X;\mu_n),\ \ \Psi_n(x) := x,\ \ n \in \mathbb{N},$$

which are well-defined linear maps by full support of the μ_n's. By Lemmas 6.13 and 6.14 C is dense in both $L^p(X;\mu)$ and $L^q(X;\mu)$. Furthermore, by weak resp. vague convergence and the fact that C is invariant under the maps $t \mapsto |t|^p$, $s \mapsto |s|^q$,

$$\lim_n \|\Phi_n(f)\|_{L^p(X;\mu_n)} = \lim_n \left(\int_X |f|^p \,\mathrm{d}\mu_n\right)^{1/p} = \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{1/p} = \|f\|_{L^p(X;\mu)}\,, \quad \forall f \in C,$$

and

$$\lim_n \|\Psi_n(f)\|_{L^q(X;\mu_n)} = \lim_n \left(\int_X |f|^q \,\mathrm{d}\mu_n\right)^{1/q} = \left(\int_X |f|^q \,\mathrm{d}\mu\right)^{1/q} = \|f\|_{L^q(X;\mu)}\,, \quad \forall f \in C.$$

We have proved that $\{(\Phi_n, C)\}$, $\{(\Psi_n, C)\}$ are linear metric approximations (i.e., (B1), (B2) and (BL) hold) for

$$\mathfrak{L}^p := \dot{\bigcup_n} L^p(X;\mu_n) \dot{\cup} L^p(X;\mu), \quad \mathfrak{L}^q := \dot{\bigcup_n} L^q(X;\mu_n) \dot{\cup} L^q(X;\mu),$$

respectively. Since C is an algebra, also

$$\begin{aligned} \lim_n {}_{L^p(X;\mu_n)}\langle \Phi_n(f), \Psi_n(g)\rangle_{L^q(X;\mu_n)} &= \lim_n \int_X fg \,\mathrm{d}\mu_n \\ &= \int_X fg \,\mathrm{d}\mu = {}_{L^p(X;\mu)}\langle f, g\rangle_{L^q(X;\mu)}, \quad \forall f, g \in C, \end{aligned}$$

which yields that $\mathfrak{L}^p$ and $\mathfrak{L}^q$ are mutually asymptotically dual. Asymptotic reflexivity is clear, too. The asymptotic Kadeč-Klee property was verified above as well as the asymptotic isometry property.

We note that we merely need that μ is regular, and not the μ_n's.

6.2.2 Varying domain

Suppose that X is either Souslinean or locally compact. Suppose that there is a sequence of Borel subsets $\{X_n\}$ of X such that $X_n \subset X_{n+1}$ for every n and $X_n \uparrow X$, directed by inclusion of sets.

Furthermore, suppose that $C_b(X)$, $C_0(X)$ resp. is the *strict inductive limit* (in the sense of locally convex linear topologies, cf. [Sch71, Ch. II, §6] or [RS80, Ch. V.4]) of $\{C_b(X_n)\}$, $\{C_0(X_n)\}$ resp., in particular, for each $f \in C_b(X)$, there exists an index $n_0 \in \mathbb{N}$ such that $f \in C_b(X_n)$ for every $n \geq n_0$ and $\bigcup_n C_b(X_n) = C_b(X)$; similarly for C_0 resp.

Set either $C_n := C_b(X_n)$, $n \in \mathbb{N}$ or $C_n := C_0(X_n)$, $n \in \mathbb{N}$. Suppose that for each $n \in \mathbb{N}$ we are given a regular positive Borel measure μ_n on $\mathscr{B}(X)$ such that $\operatorname{supp}\mu_n = \overline{X_n}$ (closure), μ_n is finite in the first case and converges weakly, resp. vaguely to a fully supported regular positive Borel measure μ on X.

Let $1 < p < \infty$, $q := p/(p-1)$. Define

$$\Phi_n : D(\Phi_n) := \widetilde{C_n} \subset L^p(X;\mu) \to L^p(X_n;\mu_n), \;\; \Phi_n(x) := x, \;\; n \in \mathbb{N},$$

and
$$\Psi_n : D(\Phi_n) := \widetilde{C_n} \subset L^q(X;\mu) \to L^q(X_n;\mu_n), \ \ \Psi_n(x) := x, \ \ n \in \mathbb{N},$$
which are well-defined linear maps by full support of the μ_n's on X_n resp.

Analogously to the preceding section we verify that $\{(\Phi_n, D(\Phi_n))\}$ and $\{(\Psi_n, D(\Psi_n))\}$ are linear metric approximations for

$$\mathfrak{L}^p := \dot{\bigcup_n} L^p(X_n;\mu_n) \dot{\cup} L^p(X;\mu), \quad \mathfrak{L}^q := \dot{\bigcup_n} L^q(X_n;\mu_n) \dot{\cup} L^q(X;\mu),$$

respectively and that $\mathfrak{L}^p$ is asymptotically dual to $\mathfrak{L}^q$. Asymptotic reflexivity is clear, too.

We note again that we merely need that μ is regular, and not the μ_n's.

6.2.3 Varying p

Let X be a Hausdorff space with Borel σ-algebra $\mathscr{B}(X)$. Denote by $\mathscr{M}_b(X)$ the topological vector space of totally finite signed Borel measures on $\mathscr{B}(X)$ which is a Banach space with the total variation norm $\|\mu\|_{\mathrm{var}} := |\mu|(X)$, where the variation $|\mu|$ of a signed measure μ is a positive measure defined by

$$|\mu|(B) := \sup\left\{ \sum_{i=1}^{k} |\mu(B_i)| \ \middle|\ k \in \mathbb{N}, \ B_i \in \mathscr{B}(X) \cap B, \text{ pairwise disjoint} \right\}, \quad B \in \mathscr{B}(X).$$

Suppose either that X is metrizable or regular and strongly Lindelöf (i.e. is a *Lindelöf space*). Then $(\mathscr{M}_b(X), \|\cdot\|_{\mathrm{var}})$ is the topological dual space of $(C_b(X), \|\cdot\|_\infty)$ with bilinear dualization
$$\langle f, \mu\rangle = \int_X f \,\mathrm{d}\mu$$
such that a Cauchy-Schwarz type inequality holds with the norms $\|\cdot\|_\infty$, $\|\cdot\|_{\mathrm{var}}$ resp., cf. [Bog07b, Corollary 6.3.5 and §7.9]. The topology of weak convergence of measures coincides with the sequential weak* topology on $\mathscr{M}_b(X)$. By duality for any $\mu \in \mathscr{M}_b(X)$

$$\|\mu\|_{\mathrm{var}} = \sup\left\{ \left| \int_X f \,\mathrm{d}\mu \right| \ \middle|\ f \in C_b(X), \ \|f\|_\infty \le 1 \right\}.$$

Suppose that we are given a sequence of fully supported positive $\{\mu_n\} \subset \mathscr{M}_b(X)$ and a fully supported regular positive Borel measure $\mu \in \mathscr{M}_b(X)$ such that

$$\lim_n \|\mu_n - \mu\|_{\mathrm{var}} = 0.$$

Suppose additionally, that X is a Souslin space.

Let $p_n \in (1,\infty)$, $n \in \mathbb{N}$, $p \in (1,\infty)$ such that $\lim_n p_n = p$. Let $q_n := p_n/(p_n - 1)$, $n \in \mathbb{N}$, $q := p/(p-1)$. Set $C := \widetilde{C_b(X)}$.

Define
$$\Phi_n : C \subset L^p(X;\mu) \to L^{p_n}(X;\mu_n), \ \ \Phi_n(x) := x, \ \ n \in \mathbb{N},$$

and

$$\Psi_n : C \subset L^q(X;\mu) \to L^{q_n}(X;\mu_n), \quad \Psi_n(x) := x, \quad n \in \mathbb{N},$$

which are well-defined linear maps by full support of the μ_n's. The Φ_n's and Ψ_n's are densely defined by Lemma 6.13.

Let $f \in C_b(X)$. If $\|f\|_\infty \le 1$, set $K_f := \|f\|_\infty^{\inf_n p_n}$ and if $\|f\|_\infty > 1$, set $K_f := \|f\|_\infty^{\sup_n p_n}$. For μ-a.a. $x \in X$, $\sup_n |f(x)|^{p_n} \le K_f \in L^1(X;\mu) \cap L^\infty(X;\mu)$. Therefore by Lebesgue's dominated convergence theorem

$$\lim_n \int_X \left||f|^{p_n} - |f|^p\right| \,\mathrm{d}\mu = 0$$

and hence

$$\begin{aligned}
&\left|\int_X |f|^{p_n} \,\mathrm{d}\mu_n - \int_X |f|^p \,\mathrm{d}\mu\right| \\
\le &\left|\int_X |f|^{p_n} \,\mathrm{d}(\mu_n - \mu)\right| + \left|\int_X |f|^{p_n} - |f|^p \,\mathrm{d}\mu\right| \\
\le &K_f \,\|\mu_n - \mu\|_{\mathrm{var}} + \int_X \left||f|^{p_n} - |f|^p\right| \,\mathrm{d}\mu
\end{aligned} \tag{6.9}$$

which clearly tends to zero as $n \to \infty$.

Combined with

Lemma 6.15. *For each $P \in [1,\infty)$, the function*

$$[1,P] \times [0,\infty) \ni (p,t) \mapsto \begin{cases} t^{1/p}, & \text{if } t > 0, \\ 0, & \text{if } t = 0, \end{cases}$$

is continuous.

Proof. Let $P \in [1,\infty)$. For $t \neq 0$, $(p,t) \mapsto \exp[(1/p)\log(t)]$ is obviously continuous. Let $p \in [1,P]$ and $\{(p_n,t_n)\}$ a sequence in $\mathbb{R}^2$ with $\lim_n (p_n,t_n) = (p,0)$. For large n, $t_n \le 1$ and therefore

$$\left|t_n^{1/p_n}\right| \le \left|t_n^{1/P}\right| \to 0$$

as $n \to \infty$. The claim is proved. □

we get that

$$\lim_n \|\Phi_n(f)\|_{L^{p_n}(X;\mu_n)} = \lim_n \left(\int_X |f|^{p_n} \,\mathrm{d}\mu_n\right)^{1/p_n} = \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{1/p} = \|f\|_{L^p(X;\mu)}.$$

Hence $\{(\Phi_n, C)\}$ is a metric approximation for

$$\mathfrak{L}^{\to p} := \bigcup_n L^{p_n}(X;\mu_n) \dot\cup L^p(X;\mu).$$

With $\{(\Psi_n, C)\}$, the case of $\mathfrak{L}^{\to q}$ works exactly the same way. Asymptotic duality is easy, as is asymptotic reflexivity.

Note that we have excluded the case that $p, q \in \{1, \infty\}$. An application with varying domains as above is easy to construct, too.

The attentive reader will have noticed that this example allows us to talk about convergence of vectors along "$L^{p_n} \to L^p$" in the special case of one fixed measure μ.

6.2.4 $p \to 1$ and $q \to \infty$

In this paragraph, we present an example that is built to fit for an application given later. With the knowledge of the preceding sections, generalizations can easily be given. An asymptotic relation on

$$\mathfrak{L}^1 := \dot{\bigcup_n} L^1(X; \mu_n) \dot{\cup} L^1(X; \mu)$$

with weakly converging measures $\mu_n \to \mu$ can be constructed as in Section 6.2.1.

Let $d \geq 1$, $\Omega \subset \mathbb{R}^d$ be a bounded open domain. Let $p_n \in (1, \infty)$, $n \in \mathbb{N}$ such that $\lim_n p_n = 1$. Let $q_n := p_n/(p_n - 1)$ for each $n \in \mathbb{N}$. Then $\lim_n q_n = \infty$. Set $C_1 := C_0(\Omega)$, $C_2 := L^\infty(\Omega; \mathrm{d}x)$.

Define

$$\Phi_n : C_1 \subset L^1(\Omega; \mathrm{d}x) \to L^{p_n}(\Omega; \mathrm{d}x), \quad \Phi_n(x) := x, \quad n \in \mathbb{N},$$

and

$$\Psi_n : C_2 = L^\infty(\Omega; \mathrm{d}x) \to L^{q_n}(\Omega; \mathrm{d}x), \quad \Psi_n(x) := x, \quad n \in \mathbb{N},$$

which both are well-defined linear operators for all $n \in \mathbb{N}$. The density $\widetilde{C_0(\Omega)} \subset L^1(\Omega; \mathrm{d}x)$ is well-known. Let $f \in C_0(\Omega)$. If $\|f\|_\infty \leq 1$, set $K_f := \|f\|_\infty$ and if $\|f\|_\infty > 1$, set $K_f := \|f\|_\infty^{\sup_n p_n}$. For $\mathrm{d}x$-a.a. $x \in \Omega$, $\sup_n |f(x)|^{p_n} \leq K_f \in L^1(\Omega; \mathrm{d}x) \cap L^\infty(\Omega; \mathrm{d}x)$. Therefore by Lebesgue's dominated convergence theorem

$$\overline{\lim_n} \left| \int_\Omega |f|^{p_n} \, \mathrm{d}x - \int_\Omega |f| \, \mathrm{d}x \right| \leq \overline{\lim_n} \int_\Omega \left| |f|^{p_n} - |f| \right| \, \mathrm{d}x = 0.$$

With Lemma 6.15 we see that

$$\lim_n \|\Phi_n(f)\|_{L^{p_n}} = \lim_n \left(\int_\Omega |f|^{p_n} \, \mathrm{d}x \right)^{1/p_n} = \int_\Omega |f| \, \mathrm{d}x = \|f\|_{L^1}.$$

On the other hand, it is well-known that for all $f \in L^\infty(\Omega; \mathrm{d}x)$ (in fact, we need only a bounded measure)

$$\lim_{q \to \infty} \left(\int_\Omega |f|^q \, \mathrm{d}x \right)^{1/q} = \|f\|_\infty,$$

see e.g. [Ran02, Exercise 8.8.5]. From this formula we see that $\{(\Phi_n, C_1)\}$, $\{(\Psi_n, C_2)\}$ respectively are metric approximations for

$$\mathfrak{L}^{\to 1} := \dot{\bigcup_n} L^{p_n}(\Omega; \mathrm{d}x) \dot{\cup} L^1(\Omega; \mathrm{d}x), \quad \mathfrak{L}^{\to \infty} := \dot{\bigcup_n} L^{q_n}(\Omega; \mathrm{d}x) \dot{\cup} L^\infty(\Omega; \mathrm{d}x),$$

respectively. Let $f \in C_1$, $g \in C_2$. Their product is in $L^\infty(\Omega; \mathrm{d}x) \subset L^1(\Omega; \mathrm{d}x)$ and therefore,

$$\lim_n {}_{L^{p_n}}\langle \Phi_n(f), \Psi_n(g) \rangle_{L^{q_n}} = \lim_n \int_\Omega fg \,\mathrm{d}x = \int_\Omega fg \,\mathrm{d}x = {}_{L^1}\langle f, g \rangle_{L^\infty},$$

which proves that $\mathfrak{L}^{\to 1}$ and $\mathfrak{L}^{\to\infty}$ are asymptotically dual. We point out that, although each L^{p_n}, L^{q_n} are reflexive, $\mathfrak{L}^{\to 1}$ and $\mathfrak{L}^{\to\infty}$ are not asymptotically reflexive. In particular, by the general theory of the preceding chapter, weak convergence on $\mathfrak{L}^{\to 1}$ and weak* convergence on $\mathfrak{L}^{\to\infty}$ make sense and are of interest.

6.3 Orlicz spaces

In the Orlicz space case, which is one of our main applications later on, convergence of norms cannot be derived that simply from the convergence of modulars as in the L^p-case. The proof below is employing duality theory of Orlicz spaces for both the Luxemburg and the Orlicz norm. Anyhow, the remaining arguments are quite similar to those above. For all of this section, fix a locally compact Hausdorff space Ω and a pair of complementary N-functions (Φ, Ψ) (in particular, $\Phi(x) = \Psi(x) = 0$ iff $x = 0$ and both functions are continuous), such that $\Phi, \Psi \in \Delta_2 \cap \nabla_2$. All necessary facts about Orlicz spaces can be found in Appendix C.

The next lemma is standard.

Lemma 6.16. *Let $\Phi \in \Delta_2$, Ω a Hausdorff locally compact space and μ a regular Borel measure on $(\Omega, \mathscr{B}(\Omega))$. Then $C_0^\infty(\Omega)$, i.e.* the infinitely often differentiable real-valued functions on Ω with compact support*, is a dense linear subspace of $L^\Phi(\Omega; \mu)$.*

Proof. We will first show that $C_0(\Omega)$, i.e. *the continuous real-valued functions on Ω with compact support*, is a dense linear subspace of $L^\Phi(\Omega; \mu)$. First note that $\widetilde{C_0(\Omega)} \subset L^1(\Omega; \mu)$, which follows from

$$\int_\Omega |f| \,\mathrm{d}\mu \le \|f\|_\infty \,\mu(\mathrm{supp}(f)) \quad \forall f \in C_0(\Omega).$$

Since $\Phi(C_0(\Omega)) \subset C_0(\Omega)$, we get that $\widetilde{C_0(\Omega)} \subset L^\Phi(\Omega; \mu)$. By [RR91, Ch. III, §4, Corollary 4, p. 77] step functions with finite Luxemburg norm are dense in $L^\Phi(\Omega; \mu)$ when $\Phi \in \Delta_2$. Since the Luxemburg norm is *absolutely continuous* (cf. [RR91, p. 84 et seq.]) when $\Phi \in \Delta_2$ and when $\Phi(x) = 0$ iff $x = 0$, it is enough to approximate functions of the type 1_A, $A \in \mathscr{B}(\Omega)$, $\mu(A) < +\infty$. Let $\varepsilon > 0$ and 1_A be such a function. Set $C := \Phi(2)$. By outer regularity of μ, there is an open set $U \supset A$ with $\mu(U \setminus A) < \varepsilon/C$. In particular, $\mu(U) < +\infty$. By inner regularity of μ, we can find a compact set $K \subset U$ with $\mu(U \setminus K) < \varepsilon/C$. By Urysohn's Lemma (cf. [Bau74, Lemma 42.2]) we can find a function $f \in C_0(\Omega)$ with $1_K \le f \le 1_U$, that is, $0 \le 1_U - f \le 1_U - 1_K$. We conclude by convexity

$$\begin{aligned}\int_\Omega \Phi(|f-1_A|)\,\mathrm{d}\mu &\le \frac{1}{2}\int_\Omega \Phi(2|1_U-f|)\,\mathrm{d}\mu + \frac{1}{2}\int_\Omega \Phi(2|1_U-1_A|)\,\mathrm{d}\mu \\ &\le \frac{1}{2}\int_\Omega \Phi(2|1_U-1_K|)\,\mathrm{d}\mu + \frac{1}{2}\int_\Omega \Phi(2|1_U-1_A|)\,\mathrm{d}\mu \le \frac{C}{2}\left[\mu(U\setminus K)+\mu(U\setminus A)\right] < \varepsilon\end{aligned}$$

which yields the density of $C_0(\Omega)$ by Lemma C.16.

It is easily seen that $C_0^\infty(\Omega)$ separates the points of Ω (consider e.g. compositions of a suitable $C_0^\infty(\Omega)$ function, obtained by smoothing mollifiers, and $x \mapsto \sin(\alpha x)$ for some suitable $\alpha \in \mathbb{R}$) and contains for each $x \in \Omega$ a function $g \in C_0^\infty(\Omega)$ with $g(x) \neq 0$. By a locally compact version of the Stone-Weierstraß Theorem $C_0^\infty(\Omega)$ is dense in $C_0(\Omega)$ w.r.t. uniform convergence (cf. [Cho69, p. 28 (iii)]). Let $\varepsilon > 0$. Let $g \in C_0^\infty(\Omega)$ with $\|f-g\|_\infty < \varepsilon$. We see that

$$\int_\Omega \Phi(|f-g|)\,\mathrm{d}\mu \le \Phi(\|f-g\|_\infty)\mu(\operatorname{supp}(f-g)) \le \Phi(\varepsilon)\mu(\operatorname{supp}(f-g)).$$

Recall that μ is a Borel measure, hence $\mu(\operatorname{supp}(f-g)) < +\infty$. Φ is continuous, increasing and $\Phi(x) = 0$ iff $x = 0$. Therefore we get the desired density by Lemma C.16. $\square$

Let $\Phi \in \Delta_2 \cap \nabla_2$. Ω a locally compact Hausdorff space and $\{\mu_n\}$ a sequence of fully supported positive Borel measures with a vague limit μ, which is assumed to be a fully supported regular positive Borel measure. Clearly for each $f \in C_0(\Omega)$

$$\lim_n \int_\Omega \Phi(f)\,\mu_n(\mathrm{d}x) = \int_\Omega \Phi(f)\,\mu(\mathrm{d}x). \tag{6.10}$$

The above equation holds for Φ replaced by Ψ as well.

Lemma 6.17. *For each $f \in C_0(\Omega)$*

$$\lim_n \|f\|_{(\Phi,\mu_n)} = \|f\|_{(\Phi,\mu)}\,, \tag{6.11}$$

that is, the Luxemburg norms converge. A similar statement holds for Φ replaced by Ψ.

Proof. For all of the proof fix a function $f \in C_0(\Omega)$.

Let $\alpha := \|f\|_{(\Phi,\mu)}$. Suppose that $\alpha \neq 0$. Let $\varepsilon > 0$ and $\delta > 0$. Since $\Phi \in \Delta_2$ there is a constant $C > 2$ with $\Phi(2x) \le C\Phi(x)$. Therefore $\Phi(x/(1+\delta)) \le C^s\Phi(x)$ when $s \ge -\log_2(1+\delta)$. But s can be chosen negative. Therefore there is a constant $C^s =: C(\Phi,\delta) \in (0,1)$ depending only on Φ and δ such that large n

$$\int_\Omega \Phi\left(\frac{f}{(1+\delta)\alpha}\right)\mu_n(\mathrm{d}x) \le C(\Phi,\delta)\int_\Omega \Phi\left(\frac{f}{\alpha}\right)\mu_n(\mathrm{d}x) \le C(\Phi,\delta)\int_\Omega \Phi\left(\frac{f}{\alpha}\right)\mu(\mathrm{d}x) + \varepsilon.$$

Choosing ε small enough we see that the right hand side is smaller or equal to 1. Thus for large n

$$\|f\|_{(\Phi,\mu_n)} \le (1+\delta)\alpha.$$

Since δ was arbitrary,

$$\overline{\lim_n} \|f\|_{(\Phi,\mu_n)} \leq \|f\|_{(\Phi,\mu)} . \tag{6.12}$$

If $\alpha = 0$, (6.12) holds by a simple modification of the proof of Lemma C.16.

Let $\beta := \underline{\lim}_n \|f\|_{(\Phi,\mu_n)}$. Suppose that $0 < \beta < \infty$. Let $\varepsilon > 0$ and $\delta > 0$. Extract a convergent subsequence such that $(1+\delta)\beta \geq \|f\|_{(\Phi,\mu_n)}$ for large n (if necessary). Then for large n we have that

$$\int_\Omega \Phi\left(\frac{f}{(1+\delta)\beta}\right) \mu(\mathrm{d}x) \leq \int_\Omega \Phi\left(\frac{f}{(1+\delta)\beta}\right) \mu_n(\mathrm{d}x) + \varepsilon \leq \int_\Omega \Phi\left(\frac{f}{\|f\|_{(\Phi,\mu_n)}}\right) \mu_n(\mathrm{d}x) + \varepsilon.$$

The right hand side is smaller or equal to $1+\varepsilon$. Since ε was arbitrary,

$$\|f\|_{(\Phi,\mu)} \leq (1+\delta)\beta,$$

and hence, since δ was arbitrary,

$$\underline{\lim_n} \|f\|_{(\Phi,\mu_n)} \geq \|f\|_{(\Phi,\mu)} . \tag{6.13}$$

If $\beta = \infty$, (6.13) holds trivially. If $\beta = 0$, (6.13) can be proved with the same arguments as in the proof of Lemma C.16. (6.12) and (6.13) together give (6.11). □

The part with the Luxemburg norms was the easier one. For the Orlicz norm-part we need some preparation.

Lemma 6.18. *Let $\Phi \in \Delta_2 \cap \nabla_2$. Let $f \in L^\Phi(\Omega, \mu)$ with $\|f\|_{\Phi,\mu} = 1$. The set $K \subset (1, \infty)$ such that $k \in K$ iff k is a solution to*

$$k - 1 = \int_\Omega \Phi(k|f|) \, \mu(\mathrm{d}x)$$

is bounded above by a constant solely depending on Φ (and not on μ).

Proof. The idea for the proof is borrowed from S. T. Chen, [Che96, Theorem 1.35]. By $\Phi \in \nabla_2$ there exists $\delta > 0$ satisfying

$$\Phi(2x) \geq (2+\delta)\Phi(x),$$

compare e.g. [RR91, Ch. II.3, Theorem 3, p. 22]. Set $D := 1 + \delta/2 > 1$, such that we have

$$\Phi(2x) \geq 2D\Phi(x).$$

By Lemma C.14

$$1 = \|f\|_{\Phi,\mu} \leq \frac{1}{2}\left[1 + \int_\Omega \Phi(2|f|) \, \mathrm{d}\mu\right],$$

which yields $\int_\Omega \Phi(2|f|)\,\mathrm{d}\mu \geq 1$. For any $k \in K$, if $k > 4$, there exists an integer $i \in \mathbb{N}$ such that $2^i < 2^{-1}k \leq 2^{i+1}$. Since $\Phi(2^i x) \geq 2^i D^i \Phi(x)$, we have that (using Lemma C.14 in the second equality)

$$\begin{aligned}1 = \|f\|_{\Phi,\mu} = \frac{1}{k}\left[1 + \int_\Omega \Phi(k|f|)\,\mathrm{d}\mu\right] &\geq \frac{1}{k}\int_\Omega \Phi(2^{-1}k2|f|)\,\mu(\mathrm{d}x)\\ &\geq \frac{1}{k}D^i 2^i \int_\Omega \Phi(2|f|)\,\mu(\mathrm{d}x) \geq \frac{2}{k}2^{i-1}D^i \geq \frac{D^i}{2}.\end{aligned}$$

Hence $D^i \leq 2$ which means that $k \leq 2^{2+\log_D 2}$ is an upper bound for K. Note that D does only depend on Φ and not on μ. The proof is complete. □

Lemma 6.19. *Suppose that $\Phi \in C^1(\mathbb{R})$ such that $\varphi = \Phi'$ (the gauge of Φ) is continuous. Suppose that $\varphi(t) > 0$ if $t > 0$. For each $f \in C_0(\Omega)$*

$$\lim_n \|f\|_{\Phi,\mu_n} = \|f\|_{\Phi,\mu}\,, \tag{6.14}$$

that is, the Orlicz norms converge. A similar statement holds for Φ replaced by Ψ.

Proof. For all of the proof fix a function $f \in C_0(\Omega)$. As usually, $\varphi(t) = \varphi(|t|)\,\mathrm{sign}(t)$ for $t \in \mathbb{R}$.

Suppose that $\|f\|_{\Phi,\mu} \neq 0$. Let $k > 1$ be the solution to

$$k - 1 = \int_\Omega \Phi\left(k\frac{f}{\|f\|_{\Phi,\mu}}\right)\,\mu(\mathrm{d}x).$$

Let $\varepsilon > 0$ and $\delta > 0$. Since $\Psi \in \Delta_2$ there is a constant $C > 2$ with $\Psi(2x) \leq C\Psi(x)$. Therefore $\Psi(x/(1+\delta)) \leq C^s\Psi(x)$ when $s \geq -\log_2(1+\delta)$. But s can be chosen negative. Therefore there is a constant $C^s =: C(\Psi,\delta) \in (0,1)$ depending only on Ψ and δ such that large n

$$\begin{aligned}&\int_\Omega \Psi\left(\frac{1}{1+\delta}\varphi\left(k\frac{f}{\|f\|_{\Phi,\mu}}\right)\right)\,\mu_n(\mathrm{d}x)\\ \leq& C(\Psi,\delta)\int_\Omega \Psi\left(\varphi\left(k\frac{f}{\|f\|_{\Phi,\mu}}\right)\right)\,\mu_n(\mathrm{d}x)\\ \leq& C(\Psi,\delta)\int_\Omega \Psi\left(\varphi\left(k\frac{f}{\|f\|_{\Phi,\mu}}\right)\right)\,\mu(\mathrm{d}x) + \varepsilon.\end{aligned}$$

Choosing ε small enough we see that the right hand side is smaller or equal to 1 by Lemma C.23. Thus for large n

$$\int_\Omega f\varphi\left(k\frac{f}{\|f\|_{\Phi,\mu}}\right)\,\mu_n(\mathrm{d}x) \leq (1+\delta)\,\|f\|_{\Phi,\mu_n}\,.$$

Since δ was arbitrary,

$$\int_\Omega f\varphi\left(k\frac{f}{\|f\|_{\Phi,\mu}}\right)\mu_n(\mathrm{d}x) \leq \|f\|_{\Phi,\mu_n}. \tag{6.15}$$

Let $\varepsilon > 0$. For large n we have

$$\int_\Omega f\varphi\left(k\frac{f}{\|f\|_{\Phi,\mu}}\right)\mu(\mathrm{d}x) - \varepsilon \leq \int_\Omega f\varphi\left(k\frac{f}{\|f\|_{\Phi,\mu}}\right)\mu_n(\mathrm{d}x). \tag{6.16}$$

It is well-known that the normalized duality map $J : L^\Phi \to L^\Psi$ is equal to the Gâteaux derivative of $f \mapsto \frac{1}{2}\|f\|^2$ (cf. [Phe89]). But we know the Gâteaux derivative of the norm by Lemma C.23. Therefore,

$$\|f\|_{\Phi,\mu}\int_\Omega \frac{f}{\|f\|_{\Phi,\mu}}\varphi\left(k\frac{f}{\|f\|_{\Phi,\mu}}\right)\mu(\mathrm{d}x) = \|f\|_{\Phi,\mu}. \tag{6.17}$$

Combining (6.15), (6.16) and (6.17) we get

$$\underline{\lim}_n \|f\|_{\Phi,\mu_n} \geq \|f\|_{\Phi,\mu}. \tag{6.18}$$

If $\|f\|_{\Phi,\mu} = 0$, (6.18) can be proved with the same arguments as in the proof of Lemma C.16.

Let us prove the other limit-estimate. For each n, let $k_n > 1$ be a solution to

$$k_n - 1 = \int_\Omega \Phi\left(k_n\frac{|f|}{\|f\|_{\Phi,\mu_n}}\right)\mu_n(\mathrm{d}x).$$

By Lemma 6.18 $[\inf_n k_n, \sup_n k_n] \subset [1, C(\Phi)]$ where $C(\Phi) \geq 4$ is a constant only depending on the ∇_2 condition of Φ.

Let $\alpha := \overline{\lim}_n k_n/\|f\|_{\Phi,\mu_n}$. Suppose that $0 < \alpha < \infty$, which by the above remark is satisfied if and only if $0 < \overline{\lim}_n \|f\|_{\Phi,\mu_n} < \infty$.

Let $\varepsilon > 0$, $\delta > 0$. Extract a convergent subsequence (if necessary) such that

$$\frac{1}{1+\delta}\varphi(\alpha|f|) = \frac{1}{1+\delta}\overline{\lim_n}\,\varphi\left(k_n|f|/\|f\|_{\Phi,\mu_n}\right) \leq \varphi\left(k_n|f|/\|f\|_{\Phi,\mu_n}\right)$$

for large n, where we have used that φ is continuous.

Then for large n we have that

$$\begin{aligned}&\int_\Omega \Psi\left(\frac{1}{1+\delta}\varphi(\alpha|f|)\right)\mu(\mathrm{d}x)\\ \leq &\int_\Omega \Psi\left(\frac{1}{1+\delta}\varphi(\alpha|f|)\right)\mu_n(\mathrm{d}x) + \varepsilon\\ \leq &\int_\Omega \Psi\left(\varphi\left(k_n\frac{|f|}{\|f\|_{\Phi,\mu_n}}\right)\right)\mu_n(\mathrm{d}x) + \varepsilon.\end{aligned}$$

The right hand side is smaller than $1+\varepsilon$ by Lemma C.23. Hence, since ε was arbitrary,

$$\left|\int_\Omega f\varphi\left(\alpha|f|\right)\,\mu(\mathrm{d}x)\right| \leq (1+\delta)\,\|f\|_{\Phi,\mu}\,.$$

Hence, since δ was arbitrary,

$$\left|\int_\Omega f\varphi\left(\alpha|f|\right)\,\mu(\mathrm{d}x)\right| \leq \|f\|_{\Phi,\mu}\,. \tag{6.19}$$

We have for each fixed $m \in \mathbb{N}$

$$\lim_n \left|\int_\Omega f\varphi\left(k_m \frac{|f|}{\|f\|_{\Phi,\mu_m}}\right)\,\mu_n(\mathrm{d}x)\right| = \left|\int_\Omega f\varphi\left(k_m \frac{|f|}{\|f\|_{\Phi,\mu_m}}\right)\,\mu(\mathrm{d}x)\right|. \tag{6.20}$$

Also

$$\lim_n \left|\int_\Omega f\varphi\left(\alpha|f|\right)\,\mu_n(\mathrm{d}x)\right| = \left|\int_\Omega f\varphi\left(\alpha|f|\right)\,\mu(\mathrm{d}x)\right|. \tag{6.21}$$

Since the term inside the integral is bounded above and since φ is upper semi-continuous, by Fatou's Lemma (cf. [Yos78, p. 17]), for each fixed $n \in \mathbb{N}$,

$$\overline{\lim_m} \left|\int_\Omega f\varphi\left(k_m \frac{|f|}{\|f\|_{\Phi,\mu_m}}\right)\,\mu_n(\mathrm{d}x)\right| \leq \left|\int_\Omega f\varphi\left(\alpha|f|\right)\,\mu_n(\mathrm{d}x)\right| \tag{6.22}$$

and similarly

$$\overline{\lim_m} \left|\int_\Omega f\varphi\left(k_m \frac{|f|}{\|f\|_{\Phi,\mu_m}}\right)\,\mu(\mathrm{d}x)\right| \leq \left|\int_\Omega f\varphi\left(\alpha|f|\right)\,\mu(\mathrm{d}x)\right|. \tag{6.23}$$

Define

$$a := \left|\int_\Omega f\varphi\left(\alpha|f|\right)\,\mu(\mathrm{d}x)\right|$$

and for $n, m \in \mathbb{N}$

$$a_{n,m} := \left|\int_\Omega f\varphi\left(k_m \frac{|f|}{\|f\|_{\Phi,\mu_m}}\right)\,\mu_n(\mathrm{d}x)\right|.$$

By (6.20), (6.21), (6.22) and (6.23)

$$\overline{\lim_m}\,\overline{\lim_n}\, a_{n,m} \leq a \quad \text{and} \quad \overline{\lim_n}\,\overline{\lim_m}\, a_{n,m} \leq a$$

and hence

$$\overline{\lim_m}\,\overline{\lim_n}\, \frac{1}{2}[a_{n,m} + a_{m,n}] \leq a. \tag{6.24}$$

Let $\delta_{n,m}$ be the *Kronecker delta*. Clearly

$$\delta_{n,m} a_{n,m} \leq a_{n,m} \quad \forall n, m \in \mathbb{N},$$

which together with (6.24) gives

$$\overline{\lim_n}\, a_{n,n} \leq a. \tag{6.25}$$

It is well-known that the normalized duality map $J : L^\Phi \to L^\Psi$ is equal to the Gâteaux derivative of $f \mapsto \frac{1}{2}\|f\|^2$ (cf. [Phe89]). But we know the Gâteaux derivative of the norm by Lemma C.23. Therefore for any $n \in \mathbb{N}$,

$$\|f\|_{\Phi,\mu_n} \left| \int_\Omega \frac{f}{\|f\|_{\Phi,\mu_n}} \varphi\left(k_n \frac{|f|}{\|f\|_{\Phi,\mu_n}}\right) \mu_n(\mathrm{d}x) \right| = \|f\|_{\Phi,\mu_n},$$

which equal to $a_{n,n}$. Hence by (6.19) and (6.25)

$$\overline{\lim_n}\, \|f\|_{\Phi,\mu_n} \leq \|f\|_{\Phi,\mu}. \tag{6.26}$$

If $\overline{\lim}_n \|f\|_{\Phi,\mu_n} = 0$, (6.26) holds by a simple modification of the proof of Lemma C.16. The case of $\overline{\lim}_n \|f\|_{\Phi,\mu_n} = +\infty$ does not occur by boundedness of modulars (6.10) and a argument similar to that in the proof of Lemma C.17.

(6.26) and (6.18) together give (6.14) which completes the proof. □

Finally, it is an easy consequence of the vague convergence of measures, that

$$\lim_n \int_\Omega fg\, \mu_n(\mathrm{d}x) = \int_\Omega fg\, \mu(\mathrm{d}x) \tag{6.27}$$

for all $f, g \in C_0(\Omega)$.

Now set $C := D := C_0(\Omega)$, $\Theta_n : C \subset L^\Phi(\Omega;\mu) \to L^\Phi(\Omega;\mu_n)$, $\Theta_n^* : D \subset L^\Psi(\Omega;\mu) \to L^\Psi(\Omega;\mu_n)$, $\Theta_n(f) := f$, $\Theta_n^*(g) := g$, $n \in \mathbb{N}$.

Lemma 6.16 guarantees that C is dense in $L^\Phi(\Omega;\mu)$ and D is dense in $L^\Psi(\Omega;\mu)$. Full support of measures guarantees that each Θ_n, Θ_n^* is a well-defined linear operator. (6.11) and (6.14) prove that $\{\Theta_n\}$ and $\{\Theta_n^*\}$ are metric approximations in either norm. (6.27) tells that they are in duality. Therefore $\{\Theta_n\}$ and $\{\Theta_n^*\}$ generate linear asymptotic relations asymptotically in duality on

$$\mathfrak{L}^\Phi := \bigcup_n L^\Phi(\Omega;\mu_n) \dot\cup L^\Phi(\Omega;\mu),$$

and

$$\mathfrak{L}^\Psi := \bigcup_n L^\Psi(\Omega;\mu_n) \dot\cup L^\Psi(\Omega;\mu),$$

respectively equipped with either type of norm. For the asymptotic duality note that the functional norm of the dual space is given by the Hölder inequality (C.1). This means that if we consider $\mathfrak{L}^\Phi$ with the Luxemburg norm, we will consider $\mathfrak{L}^\Psi$ with the Orlicz norm and vice versa.

6.4 Scales of Banach spaces

For this section we refer to any comprehensive textbook on topological vector spaces, e.g. [Sch71].

Definition 6.20. *A* projective scale of Banach spaces *is a collection* $\{X_n\}_{n\in\mathbb{N}}$ *of Banach spaces such that* $X_k \hookrightarrow X_n$ *densely and continuously for all* $k \geq n$, $k, n \in \mathbb{N}$. *Define* $X_\infty := \bigcap_n X_n$.

A inductive scale of Banach spaces *is a collection* $\{Y_n\}_{n\in\mathbb{N}}$ *of Banach spaces such that* $Y_k \hookrightarrow Y_n$ *densely and continuously for all* $k \leq n$, $k, n \in \mathbb{N}$. *Define* $Y_\infty := \bigcup_n Y_n$.

X_∞ is clearly a linear space (called the *projective limit*). It inherits a locally convex topology from the family $\{X_n\}$ called the *projective topology*, that is, the coarsest locally convex topology such that each embedding $X_\infty \hookrightarrow X_n$ is continuous.

It is well-known that X_∞ is dense in each of X_n, $n \in \mathbb{N}$, cf. [Cap02, Lemma 2.1.2].

One can equivalently renorm each X_n, $n \in \mathbb{N}$ such that

$$\cdots \geq \|\cdot\|_{X_{n+1}} \geq \|\cdot\|_{X_n} \geq \cdots \geq \|\cdot\|_{X_2} \geq \|\cdot\|_{X_1},$$

cf. [GS68, Chapter I, Section 3]. For $x, y \in X_n$ set

$$d_n(x,y) := \sum_{k=1}^{n} \frac{1}{2^k} \frac{\|x-y\|_{X_k}}{1+\|x-y\|_{X_k}} + \frac{\|x-y\|_{X_n}}{1+\|x-y\|_{X_n}} \cdot \sum_{k=n+1}^{\infty} \frac{1}{2^k}.$$

It is an easy exercise to see that $d_n(\cdot,\cdot)$ generates the same uniformity on X_n as $\|\cdot\|_{X_n}$ and that

$$\cdots \geq d_{n+1}(\cdot,\cdot) \geq d_n(\cdot,\cdot) \geq \cdots \geq d_2(\cdot,\cdot) \geq d_1(\cdot,\cdot).$$

For $x, y \in X_\infty$ set

$$d_\infty(x,y) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|x-y\|_{X_k}}{1+\|x-y\|_{X_k}},$$

which is a metric generating the projective topology (and uniformity) on X_∞.

We see that

$$\lim_n d_n(x,y) = d_\infty(x,y) \quad \text{for all } x, y \in X_\infty. \tag{6.28}$$

Therefore the embeddings defined on X_∞ for each n define a sequence of linear metric approximations in the sense of the original Definition 5.26 of Kuwae and Shioya for metric spaces.

Y_∞ is a linear space as well (called the *inductive limit*). It inherits a locally convex topology from the family $\{Y_n\}$ called the *inductive topology*, that is, the finest locally convex topology such that each embedding $Y_n \hookrightarrow Y_\infty$ is continuous. Nevertheless, we will not concern ourselves with inductive limits of Banach spaces. The reason for this is, that they never admit a metric, as was proved in [NS86, Corollary 3]. Inductive limits of metrizable Fréchet spaces again might be metrizable. If one wants to include inductive scales of Banach spaces into the framework of asymptotic relations, one has to develop a more general asymptotic topology than the metric one (e.g. with approximations for uncountable families of semi-norms).

6.5 Finite dimensional approximation

6.5.1 Projection schemes

Let E be an infinite dimensional Banach space which *admits a projection scheme*, that is, there exists a family $\{E_n\}_{n\in\mathbb{N}}$ of finite dimensional subspaces such that $E_n \subset E_{n+1}$, $\dim E_n = n$, $n \in \mathbb{N}$, $\bigcup_n E_n$ is dense in E and there exist continuous linear projections $P_n : E \to E_n$ (onto) with $\|P_n\|_{\mathscr{L}(E)} = 1$, $n \in \mathbb{N}$ such that

$$\lim_n \|P_n(x) - x\|_E = 0 \quad \text{for any } \; x \in E. \tag{6.29}$$

If E admits a projection scheme and if $P_nP_m = P_mP_n = P_{m\wedge n}$ for all $n, m \in \mathbb{N}$, it follows from Lemma B.23 that E has a monotone Schauder basis.

Even more can be said, E admits a projection scheme if and only if E has a monotone Schauder basis, compare [Sin70a, Ch. II, Proposition 1.3].

It is straightforward that any separable Hilbert space admits a projection scheme. Any $L^p(\Omega, \mathscr{F}, \mu)$-space, $1 \le p < \infty$, $(\Omega, \mathscr{F}, \mu)$ is σ-finite and separable, admits a projection scheme by [Cio90, Ch. II, Proposition 5.14]. Also $L^\Phi[0,1]$, $\Phi \in \Delta_2$, and all Banach spaces isometric to it, admit a projection scheme by [RR91, Ch. VI, Corollary 2, p. 260] and [LT79, Proposition 2.c.1].

By [Cio90, Ch. II, Proposition 5.16] if E is reflexive, E^* admits a projection scheme with subspaces $\{E_n^* := P_n^*(E^*)\}$ and projections $\{P_n^*\}$ where P_n^* is the linear adjoint of $P_n : E \to E$, $n \in \mathbb{N}$.

Setting $\mathfrak{E} := \dot{\bigcup}_n E_n \dot{\cup} E$, $\mathfrak{E}^* := \dot{\bigcup}_n E_n^* \dot{\cup} E^*$ (with norms inherited from E, E^* resp.) we see that $\Phi_n := P_n$, $\Phi_n^* := P_n^*$, $n \in \mathbb{N}$ are linear metric approximations and, indeed, asymptotically continuous by (6.29). If E is reflexive, by the result cited above for $x \in E$, $y \in E^*$,

$$\lim_n {}_{E_n^*}\langle P_n^* y, P_n x\rangle_{E_n} = \lim_n {}_{E^*}\langle y, P_nP_n x\rangle_E = \lim_n {}_{E^*}\langle y, P_n x\rangle_E = {}_{E^*}\langle y, x\rangle_E,$$

meaning that $(\mathfrak{E}, \mathfrak{E}^*)$ are asymptotically dual.

6.5.2 Conditional expectations

Let $1 \le p < \infty$. Let $(\Omega, \mathscr{F}, \mu)$ be separable probability space (with expectation $\mathbb{E}$) such that $\mathscr{F} = \sigma\{A_n \mid n \in \mathbb{N}\}$. Set $\mathscr{F}_n := \sigma\{A_k \mid k \le n\}$. Let $f \in L^1(\mu)$. Denote by $\mathbb{E}_n[f] := \mathbb{E}[f \mid \mathscr{F}_n]$ the conditional expectation w.r.t. the sub-σ-algebra $\mathscr{F}_n$ (cf. [Bau74, Ch. X, §54]). Let $f \in L^p(\Omega, \mathscr{F}, \mu)$. By Jensen's inequality $\mathbb{E}_n[|f|]^p \le \mathbb{E}_n[|f|^p]$ for any n and hence $\|\mathbb{E}_n[f]\|_{L^p(\Omega,\mathscr{F},\mu)} \le \|f\|_{L^p(\Omega,\mathscr{F},\mu)}$ by the projectivity of the conditional expectation. By the same argument, if $f \in L^p(\Omega, \mathscr{F}_n, \mu)$, $\|\mathbb{E}_m[f]\|_{L^p(\Omega,\mathscr{F}_m,\mu)} \le \|f\|_{L^p(\Omega,\mathscr{F}_n,\mu)}$ whenever $m \le n$. Hence

$$\|\mathbb{E}_m[f]\|_{L^p(\Omega,\mathscr{F}_m,\mu)} = \|\mathbb{E}_m[\mathbb{E}_n[f]]\|_{L^p(\Omega,\mathscr{F}_m,\mu)} \le \|\mathbb{E}_n[f]\|_{L^p(\Omega,\mathscr{F}_n,\mu)}$$

whenever $m \le n$. We see that

$$\|\mathbb{E}_n[f]\|_{L^p(\Omega,\mathscr{F}_n,\mu)} \uparrow \|f\|_{L^p(\Omega,\mathscr{F},\mu)}.$$

Since $L^p(\Omega, \mathscr{F}_n, \mu)$ is finite dimensional, we get a projection scheme as in the previous subsection, except that, $\dim L^p(\Omega, \mathscr{F}_n, \mu) = n$ only when all A_n's are atoms. Therefore we can easily construct a linear metric approximation for

$$\dot{\bigcup_n} L^p(\Omega, \mathscr{F}_n, \mu) \dot{\cup} L^p(\Omega, \mathscr{F}, \mu)$$

in the analog way. Note also that, contrary to the above, each L^p space carries its own norm. Asymptotic duality for $1 < p < \infty$, $q := p/(p-1)$ follows by the remark that $L^q(\Omega, \mathscr{F}_n, \mu)$ is the dual space of $L^p(\Omega, \mathscr{F}_n, \mu)$.

6.6 Two-scale convergence

The study of homogenization of convex functionals involves the employment of the so-called two-scale convergence. In this section we prove that a two-scale convergence is always a linear asymptotic relation convergence.

For the next definition see [Ngu89, All92, Nec01, LNW02, Zhi04].

Definition 6.21 (Nguetseng)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain. Let $\square := [0,1)^d$ be the unit cube in $\mathbb{R}^d$. Let $1 < p < \infty$, $q := p/(p-1)$. Let $\{\varepsilon\} = \{\varepsilon_n\}$ be a fixed sequence of positive real numbers converging to zero. A sequence $\{u_\varepsilon\}$ of functions in $L^p(\Omega)$ is said to* two-scale converge (weakly) *to a limit $u \in L^p(\Omega \times \square)$ if*

$$\int_\Omega u_\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x \to \int_\Omega \int_\square u(x,y)\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x \quad \text{as } \varepsilon \searrow 0 \tag{6.30}$$

for every $\varphi \in L^q(\Omega \to C_{\mathrm{per}}(\square))$, where $C_{\mathrm{per}}(\square)$ is the function space of continuous $\square$-periodic real-valued functions on $\square$ that have continuous continuations on $\overline{\square} = [0,1]^d$.

If, in addition to (6.30),

$$\lim_{\varepsilon \searrow 0} \left\| u_\varepsilon(x) - u\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^p(\Omega)} = 0, \tag{6.31}$$

we say that $\{u_\varepsilon\}$ strongly two-scale converges *to u.*

For the next definition see [LW05].

Definition 6.22 (Lukkassen-Wall)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain. Let $\square := [0,1)^d$ be the unit cube in $\mathbb{R}^d$. Let μ be a $\square$-periodic positive Borel measure on $\mathbb{R}^d$ such that $\mu(\square) = 1$ and $\mu(\partial\square) = 0$. Let $1 < p < \infty$, $q := p/(p-1)$. Let $\{\varepsilon\} = \{\varepsilon_n\}$ be a fixed sequence of positive real numbers converging to zero. Set $\mu_\varepsilon(B) := \varepsilon^d \mu(\varepsilon^{-1}B)$ for any Borel set $B \subset \mathbb{R}^d$.*

A norm-bounded sequence $\{u_\varepsilon\}$ of functions, such that $u_\varepsilon \in L^p(\Omega, \mathrm{d}\mu_\varepsilon)$ is said to two-scale converge weakly *to a limit $u \in L^p(\Omega \times \square, \mathrm{d}x \otimes \mathrm{d}\mu)$ if*

$$\int_\Omega u_\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) \mu_\varepsilon(\mathrm{d}x) \to \int_\Omega \int_\square u(x,y)\varphi(x,y)\mu(\mathrm{d}y)\mathrm{d}x \quad \text{as } \varepsilon \searrow 0 \tag{6.32}$$

for every $\varphi(x,y) = \psi(x)\sigma(y)$*,* $\psi \in C_0^\infty(\Omega)$*,* $\sigma \in C_{\mathrm{per}}^\infty(\square)$ *where* $C_{\mathrm{per}}^\infty(\square)$ *is the function space of* $\square$*-periodic real-valued infinitely often continuously differentiable functions on* $\square$ *that have continuous and infinitely often continuously differentiable continuations on* $\overline{\square} = [0,1]^d$*.*

If, in addition to (6.32)*, for every bounded sequence* $\{v_\varepsilon\}$*,* $v_\varepsilon \in L^q(\Omega, \mathrm{d}\mu_\varepsilon)$*, every* $v \in L^q(\Omega \times \square, \mathrm{d}x \otimes \mathrm{d}\mu)$*, such that* $\{v_\varepsilon\}$ *weakly two-scale converges to* v *we have that*

$$\int_\Omega u_\varepsilon(x)v_\varepsilon(x)\mu_\varepsilon(\mathrm{d}x) \to \int_\Omega \int_\square u(x,y)v(x,y)\mu(\mathrm{d}y)\mathrm{d}x \quad \text{as } \varepsilon \searrow 0 \tag{6.33}$$

we say that $\{u_\varepsilon\}$ strongly two-scale converges *to* u*.*

Note that V. V. Zhikov introduced the same definition for $p = 2$ in [Zhi00].

Set now $E_\varepsilon := L^p(\Omega, \mathrm{d}\mu_\varepsilon)$, $E := L^p(\Omega \times \square, \mathrm{d}x \otimes \mathrm{d}\mu)$. $\mathfrak{E} := \dot{\bigcup}_\varepsilon E_\varepsilon \dot{\cup} E$. Set $E \supset C := C_0^\infty(\Omega) \otimes C_{\mathrm{per}}^\infty(\square)$ (where we only take finite linear combinations of tensors), and $\Phi_\varepsilon : C \to E_n$ is defined by

$$\Phi_\varepsilon(f \otimes g)(x) := (f \otimes g)\left(x, \frac{x}{\varepsilon}\right)$$

and linear extension. (B1) and (BL) are satisfied.

Let $\varphi = f \otimes g$, $f \in C_0^\infty(\Omega)$, $g \in C_{\mathrm{per}}^\infty(\square)$. By [LW05, Theorem 1, Theorem 2]

$$\lim_{\varepsilon \searrow 0} \int_\Omega \left|\varphi\left(x, \frac{x}{\varepsilon}\right)\right|^p \mu_\varepsilon(\mathrm{d}x) = \int_\Omega \int_\square |\varphi(x,y)|^p \mu(\mathrm{d}y)\mathrm{d}x.$$

By linear extension we see that (B2) is satisfied.

We repeat the above for $E_\varepsilon^* := L^q(\Omega, \mathrm{d}\mu_\varepsilon)$, $E^* := L^q(\Omega \times \square, \mathrm{d}x \otimes \mathrm{d}\mu)$. $\mathfrak{E}^* := \dot{\bigcup}_\varepsilon E_\varepsilon^* \dot{\cup} E^*$. $\mathfrak{E}$ and $\mathfrak{E}^*$ are asymptotically dual again by [LW05, Theorem 1, Theorem 2].

Remark 6.23. *If* $\mu(\mathrm{d}y) := \mathrm{d}y$ *is the Lebesgue measure on* $\square$*, Definitions 6.21 and 6.22 are equivalent by an application of* [LNW02, Theorem 11] *and* [LW05, Theorem 5].

Lemma 6.24. $\{u_\varepsilon\}$ *converges strongly (weakly) two-scale to* u *if and only if it converges strongly (weakly) in the asymptotic relation sense.*

Proof. Since we consider L^p-spaces for $1 < p < \infty$, we have the asymptotic Kadeč-Klee property by Theorem 5.57 and (6.5). The equivalence follows now from Lemmas 5.35, 5.60. $\square$

Many properties following from our asymptotic relation theory have been proved within the framework of two-scale convergence. We mention for instance the Kadeč-Klee property, weak sequential relative-compactness of bounded set, the characterization of strong and weak convergence in section and the attainability of limits points verified in (A2).

7 Variational convergence of operators and forms

In this chapter, the abstract functional analytic theory of variational convergence for operators and forms is treated. This is done in the varying-Banach-space-framework. We note that the proofs are similar to the classical ones in the case of one fixed space. Our theory from Chapter 5 hence provides rich enough in order to carry over the arguments.

7.1 Lifting via asymptotic isometry

As was noticed by a referee in 2007, Theorem 5.68 can be used to transfer classical results in a single space to the case of an asymptotically isometric linear asymptotic relation. Since the requirements for Theorem 5.68 are always satisfied in the case of separable Hilbert spaces and non-atomic separable L^p-spaces, we find that numerous classical results will hold in the case of asymptotic relations.

The implications of Proposition 7.2 below are enormous. The resulting method is demonstrated as an example in the proof of Theorem 7.10 below.

Fix a linear asymptotic relation $\mathfrak{H}$ consisting of separable Hilbert spaces H_n, $n \in \mathbb{N}$, H. Let $\{\Psi_n : H \to H_n\}$ be as in Theorem 5.68 (which always holds for separable Hilbert spaces). Recall that by Corollary 5.72 for $u_n \in H_n$, $n \in \mathbb{N}$, $u \in H$ it holds that:

$$\begin{aligned} u_n \to (\rightharpoonup) u \ \text{ strongly (weakly) in } \ \mathfrak{H} \\ \text{if and only if} \\ \Psi_n^{-1} u_n \to (\rightharpoonup) u \ \text{ strongly (weakly) in } \ H. \end{aligned} \tag{7.1}$$

Definition 7.1. *Let $n \in \mathbb{N}$. If $x_n \in H_n$ is a vector, $F_n : D(F_n) \subset H_n \to \overline{\mathbb{R}}$ a functional with domain $D(F_n)$, $\mathscr{E}_n : D(\mathscr{E}_n) \subset H_n \times H_n \to \overline{\mathbb{R}}$ a bivariate functional with domain $D(\mathscr{E}_n)$ or $A_n : H_n \to 2^{H_n}$ a possibly multi-valued operator with (effective) domain $D(A^n)$. We shall define the* lifting transformations *$x_n^\Psi := \Psi_n^{-1}(x_n)$, $F_n^\Psi(\cdot) := F_n(\Psi_n(\cdot))$ with domain $D(F_n^\Psi) := \{x \in H \mid \Psi_n(x) \in D(F_n)\}$, $\mathscr{E}_n^\Psi(\cdot,\cdot) := \mathscr{E}_n(\Psi_n(\cdot), \Psi_n(\cdot))$ with domain $D(\mathscr{E}_n) := \{(x,y) \in H \times H \mid (\Psi_n(x), \Psi_n(y)) \in D(\mathscr{E}_n)\}$ or $A_n^\Psi := \Psi_n^{-1} \circ A_n \circ \Psi_n$ with (effective) domain $D(A_n^\Psi) := \{x \in H \mid \Psi_n(x) \in D(A_n)\}$ which are objects defined in/on H. Define the* reverse lifting transformations *$x_n^{\Psi^{-1}}$ for $x_n \in H$, etc. in the obvious way.*

Proposition 7.2 (Lifting method)**.** *Let $\mathscr{V} = \mathscr{C}^1$, $\mathscr{F} = \mathscr{C}^2$, $\mathscr{F}^2 = \mathscr{C}^3$, $\mathscr{O} = \mathscr{C}^4$ be classes of vectors, functionals, bivariate functionals, operators respectively in the Hilbert space H. Suppose that there exist one-to-one and onto maps $\mathscr{I}^{i,j} : \mathscr{C}^i \to \mathscr{C}^j$, $1 \le i, j \le 4$.*

Suppose that $(\mathbf{P})$ *is a statement about strong and weak convergence of sequences in these classes depending only on the classes, the maps and the Hilbert space* H. *Suppose that for any* $n \in \mathbb{N}$ *we are given classes (of such objects)* $\mathscr{V}_n = \mathscr{C}_n^1$, $\mathscr{F}_n = \mathscr{C}_n^2$, $\mathscr{F}_n^2 = \mathscr{C}_n^3$ *and* $\mathscr{O}_n = \mathscr{C}_n^4$ *in the Hilbert space* H_n *with one-to-one onto maps* $\mathscr{I}_n^{i,j} : \mathscr{C}_n^i \to \mathscr{C}_n^j$, $1 \le i, j \le 4$.

If for the lifted classes (as in Definition 7.1) and every $n \in \mathbb{N}$ *it holds that* $\mathscr{V}_n^\Psi \subset \mathscr{V}$, $\mathscr{F}_n^\Psi \subset \mathscr{F}$, $(\mathscr{F}_n^2)^\Psi \subset \mathscr{F}^2$ *and* $\mathscr{O}_n^\Psi \subset \mathscr{O}$ *and if for* $1 \le i, j \le 4$ *and for all* $O_n^i \in \mathscr{C}_n^i$, $O_n^j \in \mathscr{C}_n^j$ *it holds that*

$$\mathscr{I}^{i,j}((O_n^i)^\Psi) = (O_n^j)^\Psi \quad \text{if} \quad \mathscr{I}_n^{i,j}(O_n^i) = O_n^j, \tag{7.2}$$

then an asymptotic version of $(\mathbf{P})$ *holds with the original classes replaced by* $\dot{\bigcup}_n \mathscr{V}_n \dot{\cup} \mathscr{V}$, $\dot{\bigcup}_n \mathscr{F}_n \dot{\cup} \mathscr{F}$, $\dot{\bigcup}_n \mathscr{F}_n^2 \dot{\cup} \mathscr{F}^2$ *and* $\dot{\bigcup}_n \mathscr{O}_n \dot{\cup} \mathscr{O}$ *respectively and* H *replaced by* $\mathfrak{H}$ *with strong and weak convergence replaced as well.*

If the classes coincide and "if" in (7.2) *is replaced by "iff",* $(\mathbf{P})$ *is equivalent to its asymptotic version. The lifting method extends to families of classes in the obvious way. A reverse statement with the reverse lifting transformations also holds.*

Proof. Straightforward from (7.1). □

The next result was noted by a referee. For the notions of strong and weak convergence of bounded linear operators see Definition 7.4 below or [KS03].

Corollary 7.3 (A referee)**.** *Let* $A_n : H_n \to H_n$, $n \in \mathbb{N}$, $A : H \to H$ *be continuous linear operators. Then:*
$A_n \to A$ *strongly in* $\mathfrak{H}$ *if and only if* $A_n^\Psi \to A$ *strongly in* H.
$A_n \rightharpoonup A$ *weakly in* $\mathfrak{H}$ *if and only if* $A_n^\Psi \rightharpoonup A$ *weakly in* H.

Equivalent statements holds for a sequence $\{A_n\} \subset \mathscr{L}(H)$ *and* $A \in \mathscr{L}(H)$ *relative to* $\{A_n^{\Psi^{-1}}\}$ *and* A.

Proof. Apply Proposition 7.2 with property

$$A_n x_n \to Ax \text{ for every } x \in H, \ \{x_n\} \subset H, \ x_n \to x \text{ strongly.} \tag{$\mathbf{P}$}$$

Furthermore, $\mathscr{V} = H$, $\mathscr{V}_n = H_n$, $\mathscr{O} = \mathscr{L}(H)$, $\mathscr{O}_n = \mathscr{L}(H_n)$, $n \in \mathbb{N}$. Clearly, $\mathscr{V} = \mathscr{V}_n^\Psi$ and $\mathscr{O} = \mathscr{O}_n^\Psi$ for each $n \in \mathbb{N}$ by isometry. The proof is concluded by the remark that $(\mathbf{P})$ is equivalent to strong operator convergence by the Uniform Boundedness Principle (cf. [Yos78, pp. 68–69, Corollary II.1.1]). The weak convergence case works similarly (also using the Uniform Boundedness Principle). □

As the reader has certainly noticed, we can easily extend all of the above to a dual pair of asymptotically isometric linear asymptotic relations $(\mathfrak{E}, \mathfrak{E}^*)$ consisting of dual pairs of separable reflexive Banach spaces (E_n, E_n^*), $n \in \mathbb{N}$, (E, E^*). To do this one, we need to adapt to the dual pair of isometries (Ψ_n, Ψ_n^*). As a result, we can e.g. include operators of the type $A_n : E_n \to 2^{E_n^*}$.

7.2 Convergence of bounded linear operators

For this section fix two pairs of asymptotically dual linear strong asymptotic relations $(\mathfrak{E}, \mathfrak{E}^*)$, $(\mathfrak{F}, \mathfrak{F}^*)$.

Definition 7.4 (Convergence of bounded operators)**.** *$\{B_n\}$, $B_n \in \mathscr{L}(E_n, F_n)$, $n \in \mathbb{N}$, is said to* strongly converge *to $B \in \mathscr{L}(E, F)$ if for every sequence $\{u_n\}$, $u_n \in E_n$ strongly converging to $u \in E$ in $\mathfrak{E}$, $\{B_n u_n\}$ strongly converges to Bu in $\mathfrak{F}$.*

$\{B_n\}$, $B_n \in \mathscr{L}(E_n, F_n)$, $n \in \mathbb{N}$ is said to weakly converge *to $B \in \mathscr{L}(E, F)$ if for every sequence $\{u_n\}$, $u_n \in E_n$ weakly converging to $u \in E$ in $\mathfrak{E}$, $\{B_n u_n\}$ weakly converges to Bu in $\mathfrak{F}$.*

$\{B_n\}$, $B_n \in \mathscr{L}(E_n, F_n)$, $n \in \mathbb{N}$, is said to demi converge *to $B \in \mathscr{L}(E, F)$ if for every sequence $\{u_n\}$, $u_n \in E_n$ strongly converging to $u \in E$ in $\mathfrak{E}$, $\{B_n u_n\}$ weakly converges to Bu in $\mathfrak{F}$.*

$\{B_n\}$, $B_n \in \mathscr{L}(E_n, F_n)$, $n \in \mathbb{N}$, is said to compactly converge *to $B \in \mathscr{L}(E, F)$ if for every sequence $\{u_n\}$, $u_n \in E_n$ weakly converging to $u \in E$ in $\mathfrak{E}$, $\{B_n u_n\}$ strongly converges to Bu in $\mathfrak{F}$.*

Clearly, for a sequence of bounded operators following implications hold:

$$\begin{tikzcd} & \text{strong convergence} \arrow[dr] & \\ \text{compact convergence} \arrow[ur] \arrow[dr] & & \text{demi convergence} \\ & \text{weak convergence} \arrow[ur] & \end{tikzcd}$$

Note that, in contrary to the case of one fixed space, in the case of asymptotic relations strong operator convergence does not necessarily imply weak operator convergence. Also uniform operator convergence does not make any sense.

Lemma 7.5. *Let $\{B_n\}$ be a sequence of bounded operators, $B_n \in \mathscr{L}(E_n, F_n)$, $n \in \mathbb{N}$, $B \in \mathscr{L}(E, F)$. Then we have:*

(i) $B_n \to B$ demi if and only if

$$\lim_n \, {}_{F_n^*}\langle v_n, B_n u_n \rangle_{F_n} = {}_{F^*}\langle v, Bu \rangle_F \tag{7.3}$$

for any $\{u_n\}$, $\{v_n\}$, u, v such that $u_n \to u$ strongly in $\mathfrak{E}$ and $v_n \to^* v$ strongly in $\mathfrak{F}^*$.

(ii) $B_n \to B$ weakly if and only if (7.3) holds for any $\{u_n\}$, $\{v_n\}$, u, v such that $u_n \rightharpoonup u$ weakly in $\mathfrak{E}$ and $v_n \to^* v$ strongly in $\mathfrak{F}^*$.

If, additionally, $\mathfrak{F}$ and $\mathfrak{F}^*$ both possess the asymptotic Kadeč-Klee property and are separable and asymptotically reflexive,

(iii) $B_n \to B$ strongly if and only if (7.3) holds for any $\{u_n\}$, $\{v_n\}$, u, v such that $u_n \to u$ strongly in $\mathfrak{E}$ and $v_n \rightharpoonup^* v$ weakly in $\mathfrak{F}^*$.

(iv) $B_n \to B$ *compactly if and only if* (7.3) *holds for any* $\{u_n\}$, $\{v_n\}$, u, v *such that* $u_n \rightharpoonup u$ *weakly in* $\mathfrak{E}$ *and* $v_n \rightharpoonup^* v$ *weakly in* $\mathfrak{F}^*$.

Proof. The assertion follows from the definition and Lemma 5.60. □

Lemma 7.6. *Let* $\{B_n\}$ *be a sequence of bounded operators,* $B_n \in \mathscr{L}(E_n, F_n)$, $n \in \mathbb{N}$, $B \in \mathscr{L}(E, F)$. *Then:*

(i) If $B_n \to B$ *strongly, then*

$$\varliminf_n \|B_n\|_{\mathscr{L}(E_n,F_n)} \geq \|B\|_{\mathscr{L}(E,F)} .$$

(ii) If $\mathfrak{E}$ *is asymptotically reflexive and separable, and if* $B_n \to B$ *compactly, then*

$$\lim_n \|B_n\|_{\mathscr{L}(E_n,F_n)} = \|B\|_{\mathscr{L}(E,F)} .$$

Proof. (i): For any $\varepsilon > 0$ there is a unit vector $u \in E$ such that $\|Bu\|_F > \|B\|_{\mathscr{L}(E,F)} - \varepsilon$. Let $\{u_n\}$, $u_n \in E_n$, $n \in \mathbb{N}$ be strongly convergent to u. By (A3) $\|u_n\|_{E_n} \to 1$. Since $B_n \to B$ strongly, we have by (A3) $\|B_n u_n\|_{F_n} \to \|Bu\|_F$ and therefore,

$$\varliminf_n \|B_n\|_{\mathscr{L}(E_n,F_n)} \geq \varliminf_n \frac{\|B_n u_n\|_{F_n}}{\|u_n\|_{E_n}} = \|Bu\|_F > \|B\|_{\mathscr{L}(E,F)} - \varepsilon,$$

which gives the desired statement.

(ii): There is a sequence of unit vectors $\{u_n\}$, $u_n \in E_n$, $n \in \mathbb{N}$ such that

$$\left| \|B_n\|_{\mathscr{L}(E_n,F_n)} - \|B_n u_n\|_{F_n} \right| \to 0.$$

Replacing with a subsequence, we assume that u_n weakly converges to a vector $u \in E$ with $\|u\|_E \leq 1$ (cf. Lemmas 5.46 and 5.53). Since $B_n u_n \to Bu$ strongly by the assumption, we have

$$\|B\|_{\mathscr{L}(E,F)} \geq \frac{\|Bu\|_F}{\|u\|_E} \geq \|Bu\|_F = \lim_n \|B_n u_n\|_{F_n} = \lim_n \|B_n\|_{\mathscr{L}(E_n,F_n)} ,$$

which together with (i) completes the proof.

□

The following is a direct consequence of Lemmas 5.60 and 7.5.

Corollary 7.7. *Let* $\{B_n\}$ *be a sequence of bounded operators,* $B_n \in \mathscr{L}(E_n, F_n)$, $n \in \mathbb{N}$, $B \in \mathscr{L}(E, F)$. *Let* $B_n^* \in \mathscr{L}(F_n^*, E_n^*)$, $n \in \mathbb{N}$, $B^* \in \mathscr{L}(F^*, E^*)$ *be their linear adjoints. Suppose that* $\mathfrak{E}$ *and* $\mathfrak{E}^*$ *have the asymptotic Kadeč-Klee property,* $\mathfrak{E}$ *is asymptotically reflexive and separable. Then:*

(i) $B_n \to B$ *weakly if and only if* $B_n^* \to B^*$ *strongly.*

Suppose that $\mathfrak{F}$ and $\mathfrak{F}^*$ have the asymptotic Kadeč-Klee property, $\mathfrak{E}$ is asymptotically reflexive and separable. Then:

(ii) $B_n \to B$ strongly if and only if $B_n^* \to B^*$ weakly.

Suppose that $\mathfrak{E}, \mathfrak{E}^*, \mathfrak{F}, \mathfrak{F}^*$ have the asymptotic Kadeč-Klee property, $\mathfrak{E}$ is asymptotically reflexive and separable. Then:

(iii) $B_n \to B$ compactly if and only if $B_n^* \to B^*$ compactly.

If $\mathfrak{E} = \mathfrak{F} = \mathfrak{H}$, where $\mathfrak{H}$ consists of Hilbert spaces, then B_n^* in the above corollary can be replaced by the Hilbert space adjoint $\widehat{B}_n$, since the Riesz maps preserve strong and weak convergence (see Section 6.1).

Lemma 7.8. *Suppose that $\mathfrak{E}$, $\mathfrak{E}^*$, $\mathfrak{F}$, $\mathfrak{F}^*$ have the asymptotic Kadeč-Klee property. Suppose that $\mathfrak{E}$ is asymptotically reflexive and separable. If $B_n \to B$ compactly, then B and B^* are compact operators.*

Proof. By Lemma 5.60 it is enough to prove that B^* is compact.

Let $\{u_m\} \subset F^*$, be a sequence of vectors in F^* weakly converging to a vector $u \in F^*$. It suffices to prove that $B^* u_m \to^* B^* u$ E^*-strongly. We easily see by continuity of B that $B^* u_m \rightharpoonup^* B^* u$ E^*-weakly. For each $m \in \mathbb{N}$, pick a sequence u_n^m with $u_n^m \in F_n^*$, $n \in \mathbb{N}$ such that $u_n^m \to^* u_m$ $\mathfrak{F}^*$-strongly. Since $B_n^* \to B^*$ strongly, we have $B_n^* u_n^m \to^* B^* u_m$ $\mathfrak{E}^*$-strongly for every m. By Lemma 5.27 there is a sequence of natural numbers $m_n \uparrow \infty$ such that

$$\lim_n u_n^{m_n} = u \quad \mathfrak{F}^*\text{-weakly}, \tag{7.4}$$

$$\lim_n \Big| \|B_n^* u_n^{m_n}\|_{E_n^*} - \|B^* u_{m_n}\|_{E^*} \Big| = 0. \tag{7.5}$$

The compact convergence $B_n^* \to B^*$ (cf. Lemma 7.7 (iii)) and (7.4) together show that $B_n^* u_n^{m_n} \to^* B^* u$ $\mathfrak{E}^*$-strongly. Hence, $\|B_n^* u_{m_n}\|_{E_n^*} \to \|B^* u\|_{E^*}$ and so by (7.5) $\|B^* u_{m_n}\|_{E^*} \to^* \|B^* u\|_{E^*}$. Since we can repeat the arguments for any subsequence, $B^* u_m \to^* B^* u$ E^*-strongly by the Kadeč-Klee property. □

The next Lemma shows that strong convergence can be checked on a dense subset if the operator norms are uniformly bounded. We remark that the proof also works for nonlinear Lipschitz operators (as resolvents and semigroups in Section 7.4 below).

Lemma 7.9. *Let $B_n \in \mathscr{L}(E_n, F_n)$, $n \in \mathbb{N}$, $B \in \mathscr{L}(E, F)$, such that*

$$M := \sup_n \|B_n\|_{\mathscr{L}(E_n, F_n)} < \infty.$$

Let C be a strongly dense subset of E. If $B_n v_n \to Bv$ $\mathfrak{F}$-strongly for every $v \in C$ and some $v_n \in E_n$, $n \in \mathbb{N}$ such that $v_n \to v$ $\mathfrak{E}$-strongly, then $B_n \to B$ strongly.

In particular, for uniformly norm bounded sequences of operators it is enough to check strong convergence along metric approximations.

Proof. Let $u \in E$ and $u_n \in E_n$, $n \in \mathbb{N}$ with $u_n \to u$ $\mathfrak{E}$-strongly. Pick $v_m \to u$ strongly in E with $v_m \in C$ for every m. Pick $v_n^m \in E_n$, $n \in \mathbb{N}$ with $v_n^m \to v_m$ strongly in $\mathfrak{E}$ and $B_n v_n^m \to B v_m$ strongly in $\mathfrak{F}$. Let $f_n \in F_n$, $n \in \mathbb{N}$ with $f_n \to Bu$ strongly in $\mathfrak{F}$, let for each $m \in \mathbb{N}$, $g_n^m \in F_n$, $n \in \mathbb{N}$ with $g_n^m \to Bv_m$ strongly in $\mathfrak{F}$.

Clearly,

$$\begin{aligned}
&\|B_n u_n - f_n\|_{F_n} \\
&\le \|B_n u_n - B_n v_n^m\|_{F_n} + \|B_n v_n^m - g_n^m\|_{F_n} + \|g_n^m - f_n\|_{F_n} \\
&\le M\,\|u_n - v_n^m\|_{E_n} + \|B_n v_n^m - g_n^m\|_{F_n} + \|g_n^m - f_n\|_{F_n} \\
&\xrightarrow[n\to\infty]{} M\,\|u - v_m\|_E + \|Bv_m - Bv_m\|_F + \|Bv_m - Bu\|_F \\
&\xrightarrow[m\to\infty]{} 0.
\end{aligned} \tag{7.6}$$

This proves the assertion by (A4). □

7.2.1 A convergence theorem

For this section fix a linear asymptotic relation $\mathfrak{H}$ consisting of separable Hilbert spaces H_n, $n \in \mathbb{N}$, H. The next Theorem is a linear version of Theorem 7.23 below. Classically, it is referred to as the Trotter-Neveu-Kato Theorem as in [Tro58, Kat66]. See Section 2.2 for the terminology. As was noted by a referee, it is an easy exercise to verify that, in the next theorem, the corresponding objects involved are invariant under lifting.

Theorem 7.10. *For each $n \in \mathbb{N}$ let A^n be the infinitesimal generator of a contractive C_0-semigroup $(T_t^n)_{t\ge 0}$ on H_n. Let $(G_\alpha^n)_{\alpha>0}$ be the contractive C_0-resolvent of A^n on H_n. Let A be a generator of a contractive C_0-semigroup $(T_t)_{t\ge 0}$ on H. Let $(G_\alpha)_{\alpha>0}$ be the contractive C_0-resolvent of A on H. Then the following statements are equivalent:*

(i) $A^n \xrightarrow{G} A$ *in the strong graph sense as $n \to \infty$ (see Definition 7.20 below).*

(ii) $G_{\alpha_0}^n \to G_{\alpha_0}$ *strongly as $n \to \infty$ for some $\alpha_0 > 0$.*

(iii) $G_\alpha^n \to G_\alpha$ *strongly as $n \to \infty$ for each $\alpha > 0$.*

(iv) $T_{t_0}^n \to T_{t_0}$ *strongly as $n \to \infty$ for some $t_0 > 0$.*

(v) $T_t^n \to T_t$ *strongly as $n \to \infty$ for each $t \ge 0$.*

Proof. In order to demonstrate the method, we shall only give an example. See [Töl06, §2.1.4 and §2.4] for a proof without lifting. For $n \in \mathbb{N}$ let (T_t^n) be the C_0-contraction semigroup associated to the C_0-contraction resolvent (G_α^n); let (T_t) be the C_0-contraction semigroup associated to the C_0-contraction resolvent (G_α). Let $n \in \mathbb{N}$. $\Psi_n^{-1} T_t^n \Psi_n$ is clearly a contraction for each $t \ge 0$. Also $\lim_{t\to 0} \Psi_n^{-1} T_t^n \Psi_n u = \Psi_n^{-1}\Psi_n u = u$ and $\Psi_n^{-1} T_t^n \Psi_n \Psi_n^{-1} T_s^n \Psi_n = \Psi_n^{-1} T_t^n T_s^n \Psi_n = \Psi_n^{-1} T_{t+s}^n \Psi_n$ for $t, s > 0$. Therefore $(T_t^{\Psi_n})$ is a C_0-contraction semigroup on H for each $n \in \mathbb{N}$. Furthermore, $\alpha \Psi_n^{-1} G_\alpha^n \Psi_n$ is clearly a contraction for each $\alpha > 0$. Also $\lim_{\alpha\to\infty} \Psi_n^{-1} \alpha G_\alpha^n \Psi_n u = \Psi_n^{-1}\Psi_n u = u$ and $\Psi_n^{-1} G_\alpha^n \Psi_n -$

$\Psi_n^{-1}G_\beta^n\Psi_n = \Psi_n^{-1}(G_\alpha^n - G_\beta^n)\Psi_n = (\beta-\alpha)\Psi_n^{-1}G_\alpha^n G_\beta^n\Psi_n = (\beta-\alpha)\Psi_n^{-1}G_\alpha^n\Psi_n\Psi_n^{-1}G_\beta^n\Psi_n$ for $\alpha,\beta>0$. Therefore $(G_\alpha^{\Psi_n})$ is a C_0-contraction resolvent on H for each $n\in\mathbb{N}$. By [MR92, Proposition I.1.10]

$$G_\alpha^n u = \int_0^\infty e^{-\alpha s}T_s^n u\,\mathrm{d}s, \quad \forall u\in H_n,\ \alpha>0.$$

It is well-known that Bochner integrals interchange with continuous linear operators (cf. [Yos78, Ch. V.5, Corollary 2]). Therefore ($\alpha>0$, $v\in H$)

$$\Psi_n^{-1}G_\alpha^n\Psi_n v = \Psi_n^{-1}\int_0^\infty e^{-\alpha s}T_s^n\Psi_n v\,\mathrm{d}s = \int_0^\infty e^{-\alpha s}\Psi_n^{-1}T_s^n\Psi_n v\,\mathrm{d}s,$$

which proves that $(G_\alpha^{\Psi_n})$ is the resolvent associated to $(T_t^{\Psi_n})$.

Now we can apply the classical Trotter-Neveu-Kato Theorem (see e.g. the survey paper of Simeon Reich [Rei82] and the references therein) with Proposition 7.2 to get (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v). The equivalence with (i) is proved in the same way. The general method should be clear. □

Quite recently, Zhikov and Pastukhova [ZP07] proved a Trotter-Kato-type theorem in variable spaces. Please compare with Section 5.2.1. Anyhow, even if not taking advantage of the lifting method demonstrated above, we claim that the general features of linear asymptotic relations of Banach spaces suffice to transfer many classical results. We will do so in the case of nonlinear operators below.

7.3 Convergence of bilinear forms

7.3.1 Symmetric forms

For this section fix a linear asymptotic relation $\mathfrak{H}$ consisting of separable Hilbert spaces H_n, $n\in\mathbb{N}$, H.

Definition 7.11. *A sequence $\{\mathscr{E}^n : H_n\to\overline{\mathbb{R}}_+\}$ of non-negative, closed, quadratic forms is said to Mosco converge to a non-negative, closed, quadratic form $\mathscr{E}: H\to\overline{\mathbb{R}}_+$ if the following two conditions hold:*

(M1) *If $\{u_n\}$, $u_n\in H_n$, $n\in\mathbb{N}$ weakly converges to $u\in H$ then*

$$\mathscr{E}(u)\le\varliminf_n \mathscr{E}^n(u_n).$$

(M2) *For every $u\in H$ there exists a strongly convergent sequence $u_n\to u$, $u_n\in H_n$, $n\in\mathbb{N}$ such that*

$$\mathscr{E}(u)=\lim_n\mathscr{E}^n(u_n).$$

The following theorem is classical, compare [Mos94].

Theorem 7.12 (Mosco, Kuwae, Shioya)**.** *Let $\{\mathscr{E}^n : H_n \to \overline{\mathbb{R}}_+\}$ be a sequence of non-negative, closed, quadratic forms and let $\mathscr{E} : H \to \overline{\mathbb{R}}_+$ be a non-negative, closed, quadratic form. The following statements are equivalent:*

(i) $\{\mathscr{E}^n\}$ *Mosco converges to* $\mathscr{E}$.

(ii) $\{G^n_\alpha\}$ *strongly converges to* G_α *for all* $\alpha > 0$.

(iii) $\{T^n_t\}$ *strongly converges to* T_t *for all* $t \geq 0$.

(iv) $A^n \xrightarrow{G} A$ *in the graph sense.*

Proof. See [KS03, Theorem 2.4] or [Kol05, Theorem 2.8, Appendix] and Theorem 7.10. □

7.3.2 Non-symmetric forms

Consider a generalized form $\mathscr{E}$ on the Hilbert space H which is associated to a coercive closed form $(\mathscr{A}, \mathscr{V})$ (with sector constant $K \geq 1$) and an operator $(\Lambda, \mathscr{F})$. See Subsection 2.2.3 for the terminology.

Define a functional, which measures the rate of asymmetry of our form $\mathscr{E}$, and is, in fact, an equivalent norm to $\|\cdot\|_{\widehat{\mathscr{F}}}$ (cf. Lemma 7.13 below). So let

$$\Theta(u) := \sup_{v \in \mathscr{V},\, \|v\|_{\mathscr{V}}=1} |\mathscr{E}_1(v,u)| = \|\mathscr{E}_1(\cdot,u)\|_{\mathscr{V}^*}, \quad \text{for } u \in \widehat{\mathscr{F}},$$

which is finite. If $u \in H \setminus \widehat{\mathscr{F}}$, we extend Θ to a functional on H with values in $\overline{\mathbb{R}}_+$ by setting $\Theta(u) := +\infty$.

Lemma 7.13. *For* $u \in \widehat{\mathscr{F}}$, *we have*

(i) $\Theta(u) \leq K \|u\|_{\widehat{\mathscr{F}}}$,

(ii) $\|u\|_{\mathscr{V}} \leq \Theta(u)$,

(iii) $\left\|\widehat{\Lambda} u\right\|_{\mathscr{V}^*} \leq (K+1)\Theta(u)$,

(iv) In particular, $\|\cdot\|_{\widehat{\mathscr{F}}} \sim \Theta(\cdot)$ *on* $\widehat{\mathscr{F}}$.

Proof. See [Hin98, Lemma 2.2]. □

For each $n \in \mathbb{N}$ let $\mathscr{E}^n$ be a generalized form on the Hilbert space H_n which is associated to a coercive closed form $(\mathscr{A}^n, \mathscr{V}_n)$ (with sector constant $K_n \geq 1$) and an operator $(\Lambda^n, \mathscr{F}_n)$. Also Θ^n as above. The following conditions have been introduced by Masanori Hino in [Hin98, §3]. Compare Silvia Mataloni's work [Mat99] for the sectorial case.

Definition 7.14. *Consider the following conditions:*

(F1) *If a sequence* $\{u_n\}$ *weakly convergent to* u *in* $\mathfrak{H}$ *satisfies* $\underline{\lim}_n \Theta^n(u_n) < \infty$*, then* $u \in \mathscr{V}$*.*

(F2) *For any* $w \in \mathscr{F}$ *and any* $u \in \mathscr{V}$ *and any sequence* $\{u_n\}$ *weakly convergent to* u *in* $\mathfrak{H}$*,* $u_n \in \mathscr{V}_n$*,* $n \in \mathbb{N}$*, there exists* $\{w_n\}$*,* $w_n \in H_n$*,* $n \in \mathbb{N}$ *converging to* w *strongly in* $\mathfrak{H}$ *such that* $\lim_n \mathscr{E}^n(w_n, u_n) = \mathscr{E}(w, u)$*.*

(F2') *There exists a linear subspace* C *of* H *such that* $C \subset \mathscr{F}$ *densely w.r.t* $\| \|_{\mathscr{F}}$ *and for any sequence* $n_k \uparrow \infty$ *and every* $w \in C$*,* $u \in \mathscr{V}$ *and any sequence* $\{u_k\}$*,* $u_k \in H_{n_k}$*,* $k \in \mathbb{N}$ *converging weakly to* u *in* $\mathfrak{H}$ *and satisfying* $\sup_k \Theta^{n_k}(u_k) < +\infty$ *one has a sequence* $\{w_k\}$*,* $w_k \in H_{n_k}$*,* $k \in \mathbb{N}$ *converging* $\mathfrak{H}$*-strongly to* w *with* $\underline{\lim}_k \mathscr{E}^{n_k}(w_k, u_k) \le \mathscr{E}(w, u)$*.*

(R) $\{G^n_\alpha\}$ *converges to* G_α *strongly for all* $\alpha > 0$*.*

Define also (F1a) *(resp.* (F1b)*) by replacing* $\Theta^n(u_n)$ *by* $\|u_n\|_{\widehat{\mathscr{F}}_n}$ *(resp.* $\|u_n\|_{\mathscr{V}_n}$*) in* (F1) *and* (F2'a) *(resp.* (F2'b)*) by replacing* $\Theta^{n_k}(u_k)$ *by* $\|u_k\|_{\widehat{\mathscr{F}}_{n_k}}$ *(resp.* $\|u_k\|_{\mathscr{V}_{n_k}}$*) in* (F2')*.*

We have (F1b) $\Rightarrow$ (F1) and (F2'b) $\Rightarrow$ (F2') by Lemma 7.13 (ii).

The following Theorem for a single Hilbert space H was proved by Masanori Hino in [Hin98, Theorem 3.1].

Theorem 7.15.

$$(\text{F2}) \;\Rightarrow\; (\text{F2'}),$$

$$(\text{F1})\,\&\,(\text{F2'}) \;\Leftrightarrow\; (\text{F1})\,\&\,(\text{F2}) \;\Leftrightarrow\; (\text{R}).$$

Proof. As was proposed by a referee in 2007, it is an easy exercise to check that generalized forms, Θ and resolvents are invariant under lifting. Hence we can apply Proposition 7.2 to [Hin98, Theorem 3.1]. □

Corollary 7.16. *(i)* (F1b),(F2'b) $\Rightarrow$ (R),

(ii) If the sector constants K_n *of the* $\mathscr{A}^n$*'s are uniformly bounded, then* (F1a),(F2'a) $\Leftrightarrow$ (R).

Proof. (i): This is trivial, since clearly (F1b) $\Rightarrow$ (F1) and (F2'b) $\Rightarrow$ (F2') by Lemma 7.13 (ii).

(ii): This is an consequence of Theorem 7.15 and Lemma 7.13.

□

Remark 7.17. *According to Theorem 7.10, corresponding statements to (R) hold also for the associated semigroups* $(T^n_t)_{t\ge 0}$*,* $n \in \mathbb{N}$*,* $(T_t)_{t\ge 0}$ *resp. And for the associated co-semigroups* $(\widehat{T}^n_t)_{t\ge 0}$*,* $n \in \mathbb{N}$*,* $(\widehat{T}_t)_{t\ge 0}$ *resp. and co-resolvents* $(\widehat{G}^n_\alpha)_{\alpha>0}$*,* $n \in \mathbb{N}$*,* $(\widehat{G}_\alpha)_{\alpha>0}$ *with strong operator-convergence replaced by weak operator-convergence by Lemma 7.7.*

The next Proposition can easily be proved with the help of "diagonal-lemma" 5.27. See [Töl06, Proposition 2.44]. Compare also [Mat99, Proposition 2.5] for one fixed space H.

Proposition 7.18. *Let $\mathscr{E}^n$, $n \in \mathbb{N}$, $\mathscr{E}$ be as in Theorem 7.15. Suppose that there is a dense linear subset C of $\mathscr{F}$ ($\mathscr{V}$ resp.), dense in $\|\cdot\|_{\mathscr{F}}$-norm ($\|\cdot\|_{\mathscr{V}}$-norm resp.) Then the following weaker version of* (F2) *(*(F2b) *resp.) is equivalent to it:*

For any $w \in C$ and any $u_n \in \mathscr{V}_n$, $n \in \mathbb{N}$, $u \in \mathscr{V}$ with $u_n \to u$ $\mathscr{H}$-weakly there exists a sequence $\{w_n\}$, $w_n \in H_n$, $n \in \mathbb{N}$, $w_n \to w$ $\mathscr{H}$-strongly such that

$$\lim_n \mathscr{E}^n(w_n, u_n) = \mathscr{E}(w, u).$$

This result also extends to (F2') *and* (F2'b) *in the obvious way.*

Remark 7.19. *(i) It is clear by Theorem 7.12 and Theorem 7.15 that in case of symmetric forms* (F1) *and* (F2) *are just another characterization of Mosco convergence.*

(ii) Obviously, if $\mathscr{A}^n \equiv 0$, $\mathscr{V}_n = H_n$, $n \in \mathbb{N}$, $\mathscr{A} \equiv 0$, $\mathscr{V} = H$, i.e., our generalized forms $\{\mathscr{E}^n\}$ depend only on operators $(\Lambda_n, D(\Lambda_n, H_n)) = (L_n, D(L_n))$, $n \in \mathbb{N}$ (see [Sta99b, Remark I.4.10] *for details), condition* (F1) *can be omitted.*

7.4 G-convergence of nonlinear operators

Fix two linear strong asymptotic relations $\mathfrak{X}, \mathfrak{Y}$ consisting of Banach spaces X_n, Y_n, $n \in \mathbb{N}$, X, Y. For the terminology of monotone operators, resolvents etc. we refer to Chapter 2.

Let $A_n : X_n \to 2^{Y_n}$, $n \in \mathbb{N}$, $A : X \to 2^Y$ be multi-valued operators (graphs).

Definition 7.20. *We say that $A_n \xrightarrow[n\to\infty]{G} A$ ($\{A_n\}$ G-*converges *to A, resp. $\{A_n\}$* converges strong graph *to A) if for all $[x, y] \in A$ there are $[x_n, y_n] \in A_n$, $n \in \mathbb{N}$, such that $x_n \to x$ strongly in $\mathfrak{X}$ and $y_n \to y$ strongly in $\mathfrak{Y}$.*

*We say that $A_n \xrightarrow[n\to\infty]{G(w,s)} A$ ($\{A_n\}$ $G(w,s)$-*converges *to A) if for all $[x, y] \in A$ there are $[x_n, y_n] \in A_n$, $n \in \mathbb{N}$, such that $x_n \rightharpoonup x$ weakly in $\mathfrak{X}$ and $y_n \to y$ strongly in $\mathfrak{Y}$.*

*We say that $A_n \xrightarrow[n\to\infty]{G(s,w)} A$ ($\{A_n\}$ $G(s,w)$-*converges *to A) if for all $[x, y] \in A$ there are $[x_n, y_n] \in A_n$, $n \in \mathbb{N}$, such that $x_n \to x$ strongly in $\mathfrak{X}$ and $y_n \rightharpoonup y$ weakly in $\mathfrak{Y}$.*

*$G(w^*, s)$- and $G(s, w^*)$-*convergence *are defined analogously with weak convergence replaced by weak** convergence*.*

Remark 7.21. *(i) G-convergence both implies $G(w,s)$- and $G(s,w)$-convergence.*

(ii) If A_n, $n \in \mathbb{N}$, A are linear continuous operators with domains $D(A_n) = X_n$, $n \in \mathbb{N}$, $D(A) = X$, $A_n \to A$ strongly clearly implies $A_n \xrightarrow{G} A$ in the G-sense. $A_n \to A$ demi implies $G(s,w)$-convergence.

From now on assume that $(\mathfrak{X}, \mathfrak{X}^*)$ is a pair of asymptotically dual strong linear asymptotic relations consisting of Banach spaces X_n, $n \in \mathbb{N}$, X and their duals respectively.

Lemma 7.22 ("Monotonicity trick"). *Let $A^n : D(A^n) \subset X_n \to 2^{X_n^*}$, $n \in \mathbb{N}$, $A : D(A) \subset X \to 2^{X^*}$ be a G-convergent sequence of monotone operators. Suppose that A is maximal monotone. Let $[x_n, y_n] \in A^n$, $n \in \mathbb{N}$, $x \in D(A)$, $y \in X^*$ such that*

$$x_n \rightharpoonup x \quad \text{weakly in } \mathfrak{X} \text{ as } n \to \infty, \tag{7.7}$$

$$y_n \rightharpoonup^* y \quad \text{weakly in } \mathfrak{X}^* \text{ as } n \to \infty, \tag{7.8}$$

$$\varlimsup_n \, {}_{X_n^*}\langle y_n, x_n \rangle_{X_n} \leq {}_{X^*}\langle y, x \rangle_X \,. \tag{7.9}$$

Then $y \in A(x)$.

Proof. Let $[u, v] \in A$. Let $[u_n, v_n] \in A^n$ with $u_n \to u$, $v_n \to^* v$ strongly, which exist by the G-convergence. Then

$${}_{X_n^*}\langle y_n, x_n \rangle_{X_n} - {}_{X_n^*}\langle v_n, x_n \rangle_{X_n} - {}_{X_n^*}\langle y_n - v_n, u_n \rangle_{X_n} = {}_{X_n^*}\langle y_n - v_n, x_n - u_n \rangle_{X_n} \geq 0.$$

Passing on to the limit,

$${}_{X^*}\langle y, x \rangle_X - {}_{X^*}\langle v, x \rangle_X - {}_{X^*}\langle y - v, u \rangle_X \geq 0,$$

so

$${}_{X^*}\langle y - v, x - u \rangle_X \geq 0,$$

which is true for all $[u, v] \in A$. Hence by the maximality $[x, y] \in A$. □

The subsequent theorem is proved essentially by the ideas of H. Attouch in [Att84, Proposition 3.60]. Our proof illustrates profoundly that the concepts of asymptotic reflexivity and asymptotic Kadeč-Klee property are excellent substitutes when making the transition from a single space to an asymptotic relation.

Theorem 7.23. *Suppose that $\mathfrak{X}$ as well as $\mathfrak{X}^*$ is asymptotically reflexive and separable and possesses the asymptotic Kadeč-Klee property. Suppose that X_n, $n \in \mathbb{N}$, X as well as their duals are strictly convex. Let $A^n : X_n \to 2^{X_n^*}$, $n \in \mathbb{N}$, $A : X \to 2^{X^*}$ be maximal monotone. Denote by $(A_\lambda^n)_{\lambda > 0}$, $n \in \mathbb{N}$, $(A_\lambda)_{\lambda > 0}$ the associated Yosida-approximations and denote by $(R_\lambda^n)_{\lambda > 0}$, $n \in \mathbb{N}$, $(R_\lambda)_{\lambda > 0}$ the associated resolvents. See Paragraph 2.3.1 for the terminology.*

Then the following statements are equivalent:

(i) $A_n \xrightarrow{G} A$.

(ii) $R_\lambda^n x_n \to R_\lambda x$ *strongly in $\mathfrak{X}$ for each $\lambda > 0$ and for each $x_n \in X_n$, $n \in \mathbb{N}$, $x \in X$ with $x_n \to x$ strongly in $\mathfrak{X}$.*

(iii) $A_\lambda^n x_n \to^* A_\lambda x$ *strongly in $\mathfrak{X}^*$ for each $\lambda > 0$ and for each $x_n \in X_n$, $n \in \mathbb{N}$, $x \in X$ with $x_n \to x$ strongly in $\mathfrak{X}$.*

(iv) $A^n_{\lambda_0} x_n \to^* A_{\lambda_0} x$ *strongly in* $\mathfrak{X}^*$ *for some* $\lambda_0 > 0$ *and for each* $x_n \in X_n$*,* $n \in \mathbb{N}$*,* $x \in X$ *with* $x_n \to x$ *strongly in* $\mathfrak{X}$*.*

(v) $R^n_{\lambda_0} x_n \to R_{\lambda_0} x$ *strongly in* $\mathfrak{X}$ *for some* $\lambda_0 > 0$ *and for each* $x_n \in X_n$*,* $n \in \mathbb{N}$*,* $x \in X$ *with* $x_n \to x$ *strongly in* $\mathfrak{X}$*.*

Proof. Let us prove the sequence of implications: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (iv) is trivial. Recall that by Proposition 5.58 and strict convexity (see appendix)

$$J_n(x_n) \to^* J(x), \quad J_n^{-1}(y_n) \to J^{-1}(y) \tag{7.10}$$

for all $x_n \to x$ in $\mathfrak{X}$ and all $y_n \to^* y$ in $\mathfrak{X}^*$. From this we infer (ii) $\Leftrightarrow$ (iii) and (iv) $\Leftrightarrow$ (v). We are left to prove (i) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii):

Let $x_n \in X_n$, $n \in \mathbb{N}$, $x \in X$ with $x_n \to x$ strongly in $\mathfrak{X}$. Let $\lambda > 0$. By G-convergence let $[x_0, y_0] \in A$, $[x_n^0, y_n^0] \in A^n$, $n \in \mathbb{N}$ with $x_n^0 \to x_0$ strongly in $\mathfrak{X}$ and $y_n^0 \to^* y_0$ strongly in $\mathfrak{X}^*$. By (2.8)

$$\|x_n - R^n_\lambda x_n\|_{X_n} \le 2\left[\left\|x_n - x_n^0\right\|_{X_n} + \lambda \left\|y_n^0\right\|_{X_n^*}\right]$$

so that $\left\{\|R^n_\lambda x_n\|_{X_n}\right\}$ remains bounded by (A3'). Indeed, so does $\left\{\|A^n_\lambda x_n\|_{X_n^*}\right\}$, recalling that $A^n_\lambda x_n = \frac{1}{\lambda} J_n(x_n - R^n_\lambda x_n)$.

Let $\{n_k\}$ be any subsequence of $\{n\}$. By asymptotic reflexivity and Lemma 5.53 find a subsequence $\{n_{k_l}\}$ of $\{n_k\}$ such that both

$$\begin{aligned} &R_\lambda^{n_{k_l}} x_{n_{k_l}} \rightharpoonup f \text{ weakly in } \mathfrak{X} \text{ as } l \to \infty, \\ &A_\lambda^{n_{k_l}} x_{n_{k_l}} \rightharpoonup^* g \text{ weakly in } \mathfrak{X}^* \text{ as } l \to \infty, \end{aligned}$$

for some $f \in X$ and some $g \in X^*$. For notational convenience we write $\{n\}$ for $\{n_{k_l}\}$ in the sequel. By G-convergence pick $[u, v] \in A$, $[u_n, v_n] \in A^n$, $n \in \mathbb{N}$ with $u_n \to u$ strongly in $\mathfrak{X}$ and $v_n \to^* v$ strongly in $\mathfrak{X}^*$. By monotonicity and $[R^n_\lambda x_n, A^n_\lambda x_n] \in A^n$

$$_{X_n^*}\langle A^n_\lambda x_n - v_n, R^n_\lambda x_n - u_n\rangle_{X_n} \ge 0 \tag{7.11}$$

for each $n \in \mathbb{N}$. Now by weak convergence and Lemma 5.46

$$\varliminf_n \, _{X_n^*}\langle A^n_\lambda x_n, x_n - R^n_\lambda x_n\rangle_{X_n} = \frac{1}{\lambda} \|x_n - R^n_\lambda x_n\|^2_{X_n} \ge \frac{1}{\lambda} \|x - f\|^2_X . \tag{7.12}$$

But also

$$\varliminf_n \, _{X_n^*}\langle A^n_\lambda x_n, x_n - R^n_\lambda x_n\rangle_{X_n} = \lambda \|A^n_\lambda x_n\|^2_{X_n^*} \ge \lambda \|g\|^2_{X^*} . \tag{7.13}$$

Now multiply (7.12) and (7.13), take the square root and multiply with -1:

$$\varlimsup_n - \, _{X_n^*}\langle A^n_\lambda x_n, x_n - R^n_\lambda x_n\rangle_{X_n} \le -\|g\|_{X^*} \|x - f\|_X \le - \, _{X^*}\langle g, x - f\rangle_X . \tag{7.14}$$

The last equation yields

$$\begin{aligned}&\overline{\lim_n}\ {}_{X_n^*}\langle A_\lambda^n x_n, R_\lambda^n x_n\rangle_{X_n} = \overline{\lim_n}\Big[-\ {}_{X_n^*}\langle A_\lambda^n x_n, x_n - R_\lambda^n x_n\rangle_{X_n} + {}_{X_n^*}\langle A_\lambda^n x_n, x_n\rangle_{X_n}\Big] \\ &\le -\ {}_{X^*}\langle g, x-f\rangle_X + {}_{X^*}\langle g, x\rangle_X = {}_{X^*}\langle g, f\rangle_X\,.\end{aligned} \tag{7.15}$$

Hence we can apply Lemma 7.22 and obtain that $[f,g] \in A$. Choosing u, v in (7.11) equal to f, g yields

$$\underline{\lim_n}\ {}_{X_n^*}\langle A_\lambda^n x_n, R_\lambda^n x_n\rangle_{X_n} \ge {}_{X^*}\langle g, f\rangle_X\,. \tag{7.16}$$

Combining (7.15) and (7.16) yields

$$\lim_n\ {}_{X_n^*}\langle A_\lambda^n x_n, R_\lambda^n x_n\rangle_{X_n} = {}_{X^*}\langle g, f\rangle_X \tag{7.17}$$

or equivalently

$$\lim_n\ {}_{X_n^*}\left\langle A_\lambda^n x_n, \frac{1}{\lambda}\left(x_n - R_\lambda^n x_n\right)\right\rangle_{X_n} = {}_{X^*}\left\langle g, \frac{1}{\lambda}(x-f)\right\rangle_X\,. \tag{7.18}$$

But $A_\lambda^n x_n = \frac{1}{\lambda}J_n\left(x_n - R_\lambda^n x_n\right)$. Hence by Lemma 5.59 and the asymptotic Kadeč-Klee property of $\mathfrak{X}$ and $\mathfrak{X}^*$ respectively

$$\begin{aligned}&R_\lambda^{n_{k_l}} x_{n_{k_l}} \to f \ \text{ strongly in } \mathfrak{X} \ \text{ as } \ l\to\infty,\\ &A_\lambda^{n_{k_l}} x_{n_{k_l}} \to^* g \ \text{ strongly in } \mathfrak{X}^* \ \text{ as } \ l\to\infty.\end{aligned}$$

By strict convexity of X^* we have that J is continuous w.r.t. the strong topologies of X and X^*. Combined with (7.10) we get that $g = \frac{1}{\lambda}J(x-f)$. Therefore $[f,g]\in A$ implies $f = R_\lambda x$ and $g = A_\lambda x$. Since the argument works for arbitrary subsequences with the same limit points we see that the whole sequences converge strongly to these points. (ii) is verified (as is (iii)).

(v) $\Rightarrow$ (i):

Let $[u,v]\in A$. Pick $\widetilde{u}_n \in X_n$, $\widetilde{v}_n \in X_n^*$ with $\widetilde{u}_n \to u$, $\widetilde{v}_n \to^* v$. Set

$$x := u + \lambda_0 J^{-1} v, \quad x_n := \widetilde{u}_n + \lambda_0 J_n^{-1}\widetilde{v}_n, \ \ n\in\mathbb{N}.$$

Clearly, $x_n \to x$ strongly in $\mathfrak{X}$ by (7.10). From $v = \frac{1}{\lambda_0}J(x-u)$ and $[u,v]\in A$ it follows by definition of R_{λ_0} that

$$R_{\lambda_0}x = u \quad \text{and} \quad A_{\lambda_0}x = v.$$

By assumption (v) (and (iv)) $R_{\lambda_0}^n x_n \to R_{\lambda_0}x$ in $\mathfrak{X}$ (and $A_{\lambda_0}^n x_n \to^* A_{\lambda_0}x$ in $\mathfrak{X}^*$). Set

$$u_n := R_{\lambda_0}^n x_n, \quad v_n := A_{\lambda_0}^n x_n, \ \ n\in\mathbb{N}.$$

Clearly, $[u_n, v_n]\in A^n$ and $u_n \to u$, $v_n \to^* v$: the G-convergence (i). □

The next theorem works only in Hilbert spaces, however, if a sequence $\{A_n\}$ of maximal monotone operators is defined on a linear asymptotic relation of Banach spaces $\mathfrak{V}$, it is sometimes possible to consider it as defined on $\mathfrak{H}$, where $\mathfrak{H}$ is a linear asymptotic relation of Hilbert spaces such that $\mathfrak{V}\hookrightarrow\mathfrak{H}$ is an asymptotically strong embedding as in Section 5.11.

Theorem 7.24. *Let $\mathfrak{H}$ be a linear asymptotic relation consisting of infinite dimensional separable Hilbert spaces. For each $n \in \mathbb{N}$ let $A_n : D(A_n) \subset H_n \to 2^{H_n}$ be a maximal monotone operator. Let $(S_t^n)_{t\geq 0}$ be the non-expansive semigroup associated to $-A_n$ via Theorems 2.18, 2.19. Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator. Let $(S_t)_{t\geq 0}$ be the non-expansive semigroup associated to $-A$. Then the following are equivalent:*

(i) $A_n \xrightarrow{G} A$.

(ii) $S_t^n x_n \to S_t x$ *strongly in* $\mathfrak{H}$ *for all* $t \geq 0$ *and for each* $x_n \in H_n$, $n \in \mathbb{N}$, $x \in H$ *with* $x_n \to x$ *strongly in* $\mathfrak{H}$.

Proof. As verified in Section 6.1, $\mathfrak{H}$ is asymptotically isometric. For a fixed space, the result above is classical, see [Bré73, Théorème 3.16, Théorème 4.2], Theorem 7.23 above and [Bré75, Theorem 11] (see also [BP72]). By Theorem 2.19 the strong limit

$$\lim_{t\searrow 0} \frac{S_t x - x}{t} = -A^0 x \quad \forall x \in D(A), \tag{7.19}$$

where $A^0 x := \mathrm{Proj}_{Ax}(0)$, $x \in D(A)$ is the principal section of A and A is uniquely characterized by this limit. A converse characterization and uniqueness also holds, see Theorem 2.18. Therefore, according to Proposition 7.2 we need only check that this one-to-one correspondence as well as the classes are invariant under lifting. But it is an easy exercise to check that the semigroup-axioms in Definition 2.17 as well as the axioms for A (namely: monotonicity and $\mathrm{ran}(A + \mathrm{Id}) = H$) and the properties of the domains and principal section are preserved. Limit (7.19) is clearly also invariant under lifting. For details on semigroups, we refer to Paragraph 2.3.2. □

7.5 Γ, Mosco and slice convergence

Recall that $\mathbb{R}_\infty := (-\infty, +\infty]$. For this section assume that $(\mathfrak{X}, \mathfrak{X}^*)$ is a pair of asymptotically dual strong linear asymptotic relations consisting of Banach spaces X_n, $n \in \mathbb{N}$, X and their duals respectively.

Definition 7.25 (Γ-convergence)**.** *For each $n \in \mathbb{N}$ let $f_n : X_n \to \mathbb{R}_\infty$ be a proper convex functional and let $f : X \to \mathbb{R}_\infty$ be a proper convex functional. We say that $\{f_n\}$* Γ-converges *to f ($f_n \xrightarrow[n\to\infty]{\Gamma} f$) as $n \to \infty$ if the following two conditions are satisfied:*

$$\forall x \in X\ \forall x_n \in X_n,\ n \in \mathbb{N},\ x_n \to x \text{ strongly}: \quad \varliminf_n f_n(x_n) \geq f(x). \tag{$\Gamma1$}$$

$$\forall y \in X\ \exists y_n \in X_n,\ n \in \mathbb{N},\ y_n \to y \text{ strongly}: \quad \varlimsup_n f_n(y_n) \leq f(y). \tag{$\Gamma2$}$$

Definition 7.26 (Mosco convergence)**.** *For each $n \in \mathbb{N}$ let $f_n : X_n \to \mathbb{R}_\infty$ be a proper convex functional and let $f : X \to \mathbb{R}_\infty$ be a proper convex functional. We say that $\{f_n\}$*

Mosco converges *to* f *(*$\{f_n\}$ M-converges *to* f*;* $f_n \xrightarrow[n\to\infty]{M} f$*) as* $n \to \infty$ *if the following two conditions are satisfied:*

$$\forall x \in X\ \forall x_n \in X_n,\ n \in \mathbb{N},\ x_n \rightharpoonup x \text{ weakly}: \quad \varliminf_n f_n(x_n) \geq f(x). \tag{M1}$$

$$\forall y \in X\ \exists y_n \in X_n,\ n \in \mathbb{N},\ y_n \to y \text{ strongly}: \quad \varlimsup_n f_n(y_n) \leq f(y). \tag{M2}$$

Definition 7.27 (Slice convergence)**.** *For each* $n \in \mathbb{N}$ *let* $f_n : X_n \to \mathbb{R}_\infty$ *be a proper convex functional and let* $f : X \to \mathbb{R}_\infty$ *be a proper convex functional. We say that* $\{f_n\}$ slice converges *to* f *(*$\{f_n\}$ S-converges *to* f*;* $f_n \xrightarrow[n\to\infty]{S} f$*) as* $n \to \infty$ *if*

$$f_n \xrightarrow[n\to\infty]{M} f \quad \text{and} \quad f_n^* \xrightarrow[n\to\infty]{M^*} f^*,$$

where M^*-convergence *is defined to be Mosco convergence with "weakly" in (M1) replaced by "weakly*".* f^* *denotes the* Legendre transform*, cf. Section 2.4.3.*

Remark 7.28. *Obviously,*

$$\left(f_n \xrightarrow{S} f\right) \Rightarrow \left(f_n \xrightarrow{M} f\right) \Rightarrow \left(f_n \xrightarrow{\Gamma} f\right).$$

As we shall see below, when $\mathfrak{X}$ *is asymptotically reflexive and separable,*

$$\left(f_n \xrightarrow{S} f\right) \Leftrightarrow \left(f_n \xrightarrow{M} f\right).$$

This is a consequence of Theorem 7.38 below.

Proposition 7.29 (Asymptotic properties)**.** *Suppose that for each* $n \in \mathbb{N}$*,* $f_n : X_n \to \mathbb{R}_\infty$ *is a proper functional and* $f : X \to \mathbb{R}_\infty$ *is a proper functional such that* $f_n \xrightarrow[n\to\infty]{\Gamma} f$*.*

(i) f *is l.s.c.*

(ii) If $f_n \geq 0$*,* $n \in \mathbb{N}$*, then* $f \geq 0$*.*

(iii) If f_n*,* $n \in \mathbb{N}$ *are convex, then* f *is convex.*

(iv) If $f_n \geq 0$*,* $n \in \mathbb{N}$ *are positively homogeneous with degree* p_n *and* $p_n \to p$ *as* $n \to \infty$ *for some* p*, then* f *is positively homogeneous with degree* p*.*

(v) If f_n*,* $n \in \mathbb{N}$ *are quadratic forms, then* f *is a quadratic form.*

Proof. The proof for the classical case can be found in [DM93, Propositions 6.7 and 6.8, Theorem 11.1, Proposition 11.6].

(i): Let $x \in X$. Let $x_m \in X$, $m \in \mathbb{N}$ with $\lim_m \|x_m - x\|_X = 0$ and the property that $\lim_m f(x_m)$ exists and is less than $+\infty$. By Lemma 2.22 it is enough to prove that $\lim_m f(x_m) \geq f(x)$. Let $x_n^m \in X_n$, $n \in \mathbb{N}$ with $x_n^m \to x_m$ for each $m \in \mathbb{N}$ and

$$\lim_n f_n(x_n^m) = f(x_m)$$

by (Γ1) and (Γ2). By Lemma 5.27 there is a subsequence $\{m_n\}$ of $\{m\}$ such that $x_n^{m_n} \to x$ in $\mathfrak{X}$ and

$$\lim_m f(x_m) = \lim_m \lim_n f_n(x_n^m) \geq \overline{\lim_n} f_n(x_n^{m_n}) \geq \underline{\lim_n} f_n(x_n^{m_n}) \geq f(x)$$

where we have used (Γ1).

(ii): Clear by (Γ2).

(iii): Let $x, y \in X$ with $f(x) < \infty$, $f(y) < \infty$, $t \in (0,1)$. By (Γ2) there exist $x_n, y_n \in X_n$, $n \in \mathbb{N}$ with $x_n \to x$, $y_n \to y$ and $\overline{\lim}_n f_n(x_n) \leq f(x)$, $\overline{\lim}_n f_n(y_n) \leq f(y)$. By (AL) $tx_n + (1-t)y_n \to tx + (1-t)y$. By ($\Gamma$1) and the assertion

$$\begin{aligned} f(tx + (1-t)y) &\leq \underline{\lim_n} f_n(tx_n + (1-t)y_n) \\ &\leq \overline{\lim_n} f_n(tx_n + (1-t)y_n) \leq \overline{\lim_n} \left[t f_n(x_n) + (1-t) f_n(y_n)\right] \\ &\leq t \overline{\lim_n} f_n(x_n) + (1-t) \overline{\lim_n} f_n(y_n) \leq t f(x) + (1-t) f(y). \end{aligned}$$

(iv): Let $x \in X$ with $0 \leq f(x) < \infty$, $t > 0$ (by (ii)). Let $\{p_n\}$, p be as in the assertion. By (Γ1), (Γ2) there are x_n, $n \in \mathbb{N}$, $x_n \to x$, $\overline{\lim}_n f_n(x_n) \leq f(x)$. Then

$$f(tx) \leq \underline{\lim_n} f_n(tx_n) = \underline{\lim_n} t^{p_n} f_n(x_n) \leq \overline{\lim_n} t^{p_n} \overline{\lim_n} f_n(x_n) \leq t^p f(x).$$

But also for tx there are $\widetilde{x}_n \in X_n$, $n \in \mathbb{N}$, $\widetilde{x}_n \to tx$ such that $\overline{\lim}_n f_n(\widetilde{x}_n) \leq f(tx)$. Furthermore,

$$f(tx) \geq \overline{\lim_n} f_n(\widetilde{x}_n) \geq \underline{\lim_n} f_n\left(\frac{t}{t}\widetilde{x}_n\right) \geq \underline{\lim_n} t^{p_n} \underline{\lim_n} f_n\left(\frac{1}{t}\widetilde{x}_n\right) \geq t^p f(x).$$

(v): By (Γ2) there are $x_n \in X_n$, $n \in \mathbb{N}$ with $x_n \to 0$ and

$$f(0) \geq \overline{\lim_n} f_n(x_n) \geq \overline{\lim_n} 0 = 0.$$

We are left to verify (i)–(iii) in Proposition 2.4. By (Γ1) and the assertion

$$0 = \underline{\lim_n} 0 = \underline{\lim_n} f_n(0) \geq f(0),$$

hence $f(0) = 0$, which is (i) in Proposition 2.4. (ii) in Proposition 2.4 follows from (iv) of this Proposition and the assertion. Let us prove (iii) of Proposition 2.4. Let

$x, y \in X$ with $f(x) < \infty$, $f(y) < \infty$. Let $x_n, y_n \in X_n$, $n \in \mathbb{N}$ such that $x_n \to x$, $y_n \to y$ and $\overline{\lim}_n f_n(x_n) \leq f(x)$, $\overline{\lim}_n f_n(y_n) \leq f(y)$. By (AL) $x_n + y_n \to x + y$ and $x_n - y_n \to x - y$. Now by (Γ1) and the assertion

$$\begin{aligned} f(x+y) + f(x-y) &\leq \underline{\lim}_n \, [f_n(x_n + y_n) + f_n(x_n - y_n)] \\ &\leq \overline{\lim}_n \, [f_n(x_n + y_n) + f_n(x_n - y_n)] \leq \overline{\lim}_n \, [2f_n(x_n) + 2f_n(y_n)] \\ &\leq 2 \, \overline{\lim}_n \, f_n(x_n) + 2 \, \overline{\lim}_n \, f_n(y_n) \leq 2f(x) + 2f(y). \end{aligned}$$

□

Remark 7.30. *If $f_n \xrightarrow{\Gamma} f$ as in the above proposition, then the properness of each f_n, $n \in \mathbb{N}$ does not make f proper necessarily. Consider the counter-example: $f_n : \mathbb{R} \to \mathbb{R}_\infty$, $n \in \mathbb{N}$, $f : \mathbb{R} \to \mathbb{R}_\infty$*

$$f_n(t) := \begin{cases} n, & \text{if} \quad t = n, \\ +\infty, & \text{if} \quad t \neq n, \end{cases} \qquad f :\equiv +\infty.$$

Then each f_n, $n \in \mathbb{N}$ is proper and $f_n \xrightarrow{\Gamma} f$.

Theorem 7.31. *Let X_n, $n \in \mathbb{N}$, X be separable. Any sequence of functionals $\{f_n : X_n \to \mathbb{R}_\infty\}$ has a Γ-convergent subsequence whose Γ-limit $\widetilde{f}$ is a functional on X. If each f_n, $n \in \mathbb{N}$ is proper and there is a sequence $u_n \in D(f_n)$, $n \in \mathbb{N}$ such that $u_n \to u$ for some $u \in X$, then $u \in D(\widetilde{f})$.*

Proof. Recall that $\mathfrak{X}$ is second countable by Lemma 5.15 whenever each of X_n, $n \in \mathbb{N}$, X is separable. For each $n \in \mathbb{N}$ define

$$F_n : \mathfrak{X} \to \mathbb{R}_\infty, \quad F_n(x) := \begin{cases} f_n(x), & \text{if } x \in X_n, \\ +\infty, & \text{otherwise.} \end{cases}$$

Since $\mathfrak{X}$ has a countable base, we can apply [DM93, Theorem 8.5] and find that a subsequence $\{F_{n_k}\}$ of $\{F_n\}$ Γ-converges to some $\widetilde{F} : \mathfrak{X} \to \mathbb{R}_\infty$. Suppose that $x \in X_{n_{k_0}}$ for some $k_0 \in \mathbb{N}$. Then $\widetilde{F}(x) = +\infty$ by (Γ2) and the definition of the F_n's. Therefore $\{f_{n_k}\}$ Γ-converges to $\widetilde{f} := \widetilde{F}\restriction_X$.

Let $u_k \in X_{n_k} \cap D(f_{n_k})$, $u_k \to u \in X$. Then by (Γ1) $\widetilde{f}(u) = \widetilde{F}(u) < \infty$. □

7.5.1 An excursion into epilimits and set convergence

Definition 7.32. *Let (T, τ) be a (Fréchet) topological space. Let $F_n : T \to \mathbb{R}_\infty$, $n \in \mathbb{N}$. The* sequential epilimit inferior *of the sequence $\{F_n\}$ w.r.t. the topology τ is a functional $\tau\text{-}\mathrm{Li}_e\, F_n$ defined by*

$$(\tau\text{-}\mathrm{Li}_e\, F_n)(x) := \inf_{x_n \xrightarrow[n]{\tau} x} \underline{\lim}_n \, F_n(x_n).$$

The sequential epilimit superior *of the sequence* $\{F_n\}$ *w.r.t. the topology* τ *is a functional* $\tau\text{-}\operatorname{Ls}_e F_n$ *defined by*

$$(\tau\text{-}\operatorname{Ls}_e F_n)(x) := \inf_{x_n \xrightarrow[n]{\tau} x} \overline{\lim_n} F_n(x_n).$$

If $\tau\text{-}\operatorname{Li}_e F_n = \tau\text{-}\operatorname{Ls}_e F_n$, $\{F_n\}$ *is said to be* epi-convergent *with* epilimit

$$\tau\text{-}\operatorname{Lm}_e F_n := \tau\text{-}\operatorname{Li}_e F_n = \tau\text{-}\operatorname{Ls}_e F_n.$$

Let $\{C_n\}$ *be a sequence of subsets of* T. *The* sequential limit inferior *of* $\{C_n\}$ *w.r.t. the topology* τ *is a set* $\tau\text{-}\operatorname{Li} C_n$ *defined by*

$$\tau\text{-}\operatorname{Li} C_n := \{x \in T \mid \exists x_n \in C_n,\ n \in \mathbb{N},\ x_n \xrightarrow[n]{\tau} x\}.$$

The sequential limit superior *of* $\{C_n\}$ *w.r.t. the topology* τ *is a set* $\tau\text{-}\operatorname{Ls} C_n$ *defined by*

$$\tau\text{-}\operatorname{Ls} C_n := \{x \in T \mid \exists \text{ subsequence } \{n_k\} \exists x_k \in C_{n_k},\ n \in \mathbb{N},\ x_k \xrightarrow[k]{\tau} x\}.$$

A sequence of sets $\{C_n\}$ *is said to* Painlevé–Kuratowski converge *(*PK-converge*) to a set* C *w.r.t. the topology* τ *if*

$$\tau\text{-}\operatorname{Ls} C_n \subset C \subset \tau\text{-}\operatorname{Li} C_n.$$

Remark 7.33. *If one substitutes* inf *in the definition by* sup, *one defines the so-called* hypolimits. *We shall not make use of them in the latter. There is a theory about epi- and hypolimits of sets, functionals and bivariate functionals on general topological spaces with broad contributions by H. Attouch et. al. Cf.* [AW83a, AW83b, Att84, AAW88].

Lemma 7.34. *Let* (T, τ) *be a Fréchet topological space. Let* $F_n : T \to \mathbb{R}_\infty$, $n \in \mathbb{N}$. *Then*

$$\tau\text{-}\operatorname{Li}_e F_n \le \tau\text{-}\operatorname{Ls}_e F_n.$$

Suppose that σ *is another Fréchet topology on* T *such that* σ *is weaker than* τ. *Then*

$$\sigma\text{-}\operatorname{Li}_e F_n \le \tau\text{-}\operatorname{Li}_e F_n, \quad \sigma\text{-}\operatorname{Ls}_e F_n \le \tau\text{-}\operatorname{Ls}_e F_n,$$

and if $\sigma\text{-}\operatorname{Lm}_e F_n$ *and* $\tau\text{-}\operatorname{Lm}_e F_n$ *exist,*

$$\sigma\text{-}\operatorname{Lm}_e F_n \le \tau\text{-}\operatorname{Lm}_e F_n.$$

Proof. Straightforward from the definition. □

Lemma 7.35. *Let* $f_n : X_n \to \mathbb{R}_\infty$ *for each* $n \in \mathbb{N}$, $f : X \to \mathbb{R}_\infty$. *For each* $n \in \mathbb{N}$ *define*

$$F_n : \mathfrak{X} \to \mathbb{R}_\infty, \quad F_n(x) := \begin{cases} f_n(x), & \text{if } x \in X_n, \\ +\infty, & \text{otherwise.} \end{cases}$$

and define F *in the same way.*

Write w and s for the weak and strong topology on $\mathfrak{X}$ resp. $f_n \to f$ in the Γ-sense if and only if

$$s\text{-}\mathrm{Ls}_e\, F_n \le F \le s\text{-}\mathrm{Li}_e\, F_n$$

and $f_n \to f$ in the Mosco sense if and only if

$$s\text{-}\mathrm{Ls}_e\, F_n \le F \le w\text{-}\mathrm{Li}_e\, F_n.$$

Proof. Follows from the definitions. □

Lemma 7.36. *Let $A_n : X_n \to 2^{X_n^*}$, $n \in \mathbb{N}$, $A : X \to 2^{X^*}$ be maximal monotone. For each $n \in \mathbb{N}$ define*

$$B_n : \mathfrak{X} \to 2^{\mathfrak{X}^*}, \quad B_n(x) := \begin{cases} A_n(x), & \text{if } x \in D(A_n), \\ \varnothing, & \text{otherwise.} \end{cases}$$

and define B in the same way. Then $A_n \to A$ in the G-sense if and only if $B_n \to B$ (understood as graphs) in the PK-sense w.r.t. to the product topology of strong $\mathfrak{X}$ and strong $\mathfrak{X}^*$.

Proof. Let τ be the product topology of strong $\mathfrak{X}$ and strong $\mathfrak{X}^*$. Let $A_n \to A$ in the G-sense. $B \subset \tau\text{-}\mathrm{Li}\, B_n$ follows from the definition. But by Lemma 7.22 and the monotonicity also $\tau\text{-}\mathrm{Ls}\, B_n \subset B$. Therefore $B_n \to B$ in the PK-sense. The converse follows from the definition. □

Lemma 7.37. *Let τ be either the strong or the weak topology on $\mathfrak{X}$. For each $n \in \mathbb{N}$ let $f_n : X_n \to \mathbb{R}_\infty$ be convex and proper. Then $\tau\text{-}\mathrm{Ls}_e\, f_n$ is convex on X.*

Proof. For the classical case of one topological vector space X compare [Att84, Proposition 3.1]. $\tau\text{-}\mathrm{Ls}_e\, f_n$ is a map effectively defined on a subset of X by similar arguments as in the proof of Theorem 7.31. Suppose that $\tau\text{-}\mathrm{Ls}_e\, f_n$ is proper, otherwise there is nothing to prove. Let $t \in (0,1)$, $x, y \in X$ with $(\tau\text{-}\mathrm{Ls}_e\, f_n)(x) < +\infty$, $(\tau\text{-}\mathrm{Ls}_e\, f_n)(y) < +\infty$. In the subsequent inequality let $x = \tau\text{-}\lim_n x_n$, $y = \tau\text{-}\lim_n y_n$, $tx + (1-t)y = \tau\text{-}\lim_n z_n$ where $x, y \in X$ and $x_n, y_n, z_n \in X_n$, $n \in \mathbb{N}$.

$$\begin{aligned}
(\tau\text{-}\mathrm{Ls}_e\, f_n)(tx + (1-t)y) &= \inf_{z_n \xrightarrow{\tau} tx+(1-t)y} \overline{\lim_n}\, f_n(z_n) \\
&\le \inf_{x_n \xrightarrow{\tau} x, y_n \xrightarrow{\tau} y} \overline{\lim_n}\, f_n(tx_n + (1-t)y_n) \\
&\le \inf_{x_n \xrightarrow{\tau} x} \inf_{y_n \xrightarrow{\tau} y} \overline{\lim_n}\, [t f_n(x_n) + (1-t) f_n(y_n)] \\
&\le \inf_{x_n \xrightarrow{\tau} x} \inf_{y_n \xrightarrow{\tau} y} \Big[t\, \overline{\lim_n}\, f_n(x_n) + (1-t)\, \overline{\lim_n}\, f_n(y_n) \Big] \\
&\le t \inf_{x_n \xrightarrow{\tau} x} \overline{\lim_n}\, f_n(x_n) + (1-t) \inf_{y_n \xrightarrow{\tau} y} \overline{\lim_n}\, f_n(y_n) \\
&= t(\tau\text{-}\mathrm{Ls}_e\, f_n)(x) + (1-t)(\tau\text{-}\mathrm{Ls}_e\, f_n)(y).
\end{aligned}$$

□

7.5.2 Bicontinuity of the Legendre transform

In this subsection we shall prove that the map $f \mapsto f^*$ is bicontinuous w.r.t. the Mosco topology (one can define a sequential topology on the class of all convex, proper maps from $\mathfrak{X}$ to $\mathbb{R}_\infty$ such that its convergence coincides with Mosco convergence). The proofs in this section might seem a bit technical, but there seems to be no way to avoid a regularization argument (more precisely, a coercification argument). We remind the reader that the original Mosco Theorem [Mos94, Theorem 2.4.1] uses a similar argument with regularized Dirichlet forms using the Moreau-Yosida-Deny-approximations. We shall need Theorem 7.38 for our main result Theorem 7.43 below.

In what follows, we implicitly use the convention of Lemma 7.35, which allows us to consider epilimits defined on an asymptotic relation.

Theorem 7.38. *Let $\mathfrak{X}$ be asymptotically reflexive and separable, $\mathfrak{X}$ and $\mathfrak{X}^*$ be asymptotically dual. For each $n \in \mathbb{N}$ let $f_n : X_n \to \mathbb{R}_\infty$ be convex, proper and l.s.c. Let $f : X \to \mathbb{R}_\infty$ be convex, proper and l.s.c. Suppose that the same is true for the sequence of conjugate functions $\{f_n^*\}$ and the conjugate function f^*.*

Then the following statements are equivalent:

(i) $f_n \xrightarrow{M} f$.

(ii) $f = w\text{-}\mathrm{Lm}_e\, f_n = s\text{-}\mathrm{Lm}_e\, f_n$.

(iii) $s\text{-}\mathrm{Ls}_e\, f_n \le f \le w\text{-}\mathrm{Li}_e\, f_n$.

(iv) $f_n^* \xrightarrow{M} f^*$.

(v) $f^* = w\text{-}\mathrm{Lm}_e\, f_n^* = s\text{-}\mathrm{Lm}_e\, f_n^*$.

(vi) $s\text{-}\mathrm{Ls}_e\, f_n^* \le f^* \le w\text{-}\mathrm{Li}_e\, f_n^*$.

(vii)

$$\begin{aligned} &\forall x \in X\ \exists \{x_n\},\ x_n \in X_n,\ x_n \to x \text{ strongly}:\ f_n(x_n) \to f(x), \\ &\forall y \in X^*\ \exists \{y_n\},\ y_n \in X_n^*,\ y_n \to^* y \text{ strongly}:\ f_n^*(y_n) \to f^*(y). \end{aligned} \tag{7.20}$$

(viii) $f_n \xrightarrow{S} f$.

(ix) $f_n^* \xrightarrow{S} f^*$.

We shall need another result (similar to Attouch's [Att84, Theorem 3.7]) for the proof, whose proof in turn is postponed to the end of this section.

Definition 7.39. *For $n \in \mathbb{N}$ let $f_n : X_n \to \mathbb{R}_\infty$ be proper. We say that the sequence $\{f_n\}$ is* uniformly proper *if there exists a norm-bounded sequence $\{x_n\}$, $x_n \in X_n$ such that* $\sup_n f_n(x_n) < +\infty$.

Theorem 7.40. *Let $\mathfrak{X}$ be asymptotically reflexive and separable. For each $n \in \mathbb{N}$ let $f_n : X_n \to \mathbb{R}_\infty$ be convex, proper and l.s.c. such that $\{f_n\}$ is uniformly proper. Then*

$$[w\text{-}\operatorname{Li}_e f_n]^* = s\text{-}\operatorname{Ls}_e f_n^*. \tag{7.21}$$

Proof of Theorem 7.38. "(i) $\Leftrightarrow$ (iii)" and "(iv) $\Leftrightarrow$ (vi)" result from Lemma 7.35.

We shall prove "(ii) $\Leftrightarrow$ (iii)". (ii) is equivalent to $f = w\text{-}\operatorname{Li}_e f_n = w\text{-}\operatorname{Ls}_e f_n = s\text{-}\operatorname{Li}_e f_n = s\text{-}\operatorname{Ls}_e f_n$. This sequence of equalities is clearly equivalent to the equality smallest and the largest of these four functions, which are by Lemma 7.34 respectively $w\text{-}\operatorname{Li}_e f_n$ and $s\text{-}\operatorname{Ls}_e f_n$. This is equivalent to (iii). "(v) $\Leftrightarrow$ (vi)" is proved in the same way.

We prove "(iii) $\Leftrightarrow$ (vi)". Suppose that (iii) holds. By conjugation, the inequalities are reversed, see Section 2.4.3:

$$[w\text{-}\operatorname{Li}_e f_n]^* \le f^* \le [s\text{-}\operatorname{Ls}_e f_n]^* .$$

Note that by properness of f and $f \ge s\text{-}\operatorname{Ls}_e f_n$, $\{f_n\}$ is uniformly proper. Therefore by Theorem 7.40 $[w\text{-}\operatorname{Li}_e f_n]^* = s\text{-}\operatorname{Ls}_e f_n^*$. By properness of f^* and $f^* \ge s\text{-}\operatorname{Ls}_e f_n^*$, $\{f_n^*\}$ is uniformly proper as well. Hence by Theorem 7.40 again and by conjugation, $[w\text{-}\operatorname{Li}_e f_n^*]^{**} = [s\text{-}\operatorname{Ls}_e f_n^{**}]^*$. Using that $f_n = f_n^{**}$ by Theorem 2.39, we get

$$s\text{-}\operatorname{Ls}_e f_n^* \le f^* \le [w\text{-}\operatorname{Li}_e f_n^*]^{**} .$$

Noticing that for any function g (not necessarily convex) the inequality $g^{**} \le g$ holds, we obtain (vi). One can prove "(vi) $\Rightarrow$ (iii)" by repeating the arguments and taking into account that $f_n = f_n^{**}$ for each n and $f = f^{**}$ by Theorem 2.39.

(i) together with (iv) gives (vii) easily. We would like to prove "(vii) $\Rightarrow$ (i)". Suppose that (vii) holds. (M2) is verified. We would like to verify (M1). Let $x_n \in X_n$, $n \in \mathbb{N}$, $x \in X$ such that $x_n \rightharpoonup x$ weakly in $\mathfrak{X}$. Let $y \in X^*$ be any point, $y_n \in X_n^*$, $n \in \mathbb{N}$ be such that $y_n \to^* y$ strongly in $\mathfrak{X}^*$ and $f_n^*(y_n) \to f^*(y)$. From $f_n = f_n^{**}$ for every n

$$f_n(x_n) \ge \ {}_{X_n^*}\langle y_n, x_n\rangle_{X_n} - f_n^*(y_n).$$

Passing on to the limit inferior, we see that

$$\varliminf_n f_n(x_n) \ge \ {}_{X^*}\langle y, x\rangle_X - f^*(y).$$

This being true for any $y \in X^*$, noticing that $f = f^{**}$:

$$\varliminf_n f_n(x_n) \ge f(x)$$

and (M1) follows.

"(i) & (iv) $\Leftrightarrow$ (viii) $\Leftrightarrow$ (ix)" is clear by reflexivity. $\square$

Before we prove Theorem 7.40, we need a technical lemma (cf. [Att84, Lemma 3.8]).

Lemma 7.41 (Uniform minorization). *For $n \in \mathbb{N}$ let $f_n : X_n \to \mathbb{R}_\infty$ be proper, convex and l.s.c. such that $\{f_n\}$ is uniformly proper. Then*

$$(w\text{-}\operatorname{Li}_e f_n)(x) > -\infty \tag{7.22}$$

for each $x \in X$ implies that there is a real number $R \geq 0$ such that for each $n \in \mathbb{N}$, $x_n \in X_n$

$$f_n(x_n) + R(\|x_n\|_{X_n} + 1) \geq 0. \tag{7.23}$$

Proof. Suppose that (7.23) is false. Then there is a map $k \mapsto n_k$ and there are $x_k \in X_{n_k}$ such that

$$f_{n_k}(x_k) + k(\|x_k\|_{X_{n_k}} + 1) < 0. \tag{7.24}$$

W.l.o.g. $n_k \uparrow +\infty$ as $k \to \infty$. There are two possible situations:
1. $\{\|x_k\|\}$ is bounded:
By Lemma 5.53 there is $z \in X$ such that a subsequence of $\{x_k\}$ (still denoted by $\{x_k\}$) converges weakly to z. By definition of $w\text{-}\operatorname{Li}_e$

$$w\text{-}\operatorname{Li}_e f_n(z) \leq \varliminf_k \Big[-k \,\|x_k\|_{X_{n_k}} - k\Big] = -\infty,$$

a contradiction.
2. $\{\|x_k\|\}$ is unbounded:
Extract a subsequence (still denoted by $\{x_k\}$) such that $\lim_k \|x_k\|_{X_{n_k}} = +\infty$. By uniform properness there is a sequence $\{y_k\}$, $y_k \in X_{n_k}$ with $\sup_k \|y_k\|_{X_{n_k}} \leq C < +\infty$ and $f_{n_k}(y_k) < +\infty$. Extract another subsequence (still denoted by $\{y_k\}$) such that $y_k \rightharpoonup z$ as $k \to \infty$, for some $z \in X$. W.l.o.g. $\|x_k - y_k\|_{X_{n_k}} > 1$ for $k \in \mathbb{N}$. Set

$$z_k := t_k x_k + (1 - t_k) y_k$$

with $t_k := \frac{1}{\sqrt{k}\|x_k - y_k\|_{X_{n_k}}}$ which lies in $(0, 1)$ for each $k \in \mathbb{N}$. $t_k \to 0$ as $k \to \infty$ because $\|x_k - y_k\|_{X_{n_k}} \to +\infty$ as $k \to \infty$. Also,

$$\|t_k x_k\|_{X_{n_k}} \leq \frac{1}{\sqrt{k}} \cdot \frac{\|x_k\|_{X_{n_k}}}{\Big|\|x_k\|_{X_{n_k}} - \|y_k\|_{X_{n_k}}\Big|} = \frac{1}{\sqrt{k}} \cdot \frac{1}{\left|1 - \frac{\|y_k\|_{X_{n_k}}}{\|x_k\|_{X_{n_k}}}\right|} \to 0,$$

as $k \to \infty$, because $\|x_k\|_{X_{n_k}} \to +\infty$ as $k \to \infty$ and $\{\|y_k\|_{X_{n_k}}\}$ is bounded. Therefore $t_k x_k \to 0 \in X$ by Lemma 5.13. By (WL') $z_k \rightharpoonup z$ weakly as $k \to \infty$. By convexity of

f_{n_k}:

$$\begin{aligned} f_{n_k}(z_k) &\le t_k f_{n_k}(x_k) + (1-t_k) f_{n_k}(y_k) \\ &\le \frac{1}{\sqrt{k}\,\|x_k - y_k\|_{X_{n_k}}}\left[-k\,\|x_k\|_{X_{n_k}} - k\right] + (1-t_k) f_{n_k}(y_k) \\ &\le -\sqrt{k}\frac{\|x_k\|_{X_{n_k}}}{\|x_k - y_k\|_{X_{n_k}}} + C \le -\sqrt{k}\frac{\|x_k\|_{X_{n_k}}}{\|x_k\|_{X_{n_k}} + \|y_k\|_{X_{n_k}}} + C \\ &\le -\sqrt{k}\frac{1}{1 + \frac{C}{\|x_k\|_{X_{n_k}}}} + C. \end{aligned}$$

Thus,

$$(w\text{-}\operatorname{Li}_e f_n)(z) \le \varliminf_k f_{n_k}(z_k) = -\infty,$$

again a contradiction. □

Proof of Theorem 7.40. Suppose that $\{f_n\}$ is uniformly proper.

We first verify

$$[w\text{-}\operatorname{Li}_e f_n]^* \le s\text{-}\operatorname{Ls}_e f_n^*,$$

which is the same thing as

$$\forall y \in X^*,\ \forall y_n \in X_n^*,\ n \in \mathbb{N},\ y_n \to^* y \text{ strongly} : [w\text{-}\operatorname{Li}_e f_n]^*(y) \le \varlimsup_n f_n^*(y_n).$$

By definition of $w\text{-}\operatorname{Li}_e f_n$, for every $\varepsilon > 0$ and for every $x \in X$, there exist $x_n \in X_n$, $n \in \mathbb{N}$ with $x_n^{(\varepsilon)} \rightharpoonup x$ weakly as $n \to \infty$ such that

$$(w\text{-}\operatorname{Li}_e f_n(x) + \varepsilon) \vee -\frac{1}{\varepsilon} \ge \varliminf_n f_n(x_n^{(\varepsilon)}).$$

By definition of f_n^*, for any $y_n \in X_n^*$, $n \in \mathbb{N}$, $y \in X$ with $y_n \to^* y$ strongly,

$$f_n^*(y_n) \ge {}_{X_n^*}\left\langle y_n, x_n^{(\varepsilon)} \right\rangle_{X_n} - f_n(x_n^{(\varepsilon)}). \tag{7.25}$$

Passing on to the limit superior,

$$\varlimsup_n f_n^*(y_n) \ge {}_{X^*}\langle y, x\rangle_X - \varliminf_n f_n(x_n^{(\varepsilon)}).$$

Together with (7.25),

$$\varlimsup_n f_n^*(y_n) \ge {}_{X^*}\langle y, x\rangle_X - \left[(w\text{-}\operatorname{Li}_e f_n(x) + \varepsilon) \vee -\frac{1}{\varepsilon}\right].$$

Being true for any $\varepsilon > 0$ and $x \in X$, we get

$$\varlimsup_n f_n^*(y_n) \ge \sup_{x \in X}\left[{}_{X^*}\langle y, x\rangle_X - w\text{-}\operatorname{Li}_e f_n(x)\right] = [w\text{-}\operatorname{Li}_e f_n]^*(x).$$

The harder part is to prove

$$s\text{-}\operatorname{Ls}_e f_n^* \le [w\text{-}\operatorname{Li}_e f_n]^*$$

which is,

$$\forall y \in X^*,\ \exists y_n \in X_n^*,\ n \in \mathbb{N},\ y_n \to^* y \text{ strongly} : \overline{\lim_n}\, f_n^*(y_n) \le [w\text{-}\operatorname{Li}_e f_n]^*(y). \tag{7.26}$$

Suppose for a while that $w\text{-}\operatorname{Li}_e f_n(x) > -\infty$ for any $x \in X$. Then by Lemma 7.41 there is a real number $R \ge 0$ such that

$$f_n(x_n) + R(\|x_n\|_{X_n} + 1) \ge 0 \tag{7.27}$$

for all $x_n \in X_n$, $n \in \mathbb{N}$.

Set $f := w\text{-}\operatorname{Li}_e f_n$. For each $\lambda > 0$, define *coercified* functionals

$$f_n^\lambda(x_n) := f_n(x_n) + \frac{\lambda}{2}\|x_n\|_{X_n}^2, \quad x_n \in X_n,\ n \in \mathbb{N}$$

and

$$f^\lambda(x) := f(x) + \frac{\lambda}{2}\|x\|_X^2, \quad x \in X.$$

$\{f_n^\lambda\}$ is *equi-coercive* in the following sense:

Each f_n^λ, $n \in \mathbb{N}$ is weakly coercive and:
whenever $x_n \in X_n$, $n \in \mathbb{N}$ such that $\sup_n f_n^\lambda(x_n) < +\infty$, then $\sup_n \|x_n\|_{X_n} < +\infty$. (7.28)

This follows from (7.27). Let $y \in X^*$, $y_n \in X_n^*$, $n \in \mathbb{N}$ and $y_n \to^* y$ strongly in $\mathfrak{X}^*$. (7.26) for $\{(f_n^\lambda)^*\}$ and $(f^\lambda)^*$ is satisfied if

$$\underline{\lim_n} \inf_{x_n \in X_n}\left[f_n^\lambda(x_n) - {}_{X_n^*}\langle y_n, x_n\rangle_{X_n}\right] \ge \inf_{x \in X}\left[f^\lambda(x) - {}_{X^*}\langle y, x\rangle_X\right].$$

By (7.28) and Proposition 2.43 the infimum $\inf_{x_n \in X_n}\left[f_n^\lambda(x_n) - {}_{X_n^*}\langle y_n, x_n\rangle_{X_n}\right]$ is attained for each $n \in \mathbb{N}$ at some $\overline{x}_n \in X_n$ and the sequence of minimizers $\{\|\overline{x}_n\|_{X_n}\}$ is bounded by (7.28) and uniform properness. Applying Lemma 5.53 a subsequence of $\{\overline{x}_n\}$ converges weakly in $\mathfrak{X}$. Using that for all $x \in X$ by Lemma 5.46

$$f^\lambda(x) \le f(x) + \frac{\lambda}{2}\|x\|_X^2 \le w\text{-}\operatorname{Li}_e f_n(x) + \frac{\lambda}{2}\|x\|_X^2 \le w\text{-}\operatorname{Li}_e f_n^\lambda(x),$$

we get that

$$\overline{\lim_n}(f_n^\lambda)^*(y_n) \le \left[w\text{-}\operatorname{Li}_e f_n^\lambda\right]^*(y) \le (f^\lambda)^*(y). \tag{7.29}$$

The function $x \mapsto \frac{\lambda}{2}\|x\|_X^2$ being continuous on X, from Lemma 2.45 and (2.22) for every l.s.c. proper convex function g

$$\left[g + \frac{\lambda}{2}\|\cdot\|_X^2\right]^* = g^* \diamond \frac{1}{2\lambda}\|\cdot\|_{X^*}^2 = (g^*)_\lambda,$$

where $\diamond$ denotes the infimal convolution and $(\ )_\lambda$ the Moreau-Yosida approximation. Applying this formula to (7.29) we obtain

$$\overline{\lim_n}(f_n^*)_\lambda(y_n) \leq (f^\lambda)^*(y).$$

This being true for any $\lambda > 0$,

$$\overline{\lim_{\lambda\to 0}}\,\overline{\lim_n}(f_n^*)_\lambda(y_n) \leq \sup_{\lambda>0}(f^\lambda)^*(y). \tag{7.30}$$

The right hand side of (7.30) is equal to

$$\sup_{\lambda>0}\sup_{x\in X}\left[{}_{X^*}\langle y,x\rangle_X - f(x) - \frac{\lambda}{2}\|x\|_X^2\right] = \sup_{x\in X}\sup_{\lambda>0}\left[{}_{X^*}\langle y,x\rangle_X - f(x) - \frac{\lambda}{2}\|x\|_X^2\right] = f^*(y).$$

(7.30) can be rewritten

$$\overline{\lim_{\lambda\to 0}}\,\overline{\lim_n}(f_n^*)_\lambda(y_n) \leq f^*(y).$$

By Lemma 5.27 there is a map $n \mapsto \lambda(n)$ with $\lambda(n) \downarrow 0$ as $n \to \infty$ such that

$$\overline{\lim_n}(f_n^*)_{\lambda(n)}(y_n) \leq \overline{\lim_{\lambda\to 0}}\,\overline{\lim_n}(f_n^*)_\lambda(y_n).$$

Thus,

$$\overline{\lim_n}(f_n^*)_{\lambda(n)}(y_n) \leq f^*(y).$$

Let us examine the quantity

$$(f_n^*)_{\lambda(n)}(y_n) = \inf_{u_n\in X_n^*}\left[f_n^*(u_n) + \frac{1}{2\lambda(n)}\|y_n - u_n\|_{X_n^*}^2\right].$$

By Theorem 2.47 and reflexivity the Moreau-Yosida approximation is attained at the unique minimizer of this function which is equal to the resolvent $\overline{y}_n := R_\lambda^{f_n^*}(y_n) := R_\lambda^{\partial f_n^*}(y_n) \in X^*$. Hence

$$f^*(y) \geq \overline{\lim_n}\left[f_n^*(\overline{y}_n) + \frac{1}{2\lambda(n)}\|y_n - \overline{y}_n\|_{X_n^*}^2\right]. \tag{7.31}$$

The term $\frac{1}{2\lambda(n)}\|y_n - \overline{y}_n\|_{X_n^*}^2$ being positive, it follows that

$$f^*(y) \geq \overline{\lim_n} f_n^*(\overline{y}_n)$$

which implies (7.26) if we can prove that $\overline{y}_n \to^* y$ strongly in $\mathfrak{X}^*$.

By the uniform properness assumption on the $\{f_n\}$ there are $x_{n,0} \in X_n$, $n \in \mathbb{N}$ and a constant $C > 0$ with

$$f_n^*(\overline{y}_n) \geq {}_{X_n^*}\langle \overline{y}_n, x_{n,0}\rangle_{X_n} - f_n(x_{n,0}) \geq -C\left[1 + \|\overline{y}_n\|_{X_n^*}\right].$$

Returning to (7.31), for sufficiently large n

$$f^*(y) + 1 \geq -C \left[1 + \|\overline{y}_n\|_{X_n^*}\right] + \frac{1}{2\lambda(n)} \|y_n - \overline{y}_n\|^2_{X_n^*} .$$

If $f^*(y) = +\infty$, (7.26) holds trivially. Otherwise, the above inequality clearly implies that $\overline{y}_n \to^* y$ by (A4) (since $\lambda(n) \downarrow 0$ as $n \to \infty$).

In order to complete the proof, suppose that there is $x_0 \in X$ such that $w\text{-}\operatorname{Li}_e f_n(x_0) = -\infty$. Therefore, for any $y \in X^*$

$$\begin{aligned}[w\text{-}\operatorname{Li}_e f_n]^* (y) &= \sup_{x\in X} \left[{}_{X^*}\langle y, x\rangle_X - w\text{-}\operatorname{Li}_e f_n(x)\right] \\ &\geq {}_{X^*}\langle y, x_0\rangle_X - w\text{-}\operatorname{Li}_e f_n(x_0) \geq +\infty.\end{aligned}$$

On the other hand, since $w\text{-}\operatorname{Li}_e f_n(x_0) = -\infty$, there exists a sequence $\{x_n\}$, $x_n \in X_n$, $n \in \mathbb{N}$ for which $x_n \rightharpoonup x_0$ weakly and which satisfies

$$\underline{\lim_n} f_n(x_n) = -\infty.$$

It follows that, for any sequence $\{y_n\}$, $y_n \in X_n^*$, $n \in \mathbb{N}$ and for any $y \in X^*$ such that $y_n \to^* y$ strongly,

$$\overline{\lim_n} \sup_{z_n\in X_n} \left[{}_{X_n^*}\langle y_n, z_n\rangle_{X_n} - f_n(z_n)\right] \geq \overline{\lim_n} \left[{}_{X_n^*}\langle y_n, x_n\rangle_{X_n} - f_n(x_n)\right] \geq +\infty.$$

Thus $s\text{-}\operatorname{Ls}_e f_n^* \equiv +\infty$ and $[w\text{-}\operatorname{Li}_e f_n]^* = s\text{-}\operatorname{Ls}_e f_n^* \equiv +\infty$. So, equality (7.21) is always satisfied.

The proof is complete. □

7.6 G-convergence of subdifferentials

Definition 7.42. *For each $n \in \mathbb{N}$ let $f_n : X_n \to \mathbb{R}_\infty$ be a proper convex functional and let $f : X \to \mathbb{R}_\infty$ be a proper convex functional. We say that the* normalization (N) *holds for the sequence $\{f_n\}$ relative to f if*

$$\begin{aligned}&\exists [u, v] \in \partial f,\ \exists [u_n, v_n] \in \partial f_n,\ n \in \mathbb{N} \text{ such that:} \\ &u_n \to u \text{ strongly},\ v_n \to^* v \text{ strongly},\ f_n(u_n) \to f(u) \text{ and } f_n^*(v_n) \to f^*(v).\end{aligned} \tag{N}$$

By Proposition 2.40 either of $f_n(u_n) \to f(u)$ and $f_n^*(v_n) \to f^*(v)$ in (N) implies the other.

Theorem 7.43. *Let $\mathfrak{X}$ be asymptotically reflexive and separable. For each $n \in \mathbb{N}$ let $f_n : X_n \to \mathbb{R}_\infty$ be a proper, l.s.c., convex functional and let $f : X \to \mathbb{R}_\infty$ be a proper, l.s.c., convex functional.*

The following two statements are equivalent:

(i) $f_n \xrightarrow{M} f$ *as* $n \to \infty$.

(ii) $\partial f_n \xrightarrow{G} \partial f$ *as* $n \to \infty$ *and* (N) *holds for* $\{f_n\}$ *relative to* f.

The original proof of Theorem 7.43 in the case of a single Hilbert space can be found in [Att77]. Compare [Att84, Theorem 3.66] for the proof in the case of a single reflexive Banach space. The proof depends heavily on the integration formula (2.13) for subdifferentials by R. T. Rockafellar. By Theorem 7.38, Theorem 7.43 is a special case of Theorem 7.46 below. We follow the proof of C. Combari and L. Thibault in [CT98] which depends on the alternative integration formula given in the following lemma.

Lemma 7.44. *Let* $g : X \to \mathbb{R}_\infty$ *be l.s.c. and* $f : X \to \mathbb{R}_\infty$ *be proper, l.s.c. and convex. If* $\partial f \subset \partial^{\mathrm{Fen}} g$*, then* $f = g + C$*, where* C *is some constant real number. Here the* Fenchel subgradient ∂^{Fen} *is defined for any (not necessarily convex)* g *by*

$$\partial^{\mathrm{Fen}} g := \{x^* \in X^* \mid {}_{X^*}\langle x^*, u - x\rangle_X + g(x) \le g(u), \ \ \forall u \in X\}$$

if $x \in D(g)$ *and by* $\partial^{\mathrm{Fen}} g(x) = \varnothing$ *if* $x \notin D(g)$.

Define also ∂^*, $(\partial^{\mathrm{Fen}})^*$ *(mapping from* X^* *to* X *and not to* X^{**}*) in the obvious way. Then the above statement holds similarly for functions* $g^* : X^* \to \mathbb{R}_\infty$, $f^* : X^* \to \mathbb{R}_\infty$.

Proof. See [CT98, Lemma 3.3]. The proof is heavily inspired by techniques in [Roc70]. □

We need another lemma. The proof is similar to that in [CT98]. Recall that by Lemma 7.37 $s\text{-}\mathrm{Ls}_e\, f_n$ is convex, whenever $f_n : X_n \to \mathbb{R}_\infty$ are convex.

Lemma 7.45. *For each* $n \in \mathbb{N}$ *let* $f_n : X_n \to \mathbb{R}_\infty$ *be a proper, l.s.c., convex functional. Suppose that* $s\text{-}\mathrm{Li}_e\, f_n > -\infty$ *and that there exist* $u_n \in X_n$, $n \in \mathbb{N}$, $u_n \to u$ *strongly and*

$$\overline{\lim_n}\, f_n(u_n) < +\infty. \tag{7.32}$$

Then for any sequence $\{[x_n, y_n]\}$ *with* $[x_n, y_n] \in \partial f_n$ *and* $x_n \to x$ *strongly,* $y_n \to^* y$ *strongly one has*

$$(w\text{-}\mathrm{Li}_e\, f_n)(x) = \underline{\lim_n}\, f_n(x_n) \quad \textit{and} \quad (s\text{-}\mathrm{Ls}_e\, f_n)(x) = \overline{\lim_n}\, f_n(x_n)$$

and

$$[x, y] \in \partial^{\mathrm{Fen}}(w\text{-}\mathrm{Li}_e\, f_n) \cap \partial(s\text{-}\mathrm{Ls}_e\, f_n).$$

So,

$$\mathrm{Li}\, \partial f_n \subset \partial^{\mathrm{Fen}}(w\text{-}\mathrm{Li}_e\, f_n) \cap \partial(s\text{-}\mathrm{Ls}_e\, f_n).$$

The analog statements hold for a sequence $\{f_n^* : X_n^* \to \mathbb{R}_\infty\}$ *with* w *(weak) replaced by* w^* *(weak*) and* ∂, ∂^{Fen} *replaced by* ∂^*, $(\partial^{\mathrm{Fen}})^*$ *(mapping from* X^* *to* X *and not to* X^{**}*).*

Proof. Fix any sequence $\{[x_n, y_n]\}$ with $[x_n, y_n] \in \partial f_n$ and $x_n \to x$ strongly, $y_n \to^* y$ strongly. Let $\widetilde{x} \in X$, $\widetilde{x}_n \in X_n$, $n \in \mathbb{N}$, $\widetilde{x}_n \rightharpoonup \widetilde{x}$ weakly. By subdifferentiability

$${}_{X_n^*}\langle y_n, \widetilde{x}_n - x_n\rangle_{X_n} + f_n(x_n) \le f_n(\widetilde{x}_n).$$

Passing on to the limit we find that

$$ {}_{X^*}\langle y, \widetilde{x} - x\rangle_X + \underline{\lim}_n f_n(x_n) \le \underline{\lim}_n f_n(\widetilde{x}_n). $$

If we take $\widetilde{x}_n = u_n$, $n \in \mathbb{N}$, $\widetilde{x} = u$ as in (7.32) we get that $\underline{\lim}_n f_n(\widetilde{x}_n) < +\infty$ and hence $\underline{\lim}\, f_n(x_n) \in \mathbb{R}$ by $s\text{-}\mathrm{Li}_e\, f_n > -\infty$. By taking the infimum over all weakly converging sequences $\{\widetilde{x}_n\}$, we find

$$ {}_{X^*}\langle y, \widetilde{x} - x\rangle_X + \underline{\lim}_n f_n(x_n) \le w\text{-}\mathrm{Li}_e\, f_n(\widetilde{x}), \tag{7.33} $$

and, setting $\widetilde{x} = x$

$$ \underline{\lim}_n f_n(x_n) \le (w\text{-}\mathrm{Li}_e\, f_n)(x) $$

which ensures (by definition)

$$ \underline{\lim}_n f_n(x_n) = (w\text{-}\mathrm{Li}_e\, f_n)(x). $$

Going back to (7.33) we see that $[x, y] \in \partial^{\mathrm{Fen}}(w\text{-}\mathrm{Li}_e\, f_n)$. By similar arguments we prove that

$$ {}_{X^*}\langle y, \widetilde{x} - x\rangle_X + \overline{\lim}_n f_n(x_n) \le s\text{-}\mathrm{Ls}_e\, f_n(\widetilde{x}), \tag{7.34} $$

and

$$ \overline{\lim}_n f_n(x_n) = (s\text{-}\mathrm{Ls}_e\, f_n)(x), $$

proving $[x, y] \in \partial(s\text{-}\mathrm{Ls}_e\, f_n)$. The last part of the statement follows by similar arguments on the conjugate space. □

Now we are ready to formulate and prove the nonlinear Mosco Theorem in all its generality.

Theorem 7.46. *For each $n \in \mathbb{N}$ let $f_n : X_n \to \mathbb{R}_\infty$ be a proper, l.s.c., convex functional and let $f : X \to \mathbb{R}_\infty$ be a proper, l.s.c., convex functional.*

The following two statements are equivalent:

(i) $f_n \xrightarrow{S} f$.

(ii) $\partial f_n \xrightarrow{G} \partial f$ *and* (N) *holds for $\{f_n\}$ relative to f.*

Proof. Suppose that (i) holds. Fix $[x, y] \in \partial f$. By (M2) there exist $u_n \in X_n$, $v_n \in X_n^*$, $n \in \mathbb{N}$ with $u_n \to x$, $v_n \to^* y$ such that $f_n(u_n) \to f(x)$ and $f_n^*(v_n) \to f^*(y)$. Set

$$ \varepsilon_n := f_n^*(v_n) + f_n(u_n) - {}_{X_n^*}\langle v_n, u_n\rangle_{X_n} \ge 0. $$

Then $\{\varepsilon_n\}$ converges to $f^*(y) + f(x) - {}_{X^*}\langle y, x\rangle_X = 0$ and $v_n \in \partial_{\varepsilon_n} f_n(u_n)$, see Definition 2.29. By Lemma 2.31 there exist $[x_n, y_n] \in \partial f_n$ such that

$$ \|x_n - u_n\|_{X_n} \le \sqrt{\varepsilon_n} \quad \text{and} \quad \|y_n - v_n\|_{X_n^*} \le \sqrt{\varepsilon_n}. $$

Obviously, the requirements of Lemma 7.45 are satisfied. Combined with the Mosco convergence we get

$$\underline{\lim}_n f_n(x_n) \geq f(x) \geq \overline{\lim}_n f_n(x_n)$$

which yields the normalization condition (N). (ii) is proved.

Suppose that (ii) holds. From (N) it follows that the requirements of Lemma 7.45 are satisfied. Therefore by Lemma 7.45 and the G-convergence

$$\partial f \subset \operatorname{Li} \partial f_n \subset \partial^{\text{Fen}}(w\text{-}\operatorname{Li}_e f_n) \cap (s\text{-}\operatorname{Ls}_e f_n)$$

and by the "integration-lemma" 7.44 we deduce

$$f = w\text{-}\operatorname{Li}_e f_n + C_1 = s\text{-}\operatorname{Ls}_e f_n + C_2 \tag{7.35}$$

with two constant real numbers C_1, C_2. By condition (N) (with u and v) and Lemma 7.45 again we see that

$$(w\text{-}\operatorname{Li}_e f_n)(u) = f(u) = (s\text{-}\operatorname{Ls}_e f_n)(u) \tag{7.36}$$

in $\mathbb{R}$ which shows that $C_1 = C_2 = 0$ and hence (7.35) becomes $f_n \xrightarrow{M} f$ by Lemma 7.35.

To show $f_n^* \xrightarrow{M^*} f^*$ we note that by equation (2.19)

$$\partial f_n \xrightarrow{G} \partial f \quad \Leftrightarrow \quad \partial^* f_n^* \xrightarrow{G} \partial^* f^*$$

and (N) includes the conjugate statement. So we can repeat all of the above arguments for the conjugate case. □

8 Examples of Mosco and slice convergence

As announced in the introduction, we establish Mosco and slice approximations for four types of convex functionals within the theory of varying spaces. In all of this chapter, denote by ∂_i, ∇ the ordinary or weak i-th partial derivative and gradient resp. and denote by D_i, D the i-th partial derivative in the sense of Schwartz distributions and the gradient in the sense of Schwartz distributions resp. We remark that in in Sections 8.1, 8.2, our domain is all of $\mathbb{R}^d$, whereas, in Section 8.3, due to a more specific situation, we are working on a bounded subdomain $\Omega \subset \mathbb{R}^d$. In Section 8.4, we consider an abstract Souslin space E.

8.1 Convergence of weighted Φ-Laplace operators

Let (Φ, Ψ) be a pair of complementary N-functions with pair of gauges (φ, ψ). Suppose that $\Phi, \Psi \in C^1(\mathbb{R})$ and that $\Phi, \Psi \in \Delta_2 \cap \nabla_2$. Let $d \in \mathbb{N}$. Let $w \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $w \geq 0$ satisfy condition (S1) in Definition 3.2 (with Φ), that is, $w = 0$ $\mathrm{d}x$-a.e. on $\mathbb{R}^d \setminus R^\Phi(w)$, where

$$R^\Phi(w) := \left\{ x \in \mathbb{R}^d \;\middle|\; \int_{B(x,\varepsilon)} \psi\left(\frac{1}{w(y)}\right) \mathrm{d}y < \infty \;\text{ for some } \varepsilon > 0 \right\}.$$

Let $H^{1,\Phi}_w$, $W^{1,\Phi}_w$, resp. be the strong, resp. weak weighted Φ-Sobolev spaces of first order with norms $\|\cdot\|_{(1,\Phi,w,0)}$, $\|\cdot\|_{1,\Phi,w,0}$, $\|\cdot\|_{(1,\Phi,w)}$ and $\|\cdot\|_{1,\Phi,w}$ resp., cf. Section 3.1.

Define the Φ-*energy functionals* $G_{\Phi,w,0} : H^{1,\Phi}_w \to \mathbb{R}_+$, $G_{\Phi,w} : W^{1,\Phi}_w \to \mathbb{R}_+$ by

$$G_{\Phi,w,0}(u) := \int \Phi(|\nabla u|)\, w \,\mathrm{d}x, \quad u \in H^{1,\Phi}_w,$$

and

$$G_{\Phi,w}(u) := \int \Phi(|\nabla u|)\, w \,\mathrm{d}x, \quad u \in W^{1,\Phi}_w.$$

Denote the dual space of $H^{1,\Phi}_w$, $W^{1,\Phi}_w$ resp. by $H^{-1,\Psi}_w$, $W^{-1,\Psi}_w$. The following lemma was proved in the unweighted case in [GHLMS99, Lemma 3.4].

Lemma 8.1. *$G_{\Phi,w,0}$ and $G_{\Phi,w}$ are strongly continuous, convex and Gâteaux differentiable and the Gâteaux derivative is given by*

$$_{H^{-1,\Psi}_w}\langle \nabla_G G_{\Phi,w,0}(u), h\rangle_{H^{1,\Phi}_w} = \int \langle \varphi(|\nabla u|)\operatorname{sign}(\nabla u), \nabla h\rangle \; w \,\mathrm{d}x, \;\; u, h \in H^{1,\Phi}_w. \tag{8.1}$$

A similar formula holds for $\nabla_G G_{\Phi,w}$.

Proof. Strong continuity follows from Lemma C.16. Convexity is clear. We have for $u, h \in H_w^{1,\Phi}$ and $t > 0$

$$\frac{1}{t}\left[G_{\Phi,w,0}(u+th) - G_{\Phi,w,0}(u)\right] = \int \frac{1}{t}\int_{|\nabla u|}^{|\nabla u + t\nabla h|} \varphi(s)\,\mathrm{d}s\, w\,\mathrm{d}x.$$

On the other hand, as $t \to 0$,

$$|\nabla u + t\nabla h| \to |\nabla u|, \quad \mathrm{d}x\text{-a.e.}$$

and hence $w\,\mathrm{d}x$-a.e. We see that for $t \le 1$

$$\left|\frac{1}{t}\int_{|\nabla u|}^{|\nabla u + t\nabla h|} \varphi(s)\,\mathrm{d}s\right| \le \left|\frac{1}{t}\int_0^{t|\nabla h|} \varphi(s + |\nabla u|)\,\mathrm{d}s\right| \le \varphi(|\nabla u| + |\nabla h|)|\nabla h|$$

with $\varphi(|\nabla u| + |\nabla h|) \in L_w^\Psi$ (by Lemma C.21) and $|\nabla h| \in L_w^\Phi$ and hence $\varphi(|\nabla u| + |\nabla h|)|\nabla h| \in L_w^1$ by the Hölder inequality (cf. Theorem C.15). We calculate the directional derivative

$$\lim_{t\to 0}\frac{1}{t}\left[|\nabla u + t\nabla h| - |\nabla u|\right] = \frac{\langle \nabla u, \nabla h\rangle}{|\nabla u|},$$

where we have used that

$$\nabla(|\nabla u|) = \begin{cases} \frac{\nabla u}{|\nabla u|}, & \text{if } \nabla u \neq 0,\\ 0, & \text{if } \nabla u = 0.\end{cases}$$

Thus, by the dominated convergence theorem and the fundamental theorem of calculus

$$\begin{aligned}&\lim_{t\to 0}\frac{1}{t}\left[G_{\Phi,w,0}(u+th) - G_{\Phi,w,0}(u)\right]\\ &= \int \varphi(|\nabla u|)\frac{\langle \nabla u, \nabla h\rangle}{|\nabla u|} w\,\mathrm{d}x = \int \langle \varphi(|\nabla u|)\operatorname{sign}(\nabla u), \nabla h\rangle\, w\,\mathrm{d}x.\end{aligned}$$

The proof for $G_{\Phi,w}$ works in the same way. □

By Lemma 3.20, $H_w^{1,\Phi}$, $W_w^{1,\Phi}$ are reflexive and separable. The continuity of the embeddings $H_w^{1,\Phi} \hookrightarrow L_w^\Phi$, $W_w^{1,\Phi} \hookrightarrow L_w^\Phi$ with continuity constant equal to 1 is straightforward. Note also that the embeddings have dense range, since the $w\,\mathrm{d}x$ classes of $C_0^\infty(\mathbb{R}^d)$ are contained in both Sobolev spaces and are known to be dense in L_w^Φ, cf. Lemma 6.16.

Define the Φ-energy on L_w^Φ by

$$F_{\Phi,w,0}(u) := \begin{cases} G_{\Phi,w,0}(u), & \text{if } u \in H_w^{1,\Phi},\\ +\infty, & \text{if } u \in L_w^\Phi \setminus H_w^{1,\Phi},\end{cases}$$

and

$$F_{\Phi,w}(u) := \begin{cases} G_{\Phi,w}(u), & \text{if } u \in W_w^{1,\Phi},\\ +\infty, & \text{if } u \in L_w^\Phi \setminus W_w^{1,\Phi}.\end{cases}$$

Lemma 8.2. *$F_{\Phi,w,0}$ and $F_{\Phi,w}$ are proper, convex and l.s.c. Furthermore, the subdifferentials $\partial F_{\Phi,w,0} \subset L^\Phi_w \times L^\Psi_w$ and $\partial F_{\Phi,w} \subset L^\Phi_w \times L^\Psi_w$ are single-valued.*

Proof. Properness and convexity are obvious. Recall that $G_{\Phi,w,0}$ is continuous and Gâteaux differentiable by Lemma 8.1. Let $S \subset H^{1,\Phi}_w$. Note that $\sup_{u\in S}\|u\|_{\Phi,w} < +\infty$ and $\sup_{u\in S} F_{\Phi,w,0}(u) < +\infty$ imply that $\sup_{u\in S}\|u\|_{1,\Phi,w} < +\infty$ by Lemma C.17. Therefore, $F_{\Phi,w,0}$ is l.s.c. by Lemma 2.52. $\partial F_{\Phi,w,0}$ is single-valued by Corollary 2.51. Repeat the steps for $F_{\Phi,w}$. □

Remark 8.3. *By Proposition 2.50 and (8.1), the subdifferential $\partial F_{\Phi,w,0} : L^\Phi_w \to L^\Psi_w$ has the representation ($u \in D(\partial F_{\Phi,w,0}) \subset H^{1,\Phi}_w$, $h \in H^{1,\Phi}_w$)*

$$ {}_{L^\Psi_w}\langle \partial F_{\Phi,w,0}(u), h\rangle_{L^\Phi_w} = \int \varphi(|\nabla u|)\,\langle \operatorname{sign}(\nabla u), \nabla h\rangle\; w\,\mathrm{d}x. $$

Suppose that (S3) *and* (S4) *holds for w. Then by integration by parts for $u, h \in C_0^\infty$, we get the formula,*

$$ {}_{L^\Psi_w}\langle \partial F_{\Phi,w,0}(u), h\rangle_{L^\Phi_w} = -\int \operatorname{div}\left[w\varphi(|\nabla u|)\operatorname{sign}(\nabla u)\right] h\,\mathrm{d}x, $$

which extends to all $h \in L^\Phi_w$ by the Hahn-Banach theorem. Moreover, if

$$ \beta := (\beta_1, \ldots, \beta_d) := \frac{\mathrm{D}w}{w}, $$

it is easily seen that

$$ \begin{aligned} {}_{L^\Psi_w}\langle \partial F_{\Phi,w,0}(u), h\rangle_{L^\Phi_w} &= -\int \operatorname{div}\left[\varphi(|\nabla u|)\operatorname{sign}(\nabla u)\right] h\,w\,\mathrm{d}x, \\ &\quad -\int \langle \varphi(|\nabla u|)\operatorname{sign}(\nabla u), \beta\rangle\; h\,w\,\mathrm{d}x. \end{aligned} $$

We arrive at the main result of this section.

Theorem 8.4. *Let (Φ,Ψ) be a pair of complementary N-functions, $\Phi,\Psi \in \Delta_2 \cap \nabla_2$ and $\Phi,\Psi \in C^1(\mathbb{R})$. Denote by (φ,ψ) the pair of associated gauges. Let $w_n \in L^1_{\mathrm{loc}}(\mathrm{d}x)$, $w_n \ge 0$ be weights, such that* (S1) *holds with Φ for $n \in \mathbb{N}$. Let $w \in L^1_{\mathrm{loc}}(\mathrm{d}x)$, $w \ge 0$ be a weight, such that* (S1) *holds with Φ. Suppose that the measures $w_n\,\mathrm{d}x$, $n \in \mathbb{N}$, $w\,\mathrm{d}x$ have full support. Let $F_n := F_{\Phi,w_n}$, $n \in \mathbb{N}$, $F := F_{\Phi,w}$. $F_{n,0} := F_{\Phi,w_n,0}$, $n \in \mathbb{N}$, $F_0 := F_{\Phi,w,0}$. Suppose that the following two conditions hold true:*

$$ w_n \to w \quad \text{weakly in } L^1_{\mathrm{loc}}(\mathrm{d}x), \tag{8.2} $$

$$ H^{1,\Phi}_w = W^{1,\Phi}_w, \tag{8.3} $$

which, in particular, implies that $F_0 = F$. Let

$$ \mathfrak{L}^\Phi := \dot{\bigcup_n} L^\Phi_{w_n} \dot{\cup} L^\Phi_w, \quad \mathfrak{L}^\Psi := \dot{\bigcup_n} L^\Psi_{w_n} \dot{\cup} L^\Psi_w. $$

Suppose furthermore that one of the following conditions holds true.

Case 1:

$$\begin{aligned}
&w_n,\ n \in \mathbb{N},\ w \quad \textit{satisfy } (\mathrm{S2}) \textit{ with } \Phi,\\
&\sup_n \int_B \psi\left(\frac{1}{w_n}\right) \mathrm{d}x < \infty \quad \textit{for all balls } B \subset \mathbb{R}^d,\\
&\frac{\eta}{w_n} \to \frac{\eta}{w} \textit{ strongly in } \mathfrak{L}^\Psi \textit{ for each } \eta \in C_0.
\end{aligned} \tag{8.4}$$

Case 2:

$$\begin{aligned}
&w_n,\ n \in \mathbb{N},\ w \quad \textit{satisfy } (\mathrm{S3}) \textit{ and } (\mathrm{S4}),\\
&\mathrm{D}_i w_n \to \mathrm{D}_i w \quad \textit{weakly in } L^1_{\mathrm{loc}} \textit{ for each } 1 \le i \le d,\\
&\frac{\mathrm{D}_i w_n}{w_n} \to \frac{\mathrm{D}_i w}{w} \textit{ strongly in } \mathfrak{L}^\Psi \textit{ for each } 1 \le i \le d.
\end{aligned} \tag{8.5}$$

Then $F_n \to F = F_0$ Mosco and $F_{n,0} \to F_0 = F$ Mosco in $\mathfrak{L}^\Phi$.

It follows from (8.2), that the measures $w_n \,\mathrm{d}x \to w \,\mathrm{d}x$ converge in the vague sense. Also, $w \,\mathrm{d}x$ is a regular measure. Hence by full support and the results of Section 6.3, $\{\mathrm{Id} : C_0 \subset L^\Phi_w \to L^\Phi_{w_n}\}_{n\in\mathbb{N}}$ is a linear metric approximation for both norms.

Sufficient conditions on w for (8.3) to hold can be found in Section 4.1. Instead, one could just consider the statement of the above theorem for $F_{n,0}$, $n \in \mathbb{N}$, F_0.

Lemma 8.5. *Suppose that (8.2) holds. Suppose that $\Psi \in \Delta_2$, $\Psi \in C^1(\mathbb{R})$. Suppose additionally that Ψ is strictly convex and that for each $\varepsilon > 0$ there exist $K_\varepsilon > 1$ such that $\psi((1+\varepsilon)t) \ge K_\varepsilon \psi(t)$ for all $t \ge 0$. Then a sufficient condition for*

$$\frac{\eta}{w_n} \to \frac{\eta}{w} \textit{ strongly in } \mathfrak{L}^\Psi \textit{ for each } \eta \in C_0, \tag{8.6}$$

is

$$\int \Psi\left(\frac{\eta}{w_n}\right) w_n \,\mathrm{d}x \to \int \Psi\left(\frac{\eta}{w}\right) w \,\mathrm{d}x. \tag{8.7}$$

Furthermore, a sufficient condition for

$$\frac{\mathrm{D}_i w_n}{w_n} \to \frac{\mathrm{D}_i w}{w} \textit{ strongly in } \mathfrak{L}^\Psi \textit{ for each } 1 \le i \le d, \tag{8.8}$$

is

$$\mathrm{D}_i w_n \to \mathrm{D}_i w \quad \textit{weakly in } L^1_{\mathrm{loc}} \textit{ for each } 1 \le i \le d, \tag{8.9}$$

and

$$\int \Psi\left(\frac{\mathrm{D}_i w_n}{w_n}\right) w_n \,\mathrm{d}x \to \int \Psi\left(\frac{\mathrm{D}_i w}{w}\right) w \,\mathrm{d}x. \tag{8.10}$$

Proof. Under the condition imposed, each $L^{\Psi}_{w_n}$, $n \in \mathbb{N}$, L^{Ψ}_{w} is uniformly convex by Lemma C.24. The moduli of convexity do not depend on the weights. As a result, $\mathfrak{L}^{\Psi}$ is asymptotically uniformly convex, see Definition 5.56. By Theorem 5.57, $\mathfrak{L}^{\Psi}$ possesses the asymptotic Kadeč-Klee property.

(8.7) applied to the proof of Lemma 6.19, gives that

$$\lim_n \left\| \frac{\eta}{w_n} \right\|_{\Psi,w_n} = \left\| \frac{\eta}{w} \right\|_{\Psi,w}.$$

Clearly, for $\eta, \zeta \in C_0$,

$$\lim_n \int \zeta \frac{\eta}{w_n} \, w_n \, \mathrm{d}x = \lim_n \int \zeta\eta \, \mathrm{d}x = \int \zeta \frac{\eta}{w} \, w \, \mathrm{d}x.$$

Now by Lemma 5.35 and asymptotic duality, we get that

$$\frac{\eta}{w_n} \rightharpoonup \frac{\eta}{w}$$

weakly in $\mathfrak{L}^{\Psi}$. By the asymptotic Kadeč-Klee property, (8.6) holds.

Fix $1 \leq i \leq d$. (8.10) applied to the proof of Lemma 6.19, gives that

$$\lim_n \left\| \frac{\mathrm{D}_i w_n}{w_n} \right\|_{\Psi,w_n} = \left\| \frac{\mathrm{D}_i w}{w} \right\|_{\Psi,w}.$$

Clearly by (8.9), for $\zeta \in C_0$,

$$\lim_n \int \zeta \frac{\mathrm{D}_i w_n}{w_n} \, w_n \, \mathrm{d}x = \lim_n \int \zeta \mathrm{D}_i w_n \, \mathrm{d}x = \int \zeta \mathrm{D}_i w \, \mathrm{d}x = \int \zeta \frac{\mathrm{D}_i}{w} \, w \, \mathrm{d}x.$$

Now by Lemma 5.35 and asymptotic duality, we get that

$$\frac{\mathrm{D}_i w_n}{w_n} \rightharpoonup \frac{\mathrm{D}_i w}{w}$$

weakly in $\mathfrak{L}^{\Psi}$. By the asymptotic Kadeč-Klee property, (8.8) holds. □

In the weighted Orlicz space setting, different from L^p-spaces, norm inequalities as well as convergence of norms turn out to be weaker than the corresponding modular statements. This pitfall can be avoided by the following idea due to Bloom and Kerman in [BK94, Proof of Proposition 2.5].

Lemma 8.6. *Suppose that the following holds true. If $f_n \in L^{\Phi}_{w_n}$, $n \in \mathbb{N}$, $f \in L^{\Phi}_{w}$, $f_n \rightharpoonup f$ weakly in $\mathfrak{L}^{\Phi}$, then for all $\varepsilon > 0$*

$$\varliminf_n \|\nabla f_n\|_{(\Phi,\varepsilon w_n)} \geq \|\nabla f\|_{(\Phi,\varepsilon w)} \tag{8.11}$$

where we set $\|\nabla f\| := +\infty$ if $f \notin W^{1,\Phi}_{w}$ (and so for n). Then condition (M1) *of the Mosco convergence holds for the sequence of modulars, i.e.,*

$$\varliminf_n \int \Phi(|\nabla f_n|) \, w_n \, \mathrm{d}x \geq \int \Phi(|\nabla f|) \, w \, \mathrm{d}x$$

for all $f_n \rightharpoonup f$ weakly in $\mathfrak{L}^{\Phi}$.

Proof. Suppose that $0 < \underline{\lim}_n \int \Phi(|\nabla f_n|)\, w_n \,\mathrm{d}x < \infty$. Set

$$\varepsilon := \left(\underline{\lim}_n \int \Phi(|\nabla f_n|)\, w_n \,\mathrm{d}x\right)^{-1}.$$

Consider

$$K_n := \inf\left\{ k_n \geq 0 \;\middle|\; \frac{\int \Phi\left(\frac{|\nabla f|}{k_n}\right) w_n \,\mathrm{d}x}{\underline{\lim}_n \int \Phi\left(|\nabla f|\right)\, w_n \,\mathrm{d}x} \leq 1 \right\}.$$

Let $\delta > 0$. By $\Phi \in \Delta_2$ there is a constant $C > 2$ with $\Phi(2x) \leq C\Phi(x)$. Therefore $\Phi(x/(1+\delta)) \leq C^s\Phi(x)$ when $s \geq -\log_2(1+\delta)$. But s can be chosen negative. Therefore there is a constant $C^s =: C(\Phi,\delta) \in (0,1)$ depending only on Φ and δ such that for all n

$$\int \Phi\left(\frac{|\nabla f_n|}{1+\delta}\right) w_n \,\mathrm{d}x \leq C(\Phi,\delta) \int \Phi(|\nabla f_n|) w_n \,\mathrm{d}x.$$

For large n the right hand side is smaller or equal to $\underline{\lim}_n \int \Phi(|\nabla f_n|) w_n \,\mathrm{d}x$. Therefore $K_n \leq 1+\delta$ for large n and hence $\underline{\lim}_n K_n \leq 1$ since δ was arbitrary. We conclude that

$$\underline{\lim}_n \|\nabla f_n\|_{(\Phi,\varepsilon w_n)} \leq 1.$$

Thus $\|\nabla f\|_{(\Phi,\varepsilon w)} \leq 1$ by the assertion and hence

$$\int \Phi(|\nabla f|) w \,\mathrm{d}x \leq \frac{1}{\varepsilon} = \underline{\lim}_n \int \Phi(|\nabla f_n|) w_n \,\mathrm{d}x.$$

The case of $\underline{\lim}_n \int \Phi(|\nabla f_n|) w_n \,\mathrm{d}x = +\infty$ is trivial. Suppose that $\underline{\lim}_n \int \Phi(|\nabla f_n|) w_n \,\mathrm{d}x = 0$. Then by an easy modification of the proof of Lemma C.16, we have for a subsequence $\lim_l \|\nabla f_{n_l}\|_{(\Phi,w_{n_l})} = 0$. By (8.11) with $\varepsilon = 1$, $\|\nabla f\|_{(\Phi,w)} = 0$ and hence $\int \Phi(|\nabla f|) w \,\mathrm{d}x = 0$. □

As seen from the above proof, we have that:

Corollary 8.7. *Let μ_n, $n \in \mathbb{N}$, μ be finite measures on some measurable space $(E, \mathscr{B})$. Suppose that $\mathfrak{L}^\Phi := \dot\bigcup_n L^\Phi(\mu_n) \dot\cup L^\Phi(\mu)$ has an asymptotic relation which is asymptotically dual to $\mathfrak{L}^\Psi$ (defined similarly). Suppose that the following holds true. If $f_n \in L^\Phi(\mu_n)$, $n \in \mathbb{N}$, $f \in L^\Phi(\mu)$, $f_n \rightharpoonup f$ weakly in $\mathfrak{L}^\Phi$, then for all $\varepsilon > 0$*

$$\underline{\lim}_n \|f_n\|_{(\Phi,\varepsilon\mu_n)} \geq \|f\|_{(\Phi,\varepsilon\mu)} \tag{8.12}$$

where we set $\|f\| := +\infty$ if $f \notin L^\Phi(\mu)$ (and so for n). Then condition (M1) *of the Mosco convergence holds for the sequence of modulars, i.e.,*

$$\underline{\lim}_n \int \Phi(|f_n|)\,\mathrm{d}\mu_n \geq \int \Phi(|f|)\,\mathrm{d}\mu$$

for all $f_n \rightharpoonup f$ weakly in $\mathfrak{L}^\Phi$.

We note that the following proof is somewhat similar to those of [Kol05, Theorem 1.1] and [Kol06, Proposition 4.1] by Alexander V. Kolesnikov. Anyhow, we have to cope with the technicalities caused by the Orlicz spaces.

Proof of Theorem 8.4. We would like to prove (8.11). Let $\varepsilon > 0$.

Let $\{u_n\}$, $u_n \in L^\Phi_{w_n}$, $n \in \mathbb{N}$ be a sequence with $\underline{\lim}_n \|\nabla u_n\|_{(\Phi,\varepsilon w_n)} < +\infty$. Extract a subsequence (also denoted by $\{u_n\}$) such that $\lim_n \|\nabla u_n\|_{(\Phi,\varepsilon w_n)} = \underline{\lim}_n \|\nabla u_n\|_{(\Phi,\varepsilon w_n)} =: C$ and $u_n \in W^{1,\Phi}_{w_n}$ for $n \in \mathbb{N}$.

Suppose that (8.4) holds. Fix a ball $B \subset \mathbb{R}^d$. By Hölder's inequality,

$$\int_B |u_n| \,\mathrm{d}x \le \|u_n\|_{(\Phi,\varepsilon w_n)} \left\| \frac{1_B}{\varepsilon w_n} \right\|_{\Psi,\varepsilon w_n}$$

and

$$\int_B |\nabla u_n| \,\mathrm{d}x \le \|\nabla u_n\|_{(\Phi,\varepsilon w_n)} \left\| \frac{1_B}{\varepsilon w_n} \right\|_{\Psi,\varepsilon w_n}.$$

Suppose that

$$\sup_n \|u_n\|_{(\Phi,\varepsilon w_n)} =: c < +\infty. \tag{8.13}$$

By $\Psi \in \Delta_2$, $\left\| \frac{1_B}{\varepsilon w_n} \right\|_{\Psi,\varepsilon w_n}$ is finite if and only if $\int_B \psi(1/w_n)(1/w_n)\, w_n \,\mathrm{d}x$ is, which in turn is finite if and only if $\int_B \psi(1/w_n)\,\mathrm{d}x$ is. Hence by (8.4), $\{u_n\}$ and $\{\nabla u_n\}$ are bounded in L^1_{loc}. Hence the sequences of measures $\{u_n \,\mathrm{d}x\}$, $\{\partial_i u_n \,\mathrm{d}x\}$, $1 \le i \le d$ are vaguely bounded and hence vaguely relatively compact, see [Bau74, 46.1, 46.2]. Extract a subsequence such that $\{u_n \,\mathrm{d}x\}$ tends vaguely to some locally finite Radon measure m. Also for each $1 \le i \le d$, we can extract a subsequence of $\{\partial_i u_n \,\mathrm{d}x\}$ that converges vaguely to some locally finite Radon measure m_i. For any $\eta \in C_0(\mathbb{R}^d)$,

$$\int \eta \,\mathrm{d}m = \lim_n \int \eta u_n \,\mathrm{d}x \le \sup_n \left[\|u_n\|_{(\Phi,\varepsilon w_n)} \left\| \frac{\eta}{\varepsilon w_n} \right\|_{\Psi,\varepsilon w_n} \right],$$

which is finite uniformly in η by (8.4) and (8.13). Hence m is absolutely continuous w.r.t. the Lebesgue measure. Let $f := \mathrm{d}m/\mathrm{d}x$.

Also, by (8.4), for any $\eta \in C_0(\mathbb{R}^d \to \mathbb{R}^d)$,

$$\begin{aligned}\sum_{i=1}^d \int \eta_i \,\mathrm{d}m_i &= \lim_n \int \langle \eta, \nabla u_n \rangle \,\mathrm{d}x \\ &\le \underline{\lim}_n \left[\|\nabla u_n\|_{(\Phi,\varepsilon w_n)} \left\| \frac{\eta}{\varepsilon w_n} \right\|_{\Psi,\varepsilon w_n} \right] \le C \left\| \frac{\eta}{\varepsilon w} \right\|_{\Psi,\varepsilon w},\end{aligned}$$

which implies that each m_i is absolutely continuous w.r.t. the Lebesgue measure. Let $g_i := \mathrm{d}m_i/\mathrm{d}x$. By vague convergence and integration by parts (choosing a version of each u_n which is absolutely continuous on almost all parallel axes to the i-coordinate, compare with Corollary 3.16)

$$\int \eta g_i \,\mathrm{d}x = \lim_n \int \eta \partial_i u_n \,\mathrm{d}x = -\lim_n \int \partial_i \eta u_n \,\mathrm{d}x = -\int \partial_i \eta f,$$

for every $\eta \in C_0^\infty$. Hence $g_i = \mathrm{D}_i f$ (the distributional derivative). For $\eta \in C_0(\mathbb{R}^d \to \mathbb{R}^d)$

$$\int \langle \eta, \nabla u_n \rangle \,\mathrm{d}x \le \|\nabla u_n\|_{(\Phi,\varepsilon w_n)} \left\| \frac{\eta}{\varepsilon w_n} \right\|_{\Psi,\varepsilon w_n}.$$

Taking the limit inferior, using (8.4), gives that

$$\int \langle \eta, \mathrm{D}f \rangle \frac{\varepsilon w}{\varepsilon w} \,\mathrm{d}x \le \varliminf_n \|\nabla u_n\|_{(\Phi,\varepsilon w_n)} \left\| \frac{\eta}{\varepsilon w} \right\|_{\Psi,\varepsilon w}.$$

Take the supremum over all $\eta \in C_0^\infty$ such that $\left\| \frac{\eta}{\varepsilon w} \right\|_{\Psi,\varepsilon w} \le 1$. Using the l.s.c. property of the supremum, we get that

$$\|\mathrm{D}f\|_{\Phi,\varepsilon w} \le \varliminf_n \|\nabla u_n\|_{(\Phi,\varepsilon w_n)}.$$

But $\|\mathrm{D}f\|_{\Phi,\varepsilon w} \ge \|\mathrm{D}f\|_{(\Phi,\varepsilon w)}$, see Lemma C.13. Hence

$$\|\mathrm{D}f\|_{(\Phi,\varepsilon w)} \le \varliminf_n \|\nabla u_n\|_{(\Phi,\varepsilon w_n)}. \tag{8.14}$$

Suppose now that $\{u_n\}$ converges weakly in $\mathfrak{L}^\Phi$ to u. Thereby (8.13) is justified by Lemma 5.36. Hence by (8.4), for all $\eta \in C_0$,

$$\int \eta f \,\mathrm{d}x = \lim_n \int \eta u_n \,\mathrm{d}x = \lim_n \int \frac{\eta}{w_n} u_n \, w_n \,\mathrm{d}x = \int \frac{\eta}{w} u \, w \,\mathrm{d}x = \int \eta u \,\mathrm{d}x,$$

and therefore $f = u$ $\mathrm{d}x$-a.s. By the results of Paragraph 4.3, (S2) and (8.3) imply that $\mathrm{D}f = \mathrm{D}u = \nabla u$ $\mathrm{d}x$-a.s. Since our arguments work for any subsequence of $\{u_n\}$, (8.14) holds for the whole sequence $\{u_n\}$ and $\mathrm{D}f$ replaced by ∇u. (8.11) is proved. (M1) follows by Lemma 8.6.

Suppose now that (8.5) holds. Fix a ball $B \subset \mathbb{R}^d$. By Hölder's inequality,

$$\int_B |u_n| \, \varepsilon w_n \,\mathrm{d}x \le \|u_n\|_{(\Phi,\varepsilon w_n)} \|1_B\|_{\Psi,\varepsilon w_n}$$

and

$$\int_B |\nabla u_n| \, \varepsilon w_n \,\mathrm{d}x \le \|\nabla u_n\|_{(\Phi,\varepsilon w_n)} \|1_B\|_{\Psi,\varepsilon w_n}.$$

Suppose again that condition (8.13) holds. Then both inequalities combined with (8.2) give that the sequences $\{u_n \, \varepsilon w_n\}$, $\{\partial_i u_n \, \varepsilon w_n\}$, $1 \le i \le d$ are bounded in L^1_{loc} and hence the measures $\{u_n \, \varepsilon w_n \,\mathrm{d}x\}$, $\{\partial_i u_n \, \varepsilon w_n \,\mathrm{d}x\}$, $1 \le i \le d$ are vaguely bounded and hence vaguely relatively compact, see [Bau74, 46.1, 46.2]. Hence we can extract a subsequence of $\{u_n \, \varepsilon w_n \, \mathrm{d}x\}$ that converges vaguely to some locally finite Radon measure m. Also for each $1 \le i \le d$ we can extract a subsequence of $\{\partial_i u_n \, \varepsilon w_n \, \mathrm{d}x\}$ that converges vaguely to some locally finite Radon measure m_i. For any $\eta \in C_0(\mathbb{R}^d \to \mathbb{R}^d)$,

$$\sum_{i=1}^{d} \int \eta_i \, \mathrm{d}m_i = \lim_n \int \langle \eta, \nabla u_n \rangle \, \varepsilon w_n \, \mathrm{d}x$$
$$\le \underline{\lim}_n \|\nabla u_n\|_{(\Phi,\varepsilon w_n)} \lim_n \|\eta\|_{\Psi,\varepsilon w_n} \le C \|\eta\|_{\Psi,\varepsilon w},$$

where in the last step we have used the fact that $\mathfrak{L}^\Psi$ is an asymptotic relation. Thus we have proved that m_i is absolutely continuous w.r.t. $\varepsilon w \, \mathrm{d}x$. Let f be such that $\mathrm{d}m_i = f \, \varepsilon w \, \mathrm{d}x$.

Suppose now that $u_n \rightharpoonup u$ weakly in $\mathfrak{L}^\Phi$. This justifies (8.13) by Lemma 5.36. By Lemma 5.35 and vague convergence,

$$\int \eta \, \mathrm{d}m = \lim_n \int \eta u_n \, \varepsilon w_n \, \mathrm{d}x = \int \eta u \, \varepsilon w \, \mathrm{d}x,$$

which holds for all $\eta \in C_0$, hence $\mathrm{d}m = u \, \varepsilon w \, \mathrm{d}x$.

Upon setting $\beta_i^n := \mathrm{D}_i w_n / w_n$, $n \in \mathbb{N}$, $\beta_i := \mathrm{D}_i w / w$, $1 \le i \le d$, by integration by parts (3.4) and condition (8.5), for all $\eta \in C_0^\infty$, $1 \le i \le d$,

$$\begin{aligned}\int \eta f \, \varepsilon w \, \mathrm{d}x &= \lim_n \int \eta \partial_i u_n \, \varepsilon w_n \, \mathrm{d}x \\ &= -\lim_n \left[\int \partial_i \eta u_n \, \varepsilon w_n \, \mathrm{d}x + \int \eta u_n \beta_i^n \, \varepsilon w_n \, \mathrm{d}x \right] \\ &= -\int \partial_i \eta u \, \varepsilon w \, \mathrm{d}x - \int \eta u \beta_i \, \varepsilon w \, \mathrm{d}x,\end{aligned}$$

where we have used strong convergence $\beta_i^n \to \beta_i$ in $\mathfrak{L}^\Psi$, weak convergence $u_n \rightharpoonup u$ in $\mathfrak{L}^\Phi$ and asymptotic duality. Since $f \in L_w^\Phi$, by (3.4), $f = \nabla u$ in $L_w^\Phi(\mathbb{R}^d \to \mathbb{R}^d)$.

For $\eta \in C_0(\mathbb{R}^d \to \mathbb{R}^d)$

$$\int \langle \eta, \nabla u_n \rangle \, \varepsilon w_n \, \mathrm{d}x \le \|\nabla u_n\|_{(\Phi,\varepsilon w_n)} \|\eta\|_{\Psi,\varepsilon w_n}.$$

Taking the limit inferior, using that $\mathfrak{L}^\Psi$ is an asymptotic relation, gives

$$\int \langle \eta, \nabla u \rangle \, \varepsilon w \, \mathrm{d}x \le \underline{\lim}_n \|\nabla u_n\|_{(\Phi,\varepsilon w_n)} \|\eta\|_{\Psi,\varepsilon w}.$$

Take the supremum over all η such that $\|\eta\|_{\Psi,\varepsilon w} \le 1$. Using the l.s.c. property of the supremum, we get

$$\|\nabla u\|_{\Phi,\varepsilon w} \le \underline{\lim}_n \|\nabla u_n\|_{(\Phi,\varepsilon w_n)},$$

and hence by Lemma C.13

$$\|\nabla u\|_{(\Phi,\varepsilon w)} \le \underline{\lim}_n \|\nabla u_n\|_{(\Phi,\varepsilon w_n)}.$$

(8.11) is proved. (M1) follows by Lemma 8.6.

Let us prove (M2). Let $u \in W_w^{1,\Phi}$ (the case $u \in L_w^\Phi \setminus W_w^{1,\Phi}$ is trivial). Since C_0^∞ is dense in $W_w^{1,\Phi}$ by (8.3), we find a sequence $\{\eta_m\} \subset C_0^\infty$ with $\lim_m \|u - \eta_m\|_{1,\Phi,w} = 0$. Hence by Lemma C.16

$$\lim_m \int_E \Phi(|\nabla \eta_m|)\, w \, \mathrm{d}x = \int_E \Phi(|\nabla u|)\, w \, \mathrm{d}x.$$

By Lemma 3.14, $\eta_m \in W_{w_n}^{1,\Phi}$ for each n. Clearly $\eta_m \to \eta_m$ strongly in $\mathfrak{L}^\Phi$. Also, by (8.2), for each m,

$$\lim_n \int_E \Phi(|\nabla \eta_m|)\, w_n \, \mathrm{d}x = \int_E \Phi(|\nabla \eta_m|)\, w \, \mathrm{d}x.$$

By Lemma 5.27, there exists a sequence of natural numbers $\{m_n\}$, $m_n \uparrow \infty$ as $n \to \infty$ such that $\eta_{m_n} \to u$ in $\mathfrak{L}^\Phi$ (indeed, $\lim_m \overline{\lim}_n \|\eta_{m_n} - \eta_m\|_{(\Phi,w_n)} = 0$) and such that

$$\begin{aligned}\overline{\lim_n} \int_E \Phi(|\nabla \eta_{m_n}|)\, w_n \, \mathrm{d}x &\le \overline{\lim_m}\,\overline{\lim_n} \int_E \Phi(|\nabla \eta_m|)\, w_n \, \mathrm{d}x \\ &= \overline{\lim_m} \int_E \Phi(|\nabla \eta_m|)\, w \, \mathrm{d}x = \int_E \Phi(|\nabla u|)\, w \, \mathrm{d}x.\end{aligned}$$

But $\eta_{m_n} \to u$ in $\mathfrak{L}^\Phi$ as $n \to \infty$. (M2) is proved.

We conclude the proof by noting that one can repeat all steps for the sequence of functionals $\{F_{n,0}\}$. $\square$

Combining the previous result with Theorem 7.43, we immediately get:

Corollary 8.8. *Under the assumptions of Theorem 8.4 (recall that then, $F_0 = F$), the subdifferential operators $\partial F_n \to \partial F$ and $\partial F_{n,0} \to \partial F_0$ in the G-sense (the strong graph sense).*

8.2 Convergence of weighted p-Laplace operators

Although the operators in this section are a special case of those in the previous section, we are able to weaken our conditions due to a more specific situation. Moreover, we are also varying p. As a special feature of the weighted p-Laplace case, in contrast to the Φ-Laplacian, we have given an explicit new condition (HW) on the weight such that the uniqueness of Sobolev spaces $H = W$ holds (different from the Muckenhoupt condition). See Chapter 4 for the results on uniqueness. As mentioned already in the introduction, the uniqueness of the Sobolev space is needed to identify the Mosco limit, see condition (8.18) below. In some sense, $H = W$ implies “attainability” of a gradient, which is linked to the Mosco condition (M2) in Definition 7.26. We note that the idea of proof is analog to the proof of Theorem 8.4.

Let $1 < p < \infty$, $q := p/(p-1)$, $d \in \mathbb{N}$. Let $w \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $w \geq 0$ satisfy condition (S1) in Definition 3.2 (with p), that is, $w = 0$ $\mathrm{d}x$-a.e. on $\mathbb{R}^d \setminus R^p(w)$, where

$$R^p(w) := \left\{ x \in \mathbb{R}^d \;\middle|\; \int_{B(x,\varepsilon)} w^{-1/(p-1)} \,\mathrm{d}y < \infty \text{ for some } \varepsilon > 0 \right\}.$$

Let $H^{1,p}_w$, $W^{1,p}_w$, resp. be the strong, resp. weak weighted p-Sobolev spaces of first order with norm

$$\|u\|_{1,p,w} := \left(\int |u|^p \, w \,\mathrm{d}x \right)^{1/p} + \left(\int |\nabla u|^p \, w \,\mathrm{d}x \right)^{1/p},$$

cf. Chapter 3. We implicitly assume that (S3) and (S4) hold, when we define the weak Sobolev space. In Chapter 3, we have used the suggestive notation $\overline{\partial}_i$ and $\overline{\nabla}$ for the gradient in $W^{1,p}_w$. We do not use it here, anyhow, caution is needed, since, even if the gradients are in L^1_{loc}, they are generally not equal to the ordinary weak derivatives nor to the distributional derivatives!

Define the *p-energy functionals* $G_{p,w,0} : H^{1,p}_w \to \mathbb{R}_+$, $G_{p,w} : W^{1,p}_w \to \mathbb{R}_+$ by

$$G_{p,w,0}(u) := \frac{1}{p} \int |\nabla u|^p \, w \,\mathrm{d}x, \quad u \in H^{1,p}_w,$$

and

$$G_{p,w}(u) := \frac{1}{p} \int |\nabla u|^p \, w \,\mathrm{d}x, \quad u \in W^{1,p}_w.$$

Denote the dual space of $H^{1,p}_w$, $W^{1,p}_w$, resp. by $H^{-1,q}_w$, $W^{-1,q}_w$, resp.

Lemma 8.9. *$G_{p,w,0}$ and $G_{p,w}$ are strongly continuous, convex and Gâteaux differentiable and the Gâteaux derivative is given by*

$$_{H^{-1,q}_w}\langle \nabla_G G_{p,w,0}(u), h \rangle_{H^{1,p}_w} = \int |\nabla u|^{p-1} \langle \operatorname{sign}(\nabla u), \nabla h \rangle \; w \,\mathrm{d}x, \quad u, h \in H^{1,p}_w. \tag{8.15}$$

A similar formula holds for $\nabla_G G_{p,w}$.

Proof. Strong continuity and convexity are clear; the Gâteaux differentiability follows exactly as in Lemma 8.1 with $\Phi(\cdot)$ replaced by $(1/p)|\cdot|^p$ and $\varphi(\cdot)$ replaced by $|\cdot|^{p-1} \operatorname{sign}(\cdot)$. □

By Lemma 3.20, $H^{1,p}_w$, $W^{1,p}_w$ are reflexive and separable. The continuity of the embeddings $H^{1,p}_w \hookrightarrow L^p_w$, $W^{1,p}_w \hookrightarrow L^p_w$ with continuity constant equal to 1 is straightforward. Note also that the embeddings have dense range, since the $w\,\mathrm{d}x$ classes of $C_0^\infty(\mathbb{R}^d)$ are contained in both Sobolev spaces and are known to be dense in L^p_w, cf. Lemma 6.16.

Define the p-energy on L^p_w by

$$F_{p,w,0}(u) := \begin{cases} G_{p,w,0}(u), & \text{if } u \in H^{1,p}_w, \\ +\infty, & \text{if } u \in L^p_w \setminus H^{1,p}_w, \end{cases}$$

and

$$F_{p,w}(u) := \begin{cases} G_{p,w}(u), & \text{if } u \in W^{1,p}_w, \\ +\infty, & \text{if } u \in L^p_w \setminus W^{1,p}_w. \end{cases}$$

Lemma 8.10. *$F_{p,w,0}$ and $F_{p,w}$ are proper, convex and l.s.c. Furthermore, the subdifferentials $\partial F_{p,w,0} \subset L^p_w \times L^q_w$ and $\partial F_{p,w} \subset L^p_w \times L^q_w$ are single-valued.*

Proof. Properness and convexity are obvious. Recall that $G_{p,w,0}$ is continuous and Gâteaux differentiable by Lemma 8.9. Let $S \subset H^{1,p}_w$. Note that $\sup_{u\in S} \left(\int |u|^p \, w \,\mathrm{d}x\right)^{1/p} < +\infty$ and $\sup_{u\in S} F_{p,w,0}(u) < +\infty$ imply that $\sup_{u\in S} \|u\|_{1,p,w} < +\infty$. Therefore, $F_{p,w,0}$ is l.s.c. by Lemma 2.52. $\partial F_{p,w,0}$ is single-valued by Corollary 2.51. Repeat the steps for $F_{p,w}$. □

Remark 8.11. *By Proposition 2.50 and* (8.15)*, the subdifferential $\partial F_{p,w,0} : L^p_w \to L^q_w$ has the representation ($u \in D(\partial F_{p,w,0}) \subset H^{1,p}_w$, $h \in H^{1,p}_w$)*

$$_{L^q_w}\langle \partial F_{p,w,0}(u), h\rangle_{L^p_w} = \int |\nabla u|^{p-1} \langle \mathrm{sign}(\nabla u), \nabla h\rangle \; w \,\mathrm{d}x.$$

Suppose that (S3) *and* (S4) *holds for w. Then by integration by parts for $u, h \in C_0^\infty$, we get the forumla,*

$$_{L^q_w}\langle \partial F_{p,w,0}(u), h\rangle_{L^p_w} = -\int \mathrm{div}\left[w|\nabla u|^{p-1}\,\mathrm{sign}(\nabla u)\right] h \,\mathrm{d}x,$$

which extends to all $h \in L^p_w$ by the Hahn-Banach theorem. Moreover, if

$$\beta := (\beta_1, \ldots, \beta_d) := \frac{\mathrm{D}w}{w},$$

it is easily seen that

$$\begin{aligned} _{L^q_w}\langle \partial F_{p,w,0}(u), h\rangle_{L^p_w} = &-\int \mathrm{div}\left[|\nabla u|^{p-1}\,\mathrm{sign}(\nabla u)\right] h \, w \,\mathrm{d}x, \\ &-\int \left\langle |\nabla u|^{p-1}\,\mathrm{sign}(\nabla u), \beta\right\rangle \, h \, w \,\mathrm{d}x. \end{aligned}$$

We are ready to formulate the main result of this section.

Theorem 8.12. *Let $1 < p < \infty$, $q := p/(p-1)$ and $\{p_n\} \subset (1,\infty)$, $q_n := p_n/(p_n - 1)$. Let $w_n \in L^1_{\mathrm{loc}}(\mathrm{d}x)$, $w_n \geq 0$ be weights, such that* (S1) *holds with p_n for $n \in \mathbb{N}$. Let $w \in L^1_{\mathrm{loc}}(\mathrm{d}x)$, $w \geq 0$ be a weight, such that* (S1) *holds with p. Suppose that the measures $w_n \,\mathrm{d}x$, $n \in \mathbb{N}$, $w \,\mathrm{d}x$ have full support. Let $F_n := F_{p_n,w_n}$, $n \in \mathbb{N}$, $F := F_{p,w}$. $F_{n,0} := F_{p_n,w_n,0}$, $n \in \mathbb{N}$, $F_0 := F_{p,w,0}$. Suppose that the following three conditions hold true:*

$$\lim_n |p_n - p| = 0, \tag{8.16}$$

$$w_n \to w \quad \textit{weakly in } L^1_{\mathrm{loc}}(\mathrm{d}x), \tag{8.17}$$

$$H^{1,p}_w = W^{1,p}_w, \tag{8.18}$$

which, in particular, implies that $F_0 = F$. Define

$$\mathfrak{L}^{\to p} := \dot{\bigcup_n} L^{p_n}_{w_n} \dot{\cup} L^p_w, \quad \mathfrak{L}^{\to q} := \dot{\bigcup_n} L^{q_n}_{w_n} \dot{\cup} L^q_w$$

Suppose furthermore that one of the following conditions holds true.

Case 1:

$$\begin{aligned} &w_n,\ n \in \mathbb{N},\ w \quad \textit{satisfy (S2) with } p_n,\ n \in \mathbb{N},\ p \textit{ resp.},\\ &\sup_n \left[\int_B {w_n}^{1-q_n}\,\mathrm{d}x \right]^{(p_n-1)/p_n} < \infty \quad \textit{for all balls } B \subset \mathbb{R}^d,\\ &w_n^{1-q_n} \to w^{1-q} \quad \textit{weakly in } L^1_{\mathrm{loc}}(\mathrm{d}x), \end{aligned} \tag{8.19}$$

Case 2:

$$\begin{aligned} &w_n,\ n \in \mathbb{N},\ w \quad \textit{satisfy (S3) and (S4)},\\ &\mathrm{D}_i w_n \to \mathrm{D}_i w \quad \textit{weakly in } L^1_{\mathrm{loc}} \textit{ for each } 1 \le i \le d,\\ &\left(\int \left| \frac{\mathrm{D}_i w_n}{w_n} \right|^{q_n} w_n\,\mathrm{d}x \right)^{1/q_n} \to \left(\int \left| \frac{\mathrm{D}_i w}{w} \right|^{q} w\,\mathrm{d}x \right)^{1/q} \quad \textit{for each } 1 \le i \le d. \end{aligned} \tag{8.20}$$

Then $F_{n,0} \to F_0 = F$ Mosco in $\mathfrak{L}^{\to p}$. If (S3) and (S4) hold for w_n, $n \in \mathbb{N}$, w, also $F_n \to F = F_0$ Mosco in $\mathfrak{L}^{\to p}$.

We would like to construct a linear strong asymptotic relation on $\mathfrak{L}^{\to p}$ similar to that in the example in Paragraph 6.2.3. But, in the case with densities, measure-convergence in total variation norm is too strong. It follows from (8.17), that the measures $w_n\,\mathrm{d}x \to w\,\mathrm{d}x$ converge in the vague sense. Anyhow, weak convergence in L^1_{loc} is stronger and sufficient for the estimate (6.9) (with C_b replaced by C_0). Hence by full support and arguments analog to those in Paragraph 6.2.3, $\{\mathrm{Id} : C_0 \subset L^p_w \to L^{p_n}_{w_n}\}_{n\in\mathbb{N}}$ is a linear metric approximation.

As mentioned above, condition (8.18) is fundamental for proving (M2). With its help, we can identify the limit and attain a given gradient by a sequence of smooth functions, which naturally is linked to (M2) being an attainability condition, too. Sufficient conditions on w for (8.18) to hold are discussed in Chapter 4.

Condition (8.19) can be regarded as a uniform (S2) condition, which in some sense can be regarded a uniform lower semi-continuity of the gradient forms. The connection to (M1) then is natural.

Condition (8.20) is used to prove (M1) with an alternative method, via strong $\mathfrak{L}^{\to q}$ convergence of logarithmic derivatives and integration by parts. This method is the more straightforward one.

When in Theorem 8.12, $p_n = p = 2$ for any $n \in \mathbb{N}$, it reduces to [Kol05, Theorem 1.1] and [Kol06, Proposition 4.1] by Alexander V. Kolesnikov. Our proofs have an analog argument structure.

Proof of Theorem 8.12. We have to verify (M1), (M2) in Definition 7.26. We start with (M2) which is a consequence of (8.16), (8.17) and (8.18). It follows that C_0^∞ is dense in $W_w^{1,p}$. Since by full support $\varphi \in L_{w_n}^{p_n}(\mathbb{R}^d)$, $\nabla\varphi \in L_{w_n}^{p_n}(\mathbb{R}^d \to \mathbb{R}^d)$ for any $n \in \mathbb{N}$ and any $\varphi \in C_0^\infty$, $C_0^\infty \subset W_{w_n}^{1,p_n}$ for any $n \in \mathbb{N}$. Hence repeating the arguments in Paragraph 6.2.3 with C_b replaced by C_0^∞ and $\|\cdot\|_{p_n,w_n}$ replaced by $\|\cdot\|_{1,p_n,w_n}$ for each n (we can consider the gradient and the zero order norms separately) we infer that $\{\mathrm{Id} : C_0^\infty \subset W_w^{1,p} \to W_{w_n}^{1,p_n}\}_{n\in\mathbb{N}}$ is a linear metric approximation for

$$\mathfrak{W}^{1,\to p} := \dot{\bigcup_n} W_{w_n}^{1,p_n} \dot{\cup} W_w^{1,p}.$$

Since $C_0^\infty \subset C_0$, C_0^∞ is dense in L_w^p (Lemma 6.16) and $\|u\|_{p_n,w_n} \leq \|u\|_{1,p_n,w_n}$ for any $u \in W_{w_n}^{1,p_n}$ by our choice of the Sobolev norm, condition (5.36) (in Paragraph 5.11.1) is satisfied. Hence by Lemma 5.66, $\mathfrak{W}^{1,\to p} \hookrightarrow \mathfrak{L}^{\to p}$ is an asymptotically strong embedding.

Let $u \in L_w^p$ be any vector. If $F(u) = +\infty$, (M2) is trivial. Hence we can assume that $u \in W_w^{1,p}$. Let $u_n \in W_{w_n}^{1,p_n}$, $n \in \mathbb{N}$ be such that $u_n \to u$ strongly in $\mathfrak{W}^{1,\to p}$, which exists by (A2). By (A3),

$$\lim_n \|u_n\|_{1,p_n,w_n} = \|u\|_{1,p,w}\,.$$

But $\mathfrak{W}^{1,\to p} \hookrightarrow \mathfrak{L}^{\to p}$ is an asymptotically strong embedding, hence $u_n \to u$ strongly in $\mathfrak{L}^{\to p}$, and as a consequence,

$$\lim_n \left(\int |\nabla u_n|^{p_n}\, w_n \,\mathrm{d}x\right)^{1/p_n} = \left(\int |\nabla u|^{p}\, w \,\mathrm{d}x\right)^{1/p}.$$

(M2) follows now with the help of

Lemma 8.13. *For each $P \in [1,\infty)$, the function*

$$[1,P] \times [0,\infty) \ni (p,t) \mapsto \begin{cases} t^p, & \text{if } t > 0,\\ 0, & \text{if } t = 0,\end{cases}$$

is continuous.

Proof. Let $P \in [1,\infty)$. For $t \neq 0$, $(p,t) \mapsto \exp[p\log(t)]$ is obviously continuous. Let $p \in [1,P]$ and $\{(p_n,t_n)\}$ a sequence in $\mathbb{R}^2$ with $\lim_n(p_n,t_n) = (p,0)$. For large n, $t_n \leq 1$ and therefore

$$|t_n^{p_n}| \leq |t_n| \to 0$$

as $n \to \infty$. The claim is proved. $\square$

Let us prove (M1). Let $\{u_n\}$, $u_n \in L_{w_n}^{p_n}$, $n \in \mathbb{N}$ be a sequence with $\underline{\lim}_n F_n(u_n) < +\infty$. Extract a subsequence (also denoted by $\{u_n\}$) such that

$$\lim_n F_n(u_n) = \underline{\lim}_n F_n(u_n) =: C$$

and $u_n \in W_{w_n}^{1,p_n}$ for $n \in \mathbb{N}$.

Suppose that (8.19) holds. Fix a ball $B \subset \mathbb{R}^d$. By Hölder's inequality,

$$\int_B |u_n| \,\mathrm{d}x \le \left(\int_B |u_n|^{p_n} w_n \mathrm{d}x\right)^{1/p_n} \left(\int_B w_n^{1-q_n} \,\mathrm{d}x\right)^{(p_n-1)/p_n}$$

and

$$\int_B |\nabla u_n| \,\mathrm{d}x \le \left(\int_B |\nabla u_n|^{p_n} w_n \mathrm{d}x\right)^{1/p_n} \left(\int_B w_n^{1-q_n} \,\mathrm{d}x\right)^{(p_n-1)/p_n}.$$

Suppose that

$$\sup_n \|u_n\|_{L^{p_n}_{w_n}} = c < +\infty. \tag{8.21}$$

Then both inequalities combined with (8.19) give that $\{u_n\}$ and $\{\nabla u_n\}$ are bounded in L^1_{loc}. Hence the sequences of measures $\{u_n \,\mathrm{d}x\}$, $\{\partial_i u_n \,\mathrm{d}x\}$, $1 \le i \le d$ are vaguely bounded and hence vaguely relatively compact, see [Bau74, 46.1, 46.2]. Extract a subsequence such that $\{u_n \,\mathrm{d}x\}$ tends vaguely to some locally finite Radon measure m. Also for each $1 \le i \le d$, we can extract a subsequence of $\{\partial_i u_n \,\mathrm{d}x\}$ that converges vaguely to some locally finite Radon measure m_i. For any $\varphi \in C_0(\mathbb{R}^d)$, by (8.19) and (8.21),

$$\int \varphi \,\mathrm{d}m = \lim_n \int \varphi u_n \,\mathrm{d}x \le \sup_n \left[\|u_n\|_{L^{p_n}_{w_n}} \left(\int |\varphi|^{q_n} w_n^{1-q_n} \,\mathrm{d}x\right)^{1/q_n}\right],$$

which implies that m is absolutely continuous w.r.t. the Lebesgue measure. Let $f := \mathrm{d}m/\mathrm{d}x$.

Also, for any $\varphi \in C_0(\mathbb{R}^d \to \mathbb{R}^d)$,

$$\begin{aligned}\sum_{i=1}^d \int \varphi_i \,\mathrm{d}m_i &= \lim_n \int \langle \varphi, \nabla u_n \rangle \,\mathrm{d}x \\ &\le \varliminf_n \left[(p_n F(u_n))^{1/p_n} \left(\int |\varphi|^{q_n} w_n^{1-q_n} \,\mathrm{d}x\right)^{(p_n-1)/p_n}\right] \\ &\le (pC)^{1/p} \left(\int |\varphi|^q w^{1-q} \,\mathrm{d}x\right)^{(p-1)/p},\end{aligned}$$

which implies that each m_i is absolutely continuous w.r.t. the Lebesgue measure. The second convergence can be proved with the help of (8.19) as in Paragraph 6.2.3. Let $g_i := \mathrm{d}m_i/\mathrm{d}x$. By vague convergence and integration by parts (choosing a version of each u_n which is absolutely continuous on almost all parallel axes to the i-coordinate, compare with Corollary 3.16)

$$\int \varphi g_i \,\mathrm{d}x = \lim_n \int \varphi \partial_i u_n \,\mathrm{d}x = -\lim_n \int \partial_i \varphi u_n \,\mathrm{d}x = -\int \partial_i \varphi f,$$

for every $\varphi \in C_0^\infty$. Hence $g_i = \mathrm{D}_i f$ (the distributional derivative). For $\varphi \in C_0(\mathbb{R}^d \to \mathbb{R}^d)$

$$\left(\int \langle \varphi, \nabla u_n \rangle \,\mathrm{d}x\right)^{p_n} \le p_n F_n(u_n) \left(\int |\varphi|^{q_n} w_n^{1-q_n} \,\mathrm{d}x\right)^{p_n/q_n}.$$

Taking the limit inferior, using (8.19), gives

$$\frac{1}{p}\left(\int \langle \varphi, \mathrm{D}f\rangle \,\mathrm{d}x\right)^{p} \leq \varliminf_{n} F_n(u_n)\left(\int |\varphi|^q\, w^{1-q}\,\mathrm{d}x\right)^{p-1}.$$

Pick a sequence $\{h_k^N\}$ in C_0 in order to approximate

$$h^N := |\mathrm{D}f|^{p-1}\operatorname{sign}(\mathrm{D}f)\, w 1_{\{|\mathrm{D}f|^{p-1}\, w\leq N\}} \in L^\infty, \quad N\in\mathbb{N}$$

in $L^q(w^{1-q}\,\mathrm{d}x)$-norm. Since $w^{1-q}\in L^1_{\mathrm{loc}}$, a subsequence converges $\mathrm{d}x$-a.s. Cutting off, if necessary, and noting that $|h_k^N \mathrm{D}f| \leq N|\mathrm{D}f| \in L^1_{\mathrm{loc}}$, we get by Lebesgue's dominated convergence theorem that

$$\frac{1}{p}\int_{\{|\mathrm{D}f|^{p-1}\, w\leq N\}} |\mathrm{D}f|^p w\,\mathrm{d}x \leq \varliminf_{n} F_n(u_n),$$

which is true for any N. By the monotone convergence theorem,

$$\frac{1}{p}\int |\mathrm{D}f|^p w\,\mathrm{d}x \leq \varliminf_{n} F_n(u_n). \tag{8.22}$$

Suppose now that $\{u_n\}$ converges weakly in $\mathfrak{L}^{\to p}$ to u. This justifies (8.21) by Lemma 5.36. Let $\varphi\in C_0$. Define $v_n := \varphi/w_n$, $n\in\mathbb{N}$, $v := \varphi/w$. Let $\psi\in C_0$.

$$\lim_n \int \psi v_n\, w_n\,\mathrm{d}x = \lim_n \int \psi\varphi\,\mathrm{d}x = \int \psi v\, w\,\mathrm{d}x.$$

Also, by (8.19),

$$\lim_n \int |v_n|^{q_n}\, w_n\,\mathrm{d}x = \lim_n \int |\varphi|^{q_n}\, w_n^{1-q_n}\,\mathrm{d}x = \int |\varphi|^q\, w^{1-q}\,\mathrm{d}x = \int |v|^q\, w\,\mathrm{d}x,$$

where the convergence follows exactly like in Paragraph 6.2.3. Hence $\lim_n \|v_n\|_{L^{q_n}_{w_n}} = \|v\|_{L^q_w}$. By Lemma 5.35, $v_n \rightharpoonup v$ weakly in $\mathfrak{L}^{\to q}$. Since $\{p_n\}$ and hence $\{q_n\}$ is contained in a compact interval away from one and infinity, by (6.5) in Paragraph 6.2, $\mathfrak{L}^{\to q}$ is uniformly asymptotically convex (see Definition 5.56) and hence possesses the asymptotic Kadeč-Klee property by Theorem 5.57. We conclude that $v_n\to v$ strongly in $\mathfrak{L}^{\to q}$. Hence for all $\varphi\in C_0$,

$$\int \varphi f\,\mathrm{d}x = \lim_n \int \varphi u_n\,\mathrm{d}x = \lim_n \int v_n u_n\, w_n\,\mathrm{d}x = \int v u\, w\,\mathrm{d}x = \int \varphi u\,\mathrm{d}x,$$

and therefore $f = u$ $\mathrm{d}x$-a.s. By the results of Paragraph 4.3, (S2) and (8.18) imply that $\mathrm{D}f = \mathrm{D}u = \nabla u$ $\mathrm{d}x$-a.s. Since our arguments work for any subsequence of $\{u_n\}$, (8.22) holds for the whole sequence $\{u_n\}$ and $\mathrm{D}f$ replaced by ∇u. (M1) is proved.

Suppose now that (8.20) holds. Fix a ball $B\subset\mathbb{R}^d$. By Hölder's inequality,

$$\int_B |u_n|\, w_n\,\mathrm{d}x \leq \left(\int |u_n|^{p_n}\, w_n \mathrm{d}x\right)^{1/p_n}\left(\int 1_B w_n\,\mathrm{d}x\right)^{(p_n-1)/p_n}$$

and

$$\int_B |\nabla u_n|\, w_n \,\mathrm{d}x \le \left(\int |\nabla u_n|^{p_n}\, w_n \mathrm{d}x\right)^{1/p_n} \left(\int 1_B w_n \,\mathrm{d}x\right)^{(p_n-1)/p_n}.$$

Suppose that condition (8.21) holds. Then both inequalities combined with (8.17) give that the sequences $\{u_n\, w_n\}$, $\{\partial_i u_n\, w_n\}$, $1 \le i \le d$ are bounded in L^1_{loc} and hence the measures $\{u_n\, w_n \,\mathrm{d}x\}$, $\{\partial_i u_n\, w_n \,\mathrm{d}x\}$, $1 \le i \le d$ are vaguely bounded and hence vaguely relatively compact, see [Bau74, 46.1, 46.2]. Hence we can extract a subsequence of $\{u_n\, w_n \,\mathrm{d}x\}$ that converges vaguely to some locally finite Radon measure m. Also for each $1 \le i \le d$ we can extract a subsequence of $\{\partial_i u_n\, w_n \,\mathrm{d}x\}$ that converges vaguely to some locally finite Radon measure m_i. For any $\varphi \in C_0(\mathbb{R}^d \to \mathbb{R}^d)$,

$$\begin{aligned}\sum_{i=1}^{d} \int \varphi_i \,\mathrm{d}m_i &= \lim_n \int \langle \varphi, \nabla u_n \rangle\, w_n \,\mathrm{d}x \\ &\le \varliminf_n (p_n F(u_n))^{1/p_n} \lim_n \left(\int |\varphi|^{q_n}\, w_n \,\mathrm{d}x\right)^{(p_n-1)/p_n} \\ &\le (pC)^{1/p} \left(\int |\varphi|^{q}\, w \,\mathrm{d}x\right)^{(p-1)/p},\end{aligned}$$

where in the last step we have used the fact that $\mathfrak{L}^{\to q}$ is an asymptotic relation. Thus we have proved that m_i is absolutely continuous w.r.t. $w \,\mathrm{d}x$. Let f be such that $\mathrm{d}m_i = f\, w \,\mathrm{d}x$.

Suppose now that $u_n \rightharpoonup u$ weakly in $\mathfrak{L}^{\to p}$. This justifies (8.21) by Lemma 5.36. By Lemma 5.35 and vague convergence,

$$\int \varphi \,\mathrm{d}m = \lim_n \int \varphi u_n\, w_n \,\mathrm{d}x = \int \varphi u\, w \,\mathrm{d}x,$$

which holds for all $\varphi \in C_0$, hence $\mathrm{d}m = u\, w \,\mathrm{d}x$.

Upon setting $\beta_i^n := \mathrm{D}_i w_n / w_n$, $n \in \mathbb{N}$, $\beta_i := \mathrm{D}_i w / w$, $1 \le i \le d$ by (8.20),

$$\lim_n \int \varphi \beta_i^n\, w_n \,\mathrm{d}x = \int \varphi \beta_i\, w \,\mathrm{d}x \quad \forall \varphi \in C_0, \forall 1 \le i \le d.$$

Also, by (8.20)

$$\lim_n \|\beta_i^n\|_{L^{q_n}_{w_n}} = \|\beta_i\|_{L^q_w} \quad \forall 1 \le i \le d.$$

By Lemma 5.35 and the density $C_0 \subset L^q_w$ (Lemma 6.16) $\beta_n \rightharpoonup \beta$ weakly in $\mathfrak{L}^{\to q}$. Since $\{p_n\}$ and hence $\{q_n\}$ is contained in a compact interval away from one and infinity, by (6.5) in Paragraph 6.2, $\mathfrak{L}^{\to q}$ is uniformly asymptotically convex (see Definition 5.56) and hence possesses the asymptotic Kadeč-Klee property by Theorem 5.57. We conclude that $\beta_i^n \to \beta_i$ strongly for $1 \le i \le d$ in $\mathfrak{L}^{\to q}$.

By integration by parts (3.4) and condition (8.20), for all $\varphi \in C_0^\infty$, $1 \le i \le d$,

$$\begin{aligned}\int \varphi f\, w \,\mathrm{d}x &= \lim_n \int \varphi \partial_i u_n\, w_n \,\mathrm{d}x \\ &= -\lim_n \left[\int \partial_i \varphi u_n\, w_n \,\mathrm{d}x + \int \varphi u_n \beta_i^n\, w_n \,\mathrm{d}x \right] \\ &= -\int \partial_i \varphi u\, w \,\mathrm{d}x - \int \varphi u \beta_i\, w \,\mathrm{d}x,\end{aligned}$$

where we have used strong convergence $\beta_i^n \to \beta_i$ in $\mathfrak{L}^{\to q}$, weak convergence $u_n \rightharpoonup u$ in $\mathfrak{L}^{\to p}$ and asymptotic duality. Since $f \in L_w^p$, by (3.4), $f = \nabla u$ in $L_w^p(\mathbb{R}^d \to \mathbb{R}^d)$.

For $\varphi \in C_0(\mathbb{R}^d \to \mathbb{R}^d)$

$$\left(\int \langle \varphi, \nabla u_n \rangle\, w_n \,\mathrm{d}x \right)^{p_n} \le p_n F_n(u_n) \left(\int |\varphi|^{q_n}\, w_n \,\mathrm{d}x \right)^{p_n/q_n}.$$

Taking the limit inferior, using (8.19), gives

$$\frac{1}{p} \left(\int \langle \varphi, \nabla u \rangle\ w \,\mathrm{d}x \right)^{p} \le \varliminf_n F_n(u_n) \left(\int |\varphi|^{q}\, w \,\mathrm{d}x \right)^{p-1}.$$

Finally, pick a sequence in C_0 approximating $|\nabla u|^{p-1} \operatorname{sign}(\nabla u) \in L_w^q$ in L_w^q-norm. Since $\nabla u \in L_w^p$, we can pass to the limit by Hölder's inequality and get that

$$F(u) = \frac{1}{p} \int |\nabla u|^p w \,\mathrm{d}x \le \varliminf_n F_n(u_n).$$

(M1) is proved.

We conclude the proof by noting that one can repeat all steps for the sequence of functionals $\{F_{n,0}\}$. □

Combining the previous result with Theorem 7.43, we immediately get:

Corollary 8.14. *Under the assumptions of Theorem 8.12 (recall that then, $F_0 = F$), the subdifferential operators $\partial F_n \to \partial F$ and $\partial F_{n,0} \to \partial F_0$ in the G-sense (the strong graph sense).*

For the sake of completeness, we add the following result, for which we do not give explicit sufficient conditions. Anyhow, it particularly holds, if all spaces involved are Hilbert spaces.

Proposition 8.15. *In the above situation, consider*

$$\mathfrak{W}^{1,\to p} = \dot{\bigcup_n} W_{w_n}^{1,p_n} \dot\cup W_w^{1,p}, \quad \mathfrak{W}^{-1,\to q} = \dot{\bigcup_n} W_{w_n}^{-1,-q_n} \dot\cup W_w^{-1,q}.$$

Suppose that $\mathfrak{W}^{1,\to p}$ and $\mathfrak{W}^{-1,\to q}$ have linear asymptotic relations which are asymptotically dual. Suppose furthermore that both the embeddings $\mathfrak{W}^{1,\to p} \hookrightarrow \mathfrak{L}^{\to p}$ and $\mathfrak{L}^{\to q} \hookrightarrow \mathfrak{W}^{-1,\to q}$ are asymptotically strong. Set $G_n := G_{p_n,w_n}$, $n \in \mathbb{N}$, $G := G_{p,w}$. Then $G_n \to G$ Mosco.

Proof. Let $u_n \in W^{1,p_n}_{w_n}$, $n \in \mathbb{N}$, $u \in W^{1,p}_w$ such that $u_n \rightharpoonup u$ weakly in $\mathfrak{W}^{1,\to p}$. Since $\mathfrak{L}^{\to q} \hookrightarrow \mathfrak{W}^{-1,\to q}$ is asymptotically strong, $\mathfrak{W}^{1,\to p} \hookrightarrow \mathfrak{L}^{\to p}$ is asymptotically weak by Lemma 5.63. Hence $u_n \rightharpoonup u$ weakly in $\mathfrak{L}^{\to p}$. By Lemma 5.46,

$$\varliminf_n \|u_n\|_{1,p_n,w_n} \geq \|u\|_{1,p,w}$$

and

$$\varliminf_n \|u_n\|_{p_n,w_n} \geq \|u\|_{p,w}\,,$$

and hence

$$\varliminf_n G_n(u_n) \geq G(u)$$

by Lemma 8.13. (M1) is proved.

(M2) follows from (A2), the strong embedding $\mathfrak{W}^{1,\to p} \hookrightarrow \mathfrak{L}^{\to p}$ and Lemma 8.13. □

Obviously, a version of the above proposition also holds for the strong Sobolev spaces $H^{1,p_n}_{w_n}$ and the functionals $G_{p_n,w_n,0}$.

8.3 $p \to 1$: The critical case

Let $1 < p < \infty$, $q := p/(p-1)$, $d \in \mathbb{N}$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Let $W^{1,p}(\Omega)$ be the standard p-Sobolev space of first order. Let $W^{1,p}_0(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ w.r.t. the Sobolev norm. Denote the weak partial derivative in direction i by ∂_i and denote the weak gradient by ∇. Denote the distributional derivative in direction i by D_i and denote the distributional gradient by D.

For $u \in W^{1,p}(\Omega)$, define the *trace* $\gamma(f)$ to $\partial\Omega$ by

$$\int_\Omega f \operatorname{div} \eta \,\mathrm{d}x = -\int_\Omega \langle \eta, \nabla f\rangle \,\mathrm{d}x + \int_{\partial\Omega} \gamma(f)\, \langle \eta, \nu\rangle \,\mathrm{d}\mathscr{H}^{d-1} \quad \forall \eta \in C^1(\overline{\Omega} \to \mathbb{R}^d), \quad (8.23)$$

where ν is the outward normal and $\mathrm{d}\mathscr{H}^{d-1}$ is the Hausdorff measure on $\partial\Omega$. We have that $\gamma(f) \in L^p(\partial\Omega; \mathrm{d}\mathscr{H}^{d-1})$. See [EG92, §4.3]. By [Eva98, §5.5, Theorem 2], $W^{1,p}_0(\Omega)$ are precisely the elements of $W^{1,p}(\Omega)$ with zero trace.

Define

$$F_0^p(u) := \begin{cases} \frac{1}{p} \int |\nabla u|^p \,\mathrm{d}x, & \text{if } u \in W^{1,p}_0(\Omega), \\ +\infty, & \text{if } u \in L^p(\Omega) \setminus W^{1,p}_0(\Omega). \end{cases}$$

Clearly, F_0^p is convex and proper. By Lemma 2.52, it is l.s.c.

The corresponding subgradient ∂F_0^p is a realization of the p-Laplace operator on Ω with Dirichlet boundary conditions. On smooth functions $\varphi \in C_0^\infty(\Omega)$, it has the representation

$$(\partial F_0^p)(\varphi) = -\operatorname{div}\left[|\nabla\varphi|^{p-1} \operatorname{sign}(\nabla\varphi)\right],$$

see e.g. Remark 8.11 with $w \equiv 1$.

Denote by $(F_0^p)^*$ the Legendre transforms of F_0^p, see Paragraph 2.4.3 for the definition. We shall give a functional representation for it.

Lemma 8.16. *Let*

$$G_0^q(v) := \frac{1}{q}\left[\sup\left\{\int_\Omega uv\,\mathrm{d}x \;\middle|\; u \in W_0^{1,p}(\Omega),\ \|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)} \le 1\right\}\right]^q, \quad v \in L^q(\Omega). \tag{8.24}$$

Then $G_0^q = (F_0^p)^*$.

Proof. Let $v \in L^q(\Omega)$. If $v = 0$, clearly $G_0^q(v) = 0 = (F_0^p)^*(v)$. Thus, let $v \in L^q(\Omega) \setminus \{0\}$. Set $C := \{u \in W_0^{1,p}(\Omega) \mid \|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)} = 0\}$. Clearly, $C = \{0\}$. Hence,

$$\begin{aligned}
(F_0^p)^*(v) &= \sup_{u\in W_0^{1,p}(\Omega)}\left[\int_\Omega uv\,\mathrm{d}x - \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x\right]\\
&= \sup_{u\in C}\int_\Omega uv\,\mathrm{d}x \vee \sup_{u\in W_0^{1,p}(\Omega)\setminus C}\left[\int_\Omega uv\,\mathrm{d}x - \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x\right]\\
&= \sup_{\alpha>0}\ \sup_{u:\|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)}=\alpha}\left[\alpha\int_\Omega \frac{u}{\alpha}v\,\mathrm{d}x - \frac{\alpha^p}{p}\right]\\
&\le \sup_{u:\|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)}=\alpha}\ \sup_{\alpha>0}\left[\frac{\alpha^p}{p} + \frac{1}{q}\left(\int_\Omega \frac{u}{\alpha}v\,\mathrm{d}x\right)^q - \frac{\alpha^p}{p}\right]\\
&= \sup_{u\in W_0^{1,p}(\Omega)\setminus C}\frac{1}{q}\left(\int_\Omega \frac{u}{\|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)}}v\,\mathrm{d}x\right)^q\\
&= G_0^q(v),
\end{aligned}$$

and the Young inequality is attained at $\alpha_0 := \left(\int_\Omega uv\,\mathrm{d}x\right)^{q/(p+q)}$ and hence becomes an equality. The claim is proved. □

We shall continue investigating the limit case $p = 1$.

Definition 8.17. *A function $f \in L^1_{\mathrm{loc}}(\Omega)$ is said to be of* bounded variation *if*

$$\|\mathrm{D}f\|(\Omega) := \sup\left\{\int_\Omega f\,\mathrm{div}\,\eta\,\mathrm{d}x \;\middle|\; \eta \in C_0^\infty(\Omega \to \mathbb{R}^d),\ \|\eta\|_\infty \le 1\right\} < +\infty.$$

The value $\|\mathrm{D}f\|(\Omega)$ is called the total variation *of f. The space of all classes of functions in $L^1(\Omega)$ that are of bounded variation is denoted by $BV(\Omega)$.*

Lemma 8.18. *$BV(\Omega)$ is a Banach space with norm*

$$\|f\|_{BV(\Omega)} := \|f\|_{L^1(\Omega)} + \|\mathrm{D}f\|(\Omega).$$

Proof. See [AFP00, Ch. 3]. □

Let $f \in BV(\Omega)$. Then there is a Radon measure μ_f on $\overline{\Omega}$ and a measurable function $\sigma_f : \Omega \to \mathbb{R}^d$ such that $|\sigma_f| = 1$ μ_f-a.e. and

$$\int_\Omega f\,\mathrm{div}\,\eta\,\mathrm{d}x = -\int_\Omega \langle \eta, \sigma_f\rangle\,\mathrm{d}\mu_f \quad \forall \eta \in C_0^1(\Omega \to \mathbb{R}^d). \tag{8.25}$$

By [EG92, §5.1], $\|\mathrm{D}f\|(\Omega) = \mu_f(\Omega)$. Set $\mathrm{d}[\mathrm{D}f] := \sigma_f \mu_f(\mathrm{d}x)$, which is a $\mathbb{R}^d$-valued Radon measure on Ω. Hence (8.25) becomes

$$\int_\Omega f \operatorname{div} \eta \, \mathrm{d}x = -\int_\Omega \langle \varphi, \mathrm{d}[\mathrm{D}f] \rangle \quad \forall \eta \in C_0^1(\Omega \to \mathbb{R}^d). \tag{8.26}$$

For each $f \in BV(\Omega)$ there is the *trace* $\gamma(f)$ to $\partial\Omega$ defined by

$$\int_\Omega f \operatorname{div} \eta \, \mathrm{d}x = -\int_\Omega \langle \eta, \mathrm{d}[\mathrm{D}f] \rangle + \int_{\partial\Omega} \gamma(f) \langle \eta, \nu \rangle \, \mathrm{d}\mathscr{H}^{d-1} \quad \forall \eta \in C^1(\overline{\Omega} \to \mathbb{R}^d), \tag{8.27}$$

where ν is the outward normal and $\mathrm{d}\mathscr{H}^{d-1}$ is the Hausdorff measure on $\partial\Omega$. We have that $\gamma(f) \in L^1(\partial\Omega; \mathrm{d}\mathscr{H}^{d-1})$. We refer to [EG92, Ch. 5.3] and [AFP00, Ch. 3] for details.

Definition 8.19. *Let $BV_0(\Omega)$ be the space of functions in $BV(\Omega)$ with vanishing trace on $\partial\Omega$.*

We have that $W_0^{1,1}(\Omega) \subset BV_0(\Omega)$ and

$$\int_\Omega |\nabla u| \, \mathrm{d}x = \|\mathrm{D}u\|(\Omega) \quad \forall u \in W_0^{1,1}(\Omega), \tag{8.28}$$

see [EG92, Ch. 5.1, Example 1].

Remark 8.20. *Suppose that $\partial\Omega$ is Lipschitz. Let $u \in BV(\Omega)$. Then*

$$\|\mathrm{D}u\|(\mathbb{R}^d) = \|\mathrm{D}u\|(\Omega) + \int_{\partial\Omega} |\gamma(u)| \, \mathrm{d}\mathscr{H}^{d-1}$$

where $\gamma(u) \in L^1(\partial\Omega, \mathscr{H}^{d-1})$ is the trace of u. We refer to [AFP00, Theorem 3.87].

Define

$$F_0^1(u) := \begin{cases} \|\mathrm{D}u\|(\Omega), & \text{if } u \in BV_0(\Omega), \\ +\infty, & \text{if } u \in L^1(\Omega) \setminus BV_0(\Omega). \end{cases}$$

F_0^1 is proper, convex and l.s.c. on $L^1(\Omega)$ by [AFP00, Proposition 3.6].

Definition 8.21. *Define the* multi-valued sign function *on $\mathbb{R}^d$ by*

$$\overline{\operatorname{sign}}(v) := \begin{cases} v/|v|, & \text{if } v \in \mathbb{R}^d \setminus \{0\}, \\ \overline{B}(0,1), & \text{if } v = 0 \in \mathbb{R}^d. \end{cases}$$

Remark 8.22. *$\partial F_0^1 \subset L^1(\Omega) \times L^\infty(\Omega)$ is multi-valued and difficult to describe. Nevertheless, when $u \in W_0^{1,1}(\Omega)$, then $\|\mathrm{D}u\|(\Omega) = \|\nabla u\|_{L^1(\Omega)}$ and if*

$$v = -\operatorname{div}\left[\overline{\operatorname{sign}}(\nabla u)\right] \in L^\infty(\Omega),$$

then $[u, v] \in \partial F_0^1$. Compare with [BDPR09]. *∂F_0^1 is called the* 1-Laplacian *on Ω with Dirichlet boundary conditions.*

Denote by $(F_0^1)^*$ the Legendre transform of F_0^1. It has the following representation:

Lemma 8.23. *For $v \in L^\infty(\Omega)$, let*

$$G_0^\infty(v) := \begin{cases} 0, & \text{if } \sup\left\{\int_\Omega uv \mid u \in BV_0(\Omega),\ \|\mathrm{D}u\|(\Omega) \le 1\right\} \le 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then $G_0^\infty = (F_0^1)^*$.

Proof. It is straightforward that $G_0^\infty(0) = 0 = (F_0^1)^*(0)$. Therefore, let $v \in L^\infty(\Omega) \setminus \{0\}$. Set $C := \{u \in BV_0(\Omega) \mid \|\mathrm{D}u\|(\Omega) = 0\}$. By Poincaré's inequality [EG92, §5.6.1, Theorem 1 (i)] and Remark 8.20, $C = \{0\}$. Now,

$$\begin{aligned}
(F_0^1)^*(v) &= \sup_{u \in BV_0(\Omega)} \left[\int_\Omega uv \,\mathrm{d}x - \|\mathrm{D}u\|(\Omega)\right] \\
&= \sup_{u \in C} \int_\Omega uv \,\mathrm{d}x \vee \sup_{u \in BV_0(\Omega)\setminus C} \left[\int_\Omega uv \,\mathrm{d}x - \|\mathrm{D}u\|(\Omega)\right] \\
&= \sup_{\alpha > 0} \sup_{u : \|\mathrm{D}u\|(\Omega) = \alpha} \left[\alpha \int_\Omega \frac{u}{\alpha} v \,\mathrm{d}x - \alpha\right] \\
&= \sup_{\alpha > 0} \alpha \left[\sup\left\{\int_\Omega uv \,\mathrm{d}x \mid u \in BV_0(\Omega),\ \|\mathrm{D}u\|(\Omega) \le 1\right\} - 1\right] \\
&= G_0^\infty(v).
\end{aligned}$$

The proof is complete. □

We are ready to formulate the main result of this section. Fix a sequence $\{p_n\} \subset (1, \infty)$ with $\lim_n p_n = 1$. Set $q_n := p_n/(p_n - 1)$. Then $\lim_n q_n = +\infty$. Denote by

$$\mathfrak{L}^{\to 1} = \dot{\bigcup_{n \in \mathbb{N}}} L^{p_n}(\Omega) \dot{\cup} L^1(\Omega), \quad \mathfrak{L}^{\to \infty} = \dot{\bigcup_{n \in \mathbb{N}}} L^{q_n}(\Omega) \dot{\cup} L^\infty(\Omega)$$

the asymptotic relations constructed in Paragraph 6.2.4.

Theorem 8.24. *Suppose that $p_n \to 1$. Then $F_0^{p_n} \to F_0^1$ in the Mosco sense in $\mathfrak{L}^{\to 1}$.*

Theorem 8.25. *Suppose that $p_n \to 1$. Let $q_n := p_n/(p_n - 1)$. Then $G_0^{q_n} \to G_0^\infty$ in the Mosco* sense in $\mathfrak{L}^{\to \infty}$.*

Combining both results, we immediately obtain:

Corollary 8.26. *$F_0^{p_n} \to F_0^1$ in the slice sense in $\mathfrak{L}^{\to 1}$.*

The above result implies that $\partial F_0^{p_n} \to \partial F_0^1$ in the strong graph sense by Theorem 7.46.

Before we prove Theorem 8.24, we need an approximation result for $BV_0(\Omega)$ functions. We shall use an approximation similar to that in [BDPR09].

Lemma 8.27. *Let $u \in BV_0(\Omega)$. Then there is a sequence $\{u_k\}$ of elements in $W_0^{1,2}(\Omega)$ such that*

$$\|u_k - u\|_{L^1(\Omega)} \to 0$$

and

$$\|\mathrm{D}u_k\|(\Omega) \to \|\mathrm{D}u\|(\Omega).$$

This mode of convergence is called strict convergence*, see* [AFP00, Definition 3.14].

An application of Lemma 5.27 gives:

Corollary 8.28. *Let $u \in BV_0(\Omega)$. Then there is a sequence $\{\eta_k\}$ of elements in $C_0^\infty(\Omega)$ such that $\{\eta_k\}$ converges strictly to u.*

Before we prove Lemma 8.27, we need some preparations.

Lemma 8.29. *Assume that $\partial\Omega$ is Lipschitz. Let $u \in BV(\Omega)$. Then there is a sequence $\{\zeta_k\}$ of elements in $C^\infty(\overline{\Omega})$ such that $\{\zeta_k\}$ converges strictly to u in $BV(\Omega)$.*

Proof. By [EG92, §5.2, Theorem 2] there is a sequence $\{v_k\} \subset C^\infty(\Omega) \cap BV(\Omega)$ converging strictly to u. Every v_k belongs particularly to $W^{1,1}(\Omega)$ and can hence be approximated by $\{w_m^k\} \subset C^\infty(\overline{\Omega}) \cap W^{1,1}(\Omega)$ in $W^{1,1}$-norm (see e.g. [EG92, §4.2, Theorem 3]). In particular, $\left\|\mathrm{D}w_m^k\right\|(\Omega) \to \|\mathrm{D}v_k\|(\Omega)$ by [EG92, Ch. 5.1, Example 1]. The claim is obtained by Lemma 5.27. □

Remark 8.30. *By Remark 8.20 and continuity of the trace operator from $BV(\Omega)$ (equipped with the strict convergence) to $L^1(\partial\Omega, \mathscr{H}^{d-1})$, see* [AFP00, Theorem 3.88]*, the statement of Lemma 8.29 can be extended to*

" $\{\zeta_k\}$ converges strictly to u in $BV(\mathbb{R}^d)$ ".

Proof of Lemma 8.27. Let $z \in BV_0(\Omega)$. Let $\{\zeta_k\}$ be the approximating sequence for z in $C^\infty(\overline{\Omega})$ given by Lemma 8.29. Fix $y := \zeta_k$. We shall prove that

$$\overline{\lim_{\varepsilon \to 0}} \|\nabla J_\varepsilon(y)\|_{L^1(\Omega)} \le \|\mathrm{D}y\|(\Omega).$$

Here $J_\varepsilon(y) := (1 - \varepsilon\Delta)^{-1} y$ is the resolvent of the Dirichlet Laplacian $-\Delta$ on Ω. It is well-known that $\lim_{\varepsilon \searrow 0} J_\varepsilon(y) = y$ in $L^2(\Omega)$ and hence in $L^1(\Omega)$. Let $\psi \in C_0^\infty(\Omega; \mathbb{R}^d)$ such that $\|\psi\|_\infty \le 1$. By definition of the resolvent and self-adjointness of $-\Delta$, we have that

$$\int_\Omega J_\varepsilon(y) \operatorname{div} \psi \, \mathrm{d}x \le \left| \int_\Omega y \operatorname{div} \psi \, \mathrm{d}x \right| + \varepsilon \left| \int_\Omega J_\varepsilon(y) \Delta \operatorname{div} \psi \, \mathrm{d}x \right|.$$

After integrating by parts and taking the limit $\varepsilon \to 0$ we get that

$$\overline{\lim_{\varepsilon \to 0}} \int_\Omega \langle \nabla J_\varepsilon(y), \psi \rangle_d \, \mathrm{d}x \le \|\mathrm{D}y\|(\Omega). \tag{8.29}$$

Define

$$L_\varepsilon(\psi) := \int_\Omega \langle \nabla J_\varepsilon(y), \psi \rangle_d \, \mathrm{d}x \quad \psi \in C_0(\Omega; \mathbb{R}^d).$$

L_ε is a continuous linear functional on $C_0(\Omega;\mathbb{R}^d)$ and hence a Radon measure. By [AFP00, Proposition 3.6],

$$\|L_\varepsilon\|_{(C_0(\Omega;\mathbb{R}^d))^*} = \|\mathrm{D}J_\varepsilon(y)\|(\Omega) = \|\nabla J_\varepsilon(y)\|_{L^1(\Omega;\mathbb{R}^d)}\,. \tag{8.30}$$

Now, by the Bishop-Phelps Theorem B.16, for each $\varepsilon > 0$, we can pick a norm attaining continuous linear functional $\widetilde{L}_\varepsilon$ on $C_0(\Omega;\mathbb{R}^d)$ such that $\|L_\varepsilon - \widetilde{L}_\varepsilon\|_{(C_0(\Omega;\mathbb{R}^d))^*} < \varepsilon$. Add $\overline{\lim}_{\varepsilon\to 0}(\widetilde{L}_\varepsilon - L_\varepsilon)(\psi)$ to (8.29) and get

$$\overline{\lim_{\varepsilon\to 0}}\,\widetilde{L}_\varepsilon(\psi) \le \|\mathrm{D}y\|(\Omega) + \overline{\lim_{\varepsilon\to 0}}\,\varepsilon\,\|\psi\|_\infty\,.$$

For each $\varepsilon > 0$, let $\psi_\varepsilon \in C_0(\Omega;\mathbb{R}^d)$, $\|\psi_\varepsilon\|_\infty = 1$ be the element, where $\widetilde{L}_\varepsilon$ attains its norm. Plugging into the inequality, gives

$$\overline{\lim_{\varepsilon\to 0}}\left[\|L_\varepsilon\|_{(C_0(\Omega;\mathbb{R}^d))^*} - \varepsilon\right] \le \overline{\lim_{\varepsilon\to 0}}\|\widetilde{L}_\varepsilon\|_{(C_0(\Omega;\mathbb{R}^d))^*} \le \|\mathrm{D}y\|(\Omega).$$

Finally, by (8.30), the claim is proved. Taking into account that by [AFP00, Proposition 3.6],

$$\underline{\lim_{\varepsilon\to 0}}\,\|\nabla J_\varepsilon(y)\|_{L^1(\Omega)} \ge \|\mathrm{D}y\|(\Omega),$$

the proof can be completed by Lemma 5.27. □

Proof of Theorem 8.24. We start with proving (M1). Suppose that $u_n \in L^{p_n}(\Omega)$ with

$$\underline{\lim_{n}}\,F_0^{p_n}(u_n) < +\infty.$$

Extract a subsequence (also denoted by $\{u_n\}$) such that

$$\lim_n F_0^{p_n}(u_n) = \underline{\lim_n}\,F_0^{p_n}(u_n) =: C.$$

Since (for large n) $u_n \in W_0^{1,p_n}(\Omega)$ and hence $\gamma(u_n) = 0$, we can extend u_n to $\mathbb{R}^d$ by zero outside Ω (denoted also by u_n) and get that $u_n \in W^{1,p_n}(\mathbb{R}^d)$ (cf. [Leo09, Exercise 15.26]).

Fix a ball $B \subset \mathbb{R}^d$. Then by Hölder's inequality,

$$\int_B |u_n|\,\mathrm{d}x \le \left(\int_B |u_n|^{p_n}\,\mathrm{d}x\right)^{1/p_n}(\operatorname{vol} B)^{1/q_n},$$

and

$$\int_B |\nabla u_n|\,\mathrm{d}x \le \left(p_n F_0^{p_n}(u_n)\right)^{1/p_n}(\operatorname{vol} B)^{1/q_n}. \tag{8.31}$$

Suppose that

$$\sup_n \|u_n\|_{L^{p_n}(\mathbb{R}^d)} < \infty. \tag{8.32}$$

Hence $\{u_n\}$ is bounded in $W^{1,1}(B)$. By compactness of the embedding $W^{1,1}(B) \subset L^1(B)$, a subsequence of $\{u_n\}$ converges in $L^1(B)$ (see e.g. [Maz85, §1.4.6, Lemma] or

[Ada75, Theorem 6.2]). By a diagonal argument, we can extract a subsequence such that $\{u_n\}$ converges strongly to some $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. W.l.o.g. $u_n \to f$ $\mathrm{d}x$-a.s., by extracting another subsequence, if necessary. Also, the measures $\{\partial_i u_n \, \mathrm{d}x\}$ are vaguely bounded and hence vaguely relatively compact, see [Bau74, 46.1, 46.2]. For each $1 \le i \le d$, we can extract a subsequence, such that $\{\partial_i u_n \, \mathrm{d}x\}$ converges to some locally finite Radon measure m_i on $\mathbb{R}^d$. By vague convergence and integration by parts,

$$\int \varphi \, \mathrm{d}m_i = \lim_n \int \varphi \partial_i u_n \, \mathrm{d}x = -\lim_n \int \partial_i \varphi u_n \, \mathrm{d}x = -\int \partial_i \varphi f \, \mathrm{d}x,$$

for every $\varphi \in C_0^\infty(\mathbb{R}^d)$ and every $1 \le i \le d$. Hence $m_i = \mathrm{D}_i f$. Furthermore, for every $\varphi \in C_0^\infty(\mathbb{R}^d \to \mathbb{R}^d)$,

$$\frac{1}{p_n} \left| \int u_n \operatorname{div} \varphi \, \mathrm{d}x \right|^{p_n} = \frac{1}{p_n} \left| \int \langle \nabla u_n, \varphi \rangle \, \mathrm{d}x \right|^{p_n} \le F_0^{p_n}(u_n) \, \|\varphi\|_\infty \, (\mathrm{vol}(\operatorname{supp} \varphi))^{p_n/q_n} . \tag{8.33}$$

Upon taking the limit,

$$\left| \int f \operatorname{div} \varphi \, \mathrm{d}x \right| \le \varliminf_n F_0^{p_n}(u_n) \, \|\varphi\|_\infty . \tag{8.34}$$

Taking the supremum over all φ with $\|\varphi\|_\infty \le 1$ yields

$$\|\mathrm{D}f\| (\mathbb{R}^d) \le \varliminf_n F_0^{p_n}(u_n).$$

Suppose now that $u_n \rightharpoonup u$ weakly in $\mathfrak{L}^{\to 1}$. This justifies (8.32) by Lemma 5.36. Clearly, for all $\varphi \in C_0(\mathbb{R}^d)$,

$$\int u \varphi \, \mathrm{d}x = \lim_n \int u_n \varphi \, \mathrm{d}x = \int f \varphi \, \mathrm{d}x,$$

hence $u = f$ $\mathrm{d}x$-a.e. and $\mathrm{D}u = \mathrm{D}f$. We are left to prove that $u \in BV_0(\Omega)$, because then by Remark 8.20

$$\|\mathrm{D}u\| (\mathbb{R}^d) = \|\mathrm{D}u\| (\Omega) = F_0^1(u) < +\infty.$$

We have that $u_n \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and that $\partial_i u_n \, \mathrm{d}x \to \mathrm{D}_i u$ in the vague sense on $\mathbb{R}^d$. Hence by the definition of the trace (8.23), (8.27),

$$\lim_n \int_{\partial\Omega} \gamma(u_n) \langle \varphi, \nu \rangle \, \mathrm{d}\mathscr{H}^{d-1} = \int_{\partial\Omega} \gamma(u) \langle \varphi, \nu \rangle \, \mathrm{d}\mathscr{H}^{d-1} \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d).$$

Let $\varphi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$ such that $\|\varphi\|_\infty \le 1$. Then, analog to (8.33) and (8.34), by taking the supremum over all such φ,

$$\int_{\partial\Omega} |\gamma(u)| \, \mathrm{d}\mathscr{H}^{d-1} \le \varliminf_n \int_{\partial\Omega} |\gamma(u_n)|^{p_n} \, \mathrm{d}\mathscr{H}^{d-1} = 0.$$

Compare with [AFP00, Lemma 3.90]. We get that $\gamma(u) = 0$ $\mathscr{H}^{d-1}$-a.s. and hence $u \in BV_0(\Omega)$. Since we can repeat the steps for any subsequence of $\{u_n\}$, we have proved (M1).

Let us prove (M2). Let $u \in BV_0(\Omega)$. Then by Corollary 8.28 there is a sequence $\{u_m\} \subset C_0^\infty(\Omega) \subset BV_0(\Omega)$ with

$$u_m \to u \text{ in } L^1(\Omega) \text{ and } \|\mathrm{D}u_m\|(\Omega) \to \|\mathrm{D}u\|(\Omega) \text{ as } m \to \infty.$$

But by the example in Paragraph 6.2.4 and (8.28) for each $m \in \mathbb{N}$

$$\frac{1}{p_n}\int_\Omega |\nabla u_m|^{p_n}\,\mathrm{d}x \to \int_\Omega |\nabla u_m|\,\mathrm{d}x = \|\mathrm{D}u_m\|(\Omega) \text{ as } n \to \infty.$$

An application of Lemma 5.27 shows that there exists a sequence $\{m_n\}$, $m_n \uparrow \infty$, such that

$$\lim_n F_0^{p_n}(u_{m_n}) = \lim_n \frac{1}{p_n}\int_\Omega |\nabla u_{m_n}|^{p_n}\,\mathrm{d}x = \|\mathrm{D}u\|(\Omega) = F_0^1(u),$$

which proves (M2). □

Proof of Theorem 8.25. We observe that (M1)* follows from (M2), as proved above. Indeed, let $u_n \in L^{q_n}(\Omega)$, $n \in \mathbb{N}$, $u \in L^\infty(\Omega)$ such that $u_n \rightharpoonup^* u$ weakly* in $\mathfrak{L}^{\to\infty}$ (see Paragraph 6.2.4). Let $v \in L^1(\Omega)$. By (M2), there are $v_n \in L^{p_n}(\Omega)$ such that $v_n \to v$ strongly in $\mathfrak{L}^{\to 1}$ and $\overline{\lim}_n F_0^{p_n}(v_n) \le F_0^1(v)$. Clearly,

$$\underline{\lim}_n (F_0^{p_n})^*(u_n) \ge \underline{\lim}_n \left[{}_{L^{q_n}}\langle u_n, v_n\rangle_{L^{p_n}} - F_0^{p_n}(v_n)\right] \ge {}_{L^\infty}\langle u, v\rangle_{L^1} - F_0^1(v).$$

Since $v \in L^1(\Omega)$ was arbitrary, we get that

$$\underline{\lim}_n (F_0^{p_n})^*(u_n) \ge (F_0^1)^*(u),$$

which is (M1)*.

Let us prove (M2)*. Let $v \in L^\infty(\Omega)$. For $v \in L^\infty(\Omega) \setminus D(G_0^\infty)$, (M2)* is trivial. So we can assume that $v \in D(G_0^\infty)$. Set

$$H(v) := \sup\left\{ \int_\Omega uv \mid u \in BV_0(\Omega),\ \|\mathrm{D}u\|(\Omega) \le 1 \right\}.$$

Let $u \in W_0^{1,p_n}(\Omega)$ with $0 < \|\nabla u\|_{L^{p_n}(\Omega;\mathbb{R}^d)} \le 1$. Note that by Jensen's inequality,

$$\|\nabla u\|^{-1}_{L^{p_n}(\Omega;\mathbb{R}^d)} \le \|\nabla u\|^{-1}_{L^1(\Omega;\mathbb{R}^d)} (\mathrm{vol}(\Omega))^{1/q_n}.$$

Therefore,

$$\frac{1}{q_n}\left|\int_\Omega uv\,\mathrm{d}x\right|^{q_n} \le \frac{1}{q_n}\|\nabla u\|^{q_n}_{L^{p_n}(\Omega;\mathbb{R}^d)} H(v)^{q_n}\,\mathrm{vol}(\Omega) \le \frac{1}{q_n} H(v)^{q_n}\,\mathrm{vol}(\Omega).$$

If $\|\nabla u\|_{L^{p_n}(\Omega;\mathbb{R}^d)} = 0$, then $u = 0$ and the inequality also holds. Hence, taking the supremum over all $u \in W_0^{1,p_n}(\Omega)$ with $\|\nabla u\|_{L^{p_n}(\Omega;\mathbb{R}^d)} \le 1$ yields

$$G_0^{q_n}(v) \le \frac{1}{q_n} H(v)^{q_n}\,\mathrm{vol}(\Omega),$$

but since $v \in D(G_0^\infty)$, $H(v) \le 1$ and hence taking the limit superior gives

$$\overline{\lim_n}\, G_0^{q_n}(v) \le 0 = G_0^\infty(v).$$

(M2)* is proved. □

8.4 Convergence of generalized porous medium and fast diffusion operators

We follow the framework of J. Ren, M. Röckner and F.-Y. Wang in [RRW07]. See also [RW08] for the non-monotone case. The reader should not worry about the stochastic statements of these works, for we are merely using their functional analytic parts.

Let $(E, \mathscr{B}, \mu)$ be a separable σ-finite measure space. Let $(L, D(L))$ be a negative definite self-adjoint operator on $L^2(\mu)$ with $\ker L = \{0\}$. Let $(\mathscr{E}, \mathscr{F})$ be the quadratic form on $L^2(\mu)$ associated to $(L, D(L))$, i.e.

$$\mathscr{F} := D(\sqrt{-L}) \quad \text{and} \quad \mathscr{E}(u,v) := (\sqrt{-L}u, \sqrt{-L}v)_{L^2(\mu)}, \ u, v \in \mathscr{F}.$$

Let $\mathscr{F}_e$ be the abstract completion of $\mathscr{F}$ with respect to the pre-norm

$$\|\cdot\|_{\mathscr{F}_e} := \sqrt{\mathscr{E}(\cdot,\cdot)}$$

and let $\mathscr{F}_e^*$ be its dual space. Obviously, $\mathscr{F} \subset \mathscr{F}_e$.

Definition 8.31. *Let X, Y be Banach spaces. Define norms on $X \cap Y$ by*

$$\|z\|_{X \cap Y, 1} := \|z\|_X + \|z\|_Y$$

or

$$\|z\|_{X \cap Y, \infty} := \max\left[\|z\|_X, \|z\|_Y\right].$$

Define $W \subset X \oplus Y$ by $W := \{(x,y) \,|\, x = -y\}$. Define $X + Y := (X \oplus Y)/W$ to be the quotient space. Define norms on $X + Y$ by

$$\|u\|_{X+Y,1} := \inf_{x+y=u} \left[\|x\|_X + \|y\|_Y\right]$$

or

$$\|u\|_{X+Y,\infty} := \inf_{x+y=u} \max\left[\|x\|_X, \|y\|_Y\right].$$

Lemma 8.32. *Let X, Y be Banach spaces. If*

$$\{u_n\} \subset X \cap Y, \ \|u_n - x\|_X \to 0, \ \|u_n - y\|_Y \to 0 \text{ always implies } x = y \in X \cap Y, \quad (8.35)$$

then $X \cap Y$ is a Banach space with either norm $\|\cdot\|_{X \cap Y, 1}$ or norm $\|\cdot\|_{X \cap Y, \infty}$. Also $X + Y$ is a Banach space with either norm $\|\cdot\|_{X+Y,1}$ or norm $\|\cdot\|_{X+Y,\infty}$.

Proof. See [AG65, §2]. □

Lemma 8.33. *Let X, Y be Banach spaces such that* (8.35) *holds. Suppose that $X \cap Y$ is dense in both X and Y. Then $(X + Y)^* = X^* \cap Y^*$ and $(X \cap Y)^* = X^* + Y^*$. In the functional norms related to the norms as in Definition 8.31,* 1 *becomes* ∞ *and vice versa. Furthermore, $x^* = y^*$ in $X^* \cap Y^*$ if and only if*

$${}_{X^*}\langle x^*, z\rangle_X = {}_{Y^*}\langle y^*, z\rangle_Y \quad \forall z \in X \cap Y.$$

Proof. See [AG65, §8]. □

Let (Φ, Ψ) be a pair of complementary N-functions with gauges (φ, ψ). Suppose that $\Phi, \Psi \in C^1(\mathbb{R})$ and $\Phi, \Psi \in \Delta_2 \cap \nabla_2$. Let $L^\Phi := L^\Phi(E, \mathscr{B}, \mu)$, $L^\Psi := L^\Psi(E, \mathscr{B}, \mu)$ be corresponding Orlicz spaces. Set $H := \mathscr{F}_e^*$ with inner product $(\cdot,\cdot)_H := \mathscr{E}(-\overline{L}^{-1}\cdot, -\overline{L}^{-1}\cdot)$, where $-\overline{L}v := \mathscr{E}(v,\cdot)$, $v \in \mathscr{F}_e$ is the Riesz map of $\mathscr{F}_e$, which is the canonical extension of $-L : D(L) \to L^2(\mu)$. Furthermore, set $V := H \cap L^\Phi$ with norm $\|u\|_V := \|u\|_{H\cap L^\Phi,1}$ as in Definition 8.31. In this choice, we shall select the Orlicz norm.

Definition 8.34. *Consider the following conditions:*

(1) *There exists a strictly positive $g \in L^1(\mu) \cap L^\infty(\mu)$ such that $\mathscr{F}_e \subset L^1(g\cdot\mu)$ continuously.*

(2) *$\mathscr{F}_e \cap L^\Psi$ is a dense linear subset of both $\mathscr{F}_e$ and L^Ψ.*

(3) *V is a dense linear subset of both H and L^Φ.*

Proposition 8.35 ([RRW07]). *Suppose that $\Phi, \Psi \in \Delta_2 \cap \nabla_2$ and that $(\mathscr{E}, \mathscr{F})$ is a transient Dirichlet space in the sense of* [FOT94, §1.5]*. Then conditions* (1)–(3) *of Definition 8.34 hold.*

Proof. See [RRW07, Proposition 3.1]. □

As a consequence, V is a well-defined dense linear subspace of H, which gives (8.35). V is a Banach space with norm $\|\cdot\|_V$ by Lemma 8.32. By Lemma 8.33 and condition (2),

$$V = \left\{ u \in L^\Phi \;\middle|\; \text{the map } \mathscr{F}_e \cap L^\Psi \ni v \mapsto \int_E uv \,\mathrm{d}\mu \text{ is in } \mathscr{F}_e^* \right\}.$$

Since the map $V \ni u \mapsto (u, \mu(u\cdot)) \in L^\Phi \times H$ is an isomorphism from V to a closed subspace of $L^\Phi \times H$ (which is a reflexive Banach space), V itself is reflexive.

Define $G^{\Phi,\mu} : V \to [0, +\infty)$ by

$$G^{\Phi,\mu}(u) := \int_E \Phi(|u|)\,\mathrm{d}\mu. \tag{8.36}$$

Lemma 8.36. *$G^{\Phi,\mu}$ is strongly continuous on V, convex and Gâteaux differentiable and the Gâteaux derivative is given by*

$${}_{V^*}\left\langle \nabla_G G^{\Phi,\mu}(u), h \right\rangle_V = \int_E \varphi(|u|)\operatorname{sign}(u)\, h\,\mathrm{d}\mu.$$

Proof. Convexity is clear. Strong continuity follows from the definition of the V-norm and Lemma C.16. Compare with the proof of Lemma 8.1. We have for $u, h \in V$ and $t > 0$

$$\frac{1}{t}\left[G^{\Phi,\mu}(u+th) - G^{\Phi,\mu}(u)\right] = \int_E \frac{1}{t}\int_{|u|}^{|u+th|} \varphi(s)\,\mathrm{d}s\,\mathrm{d}\mu.$$

On the other hand, as $t \to 0$,

$$|u + th| \to |u|, \quad \mu\text{-a.e.}$$

and hence μ-a.e. We see that for $t \le 1$

$$\left| \frac{1}{t} \int_{|u|}^{|u+th|} \varphi(s)\,\mathrm{d}s \right| \le \left| \frac{1}{t} \int_0^{t|h|} \varphi(s + |u|)\,\mathrm{d}s \right| \le \varphi(|u| + |h|)|h|$$

with $\varphi(|u| + |h|) \in L^{\Psi}(\mu)$ (by Lemma C.21) and $|h| \in L^{\Phi}(\mu)$ and hence $\varphi(|u| + |h|)|h| \in L^1(\mu)$ by the Hölder inequality C.15. We calculate the directional derivative

$$\lim_{t \to 0} \frac{1}{t} \left[|u + th| - |u| \right] = \frac{uh}{|u|},$$

where we have used that

$$\nabla |u| = \begin{cases} \frac{u}{|u|}, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases}$$

Thus, by the dominated convergence theorem and the fundamental theorem of calculus

$$\lim_{t \to 0} \frac{1}{t} \left[G^{\Phi,\mu}(u + th) - G^{\Phi,\mu}(u) \right] = \int_E \varphi(|u|) \frac{uh}{|u|}\,\mathrm{d}\mu = \int_E \varphi(|u|)\,\mathrm{sign}(u) h\,\mathrm{d}\mu.$$

□

Define

$$F^{\Phi,\mu}(u) := \begin{cases} G^{\Phi,\mu}(u), & \text{if } u \in V, \\ +\infty, & \text{if } u \in H \setminus V. \end{cases} \tag{8.37}$$

Lemma 8.37. $F^{\Phi,\mu} : H \to [0, +\infty]$, *as defined above, is a proper, convex and l.s.c. functional. Furthermore, the subdifferential* $\partial F^{\Phi,\mu} \subset H \times H^*$ *is single-valued.*

Proof. Properness and convexity are obvious. Recall that, by Lemma 8.36, $G^{\Phi,\mu}$ is continuous and Gâteaux differentiable. Let $S \subset V$. Note that by Lemma C.17, if both $\sup_{u \in S} \|u\|_H < +\infty$ and $\sup_{u \in S} F^{\Phi,\mu}(u) < +\infty$, then we have that $\sup_{u \in S} \|u\|_{L^{\Phi}(\mu)} < +\infty$ and hence that $\sup_{u \in S} \|u\|_V < +\infty$. Therefore, $F^{\Phi,\mu}$ is l.s.c. by Lemma 2.52. $\partial F^{\Phi\mu}$ is single-valued by Corollary 2.51. □

We are ready to define the fast diffusion operator (if $\Phi \prec |\cdot|^2$) respectively the porous medium operator (if $\Phi \succ |\cdot|^2$). Set

$$\widetilde{A} := \partial F^{\Phi,\mu} : D(\partial F^{\Phi,\mu}) \subset V \subset \mathscr{F}_e^* \to \mathscr{F}_e,$$

When considered as an operator on the Hilbert space $\mathscr{F}_e^* = H$, we concatenate the Riesz map $-\overline{L} : \mathscr{F}_e \to \mathscr{F}_e^*$ (which is equivalent to an alternative definition the subgradient in Hilbert spaces using the scalar product instead of the dualization) and obtain

$$A : D(A) \subset V \subset H \to H, \;\; Au := -\overline{L}\widetilde{A}u$$

which is an operator having the variational representation

$$(Au, v)_{\mathscr{F}_e^*} = -\int_E \overline{L}\,[\varphi(|u|)\,\mathrm{sign}(u)]\,v\,\mathrm{d}\mu$$

and

$$_{\mathscr{F}_e}\left\langle \widetilde{A}u, v\right\rangle_{\mathscr{F}_e^*} = -\int_E \varphi(|u|)\,\mathrm{sign}(u)v\,\mathrm{d}\mu,$$

which by the Hahn-Banach Theorem holds for all $v \in L^{\Psi}(\mu) \subset V^*$, $u \in D(A)$. We would like to point out, that the operator A defined in this way differs from the one defined in [RRW07], wherein an operator defined on all of V mapping into V^* is considered.

Suppose now that for each $n \in \mathbb{N}$ we are given operators L_n with the properties stated above. Associate quadratic forms $(\mathscr{E}_n, \mathscr{F}_n)$, symmetrizing measures μ_n and spaces V_n, H_n. Set

$$\mathfrak{V} := \dot{\bigcup_n} V_n \dot{\cup} V, \quad \mathfrak{H} := \dot{\bigcup_n} H_n \dot{\cup} H.$$

We are ready to formulate the main theorem of this section. For concrete examples satisfying its conditions, we refer to [FOT94, Ebe99]. For $d \geq 3$, it is verified that the Laplacian $-\Delta$ on all of $\mathbb{R}^d$ has a transient Dirichlet form and satisfies the assumptions on L in Theorem 8.38 below (in particular, it has a core consisting of continuous functions). Here $\mu := \mathrm{d}x$. We refer to [FOT94, Example 1.5.2, p. 44]. Hence the classical porous medium operator is included as a possible limit, see Introduction. For the notion of a *core* for symmetric operators we refer to [RS80, Ch. VIII.2].

Theorem 8.38. *Suppose that the following holds true. E is a Souslin space, μ_n, $n \in \mathbb{N}$, μ are finite Borel measures with full support in E, $\mu_n \rightharpoonup^w \mu$ weakly, μ is a regular measure, L possesses a core $C \subset C_b(E)$ such that C is dense in $\mathscr{F}_e$ and $C \subset \mathscr{F}_{e,n}$, $D := \overline{L}(C) \subset \mathscr{F}_{e,n}^*$ for $n \in \mathbb{N}$. Suppose that $D \subset C_b(E)$ densely w.r.t. the uniform norm. At last, suppose that*

$$\lim_n \mathscr{E}_n(u, u) = \mathscr{E}(u, u) \quad \forall u \in C. \tag{8.38}$$

Set $F_n := F^{\Phi,\mu_n}$, $n \in \mathbb{N}$, $F := F^{\Phi,\mu}$.

Then $\mathfrak{H}$ has an asymptotic relation and $F_n \to F$ Mosco in $\mathfrak{H}$.

Proof. Firstly, by the results of Paragraph 6.2.1, $\mathfrak{L}^2 := \dot{\bigcup}_n L^2(\mu_n)\dot{\cup}L^2(\mu)$ has a linear asymptotic relation. Because $C \subset \mathscr{F}_e$ densely, and by (8.38), $\mathfrak{H}^* = \dot{\bigcup}_n \mathscr{F}_{e,n}\dot{\cup}\mathscr{F}_e$ and hence $\mathfrak{H}$ have linear asymptotic relations (with an obvious candidate for the metric approximation). By the results of Paragraph 6.3, $\mathfrak{L}^{\Phi} := \dot{\bigcup}_n L^{\Phi}(\mu_n)\dot{\cup}L^{\Phi}(\mu)$ have linear asymptotic relations. By our assumptions, D is dense in $L^{\Phi}(\mu)$ and also in $\mathscr{F}_e^*$. Now, $\overline{L}_n \to \overline{L}$ strongly in the sense of Definition 7.4 (with the topologies of $\mathfrak{H}^*$, $\mathfrak{H}$ resp.) because of being the Riesz maps, cf. Section 6.1. We get that

$$\lim_n \|u\|_{H_n} = \|u\|_H \quad \forall u \in D.$$

Furthermore, by arguments similar to those in Paragraph 6.3,

$$\lim_n \|u\|_{L^\Phi(\mu_n)} = \|u\|_{L^\Phi(\mu)} \quad \forall u \in D.$$

Now, since $D \cap V$ is dense in V,

$$\lim_n \|u\|_{V_n} = \|u\|_V \quad \forall u \in D.$$

Hence $\mathfrak{V}$ has a linear strong asymptotic relation. As the embedding constants are all equal to 1 and D is dense both in V and H, it follows by Lemma 5.66 that the embedding $\mathfrak{V} \hookrightarrow \mathfrak{H}$ is asymptotically strong.

Let us prove (M1). We would like to apply Corollary 8.7. Fix $\varepsilon > 0$. Let $\{u_n\}$ be a sequence in $\mathfrak{H}$, $u_n \in H_n$, $n \in \mathbb{N}$. Suppose that $\underline{\lim}_n \|u_n\|_{(\Phi,\varepsilon\mu_n)}$ is finite. Extract a subsequence, also denoted by $\{u_n\}$, such that

$$\lim_n \|u_n\|_{(\Phi,\varepsilon\mu_n)} = \underline{\lim}_n \|u_n\|_{(\Phi,\varepsilon\mu_n)} =: C.$$

By the proof above,

$$\mathfrak{L}^\Phi_\varepsilon := \dot{\bigcup_n} L^\Phi(\varepsilon\mu_n) \dot{\cup} L^\Phi(\varepsilon\mu)$$

has a asymptotically reflexive asymptotic relation, asymptotically dual to $\mathfrak{L}^\Phi_\varepsilon$. Hence by Lemma 5.53 there is a $\mathfrak{L}^\Phi_\varepsilon$-weakly convergent subsequence $\{u_n\}$ with $u_n \rightharpoonup \widetilde{u} \in L^\Phi(\mu)$. By Lemma 5.46,

$$\underline{\lim}_n \|u_n\|_{(\Phi,\varepsilon\mu_n)} \geq \|\widetilde{u}\|_{(\Phi,\varepsilon\mu)}.$$

Suppose now that $u_n \rightharpoonup u$ weakly in $\mathfrak{H}$. Then

$$\int_E \eta u \,\mathrm{d}\mu = \lim_n \int_E \eta u_n \,\mathrm{d}\mu_n = \int_E \eta \widetilde{u} \,\mathrm{d}\mu,$$

for all $\eta \in D$. But D is dense both in H and $L^\Phi(\varepsilon\mu)$. Hence $u = \widetilde{u}$ in H and in $L^\Phi(\varepsilon\mu)$, and hence

$$\underline{\lim}_n \|u_n\|_{(\Phi,\varepsilon\mu_n)} \geq \|u\|_{(\Phi,\varepsilon\mu)}.$$

Since the arguments work for any subsequence of $\{u_n\}$ we have proved (8.12). (M1) follows now from Corollary 8.7.

Let us prove (M2). Let $u \in V$ (the case $u \in H \setminus V$ is trivial). Since $D \subset C_b(E)$ is dense in V, we find a sequence $\{\eta_m\} \subset D$ with $\lim_m \|u - \eta_m\|_V = 0$. Hence by Lemma C.16

$$\lim_m \int_E \Phi(|\eta_m|) \,\mathrm{d}\mu = \int_E \Phi(|u|) \,\mathrm{d}\mu.$$

By the proof above, $\eta_m \in V_n$ for each n and $\eta_m \to \eta_m$ strongly in $\mathfrak{V}$. Also, by weak convergence of measures and $\Phi \in C^1(\mathbb{R})$, for each m,

$$\lim_n \int_E \Phi(|\eta_m|) \,\mathrm{d}\mu_n = \int_E \Phi(|\eta_m|) \,\mathrm{d}\mu.$$

By Lemma 5.27 there exists a sequence of natural numbers $\{m_n\}$, $m_n \uparrow \infty$ as $n \to \infty$ such that $\eta_{m_n} \to u$ in $\mathfrak{V}$ (indeed, $\lim_m \overline{\lim}_n \|\eta_{m_n} - \eta_m\|_{V_n} = 0$) and such that

$$\overline{\lim_n} \int_E \Phi(|\eta_{m_n}|)\,\mathrm{d}\mu_n \le \overline{\lim_m}\,\overline{\lim_n} \int_E \Phi(|\eta_m|)\,\mathrm{d}\mu_n = \overline{\lim_m} \int_E \Phi(|\eta_m|)\,\mathrm{d}\mu = \int_E \Phi(|u|)\,\mathrm{d}\mu.$$

But we have proved above that the embedding $\mathfrak{V} \hookrightarrow \mathfrak{H}$ is asymptotically strong. Hence $\eta_{m_n} \to u$ in $\mathfrak{H}$ as $n \to \infty$ and (M2) is proved. □

Combining the previous result with Theorem 7.43, we immediately get:

Corollary 8.39. *Under the assumptions of Theorem 8.38, the subdifferential operators* $\partial F_n \to \partial F$ *in the G-sense (the strong graph sense).*

A Facts from general topology

A.1 Nets

Lemma A.1. *Let $(T, \mathscr{T})$ be a topological space.*

(i) We assume that the reader is familiar with the notions net, subnet, directed set, cofinal, to be frequently in a set, to be eventually in a set. *They are e.g. explained in* [Kel75, Eng89].

(ii) A set A included in T is open if and only if every net $\{x_i\}_{i \in I}$ which converges to a point $x \in A$ is eventually in A.

(iii) A set A included in T is closed if and only if it contains with any net all its possible limits, or equivalently, no net included in A converges to a point in $T \setminus A$.

(iv) For a set A included in T one defines the relative topology of A in T *by $\mathscr{T}_A := \{O \cap A \mid O \in \mathscr{T}\}$. $B \subset A$ is called* relatively open *if $B \in \mathscr{T}_A$ and $B \subset A$ is called* relatively closed *if $A \setminus B \in \mathscr{T}_A$.*

(v) Consider a directed set $(\mathscr{N}, \succeq)$. The identity $\nu \mapsto \nu_\nu$ on $\mathscr{N}$ is a net $\{\nu_\nu\}_{\nu \in \mathscr{N}} = \{\nu\}_{\nu \in \mathscr{N}}$ directed by $\succeq$. Consider another directed set $(\mathscr{M}, \succsim)$. A net $\{\nu_\mu\}_{\mu \in \mathscr{M}}$ of elements in $\mathscr{N}$ directed by $\succsim$ is is a subnet of $\{\nu_\nu\}_{\nu \in \mathscr{N}}$ if and only if there exists a map $\mathscr{M} \ni \mu \mapsto \widetilde{\nu}_\mu \in \mathscr{N}$ such that

$$\begin{aligned} &\text{for every } \nu_0 \in \mathscr{N} \text{ there exists a } \mu_0 \in \mathscr{M} \\ &\text{such that } \widetilde{\nu}_\mu \succeq \nu_0 \text{ whenever } \mu \succsim \mu_0 \end{aligned} \tag{A.1}$$

and

$$\nu_{\widetilde{\nu}_\mu} = \nu_\mu \text{ for all } \mu \in \mathscr{M}. \tag{A.2}$$

Since $\{\nu_\nu\}_{\nu \in \mathscr{N}} = \{\nu\}_{\nu \in \mathscr{N}}$ is the identity, (A.2) *yields $\{\nu_\mu\}_{\mu \in \mathscr{M}} = \{\widetilde{\nu}_\mu\}_{\mu \in \mathscr{M}}$. Thus $\{\nu_\mu\}_{\mu \in \mathscr{M}}$ is a subnet of $\{\nu\}_{\nu \in \mathscr{N}}$ if and only if it has property* (A.1)*. A famous special case of* (A.1) *is well-known:*

$$\begin{aligned} &\mu \mapsto \nu_\mu \text{ is non-decreasing,} \\ &\text{i.e., } \nu_{\mu_1} \succeq \nu_{\mu_2} \text{ whenever } \mu_1 \succsim \mu_2 \end{aligned} \tag{A.3}$$

and the image $\{\nu_\mu \mid \mu \in \mathscr{M}\}$ is a cofinal subset of $\mathscr{N}$.

A.2 Sequential spaces and convergence

Recall the following basic definition.

Definition A.2. *A* (Kuratowski) closure operator *on a set S is a mapping* $\overline{} : 2^S \to 2^S$ *such that the* Kuratowski closure axioms

(K1) $\overline{\varnothing} = \varnothing$,

(K2) *for each* $A \in 2^S$*:* $A \subset \overline{A}$,

(K3) *for each* $A, B \in 2^S$*:* $\overline{A \cup B} = \overline{A} \cup \overline{B}$,

(K4) *for each* $A \in 2^S$*:* $\overline{(\overline{A})} = \overline{A}$,

hold.

If X is a topological space, we define for any subset $A \subset X$ the *closure* w.r.t. the topology of X as $\overline{A} := \bigcap_{\substack{B \supset A, \\ B \text{ closed}}} B$. It satisfies the Kuratowski closure axioms. Conversely, a Kuratowski closure operator on a set S defines a topology on S by saying $A \subset S$ is closed if $\overline{A} = A$. Then the Kuratowski closure operator coincides with the closure w.r.t. to the topology it generates. See e.g. [Kel75, Chapter 1]. We also define $\partial A := \overline{A} \cap \overline{X \setminus A}$ and $\operatorname{int} A := A \setminus \partial A$.

Definition A.3. *A topological space X is called* sequential space *if a set $A \subset X$ is closed if and only if together with any sequence it contains all its limits. A topological space X is called a* (topological) Fréchet space *or* Fréchet-Urysohn space *if for every $A \subset X$ and every $x \in \overline{A}$ there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ of points of A converging to x.*

We assume that the reader is familiar with the terms *first countable*, *second countable*, *compact*, *sequentially compact* and *countably compact*.

Lemma A.4. *(i) Every first-countable space is a Fréchet space and every Fréchet space is a sequential space.*

(ii) Any subspace of a Fréchet space is itself a Fréchet space; in particular, being Fréchet is a topological invariant.

(iii) Any closed subspace of sequential space is itself a sequential space.

(iv) A mapping F of a sequential space X to a topological space Y is continuous if and only if $F(\lim_{n\to\infty} x_n) \subset \lim_{n\to\infty} F(x_n)$ for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in the space X.

(v) Sequential compactness and countable compactness are equivalent in the class of sequential spaces.

(vi) In a sequential space the characterization of open and closed sets found as in Lemma A.1 (ii), (iii) hold with nets replaced by sequences.

Proof. (i): [Eng89, Theorem 1.6.14], (ii),(iii): [Eng89, Exercise 2.1.H], (iv): [Eng89, Proposition 1.6.15], (v): [Eng89, Theorem 3.10.31], (vi): clear from the definition. □

Definition A.5. *Let S be a set. A $\mathscr{L}^*$-*sequential convergence *or $\mathscr{L}^*$-*(sequential) limit operator *$\mathscr{C}$ on S is a relation between sequences $\{s_n\}_{n\in\mathbb{N}}$ of members of S and members s of S, denoted $s_n \xrightarrow[n\to\infty]{\mathscr{C}} s$ (in words: $\{s_n\}$ $\mathscr{C}$-*converges to *s), such that:*

(L1) *If $s_n = s$ for each $n \in \mathbb{N}$, then $s_n \xrightarrow[n\to\infty]{\mathscr{C}} s$.*

(L2) *If $s_n \xrightarrow[n\to\infty]{\mathscr{C}} s$, then $s_{n_k} \xrightarrow[k\to\infty]{\mathscr{C}} s$ for every subsequence $\{s_{n_k}\}$ of $\{s_n\}$.*

(L3) *If $s_n \not\xrightarrow[n\to\infty]{\mathscr{C}} s$, then $\{s_n\}$ contains a subsequence $\{s_{n_k}\}$ such that no subsequence of $\{s_{n_k}\}$ converges to s.*

*$\mathscr{C}$ is called a $\mathscr{S}^*$-*sequential convergence *or $\mathscr{S}^*$-*(sequential) limit operator *if additionally:*

(L4) *If $s_n \xrightarrow[n\to\infty]{\mathscr{C}} s$ and $s^n_m \xrightarrow[m\to\infty]{\mathscr{C}} s_n$ for each $n \in \mathbb{N}$, then there exist increasing sequences of positive integers $n_1, n_2, \dots$ and $m_1, m_2, \dots$ such that $s^{n_k}_{m_k} \xrightarrow[k\to\infty]{\mathscr{C}} s$.*

*The pair $(S, \mathscr{C})$ is called $\mathscr{L}^*$-*space *($\mathscr{S}^*$-*space *respectively).*

*For a subset $A \subset S$ of a $\mathscr{L}^*$-space we define the $\mathscr{C}$-*closure *$\overline{A}^{\mathscr{C}} \subset S$ by the convention $s \in \overline{A}^{\mathscr{C}}$ if and only if there is a sequence $\{s_n\}$ included in A $\mathscr{C}$-converging to s.*

Theorem A.6. *The $\mathscr{C}$-closure of an $\mathscr{L}^*$-sequential convergence $\mathscr{C}$ on a $\mathscr{L}^*$-space S fulfills the first three of the Kuratowski closure axioms (*(K1)–(K3)*).* (K4) *holds in addition if $\mathscr{C}$ is a $\mathscr{S}^*$-sequential convergence.*

*In an $\mathscr{S}^*$-space S with convergence $\mathscr{C}$ the topology τ generated by the $\mathscr{C}$-closure is T_1. τ-$\lim_{n\to\infty} s_n = s$ if and only if $s_n \xrightarrow[n\to\infty]{\mathscr{C}} s$, that is, convergence *a posteriori *is equivalent to the convergence *a priori.

Alternatively, if we impose the convention that a set $A \subset S$ is closed if and only if it contains all convergent sequences together with all their limits, this defines a T_1-topology with the property that convergence a priori is identical to convergence a posteriori even in the cases of an $\mathscr{L}^*$-space. If S is an $\mathscr{S}^*$-space this topology coincides with the one coming from the closure defined above.

A topology coming from an $\mathscr{L}^*$-convergence in the above sense is a sequential topology in the sense of Definition A.5. A topology coming from an $\mathscr{S}^*$-convergence in the above sense is a Fréchet-topology in the sense of Definition A.5.

Conversely, the usual convergence of sequences in a (topological) sequential space is an $\mathscr{L}^*$-convergence and the usual convergence of sequences in a (topological) Fréchet space is an $\mathscr{S}^*$-convergence.

Proof. Cf. [Eng89, Problems 1.7.18–1.7.20] and the references therein for the proof. □

If in Definition A.5 we replace sequences consistently by nets, we get the classical Definition of *convergence classes* as proposed by J. L. Kelley in [Kel75, p. 73 et seq.].

Definition A.7 (Convergence class)**.** *Let S be a set. We call a relation $\mathscr{C}$ between nets $\{s_\nu\}_{\nu\in\mathscr{N}}$ of members of S and points $s \in S$* convergence class *and denote it by $s_\nu \xrightarrow[\nu\in\mathscr{N}]{\mathscr{C}} s$ (in words, $\{s_\nu\}_{\nu\in\mathscr{N}}$ $\mathscr{C}$*-converges *to s), if it satisfies the following four conditions.*

(i) If $\{s_\nu\}_{\nu\in\mathscr{N}}$ is a net such that $s_\nu = s$ for each $\nu \in \mathscr{N}$, then $\{s_\nu\}_{\nu\in\mathscr{N}}$ $\mathscr{C}$-converges to s.

(ii) If $\{s_\nu\}_{\nu\in\mathscr{N}}$ $\mathscr{C}$-converges to s, so does every subnet of $\{s_\nu\}_{\nu\in\mathscr{N}}$.

(iii) If $\{s_\nu\}_{\nu\in\mathscr{N}}$ does not $\mathscr{C}$-converge to s, then there is a subnet of $\{s_\nu\}_{\nu\in\mathscr{N}}$, no subnet of which $\mathscr{C}$-converges to s.

(iv) Let $\mathscr{N}$ be a directed set, let $\mathscr{M}_\nu$ be a directed set for each $\nu \in \mathscr{N}$, let $\mathscr{K} := \mathscr{N} \times \bigtimes_{\nu\in\mathscr{N}} \mathscr{M}_\nu$ be the product directed set (with the product order *$(\nu, f) \succeq (\nu', f')$ iff $\nu \succeq \nu'$ and $f(\nu) \succeq f'(\nu)$ for all $\nu \in \mathscr{N}$). Suppose that $s_{\nu,\mu} \xrightarrow[\mu\in\mathscr{M}_\nu]{\mathscr{C}} s_\nu$ for each $\nu \in \mathscr{N}$ and $s_\nu \xrightarrow[\nu\in\mathscr{N}]{\mathscr{C}} s$. Then $s_{\nu,f(\nu)} \xrightarrow[(\nu,f)\in\mathscr{K}]{\mathscr{C}} s$.*

It is classical that a convergence coming from an arbitrary topology satisfies the above axioms (cf. [Kel75, Ch. 2, Theorem 4, p. 69 and p. 74]).

We quote the related result [Kel75, Ch. 2, Theorem 9, p.74] as follows. See also [Eng89, Exercises 1.6.B and 1.7.21].

Theorem A.8. *Let $\mathscr{C}$ be a convergence class for a set S, and for each subset A of S let $\overline{A}^{\mathscr{C}}$ be the set of all points s such that, for some net $\{s_\nu\}_{\nu\in\mathscr{N}}$ included in A, $\{s_\nu\}_{\nu\in\mathscr{N}}$ $\mathscr{C}$-converges to s. Then $\overline{}^{\mathscr{C}}$ is a Kuratowski closure operator, and $s_\nu \xrightarrow[\nu\in\mathscr{N}]{\mathscr{C}} s$ if and only if $\{s_\nu\}_{\nu\in\mathscr{N}}$ converges to s relative to the topology associated with $\overline{}^{\mathscr{C}}$; in other words, convergence a priori is equivalent to convergence a posteriori.*

B The geometry of Banach spaces

Let H be a Hilbert space, E or X be a Banach space. In the sequel we will use the notation $\mathscr{R}_H : H \to H^*$ for the *Riesz map* of H, which is a linear isometric isomorphism, and the notation $\iota_E : E \to E^{**}$ for the canonical linear isometry of E into its *bidual* E^{**} defined by $\iota_E : x \mapsto {}_{E^*}\langle \cdot, x\rangle_E$, which is onto if and only if E is reflexive. Clearly, ${}_{E^*}\langle f, x\rangle_E = {}_{E^{**}}\langle \iota_E(x), f\rangle_{E^*}$ for $f \in E^*, x \in E$.

B.1 Convexity and smoothness

Definition B.1. *A Banach space E is called* strictly convex *or* rotund *if, whenever $x, y \in E$ are linearly independent, then:*

$$\|x + y\|_E < \|x\|_E + \|y\|_E .$$

A Banach space E is called smooth, *if whenever $x, y \in E$ are linearly independent, then the function*

$$t \mapsto \|x + ty\|_E$$

is differentiable for all values of $t \in \mathbb{R}$.

Lemma B.2. *A Banach space E is smooth if and only if the map $x \mapsto \|x\|_E$ is Gâteaux differentiable for any $x \neq 0$.*

Proof. See [Bea85, Part 3, Ch. I, §2, Proposition 2]. □

Lemma B.3. *If a Banach space E is smooth, then the normalized duality map $J_E := \left\{ f \in E^* \;\middle|\; {}_{E^*}\langle f, x\rangle_E = \|f\|_{E^*}^2 = \|x\|_E^2 \right\}$ is single valued and norm-weak*-continuous at all points except zero.*

Proof. See [Bea85, Part 3, Ch. I, §2, Proposition 1]. □

Lemma B.4. *If E^* is smooth, E is strictly convex. If E^* is strictly convex, E is smooth.*

Proof. See See [Bea85, Part 3, Ch. I, §3, Proposition 1]. □

Definition B.5. *(i) A Banach space E is called* uniformly convex *(or* uniformly rotund*) if for each $\varepsilon \in (0, 2)$ there exists $\delta_E(\varepsilon) > 0$ for which*

$$\|x\|_E \leq 1,\ \|y\|_E \leq 1 \quad \textit{and} \quad \|x - y\|_E \geq \varepsilon \quad \textit{imply}$$
$$\left\| \frac{1}{2}(x + y) \right\|_E \leq 1 - \delta_E(\varepsilon).$$

The map $\varepsilon \mapsto \delta_E(\varepsilon)$ is called modulus of convexity *(*rotundity*).*

(ii) A Banach space E is called uniformly smooth *if for each $\varepsilon > 0$ the exists $\eta_E(\varepsilon) > 0$ for which*

$$\|x\|_E = 1 \quad \text{and} \quad \|y\|_E \le \eta_E(\varepsilon) \quad \text{always implies}$$
$$\|x+y\|_E + \|x-y\|_E < 2 + \varepsilon \|y\|_E .$$

The map $\varepsilon \mapsto \eta_E(\varepsilon)$ is called modulus of smoothness.

δ and η are unique up to asymptotic equivalence in zero.

Clearly uniform convexity implies strict convexity and uniform smoothness implies smoothness.

Lemma B.6. *A Banach space E is uniformly convex if and only if E^* is uniformly smooth. E is uniformly smooth if and only if E^* is uniformly convex.*

Proof. See [Bea85, Part 3, Ch. II, §2, Proposition 2]. □

If E is uniformly convex, we have $\eta_{E^*} \simeq \delta_E$ (where "$\simeq$" denotes asymptotic equivalence). If E is uniformly smooth, we have $\delta_{E^*} \simeq \eta_E$. Uniformly convex Banach spaces are reflexive. Therefore uniformly smooth Banach spaces are reflexive. See [BP86, Chapter 1, §2] or [Bea85, Part 3, Ch. II, §1, Proposition 6].

Uniform convexity of E is equivalent to following condition:
For any two sequences $\{x_n\}, \{y_n\} \subset E$ with $\|x_n\|_E = \|y_n\|_E = 1$ for each $n \in \mathbb{N}$ such that $\lim_n \|x_n + y_n\|_E = 2$ we have that $\lim_n \|x_n - y_n\|_E = 0$.

Lemma B.7 (Mazur)**.** *Let X be a normed space, $\{u_n\} \subset X$, $u \in X$. Let $u_n \rightharpoonup u$ weakly in X. Then there is a sequence of convex combinations $\{v_n\}$:*

$$v_n := \sum_{k=n}^{N_n} \lambda_k^{(n)} u_k, \quad \text{with} \quad \sum_{k=n}^{N_n} \lambda_k^{(n)} = 1, \quad \lambda_k^{(n)} \ge 0, \quad n \le k \le N_n, \tag{B.1}$$

such that $v_n \to u$ strongly, that is, in norm. Moreover, if $u_n \to u$ strongly, then any convex combination $\{v_n\}$ similar to (B.1) *converges also strongly to u. Also, if $\{a_n\}$ is a convergent sequence of real numbers, then*

$$\lim_n a_n \ge \lim_n \sum_{k=n}^{N_n} \lambda_k^{(n)} a_k. \tag{B.2}$$

Proof. See [ET76, Ch. I.1] or [Yos78, Ch. V.1, Theorem 2] for the first part. We shall verify (B.2) first. Clearly,

$$\lim_n a_n = \overline{\lim_n}\, a_n = \lim_n \sup_{k \ge n} a_k \ge \lim_n \sup_{N_n \ge k \ge n} a_k \ge \lim_n \sum_{k=n}^{N_n} \lambda_k^{(n)} a_k.$$

For the remaining part just note that $x \mapsto \|x - y\|_X$ is a convex function for all $y \in X$ and apply (B.2). □

B.2 Gauges and the duality map

Definition B.8. *We call a function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *a* gauge *if it is continuous, strictly increasing,* $\varphi(0) = 0$ *and* $\varphi(t) \to +\infty$ *as* $t \to +\infty$.

It is clear that a gauge φ is bijective, so there exists $\psi := \varphi^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(\psi(t)) = \psi(\varphi(t)) = t$ for all $t \geq 0$. One verifies that ψ is again a gauge.

Definition B.9. *For a normed space* X *with dual* X^* *the (multi-valued) map* $J_X : X \to X^*$,

$$J_X(x) := \left\{ f \in X^* \;\middle|\; {}_{X^*}\langle f, x\rangle_X = \|f\|_{X^*}^2 = \|x\|_X^2 \right\}$$

is called the normalized duality map.

For a gauge φ *the (multi-valued) map* $J_X^\varphi : X \to X^*$,

$$J_X^\varphi(x) := \left\{ f \in X^* \;\middle|\; {}_{X^*}\langle f, x\rangle_X = \|f\|_{X^*} \|x\|_X \,,\; \|f\|_{X^*} = \varphi(\|x\|_X) \right\}$$

is called the duality map *with* gauge φ.

Clearly $J_X = J_X^{\mathrm{Id}_{\mathbb{R}_+}}$. If H is a Hilbert space, J_H is the Riesz isometry. J_X is linear if and only if X is a Hilbert space. It is an easy application of the Hahn-Banach Theorem that $J_X^\varphi(x) \neq \varnothing$ for all $x \in X$. For the next proposition see [Sho97, Propositions 8.6,8.7,8.8].

Proposition B.10. *Let* E *be a Banach space and* φ *a gauge. Suppose that* E *is uniformly smooth with modulus of smoothness* η_E. *Then* J_E^φ *is single-valued, monotone, and uniformly continuous on bounded sets in* E *with modulus of continuity equal to* η. *More precisely,* J_E^φ *is a function,*

$${}_{E^*}\big\langle J_E^\varphi(x) - J_E^\varphi(y), x - y\big\rangle_E \geq \big(\varphi(\|x\|_E) - \varphi(\|y\|_E)\big)(\|x\|_E - \|y\|_E) \geq 0$$

and for any $\varepsilon \in (0, 2]$ *and* $M > 0$ *we have that*

$$\begin{aligned} &\|x\|_E \,, \|y\|_E \leq M \text{ and } \|x - y\|_E < \eta_E(\varepsilon) \\ &\text{imply} \quad \big\|J_E^\varphi(x) - J_E^\varphi(y)\big\|_{E^*} < \varepsilon \end{aligned}$$

In [DP71] it has been proved:

Proposition B.11. *Let* E *be a Banach space and* φ *a gauge. Let* $\psi := \varphi^{-1}$. *Suppose that* E *is uniformly convex with modulus of convexity* δ_E. *Then* J_E^φ *is strictly monotone, that is,*

$${}_{E^*}\big\langle J_E^\varphi(x) - J_E^\varphi(y), x - y\big\rangle_E > 0 \quad \text{if and only if} \quad x \neq y.$$

Then J_E^φ *is also invertible such that for* $(J_E^\varphi)^{-1} : E^* \to E$ *we have* $J_{E^*}^\psi = \iota_E \circ (J_E^\varphi)^{-1} : E^* \to E^{**}$ *and the above proposition holds for* $J_{E^*}^\psi$ *with* φ *replaced by* ψ *and* η *replaced by* δ.

One easily notices:

Lemma B.12. *Let φ be a gauge which is multiplicative, that is, $\varphi(ts) = \varphi(t)\varphi(s)$, $t, s \geq 0$. Then J_E^φ is homogeneous in a certain sense, namely,*

$$J_E^\varphi(\alpha x) = \varphi(\alpha) J_E^\varphi(x), \ \alpha \geq 0, \ x \in E.$$

Remark B.13. *$f \in J_E^\varphi$ if and only if*

$$_{E^*}\langle f, y - x\rangle_E \leq \Phi(\|y\|_E) - \Phi(\|x\|_E)$$

where $\Phi(t) := \int_0^t \varphi(s)\,\mathrm{d}s$. In other words, $f \in \partial\Phi(\|\cdot\|_E)$, where "$\partial$" denotes subgradient, *cf. Definition 2.27.*

If E is smooth, the map $x \mapsto \frac{1}{2}\|x\|_E^2$ is always Gâteaux differentiable and the Gâteaux differential is equal to the normalized duality map J_E (and to $\partial\frac{1}{2}\|\cdot\|_E^2$), which then is single valued.

Lemma B.14. *$J_E^\varphi : E \to 2^{E^*}$ is a maximal monotone operator (graph) as defined in 2.32.*

Proof. See [Phe89, Theorem 2.25] or [Zei90b, Proposition 32.21]. □

Lemma B.15. *In a reflexive Banach space E any $x^* \in E^*$ attains its norm on the unit ball, that is,*

$$_{E^*}\langle x^*, x\rangle_E = \|x^*\|_{E^*} \quad \textit{for some} \quad x \in E, \ \|x\|_E = 1.$$

Proof. This is an easy consequence of the Hahn–Banach Theorem. □

Theorem B.16 (Bishop–Phelps)**.** *Let E be a Banach space. Then the set of all functionals $x^* \in E^*$ which attain their norms on the unit ball, that is, which satisfy*

$$_{E^*}\langle x^*, x\rangle_E = \|x^*\|_{E^*} \quad \textit{for some} \quad x \in E, \ \|x\|_E = 1,$$

is strongly dense in E^*. Equivalently, the duality map J_E has strongly dense range.

Proof. See [Phe89, Theorem 3.22]. □

B.2.1 The L^p-case

Let $1 < p < \infty$, $q := p/(p-1)$ (so that $p^{-1} + q^{-1} = 1$) and $\varphi(t) := t^{p-1}$, $t \geq 0$. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. Set $E := L^p(\Omega, \mathcal{F}, \mu)$. It is well-known that $E^* = L^q(\Omega, \mathcal{F}, \mu)$ (Cf. [Yos78, Example IV.9.3]). $f \in J_{L^p}^\varphi(x)$ is characterized by

$$\int_\Omega f(\omega)x(\omega)\,\mu(\mathrm{d}\omega) = \|f\|_{L^q}\|x\|_{L^p} \quad \text{and} \quad \|f\|_{L^q} = \|x\|_{L^p}^{p-1},$$

which is equivalent to

$$f(\omega) \in |x(\omega)|^{p-1}\operatorname{sign}(x(\omega)), \quad \text{for } \mu\text{-a.e. } \omega \in \Omega,$$

for $f \in L^q(\Omega, \mathcal{F}, \mu)$, $x \in L^p(\Omega, \mathcal{F}, \mu)$. Likewise, $f \in J_{L^p}(x)$ (normalized duality map) if and only if

$$f(\omega) \in \frac{|x(\omega)|^{p-1} \operatorname{sign}(x(\omega))}{\|x\|_{L^p}^{p-2}} \quad \text{for } \mu\text{-a.e. } \omega \in \Omega.$$

One notices that we have a pointwise characterization of J_{L^p} regardless the underlying measure μ.

It is well-known that the dual space of $L^1(\Omega, \mathcal{F}, \mu)$ is $L^\infty(\Omega, \mathcal{F}, \mu)$. Let $J_{L^1} : L^1 \to 2^{L^\infty}$ be the normalized duality map. Let $x \in L^1$. Then for

$$f(\omega) := \|x\|_{L^1} \operatorname{sign}(x(\omega)), \quad \text{for } \mu\text{-a.e. } \omega \in \Omega,$$

clearly $f \in J_{L^1}(x)$.

B.3 The Kadeč-Klee property

Definition B.17. *A Banach space E is said to have the* Kadeč-Klee property *(or* property (h)*) if a sequence $\{x_n\} \subset E$ converges strongly to some $x \in E$ if and only if $\{x_n\}$ converges weakly to x and* $\lim_n \|x_n\|_E = \|x\|_E$.

Theorem B.18. *If E is locally uniformly convex, E has the Kadeč-Klee property.*

Proof. See [Mil71, Theorem 1.8]. □

B.4 Schauder bases

A standard reference on Schauder bases is given by [Sin70a]. For an introduction, see [HHZ96, Ch. 9]. All facts in this paragraph can be found there.

Definition B.19. *Let X be a Banach space with dual X^*. A subset $S \subset X^*$ is called* total *if ${}_{X^*}\langle s, e\rangle_X = 0$ for all $s \in S$ implies $e = 0$ or, equivalently, for any $e \in X \setminus \{0\}$ there is $s \in S$ such that ${}_{X^*}\langle s, e\rangle_X \neq 0$.*

A subset $S \subset X$ is called fundamental *in X if the finite linear combinations* $\operatorname{lin} S$ *are norm-dense in X.*

Lemma B.20. *If X is reflexive, a total set $S \subset X^*$ is fundamental in X^* by the Hahn-Banach Theorem. If X fails to be reflexive,* $\operatorname{lin} S$ *is still weak* dense in X^*, cf.* [HHZ96, Chapter 3, Exercise 16].

Definition B.21. *A sequence $\{e_i\}_{i\in\mathbb{N}}$ in an infinite-dimensional normed linear space X is called a* Schauder basis *of X if for every $x \in X$ there is a unique sequence of scalars $\{a_i\}_{i\in\mathbb{N}}$ such that*

$$\lim_{N\to\infty} \left\| x - \sum_{i=1}^{N} a_i e_i \right\|_X = 0.$$

Define the canonical projections

$$P_n(x) := \sum_{i=1}^{n} a_i e_i,$$

whenever $x = \sum_{i=1}^{\infty} a_i e_i$.

Lemma B.22. *A Banach space with a Schauder basis is separable.*

Lemma B.23. *If* $\{e_i\}$ *is a Schauder basis of a normed space* X*, then the canonical projections* $\{P_n\}$ *satisfy*

(i) $\dim P_n(X) = n$

(ii) $P_n P_m = P_m P_n = P_{m \wedge n}$

(iii) $\lim_n \|x - P_n x\|_X = 0$ *for every* $x \in X$,

$n, m \in \mathbb{N}$*. Conversely, if we are given a sequence of linear projections* $\{P_n\}$ *that satisfy (i), (ii) and (iii), then they are canonical projections associated with a Schauder basis of* X*. When* X *is a Banach space,* $\sup_{n\in\mathbb{N}} \|P_n\|_{\mathscr{L}(X,X)} < +\infty$.

Lemma B.24. *For a Schauder basis* $\{e_i\}_{i\in\mathbb{N}}$ *of a Banach space* X*, there is a unique sequence of continuous linear functionals* $\{e_i^*\}_{i\in\mathbb{N}} \subset X^*$ *such that*

$$x = \sum_{i=1}^{\infty} {}_{X^*}\langle e_i^*, x\rangle_X \, e_i.$$

$\{e_i^*\}_{i\in\mathbb{N}}$ *are called the* associated biorthogonal functionals *or the* associated coefficient functionals*.* Biorthogonal *means*

$${}_{X^*}\left\langle e_j^*, e_i \right\rangle_X = \delta_{i,j} \quad \forall i, j \in \mathbb{N}.$$

Lemma B.25. *If* X *is smooth and the Schauder basis is normalized, then* $e_i^* = Je_i$ *for every* $i \in \mathbb{N}$.

Lemma B.26. *When* X *is reflexive with a Schauder basis* $\{e_i\}_{i\in\mathbb{N}}$*, the coefficient functionals* $\{e_i^*\}_{i\in\mathbb{N}}$ *form a Schauder basis for* X^*.

Definition B.27. *A Schauder basis* $\{e_i\}_{i\in\mathbb{N}}$ *is called* monotone*, if* $\|P_n\|_{\mathscr{L}(X,X)} = 1$ *for every* $n \in \mathbb{N}$*, where* $\{P_n\}$ *are the associated canonical projections.*

A Schauder basis $\{e_i\}_{i\in\mathbb{N}}$ *is called* normalized*, when* $\|e_i\|_X = 1$ *for every* $i \in \mathbb{N}$.

A Schauder basis $\{e_i\}_{i\in\mathbb{N}}$ *is called* normal*, when* $\|e_i\|_X = \|e_i^*\|_{X^*} = 1$ *for every* $i \in \mathbb{N}$*, where* $\{e_i^*\}_{i\in\mathbb{N}}$ *is the sequence of associated coefficient functionals.*

B.5 Orthogonality in Banach space

Definition B.28 ([Lum61, Gil67]). *A* semi-inner product *(*s.i.p.*) on a vector space V is a real-valued bivariate form $[\cdot,\cdot]_V : V \times V \to \mathbb{R}$ satisfying:*

(i) $[x,x]_V \geq 0$ *for every* $x \in V$*;* $[x,x]_V = 0$ *if and only if* $x = 0$.

(ii) The map $x \mapsto [x,y]_V$ *is linear for each* $y \in V$.

(iii) The map $y \mapsto [x,y]_V$ *is* 1*-homogeneous for each* $x \in V$.

(iv) $|\,[x,y]_V|^2 \leq [x,x]_V \cdot [y,y]_V$ *for all* $x, y \in V$.

A s.i.p. is sometimes also called a semi-inner product in the sense of Lumer-Giles*.*

Let X be a vector space. Then a s.i.p. $[\cdot,\cdot]_X$ on X generates a norm by $\|\cdot\|_X := [\cdot,\cdot]_X{}^{1/2}$. Conversely, for each norm on X there is a s.i.p. generating it. Every s.i.p. $[\cdot,\cdot]_X$ which generates a norm $\|\cdot\|_X$ if of the form

$$[x,y]_X = {}_{X^*}\left\langle \widetilde{J}_X(y), x \right\rangle_X$$

where $\widetilde{J}_X$ is a selection of the normalized duality map J_X coming from the norm $\|\cdot\|_X$. See [Dra04, Chapter 2] for details. We infer that: if $(X, \|\cdot\|_X)$ is smooth, the s.i.p. generating $\|\cdot\|_X$ is unique.

Definition B.29 ([Bir35, Jam47]). *In a normed linear space X an element x is said to be* orthogonal *to an element y (written $x \perp y$) if*

$$\|x\|_X \leq \|x + \lambda y\|_X$$

for all scalars $\lambda \in \mathbb{R}$.

Two subsets $M, N \subset X$ are said to be orthogonal *(written $M \perp N$) if $m \perp n$ for all $m \in M$ and for all $n \in N$.*

Orthogonality is sometimes also called Birkhoff-James orthogonality*.*

Note that in general "$\perp$" is not a symmetric relation. Birkhoff-James orthogonality of a Hilbertian norm is the usual orthogonality coming from the inner product associated to this norm (cf. [Sin70b, Ch. I, §1.14]).

Lemma B.30. *Let X be a normed space. If $[\cdot,\cdot]$ generates the norm of X, then $[y,x]$ implies $x \perp y$.*

If $x \perp y$, then there is a s.i.p. $[\cdot,\cdot]$ generating the norm such that $[y,x] = 0$ (which may depend on x, y).

If X is smooth, $x \perp y$ if and only if $[y,x]_X = 0$ for the unique s.i.p. generating the norm.

Proof. See [FJ03, Propositions 1.4.3 and 1.4.4]. See also [Dra04, Proposition 32]. □

Lemma B.31. *Let X be a Banach space with a Schauder basis $\{e_i\}_{i\in\mathbb{N}}$. Then $\{e_i\}_{i\in\mathbb{N}}$ is monotone (see Definition B.27) if and only if $\operatorname{lin}(e_i)_{i=1}^n \perp \operatorname{lin}(e_i)_{i=n+1}^m$ for all $n, m \in \mathbb{N}$.*

Proof. See [Sin70a, Ch. II.I §1]. □

C Orlicz spaces

C.1 Young functions

Definition C.1. *A mapping* $\Phi : \mathbb{R} \to \mathbb{R}_+$ *is called a* Young function *if*

(i) Φ *is even and convex,*

(ii) $\Phi(0) = 0$*,*

(iii) $\lim_{x\to\infty} \Phi(x) = +\infty$*.*

Definition C.2. *A mapping* $\Phi : \mathbb{R} \to \mathbb{R}_+$ *is called an* N-function *(i.e., a* nice Young function*) if*

(i) Φ *is even, convex and continuous,*

(ii) $\Phi(x) = 0$ *if and only if* $x = 0$*,*

(iii) $\lim_{x\to 0} \frac{\Phi(x)}{x} = 0$*,* $\lim_{x\to\infty} \frac{\Phi(x)}{x} = +\infty$*.*

The *left derivative* $\varphi(t) := \lim_{h\downarrow 0} \frac{1}{h}(\Phi(t-h) - \Phi(t))$ exists and is left continuous, non-decreasing on $(0,\infty)$, satisfies $\varphi(t) \in (0,\infty)$ for $t \in (0,\infty)$, $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = +\infty$. The *left inverse* ψ of φ is defined as $\psi(s) := \inf\{t > 0 \mid \varphi(t) > s\}$ for $s > 0$, $\psi(0) := 0$. Such φ, ψ are also referred to as *generalized gauges.* Φ, Ψ given by

$$\Phi(x) = \int_0^{|x|} \varphi(t)\,\mathrm{d}t, \quad \text{and} \quad \Psi(y) = \int_0^{|y|} \psi(s)\,\mathrm{d}s$$

are called a pair of *complementary N-functions* which satisfy the *Young inequality*:

$$|xy| \le \Phi(x) + \Psi(y).$$

The N-function Ψ complementary to Φ can be equally defined via the so-called *Legendre-Fenchel transform*:

$$\Psi(y) := \sup_{x\ge 0} \left[x|y| - \Phi(x)\right], \quad y \in \mathbb{R},$$

cf. Paragraph 2.4.3. Instead of complement we shall from time to time use the term *conjugate.*

The standard example of an N-function is $x \mapsto \frac{1}{p}|x|^p$, $1 < p < \infty$ with gauge $t \mapsto t^{p-1}$ and conjugate $y \mapsto \frac{1}{q}|y|^q$ with gauge $s \mapsto s^{q-1}$ where $q := \frac{p}{p-1}$.

Definition C.3. *Φ is said to belong globally to the class Δ_2 (written $\Phi \in \Delta_2$) if it is an N-function and if there is a constant $K > 2$ such that*

$$\Phi(2x) \leq K\Phi(x) \quad \textit{for all} \quad x \geq 0;$$

and it is said to belong globally to the class ∇_2 (written $\Phi \in \nabla_2$) if it is an N-function and there is a constant $c > 1$ such that

$$\Phi(x) \leq \frac{1}{2c}\Phi(cx) \quad \textit{for all} \quad x \geq 0.$$

Lemma C.4. *Let (Φ, Ψ) be complementary N-functions with gauges (φ, ψ) respectively. Then the following statements are equivalent:*

(i) $\Phi \in \Delta_2$;

(ii) $\sup_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \infty$;

(iii) $\inf_{s>0} \frac{s\psi(s)}{\Psi(s)} > 1$;

(iv) $\Psi \in \nabla_2$.

Proof. See [RR91, Chapter II, Theorem 3, p. 23]. □

Lemma C.5. *If for an N-function $\Phi \in \Delta_2$ globally, there are constants $\alpha, \beta > 1$, $C, D > 0$ such that $\Phi(x) \leq C|x|^\alpha$ for $x \geq 0$ and $\Psi(y) \geq D|y|^\beta$ for $y \geq 0$, where Ψ denotes the complementary N-function to Φ.*

Proof. See [RR91, Chapter II, Corollary 5, p. 26]. □

Definition C.6. *An N-function Φ is said to belong globally to the class Δ' (written $\Phi \in \Delta'$) if there is a constant $a > 0$ such that*

$$\Phi(xy) \leq a\Phi(x)\Phi(y) \quad \textit{for all} \quad x, y \geq 0;$$

and it is said to to belong globally to the class ∇' (written $\Phi \in \nabla'$) if there is a constant $b > 0$ such that

$$\Phi(x)\Phi(y) \leq \Phi(bxy) \quad \textit{for all} \quad x, y \geq 0.$$

Lemma C.7. *An N-function $\Phi \in \Delta'$ if and only if there is a constant $a' > 0$ such that*

$$\Phi(axy) \leq \Phi(x)\Phi(y) \quad \textit{for all} \quad x, y \geq 0.$$

In particular, $\Delta' \subset \Delta_2$.

Proof. See [RR91, Chapter II, Lemma 8, p. 28]. □

Lemma C.8. *Let (Φ, Ψ) be complementary N-functions with gauges (φ, ψ) respectively. Then the following statements are equivalent:*

(i) $\Phi \in \Delta'$;

(ii) *there is a constant* $D > 0$ *such that* $\varphi(xy) \leq D\varphi(x)\varphi(y)$ *for all* $x, y \geq 0$;

(iii) *there is a constant* $d > 0$ *such that* $\psi(x)\psi(y) \leq \psi(dxy)$ *for all* $x, y \geq 0$;

(iv) $\Psi \in \nabla'$.

In particular, $\nabla' \subset \nabla_2$.

Proof. See [RR91, Chapter II, Theorem 11, p. 30]. □

Lemma C.9. *For an* N*-function* $\Phi \in \Delta' \cap \nabla'$ *there are* $p > 1$ *and* $c, C > 0$ *such that*

$$\Phi(cx) \leq |x|^p \leq \Phi(Cx) \quad \textit{for all} \quad x \geq 0.$$

Proof. See [RR91, Chapter II, Proposition 12, p. 31]. □

For the next definition see [Boy69].

Definition C.10. *For a Young function* Φ *define the* upper index α *and* lower index β *by*

$$\alpha := \inf_{0<s<1} -\frac{\log h(s)}{\log s} = \lim_{s \searrow 0} -\frac{\log h(s)}{\log s},$$

and

$$\beta := \sup_{1<s<\infty} -\frac{\log h(s)}{\log s} = \lim_{s\to\infty} -\frac{\log h(s)}{\log s},$$

where

$$h(s) := \sup_{t>0} \frac{\Phi^{-1}(t)}{\Phi^{-1}(st)}.$$

A simple calculation shows that if $\Phi(x) = C|x|^p$ for some $1 \leq p < \infty$ and some $C > 0$, then both the upper and lower index of Φ is p^{-1}.

Another characterization for indices of Orlicz spaces, more common in the literature nowadays, can be found in [LT79, Chapter 2.b, Proposition 2.b.5]. See also [Boy71].

C.2 Orlicz spaces with general measures

Let $(\Omega, \mathscr{F}, \mu)$ be a σ-finite measure space.

Definition C.11. *Let* Φ *be a Young function. Let* $\widetilde{\mathscr{L}}^\Phi(\Omega; \mu)$ *be the set of all* $\mathscr{F}$*-measurable functions* $f : \Omega \to \overline{\mathbb{R}}$ *such that*

$$\rho_{\Phi,\mu}(f) := \int_\Omega \Phi(|f|)\,\mathrm{d}\mu < \infty.$$

$\rho_{\Phi,\mu}$ *is called the* modular. *Let* $\mathscr{L}^\Phi(\Omega; \mu)$ *be the set of all* $\mathscr{F}$*-measurable functions* $f : \Omega \to \overline{\mathbb{R}}$ *such that* $\alpha f \in \widetilde{\mathscr{L}}^\Phi(\Omega; \mu)$ *for some* $\alpha > 0$*, that is,* $\rho_{\Phi,\mu}(\alpha f) < \infty$ *for some* $\alpha > 0$*. Let* $\widetilde{L}^\Phi(\Omega; \mu)$*,* $L^\Phi(\Omega; \mu)$ *be the sets of equivalence classes of* μ*-a.e. identical functions from* $\widetilde{\mathscr{L}}^\Phi(\Omega; \mu)$*,* $\mathscr{L}^\Phi(\Omega; \mu)$ *resp.* $L^\Phi(\Omega; \mu)$ *is called the* Orlicz space *w.r.t. the measure space* $(\Omega, \mathscr{F}, \mu)$ *and the* N*-function* Φ*. We shall also write* $L^\Phi(\mu)$.

Note that $\Phi(|\cdot|) : \mathbb{R} \to \mathbb{R}_+$ is Borel-measurable as it is continuous. If $\Phi(x) := |x|^p$ for $1 \le p < \infty$ (or $\Phi(x) = \frac{1}{p}|x|^p$), then $L^\Phi(\mu) = L^p(\mu)$.

Lemma C.12. *If $\Phi \in \Delta_2$, then $\widetilde{\mathscr{L}}^\Phi(\Omega;\mu) = \mathscr{L}^\Phi(\Omega;\mu)$ and $\widetilde{L}^\Phi(\Omega;\mu) = L^\Phi(\Omega;\mu)$.*

Lemma C.13. *Suppose that Φ is an N-function with complement Ψ. Then $L^\Phi(\mu)$ is a Banach space with either norm:*

$$\|f\|_{(\Phi,\mu)} := \inf\left\{k > 0 \;\middle|\; \rho_{\Phi,\mu}\left(\frac{f}{k}\right) \le 1\right\}, \quad f \in L^\Phi(\mu),$$

which is called the Luxemburg norm *or* gauge norm*;*

$$\|f\|_{\Phi,\mu} := \sup\left\{\left|\int_\Omega fg \,\mathrm{d}\mu\right| \;\middle|\; \rho_{\Psi,\mu}(g) \le 1\right\}, \quad f \in L^\Phi(\mu),$$

which is called the Orlicz norm*. Moreover, for every $f \in L^\Phi(\mu)$*

$$\|f\|_{(\Phi,\mu)} \le \|f\|_{\Phi,\mu} \le 2\,\|f\|_{(\Phi,\mu)}\,,$$

that is, the norms are equivalent.

Lemma C.14. *For any $f \in L^\Phi(\mu)$,*

$$\|f\|_{\Phi,\mu} = \inf_{k>0} \frac{1}{k}\left[1 + \rho_{\Phi,\mu}(k|f|)\right]$$

and the infimum is attained at some (not necessarily unique) k_f such that

$$\inf\left\{k > 0 \mid \rho_{\Psi,\mu}(\varphi(k|f|)) \ge 1\right\} \le k_f \le \sup\left\{k > 0 \mid \rho_{\Psi,\mu}(\varphi(k|f|)) \le 1\right\}.$$

In particular, if $\|f\|_{\Phi,\mu} = 1$, then a solution $k > 1$ of

$$k - 1 = \rho_{\Phi,\mu}(k|f|)$$

is such a real number k_f.

Proof. See [RR91, Ch. III.3, Theorem 13, p. 69]. □

Theorem C.15 (Hölder inequality)**.** *Let $f \in L^\Phi(\mu)$, $g \in L^\Psi(\mu)$. Then $fg \in L^1(\mu)$ and*

$$\left|\int_\Omega fg\,\mathrm{d}\mu\right| \le \|f\|_{(\Phi,\mu)}\,\|g\|_{\Psi,\mu}\,, \qquad \left|\int_\Omega fg\,\mathrm{d}\mu\right| \le \|f\|_{\Phi,\mu}\,\|g\|_{(\Psi,\mu)}\,. \tag{C.1}$$

Proof. See [RR02, Ch. I.2, Theorem 8, p. 17]. □

Lemma C.16. *Suppose that $\Phi \in \Delta_2$. Suppose that we have a sequence $\{f_n\}$ of functions in $L^\Phi(\Omega;\mu)$. Then $\{f_n\}$ is a Cauchy sequence in $\|\cdot\|_{\Phi,\mu}$-norm (or $\|\cdot\|_{(\Phi,\mu)}$-norm) if and only if*

$$\rho_{\Phi,\mu}(f_n - f_m) \to 0 \quad \text{as } n,m \to \infty$$

which we call "to be a Cauchy sequence in mean".

Suppose that $f \in L^\Phi(\Omega;\mu)$ a priori. Then

$$\rho_{\Phi,\mu}(f_n - f) \to 0 \quad \text{as } n \to \infty$$

(which we call "to converge (strongly) in mean"*) if and only if f_n converges to f in either norm.*

Note that if $\Phi \notin \Delta_2$, norm convergence is stronger than mean convergence.

Proof. Compare [RR91, Ch. III, Theorem 12, p. 83].

Let $\|f_n - f_m\|_{\Phi,\mu} \to 0$ as $n,m \to \infty$. For each $k > 0$ there is an index $n_0 \in \mathbb{N}$ such that $k\,\|f_n - f_m\|_{\Phi,\mu} \le 1$ for all $n,m \ge n_0$. Then

$$\begin{aligned}\rho_{\Phi,\mu}(k(f_n - f_m)) &= \int_\Omega \Phi(k(f_n - f_m))\,\mathrm{d}\mu \\ &\le k\,\|f_n - f_m\|_{\Phi,\mu} \int_\Omega \Phi\left(\frac{f_n - f_m}{\|f_n - f_m\|_{\Phi,\mu}}\right)\mathrm{d}\mu \le k\,\|f_n - f_m\|_{\Phi,\mu} \to 0\end{aligned}$$

as $n,m \to \infty$. Since $k(f_n - f_m) \in L^\Phi(\Omega;\mu)$ for each k, $\{f_n\}$ is a Cauchy sequence in mean.

Conversely, let $\{f_n\}$ be a Cauchy sequence in mean. Let $\varepsilon > 0$. Since $\Phi \in \Delta_2$ there is a constant $K > 2$ with $\Phi(2x) \le K\Phi(x)$; therefore $\Phi(x/\varepsilon) \le K^s\Phi(x)$ where $s \ge -\log_2(\varepsilon)$. Set $K_\varepsilon := K^s$. Hence

$$\rho_{\Phi,\mu}\left(\frac{f_n - f_m}{\varepsilon}\right) = \int_\Omega \Phi\left(\frac{f_n - f_m}{\varepsilon}\right)\mathrm{d}\mu \le K_\varepsilon \int_\Omega \Phi(f_n - f_m)\,\mathrm{d}\mu.$$

There is an index $n_0 \in \mathbb{N}$ such that $K_\varepsilon \rho_{\Phi,\mu}(f_n - f_m) \le 1$ for all $n,m \ge n_0$. Hence $\rho_{\Phi,\mu}\left(\frac{f_n - f_m}{\varepsilon}\right) \le 1$ for $n,m \ge n_0$ so that $\|f_n - f_m\|_{\Phi,\mu} \le \varepsilon$ for $n,m \ge n_0$, which proves the assertion.

The case of strong convergence works similarly. $\square$

Lemma C.17. *Suppose that $\Phi \in \Delta_2$. Let $f, g_n \in L^\Phi(\Omega;\mu)$, $n \in \mathbb{N}$. Then $\rho_{\Phi,\mu}(g_n) \to 0$ as $n \to \infty$ implies $\rho_{\Phi,\mu}(f + g_n) \to \rho_{\Phi,\mu}(f)$ as $n \to \infty$. Moreover, a subset $S \subset L^\Phi(\Omega;\mu)$ is modular bounded if and only if it is norm bounded in either norm.*

Proof. We prove only the second part. Compare [RR91, Ch. III, Corollary 15, p. 86] for the first part.

Suppose that $\sup_{f\in S}\rho_{\Phi,\mu}(f)\le K$ for some $K\ge 1$. Then for any $f\in S$, by convexity of Φ and $\Phi(0)=0$,

$$1\ge \frac{1}{K}\int_\Omega \Phi(f)\,\mathrm{d}\mu \ge \int_\Omega \Phi\left(\frac{f}{K}\right)\,\mathrm{d}\mu, \tag{C.2}$$

which implies $\|f\|_{(\Phi,\mu)}\le K$ and $\|f\|_{\Phi,\mu}\le 2K$.

Conversely, if $\sup_{f\in S}\|f\|_{(\Phi,\mu)}\le K$ and S is not modular bounded, then we can find $f_n\in S$, $n\in\mathbb{N}$ such that $\rho_{\Phi,\mu}(f_n)\ge n$, $n\in\mathbb{N}$. So (C.2) becomes, for $n\ge n_0\ge 1\vee K_0$,

$$\rho_{\Phi,\mu}(f_n)=\int_\Omega \Phi\left(\frac{n_0 f_n}{n_0}\right)\,\mathrm{d}\mu \le C_{n_0}\int_\Omega \Phi\left(\frac{f_n}{n_0}\right)\,\mathrm{d}\mu \le C_{n_0},$$

by the Δ_2-condition. But this is a contradiction. If $\sup_{f\in S}\|f\|_{\Phi,\mu}\le K$ then also $\sup_{f\in S}\|f\|_{(\Phi,\mu)}\le K$. □

Lemma C.18. *Suppose that $\Phi\in\Delta_2$. Then $L^\Phi(\mu)$ is separable if and only if $(\Omega,\mathscr{F},\mu)$ is separable.*

Proof. See [RR91, Ch. III, Theorem 1, p. 87]. □

Lemma C.19. *Suppose that $\Phi\in\Delta_2$ and $(\Omega,\mathscr{F},\mu)$ is σ-finite. Then $(L^\Phi(\mu))^*=L^\Psi(\mu)$ with dualization*

$$_{L^\Phi}\langle f,g\rangle_{L^\Psi}=\int_\Omega fg\,\mathrm{d}\mu$$

the functional norms are given by Hölder's inequality.

Proof. See [RR91, Ch. IV, Theorem 7, Corollary 9, pp. 110-111]. □

Lemma C.20. *Suppose that $\Phi\in\Delta_2\cap\nabla_2$. Then $L^\Phi(\mu)$ (and $L^\Psi(\mu)$) is reflexive.*

Proof. See [RR91, Ch. IV, Theorem 10, p. 112]. □

For the following we extend $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ to an odd function from $\mathbb{R}$ onto $\mathbb{R}$ by defining $t\mapsto\varphi(|t|)\,\mathrm{sign}(t)$.

Lemma C.21. *Let $\Phi\in\Delta_2\cap\nabla_2$. Let $f\in L^\Phi(\mu)$. The map $f\mapsto\varphi(f)$ maps the $\|\cdot\|_{\Phi,\mu}$-unit ball of $L^\Phi(\mu)$ into the $\|\cdot\|_{(\Psi,\mu)}$-unit ball of $L^\Psi(\mu)$. More generally, $\varphi(L^\Phi(\mu))\subset L^\Psi(\mu)$.*

Proof. See [RR91, Ch. VII, Proposition 1, p. 265]. □

Lemma C.22. *Let $\Phi\in\Delta_2\cap\nabla_2$. Suppose that Φ is continuous with continuous derivative $\varphi=\Phi'$ such that $\varphi(t)>0$ if $t>0$. Then the map $\|\cdot\|_{(\Phi,\mu)}:f\mapsto\|f\|$ is Gâteaux differentiable at each point of the $\|\cdot\|_{(\Phi,\mu)}$-unit sphere with*

$$_{L^\Psi(\mu)}\left\langle\left(\nabla_G\|\cdot\|_{(\Phi,\mu)}\right)(g),f\right\rangle_{L^\Phi(\mu)}=\int_\Omega f\varphi(g)\,\mu(\mathrm{d}x)$$

for all $f,g\in L^\Phi(\mu)$, $\|f\|_{(\Phi,\mu)}=\|g\|_{(\Phi,\mu)}=1$.

Proof. See [RR91, Ch. VII, Theorem 2, p. 278]. □

Lemma C.23. *Let $\Phi \in \Delta_2 \cap \nabla_2$. Suppose that Φ is continuous with continuous derivative $\varphi = \Phi'$ such that $\varphi(t) > 0$ if $t > 0$. Then the map $\|\cdot\|_{\Phi,\mu} : f \mapsto \|f\|$ is Gâteaux differentiable at each point of the $\|\cdot\|_{\Phi,\mu}$-unit sphere with*

$$_{L^\Psi(\mu)}\Big\langle \left(\nabla_G \|\cdot\|_{\Phi,\mu}\right)(g), f \Big\rangle_{L^\Phi(\mu)} = \int_\Omega f \varphi(kg)\, \mu(\mathrm{d}x)$$

for all $f, g \in L^\Phi(\mu)$, $\|g\|_{\Phi,\mu} = 1$ such that $k > 1$ solves

$$k - 1 = \int_\Omega \Phi(kg)\, \mu(\mathrm{d}x).$$

For such k,

$$\int_\Omega \Psi\left(\varphi(k|g|)\right)\, \mu(\mathrm{d}x) = 1.$$

Proof. See [RR91, Ch. VII, Theorem 5, p. 281] and [Che96, Theorem 2.51]. □

Lemma C.24. *The Orlicz spaces $(L^\Phi(\mu), \|\cdot\|_{(\Phi,\mu)})$ and $(L^\Phi(\mu), \|\cdot\|_{\Phi,\mu})$ are uniformly convex if for each $\varepsilon > 0$, there is $k_\varepsilon > 1$ such that $\varphi((1+\varepsilon)t) \geq k_\varepsilon \varphi(t)$, $t \geq 0$, and if Φ is Δ_2-regular and strictly convex.*

Proof. See [RR91, Ch. VII, Theorem 8, p. 288, Theorem 10, p. 293]. □

Bibliography

[AAW88] D. Azé, H. Attouch, and R. J.-B. Wets, *Convergence of convex-concave saddle functions: applications to convex programming and mechanics*, Ann. Inst. H. Poincaré, Section C, **5** (1988), no. 6, 537–572.

[AB93] H. Attouch and G. Beer, *On the convergence of subdifferentials of convex functions*, Arch. Math. (Basel) **60** (1993), no. 4, 389–400.

[AC84] J.-P. Aubin and A. Cellina, *Differential inclusions: set-valued maps and viability theory*, Grundlehren der mathematischen Wissenschaften, vol. 264, Springer-Verlag, Berlin–Heidelberg–New York, 1984.

[Ada75] R. A. Adams, *Sobolev spaces*, Academic Press, New York–San Francisco–London, 1975.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Clarendon Press, Oxford University Press, 2000.

[AG65] N. Aronszajn and E. Gagliardo, *Interpolation spaces and interpolation methods*, Ann. Mat. Pura Appl. **68** (1965), no. 4, 51–117.

[AK06] G. Aubert and P. Kornprobst, *Mathematical problems in image processing, partial differential equations and the calculus of variations*, 2nd ed., Applied mathematical sciences, vol. 147, Springer, Berlin–Heidelberg–New York, 2006.

[AK09] D. Aalto and J. Kinnunen, *Maximal functions in Sobolev spaces*, Sobolev spaces in mathematics I, Sobolev type inequalities (V. Maz'ya, ed.), International Mathematical Series, vol. 8, Springer and Tamara Rozhkovskaya Publisher, 2009, pp. 25–67.

[AKR90] S. Albeverio, S. Kusuoka, and M. Röckner, *On partial integration in infinite-dimensional space and applications to Dirichlet forms*, J. London Math. Soc. (2) **42** (1990), no. 1, 122–136.

[AKS86] S. Albeverio, S. Kusuoka, and L. Streit, *Convergence of Dirichlet forms and associated Schrödinger operators*, J. Funct. Anal. **68** (1986), 130–148.

[All92] G. Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal. **23** (1992), no. 6, 1482–1518.

[Alt06] H. W. Alt, *Lineare Funktionalanalysis, Eine anwendungsorientierte Einführung*, 5. revised ed., Springer-Verlag, Berlin–Heidelberg–New York, 2006, DOI: 10.1007/3-540-34187-0.

[And66] T. Ando, *Contractive projections in L^p spaces*, Pacific J. Math. **17** (1966), no. 3, 391–405.

[AR89] S. Albeverio and M. Röckner, *Classical Dirichlet forms on topological vector spaces - construction of an associated diffusion process*, Probab. Th. Rel. Fields **83** (1989), 405–434.

[AR90a] ———, *Classical Dirichlet forms on topological vector spaces - closability and a Cameron-Martin formula*, J. Funct. Anal. **88** (1990), no. 2, 395–436.

[AR90b] ———, *New developments in the theory and application of Dirichlet forms*, Stochastic processes, physics and geometry (Ascona and Locarno, 1988), World Sci. Publ., Teaneck, NJ, 1990, pp. 27–76.

[Aro86] D. G. Aronson, *The porous medium equation*, Nonlinear diffusion problems (Montecatini Terme, 1985), Lecture notes in mathematics, vol. 1224, Springer-Verlag, Berlin–Heidelberg–New York, 1986, pp. 1–46.

[ARZ93a] S. Albeverio, M. Röckner, and T.-S. Zhang, *Markov uniqueness and its applications to martingale problems, stochastic differential equations and stochastic quantization*, C. R. Math. Rep. Acad. Sci. Canada **15** (1993), no. 1, 1–6.

[ARZ93b] ———, *Markov uniqueness for a class of infinite dimensional Dirichlet operators*, Stochastic processes and optimal control, Stochastic Monographs 7, 1993, pp. 1–26.

[Att77] H. Attouch, *Convergence de fonctions convexes, des sous différentiels et semigroupes associés*, C. R. Acad. Sci. Paris, Sér. A, **284** (1977), no. 10, 539–542.

[Att84] ———, *Variational convergence for functions and operators*, Pitman, Boston–London–Melbourne, 1984.

[AW83a] H. Attouch and R. J.-B. Wets, *A convergence for bivariate functions aimed at the convergence of saddle values*, Mathematical theories of optimization, Lecture notes in mathematics, vol. 979, Springer-Verlag, Berlin–Heidelberg–New York, 1983, pp. 1–42.

[AW83b] ———, *A convergence theory for saddle functions*, Trans. Amer. Math. Soc. **280** (1983), no. 1, 1–41.

[Ban32] S. Banach, *Théorie des opérations linéaires*, Chelsea, Warsaw, 1932.

[Bar76] V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, revised and enlarged ed., Mathematics and its applications, East European series, Editura Academiei, Bucureşti, Noordhoff International Publishing, Leyden, 1976.

[Bau74] H. Bauer, *Wahrscheinlichkeitstheorie und Grundzüge der Maßtheorie*, 2. ed., de Gruyter, Berlin–New York, 1974.

[BBI01] D. Burago, J. D. Burago, and S. Ivanov, *A course in metric geometry*, Graduate studies in mathematics, vol. 33, Amer. Math. Soc., 2001.

[BDF91] E. Barvínek, I. Daler, and J. Franců, *Convergence of sequences of inverse functions*, Arch. Math. (Brno) **27** (1991), no. 2, 201–204.

[BDPR09] V. Barbu, G. Da Prato, and M. Röckner, *Stochastic nonlinear diffusion equations with singular diffusivity*, SIAM J. Math. Anal. **41** (2009), no. 3, 1106–1120.

[Bea85] B. Beauzamy, *Introduction to Banach spaces and their geometry*, 2nd revised ed., North–Holland Mathematics Studies 68, Notas de Matemática 86, North–Holland, Amsterdam–New York–Oxford, 1985.

[Bee92] G. Beer, *The slice topology: a viable alternative to Mosco convergence in nonreflexive spaces*, Nonlinear Anal. **19** (1992), no. 3, 271–290.

[BF89] J. M. Borwein and S. Fitzpatrick, *Mosco convergence and the Kadec property*, Proc. Amer. Math. Soc. **3** (1989), no. 106, 843–851.

[BH84] N. Bouleau and F. Hirsch, *Propriétés d'absolue continuité dans les espaces de Dirichlet et application aux équations différentielles stochastiques*, Séminaire de Probabilités **20** (1984), 131–161, Lecture Notes in Math., 1204, Springer, Berlin, 1986.

[Bir35] G. Birkhoff, *Orthogonality in linear metric spaces*, Duke Math. J. **1** (1935), no. 2, 169–172.

[BK94] S. Bloom and R. A. Kerman, *Weighted L_Φ integral inequalities for operators of Hardy type*, Studia Math. **110** (1994), no. 1, 35–52.

[BL74] S. J. Bernau and H. E. Lacey, *The range of a contractive projection on an L^p-space*, Pacific J. Math. **53** (1974), no. 1, 21–41.

[Bog07a] V. I. Bogachev, *Measure theory*, vol. 1, Springer-Verlag, Berlin–Heidelberg–New York, 2007.

[Bog07b] ———, *Measure theory*, vol. 2, Springer-Verlag, Berlin–Heidelberg–New York, 2007.

[Boy69] D. W. Boyd, *Indices of function spaces and their relationship to interpolation*, Canad. J. Math. **21** (1969), 1245–1254.

[Boy71] ______, *Indices for the Orlicz spaces*, Pacific J. Math. **38** (1971), no. 2, 315–323.

[BP72] H. Brézis and A. Pazy, *Convergence and approximation of semigroups of nonlinear operators in Banach spaces*, J. Funct. Anal. **9** (1972), 63–74.

[BP86] V. Barbu and Th. Precupanu, *Convexity and optimization in Banach spaces*, 2nd revised and extended ed., Mathematics and its applications, East European series, Editura Academiei, Bucureşti, 1986.

[BR65] A. Brøndsted and R. T. Rockafellar, *On the subdifferentiability of convex functions*, Proc. Amer. Math. Soc. **16** (1965), no. 4, 605–611.

[Bré73] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Publ. Co., 1973.

[Bré75] ______, *New results concerning monotone operators and nonlinear semigroups*, Functional analysis and numerical analysis for nonlinear problems, Proc. Sympos., Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1974, no. 258, Sûrikaisekikenkyûsho Kókyûroku, 1975, pp. 2–27.

[Bro76] F. E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces*, Proceedings of symposia in pure mathematics, vol. XVIII, Part 2, Amer. Math. Soc., Providence, Rhode Island, 1976.

[Cal76] O. Caligaris, *Sulla convergenza delle soluzioni di problemi di evoluzione associati a sottodifferenziali di funzioni convesse*, Boll. Un. Mat. Ital. (5) **13-B** (1976), 778–806.

[Cap02] O. Caps, *Evolution equations in scales of Banach spaces*, Teubner–Texte zur Mathematik, vol. 140, Teubner, Stuttgart–Leipzig–Wiesbaden, 2002.

[CF96] P. Cattiaux and M. Fradon, *Entropy, reversible diffusion processes, and Markov uniqueness*, J. Funct. Anal. **138** (1996), no. 1, 243–272.

[Che96] S. T. Chen, *Geometry of Orlicz spaces*, Dissertationes Mathematicæ (Rozprawy Matematyczne), vol. 356, Polish Academy of Sciences, Warsaw, 1996.

[Cho69] G. Choquet, *Lectures on analysis I*, Benjamin, London, 1969.

[Cio90] I. Ciorănescu, *Geometry of Banach spaces, duality mappings and nonlinear problems*, Mathematics and its applications, vol. 62, Kluwer Academic Publishers, 1990.

[Cio09] I. Ciotir, *A Trotter type result for the stochastic porous media equations*, Nonlinear Anal. **71** (2009), no. 11, 5606–5615.

[Cio10] ______, *A Trotter type theorem for nonlinear stochastic equations in variational formulation and homogenization*, Preprint, to appear in Differential Integral Equations (2010).

[Con90] J. B. Conway, *A course in functional analysis*, 2nd ed., Graduate Texts in Mathematics, vol. 96, Springer, Berlin–Heidelberg–New York, 1990.

[CT98] C. Combari and L. Thibault, *On the graph convergence of subdifferentials of convex functions*, Proc. Amer. Math. Soc. **126** (1998), no. 8, 2231–2240.

[Dei85] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin–Heidelberg–New York, 1985.

[DG77] E. De Giorgi, *Γ-convergenza e G-convergenza*, Boll. Un. Mat. Ital. (5) **14-A** (1977), 213–220.

[Día85] J. I. Díaz, *Nonlinear partial differential equations and free boundaries: Elliptic equations*, Research Notes in Mathematics, vol. 106, Pitman Advanced Publ. Program, Boston, 1985.

[DL54] J. Deny and J. L. Lions, *Les espaces du type Beppo Levi*, Ann. Inst. Fourier, (Grenoble) **5** (1953–54), 305–370.

[DM78] C. Dellacherie and P.-A. Meyer, *Probabilities and potential A*, North-Holland mathematics studies 29, Hermann, Paris, 1978.

[DM93] G. Dal Maso, *An introduction to Γ-convergence*, Progress in nonlinear differential equations and their applications, vol. 8, Birkäuser, Boston–Basel–Berlin, 1993.

[Doo53] J. L. Doob, *Stochastic processes*, Wiley, New York–London–Sydney, 1953.

[DP71] C. R. DePrima and W. V. Petryshyn, *Remarks on strict monotonicity and surjectivity properties of duality mappings defined on real normed linear spaces*, Math. Z. **123** (1971), 49–55.

[Dra04] S. S. Dragomir, *Semi-inner products and applications*, Nova Science Publishers, Inc., New York, 2004.

[DS57] N. Dunford and J. T. Schwartz, *Linear operators: 1. General theory*, Pure and applied mathematics, vol. VII, Interscience Publ., New York, 1957.

[Ebe99] A. Eberle, *Uniqueness and non-uniqueness of semigroups generated by singular diffusion operators*, Lecture Notes in Mathematics, vol. 1718, Springer, Berlin–Heidelberg–New York, 1999.

[EG92] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, 1992.

[Eng89] R. Engelking, *General topology*, revised and completed ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag Berlin, 1989.

[ET76] I. Ekeland and R. Temam, *Convex analysis and variational problems*, Studies in Mathematics and its Applications, vol. 1, North-Holland, American Elsevier, 1976.

[Eva98] L. C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, 1998.

[FJ03] R. J. Fleming and J. E. Jamison, *Isometries on Banach spaces: function spaces*, Monographs and Surveys in Pure and Applied Mathematics, vol. 129, Chapman & Hall/CRC, 2003.

[FOT94] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet forms and symmetric Markov processes*, de Gruyter, Berlin–New York, 1994.

[Fra97] M. Fradon, *Diffusions réfléchies réversibles dégénérées*, Potential Anal. **6** (1997), no. 4, 369–414.

[Fri03] V. Fridman, *Das Eigenwertproblem zum p-Laplace Operator für p gegen* 1, Ph.D. thesis, Universität zu Köln, 2003, URN (NBN): urn:nbn:de:hbz:38-10570.

[GHLMS99] M. García-Huidobro, V. K. Le, R. Manásevich, and K. Schmitt, *On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz-Sobolev space setting*, NoDEA. Nonlinear Differ. Equ. Appl. **6** (1999), 207–225.

[GHMS96] M. García-Huidobro, R. Manásevich, and K. Schmitt, *On principal eigenvalues of p-Laplacian-like operators*, J. Differential Equations **130** (1996), no. 1, 235–246.

[Gil67] J. R. Giles, *Classes of semi-inner product spaces*, Trans. Amer. Math. Soc. **116** (1967), 436–446.

[Giu77] E. Giusti, *Minimal surfaces and functions of bounded variation*, Notes on pure mathematics, vol. 10, Dept. of Pure Math., Australian Nat. Univ., Canberra, 1977.

[Giu84] ———, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, vol. 80, Birkhäuser, Boston–Basel–Stuttgart, 1984.

[GKR07] M. Grothaus, Yu. G. Kondratiev, and M. Röckner, *N/V-limit for stochastic dynamics in continuous particle systems*, Probab. Theory Related Fields **137** (2007), no. 1–2, 121–160.

[GM87] J.-P. Gossez and V. Mustonen, *Variational inequalities in Orlicz-Sobolev spaces*, Nonlinear Anal. **11** (1987), no. 3, 379–392.

[GM02] J.-P. Gossez and R. Manásevich, *On a nonlinear eigenvalue problem in Orlicz-Sobolev spaces*, Proc. Roy. Soc. Edinburgh Sect. A **132** (2002), no. 4, 891–909.

[Gos82] J.-P. Gossez, *Strongly nonlinear elliptic problems in Orlicz-Sobolev spaces of order one*, Cahiers Centre Études Rech. Opér. **24** (1982), no. 2–4, 221–228.

[Gos86] ———, *A strongly nonlinear elliptic problem in Orlicz-Sobolev spaces*, Nonlinear functional analysis and its applications, Part 1 (Berkeley, Calif., 1983), Proc. Sympos. Pure Math., 45, Part 1, Amer. Math. Soc., Providence, RI, 1986, pp. 455–462.

[Gro99] M. L. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in mathematics, vol. 152, Birkhäuser, Boston, 1999, based on: Structures métriques des variétés riemanniennes.

[GS68] I. M. Gel'fand and G. E. Shilov, *Generalized functions: Volume 2, Spaces of fundamental and generalized functions*, Academic Press, New York–London, 1968.

[GT77] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der mathematischen Wissenschaften, vol. 224, Springer-Verlag, Berlin–Heidelberg–New York, 1977.

[Hal74] P. R. Halmos, *Measure theory*, Graduate Texts in Mathematics, vol. 18, Springer, Berlin–Heidelberg–New York, 1974.

[Ham75] M. M. Hamza, *Détermination des formes de Dirichlet sur* $\mathbb{R}^n$, Ph.D. thesis, Université Paris-Sud 11, Orsay, 1975.

[Han56] O. Hanner, *On the uniform convexity of* L^p *and* ℓ^p, Ark. f. Mat. **3** (1956), no. 3, 239–244.

[Hed72] L. I. Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc. **36** (1972), no. 2, 505–510.

[HHZ96] P. Habala, P. Hájek, and V. Zizler, *Introduction to Banach spaces I and II*, matfyzpress, Univerzity Karlovy, Prague, 1996.

[Hin98] M. Hino, *Convergence of non-symmetric forms*, J. Math. Kyoto Univ. **38** (1998), no. 2, 329–341.

[Hin09] M. Hinz, *Approximation of jump processes on fractals*, Osaka J. Math. **46** (2009), no. 1, 141–171.

[HKM93] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Clarendon Press, Oxford University Press, 1993.

[Jam47] R. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc. **61** (1947), 265–292.

[JL01] W. B. Johnson and J. Lindenstauss, *Basic concepts in the geometry of Banach spaces*, Handbook of the geometry of Banach spaces, vol. 1, Elsevier, Amsterdam, 2001, pp. 1–84.

[JLJ98] J. Jost and X.-Q. Li-Jost, *Calculus of variations*, Cambridge studies in advanced mathematics, vol. 64, Cambridge Univesity Press, 1998.

[Kat66] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York Inc., 1966.

[Kel75] J. L. Kelley, *General topology*, reprint of the 1955 ed., Graduate Texts in Mathematics, vol. 27, Springer, Berlin–Heidelberg–New York, 1975.

[Kil97] T. Kilpeläinen, *Smooth approximation in weighted Sobolev spaces*, Comment. Math. Univ. Carolinæ **38** (1997), no. 1, 29–35.

[Kin84] S. Kinnunen, *On projections and Birkhoff-James orthogonality in Banach spaces*, Nieuw Arch. Wiskd. (4) **2** (1984), 235–255.

[KO84] A. Kufner and B. Opic, *How to define reasonably weighted Sobolev spaces*, Comment. Math. Univ. Carolinæ **25** (1984), no. 3, 537–554.

[Kol05] A. V. Kolesnikov, *Convergence of Dirichlet forms with changing speed measures on* $\mathbb{R}^d$, Forum Math. **17** (2005), no. 2, 225–259.

[Kol06] ______, *Mosco convergence of Dirichlet forms in infinite dimensions with changing reference measures*, J. Funct. Anal. **230** (2006), no. 2, 382–418.

[Kol08] ______, *Weak convergence of diffusion processes on Wiener space*, Probab. Theory Related Fields **140** (2008), no. 1–2, 1–17.

[Kom67] Y. Komura, *Nonlinear semigroups in Hilbert space*, J. Math. Soc. Japan **19** (1967), 493–507.

[KR70] D. Koehler and P. Rosenthal, *On isometries of normed linear spaces*, Studia Math. **36** (1970), 213–216.

[KS03] K. Kuwae and T. Shioya, *Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry*, Comm. Anal. Geom. **11** (2003), no. 4, 599–673.

[KS07] B. Kawohl and F. Schuricht, *Dirichlet problems for the* 1*-Laplace operator, including the eigenvalue problem*, Commun. Contemp. Math. **9** (2007), no. 4, 515–543.

[KS08] K. Kuwae and T. Shioya, *Variational convergence over metric spaces*, Trans. Amer. Math. Soc. **360** (2008), no. 1, 35–75.

[KT82] R. A. Kerman and A. Torchinsky, *Integral inequalities with weights for the Hardy maximal function*, Studia Math. **71** (1982), no. 3, 277–284.

[KU96] K. Kuwae and T. Uemura, *Weak convergence of symmetric diffusion processes II*, Probab. Th. Math. Stat. (S. Watanabe, ed.), Proc. of the 7th Japan–Russia Symp., World Scientific, Signapore, 1996, pp. 256–265.

[KU97] ———, *Weak convergence of symmetric diffusion processes*, Probab. Theory Relat. Fields **109** (1997), 159–182.

[Kuf80] A. Kufner, *Weighted Sobolev spaces*, Teubner–Texte zur Mathematik, vol. 31, Teubner, Stuttgart–Leipzig–Wiesbaden, 1980.

[Lac74] H. E. Lacey, *The isometric theory of classical Banach spaces*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 208, Springer, Berlin–Heidelberg–New York, 1974.

[Lam58] J. Lamperti, *On the isometries of certain function-spaces*, Pacific J. Math. **8** (1958), 459–466.

[Lav26] M. Lavrent'ev, *Sur quelques problèmes du calcul des variations*, Ann. Math. Pura Appl. **4** (1926), 107–124.

[Leo09] G. Leoni, *A first course in Sobolev spaces*, Graduate Studies in Mathematics, vol. 105, American Mathematical Society, 2009.

[Lin85] P.-K. Lin, *Remarks on linear selections for the metric projection*, J. Approx. Theory **43** (1985), no. 1, 64–74.

[LNW02] D. Lukkassen, G. Nguetseng, and P. Wall, *Two-scale convergence*, Int. J. Pure Appl. Math. **2** (2002), no. 1, 35–86.

[Lor88] F. Lorenz, *Lineare Algebra I*, 2nd revised ed., Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, B.I. Wissenschaftsverlag, Mannheim–Wien–Zürich, 1988.

[Lor89] ———, *Lineare Algebra II*, 2nd revised ed., Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, B.I. Wissenschaftsverlag, Mannheim–Wien–Zürich, 1989.

[LT77] J. Lindenstauss and L. Tzafriri, *Classical Banach spaces I: Sequence spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 92, Springer, Berlin–Heidelberg–New York, 1977.

[LT79] ______, *Classical Banach spaces II: Function spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 97, Springer, Berlin–Heidelberg–New York, 1979.

[Lum61] G. Lumer, *Semi-inner product spaces*, Trans. Amer. Math. Soc. **100** (1961), 29–43.

[LW05] D. Lukkassen and P. Wall, *Two-scale convergence with respect to measures and homogenization of monotone operators*, J. Funct. Spaces Appl. **3** (2005), no. 2, 125–161.

[Mat99] S. Mataloni, *On a type of convergence for non-symmetric Dirichlet forms*, Adv. Math. Sci. Appl. **9** (1999), no. 2, 749–773.

[Maz85] V. G. Maz'ja, *Sobolev spaces*, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1985.

[Mer94] P. Mertens, *Konvergenz von koërziven abgeschlossenen Formen*, Master's thesis, University of Bonn, 1994, unpublished.

[Mil70] V. D. Mil'man, *Geometric theory of Banach spaces I, the theory of basis and minimal systems*, Uspekhi Mat. Nauk, translated in Russian Math. Surveys **25** (1970), no. 3 (153), 113–174.

[Mil71] ______, *Geometric theory of Banach spaces II, geometry of the unit sphere*, Uspekhi Mat. Nauk, translated in Russian Math. Surveys **26** (1971), no. 6 (162), 73–149.

[Miz73] S. Mizohata, *The theory of partial differential equations*, Cambridge University Press, London, New York, 1973.

[MM64] D. P. Mil'man and V. D. Mil'man, *Some properties of non-reflexive Banach spaces*, Mat. Sb. (N. S.) **65 (107)** (1964), 486–497.

[MN91] P. Meyer-Nieberg, *Banach lattices*, Universitext, Springer, Berlin–Heidelberg–New York, 1991.

[Mor70] J.-J. Moreau, *Inf-convolution, sous-additivité, convexité des fonctions numériques*, J. Math. Pures Appl. (9) **49** (1970), 109–154.

[Mos69] U. Mosco, *Convergence of convex sets and of solutions of variational inequalities*, Advances in Math. **3** (1969), no. 4, 510–585.

[Mos94] ______, *Composite media and asymptotic Dirichlet forms*, J. Funct. Anal. **123** (1994), no. 2, 368–421.

[MR92] Z.-M. Ma and M. Röckner, *Introduction to the theory of (non-symmetric) Dirichlet forms*, Universitext, Springer-Verlag, Berlin–Heidelberg–New York, 1992.

[MS64] N. Meyers and J. Serrin, $H = W$, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 1055–1056.

[Nec01] L. Nechvátal, *Alternative approaches to the two-scale convergence*, Appl. Math. **49** (2001), no. 2, 97–110.

[Ngu89] G. Nguetseng, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal. **20** (1989), no. 3, 608–623.

[NS86] P. P. Narayanaswami and S. A. Saxon, *(LS)-spaces, quasi-Baire spaces and the strongest locally convex topology*, Math. Ann. **274** (1986), 627–641.

[OP75] R. I. Ovsepian and A. Pełczyński, *On the existence of a fundamental total and bounded biorthogonal sequence in every separable Banach space, and related constructions of uniformly bounded orthonormal systems in L^2*, Studia Math. **54** (1975), no. 2, 149–159.

[Par09] E. Parini, *Asymptotic behaviour of higher eigenfunctions of the p-Laplacian as p goes to* 1, Ph.D. thesis, Universität zu Köln, 2009, URN (NBN): urn:nbn:de:hbz:38-29479.

[Pas07] S. E. Pastukhova, *Degenerate equations of monotone type: Lavrent'ev phenomenon and attainability problems*, Sb. Math. **198** (2007), no. 10, 1465–1494, translated from Math. Sb. **198** (2007), no. 10, 89–118.

[Paz83] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, 2nd corr. ed., Springer-Verlag, Berlin–Heidelberg–New York, 1983.

[Phe89] R. R. Phelps, *Convex functions, monotone operators and differentiability*, Lecture Notes in Mathematics, vol. 1364, Springer-Verlag, Berlin–Heidelberg–New York, 1989, DOI: 10.1007/BFb0089089.

[Ran01] B. Randrianantoanina, *Norm one projections in Banach spaces*, Taiwanese J. Math. **5** (2001), no. 1, 35–95, arXiv:math/0110171v1.

[Ran02] I. K. Rana, *An introduction to measure and integration*, Graduate studies in mathematics, vol. 45, American Mathematical Society, 2002.

[Rei82] S. Reich, *Nonlinear semigroups, accretive operators, and applications*, Nonlinear phenomena in mathematical sciences (Arlington, Texas, 1980) (V. Lakshmikantham, ed.), Academic Press, New York, 1982, pp. 831–838.

[Rip76] B. D. Ripley, *The disintegration of invariant measures*, Math. Proc. Cambridge Philos. Soc. **79** (1976), no. 2, 337–341.

[Roc66] R. T. Rockafellar, *Characterization of the subdifferentials of convex functions*, Pacific J. Math. **17** (1966), no. 3, 497–510.

[Roc70] ———, *On the maximal monotonicity of subdifferential mappings*, Pacific J. Math. **33** (1970), no. 1, 209–216.

[RR91] M. M. Rao and Z. D. Ren, *Theory of Orlicz spaces*, 2nd revised and extended ed., Mathematics and its applications, East European series, Marcel Dekker, Inc., New York–Basel–Hong Kong, 1991.

[RR02] ———, *Applications of Orlicz spaces*, Pure and Applied Mathematics, Marcel Dekker, Inc., New York–Basel, 2002.

[RRW07] J. Ren, M. Röckner, and F.-Y. Wang, *Stochastic generalized porous media and fast diffusion equations*, J. Differential Equations **238** (2007), no. 1, 118–152.

[RS75] M. Reed and B. Simon, *Methods of modern mathematical physics II. Fourier analysis, self-adjointness*, Academic Press, New York, 1975.

[RS80] ———, *Methods of modern mathematical physics I. Functional analysis*, revised and enlarged ed., Academic Press, New York, 1980.

[RS09] D. W. Robinson and A. Sikora, *Markov uniqueness of degenerate elliptic operators*, Preprint (2009), 24 pp.

[RW08] M. Röckner and F.-Y. Wang, *Non-monotone stochastic generalized porous media equations*, J. Differential Equations **245** (2008), no. 12, 3898–3935.

[RZ92] M. Röckner and T. S. Zhang, *Uniqueness of generalized Schrödinger operators and applications*, J. Funct. Anal. **105** (1992), 187–231.

[RZ94] ———, *Uniqueness of generalized Schrödinger operators and applications II*, J. Funct. Anal. **119** (1994), 455–467.

[RZ97] ———, *Convergence of operator semigroups generated by elliptic operators*, Osaka J. Math. **34** (1997), 923–932.

[Sch71] H. H. Schaefer, *Topological vector spaces*, 3rd printing corrected ed., Graduate texts in mathematics, vol. 3, Springer, Berlin–Heidelberg–New York, 1971.

[Sch73] L. Schwartz, *Radon measures on arbitrary topological space and cylindrical measures*, Oxford University Press, London, 1973.

[Sch06] F. Schuricht, *An alternative derivation of the eigenvalue equation for the* 1-*Laplace operator*, Arch. Math. (Basel) **87** (2006), no. 6, 572–577.

[Sha88] M. T. Sharpe, *General theory of Markov processes*, Academic Press, New York, 1988.

[Sho97] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, Mathematical surveys and monographs, Amer. Math. Soc., 1997.

[Sin70a] I. Singer, *Bases in Banach spaces I*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 154, Springer, Berlin–Heidelberg–New York, 1970.

[Sin70b] ———, *Best approximation in normed linear spaces by elements of linear subspaces*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 171, Springer, Berlin–Heidelberg–New York, 1970.

[Sta98] W. Stannat, *Dirichlet forms and Markov processes: a generalized framework including both elliptic and parabolic cases*, Potential Anal. **8** (1998), no. 1, 21–60.

[Sta99a] ———, *(Nonsymmetric) Dirichlet operators on L^1: existence, uniqueness and associated markov processes*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999), no. 1, 99–140.

[Sta99b] ———, *The theory of generalized Dirichlet forms and its applications in analysis and stochastics*, Mem. Amer. Math. Soc. **142** (1999), no. 678, viii+101 pp.

[Ste70] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton mathematical series, vol. 30, Princeton University Press, Princeton, New Jersey, 1970.

[Ste93] ———, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton mathematical series, vol. 43, Princeton University Press, Princeton, New Jersey, 1993.

[Tan79] H. Tanabe, *Equations of evolution*, Monographs and studies in mathematics, vol. 6, Pitman, London, 1979.

[Töl06] J. M. Tölle, *Convergence of non-symmetric forms with changing reference measures*, Master's thesis, Universität Bielefeld, 2006, BiBoS-Preprint E06-09-234, `http://www.math.uni-bielefeld.de/~bibos/preprints/E06-09-234.pdf`.

[Tro58] H. F. Trotter, *Approximation of semi-groups of operators*, Pacific J. Math **8** (1958), no. 4, 887–919.

[Tur00] B. O. Turesson, *Nonlinear potential theory and weighted Sobolev spaces*, Lecture Notes in Mathematics, vol. 1736, Springer, Berlin–Heidelberg–New York, 2000.

[Váz06] J. L. Vázquez, *Smoothing and decay estimates for nonlinear diffusion equations: equations of porous medium type*, Oxford lecture series in mathematics and its applications, vol. 33, Oxford University Press, Oxford, 2006.

[Váz07] ______, *The porous medium equation: mathematical theory*, Oxford Mathematical Monographs, Clarendon Press, Oxford University Press, Oxford, 2007.

[Vui87] P.-A. Vuillermot, *Compact embeddings for weighted Orlicz-Sobolev spaces on* $\mathbb{R}^N$, Math. Ann. **279** (1987), no. 2, 277–296.

[Yos78] K. Yosida, *Functional analysis*, 5. ed., Springer-Verlag, Berlin–Heidelberg–New York, 1978.

[Zei90a] E. Zeidler, *Nonlinear functional analysis and its applications, II/A, Linear monotone operators*, Springer-Verlag, Berlin–Heidelberg–New York, 1990.

[Zei90b] ______, *Nonlinear functional analysis and its applications, II/B, Nonlinear monotone operators*, Springer-Verlag, Berlin–Heidelberg–New York, 1990.

[Zhi98] V. V. Zhikov, *On weighted Sobolev spaces*, Sb. Math. **189** (1998), no. 8, 1139–1170, translated from Math. Sb. **189** (1998), no. 8, 27–58.

[Zhi00] ______, *On an extension and applications of the method of two-scale convergence*, Math. Sb. **191** (2000), no. 7, 31–72, translation in Sb. Math. **191** (2000), no. 7-8, 973–1014.

[Zhi04] ______, *On two-scale convergence*, J. Math. Sci. (N.Y.) **120** (2004), no. 3, 1328–1352.

[Zhi09] ______, *On the technique for passing to the limit in nonlinear elliptic equations*, Funct. Anal. Appl. **43** (2009), no. 3, 96–112, translated from Funktsional'nyi Analiz i Ego Prilozheniya **43** (2009), no. 2, 19–38.

[Zie89] W. P. Ziemer, *Weakly differentiable functions: Sobolev spaces and functions of bounded variation*, Graduate texts in mathematics, vol. 120, Springer-Verlag, Berlin–Heidelberg–New York, 1989.

[ZP07] V. V. Zhikov and S. E. Pastukhova, *On the Trotter-Kato Theorem in a variable space*, Funct. Anal. Appl. **41** (2007), no. 4, 264–270, translated from Funktsional'nyi Analiz i Ego Prilozheniya **41** (2007), no. 4, 22–29.

[ZP08] ______, *Homogenization of degenerate elliptic equations*, Sb. Math. J. **49** (2008), no. 1, 80–101, translated from Sibirsk. Mat. Zh. **49** (2008), no. 1, 101–124.