Diploma thesis

Quasi-linear Partial Differential Equations in Dirichlet Spaces

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February 2009

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Introduction

A very interesting quasi-linear parabolic system of backward partial differential equations is the following:

$$(1) \qquad \begin{aligned} (\partial_t + L)u(t,x) + f(t,x,u,D_\sigma u) &= 0 \qquad \forall\, 0 \le t \le T \\ u(T,x) &= \phi(x), \quad x \in \mathbb{R}^d \end{aligned}$$

where L is a second order differential operator with measurable coefficients and $D_\sigma u$ is defined as a generalized gradient, which depends on some coefficients of L. In the analytic part of [BPS05] the above system is considered for an operator L associated to the bilinear form

$$\mathcal{E}(u,v) \quad = \quad \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a^{i,j}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x)\, m(dx), \;\; u,v \in \mathcal{C}_0^\infty(\mathbb{R}^d)$$

where $\mathcal{C}_0^\infty(\mathbb{R}^d)$ denotes the space of infinitely differentiable functions with compact support. Here $m(dx) := \pi_\rho(x)dx$, where π_ρ is a weight function and dx denotes Lebesgue measure, such that $m(\mathbb{R}^d) < \infty$ if $\rho > 0$, and $m(dx) = dx$ if $\rho = 0$. In Section 2 of [BPS05] the authors, V. Bally, E. Pardoux and L. Stoica, prove the existence and uniqueness of weak and also strong solutions for the linear equation

$$(2) \qquad \begin{aligned} (\partial_t + L)u(t,x) + f(t,x) &= 0 \qquad \forall\, 0 \le t \le T \\ u(T,x) &= \phi(x), \quad x \in \mathbb{R}^d \end{aligned}$$

where f is an element of $L^1([0,T]; L^2(\mathbb{R}^d, m))$ and $\phi \in L^2(\mathbb{R}^d, m)$. Moreover, they derive basic relations, which are very useful in the treatment of the nonlinear case. The third section deals with the nonlinear case (1). The authors present an existence and uniqueness proof for a weak solution in the case of Lipschitz conditions and also in the case of more general monotonicity conditions.

In this thesis we generalize the analytic part of [BPS05] to a non-symmetric case. More precisely, we consider the system (1) of BPDEs for a non-symmetric second order differential operator L, which is associated to the bilinear form

$$\begin{aligned} \mathcal{E}(u,v) \quad := \quad & \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a^{i,j}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x)\, m(dx) \\ & + \sum_{i=1}^{d} \int_{\mathbb{R}^d} \left(u(x) \frac{\partial v}{\partial x_i}(x) d_i(x) + \frac{\partial u}{\partial x_i}(x) v(x) b_i(x) \right) m(dx) \\ & + \int_{\mathbb{R}^d} u(x)v(x)c(x)\, m(dx) \end{aligned}$$

where $a^{i,j}, b_i, d_i, c \in L^1_{loc}(\mathbb{R}^d, m), 1 \leq i,j \leq d$. We solve the system (1) in the linear and also nonlinear case under general conditions on f and the coefficients of the above bilinear form. Now let us give a brief overview of this work.

In the first chapter we explain the functional analytic methods needed to understand the non-symmetric framework (Chapter 2) and to solve the system (1) of BPDEs.

In Section 1.1 we repeat the basic definitions of semigroups and Dirichlet forms and some useful lemmas and theorems from [MR92]. Moreover, we present some important properties (cf. Lemma 1.9, 1.10 and 1.11).

Section 1.2 contains a Hilbert space version of [MR92, I. Lemma 2.12].

Chapter 2 deals with the framework of this thesis.

In Section 2.1 we introduce the non-symmetric framework. We define the bilinear form (2.1) and state our basic conditions, which are

- $(\mathcal{E}, F)$ is a Dirichlet form, where F is the closure of $\mathcal{C}_0^\infty(\mathbb{R}^d)$ w.r.t. $\mathcal{E}_1$,
- $(\mathcal{E}^A, \mathcal{D}(\mathcal{E}^A))$ is a coercive closed form.

Note that there is no strong ellipticity condition on the coefficients of the bilinear form. Examples of such forms are presented in Remark 2.2.

In Section 2.2 we introduce $C_T = L^2((0,T); F) \cap C^1((0,T); L^2(\mathbb{R}^d, m))$ and its completion $\hat{F}$ w.r.t. $\| \cdot \|_T$ where $\|u\|_T^2 = \sup_{t \in [0,T]} \|u_t\|_2^2 + \int_0^T \mathcal{E}(u_t, u_t)dt$. In Lemmas 2.4 - 2.6 we prove basic properties of these spaces, which were only claimed, but not proved in [BPS05]. Moreover, we give our own proof for the statement that $b\mathcal{C}_T$ is dense in $\mathcal{C}_T$ (cf. Lemma 2.7). A characterization of $\hat{F}$ is given in Lemma 2.10. In the proof we follow the very rough idea of [BPS05, Lemma 2.1]. At the end of this chapter we give an approximation lemma for functions in $\hat{F}$.

In Chapter 3 we solve the linear system (2) in the case where $\phi \in L^2(\mathbb{R}^d, m)$ and $f \in L^1([0,T]; L^2(\mathbb{R}^d, m))$.

Section 3.1 contains our definition of weak and strong solutions for the linear case and some proofs of their important properties. Sufficient conditions for the existence and uniqueness of a strong solution are given in Proposition 3.6, which is taken from [BPS05, Proposition 2.6]. We prove it with all details for our non-symmetric framework. Existence and uniqueness of weak solutions are proved in Proposition 3.8. Here we follow the lines of arguments of [BPS05, Proposition 2.7].

In Section 3.2 we state useful relations. In Lemma 3.10 we point out an important relation for the positive part u^+ of a weak solution u:

$$\|u_{t_1}^+\|_2^2 + 2\int_{t_1}^{t_2} \mathcal{E}(u_s^+)\, ds \leq 2\int_{t_1}^{t_2} (f_s, u_s^+)\, ds + \|u_{t_2}^+\|_2^2.$$

The proof follows [BPS05, Lemma 2.8]. Note that in the symmetric framework of [BPS05] the above relation is even an equality. A modified version of this statement is Lemma 3.11. The main result of Section 3.2 is Proposition 3.12. In this proposition we verify a representation of a function $u \in \hat{F}$ satisfying the weak relation

$$\int_0^T ((u_t, \partial_t \varphi_t) + \mathcal{E}(u_t, \varphi_t))\, dt = \int_0^T (f_t, \varphi_t)\, dt + (\phi, \varphi_T) - (u_0, \varphi_0) \quad \forall \varphi \in \mathcal{C}_T$$

for certain data. Moreover, we show two very useful relations. The proof of this proposition is a rewritten version of [BPS05, Proposition 2.9].

Chapter 4 deals with the nonlinear case, where f depends on t, x, u and $D_\sigma u$, in the case of vector valued functions. This chapter contains our main results: the existence and uniqueness of a solution of (1) in a weak sense, firstly under Lipschitz conditions and secondly under monotonicity conditions on f. The general conditions in this chapter are:

$$\begin{aligned} (A1) \qquad & \tilde{A} := (\tilde{a}^{i,j})_{i,j=1,\ldots,d} \text{ is bounded and} \\ & \sum_{i,j=1}^{d} a^{i,j} \xi_i \xi_j \geq 0 \text{ for all } \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d, \\ (A2) \qquad & \mathcal{E}^A(u,u) \leq K_A\, \mathcal{E}(u,u) + C_A \|u\|_2^2 \\ & \text{ for some } K_A \in [1,2), C_A \in \mathbb{R}_+ \text{ and for all } u \in F. \end{aligned}$$

In the monotonicity case we have to assume additional conditions.

In Section 4.1 we prove

$$\tilde{\mathcal{E}}^A(u,v) = \int_{\mathbb{R}^d} \langle D_\sigma u, D_\sigma v \rangle \, dm$$

where $D_\sigma u$ is a generalized gradient. This equation is first shown for $u, v \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ (cf. Lemma 4.3) and then for $u, v \in \hat{F^A}$ (cf. Proposition 4.4). Since $\tilde{\mathcal{E}}^A$ is exactly the bilinear form considered in [BPS05], the statement of Proposition 4.4 coincides with [BPS05, Proposition 2.3]. We present a completed proof with all details, which follows the arguments of the proof in the original paper.

In Section 4.2 we give our definition for a solution of the nonlinear equation.

Section 4.3 contains the case of Lipschitz conditions. We basically follow [BPS05, Proposition 3.1] and prove thereby the existence and uniqueness of a solution under Lipschitz conditions on f.

We start Section 4.4.1 by introducing monotonicity conditions. Then we prove some properties in Lemma 4.9, which are associated to these conditions.

The aim of Section 4.4.2 is to prove two important estimates for a solution. By Lemma 4.10 we obtain an estimate in the $\|\cdot\|_T$ norm. For an estimate in the $\|\cdot\|_\infty$ norm we need additional conditions on the coefficients of the bilinear form:

$$\begin{aligned} \text{(A3)} \qquad & d_i = 0 \text{ for } i = 1, \ldots, d, \\ \text{(A4)} \qquad & c \in L^\infty(\mathbb{R}^d; \mathbb{R}_+). \end{aligned}$$

We start by proving two approximation lemmas for the data (f, ϕ) (Lemma 4.11, Lemma 4.12.), which were not explicitly stated and proved in the original paper. Next we give a version of [BPS05, Proposition 2.10] for nice data in our framework. The main arguments of this proof are analogous to the symmetric case. In general, an explicit formula of the non-symmetric Dirichlet form does not exist. Thus, our proof becomes very technical in contrast to the symmetric case. The case with general data is treated in the next corollary. In Lemma 4.16 we show two important relations, which are useful for the $\| \cdot \|_\infty$ estimate of our solution. Note that in this lemma it is essential that a Markov process is associated to our sub-Markovian semigroup. By the last lemma of this section we gain a $\| \cdot \|_\infty$ estimate for the solution.

In the first theorem of Section 4.4.3 we prove the existence of a unique solution under monotonicity conditions for the case $\rho > 0$. We follow [BPS05, Theorem 3.2]. The case, where $m(dx)$ is the Lebesgue measure, is treated in Theorem 4.21. Note that in the latter case we postulate additional conditions on the coefficients of $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$:

$(A1)'$ $\quad A = \tilde{A}$ and A is bounded,

$(A5)$ $\quad \exists\, \sigma^{-1}$ such that $\sigma\sigma^{-1} = \mathbb{1}$ and $|\sigma^{-1}(x)| < \infty$ uniformly,

$(A6)$ $\quad -\nabla \cdot b \geq 0$,

$(A7)$ $\quad b \in L^2(\mathbb{R}^d, dx)$,

$(A8)$ $\quad \mathcal{E}(u) < \infty \Rightarrow u \in F$.

We point out that an essential condition, which is independent of the non-symmetric part, is $(A8)$ (cf. Theorem 4.21 (4)). Since this condition is not stated in [BPS05], the proof in the original paper is doubtful. Note that the form of the operator $(L_\rho, \mathcal{D}(L_\rho))$, which is used in step 4 of [BPS05, Theorem 3.2], only exists under additional conditions on the coefficients. But these conditions are not stated in the original work. In our proof we do not use any explicit representation of this operator.

Finally, we prove analogous to [BPS05, Proposition 3.4] a comparison result for solutions.

In the appendix we demonstrate an important proposition for the Bochner integral and a very useful backward version of Gronwall's Lemma.

Note that in the notation we do not distinguish between m-equivalence classes of functions on $\mathbb{R}^d$ and representatives, if there is no confusion.

We expect that our results are also valid in the framework of semi-Dirichlet forms without major changes.

First of all I want to express my deepest thanks and gratitude to Prof. Dr. Michael Röckner for his guidance. Further, I would like to thank Dr. Gerald Trutnau for the discussions about open questions and problems. Finally, I am grateful for the moral support and understanding of my wife.

Chapter 1

Functional Analytic Methods

The aim of this chapter is to give an overview about the functional analytic methods, which we will need afterwards. We start by repeating some definitions and lemmas in the first section. In Section 1.2 we present a variant of [MR92, I. Lemma 2.12] in a Hilbert space form, which is an important tool in this work.

1.1 Semigroups and Dirichlet Forms

The following three definitions are taken from [MR92], (cf. [MR92, I.Definition 1.4, 1.6 and 1.8]).

Definition 1.1. [*strongly continuous contraction resolvent*]
A family $(G_\alpha)_{\alpha>0}$ of linear operators on a Banach space B with $D(G_\alpha) = B$ for all $\alpha \in]0,\infty[$ is called a strongly continuous contraction resolvent on B, if

$$(i) \quad \lim_{\alpha\to\infty} \alpha G_\alpha u = u \text{ for all } u \in B.$$
$$(ii) \quad \alpha G_\alpha \text{ is a contraction on } B \text{ for all } \alpha > 0.$$
$$(iii) \quad G_\alpha - G_\beta = (\beta - \alpha) G_\alpha G_\beta \text{ for all } \alpha, \beta > 0.$$

Definition 1.2. [*strongly continuous contraction semigroup*]
A family $(T_t)_{t>0}$ of linear operators on a Banach space B with $D(T_t) = B$ for all $t > 0$ is called a strongly continuous contraction semigroup on B, if

$$(i) \quad \lim_{t\to 0} T_t u = u \text{ for all } u \in B.$$
$$(ii) \quad T_t \text{ is a contraction on } B \text{ for all } t > 0.$$
$$(iii) \quad T_t T_s = T_{s+t} \text{ for all } t, s > 0.$$

Definition 1.3. [*infinitesimal generator of $(T_t)_{t>0}$*]
Given a strongly continuous contraction semigroup $(T_t)_{t>0}$ on a Banach space B, the linear operator $(L, \mathcal{D}(L))$ on B defined by

$$\begin{aligned} \mathcal{D}(L) &:= \{u \in B | \lim_{t\downarrow 0} \frac{1}{t}(T_t u - u) \text{ exists in } B\} \\ Lu &:= \lim_{t\downarrow 0} \frac{1}{t}(T_t u - u), u \in \mathcal{D}(L), \end{aligned}$$

is called the infinitesimal generator of $(T_t)_{t>0}$.

Notation. In this chapter we fix a Hilbert space $(\mathcal{H}, (\cdot,\cdot)_{\mathcal{H}})$. Let D be a linear subspace of $\mathcal{H}$ and $\mathcal{E}$ be a bilinear form on $D \times D$. We define

$$\mathcal{E}_1(\cdot,\star) := \mathcal{E}(\cdot,\star) + (\cdot,\star)_{\mathcal{H}}$$

and the symmetric part

$$\tilde{\mathcal{E}_1}(\cdot,\star) := \frac{1}{2}\big(\mathcal{E}(\cdot,\star) + \mathcal{E}(\star,\cdot)\big).$$

Moreover, we write $L^2(\mathbb{R}^d) := L^2(\mathbb{R}^d, m)$ with the usual inner product $(\cdot,\cdot)$ where $m(dx)$ is some σ-finite measure.

Definition 1.4. [*symmetric closed form (cf. [MR92, I.Def.2.3.])*]
A pair $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *is called a symmetric closed form on* $\mathcal{H}$*, if* $\mathcal{D}(\mathcal{E})$ *is a dense linear subspace of* $\mathcal{H}$ *and* $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{R}$ *is a positive definite bilinear form, which is symmetric and closed on* $\mathcal{H}$.

Definition 1.5. [*coercive closed form (cf. [MR92, I.Def.2.4.])*]
A pair $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *is called a coercive closed form on* $\mathcal{H}$*, if* $\mathcal{D}(\mathcal{E})$ *is a dense linear subspace of* $\mathcal{H}$ *and* $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{R}$ *is a bilinear form such that the following two conditions hold:*

- *(i)* *Its symmetric part is a symmetric closed form on* $\mathcal{H}$.
- *(ii)* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *satisfies the weak sector condition:*
  $\exists K_{\mathcal{E}} > 0$ *such that* $|\mathcal{E}_1(u,v)| \le K_{\mathcal{E}} \mathcal{E}_1(u,u)^{\frac{1}{2}} \mathcal{E}_1(v,v)^{\frac{1}{2}}$ *for all* $u, v \in \mathcal{D}(\mathcal{E})$.

Definition 1.6. [*Dirichlet form (cf. [MR92, I.Def.4.5.])*]
A coercive closed form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *on* $L^2(\mathbb{R}^d)$ *is called Dirichlet form, if for all* $u \in \mathcal{D}(\mathcal{E})$ *one has that*

$$u^+ \wedge 1 \in \mathcal{D}(\mathcal{E}) \quad \text{and} \quad \mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \ge 0$$
$$\text{and} \quad \mathcal{E}(u - u^+ \wedge 1, u + u^+ \wedge 1) \ge 0.$$

Definition 1.7. [*sub-Markovian (cf. [MR92, I.Def.4.1.])*]
Let G *be a bounded linear operator on* $L^2(\mathbb{R}^d)$ *with* $D(G) = L^2(\mathbb{R}^d)$. G *is called sub-Markovian, if for all* $f \in L^2(\mathbb{R}^d), 0 \le f \le 1$ *implies* $0 \le Gf \le 1$. *A strongly continuous contraction resolvent* $(G_\alpha)_{\alpha>0}$ *resp. semigroup* $(T_t)_{t>0}$ *is called sub-Markovian, if all* $\alpha G_\alpha, \alpha > 0$, *resp.* $T_t, t > 0$ *are sub-Markovian.*

The correspondence between $(G_\alpha)_{\alpha>0}$,$(T_t)_{t>0}$,$(L, \mathcal{D}(L))$ and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is illustrated in [MR92, Diagram 3]. Note that every sub-Markovian semigroup $(T_t)_{t>0}$ is positivity preserving, i.e. $f \ge 0 \Rightarrow T_t f \ge 0$.

Lemma 1.8. *Let* $(L, \mathcal{D}(L))$ *be the infinitesimal generator of a strongly continuous contraction semigroup* $(T_t)_{t>0}$ *on* $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$*, which is associated to a coercive closed form* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. *Then for all* $t > 0$ *and* $f \in \mathcal{H}$ *it holds*

- *(i)* $T_t f \in \mathcal{D}(L)$,
- *(ii)* $\|LT_t f\|_{\mathcal{H}} \le C \frac{\|f\|_{\mathcal{H}}}{t}$ *for some* $C \in]0, \infty[$ *independent of* f *and* t.

Proof. See [MR92, p.25]. □

Lemma 1.9. *Let $(L, \mathcal{D}(L))$ be the infinitesimal generator of a strongly continuous contraction semigroup $(T_t)_{t>0}$ on $(L^2(\mathbb{R}^d), \|\cdot\|_2)$, which is associated to a coercive closed form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Then for $u \in L^2(\mathbb{R}^d)$ it holds*

$$|\mathcal{E}(T_t u, T_t u)| \leq \frac{1}{t}\|u\|_2^2 \tilde{C}$$

where $\tilde{C} \in]0, \infty[$.

Proof. By Lemma 1.8 it holds $T_t u \in \mathcal{D}(L)$ and

$$\|LT_t u\|_2 \leq \tilde{C}\frac{\|u\|_2}{t} \text{ for some } \tilde{C} \in]0, \infty[.$$

Therefore, we conclude

$$|\mathcal{E}(T_t u, T_t u)| = |(-LT_t u, T_t u)| \leq \|LT_t u\|_2 \|T_t u\|_2 \leq \frac{\|u\|_2^2}{t}\tilde{C}.$$

□

Lemma 1.10. *Let $(T_t)_{t>0}$ be a strongly continuous contraction semigroup on $(L^2(\mathbb{R}^d), \|\cdot\|_2)$, which is sub-Markovian. Then for $f, g \in L^2(\mathbb{R}^d)$ and $t > 0$ it holds:*

$$(i) \qquad |T_t(f)| \leq T_t(|f|),$$
$$(ii) \qquad T_t(fg) \leq \frac{1}{2}T_t(|f|^2 + |g|^2).$$

Proof. (i) Since $T_t(|f| - f) \geq 0$ and $T_t(|f| + f) \geq 0$, it follows $T_t(|f|) \geq |T_t(f)|$.
(ii) Since $0 \leq T_t(|f|^2 + |g|^2 - 2fg) = T_t(|f|^2 + |g|^2) - T_t(2fg)$, the assertion follows. □

Lemma 1.11. *Let $(T_t)_{t>0}$ be a sub-Markovian strongly continuous contraction semigroup on $(L^2(\mathbb{R}^d), \|\cdot\|_2)$. Then*

$$\|T_t f\|_\infty \leq \|f\|_\infty \textit{ for all } f \in L^\infty(\mathbb{R}^d, m) \cap L^2(\mathbb{R}^d).$$

Proof. Let us first assume $f = 0$ m-a.e.. Hence, it holds $\|f\|_\infty = 0$. Since $\|T_t f\|_2 \leq \|f\|_2 = 0$, it follows $T_t f = 0$ m-a.e. and we can conclude $\|T_t f\|_\infty = 0$. Now we turn to the case $f \neq 0$ m-a.e. and define $\tilde{f} = \frac{f}{\|f\|_\infty}$. Since T is sub-Markovian and $|\tilde{f}| \leq 1$, we get

$$|T_t \tilde{f}(x)| \underset{\text{Lemma 1.10}}{\leq} T_t|\tilde{f}(x)| \leq 1 \text{ for } m\text{-a.e. } x \in \mathbb{R}^d$$

and therefore,

$$|T_t f(x)| \leq \|f\|_\infty \text{ for } m\text{-a.e. } x \in \mathbb{R}^d.$$

Finally, we obtain the assertion $\|T_t f\|_\infty \leq \|f\|_\infty$. □

Lemma 1.12. *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Dirichlet form on $L^2(\mathbb{R}^d)$. Then the symmetric part of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form. (cf. [MR92, I.Exercise 4.6])*

Proof. The assertion follows immediately from the definition of a Dirichlet form. (cf. Definition 1.6) □

Theorem 1.13. *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Dirichlet form on $L^2(\mathbb{R}^d)$ and $T : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$-function, such that $T(0) = 0$ and T' is bounded by $K \in \mathbb{R}_+$. Then we have for $u \in D(\mathcal{E})$*

$$\begin{array}{ll} (i) & T(u) \in D(\mathcal{E}), \\ (ii) & \mathcal{E}(T(u), T(u)) \leq K^2 \mathcal{E}(u, u). \end{array}$$

Proof. The assertion follows by Lemma 1.12 and [MR92, I. Theorem 4.12]. □

Corollary 1.14. *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Dirichlet form on $L^2(\mathbb{R}^d)$. Then for all $u_1, \ldots, u_n \in \mathcal{D}(\mathcal{E})$ and $u \in L^2(\mathbb{R}^d)$ such that $|u(x) - u(y)| \leq \sum_{k=1}^n |u_k(x) - u_k(y)|$ and $|u(x)| \leq \sum_{k=1}^n |u_k(x)|$ for all $x, y \in \mathbb{R}^d$, we have $u \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(u,u)^{\frac{1}{2}} \leq \sum_{k=1}^n \mathcal{E}(u_k, u_k)^{\frac{1}{2}}$.*

Proof. The assertion follows by Lemma 1.12 and [MR92, I. Corollary 4.13]. □

Corollary 1.15. *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Dirichlet form on $L^2(\mathbb{R}^d)$ and $u, v \in \mathcal{D}(\mathcal{E}), u, v$ bounded. Then $u \cdot v \in \mathcal{D}(\mathcal{E})$ and*

$$\mathcal{E}(u \cdot v, u \cdot v)^{\frac{1}{2}} \leq \|u\|_\infty \mathcal{E}(v, v)^{\frac{1}{2}} + \|v\|_\infty \mathcal{E}(u, u)^{\frac{1}{2}}.$$

Proof. See [MR92, I. Corollary 4.15]. □

1.2 A Hilbert Space Lemma

In this section we present a Hilbert space form of the well known and useful Lemma [MR92, I. Lemma 2.12]. For the proof we need the Banach-Alaoglu and Banach-Saks theorems.

Theorem 1.16. [*Banach-Alaoglu*]
Let B be a Banach space with norm $\|\cdot\|$ and B' its dual. Then the unit ball B'_1 in B' is compact in the weak* topology.

Proof. See [MR92, A. Theorem 2.1] or [RS80, Theorem IV.21]. □

Theorem 1.17. [*Banach-Saks*]
Let $\mathcal{H}$ be a real Hilbert space with inner product $(\, , \,)$ and norm $\| \, \| := (\, , \,)^{\frac{1}{2}}$. Let $u, u_n \in \mathcal{H}, n \in \mathbb{N}$, with $u_n \to u$ as $n \to \infty$ weakly in $\mathcal{H}$, then there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that the Cesáro mean

$$u_N := \frac{1}{N} \sum_{k=1}^N u_{n_k}, \qquad N \in \mathbb{N},$$

converges strongly to u in $\mathcal{H}$.

Proof. See [MR92, A. Theorem 2.2] or [NR55, Section 38]. □

Lemma 1.18. *Let $(B, \|\cdot\|_B)$ be a Banach space and u_n a sequence in B such that $u_n \to u$. Then the Cesáro mean of $(u_n)_{n \in \mathbb{N}}$ converges to u in B.*

Proof. Since $u_n \to u$ in B, for every $\varepsilon > 0$ there exists $K \in \mathbb{N}\backslash\{0\}$ such that

$$\|u_n - u\|_B \leq \varepsilon \text{ for all } n > K.$$

Hence, $\sum_{n=1}^{K} \|u_n\|_B$ is bounded by $c_K \in \mathbb{R}_+$. Therefore, we have for $N > K$:

$$\begin{aligned}
\Big\|\frac{1}{N}\sum_{n=1}^{N} u_n - u\Big\|_B &= \Big\|\frac{1}{N}\sum_{n=1}^{N}(u_n - u)\Big\|_B \\
&\leq \frac{1}{N}\left(\sum_{n=1}^{K}\|u_n - u\|_B + \sum_{n=K+1}^{N}\|u_n - u\|_B\right) \\
&\leq \frac{1}{N}\left(\sum_{n=1}^{K}(\|u_n\|_B + \|u\|_B)\right) + \frac{1}{N}\left(\varepsilon(N-(K+1))\right) \\
&= \frac{1}{N}\left(\sum_{n=1}^{K}\|u_n\|_B\right) + \frac{K}{N}\|u\|_B + \varepsilon - \varepsilon\frac{K+1}{N} \\
&\leq \frac{c_K}{N} + \frac{K}{N}\|u\|_B - \varepsilon\frac{K+1}{N} + \varepsilon \\
&\leq \frac{c_K + K\|u\|_B - \varepsilon(K+1)}{N} + \varepsilon.
\end{aligned}$$

Since K is fix, we can choose $\tilde{N} \geq N$ big enough such that

$$\frac{c_K + K\|u\|_B - \varepsilon(K+1)}{n} \leq \varepsilon \quad \text{for all } n > \tilde{N}.$$

Finally, we get

$$\Big\|\frac{1}{N}\sum_{n=1}^{N} u_n - u\Big\|_B \leq 2\varepsilon \quad \text{for all } n > \tilde{N}.$$

□

Lemma 1.19. *Let $(\mathcal{H}_0, (\ ,\ )_{\mathcal{H}_0})$ and $(\mathcal{H}, (\ ,\ )_{\mathcal{H}})$ be Hilbert spaces, $\mathcal{H}_0 \subset \mathcal{H}$, with norms $\|\cdot\|_{\mathcal{H}} := (\cdot,\cdot)_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_0} := (\cdot,\cdot)_{\mathcal{H}_0}$ such that there exists $c \in \mathbb{R}_+$ with*

$$c\|u\|_{\mathcal{H}_0} \geq \|u\|_{\mathcal{H}} \quad \textit{for all } u \in \mathcal{H}_0.$$

If $u_n \in \mathcal{H}_0, n \in \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} \|u_n\|_{\mathcal{H}_0} < \infty$$

and $u \in \mathcal{H}$ such that $u_n \to u$ in $\mathcal{H}$ as $n \to \infty$, then:

(i) $u \in \mathcal{H}_0$ and $u_n \to u$ weakly in $\mathcal{H}_0$.

(ii) There exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that its Cesáro mean $w_n := \frac{1}{n}\sum_{k=1}^{n} u_{n_k} \to u$ in $\mathcal{H}_0$ as $n \to \infty$.

(iii) $\|u\|_{\mathcal{H}_0} \leq \liminf_{n\to\infty} \|u_n\|_{\mathcal{H}_0}$.

Proof. We follow the idea from the proof of [MR92, I. Lemma 2.12].
Since $\sup_{n\in\mathbb{N}} \|u_n\|_{\mathcal{H}_0} < \infty$, we conclude by the Banach-Alaoglu theorem that there exists $v \in \mathcal{H}_0$ such that

$$u_{n_k} \to v \text{ weakly in } \mathcal{H}_0,$$

for some subsequence $(n_k)_{k\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$. With Banach-Saks we obtain that there exists a subsequence $(n_{k_l})_{l\in\mathbb{N}}$ such that

$$w_n := \frac{1}{n}\sum_{l=1}^{n} u_{n_{k_l}}, \qquad n \in \mathbb{N}$$

converges to v in $\mathcal{H}_0$. Since we have $c\|\cdot\|_{\mathcal{H}_0} \geq \|\cdot\|_{\mathcal{H}}$, it follows that w_n converges in $\mathcal{H}$ to v. By Lemma 1.18 we know, since $u_n \to u$ in $\mathcal{H}$, that the Cesáro mean w_n converges to u in $\mathcal{H}$. Therefore, we have $u = v$. Since this reasoning holds for every subsequence, we get

$$u_n \to u \text{ weakly in } \mathcal{H}_0.$$

This can be seen as follows: Assume that an element $\tilde{u}$ exists such that $(\tilde{u}, u_n)_{\mathcal{H}_0}$ does not converge to $(\tilde{u}, u)_{\mathcal{H}_0}$. Then we can find $\varepsilon > 0$ and a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$|(\tilde{u}, u_{n_k})_{\mathcal{H}_0} - (\tilde{u}, u)_{\mathcal{H}_0}| \geq \varepsilon \quad \text{for all } k \in \mathbb{N}. \tag{1.1}$$

Since the above reasoning holds for every subsequence, there exists a weakly converging subsequence of $(u_{n_k})_{k\in\mathbb{N}}$ in $\mathcal{H}_0$. But this is a contradiction to (1.1).

Now we will prove the estimate (iii). W.l.o.g. we can assume $\|u\|_{\mathcal{H}_0} > 0$. Since

$$\|u\|^2_{\mathcal{H}_0} = \lim_{n\to\infty}(u, u_n)_{\mathcal{H}_0} = \liminf_{n\to\infty}(u, u_n)_{\mathcal{H}_0} \leq \liminf_{n\to\infty}\left((u, u)^{\frac{1}{2}}_{\mathcal{H}_0}(u_n, u_n)^{\frac{1}{2}}_{\mathcal{H}_0}\right),$$

we get

$$\|u\|_{\mathcal{H}_0} \leq \liminf_{n\to\infty} \|u_n\|_{\mathcal{H}_0}.$$

□

Chapter 2

Framework

In this chapter we define the non-symmetric framework. It is a generalization of the symmetric framework in [BPS05, 2. Preliminaries], which is based on the symmetric bilinear form

$$\mathcal{E}(u,v) \quad = \quad \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a^{i,j}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x)\, m(dx), \;\; u,v \in \mathcal{C}_0^\infty(\mathbb{R}^d)$$

where $a^{i,j} = a^{j,i}$.

2.1 The Non-Symmetric Framework

Let us start by giving the definition of a weight function analogous to the symmetric case. For $\rho \in \mathbb{R}_+$ we define

$$\pi(x) := \exp[-\rho\theta(x)]$$

where

$$\theta \in \mathcal{C}^1(\mathbb{R}^d) \text{ such that } 0 \leq \theta(x) \leq |x|, \text{ if } |x| < 1 \text{ and } \theta(x) = |x|, \text{ if } |x| \geq 1.$$

For simplicity of notation let us define the measure

$$m(dx) := \pi(x)dx.$$

This will be the basic measure in this work.

Notation. From now on, L^2 denotes $L^2(\mathbb{R}^d, m)$ with the just specified density π.

Lemma 2.1. *If $\rho > 0$, then $m(\mathbb{R}^d) < \infty$.*

Proof.

$$\begin{aligned} m(\mathbb{R}^d) &= \int_{\mathbb{R}^d} \exp(-\rho\theta(x))dx \\ &= \underbrace{\int_{\{|x|<1, x\in\mathbb{R}^d\}} \exp(-\rho\theta(x))dx}_{\leq \int_{\{|x|<1,x\in R^d\}} dx \leq const} + \underbrace{\int_{\{|x|\geq 1, x\in\mathbb{R}^d\}} \exp(-\rho|x|)dx}_{=const \int_1^\infty \exp(-\rho r)\, dr \leq const} < \infty \end{aligned}$$

where we have used in the second term of the right hand side the d-dimensional polar coordinates. □

We define for $u, v \in C_0^\infty(\mathbb{R}^d)$ the non-symmetric bilinear form:

$$\begin{aligned}
\mathcal{E}(u,v) &:= \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a^{i,j}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x)\, m(dx) \qquad (2.1)\\
&\quad + \sum_{i=1}^{d} \int_{\mathbb{R}^d} \left(u(x) \frac{\partial v}{\partial x_i}(x) d_i(x) + \frac{\partial u}{\partial x_i}(x) v(x) b_i(x) \right) m(dx)\\
&\quad + \int_{\mathbb{R}^d} u(x)v(x)c(x)\; m(dx)\\
&= \int_{\mathbb{R}^d} \langle A(x)\nabla u(x), \nabla v(x)\rangle\, m(dx) + \int_{\mathbb{R}^d} u(x)\langle d(x), \nabla v(x)\rangle\, m(dx)\\
&\quad + \int_{\mathbb{R}^d} \langle b(x), \nabla u(x)\rangle v(x)\, m(dx) + \int_{\mathbb{R}^d} u(x)v(x)c(x)\, m(dx)
\end{aligned}$$

where $a^{i,j}, b_i, d_i, c \in L^1_{loc}(\mathbb{R}^d, m), 1 \le i,j \le d$ and $A := (a^{i,j})_{1\le i,j\le d}$, cf. [MR92, p.48(2.17)]. Moreover, we introduce the following notation for the symmetric and anti-symmetric part of (2.1):

$$\begin{aligned}
\tilde{\mathcal{E}}(u,v) &= \frac{1}{2}(\mathcal{E}(u,v) + \mathcal{E}(v,u)),\\
\check{\mathcal{E}}(u,v) &= \frac{1}{2}(\mathcal{E}(u,v) - \mathcal{E}(v,u)),\ u,v \in C_0^\infty(\mathbb{R}^d)
\end{aligned}$$

and analogously

$$\begin{aligned}
\tilde{a}^{i,j} &= \frac{1}{2}(a^{i,j} + a^{i,j}),\\
\check{a}^{i,j} &= \frac{1}{2}(a^{i,j} - a^{i,j}).
\end{aligned}$$

Let us denote by $(\mathcal{E}^A, C_0^\infty(\mathbb{R}^d))$ the first part of the bilinear form (2.1)

$$\mathcal{E}^A(u,v) := \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a^{i,j}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x)\, m(dx)$$

and

$$\begin{aligned}
\mathcal{E}^B(u,v) &:= \sum_{i=1}^{d} \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i}(x) v(x) b_i(x)\, m(dx),\\
\mathcal{E}^C(u,v) &:= \int_{\mathbb{R}^d} c(x)u(x)v(x)\, m(dx), \qquad u,v \in C_0^\infty(\mathbb{R}^d).
\end{aligned}$$

We will write $\mathcal{E}(u)$ instead of $\mathcal{E}(u,u)$ and $\mathcal{E}_1(u)$ instead of $\mathcal{E}_1(u,u)$.

Our basic assumptions are that $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ and $(\mathcal{E}^A, C_0^\infty(\mathbb{R}^d))$ are closable and that their closures are coercive closed forms. Furthermore we assume that $(\mathcal{E}, F)$ is a Dirichlet form where F is the closure of $C_0^\infty(\mathbb{R}^d)$ w.r.t. $\tilde{\mathcal{E}}_1^{\frac{1}{2}}$. We denote the sub-Markovian semigroup associated to $(\mathcal{E}, F)$ by $(P_t)_{t\geq 0}$ and the infinitesimal generator of the semigroup by $(L, \mathcal{D}(L))$. We say that L is non-degenerate, if there exists a constant $\nu > 0$ such that $\sum_{i,j} a^{i,j}\xi^i\xi^j \geq \nu|\xi|^2$ for all $\xi \in \mathbb{R}^d$.

A Dirichlet form is regular on $L^2(\mathbb{R}^d, m)$, if $\mathcal{C}_0(\mathbb{R}^d) \cap \mathcal{D}(\mathcal{E})$ is dense in $\mathcal{D}(\mathcal{E})$ w.r.t. $\tilde{\mathcal{E}}_1^{\frac{1}{2}}$ and in $\mathcal{C}_0(\mathbb{R}^d)$ w.r.t. the uniform norm $\| \, \|_\infty$. Easily we see that the Dirichlet form, which is associated to (2.1), is regular on $L^2(\mathbb{R}^d, m)$ and hence quasi regular (cf. [MR92, IV. Definition 3.1 and IV.4 Examples of quasi-regular Dirichlet forms, a)]). Thus, there exists a Markov process X such that $P_t f(x) = E_x[f(X_t)]$ for $f \in L^2$. For more details we refer to [MR92, IV. Markov Processes and Dirichlet Forms] and [MOR95].

Remark 2.2. *(i) Sufficient conditions for the closability of $(\mathcal{E}^A, C_0^\infty(\mathbb{R}^d))$ are given in [FOT94, Section 3.1] and [MR92, II. Section 2.2].*

(ii) [degenerate case] (cf. [RS95, Theorem 1.2] and [MR95])
Let $U \subset \mathbb{R}^d$ open, $d \geq 3, \rho, \sigma \in L^1_{loc}(U, dx), \rho, \sigma > 0$ dx-a.e. and F be the set of all functions $g \in L^1_{loc}(U, dx)$ such that the distributional derivatives $\frac{\partial g}{\partial x_i}, 1 \leq i \leq d$, are in $L^1_{loc}(U, dx)$ such that $\|\nabla g\|(g\sigma)^{-\frac{1}{2}} \in L^\infty(U, dx)$ or $\|\nabla g\|^p(g^{p+1}\sigma^{\frac{p}{q}})^{-\frac{1}{2}} \in L^d(U, dx)$ for some $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1, p < \infty$. We say that a $\mathcal{B}(U)$-measurable function f has property $(A_{\rho,\sigma})$, if one of the following conditions holds:

- $f(\rho\sigma)^{-\frac{1}{2}} \in L^\infty(U, dx)$,
- $f^p(\rho^{p+1}\sigma^{\frac{p}{q}})^{-\frac{1}{2}} \in L^d(U, dx)$ *for some* $p, q \in [1, \infty]$ *with* $\dfrac{1}{p} + \dfrac{1}{q} = 1, p < \infty$,

  and $\rho \in F$.

Suppose that

$(P1)$ $\quad \displaystyle\sum_{i,j=1}^d \tilde{a}^{i,j}\xi_i\xi_j \geq \rho\|\xi\|^2 dx$*-a.e. for all* $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$.

$(P2)$ $\quad \check{a}^{i,j}\rho^{-1} \in L^\infty(U, dx)$.

$(P3)$ $\quad$ *For all $K \subset U, K$ compact , $\mathbb{1}_K\|b + d\|$ and $\mathbb{1}_K c^{\frac{1}{2}}$ have property $(A_{\rho,\sigma})$, and $(c + \alpha_0\sigma)dx - \displaystyle\sum_{i=1}^d \frac{\partial d_i}{\partial x_i}$ is a positive measure on $\mathcal{B}(U)$ for some $\alpha_0 \in (0, \infty)$.*

$(P4)$ $\quad$ *$\|b - d\|$ has property $(A_{\rho,\sigma})$.*

$(P5)$ $\quad \|b\| \in L^1_{loc}(U; dx)$,

$$(c + \alpha_0\sigma)dx - \sum_{i=1}^d \frac{\partial b_i}{\partial x_i} \text{ is a positive measure on } \mathcal{B}(U).$$

Then $(\mathcal{E}_{\alpha_0}, \mathcal{C}_0^\infty(U))$ is closable on $L^2(U, \sigma dx)$ and its closure $(\mathcal{E}_{\alpha_0}, \mathcal{D}(\mathcal{E}_{\alpha_0}))$ is a regular Dirichlet form, where α_0 is given by $(P3)$ and $\mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \alpha(u, v)_{L^2(U,\sigma dx)}$.

(iii) [non-degenerate case] (cf. [MR92, II.Examples 2.d)])

Assume that it holds for $d \geq 3$ and $U \subset \mathbb{R}^d$, U open:

(P1) $$c\,dx - \sum_{i=1}^{d} \frac{\partial d_i}{\partial x_i} \geq 0, \qquad c\,dx - \sum_{i=1}^{d} \frac{\partial b_i}{\partial x_i} \geq 0$$
in the sense of Schwartz distributions.

(P2) *[strong ellipticity condition]*
$$\exists \nu \in]0,\infty[\text{ s.th. } \sum_{i,j=1}^{d} \tilde{a}^{i,j}\xi_i\xi_j \geq \nu\|\xi\|^2_{\mathbb{R}^d} \text{ for all } \xi = (\xi_1,\ldots,\xi_d) \in \mathbb{R}^d,$$

(P3) $\exists M \in]0,\infty[$ *s.th.* $|\check{a}^{i,j}| \leq M \,\forall\, 1 \leq i,j \leq d$.

(P4) $c \in L^{\frac{d}{2}}_{loc}(U,dx), b_i, d_i \in L^d_{loc}(U,dx)$,
$d_i - b_i \in L^d(U,dx) \cup L^\infty(U,dx), 1 \leq i \leq d$.

Then $(\mathcal{E}, C_0^\infty(U))$ is closable and its closure is a Dirichlet form on $L^2(U,dx)$.

Remark 2.3. *If the coefficients of the bilinear form (2.1) fulfill the conditions*

(D1) $$\sum_{i,j=1}^{d} a^{i,j}\xi_i\xi_j \geq 0 \quad m\text{-a.e.} \quad \text{for all } \xi = (\xi_1,\ldots,\xi_d) \in \mathbb{R}^d,$$

(D2) $$\int_{\mathbb{R}^d} \left(cu + \sum_{i=1}^{d} (d_i + b_i) \frac{\partial u}{\partial x_i} \right) dm \geq 0 \quad \text{for all } u \in \mathcal{C}_0^\infty(\mathbb{R}^d), u \geq 0,$$

then $0 \leq \mathcal{E}^A(u,u) \leq \mathcal{E}(u,u)$ for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$.

2.2 The Function Spaces $\mathcal{C}_T$ and $\hat{F}$

Let us introduce the function space (cf. [BPS05, p.21])

$$\mathcal{C}_T := \mathcal{C}^1((0,T); L^2) \cap L^2((0,T); F)$$

with the norm $\|\varphi\|_T := \left(\sup_{t\in[0,T]} \|\varphi_t\|_2^2 + \int_0^T \tilde{\mathcal{E}}(\varphi_t,\varphi_t)\,dt \right)^{\frac{1}{2}}$. We denote the completion of $\mathcal{C}_T$ w.r.t. $\|\cdot\|_T$ by $\hat{F}$. The conditions in the next lemmas are taken from [BPS05, p.21].

Notation. From now on ∂_t denotes the time derivative.

Lemma 2.4. *Let $\varphi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be measurable such that*

(i) $\varphi_t \in F$ *for almost all t,*

(ii) $\int_0^T \mathcal{E}(\varphi_t)\,dt < \infty$,

(iii) $t \mapsto \varphi_t$ *is differentiable in L^2,*

(vi) $t \mapsto \partial_t\varphi_t$ *is L^2-continuous on $[0,T]\}$.*

Then

$$\mathcal{C}_T = \left\{ \varphi : [0,T] \times \mathbb{R}^d \to \mathbb{R} \text{ such that } (i) - (iv) \text{ hold } \right\}.$$

Proof. Let $\varphi \in \mathcal{C}^1((0,T);L^2) \cap L^2((0,T);F)$. Because $\varphi \in L^2((0,T);F)$, it follows $\varphi_t \in F$ for almost all t and

$$\int_0^T \|\varphi_t\|^2_{\tilde{\mathcal{E}}_1^{\frac{1}{2}}} \, dt = \int_0^T \left(\mathcal{E}(\varphi_t) + \|\varphi_t\|_2^2\right) \, dt < \infty. \tag{2.2}$$

Since $\varphi \in \mathcal{C}^1((0,T);L^2)$,

$t \mapsto \varphi_t$ is L^2-differentiable and $t \mapsto \partial_t \varphi_t$ is L^2-continuous on $[0,T]$.

Moreover, we have:

$$\int_0^T \|\varphi_t\|_2^2 \, dt \leq \int_0^T \sup_{t\in[0,T]} \|\varphi_t\|_2^2 \, dt < \infty.$$

Hence, we can reduce (2.2) to

$$\int_0^T \mathcal{E}(\phi_t) \, dt < \infty$$

and the assertion follows. □

Lemma 2.5. *If $\alpha \in \mathcal{C}^1([0,T])$ and $u \in F$, then $\alpha(t)u(x) \in \mathcal{C}_T$.*

Proof. We define $\varphi_t(x) := \varphi(t,x) := \alpha(t)u(x)$. We have to prove $(i) - (iv)$ from Lemma 2.4.

(i) Fix $t \in [0,T]$, then $\varphi_t(x) = \underbrace{\alpha(t)}_{=:c\in\mathbb{R}} u(x) = cu(x)$ and hence $\varphi_t(x) \in F$.

(ii) $\displaystyle\int_0^T \mathcal{E}(\varphi_t) \, dt = \int_0^T \mathcal{E}(\alpha(t)u) \, dt = \mathcal{E}(u) \int_0^T |\alpha(t)|^2 \, dt \underset{\alpha\in\mathcal{C}^1([0,T])}{<} \infty.$

(iii) Since $F \subset L^2$, it holds $u \in L^2$. Therefore, we can calculate:

$$\lim_{h\to 0} \left\| \frac{\varphi_{t+h} - \varphi_t}{h} - \partial_t \varphi(t,\cdot) \right\|_2 = \lim_{h\to 0} \left\| \frac{\alpha(t+h) - \alpha(t)}{h} u - \partial_t \alpha(t) u \right\|_2$$
$$= \lim_{h\to 0} \left| \frac{\alpha(t) - \alpha(t+h)}{h} - \partial_t \alpha(t) \right| \|u\|_2 = 0,$$

(iv) $$\lim_{h\to 0} \|\partial_t(\alpha(t)u) - \partial_t(\alpha(t+h)u)\|_2 = \lim_{h\to 0} \|\big(\partial_t\alpha(t) - \partial_t\alpha(t+h)\big)u\|_2$$
$$= \lim_{h\to 0} |\partial_t \alpha(t) - \partial_t \alpha(t+h)| \, \|u\|_2 = 0.$$

□

Lemma 2.6. *If $u \in \mathcal{C}_T$ and $\varphi \in \mathcal{C}^2(\mathbb{R})$ such that $\varphi(0) = 0$ and the functions φ' and φ'' are bounded by $K \in \mathbb{R}, K \neq 0$, then $\varphi(u) \in \mathcal{C}_T$ and $\partial_t \varphi(u_t) = \varphi'(u_t)\partial_t u_t$.*

Proof. We will show $(i) - (iv)$ from Lemma 2.4.
$(i) - (ii)$. By Theorem 1.13 we obtain $\varphi(u_t) \in \mathcal{D}(\mathcal{E}) = F$ and

$$\int_0^T \mathcal{E}(\varphi(u_t)) \, dt \leq K^2 \int_0^T \mathcal{E}(u_t) \, dt < \infty.$$

(iii)

$$\begin{aligned}
& \lim_{h\to 0}\left\|\frac{\varphi(u_{t+h})-\varphi(u_t)}{h}-\varphi'(u_t)\partial_t u_t\right\|_2 \\
= \; & \lim_{h\to 0}\left\|\frac{u_{t+h}-u_t}{h}\varphi'(\xi_h)-\varphi'(u_t)\partial_t u_t\right\|_2 \\
\leq \; & \lim_{h\to 0}\left\|\left|\frac{u_{t+h}-u_t}{h}-\partial_t u_t\right| |\varphi'(\xi_h)|\right\|_2 \\
& + \lim_{h\to 0}\| |\varphi'(\xi_h)-\varphi'(u_t)| \, |\partial_t u_t| \|_2 \\
\leq \; & \lim_{h\to 0} K\left\|\frac{u_{t+h}-u_t}{h}-\partial_t u_t\right\|_2 + \lim_{h\to 0}\| |\varphi'(\xi_h)-\varphi'(u_t)|\partial_t u_t\|_2 \\
\underset{(\star)}{=} \; & 0
\end{aligned}$$

where for every fixed $x \in \mathbb{R}^d$ we have chosen by the mean value theorem $\xi_h \in [u_{t+h}, u_t]$. Note that $\xi_h \to u_t$ as $h \to 0$ in L^2:

$$\|\xi_h - u_t\|_2 \leq \|u_{h+t} - u_t\|_2 \to 0.$$

Now we will show $(\star)$. Since $u \in \mathcal{C}_T$, the first term converges to zero. Therefore, we have to examine the second term. This will be done in five steps. Let h_n be a sequence such that $h_n \to 0$.

(1.) Since $\partial_t u_t$ is an element of L^2, it follows that $(|\partial_t u_t|^2)$ is uniformly $m-$integrable.
(2.) Since φ'' is bounded by K, φ' is Lipschitz continuous with the constant K.
(3.) Let $\varepsilon > 0$.

$$\begin{aligned}
& m\left(\left\{x \in \mathbb{R}^d \middle| \, |\varphi'(\xi_{h_n}(x)) - \varphi'(u_t(x))||\partial_t u_t(x)| > \varepsilon\right\}\right) \\
\underset{(2.)}{\leq} \; & m\left(\left\{x \in \mathbb{R}^d \middle| \, |\xi_{h_n}(x) - u_t(x)||\partial_t u_t(x)| > \frac{\varepsilon}{K}\right\}\right) \\
\underset{\text{Markov's inequality}}{\leq} \; & \frac{K}{\varepsilon}\int_{\mathbb{R}^d} |\xi_{h_n}(x) - u_t(x)||\partial_t u_t(x)| \, m(dx) \\
\leq \; & \frac{K}{\varepsilon}\Big(\|\xi_{h_n} - u_t\|_2 \|\partial_t u_t\|_2\Big) \\
\to \; & 0
\end{aligned}$$

(4.) Since φ' is bounded by K and $(|\partial_t u_t|^2)$ is uniformly $m-$integrable, we conclude that $(|\varphi'(\xi_{h_n})|^2 |\partial_t u_t|^2)_{n\in\mathbb{N}}$ is uniformly $m-$integrable.
(5.) From (3.) and (4.) it follows that

$$\varphi'(\xi_{h_n})\partial_t u_t \to \varphi'(u_t)\partial_t u_t \text{ in } L^2.$$

(iv) The proof of this point will be done analogous to (iii). Let h_n be a sequence such that $h_n \to 0$. Since

$$\|\partial_t \varphi(u_t) - \partial_t \varphi(u_{t+h_n})\|_2 = \|\varphi'(u_t)\partial_t u_t - \varphi'(u_{t+h_n})\partial_t u_{t+h_n}\|_2,$$

we have to show in L^2

$$\varphi'(u_{t+h_n})\partial_t u_{t+h_n} \to \varphi'(u_t)\partial_t u_t.$$

(1.) Since we have $\partial_t u_{t+h_n} \to \partial_t u_t$ in L^2, it follows $\partial_t u_{t+h_n} \to \partial_t u_t$ in $m-$measure. It also follows that $(|\partial_t u_{t+h_n}|^2)_{n\in\mathbb{N}}$ is uniformly $m-$integrable.
(2.) Since φ'' is bounded, φ' is Lipschitz continuous with constant K.
(3.) Let $\varepsilon > 0$. Then it follows by (2.) and (iii)

$$\begin{aligned}
& m\left(\left\{x \in \mathbb{R}^d \middle| \, |\varphi'(u_t(x))\partial_t u_t(x) - \varphi'(u_{t+h_n}(x))\partial_t u_{t+h_n}(x)| > \varepsilon\right\}\right) \\
\leq\ & m\left(\left\{x \in \mathbb{R}^d \middle| \, |\varphi'(u_t(x))\partial_t u_t(x) - \varphi'(u_{t+h_n}(x))\partial_t u_t(x)| > \varepsilon\right\}\right) \\
& + m\left(\left\{x \in \mathbb{R}^d \middle| \, |\varphi'(u_{t+h_n}(x))\partial_t u_t(x) - \varphi'(u_{t+h_n}(x))\partial_t u_{t+h_n}(x)| > \varepsilon\right\}\right) \\
\leq\ & m\left(\left\{x \in \mathbb{R}^d \middle| \, |\varphi'(u_t(x)) - \varphi'(u_{t+h_n}(x))||\partial_t u_t(x)| > \varepsilon\right\}\right) \\
& + m\left(\left\{x \in \mathbb{R}^d \middle| \, K|\partial_t u_t(x) - \partial_t u_{t+h_n}(x)| > \varepsilon\right\}\right) \\
\leq\ & m\left(\left\{x \in \mathbb{R}^d \middle| \, K|u_t(x) - u_{t+h_n}(x)||\partial_t u_t(x)| > \varepsilon\right\}\right) \\
& + m\left(\left\{x \in \mathbb{R}^d \middle| \, K|\partial_t u_t(x) - \partial_t u_{t+h_n}(x)| > \varepsilon\right\}\right).
\end{aligned}$$

Since K is constant and (1.), we conclude:

$$\begin{aligned}
& m\left(\left\{x \in \mathbb{R}^d \middle| \, K|\partial_t u_t(x) - \partial_t u_{t+h_n}(x)| > \varepsilon\right\}\right) \\
=\ & m\left(\left\{x \in \mathbb{R}^d \middle| \, |\partial_t u_t(x) - \partial_t u_{t+h_n}(x)| > \frac{\varepsilon}{K}\right\}\right) \\
\to\ & 0.
\end{aligned}$$

Now we examine the other term

$$\begin{aligned}
& m\left(\left\{x \in \mathbb{R}^d \middle| \, K|u_t(x) - u_{t+h_n}(x)||\partial_t u_t(x)| > \varepsilon\right\}\right) \\
=\ & m\left(\left\{x \in \mathbb{R}^d \middle| \, |u_t(x) - u_{t+h_n}(x)||\partial_t u_t(x)| > \frac{\varepsilon}{K}\right\}\right) \\
\underset{\text{Markov's inequality}}{\leq}\ & \frac{K}{\varepsilon} \int_{\mathbb{R}^d} |u_t(x) - u_{t+h_n}(x)||\partial_t u_t(x)| \, m(dx) \\
\leq\ & \frac{K}{\varepsilon}\Big(\|u_t - u_{t+h_n}\|_2 \|\partial_t u_t\|_2\Big) \\
\to\ & 0.
\end{aligned}$$

Hence, it follows that $\partial_t(\varphi(u_{t+h_n})) \to \partial_t(\varphi(u_t))$ in $m-$measure.
(4.) Since φ' is bounded by K and $(|\partial_t u_{t+h_n}|^2)_{n\in\mathbb{N}}$ is uniformly $m-$integrable, we conclude that $(|\varphi'(u_{t+h_n}(x))\partial_t u_{t+h_n}(x)|^2)_{n\in\mathbb{N}}$ is uniformly $m-$integrable.
(5.) From (3.) and (4.) it follows that

$$\varphi'(u_{t+h_n})\partial_t u_{t+h_n} \to \varphi'(u_t)\partial_t u_t \text{ in } L^2.$$

□

We refer for detailed information about uniform integrability to [Bau92, Chapter 21].

The assertion of the next lemma is taken from [BPS05]. In the proof we will use instead of the $C^1(\mathbb{R})$ functions, which appear in the idea of the proof in the original paper, $C^2(\mathbb{R})$ functions such that we can apply the above lemma.

Lemma 2.7. *$b\mathcal{C}_T$ is dense in $\mathcal{C}_T$ w.r.t. $\|\cdot\|_T$.*

Proof. We define the function $\varphi_n \in C^2(\mathbb{R}), n \in \mathbb{N}$ by

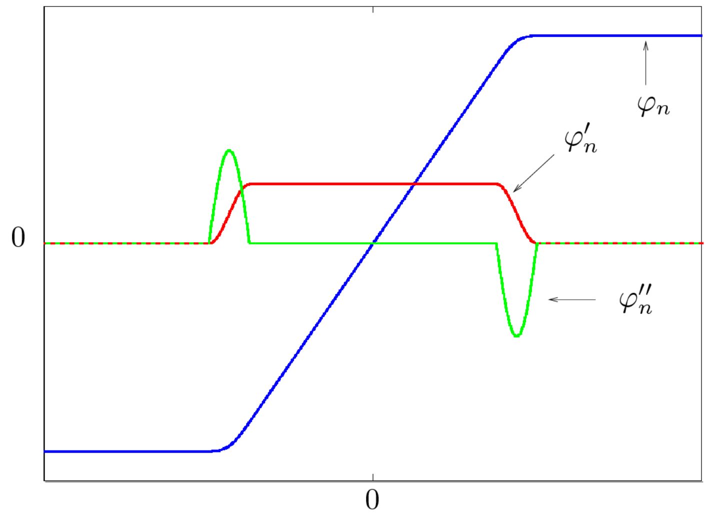
$$\varphi_n(x) := \begin{cases} x & \text{for } |x| \le n \\ \frac{\sin(\pi(n+1-x))}{2\pi} + \frac{x}{2} - \frac{n}{2} & \text{for } x \in]-(n+1), -n[\\ \frac{\sin(\pi(n+1-x))}{2\pi} + \frac{x}{2} + \frac{n}{2} & \text{for } x \in]n, n+1[\\ sign(x) \cdot (n + \frac{1}{2}) & \text{for } |x| \ge n+1 \end{cases}$$

with the derivatives

$$\varphi_n'(x) = \begin{cases} 1 & \text{for } |x| \le n \\ \frac{1}{2} - \frac{1}{2}\cos(\pi(n+1-x)) & \text{for } x \in]-(n+1), -n[\cup]n, n+1[\\ 0 & \text{for } |x| \ge n+1 \end{cases}$$

and

$$\varphi_n''(x) = \begin{cases} 0 & \text{for } |x| \le n \\ -\pi\frac{\sin(\pi(n+1-x))}{2} & \text{for } x \in]-(n+1), -n[\cup]n, n+1[\\ 0 & \text{for } |x| \ge n+1. \end{cases}$$



It is obvious that for all $n \in \mathbb{N}$ the functions φ_n, φ_n' and φ_n'' are bounded (i.e. $|\varphi_n(u_t(x))| \le n + \frac{1}{2}$ for all $t \in [0,T], x \in \mathbb{R}^d$) and $\varphi_n(0) = 0$.

Now let us prove that from $u \in \mathcal{C}_T$ follows $\varphi_n(u) \in b\mathcal{C}_T$.

$$\begin{aligned} \varphi_n' \text{ and } \varphi_n'' \text{ are bounded}, \varphi_n \in \mathcal{C}^2(\mathbb{R}), \varphi_n(0) = (0) \quad &\underset{\text{Lemma 2.6}}{\Rightarrow} \quad \varphi_n(u) \in \mathcal{C}_T, \\ \varphi_n \text{ bounded} \quad &\Rightarrow \quad \varphi_n(u) \in b\mathcal{C}_T. \end{aligned}$$

Easily we see that for all $x \in \mathbb{R}$ and $t \in [0,T]$ it holds

$$\lim_{n\to\infty} \varphi_n(u_t(x)) = u_t(x) \quad , \quad |\varphi_n(u_t(x))| \le |u_t(x)|$$

and since $|\varphi_n(t)| \leq |\varphi_{n+1}(t)|$,

$$\|u_t(x) - \varphi_n(u_t(x))\|_2 \geq \|u_t(x) - \varphi_{n+1}(u_t(x))\|_2 \text{ for all } n \in \mathbb{N}.$$

Hence

$$\psi_n(t) := \|u_t(x) - \varphi_n(u_t(x))\|_2 \searrow 0.$$

Since u and $\varphi_n(u)$ are elements of $\mathcal{C}_T$, it follows that ψ_n is continuous on $[0, T]$. By Dini's theorem we get

$$\sup_{t \in [0,T]} \|\varphi_n(u_t(x)) - u_t(x)\|_2 \to 0.$$

Next we will show for a new sequence $(\tilde{\varphi}_n(u_t(x)))_{n\in\mathbb{N}}$ where $\tilde{\varphi}_n(u_t(x)) := \frac{1}{n}\sum_{k=1}^n \varphi_{n_k}(u_t(x))$:

$$\lim_{n\to\infty} \int_0^T \mathcal{E}(\tilde{\varphi}_n(u_t) - u_t)\, dt = 0.$$

Since φ_n' is uniformly bounded and $\varphi_n(0) = 0$, we have by Theorem 1.13 for a constant K

$$\int_0^T \mathcal{E}(\varphi_n(u_t))\, dt \leq K^2 \int_0^T \mathcal{E}(u_t)\, dt.$$

Hence,

$$\sup_{n\in\mathbb{N}} \int_0^T \mathcal{E}_1(\varphi_n(u_t))\, dt < \infty.$$

By Lemma 1.19 there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ such that for the Cesáro mean $\tilde{\varphi}_n(u_t) := \frac{1}{n}\sum_{k=1}^n \varphi_{n_k}(u_t)$ it follows that

$$\lim_{n\to\infty} \int_0^T \mathcal{E}(\tilde{\varphi}_n(u_t) - u_t)\, dt = 0.$$

Since $\sup_{t\in[0,T]} \|\varphi_n(u_t(x)) - u_t(x)\|_2 \to 0$, we obtain by Lemma 1.18 that

$$\sup_{t\in[0,T]} \left\| \frac{1}{n} \sum_{n=1}^n \varphi_{n_k}(u_t) - u_t \right\|_2 \to 0.$$

Finally, we get $\|\tilde{\varphi}_n(u) - u\|_T \to 0$, where $\tilde{\varphi}_n(u) \in b\mathcal{C}_T$. □

Lemma 2.8. *With the definitions of the above proof it holds*

$$\int_0^T \|\partial_t \tilde{\varphi}_n(u_t) - \partial_t(u_t)\|_2\, dt \to 0.$$

Proof.

$$\begin{aligned} \lim_{n\to\infty} \int_0^T \|\partial_t \tilde{\varphi}_n(u_t) - \partial_t(u_t)\|_2\, dt &= \lim_{n\to\infty} \int_0^T \|\tilde{\varphi}_n'(u_t)\partial_t u_t - \partial_t(u_t)\|_2\, dt \\ &= \lim_{n\to\infty} \int_0^T \|\partial_t u_t(\tilde{\varphi}_n'(u_t) - 1)\|_2\, dt \\ &= 0 \end{aligned}$$

□

Lemma 2.9. *If $u \in \hat{F}$ and $\varphi \in F$, then we have $\int_0^T u_t\,dt \in F$ and*

$$\mathcal{E}\left(\int_0^T u_t\,dt, \varphi\right) = \int_0^T \mathcal{E}(u_t, \varphi)\,dt.$$

Proof. Let $u \in \hat{F}$. Since $\|u\|_T < \infty$, we deduce that u is an element of $L^2((0,T);F)$ and conclude

$$\int_0^T \|u\|_{\tilde{\mathcal{E}}_1^{\frac{1}{2}}}\,dt \leq T^{\frac{1}{2}} \left(\int_0^T \|u\|^2_{\tilde{\mathcal{E}}_1^{\frac{1}{2}}}\,dt\right)^{\frac{1}{2}} < \infty.$$

Let $\mathcal{E}^\varphi(u_t) := \mathcal{E}(u_t, \varphi_t)$, then $\mathcal{E}^\varphi : F \to \mathbb{R}$ is obviously linear. Moreover, we have:

$$\begin{aligned}
\|\mathcal{E}^\varphi\| &= \sup_{\|v\|_{\tilde{\mathcal{E}}_1^{\frac{1}{2}}}=1} |\mathcal{E}^\varphi(v)| \\
&\leq \sup_{\|v\|_{\tilde{\mathcal{E}}_1^{\frac{1}{2}}}=1} \left(|\mathcal{E}_1^\varphi(v)| + |(\varphi, v)|\right) \\
&\leq \sup_{\|v\|_{\tilde{\mathcal{E}}_1^{\frac{1}{2}}}=1} \left(|K_{\mathcal{E}}\mathcal{E}_1(\varphi)^{\frac{1}{2}}\mathcal{E}_1(v)^{\frac{1}{2}}| + \|\varphi\|_2\|v\|_2\right) \\
&\leq |K_{\mathcal{E}}\mathcal{E}_1(\varphi)^{\frac{1}{2}}| + \|\varphi\|_2 < \infty.
\end{aligned}$$

Hence, we get $\mathcal{E}^\varphi \in L(F, \mathbb{R})$. Now we can use (ii) of Proposition $A.1$ to conclude the assertion

$$\mathcal{E}\left(\int_0^T u_t\,dt, \varphi\right) = \int_0^T \mathcal{E}(u_t, \varphi)\,dt.$$

□

Remark: It is enough to assume $u \in L^1([0,T];F)$ in Lemma 2.9.

The next lemma presents a useful representation of $\hat{F}$. It is taken from [BPS05, Lemma 2.1], where already a very rough idea of the proof is given. Here we will give this proof with all details.

Lemma 2.10. *$\hat{F} = \mathcal{C}([0,T];L^2) \cap L^2((0,T);F)$.*

Proof. ($\subset$) Let $u \in \hat{F}$. Since $\|u\|_T < \infty$, we deduce $u \in L^2((0,T);F)$. Let $u^n \in \mathcal{C}_T$ such that $\|u^n - u\|_T \to 0$. Then we have $\sup_{t\in[0,T]} \|u_t^n - u_t\|_2^2 \to 0$ and can conclude that $u \in \mathcal{C}([0,T];L^2)$.

($\supset$) Let $u \in \mathcal{C}([0,T];L^2) \cap L^2((0,T);F)$. We have to show the existence of a sequence $u_n \in \mathcal{C}_T$ such that

$$\lim_{n\to\infty} \|u_n - u\|_T = 0.$$

Then we can deduce $u \in \hat{F}$, which proves the lemma.

Step 1: Let $\varepsilon > 0$. Define $\tilde{u}_t = u_t$ for $t \in [0,T]$ and $\tilde{u}_t = u_{2T-t}$ for $t \in [T, T+\varepsilon]$. Then set

$$u_{n,t} = n \int_0^{1/n} \tilde{u}_{t+s}\, ds, \quad t \in [0,T], n \in \mathbb{R}_+, n > \frac{1}{\varepsilon}.$$

First we will show that every u_n belongs to $\mathcal{C}^1([0,T], L^2)$. This can be seen as follows:

- $t \mapsto u_{n,t}$ is L^2-continuous:

$$\begin{aligned}
\lim_{h\to 0} \|u_{n,t} - u_{n,t+h}\|_2 &\leq \lim_{h\to 0} n \int_0^{\frac{1}{n}} \|\tilde{u}_{t+s} - \tilde{u}_{t+s+h}\|_2\, ds \\
&= n \int_0^{\frac{1}{n}} \lim_{h\to 0} \|\tilde{u}_{t+s} - \tilde{u}_{t+s+h}\|_2\, ds \\
&= 0.
\end{aligned}$$

- $t \mapsto u_{n,t}$ is L^2-differentiable:

We only consider the case $\frac{1}{n} > h > 0$. The other case $-\frac{1}{n} < h < 0$ can be treated analogously.

$$\begin{aligned}
&\left\| \frac{u_{n,t+h} - u_{n,t}}{h} - n\left(u_{t+\frac{1}{n}} - u_t\right) \right\|_2 \\
= \;& n \left\| \frac{\int_{t+h}^{t+\frac{1}{n}+h} \tilde{u}_s\, ds - \int_t^{t+\frac{1}{n}} \tilde{u}_s\, ds}{h} - \left(u_{t+\frac{1}{n}} - u_t\right) \right\|_2 \\
= \;& n \left\| \frac{-\int_t^{t+h} \tilde{u}_s\, ds + \int_{t+\frac{1}{n}}^{t+\frac{1}{n}+h} \tilde{u}_s\, ds}{h} - \left(u_{t+\frac{1}{n}} - u_t\right) \right\|_2 \\
\leq \;& n \left\| \frac{\int_t^{t+h} \tilde{u}_s\, ds}{h} - u_t \right\|_2 + n \left\| \frac{\int_{t+\frac{1}{n}}^{t+\frac{1}{n}+h} \tilde{u}_s\, ds}{h} - u_{t+\frac{1}{n}} \right\|_2 \\
= \;& n \left\| u_{\frac{1}{h},t} - u_t \right\|_2 + n \left\| u_{\frac{1}{h},t+\frac{1}{n}} - u_{t+\frac{1}{n}} \right\|_2 \\
\underset{\substack{h\searrow 0\\(\star)}}{\longrightarrow} \;& 0
\end{aligned}$$

($\star$) In step 2 we show $\lim_{h\searrow 0} \|u_{\frac{1}{h},t} - u_t\|_2 = 0$.

- $t \mapsto \partial_t(u_{n,t})$ is L^2-continuous:

$$\begin{aligned}
&\|\partial_t(u_{n,t}) - \partial_t(u_{n,t+h})\|_2 \\
= \;& \left\| n\left(u_{t+\frac{1}{n}} - u_t\right) - n\left(u_{t+h+\frac{1}{n}} - u_t\right) \right\|_2 \\
= \;& n \left\| u_{t+\frac{1}{n}} - u_{t+h+\frac{1}{n}} \right\|_2 \\
\underset{h\to 0}{\longrightarrow} \;& 0.
\end{aligned}$$

Hence, $u_n \in \mathcal{C}^1([0,T], L^2)$. Moreover, by Lemma 2.9 it follows that $u_{n,t} \in F$.

Since

$$\begin{aligned}
\int_0^T \mathcal{E}_1(u_{n,t})\,dt &= \int_0^T \mathcal{E}_1\left(n\int_0^{\frac{1}{n}} \tilde{u}_{t+s}\,ds\right)\,dt \\
&\leq \int_0^T n^2\left(\int_0^{\frac{1}{n}} \mathcal{E}_1\left(\tilde{u}_{t+s}\right)^{\frac{1}{2}}\,ds\right)^2\,dt \\
&\leq \int_0^T n\left(\int_0^{\frac{1}{n}} \mathcal{E}_1\left(\tilde{u}_{t+s}\right)\,ds\right)\,dt \\
&\leq n\int_0^{\frac{1}{n}}\left(\int_s^{T+s} \mathcal{E}_1(\tilde{u}_t)\,dt\right)\,ds \\
&= n\int_0^{\frac{1}{n}}\left(\int_s^{T} \mathcal{E}_1(\tilde{u}_t)\,dt + \int_T^{T+s} \mathcal{E}_1(\tilde{u}_t)\,dt\right)\,ds \\
&\leq 2\int_0^T \mathcal{E}_1(u_t)\,dt \\
&< \infty,
\end{aligned}$$

we finally deduce by Lemma 2.4 that $u_n \in \mathcal{C}_T$.

Step 2: In this step we will show that the Cesáro mean of a subsequence of $(u_n)_{n\in\mathbb{N}}$ converges to u in $\|\,\|_T$.

Note that $t \mapsto \tilde{u}_t$ is L^2-continuous. Let $\varepsilon > 0$, then there exists $\delta > 0$ such that:

$$\|\tilde{u}_{t+s} - \tilde{u}_t\|_2 < \varepsilon \quad \text{for all } |s| < \delta, t \in [0,T].$$

So we get for $n > \frac{1}{\delta}$

$$\begin{aligned}
\|u_{n,t} - u_t\|_2 &\leq n\int_0^{\frac{1}{n}} \|\tilde{u}_{t+s} - \tilde{u}_t\|_2\,ds \\
&\leq n\int_0^{\frac{1}{n}} \varepsilon\,ds = \varepsilon \quad \text{for all } t \in [0,T], n > N := \frac{1}{\delta}
\end{aligned}$$

and can deduce

$$\lim_{n\to\infty}\sup_{t\in[0,T]} \|u_{n,t} - u_t\|_2 = 0.$$

Next we will show by Lemma 1.19

$$\int_0^T \mathcal{E}(u_{n,t} - u_t)\,dt \quad \stackrel{!}{\to} \quad 0.$$

Let $\mathcal{H} := L^2([0,T];L^2)$ and $\mathcal{H}_0 := L^2([0,T];F)$ be Hilbert spaces. We have to check the conditions of Lemma 1.19:

Since we have $\sup_{t\in[0,T]} \|u_{n,t} - u_t\|_2 \to 0$, we deduce $u_{n,t} \to u_t$ in $\mathcal{H}$. In step 1 we have already shown $\sup_{n\in\mathbb{N}} \int_0^T \mathcal{E}_1(u_{n,t})\,dt < \infty$. Hence, it follows

$$\sup_n \|u_n\|_{\mathcal{H}_0} = \sup_n \int_0^T \|u_{n,t}\|^2_{\tilde{\mathcal{E}_1}^{\frac{1}{2}}}\,dt < \infty.$$

By Lemma 1.19 we obtain

$$\frac{1}{N}\sum_{k=1}^{N} u_{n_k} \to u \text{ in } \mathcal{H}_0.$$

Since $\lim_{n\to\infty}\sup_{t\in[0,T]}\|u_{n,t}-u_t\|_2 = 0$, it follows by Lemma 1.18 that

$$\lim_{k\to\infty}\sup_{t\in[0,T]}\left\|\frac{1}{N}\sum_{n=1}^{N} u_{n_k,t}-u_t\right\|_2 = 0.$$

□

The assertion of the following lemma is taken from the proof of [BPS05, Lemma 2.2].

Lemma 2.11. *Let $(\mathcal{E},\mathcal{C}_0^\infty(\mathbb{R}^d))$ be closable and define $\mathcal{A}(u,v) := \int_0^T \mathcal{E}(u_t,v_t)\,dt$ for $u,v\in\mathcal{C}_0^\infty([0,T]\times\mathbb{R}^d)$. Then $(\mathcal{A},\mathcal{C}_0^\infty([0,T]\times\mathbb{R}^d))$ is also closable.*

Proof. Let $u^n \in \mathcal{C}_0^\infty([0,T]\times\mathbb{R}^d)$ be a sequence such that

(*i*) $u^n \to 0$ in $L^2((0,T)\times\mathbb{R}^d,\ dt\times\ m(dx))$

(*ii*) $(u^n)_{n\in\mathbb{N}}$ is Cauchy with respect to the norm induced by $\mathcal{A}_1$,

where $\mathcal{A}_1(u,v) := \int_0^T \mathcal{E}_1(u_t,v_t)\,dt$. By (*i*) and (*ii*) we can find a subsequence $(n_k)_{k\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ such that for almost every $t\in(0,T)$

$$u_t^{n_k} \to 0 \text{ in } L^2$$

and

$(u_t^{n_k})_k$ is Cauchy with respect to the norm induced by $\mathcal{E}_1$.

Since $\mathcal{E}$ is closable, we have:

$$\mathcal{E}_1(u_t^{n_k}) \to 0 \text{ for a.e. } t.$$

Therefore, we deduce

$$\begin{aligned}
\mathcal{A}_1(u^n,u^n) &= \int_0^T \mathcal{E}_1(u_t^n,u_t^n)\,dt \\
&= \int_0^T \lim_{k\to\infty}\left(\mathcal{E}_1(u_t^{n_k}-u_t^n,u_t^{n_k}) - \mathcal{E}_1(u_t^{n_k}-u_t^n,u_t^n)\right)dt \\
&= \int_0^T \lim_{k\to\infty}\mathcal{E}_1(u_t^{n_k}-u_t^n) \\
&\underset{Fatou's\ lemma}{\leq} \liminf_{k\to\infty}\int_0^T \mathcal{E}_1(u_t^{n_k}-u_t^n)\,dt.
\end{aligned}$$

By (ii) the last term can be made arbitrarily small by choosing n large enough. □

The next lemma shows the existence of approximation sequences of $\mathcal{C}_0^\infty([0,T]\times\mathbb{R}^d)$ functions for elements in $\hat{F}$. We follow the idea of [BPS05, Lemma 2.2].

Lemma 2.12. *For every $u \in \hat{F}$ there exists a sequence $u^n \in \mathcal{C}_0^\infty([0,T] \times \mathbb{R}^d)$, $n \in \mathbb{N}$ such that $\int_0^T \mathcal{E}_1(u_t - u_t^n)\, dt \to 0$.*

Proof. Define the bilinear form $\mathcal{A}_1(u,v) := \int_0^T \mathcal{E}_1(u_t, v_t)\, dt$ and

$$Q := \overline{\mathcal{C}_0^\infty([0,T] \times \mathbb{R}^d)}^{\tilde{\mathcal{A}}_1^{\frac{1}{2}}}$$

where $\tilde{\mathcal{A}}_1$ is the symmetric part of $\mathcal{A}_1$. By Lemma 2.11 $(\mathcal{A}, \mathcal{C}_0^\infty([0,T] \times \mathbb{R}^d))$ is closable. Therefore, Q is contained in the space $H := L^2((0,T); F)$. Now we will prove that $Q = H$.

If we can check that

$$w \in H \text{ and } w \perp Q \text{ (in the sense of } \tilde{\mathcal{A}}_1^{\frac{1}{2}}) \Rightarrow w = 0,$$

the assertion will follow immediately.

Let us assume that

$$\int_0^T \tilde{\mathcal{E}}_1(w_t, \phi_t)\, dt = 0 \text{ for all } \phi \in \mathcal{C}_0^\infty([0,T] \times \mathbb{R}^d).$$

Then we obtain by replacing ϕ_t with $\alpha_t \psi$, where $\alpha \in \mathcal{C}^\infty([0,T])$ and $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, the following equation

$$\int_0^T \tilde{\mathcal{E}}_1(w_t, \alpha_t \psi)\, dt = 0 \qquad \text{for all } \alpha_t \psi.$$

Therefore, we have for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$

$$\int_0^T \alpha_t \tilde{\mathcal{E}}_1(w_t, \psi)\, dt = 0 \qquad \text{for all } \alpha \in \mathcal{C}^\infty([0,T]).$$

Hence, we deduce that for almost every t: $\tilde{\mathcal{E}}_1(w_t, \psi) = 0$ for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. Since $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is separable and $\tilde{\mathcal{E}}_1(\cdot,\cdot)$ is an inner product, it follows that $w_t = 0$ for almost every t.

We have shown that $\overline{\mathcal{C}_0^\infty([0,T] \times \mathbb{R}^d)}^{\tilde{\mathcal{A}}_1^{\frac{1}{2}}} = L^2((0,T); F)$. This means for every $u \in \hat{F} (\Rightarrow u \in L^2((0,T); F))$ there exists a sequence $(u^n)_{n \in \mathbb{N}}$, $u^n \in \mathcal{C}_0^\infty([0,T] \times \mathbb{R}^d)$, such that $\int_0^T \mathcal{E}_1(u_t - u_t^n)\, dt \to 0$. $\square$

Chapter 3

The Linear Equation

In this chapter we consider the linear equation

$$\begin{aligned} (\partial_t + L)u_t(x) + f_t(x) &= 0, \qquad \forall\, 0 \le t \le T \\ u_T(x) &= \phi(x), \quad x \in \mathbb{R}^d \end{aligned} \tag{3.1}$$

where $f \in L^1([0,T];L^2)$, $\phi \in L^2$ and the operator $(L, \mathcal{D}(L))$ is associated to the bilinear form (2.1). Section 3.1 follows the ideas of [BPS05, Section 2.1]. In Section 3.2 we present basic relations for a weak solution. The main ideas are taken from [BPS05].

3.1 Solution of the Linear Equation

We start by giving the definitions and basic properties of weak and strong solutions of the linear equation (3.1).

Definition 3.1. [*strong solution*]
A function $u \in \hat{F} \cap L^1((0,T);\mathcal{D}(L))$ is called a strong solution of equation (3.1) *with data (ϕ, f), if $t \mapsto u_t$ is L^2-differentiable on* $[0,T]$, $\partial_t u_t \in L^1((0,T);L^2)$ *and the equalities in* (3.1) *hold almost everywhere.*

Definition 3.2. [*weak solution*]
A function $u \in \hat{F}$ is called a weak solution of equation (3.1), *if the following relation holds:*

$$\int_0^T ((u_t, \partial_t \varphi_t) + \mathcal{E}(u_t, \varphi_t))\, dt = \int_0^T (f_t, \varphi_t)\, dt + (\phi, \varphi_T) - (u_0, \varphi_0) \quad \forall \varphi \in \mathcal{C}_T. \tag{3.2}$$

Note that we have not assumed $u \in L^1((0,T);\mathcal{D}(L))$ in this definition.

Lemma 3.3. *Every strong solution is a weak solution.*

Proof. Let u be a strong solution. Then the equation

$$(\partial_t + L)u_t + f_t = 0$$

holds almost everywhere and we can derive for $\varphi \in \mathcal{C}_T$ and a.e. $t \in [0,T]$

$$0 = ((\partial_t u_t + Lu_t + f_t), \varphi_t) = (\partial_t u_t, \varphi_t) + (Lu_t, \varphi_t) + (f_t, \varphi_t).$$

Then we have

$$\int_0^T ((\partial_t u_t, \varphi_t) + (Lu_t, \varphi_t) + (f_t, \varphi_t))\, dt = 0.$$

With integration by parts we deduce

$$\int_0^T (\partial_t (u_t, \varphi_t) - (u_t, \partial_t \varphi_t) + (Lu_t, \varphi_t) + (f_t, \varphi_t))\, dt = 0$$

and since $-(Lu_t, \varphi_t) = \mathcal{E}(u_t, \varphi_t)$ for all $u_t \in \mathcal{D}(L)$, we get

$$\int_0^T ((u_t, \partial_t \varphi_t) + \mathcal{E}(u_t, \varphi_t))\, dt = \int_0^T (f_t, \varphi_t)\, dt + (u_T, \varphi_T) - (u_0, \varphi_0).$$

□

Lemma 3.4. *(i) The equation* (3.2) *is equivalent to*

$$\int_{t_0}^T ((u_t, \partial_t \varphi_t) + \mathcal{E}(u_t, \varphi_t))\, dt = \int_{t_0}^T (f_t, \varphi_t)\, dt + (\phi, \varphi_T) - (u_{t_0}, \varphi_{t_0}) \tag{3.3}$$

for every $t_0 \in [0, T]$ *and every* $\varphi \in \mathcal{C}_T$.
(ii) A weak solution satisfies $u_T = \phi$.

Proof. (i) We have

$$\int_0^T ((u_t, \partial_t \varphi_t) + \mathcal{E}(u_t, \varphi_t))\, dt = \int_0^T (f_t, \varphi_t)\, dt + \int_0^T \partial_t (u_t, \varphi_t)\, dt \quad \forall \varphi \in \mathcal{C}_T.$$

Fix $t_0 \in (0, T)$. By using an integration by parts formula and approximating functions $\psi_n \in \mathcal{C}^1([0, T]; \mathbb{R})$, where $\psi_n(t) = 1$ for $t \in [0, t_0]$, $0 \le \psi_n(t) \le 1$ for $t \in [t_0, t_0 + \varepsilon_n]$, $\varepsilon_n \to 0$, $\psi_n(t) = 0$ for $t \in [t_0 + \varepsilon_n, T]$ and $\psi_n(\cdot) \to \mathbb{1}_{[0,t_0]}(\cdot)$, it is a simple matter to show that

$$\int_0^{t_0} ((u_t, \partial_t \varphi_t) + \mathcal{E}(u_t, \varphi_t))\, dt = \int_0^{t_0} (f_t, \varphi_t)\, dt + (\phi, \varphi_{t_0}) - (u_0, \varphi_0) \quad \forall \varphi \in C_T.$$

Hence, the assertion follows immediately.
(ii) Set $t_0 = T$ in (i). Then it holds $0 = (\phi, \varphi_T) - (u_T, \varphi_T)$ for all $\varphi \in \mathcal{C}_T$. □

Lemma 3.5. *If equation* (3.2) *holds for all* $\varphi \in b\mathcal{C}_T$*, then* u *is a weak solution.*

Proof. The assertion follows directly by Lemma 2.7 and 2.8. □

The next proposition shows sufficient conditions for the existence of a strong solution. The proofs of (ii) and (iii) follow the idea of [BPS05, Proposition 2.6]. Note that in the proof of (ii) in [BPS05] the function f has to be extended on $[T, T + \varepsilon]$, otherwise the appearing integrals are not well defined.

Notation. We recall that ∂_t denotes the time derivative, i.e. $\partial_t f_s = \lim_{h \to 0} \frac{f_{s+h} - f_s}{h}$.

Proposition 3.6. *(i) If $\phi \in L^2$, then $t \mapsto P_{T-t}\phi$ is L^2-continuous on $[0,T]$, L^2-differentiable on $[0,T)$ and $\partial_t P_{T-t}\phi = -LP_{T-t}\phi$.*
(ii) Let $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be a function such that $t \mapsto f_t$ is L^2-differentiable and $t \mapsto \partial_t f_t$ is L^2-continuous on $[0,T]$. Then the function

$$w_t(x) := \int_t^T P_{s-t} f_s(x)\, ds$$

is L^2-differentiable on $[0,T]$ and

$$\partial_t w_t(x) = -P_{T-t} f_T(x) + \int_t^T P_{s-t} \partial_t f_s(x)\, ds.$$

Moreover, $t \to \partial_t w(t,x)$ is L^2-continuous on $[0,T]$.
(iii) Let $\phi \in \mathcal{D}(L)$ and f satisfy the conditions of (ii). Define

$$u_t = P_{T-t}\phi + \int_t^T P_{s-t} f_s\, ds.$$

Then u is a strong solution of (3.1).

Remark 3.7. *If (i) holds, we have $\partial_t P_{T-t}\phi + LP_{T-t}\phi = 0$. This gives us a strong solution for the homogeneous linear equation*

$$(\partial_t + L)u = 0, \quad u_T = \phi.$$

On the other hand by (iii), for any f satisfying the condition of (ii) and any final condition $\phi \in \mathcal{D}(L)$, we can construct a strong solution for the linear inhomogeneous equation

$$(\partial_t + L)u + f = 0, \quad u_T = \phi.$$

Proof of Proposition 3.6. (i) Let us first note that by Lemma 1.8 $P_t\phi \in \mathcal{D}(L)$ for each $t > 0$. Hence, the term $LP_{T-t}\phi$ is well defined. We have to prove:

(1) $t \mapsto P_{T-t}\phi$ is L^2-continuous on $[0,T]$
(2) $t \mapsto P_{T-t}\phi$ is L^2-differentiable on $[0,T)$
(3) $\partial_t P_{T-t}\phi = -LP_{T-t}\phi$

(1) **case:** $h < 0$ such that $T \geq t+h \geq 0, t \in (0,T]$

$$\begin{aligned}
&\lim_{h \nearrow 0} \|P_{T-(t+h)}\phi - P_{T-t}\phi\|_2 \\
= \; &\lim_{h \nearrow 0} \|P_{T-t}[P_{-h}\phi - \phi]\|_2 \\
\leq \; &\lim_{h \nearrow 0} \|P_{-h}\phi - \phi\|_2 = 0
\end{aligned}$$

case: $h > 0$ such that $T \geq t+h \geq 0, t \in [0,T)$

$$\begin{aligned}
&\lim_{h \searrow 0} \|P_{T-(t+h)}\phi - P_{T-t}\phi\|_2 \\
= \; &\lim_{h \searrow 0} \|P_{T-t-h}\phi - P_{T-t}\phi]\|_2 \\
\leq \; &\lim_{h \searrow 0} \|P_{T-t-h}[\phi - P_h\phi]\|_2 \\
\leq \; &\lim_{h \searrow 0} \|\phi - P_h\phi\|_2 = 0
\end{aligned}$$

The assertions (2) and (3) can be shown analogously to the assertion $\partial_t P_t \phi = LP_t \phi$ in [RS75, Theorem X.52]. The idea is to use the following representation of the semigroup

$$T_z = -\frac{1}{2\pi i}\int_\Gamma e^{-\lambda z} G_\lambda d\lambda$$

where z is an element of a sector in $\mathbb{C}$. For more details we refer to [RS75].

(ii) Let $\varepsilon > 0$. We extend the function f by defining

$$f : [T, T+\varepsilon] \times \mathbb{R}^d \to \mathbb{R}$$

such that $f_s(x) = f_{2T-s}(x)$ for all $s \in [T, T+\varepsilon]$ and $x \in \mathbb{R}^d$. Therefore, we derive for $r \in (-t, T-t), |r| < \varepsilon$

$$\begin{aligned} w_{t+r} - w_t &= \int_{t+r}^T P_{s-t-r} f_s \, ds - \int_t^T P_{s-t} f_s \, ds \\ &= \int_0^{T-t-r} P_s f_{t+r+s} \, ds - \int_0^{T-t} P_s f_{t+s} \, ds \\ &= \int_0^{T-t} P_s \big(f_{t+r+s} - f_{t+s}\big) \, ds - \int_{T-t-r}^{T-t} P_s f_{t+r+s} \, ds. \end{aligned}$$

Then we have to show in $(L^2, \|\cdot\|_2)$

$$\begin{aligned} &\frac{1}{r}\left(\int_0^{T-t} P_s(f_{t+r+s} - f_{t+s}) \, ds - \int_{T-t-r}^{T-t} P_s f_{t+r+s} \, ds\right) \\ &\underset{r\to 0}{\longrightarrow} \int_t^T P_{s-t}\partial_t f_s \, ds - P_{T-t} f_T. \end{aligned}$$

This will be done in two steps. In step (a) we will show the L^2-convergence of the first term and in step (b) the L^2-convergence of the second term.

$$\begin{aligned} (a) \qquad & \left\| \frac{1}{r}\int_0^{T-t} P_s(f_{t+r+s} - f_{t+s}) \, ds - \int_t^T P_{s-t}\partial_t f_s \, ds \right\|_2 \\ = & \left\| \int_t^T \left(\frac{1}{r} P_{s-t}(f_{r+s} - f_s) - P_{s-t}\partial_t f_s\right) ds \right\|_2 \\ \leq & \int_t^T \left\| P_{s-t}\left(\frac{f_{r+s} - f_s}{r} - \partial_t f_s\right)\right\|_2 ds \\ \leq & \int_t^T \left\| \frac{f_{r+s} - f_s}{r} - \partial_t f_s \right\|_2 ds. \end{aligned}$$

Since $t \mapsto \partial_t f_t$ is L^2-continuous on $[0, T]$, the last term converges to zero by the dominated convergence theorem. A dominating function can be found by using the mean value theorem for the L^2-continuous function f.

(b) Case: $\varepsilon > r > 0,\ 0 \le t \le T - \varepsilon$

$$\begin{aligned}
& \left\| \frac{1}{r} \int_{T-t-r}^{T-t} P_s f_{t+r+s}\, ds - P_{T-t} f_T \right\|_2 \\
= & \left\| \frac{1}{r} \int_{T-t-r}^{T-t} P_s f_{t+r+s}\, ds - \frac{1}{r} \int_{-r}^{0} P_{T-t} f_T\, ds \right\|_2 \\
= & \left\| \frac{1}{r} \int_{-r}^{0} \left(P_{T-t+s} f_{T+r+s} - P_{T-t} f_T \right) ds \right\|_2 \\
= & \left\| \frac{1}{r} \int_{0}^{r} \left(P_{T-t+s-r} f_{T+s} - P_{T-t} f_T \right) ds \right\|_2 \\
\underset{r>0}{\le} & \frac{1}{r} \int_0^r \left\| P_{T-t+s-r} \left(f_{T+s} - P_{r-s} f_T \right) \right\|_2 ds \\
\le & \frac{1}{r} \int_0^r \left\| f_{T+s} - P_{r-s} f_T \right\|_2 ds \\
= & \frac{1}{r} \int_0^r \| P_{r-s} f_T - f_{T+s} + f_T - f_T \|_2\, ds \\
\le & \frac{1}{r} \int_0^r \| P_{r-s} f_T - f_T \|_2\, ds + \frac{1}{r} \int_0^r \| f_{T+s} - f_T \|_2\, ds \\
= & \frac{1}{r} \int_{-r}^0 \| P_{-s} f_T - f_T \|_2\, ds + \frac{1}{r} \int_0^r \| f_{T+s} - f_T \|_2\, ds \\
\underset{(\triangle)}{\longrightarrow} & \ 0
\end{aligned}$$

In $(\triangle)$ we have used the L^2-continuity of f on $[0, T+\varepsilon]$ and the strong continuity of $(P_t)_{t>0}$.

Case: $r < 0,\ |r| < \varepsilon,\ \varepsilon \le t \le T$

$$\begin{aligned}
& \left\| \frac{1}{r} \int_{T-t-r}^{T-t} P_s f_{t+r+s}\, ds - P_{T-t} f_T \right\|_2 \\
= & \left\| \frac{1}{r} \int_{-r}^{0} \Big(P_{T-t+s} f_{T+r+s} - P_{T-t} f_T \Big) ds \right\|_2 \\
\underset{r<0}{\le} & \frac{1}{r} \int_{-r}^0 \left\| P_{T-t} \left[P_s f_{T+r+s} - f_T \right] \right\|_2 ds \\
\le & \frac{1}{r} \int_{-r}^0 \left\| P_s f_{T+r+s} - f_T \right\|_2 ds \\
\le & \frac{1}{r} \int_{-r}^0 \Big(\| P_s f_T - f_T \|_2 + \| P_s f_{T+r+s} - P_s f_T \|_2 \Big) ds \\
\le & \frac{1}{r} \int_{-r}^0 \| P_s f_T - f_T \|_2\, ds + \frac{1}{r} \int_{-r}^0 \| f_{T+r+s} - f_T \|_2\, ds \\
\underset{(\triangle)}{\longrightarrow} & \ 0
\end{aligned}$$

In $(\triangle)$ we have used the L^2-continuity of f on $[0, T+\varepsilon]$ and the strong continuity of $(P_t)_{t>0}$.

Left to show: $t \mapsto \partial_t w(t,\cdot)$ is L^2-continuous on $[0,T]$. We will only treat the case $h<0$ where $|h|<t$ small enough. The other case can be done analogously.

$$\begin{aligned}
& \|\partial_t w_t - \partial_t w_{t+h}\|_2 \\
\leq\ & \|-P_{T-t}f_t + P_{T-t-h}f_t\|_2 \\
& + \left\| \int_t^T P_{s-t}\partial_t f_s\, ds - \int_{t+h}^T P_{s-t-h}\partial_t f_s\, ds \right\|_2 \\
\leq\ & \underbrace{\|P_{T-t}[f_t - P_{-h}f_t]\|_2}_{\leq \|f_t - P_{-h}f_t\|_2} + \int_t^T \underbrace{\|P_{s-t}\partial_t f_s - P_{s-t-h}\partial_t f_s\|_2}_{\leq \|\partial_t f_s - P_{-h}\partial_t f_s\|_2 \leq 2\sup_{s\in[0,T]}\|\partial_t f_s\|_2}\, ds \\
& + \underbrace{\int_{t+h}^t \|P_{s-t-h}\partial_t f_s\|_2\, ds}_{\leq \sup_{s\in[0,T]}\|\partial_t f_s\|_2 (t-(t+h)) \to 0} \\
\underset{(\triangle)}{\to}\ & 0
\end{aligned}$$

In $(\triangle)$ we have used the L^2-continuity of $t \mapsto \partial_t f_t$ on $[0,T+\varepsilon]$ and the strong continuity of $(P_t)_{t>0}$.

(iii) By (i) we calculate for $s>t$

$$\begin{aligned}
LP_{s-t}f_s &= -\partial_t P_{s-t}f_s = -\lim_{h\to 0}\frac{P_{s-(t+h)}f_s - P_{s-t}f_s}{h} \\
&= -\lim_{h\to 0}\frac{P_{s-h-t}f_{s-h} - P_{s-t}f_s + P_{s-(t+h)}f_s - P_{s-h-t}f_{s-h}}{h} \\
&= -\lim_{h\to 0}\left(\frac{P_{s-h-t}f_{s-h} - P_{s-t}f_s}{h} + \frac{P_{s-(t+h)}f_s - P_{s-h-t}f_{s-h}}{h}\right) \\
&\underset{(\star)}{=} -\Big(-\partial_t(P_{s-t}f_s) + P_{s-t}\partial_t f_s\Big) = \partial_t(P_{s-t}f_s) - P_{s-t}\partial_t f_s.
\end{aligned}$$

We will show $(\star)$ in the case $h<0$ ($\tilde{h} := -h$) where $|h|<t$ small enough. The other case can be done analogously.

$$\begin{aligned}
(\star) \qquad & \left\| \frac{P_{s-(t+h)}f_s - P_{s-h-t}f_{s-h}}{h} - P_{s-t}\partial_t f_s \right\|_2 \\
=\ & \left\| P_{s-t}\left(\frac{P_{-h}f_s - P_{-h}f_{s-h}}{h} - \partial_t f_s\right) \right\|_2 \\
=\ & \left\| P_{-h}\left(\frac{f_s - f_{s-h}}{h} - \partial_t f_s\right) + (P_{-h}\partial_t f_s - \partial_t f_s) \right\|_2 \\
=\ & \left\| P_{\tilde{h}}\left(\frac{f_{s+\tilde{h}} - f_s}{\tilde{h}} - \partial_t f_s\right) + \big(P_{\tilde{h}}\partial_t f_s - \partial_t f_s\big) \right\|_2 \\
\leq\ & \left\| \frac{f_{s+\tilde{h}} - f_s}{\tilde{h}} - \partial_t f_s \right\|_2 + \big\|P_{\tilde{h}}\partial_t f_s - \partial_t f_s\big\|_2 \\
\underset{h\nearrow 0}{\to}\ & 0
\end{aligned}$$

Then we deduce that

$$\begin{aligned}
Lu_t &= LP_{T-t}\phi + \int_t^T LP_{s-t}f_s\,ds \\
&= LP_{T-t}\phi + \int_t^T \Big(\partial_t(P_{s-t}f_s) - P_{s-t}\partial_t f_s\Big)\,ds \\
&\underset{(i)}{=} -\partial_t P_{T-t}\phi - f_t + P_{T-t}f_T - \int_t^T P_{s-t}\partial_t f_s\,ds \\
&\underset{(ii)}{=} -\partial_t u_t - f_t.
\end{aligned}$$

□

Next we will give an existence and uniqueness proof for a weak solution under the assumptions $f \in L^1([0,T];L^2)$ and $\phi \in L^2$. Moreover, we will prove two very useful relations. We follow the idea of [BPS05, Proposition 2.7].

Proposition 3.8. *Assume that $f \in L^1([0,T];L^2)$ and $\phi \in L^2$. Then the equation* (3.1) *has a unique weak solution $u \in \hat{F}$*

$$u_t = P_{T-t}\phi + \int_t^T P_{s-t}f_s\,ds. \tag{3.4}$$

The solution satisfies the two relations:

$$\|u_t\|_2^2 + 2\int_t^T \mathcal{E}(u_s)\,ds = 2\int_t^T (f_s,u_s)\,ds + \|\phi\|_2^2, \quad 0 \le t \le T, \tag{3.5}$$

$$\|u\|_T^2 \le 2\|\phi\|_2^2 + 3\left(\int_0^T \|f_t\|_2\,dt\right)^2. \tag{3.6}$$

Proof. [Uniqueness]
Let $v,w \in \hat{F}$ be weak solutions of (3.1). Then by Lemma 3.4(i) $u := v - w$ satisfies

$$\int_{t_0}^T \Big((u_t,\partial_t\varphi_t) + \mathcal{E}(u_t,\varphi_t)\Big)\,dt = -(u_{t_0},\varphi_{t_0}) \quad \text{for all } t_0 \ge 0, \varphi \in \mathcal{C}_T. \tag{3.7}$$

Define

$$u_t^\varepsilon = \frac{1}{\varepsilon}\int_0^\varepsilon u_{t+s}\,ds$$

where we set $u_t = 0$ for $T \le t \le T+\varepsilon$. Let us check that u^ε also fulfills (3.7).

$$
\begin{aligned}
\int_{t_0}^{T} (u_t^\varepsilon, \partial_t \varphi_t)\, dt &= \int_{t_0}^{T} \left(\frac{1}{\varepsilon} \int_0^\varepsilon u_{t+s}\, ds, \partial_t \varphi_t \right) dt \\
&= \frac{1}{\varepsilon} \int_0^\varepsilon \int_{t_0}^{T} (u_{t+s}, \partial_t \varphi_t)\, dt\, ds \\
&\underset{(\star)}{=} \frac{1}{\varepsilon} \int_0^\varepsilon \int_{t_0+s}^{T+s} (u_t, \partial_t \tilde{\varphi}_t^s)\, dt\, ds \\
&= -\frac{1}{\varepsilon} \int_0^\varepsilon \left[\int_{t_0+s}^{T+s} \mathcal{E}(u_t, \tilde{\varphi}_t^s)\, dt + (u_{t_0+s}, \tilde{\varphi}_{t_0+s}^s) \right] ds \\
&= -\frac{1}{\varepsilon} \int_0^\varepsilon \left[\int_{t_0}^{T} \mathcal{E}(u_{t+s}, \varphi_t)\, dt + (u_{t_0+s}, \varphi_{t_0}) \right] ds \\
&= -\int_{t_0}^{T} \mathcal{E}\left(\frac{1}{\varepsilon} \int_0^\varepsilon u_{t+s}\, ds, \varphi_t \right) dt - \left(\frac{1}{\varepsilon} \int_0^\varepsilon u_{t_0+s}\, ds, \varphi_{t_0} \right) \\
&= -\int_{t_0}^{T} \mathcal{E}(u_t^\varepsilon, \varphi_t)\, dt - (u_{t_0}^\varepsilon, \varphi_{t_0}).
\end{aligned}
$$

($\star$) Since $\varphi \in \mathcal{C}_T$, we can choose a function $\tilde{\varphi}^s \in \mathcal{C}_T$ for a fixed $s \in [0, \varepsilon]$ such that $\tilde{\varphi}_{t-s}^s = \varphi_t$ on $[t_0, T]$.

Since $t \mapsto u_t$ is L^2-continuous, it follows that $t \mapsto u_t^\varepsilon$ is L^2-differentiable and $t \mapsto \partial_t u_t^\varepsilon$ is L^2-continuous (cf. proof of Lemma 2.10). Therefore, we deduce that the function u^ε is an element of $\mathcal{C}_T$. Hence, the above equation holds with u_t^ε as a test function

$$
\int_{t_0}^{T} \Big((u_t^\varepsilon, \partial_t u_t^\varepsilon) + \mathcal{E}(u_t^\varepsilon, u_t^\varepsilon) \Big) dt = -(u_{t_0}^\varepsilon, u_{t_0}^\varepsilon).
$$

By the L^2-continuity of $t \mapsto \partial_t u_t^\varepsilon$ we have $\partial_t (u_t^\varepsilon, u_t^\varepsilon) = 2(u_t^\varepsilon, \partial_t u_t^\varepsilon)$. Hence, it follows that

$$
\int_{t_0}^{T} (u_t^\varepsilon, \partial_t u_t^\varepsilon)\, dt = \frac{1}{2} \int_{t_0}^{T} \partial_t (u_t^\varepsilon, u_t^\varepsilon)\, dt \underset{u_t = 0 \text{ for } T \le t \le T+\varepsilon}{=} -\frac{1}{2} (u_{t_0}^\varepsilon, u_{t_0}^\varepsilon).
$$

Therefore,

$$
\frac{1}{2} (u_{t_0}^\varepsilon, u_{t_0}^\varepsilon) + \int_{t_0}^{T} \mathcal{E}(u_t^\varepsilon, u_t^\varepsilon)\, dt = 0.
$$

Clearly the left hand side of this equation is non-negative. Thus, we deduce that $u_{t_0}^\varepsilon = 0$ for all $t_0 \in [0, T]$. Since

$$
\|u_{t_0}^\varepsilon - u_{t_0}\|_2 \le \frac{1}{\varepsilon} \int_0^\varepsilon \|u_{t_0+s} - u_{t_0}\|_2\, ds \underset{L^2\text{-continuity}}{\longrightarrow} 0, \text{ as } \varepsilon \to 0,
$$

the uniqueness follows by $0 = \lim_{\varepsilon \to 0} \|u_{t_0}^\epsilon\|_2 = \|u_{t_0}\|_2 = \|v_{t_0} - w_{t_0}\|_2$ for all $t_0 \in [0, T]$.

[Existence]
First let us assume that f satisfies the conditions of Proposition 3.6(ii) and that $\phi \in \mathcal{D}(L)$. Then we know by Proposition 3.6(iii) and the first part of this proof that the unique weak solution u is an element of $\hat{F}$ given by (3.4). By Proposition 3.6(ii) it follows that u is L^2-differentiable on $[0,T]$ and that $t \mapsto \partial_t u_t$ is L^2-continuous. Hence, actually $u \in \mathcal{C}_T$ and the weak relation holds with u as a test function.

$$\int_0^T \Big((u_t, \partial_t u_t) + \mathcal{E}(u_t, u_t) \Big)\, dt = \int_0^T (f_t, u_t)\, dt + (\phi, u_T) - (u_0, u_0).$$

Next we will show the asserted relations in the above particular situation.

[Relation (3.5)]
Let $t_0 \in [0,T]$. Since $t \mapsto \partial_t u_t$ is L^2-continuous, we have:

$$(u_t, \partial_t u_t) = \frac{1}{2}\partial_t (u_t, u_t).$$

Then for any $t_0 \in [0,T]$ we obtain by Lemma 3.4(i) and integration by parts

$$\begin{aligned}
&\|\phi\|_2^2 + 2\int_{t_0}^T (f_s, u_s)\, ds \\
= \;& \|\phi\|_2^2 + 2\left(\underbrace{\int_{t_0}^T (u_t, \partial_t u_t)\, dt}_{=\frac{1}{2}\int_{t_0}^T \partial_t (u_t,u_t)\, dt} + \int_{t_0}^T \mathcal{E}(u_t, u_t)\, dt - \underbrace{(\phi, u_T)}_{=\|\phi\|_2^2} + \underbrace{(u_{t_0}, u_{t_0})}_{=\|u_{t_0}\|_2^2} \right) \\
= \;& -\|\phi\|_2^2 + 2\|u_{t_0}\|_2^2 + \int_{t_0}^T \partial_t (u_t, u_t)\, dt + 2\int_{t_0}^T \mathcal{E}(u_t, u_t)\, dt \\
= \;& \|u_{t_0}\|_2^2 + 2\int_{t_0}^T \mathcal{E}(u_t, u_t)\, dt. \qquad (3.8)
\end{aligned}$$

[Relation (3.6)]
Since

$$\begin{aligned}
&\int_t^T (f_s, u_s)\, ds \\
\underset{\text{Prop. } 3.6(iii)}{=} \;& \int_t^T \left((f_s, P_{T-s}\phi) + \left(f_s, \int_s^T P_{r-t} f_r\, dr \right) \right) ds \\
\le \;& \int_t^T \|f_s\|_2 \|P_{T-s}\phi\|_2\, ds + \int_t^T \|f_s\|_2 \underbrace{\left\| \int_s^T P_{r-t} f_r\, dr \right\|_2}_{\le \int_s^T \|P_{r-t} f_r\|_2\, dr} ds \\
\le \;& \|\phi\|_2 \int_t^T \|f_s\|_2\, ds + \int_t^T \left(\|f_s\|_2 \int_s^T \|f_r\|_2\, dr \right) ds,
\end{aligned}$$

it holds for $t \in [0,T]$:

$$\begin{aligned}
& \|u_t\|_2^2 + \int_t^T \mathcal{E}(u_t)\,dt \\
\leq \quad & \|u_t\|_2^2 + 2\int_t^T \mathcal{E}(u_t)\,dt \\
\underset{(3.8)}{=} \quad & \|\phi\|_2^2 + 2\int_t^T (f_s, u_s)\,ds \\
\leq \quad & \|\phi\|_2^2 + 2\left(\|\phi\|_2 \int_t^T \|f_s\|_2\,ds\right) + 2\int_t^T \left(\|f_s\|_2 \int_s^T \|f_r\|_2\,dr\right) ds \\
\leq \quad & \|\phi\|_2^2 + 2\underbrace{\left(\|\phi\|_2 \int_0^T \|f_s\|_2\,ds\right)}_{\leq \frac{1}{2}\left(\|\phi\|_2^2 + \left(\int_0^T \|f_s\|_2\,ds\right)^2\right)} + 2\int_0^T \left(\|f_s\|_2 \int_s^T \|f_r\|_2\,dr\right) ds \\
\leq \quad & 2\|\phi\|_2^2 + 3\left(\int_0^T \|f_s\|_2\,ds\right)^2.
\end{aligned}$$

Hence, it follows

$$\|u\|_T^2 \leq 2\|\phi\|_2^2 + 3\left(\int_0^T \|f_r\|_2\,dr\right)^2. \tag{3.9}$$

Now we will obtain the result for general data ϕ and f. Let $(f^n)_{n\in\mathbb{N}} \subset \mathcal{C}_0^\infty([0,T]\times\mathbb{R}^d)$ such that $\int_0^T \|f_t^n - f_t\|_2\,dt \to 0$. Since all f^n and $(f^n)'$ have compact support, they satisfy the conditions of Proposition 3.6(ii). Moreover, take $(\phi^n)_{n\in\mathbb{N}} \subset \mathcal{D}(L)$ such that $\phi^n \to \phi$ in L^2. We denote the unique weak solution for the data (ϕ^n, f^n) by u^n.

By linearity it follows that $u^n - u^m$ is a unique weak solution for the data $(\phi^n - \phi^m, f^n - f^m)$. Since by relation (3.9) it holds

$$\|u^n - u^m\|_T^2 \leq 2\underbrace{\|\phi^n - \phi^m\|_2^2}_{\longrightarrow 0} + 3\left(\underbrace{\int_0^T \underbrace{\|f_t^n - f_t^m\|_2}_{\leq \|f_t^n - f_t\|_2 + \|f_t - f_t^m\|_2}\,dt}_{\longrightarrow 0}\right)^2 \to 0,$$

we can deduce that $(u^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\hat{F}$. Next we will show that the limit $u := \lim_{n\to\infty} u^n$ in $\|\cdot\|_T$ is the solution corresponding to the data (ϕ, f) and satisfies the relations (3.5) and (3.6).

More precisely we have to show that by passing to the limit in the weak relation for the data (ϕ^n, f^n) we get the weak solution u for the data (ϕ, f). The weak relation for (f^n, ϕ^n) is

$$\int_0^T \Big((u_t^n, \partial_t \varphi_t) + \mathcal{E}(u_t^n, \varphi_t)\Big)\,dt = \int_0^T (f_t^n, \varphi_t)\,dt + (\phi^n, \varphi_T) - (u_0^n, \varphi_0). \tag{3.10}$$

Since we have

$$\begin{aligned}\left|\int_0^T \mathcal{E}_1(u_t^n - u_t, \varphi_t)\,dt\right| &\leq K_{\mathcal{E}} \int_0^T \mathcal{E}_1(u_t^n - u_t)^{\frac{1}{2}} \mathcal{E}_1(\varphi_t)^{\frac{1}{2}}\,dt \\ &\leq K_{\mathcal{E}} \left(\int_0^T \mathcal{E}_1(u_t^n - u_t)\,dt\right)^{\frac{1}{2}} \left(\int_0^T \mathcal{E}_1(\varphi_t)\,dt\right)^{\frac{1}{2}} \\ &\underset{n\to\infty}{\to} 0\end{aligned}$$

and

$$\int_0^T \|u_t^n - u_t\|_2\,dt \leq T \cdot \sup_{t\in[0,T]} \|u_t^n - u_t\|_2 \underset{n\to\infty}{\to} 0,$$

it holds

$$\left|\int_0^T \mathcal{E}(u_t^n - u_t, \varphi_t)\,dt\right| \underset{n\to\infty}{\to} 0.$$

Easily we see that by Hölder's inequality it follows on the one hand that

$$\left|\int_0^T (u_t^n - u_t, \partial_t \varphi_t)\,dt\right| \underset{n\to\infty}{\to} 0$$

and on the other

$$\left|\int_0^T (f_t^n - f_t, \varphi_t)\,dt\right| \underset{n\to\infty}{\to} 0.$$

Finally, we deduce

$$\int_0^T \Big((u_t, \partial_t\varphi_t) + \mathcal{E}(u_t, \varphi_t)\Big)\,dt = \int_0^T (f_t, \varphi_t)\,dt + (\phi, \varphi_T) - (u_0, \varphi_0)$$

by passing (3.10) to the limit. Therefore, u is a weak solution for the data (ϕ, f).

The relations (3.5) and (3.6) hold for the approximating functions:

$$\begin{aligned}&\|u_t^n\|_2^2 + 2\int_t^T \mathcal{E}(u_t^n)\,ds = 2\int_t^T (f_s^n, u_s^n)\,ds + \|\phi^n\|_2^2, \quad 0 \leq t \leq T \\ &\|u^n\|_T^2 \leq 2\|\phi^n\|_2^2 + 3\left(\int_0^T \|f_t^n\|_2\,dt\right)^2.\end{aligned}$$

Since $\|u_t^n\|_T \to \|u_t\|_T$, we conclude

$$\lim_{n\to\infty} \Big| \sup_{t\in[0,T]} \|u_t^n\|_2 - \sup_{t\in[0,T]} \|u_t\|_2 \Big| \leq \lim_{n\to\infty} \sup_{t\in[0,T]} \|u_t^n - u_t\|_2 = 0$$

and therefore

$$\lim_{n\to\infty} \int_0^T \mathcal{E}(u_t^n)\,dt = \int_0^T \mathcal{E}(u_t)\,dt.$$

It is easy to see that $\lim_{n\to\infty} \int_t^T (f_s^n, u_s^n)\,ds = \int_t^T (f_s, u_s)\,ds$. The convergence of the other terms follows by the definition of f_n and u_n, and by convergence of $u_n \to u$ in $\|\cdot\|_T$. Finally, by passing to the limit in the above relations, we get (3.5) and (3.6) for general data. □

3.2 Basic Relations for the Linear Equation

In this section we will prove useful relations in the linear case. At first we present an estimate for $(\mathcal{E}, F)$.

Lemma 3.9. *Let $u \in F$, then $u^+ \in F$ and $\mathcal{E}(u, u^+) \geq \mathcal{E}(u^+, u^+)$ where $u^+ := u \vee 0$.*

Proof. Since the resolvent $(G_\alpha)_{\alpha>0}$, which is associated to the Dirichlet form $(\mathcal{E}, F)$, is sub-Markovian, we can deduce with the same arguments as in the proof of [MR92, I. Theorem 4.4 $(i) \Rightarrow (ii)$] that $u^+ \in F$ and $\mathcal{E}(u^+, u^-) \leq 0$ where $u^- := u \wedge 0$. Moreover, it follows that

$$0 \geq \mathcal{E}(u^+, u^-) = \mathcal{E}(u^+, u^+ - u)$$

and hence

$$\mathcal{E}(u^+, u) \geq \mathcal{E}(u^+, u^+).$$

Since the adjoint $(\hat{G}_\alpha)_{\alpha>0}$ is positivity preserving, it holds with analogous arguments that

$$\mathcal{E}(u, u^+) \geq \mathcal{E}(u^+, u^+).$$

□

The next lemma follows the lines of arguments of [BPS05, Lemma 2.8]. The proof will be given with all details. In the second step of the proof we will use other approximating functions as in the original paper.

Lemma 3.10. *If u is a weak solution of equation* (3.1)*, then u^+ satisfies the following relation with $0 \leq t_1 < t_2 \leq T$*

$$\|u^+_{t_1}\|_2^2 + 2\int_{t_1}^{t_2} \mathcal{E}(u^+_s)\, ds \leq 2\int_{t_1}^{t_2} (f_s, u^+_s)\, ds + \|u^+_{t_2}\|_2^2.$$

Proof. The main idea of this proof is to approximate u^+_t by test functions. In the first step we will approximate $u \in \hat{F}$ with functions $u^n \in \mathcal{C}_T$ and show that it is enough to verify

$$\|u^+_{t_1}\|_2^2 + 2\int_{t_1}^{t_2} \mathcal{E}(u_s, u^+_s)\, ds = 2\int_{t_1}^{t_2} (f_s, u^+_s)\, ds + \|u^+_{t_2}\|_2^2 \tag{3.11}$$

for $u \in \mathcal{C}_T$. In the second step we will show that (3.11) holds for all $u \in \mathcal{C}_T$, which satisfy the weak relation with data (ϕ, f) over the interval $[t_1, t_2]$ where $0 < \varepsilon \leq t_1 \leq t_2 \leq T$.

[Step 1] For $n \geq 1$ let us define

$$u^n_t = n\int_0^{\frac{1}{n}} u_{t-s}\, ds, \qquad f^n_t = n\int_0^{\frac{1}{n}} f_{t-s}\, ds, \qquad \phi^n = n\int_0^{\frac{1}{n}} u_{T-s}\, ds.$$

Analogous to the proof of Lemma 2.10 $(\supset)$ it follows that u^n is an element of $\mathcal{C}_T$. Let us show that the approximating functions satisfy the equation

$$(\partial_t + L)u^n + f^n = 0, \quad u^n_T = \phi^n$$

in the weak sense over the interval $[\frac{1}{n}, T]$ for $n \geq 1$.

- $$u_T^n = n \int_0^{\frac{1}{n}} u_{T-s}\, ds = \phi^n$$
- $$\begin{aligned}
&\int_{\frac{1}{n}}^T ((u_t^n, \partial_t \varphi_t) + \mathcal{E}(u_t^n, \varphi_t))\, dt \\
= \;& n \int_0^{\frac{1}{n}} \left[\int_{\frac{1}{n}}^T \Big((u_{t-s}, \partial_t \varphi_t) + \mathcal{E}(u_{t-s}, \varphi_t) \Big)\, dt \right] ds \\
\underset{(\star)}{=} \;& n \int_0^{\frac{1}{n}} \left[\int_{\frac{1}{n}}^T \Big((u_{t-s}, \partial_t \tilde{\varphi}^s_{t-s}) + \mathcal{E}(u_{t-s}, \tilde{\varphi}^s_{t-s}) \Big)\, dt \right] ds \\
\underset{u \text{ weak sol.}}{=} \;& n \int_0^{\frac{1}{n}} \left[\int_{\frac{1}{n}}^T (f_{t-s}, \tilde{\varphi}^s_{t-s})\, dt + (u_{T-s}, \tilde{\varphi}^s_{T-s}) - (u_{\frac{1}{n}-s}, \tilde{\varphi}^s_{\frac{1}{n}-s}) \right] ds \\
= \;& \int_{\frac{1}{n}}^T (f_t^n, \varphi_t)\, dt + (\phi^n, \varphi_T) - (u^n_{\frac{1}{n}}, \varphi_{\frac{1}{n}})
\end{aligned}$$

($\star$) Since $\varphi \in \mathcal{C}_T$, we can choose a function $\tilde{\varphi}^s \in \mathcal{C}_T$ for fixed $s \in [0, \frac{1}{n}]$ such that $\tilde{\varphi}^s_{t-s} = \varphi_t$ on $[\frac{1}{n}, T]$.

Therefore, u^n is a weak solution for the data (ϕ^n, f^n) over the interval $[\frac{1}{n}, T]$. Fix $\varepsilon > 0$. Then there exists $N_\varepsilon \in \mathbb{N}$ such that u^n satisfies the weak relation with data (ϕ^n, f^n) over the interval $[\varepsilon, t_2]$ for all $n > N_\varepsilon$ and t_2 such that $\varepsilon \leq t_2 \leq T$.

We have the following equations for each $\tilde{\varepsilon} > 0$:

$$\begin{aligned}
&(1) && \lim_{n\to\infty} \sup_{t \in [\tilde{\varepsilon}, T]} \|u_t^n - u_t\|_2 = 0, \\
&(2) && \lim_{n\to\infty} \int_{\tilde{\varepsilon}}^T \mathcal{E}(u_t^n - u_t)\, dt = 0, \\
&(3) && \lim_{n\to\infty} \int_{\tilde{\varepsilon}}^T \|f_t^n - f_t\|_2\, dt = 0, \\
&(4) && \lim_{n\to\infty} \|\phi^n - \phi\|_2 = 0.
\end{aligned}$$

The equations (1), (2) and (4) can be proved analogously to the first part of Lemma 2.10 (Step 2 of $\supset$). For (3) see [LSU68, II.Lemma 4.7].

Suppose that for $\varepsilon > 0$ it holds

$$\|(u_{t_1}^n)^+\|_2^2 + 2 \int_{t_1}^{t_2} \mathcal{E}(u_s^n, (u_s^n)^+)\, ds = 2 \int_{t_1}^{t_2} (f_s^n, (u_s^n)^+)\, ds + \|(u_{t_2}^n)^+\|_2^2 \quad (3.12)$$

where $0 < \varepsilon \leq t_1 \leq t_2 \leq T$ and $n > N_\varepsilon$. This will be shown in [Step 2] below. Then we get by Lemma 3.9

$$2 \int_{t_1}^{t_2} \mathcal{E}((u_s^n)^+)\, ds \leq -\|(u_{t_1}^n)^+\|_2^2 + 2 \int_{t_1}^{t_2} (f_s^n, (u_s^n)^+)\, ds + \|(u_{t_2}^n)^+\|_2^2. \quad (3.13)$$

We note that for $v, w : \mathbb{R}^d \to \mathbb{R}$ the relation $|(v(x))^+ - (w(x))^+| \leq |v(x) - w(x)|$ holds for all $x \in \mathbb{R}^d$. This can be verified as follows:

1. If $\exists \tilde{x} \in \mathbb{R}^d$ such that $(v(\tilde{x}))^+ = 0$ and $(w(\tilde{x}))^+ > 0$, $|(v(\tilde{x}))^+ - (w(\tilde{x}))^+| = |(w(\tilde{x}))^+| \leq |v(\tilde{x}) - w(\tilde{x})|$.
2. If $\exists \tilde{x} \in \mathbb{R}^d$ such that $(v(\tilde{x}))^+ > 0$ and $(w(\tilde{x}))^+ > 0$, $|(v(\tilde{x}))^+ - (w(\tilde{x}))^+| = |v(\tilde{x}) - w(\tilde{x})|$.
3. If $\exists \tilde{x} \in \mathbb{R}^d$ such that $(v(\tilde{x}))^+ = 0$ and $(w(\tilde{x}))^+ = 0$, $|(v(\tilde{x}))^+ - (w(\tilde{x}))^+| = 0 \leq |v(\tilde{x}) - w(\tilde{x})|$.

Now fix $\varepsilon > 0$, t_1, t_2 such that $0 < \varepsilon \leq t_1 \leq t_2 \leq T$ and define the Hilbert spaces $\mathcal{H} := L^2([t_1, t_2]; L^2)$ and $\mathcal{H}_0 := L^2([t_1, t_2]; F)$. Since

$$\lim_{n\to\infty} \sup_{t\in[\varepsilon,T]} \|(u_t^n)^+ - u_t^+\|_2 \quad \leq \quad \lim_{n\to\infty} \sup_{t\in[\varepsilon,T]} \|u_t^n - u_t\|_2 \underset{(1)}{=} 0 \tag{3.14}$$

and

$$\begin{aligned} & \left| \int_{t_1}^{t_2} (f_s^n, (u_s^n)^+)\, ds - \int_{t_1}^{t_2} (f_s, u_s^+)\, ds \right| \\ \leq \quad & \left| \int_{t_1}^{t_2} (f_s^n - f_s, (u_s^n)^+)\, ds \right| + \left| \int_{t_1}^{t_2} (f_s, (u_s^n)^+ - u_s^+)\, ds \right| \\ \leq \quad & \sup_{s\in[t_1,t_2]} \|(u_s^n)^+\|_2 \int_{t_1}^{t_2} \|f_s^n - f_s\|_2\, ds + \int_{t_1}^{t_2} \|f_s\|_2 \|(u_s^n)^+ - u_s^+\|_2\, ds \\ \leq \quad & \sup_{s\in[t_1,t_2]} \|(u_s^n)^+\|_2 \int_{t_1}^{t_2} \|f_s^n - f_s\|_2\, ds \\ & + \sup_{s\in[t_1,t_2]} \|(u_s^n)^+ - u_s^+\|_2 \int_{t_1}^{t_2} \|f_s\|_2\, ds \\ \underset{n\to\infty}{\longrightarrow} \; & 0, \end{aligned}$$

we obtain by equation (3.13)

$$\begin{aligned} & \limsup_{n\to\infty} \left[2 \int_{t_1}^{t_2} \mathcal{E}((u_s^n)^+)\, ds \right] \\ \leq \quad & \limsup_{n\to\infty} \left[-\|(u_{t_1}^n)^+\|_2^2 + 2 \int_{t_1}^{t_2} (f_s^n, (u_s^n)^+)\, ds + \|(u_{t_2}^n)^+\|_2^2 \right] \\ = \quad & \lim_{n\to\infty} \left[-\|(u_{t_1}^n)^+\|_2^2 + 2 \int_{t_1}^{t_2} (f_s^n, (u_s^n)^+)\, ds + \|(u_{t_2}^n)^+\|_2^2 \right] \\ = \quad & -\|(u_{t_1})^+\|_2^2 + 2 \int_{t_1}^{t_2} (f_s, u_s^+)\, ds + \|u_{t_2}^+\|_2^2. \end{aligned}$$

Hence, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ such that

$$\sup_{k\in\mathbb{N}} \int_{t_1}^{t_2} \mathcal{E}((u_s^{n_k})^+)\, ds < \infty.$$

Since

$$\lim_{k\to\infty}\int_{t_1}^{t_2}\|(u_s^{n_k})^+ - u_s^+\|_2^2\,ds \le \lim_{k\to\infty}\int_{t_1}^{t_2}\|u_s^{n_k} - u_s\|_2^2\,ds \underset{(1)}{=} 0,$$

it follows that

$$\sup_{k\in\mathbb{N}}\int_{t_1}^{t_2}\mathcal{E}_1((u_s^{n_k})^+)\,ds < \infty.$$

Therefore, by Lemma 1.19 we obtain $\lim_{k\to\infty}(u^{n_k})^+ = u^+$ weakly in $\mathcal{H}_0$,

$$\lim_{k\to\infty}\int_{t_1}^{t_2}\mathcal{E}_1(\varphi_s,(u_s^{n_k})^+)\,ds = \int_{t_1}^{t_2}\mathcal{E}_1(\varphi_s,u_s^+)\;ds \quad \text{for all } \varphi\in\mathcal{H}_0. \tag{3.15}$$

Finally, we make the following calculation:

$$\begin{aligned}
&\left|\int_{t_1}^{t_2}\mathcal{E}(u_s^{n_k},(u_s^{n_k})^+)\,ds - \int_{t_1}^{t_2}\mathcal{E}(u_s,u_s^+)\,ds\right| \\
\le\;& \left|\int_{t_1}^{t_2}\mathcal{E}_1(u_s^{n_k}-u_s,(u_s^{n_k})^+)\,ds + \int_{t_1}^{t_2}\mathcal{E}_1(u_s,(u_s^{n_k})^+ - u_s^+)\,ds\right| \\
&+\int_{t_1}^{t_2}\Big(\big|(u_s^{n_k}-u_s,(u_s^{n_k})^+)\big| + \big|(u_s,(u_s^{n_k})^+ - u_s^+)\big|\Big)\,ds \\
\le\;& K_{\mathcal{E}}\int_{t_1}^{t_2}\mathcal{E}_1((u_s^{n_k})^+)^{\frac12}\mathcal{E}_1(u_s^{n_k}-u_s)^{\frac12}\,ds + \left|\int_{t_1}^{t_2}\mathcal{E}_1(u_s,(u_s^{n_k})^+ - u_s^+)\,ds\right| \\
&+\int_{t_1}^{t_2}\Big(\|(u_s^{n_k})^+\|_2\|u_s^{n_k}-u_s\|_2 + \|(u_s^{n_k})^+ - u_s^+\|_2\|u_s\|_2\Big)\,ds \\
\le\;& K_{\mathcal{E}}\underbrace{\left(\int_{t_1}^{t_2}\mathcal{E}_1((u_s^{n_k})^+)\,ds\right)^{\frac12}}_{<const \text{ for all } k\in\mathbb{N}}\underbrace{\left(\int_{t_1}^{t_2}\mathcal{E}_1(u_s^{n_k}-u_s)\,ds\right)^{\frac12}}_{\underset{(1),(2)}{\to}0} \\
&+\underbrace{\left|\int_{t_1}^{t_2}\mathcal{E}_1(u_s,(u_s^{n_k})^+ - u_s^+)\,ds\right|}_{\underset{(3.15)}{\to}0} \\
&+\int_{t_1}^{t_2}\Big(\|(u_s^{n_k})^+\|_2\|u_s^{n_k}-u_s\|_2 + \|(u_s^{n_k})^+ - u_s^+\|_2\|u_s\|_2\Big)\,ds \\
\underset{k\to\infty}{\longrightarrow}\;& 0.
\end{aligned}$$

By passing k to the limit in equation (3.12) for the subsequence $(n_k)_{k\in\mathbb{N}}$ we get for $0<\varepsilon\le t_1\le t_2\le T$

$$\|u_{t_1}^+\|_2^2 + 2\int_{t_1}^{t_2}\mathcal{E}(u_s,u_s^+)\,ds = 2\int_{t_1}^{t_2}(f_s,u_s^+)\,ds + \|u_{t_2}^+\|_2^2.$$

Since $t\mapsto u_t^+$ is L^2-continuous, the case $t_1=0$ can be easily verified by passing ε to 0 in the following equation

$$\|u_\varepsilon^+\|_2^2 + 2\int_{\varepsilon}^{t_2}\mathcal{E}(u_s,u_s^+)\,ds = 2\int_{\varepsilon}^{t_2}(f_s,u_s^+)\,ds + \|u_{t_2}^+\|_2^2.$$

Finally, applying Lemma 3.9 yields the assertion.

[Step 2] Let $u \in \mathcal{C}_T$ such that u satisfies the weak relation with data (ϕ, f) over the interval $[t_1, t_2]$ where $\varepsilon \leq t_1 \leq t_2 \leq T$. We define the function $\varphi_n \in C^2(\mathbb{R}), n \in \mathbb{N}$ by:

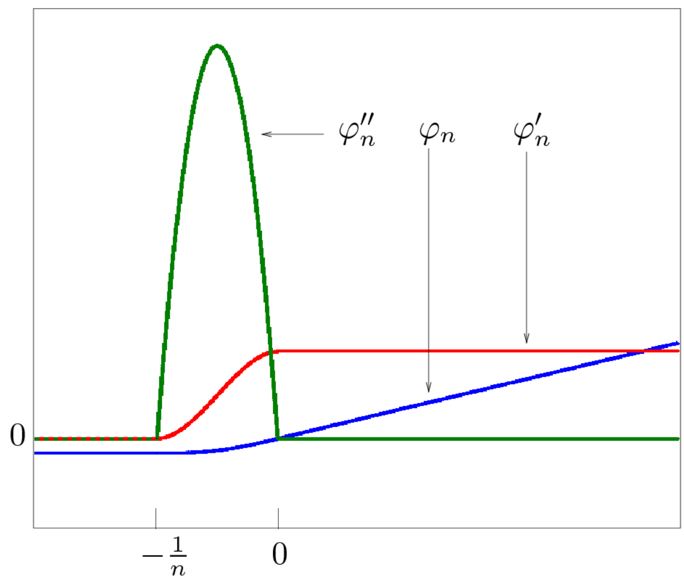
$$\varphi_n(x) := \begin{cases} x & \text{for } x \geq 0 \\ x^4\left(-\frac{1}{2}n^3\right) + x^3(-n^2) + x & \text{for } -\frac{1}{n} \leq x \leq 0 \\ -\frac{1}{2n} & \text{for } x \leq -\frac{1}{n} \end{cases}$$

with the derivatives

$$\varphi_n'(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 4x^3\left(-\frac{1}{2}n^3\right) + 3x^2(-n^2) + 1 & \text{for } -\frac{1}{n} \leq x \leq 0 \\ 0 & \text{for } x \leq -\frac{1}{n} \end{cases}$$

and

$$\varphi_n''(x) = \begin{cases} 0 & \text{for } x \geq 0 \\ 12x^2\left(-\frac{1}{2}n^3\right) + 6x(-n^2) & \text{for } -\frac{1}{n} \leq x \leq 0 \\ 0 & \text{for } x \leq -\frac{1}{n}. \end{cases}$$



It is obvious that for all $n \in \mathbb{N}$ the functions φ_n' and φ_n'' are bounded and $\varphi_n(0) = 0$. Therefore, we get by Lemma 2.6 $\varphi_n(u) \in \mathcal{C}_T$. Next we give some basic properties of φ_n:

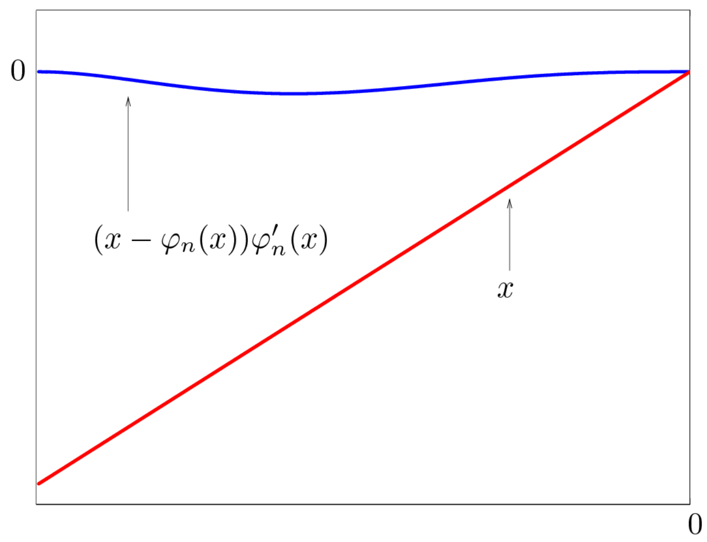
$$\varphi_n(t) = t \text{ for } t \geq 0, \quad t \vee \frac{-1}{n} \leq \varphi_n(t) \leq t \vee 0 \quad \text{and} \quad \varphi_n(t) \nearrow t \vee 0.$$

The weak relation (3.3), written with $\varphi_n(u)$ as test functions, takes the form

$$\begin{aligned} \int_{t_1}^{t_2} (u_t, \partial_t \varphi_n(u_t))\, dt &+ \int_{t_1}^{t_2} \mathcal{E}(u_t, \varphi_n(u_t))\, dt + (u_{t_1}, \varphi_n(u_{t_1})) \quad (3.16) \\ &= \int_{t_1}^{t_2} (f_t, \varphi_n(u_t))\, dt + (u_{t_2}, \varphi_n(u_{t_2})) \end{aligned}$$

where $0 < \varepsilon \le t_1 \le t_2 \le T$. The convergence of the 3rd, 4th and 5th term is easy to see. Hence, we will only check the convergence of the first two terms. Let us start by examining the first term. The integrand can be written in the form

$$(u_t, \partial_t \varphi_n(u_t)) \underset{\text{Lemma 2.6}}{=} (\varphi_n(u_t), \partial_t \varphi_n(u_t)) + (u_t - \varphi_n(u_t), \varphi_n'(u_t)\partial_t u_t). \tag{3.17}$$



The relation

$$u\mathbb{1}_{\{-\frac{1}{n}<u<0\}} \le \underbrace{(u - \varphi_n(u))\varphi_n'(u)}_{=:g(u,n)} \le 0$$

can be seen as follows:

Since $\varphi_n'(x) = 0$ for $x \le -\frac{1}{n}$, $\varphi_n(x) = x$ for $x \ge 0$, $\varphi_n'(x) \ge 0$ for all $x \in \mathbb{R}$ and $(x - \varphi_n(x)) \le 0$ for $x \in [-\frac{1}{n}, 0]$, the upper bound of $g(u,n)$ is zero. Now consider the function $\tilde{g}(x,n) := g(x,n) - x$. It is obvious that $\tilde{g}$ is a polynomial of degree 7 and $\tilde{g}$ has a root $(x,n) = (0,n)$ for all $n \in \mathbb{N}$. We only note here that $\tilde{g}$ has six other roots with imaginary parts unequal zero. It is easy to check that $\tilde{g}(-\frac{1}{n}, n) = \frac{1}{n}$. Hence, $\tilde{g}(x,n) \ge 0$ for all $x \in [-\frac{1}{n}, 0]$, $n \in \mathbb{N}$.

Now we can make the following calculation:

$$\begin{aligned}
&\lim_{n\to\infty} \left[\int_{t_1}^{t_2} (u_t, \partial_t \varphi_n(u_t))\, dt - \int_{t_1}^{t_2} (\varphi_n(u_t), \partial_t \varphi_n(u_t))\, dt \right] \\
\underset{(3.17)}{=} \quad & \lim_{n\to\infty} \int_{t_1}^{t_2} (u_t - \varphi_n(u_t), \varphi_n'(u_t)\partial_t u_t)\, dt \\
\le \quad & \lim_{n\to\infty} \int_{t_1}^{t_2} \|(u_t - \varphi_n(u_t))\varphi_n'(u_t)\|_2 \|\partial_t u_t\|_2\, dt \\
= \quad & \int_{t_1}^{t_2} \| \lim_{n\to\infty} (u_t - \varphi_n(u_t))\varphi_n'(u_t)\|_2 \|\partial_t u_t\|_2\, dt \\
= \quad & 0.
\end{aligned}$$

Since $t \mapsto \partial_t \varphi_n(u_t)$ is L^2-continuous, we have

$$\int_{t_1}^{t_2} (\varphi_n(u_t), \partial_t \varphi_n(u_t))\, dt = \frac{1}{2}\Big(\|\varphi_n(u_{t_2})\|_2^2 - \|\varphi_n(u_{t_1})\|_2^2 \Big).$$

Hence, we get for the first term of (3.16):

$$\lim_{n\to\infty} \int_{t_1}^{t_2} (u_t, \partial_t \varphi_n(u_t))\, dt = \frac{1}{2}\Big(\|u_{t_2}\|_2^2 - \|u_{t_1}\|_2^2 \Big).$$

The convergence of the second term of (3.16) will follow by Lemma 1.19. Define $\mathcal{H}_0 := L^2([t_1, t_2]; F)$. Since

$$\sup_{t\in[t_1,t_2]} \|\varphi_n(u_t)\|_2 \le \sup_{t\in[t_1,t_2]} \|u_t\|_2 < \infty$$

and

$$\int_{t_1}^{t_2} \mathcal{E}(\varphi_n(u_t))\, dt \underset{\text{Theorem 1.13}}{\le} \sup_{n\in\mathbb{N}} \left(\sup_{s\in\mathbb{R}} |\varphi_n'(s)|^2 \right) \int_{t_1}^{t_2} \mathcal{E}(u_t)\, dt < \infty,$$

it follows that

$$\sup_{n\in\mathbb{N}} \|\varphi_n(u_t)\|_{\mathcal{H}_0} < \infty.$$

The second condition of Lemma 1.19 follows from $|\varphi_n(u)| \le |u|$ easily by the dominated convergence theorem

$$\lim_{n\to\infty} \|\varphi_n(u) - u^+\|_{L^2((t_1,t_2)\times\mathbb{R}^d)} = \|\lim_{n\to\infty} \varphi_n(u) - u^+\|_{L^2((t_1,t_2)\times\mathbb{R}^d)} = 0.$$

We conclude by Lemma 1.19:

$$\varphi_n(u_t) \underset{n\to\infty}{\rightharpoonup} u_t^+ \text{ weakly in } (\mathcal{H}_0, \|\cdot\|_{\mathcal{H}_0}).$$

Hence, we get for all $\varphi \in \mathcal{C}_T$

$$\lim_{n\to\infty} \int_{t_1}^{t_2} \mathcal{E}(\varphi_t, \varphi_n(u_t))\, dt = \int_{t_1}^{t_2} \mathcal{E}(\varphi_t, u_t^+)\, dt$$

and further the convergence of the 2nd term of (3.16)

$$\lim_{n\to\infty} \int_{t_1}^{t_2} \mathcal{E}(u_t, \varphi_n(u_t))\, dt = \int_{t_1}^{t_2} \mathcal{E}(u_t, u_t^+)\, dt.$$

By passing equation (3.16) to the limit we deduce for all $0 < \varepsilon \le t_1 \le t_2 \le T$

$$\|u_{t_1}^+\|_2^2 + 2\int_{t_1}^{t_2} \mathcal{E}(u_s, u_s^+)\, ds = 2\int_{t_1}^{t_2} (f_s, u_s^+)\, ds + \|u_{t_2}^+\|_2^2.$$

□

The assertion of the next lemma will be useful in the last proposition of this section. It is a modified version of the above lemma.

Lemma 3.11. *Let $u \in \hat{F}$ be bounded and $f \in L^1(dt \times dm), f \geq 0$, be such that the weak relation (3.2) is satisfied with the test functions in $b\mathcal{C}_T$ and some function $\phi \geq 0, \phi \in L^2 \cap L^\infty$. Then u^+ satisfies the following relation with $0 \leq t_1 < t_2 \leq T$:*

$$\|u^+_{t_1}\|_2^2 + 2\int_{t_1}^{t_2} \mathcal{E}(u^+_s)\, ds \leq 2\int_{t_1}^{t_2} (f_s, u^+_s)\, ds + \|u^+_{t_2}\|_2^2.$$

Proof. First note that we can prove almost analogous to step 2 of the above proof that for each $u \in \mathcal{C}_T$, which satisfies the weak relation with data (ϕ, f) over the interval $[t_1, t_2]$, where $\varepsilon \leq t_1 \leq t_2 \leq T$ for $\varepsilon > 0$, it holds

$$\|u^+_{t_1}\|_2^2 + 2\int_{t_1}^{t_2} \mathcal{E}(u_s, u^+_s)\, ds = 2\int_{t_1}^{t_2} (f_s, u^+_s)\, ds + \|u^+_{t_2}\|_2^2. \tag{3.18}$$

Analogous to the first step of the above proof we can define approximating functions u^n and f^n and show that u^n satisfies the weak relation for the data (ϕ^n, f^n) with test functions in $b\mathcal{C}_T$ over the interval $[\varepsilon, t_2]$ where $n \geq N_\varepsilon$ and $\varepsilon \leq t_2 \leq T$. Note that by [LSU68, II.Lemma 4.7] it holds that $\lim_{n\to\infty} \int_\varepsilon^T \|f^n_t - f_t\|_1 dt = 0$ for $\varepsilon > 0$.

Fix $\varepsilon > 0$. Then it holds by (3.18) for u^n and f^n

$$\|(u^n_{t_1})^+\|_2^2 + 2\int_{t_1}^{t_2} \mathcal{E}(u^n_s, (u^n_s)^+)\, ds = 2\int_{t_1}^{t_2} (f^n_s, (u^n_s)^+)\, ds + \|(u^n_{t_2})^+\|_2^2 \tag{3.19}$$

where $0 < \varepsilon \leq t_1 \leq t_2 \leq T$ and $n > N_\varepsilon$. The convergence of all terms, which do not depend on f, follows by the same arguments as in the above proof along a subsequence. Hence, we only have to check the convergence of the term $\int_{t_1}^{t_2} (f^n_s, (u^n_s)^+)\, ds$.

Since u is bounded, it is easy to see that u^n is uniformly bounded. Now take a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} |u^{n_k}_s - u_s| = 0$ almost everywhere. Then we get

$$\begin{aligned}
& \lim_{k\to\infty} \left| \int_{t_1}^{t_2} (f^{n_k}_s, (u^{n_k}_s)^+)\, ds - \int_{t_1}^{t_2} (f_s, u^+_s)\, ds \right| \\
\leq\ & \lim_{k\to\infty} \left| \int_{t_1}^{t_2} (f^{n_k}_s - f_s, (u^{n_k}_s)^+)\, ds \right| \\
& + \lim_{k\to\infty} \left| \int_{t_1}^{t_2} (f_s, \underbrace{(u^{n_k}_s)^+ - u^+_s}_{\leq \|u\|_\infty \cdot \text{ constant}})\, ds \right| \\
\leq\ & \sup_{k\in\mathbb{N}} \|u^{n_k}\|_\infty \lim_{k\to\infty} \int_{t_1}^{t_2} \|f^{n_k}_s - f_s\|_1\, ds \\
& + \left| \int_{t_1}^{t_2} (f_s, \lim_{k\to\infty} |(u^{n_k}_s)^+ - u^+_s|)\, ds \right| \\
=\ & 0.
\end{aligned}$$

Finally, we obtain by passing a subsequence in equation (3.19) to the limit

$$\|u^+_{t_1}\|_2^2 + 2\int_{t_1}^{t_2} \mathcal{E}(u_s, u^+_s)\, ds = 2\int_{t_1}^{t_2} (f_s, u^+_s)\, ds + \|u^+_{t_2}\|_2^2$$

where $0 < \varepsilon \leq t_1 \leq t_2 \leq T$. By letting ε to 0 and applying Lemma 3.9 the assertion follows. □

The next proposition will be useful in the nonlinear case. The proof is a rewritten version of [BPS05, proof of Proposition 2.9].

Proposition 3.12. *Let $u \in \hat{F}$ be bounded and $f \in L^1(dt \times dm), f \geq 0$ be such that the weak relation (3.2) is satisfied with test functions in $b\mathcal{C}_T$ and some function $\phi \geq 0, \phi \in L^2 \cap L^\infty$. Then $u \geq 0$ and it is represented by the following relation:*

$$u_t = \int_t^T P_{s-t} f_s \, ds + P_{T-t}\phi.$$

Proof. Let $(f^n)_{n \in \mathbb{N}}$ be a sequence of bounded functions such that

$$0 \leq f^n \leq f^{n+1} \leq f, \qquad \lim_{n\to\infty} f^n = f.$$

Since f^n is bounded, we have $f^n \in L^1([0,T]; L^2)$. Next we define

$$u_t^n := \int_t^T P_{s-t} f_s^n \, ds + P_{T-t}\phi.$$

Then $u^n \in \hat{F}$ is a unique weak solution for the data (ϕ, f^n), cf. Proposition 3.8. Clearly $0 \leq u^n \leq u^{n+1}$ for all $n \in \mathbb{N}$. Define $y := u^n - u$ and $\tilde{f} := f^n - f$. Then $\tilde{f} \leq 0$ and y satisfies the weak relation for the data $(0, \tilde{f})$. Therefore, we have by Lemma 3.11 for all $t_1 \in [0,T]$

$$\|y_{t_1}^+\|_2^2 + 2\int_{t_1}^T \mathcal{E}(y_s^+)\, ds \leq 2\int_{t_1}^T (\underbrace{\tilde{f}_s}_{\leq 0}, y_s^+)\, ds.$$

Since the left hand side of this equation is positive and the right hand side is negative, we conclude that the right hand side is zero and hence $\|y_{t_1}^+\|_2^2 = 0$. Therefore, $u \geq u^n$ for all $n \in \mathbb{N}$. Set $v := \lim_{n\to\infty} u^n$. Note that we have also shown that $u \geq 0$.

Now let us write equation (3.5) for u^n and f^n

$$\|u_t^n\|_2^2 + 2\int_t^T \mathcal{E}(u_s^n)\, ds = 2\int_t^T (f_s^n, u_s^n)\, ds + \|\phi\|_2^2. \tag{3.20}$$

It is easy to see that $\lim_{n\to\infty} \|u_t^n - v_t\|_2^2 = 0$ and

$$\begin{aligned}
& \lim_{n\to\infty} \left| \int_t^T \int_{\mathbb{R}^d} (f_s^n u_s^n - f_s v_s)\, dm\, ds \right| \\
\leq \quad & \lim_{n\to\infty} \int_t^T \int_{\mathbb{R}^d} |(f_s^n u_s^n + f_s^n v_s - f_s^n v_s - f_s v_s)|\, dm\, ds \\
\leq \quad & \lim_{n\to\infty} \int_t^T \int_{\mathbb{R}^d} \underbrace{|f_s^n (u_s^n - v_s)|}_{\leq |f_s||u_s^n - v_s| \leq 2|f_s u_s|}\, dm\, ds \\
& + \lim_{n\to\infty} \int_t^T \int_{\mathbb{R}^d} \underbrace{|v_s(f_s^n - f_s)|}_{\leq 2|f_s u_s|}\, dm\, ds \\
\underset{Lebesgue}{=} \quad & 0.
\end{aligned}$$

Since $(\mathcal{E}, F)$ is a positive preserving form (i.e. if $u \in \mathcal{D}(\mathcal{E})$, then $u^+ \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(u^+, u^-) \leq 0$, cf. [MR92, I.Theorem 4.4], [Sch99, Note 2]), it follows by [Sch99, Proposition 2] that

$$\int_t^T \mathcal{E}(v_s)\, ds \leq \int_t^T \liminf_{n\to\infty} \mathcal{E}(u_s^n)\, ds$$

and hence by Fatou's lemma

$$\int_t^T \mathcal{E}(v_s)\, ds \leq \liminf_{n\to\infty} \int_t^T \mathcal{E}(u_s^n)\, ds.$$

Finally, we get for all $t \in [0, T]$

$$\begin{aligned}
\|v_t\|_2^2 + 2\int_t^T \mathcal{E}(v_s)\, ds &\leq \lim_{n\to\infty} \|u_t^n\|_2^2 + \liminf_{n\to\infty} 2\int_t^T \mathcal{E}(u_s^n)\, ds \\
&= \liminf_{n\to\infty} \left(\|u_t^n\|_2^2 + 2\int_t^T \mathcal{E}(u_s^n)\, ds \right) \\
&\underset{(3.20)}{=} \liminf_{n\to\infty} \left(2\int_t^T (f_s^n, u_s^n)\, ds + \|\phi\|_2^2 \right) \\
&= \lim_{n\to\infty} \left(2\int_t^T (f_s^n, u_s^n)\, ds + \|\phi\|_2^2 \right) \\
&= 2\int_t^T (f_s, v_s)\, ds + \|\phi\|_2^2.
\end{aligned}$$

Since the right side of this equation is finite for all $t \in [0, T]$, $t \mapsto v_t$ is L^2-continuous and by Lemma 1.19 it holds $v_t \in F$ (cf.(2) below), we obtain that $v \in \hat{F}$.

Now we present that v satisfies the weak relation for the data (ϕ, f). To conclude this we need the following three relations.

(1) Since $\varphi^n(t) := \|u_t^n - v_t\|_2$ is continuous and decreasing, we conclude by Dini's theorem

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|u_t^n - v_t\|_2 = 0$$

and therefore

$$\lim_{n\to\infty} \int_0^T \|u_t^n - v_t\|_2^2\, dt \leq \lim_{n\to\infty} \sup_{t\in[0,T]} \|u_t^n - v_t\|_2^2 \cdot T = 0.$$

(2) Since $\limsup_{n\to\infty} \int_t^T \mathcal{E}(u_s^n)\, ds \leq \frac{1}{2}\left(-\|v_t\|_2^2 + 2\int_t^T (f_s, v_s)\, ds + \|\phi\|_2^2 \right)$, there exists $K \in \mathbb{R}_+$ and a subsequence $(n_k)_{k\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ such that

$$\left| \int_0^T \mathcal{E}(u_s^{n_k})\, ds \right| \leq K \quad \text{for all } k \in \mathbb{N}.$$

Moreover, there exists $\tilde{K} \in \mathbb{N}$ such that

$$\left| \int_0^T \left(\mathcal{E}(u_s^{n_k}) + \|u_s^{n_k}\|_2^2 \right) \, ds \right| \leq \left| \int_0^T \left(\mathcal{E}(u_s^{n_k}) + \|u_s\|_2^2 \right) \, ds \right| \leq \tilde{K} \quad \text{for all } k \in \mathbb{N}.$$

By (1) we have $\lim_{k\to\infty} u_t^{n_k} = v_t$ in $\mathcal{C}([0,T];L^2)$ and therefore $\lim_{k\to\infty} u_t^{n_k} = v_t$ in $L^2([0,T];F)$. We obtain by Lemma 1.19

$$\lim_{k\to\infty} u^{n_k} = v \qquad \text{weakly in } (L^2([0,T];F), \|\cdot\|_{L^2([0,T];L^2)}).$$

Hence,

$$\lim_{k\to\infty} \int_0^T \mathcal{E}(u_s^{n_k}, \varphi_s) \, ds = \int_0^T \mathcal{E}(v_s, \varphi_s) \, ds.$$

(3)

$$\lim_{k\to\infty} \left| \int_0^T (f_t^{n_k} - f_t, \varphi_t) \, dt \right| \leq \int_0^T \lim_{k\to\infty} |(f_t^{n_k} - f_t, \varphi_t)| \, dt = 0$$

Finally, we deduce from $(1) - (3)$ by passing the weak relation for u^{n_k} associated to (ϕ, f^{n_k}) to the limit, the weak relation for v associated to (ϕ, f). Clearly $u - v$ verifies the linear equation

$$(\partial_t + L)(u - v) = 0, \qquad u_T - v_T = 0$$

in the weak sense. By Proposition 3.8 we have $u - v = 0$. Since

$$v_t = \int_t^T P_{s-t} f_s \, ds + P_{T-t}\phi,$$

the assertion follows. □

Chapter 4

The Nonlinear Equation in Dependence of $D_\sigma u$

The aim of this chapter is to generalize [BPS05, Chapter 3]. Let ϕ be an element of $L^2(\mathbb{R}^d, m; \mathbb{R}^l)$. We consider the nonlinear equation for $t \in [0,T]$

$$(\partial_t + L)u + f(\cdot,\cdot,u,D_\sigma u) = 0, \qquad u_T = \phi \tag{4.1}$$

where the nonlinear term is the measurable function

$$f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^l \times \mathbb{R}^l \otimes \mathbb{R}^k \to \mathbb{R}^l, l \in \mathbb{N}^\star.$$

Here $D_\sigma u$ is a generalized gradient, which is defined in Section 4.1 for a bounded measurable map

$$\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^k, \sigma = (\sigma_l^i), i = 1, ..., d, l = 1, ..., k$$

with the property $(\sigma\sigma^\star)^{i,j} \in L^1_{\mathrm{loc}}(\mathbb{R}^d, m)$.

From now on we assume

$$(A1) \qquad \tilde{A} := (\tilde{a}^{i,j})_{i,j=1,\ldots,d} \text{ is bounded and } \sum_{i,j=1}^{d} a^{i,j}\xi_i\xi_j \geq 0 \text{ for all } \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$$

and moreover,

$$(A2) \qquad \mathcal{E}^A(u) \leq K_A\,\mathcal{E}(u) + C_A\|u\|_2^2 \text{ for some } K_A \in [1,2), C_A \in \mathbb{R}_+ \text{ and for all } u \in F.$$

Note that we can always find σ such that $\tilde{a}^{i,j} = (\sigma\sigma^\star)^{i,j}$. We fix such a map.

Remark 4.1. *(i) The condition $K_A < 2$ in (A2) is for example necessary at the end of the proof of Proposition 4.8.*
(ii) Let A be a matrix with elements $a^{i,j} \in L^1_{loc}(\mathbb{R}^d, m), i, j = 1, \ldots, d$ such that $a^{i,j} = (\sigma\sigma^\star)^{i,j} + p^{i,j}$ where $p^{i,j} = -p^{j,i}$ and $0 \neq p^{i,j} \in L^1_{loc}(\mathbb{R}^d, m)$. Then $\tilde{a}^{i,j} = (\sigma\sigma^\star)$ and $a^{i,j} \neq a^{j,i}$.
(iii) The second part of condition (A1) is only a condition on the symmetric part of A.

Notation. We introduce the following notation:

$$\begin{aligned} f^0(t,x) &:= f(t,x,0,0), & f(t,x,y) &:= f(t,x,y,\cdot), \\ f'(t,x,y) &:= f(t,x,y,0) - f^0, & f(y,z) &:= f(\cdot,\cdot,y,z), \\ f'^{,r}(t,x) &:= \sup_{|y|\le r} |f'(t,x,y)|, \end{aligned}$$

where $r \in \mathbb{R}_+$. Let $d, k, l \in \mathbb{N}$ and $z = (z_j^i) \in \mathbb{R}^d \otimes \mathbb{R}^k$. We will denote by

$$\begin{array}{rl} |\cdot| & \text{the Euclidean norm on } \mathbb{R}^d, \\ \langle \cdot, \cdot \rangle & \text{the scalar product on } \mathbb{R}^d, \\ \langle z_1, z_2 \rangle = tr(z_1 z_2^\star) & \text{the trace scalar product on } \mathbb{R}^d \otimes \mathbb{R}^k, \\ |z| = \left(\sum_{i=1}^d \sum_{j=1}^k (z_j^i)^2 \right)^{\frac{1}{2}} & \text{the associated norm to the trace scalar product.} \end{array}$$

Moreover, we use the following notation for $\psi, \xi \in L^2(\mathbb{R}^d, m; \mathbb{R}^l)$:

$$(\psi, \phi) = \int_{\mathbb{R}^d} \langle \psi, \phi \rangle \, dm \quad \text{and} \quad \|\psi\|_2^2 \;=\; \sum_{i=1}^l \|\psi^i\|_2^2$$

where $L^2(\mathbb{R}^d, m; \mathbb{R}^l) := \left\{ \psi : \mathbb{R}^d \to \mathbb{R}^l \text{ measurable} \;\middle|\; \int_{\mathbb{R}^d} |\psi|^2 \, dm < \infty \right\}$.

4.1 The Generalized Gradient

Lemma 4.2. *There exists $\tau : \mathbb{R}^d \to \mathbb{R}^k \otimes \mathbb{R}^d$ such that*

$$\sigma\tau = \tau^\star\sigma^\star, \quad \tau\sigma = \sigma^\star\tau^\star, \quad \sigma\tau\sigma = \sigma, \quad \|\sigma\tau\| = \|\tau\sigma\| \le 1$$

where the norm is the operator norm.

Proof. See [BPS05, Lemma A.1]. □

Lemma 4.3. *Let $u, \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ then*

$$\langle \tilde{A} \nabla \varphi, \nabla u \rangle = \langle \nabla \varphi \sigma, \nabla u \sigma \rangle.$$

Proof. Let us denote $u_i := \frac{\partial u}{\partial x_i}$ and $\varphi_i := \frac{\partial \varphi}{\partial x_i}$ for $i = 1, \dots, d$.

$$\begin{aligned} \langle \tilde{A} \nabla \varphi, \nabla u \rangle &= \sum_{i=1}^d \sum_{j=1}^d \tilde{a}^{i,j} \varphi_j u_i = \sum_{i=1}^d \sum_{j=1}^d \sum_{l=1}^k \varphi_j \sigma_l^i \sigma_l^j u_i \\ &= \sum_{l=1}^k \sum_{i=1}^d \sum_{j=1}^d \varphi_j \sigma_l^i \sigma_l^j u_i = \sum_{l=1}^k \left[\left(\sum_{i=1}^d \varphi_i \sigma_l^i \right) \left(\sum_{j=1}^d u_j \sigma_l^j \right) \right] \\ &= \left\langle \left(\left(\sum_{i=1}^d \varphi_i \sigma_1^i \right), \dots, \left(\sum_{i=1}^d \varphi_i \sigma_k^i \right) \right), \left(\left(\sum_{j=1}^d u_j \sigma_1^j \right), \dots, \left(\sum_{j=1}^d u_j \sigma_k^j \right) \right) \right\rangle \\ &= \langle \nabla \varphi \sigma, \nabla u \sigma \rangle \end{aligned}$$

□

By this lemma we have the following representation of the symmetric form $(\tilde{\mathcal{E}}^A, \mathcal{C}_0^\infty(\mathbb{R}^d))$:

$$\tilde{\mathcal{E}}^A(u,v) = \int_{\mathbb{R}^d} \langle \nabla v\sigma, \nabla u\sigma \rangle \, dm \quad \text{for all } u,v \in \mathcal{C}_0^\infty(\mathbb{R}^d). \tag{4.2}$$

Notation. From now on let us write $L^2_{d,k} := L^2(\mathbb{R}^d, m; \mathbb{R}^k)$ and $L^2([0,T] \times \mathbb{R}^d) := L^2([0,T] \times \mathbb{R}^d, dt \times dm)$. We define for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ the term $D_\sigma \varphi := \nabla\varphi\sigma$. Let $V_0 := \{D_\sigma\varphi : \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)\}$ and $V := \overline{V_0}^{L^2_{d,k}}$, i.e. the closure of V_0 w.r.t. $\|\cdot\|_{L^2_{d,k}}$. Moreover, we define the spaces F^A and $\hat{F}^A$ w.r.t. $\tilde{\mathcal{E}}^A_1$ analogous to F and $\hat{F}$. Note that by $(A2)$ it holds $F^A \supset F$ and $\hat{F}^A \supset \hat{F}$.

In the next proposition we extend (4.2) to F^A. In (i) we show that (4.2) is well defined for $u \in F^A$ and in (ii) we give a representation for $u \in \hat{F}^A$. In (iii) we prove that D_σ is closable as an operator from $\hat{F}^A$ into $L^2((0,T) \times \mathbb{R}^d)$. The proof of the uniqueness in (i) and the proofs of (ii) and (iii) follow the arguments of [BPS05, Proposition 2.3].

Proposition 4.4. *(i) For every $u \in F^A$ there exists a unique element of V, which we denote by $D_\sigma u$ such that*

$$\tilde{\mathcal{E}}^A(u,\varphi) = \int_{\mathbb{R}^d} \langle D_\sigma u(x), D_\sigma\varphi(x) \rangle \, m(dx) \quad \textit{for all } \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d). \tag{4.3}$$

Moreover, the above formula (4.3) *extends*

$$\tilde{\mathcal{E}}^A(u,v) = \int_{\mathbb{R}^d} \langle D_\sigma u(x), D_\sigma v(x) \rangle \, m(dx) \quad \textit{for all } u,v \in F^A. \tag{4.4}$$

Furthermore, we have $D_\sigma u\tau\sigma = D_\sigma u$, where τ is as in Lemma 4.2.

(ii) If $u \in \hat{F}^A$, there exists a measurable function $\phi : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ such that $|\phi\sigma| \in L^2((0,T) \times \mathbb{R}^d)$ and $D_\sigma u_t = \phi_t\sigma$ for almost every $t \in [0,T]$.

(iii) Let $u^n, u \in \hat{F}^A, n \in \mathbb{N}$, such that $u^n \to u$ in $L^2((0,T)\times\mathbb{R}^d)$ and $(D_\sigma u^n)_{n\in\mathbb{N}}$ is Cauchy in $L^2([0,T] \times \mathbb{R}^d)$. Then $D_\sigma u^n \to D_\sigma u$ in $L^2((0,T) \times \mathbb{R}^d)$, i.e. D_σ is closable as an operator from $\hat{F}^A$ into $L^2((0,T) \times \mathbb{R}^d)$.

Proof. (i) [Uniqueness]: Let $v, w \in V$ such that

$$\begin{aligned} \tilde{\mathcal{E}}^A(u,\varphi) &= \int_{\mathbb{R}^d} \langle v(x), D_\sigma\varphi(x) \rangle \, m(dx) \\ &= \int_{\mathbb{R}^d} \langle w(x), D_\sigma\varphi(x) \rangle \, m(dx) \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d), \end{aligned}$$

then we have

$$0 = \int_{\mathbb{R}^d} \langle w(x) - v(x), D_\sigma\varphi(x) \rangle \, m(dx) \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d).$$

Since by definition $V_0 \subset V$ densely and $v - w \in V$, we deduce that $v = w$.

[Existence]: We have by Lemma 4.3

$$\tilde{\mathcal{E}}^A(u,\varphi) = \int_{\mathbb{R}^d} \langle D_\sigma u(x), D_\sigma\varphi(x) \rangle \, m(dx) \quad \text{for all } u,\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d).$$

Let $u \in F^A$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. Take $u_n \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $(\tilde{\mathcal{E}}_1^A)^{\frac{1}{2}}$-$\lim_{n\to\infty} u_n{=}u$. Then $\lim_{n\to\infty} \tilde{\mathcal{E}}^A(u_n,\varphi){=}\tilde{\mathcal{E}}^A(u,\varphi)$. Let $\varepsilon > 0$. Since $(u_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in $(F^A, (\tilde{\mathcal{E}}_1^A)^{\frac{1}{2}})$, there exists $N_\varepsilon \in \mathbb{N}$ such that for all $n, m > N_\varepsilon$

$$\begin{aligned}
\|D_\sigma u_n - D_\sigma u_m\|^2_{L^2_{d,k}} &= \int_{\mathbb{R}^d} |D_\sigma u_n - D_\sigma u_m|^2 \, dm \\
&= \int_{\mathbb{R}^d} |\nabla(u_n - u_m)\sigma|^2 \, dm \\
&= \tilde{\mathcal{E}}^A(u_n - u_m) \\
&< \varepsilon.
\end{aligned}$$

Hence, we deduce that $(D_\sigma u_n)_{n\in\mathbb{N}}$ is a $L^2_{d,k}$-Cauchy-sequence and define the $L_{d,k}$-limit $D_\sigma u := \lim_{n\to\infty} D_\sigma u_n$. Then

$$\begin{aligned}
\tilde{\mathcal{E}}^A(u,\varphi) = \lim_{n\to\infty} \tilde{\mathcal{E}}^A(u_n,\varphi) &\underset{\text{Lemma 4.3}}{=} \lim_{n\to\infty} \int_{\mathbb{R}^d} \langle D_\sigma u_n(x), D_\sigma \varphi(x)\rangle \, m(dx) \\
&= \int_{\mathbb{R}^d} \langle D_\sigma u(x), D_\sigma \varphi(x)\rangle \, m(dx).
\end{aligned}$$

Therefore, equation (4.3) holds. Next we will show (4.4). Let $v \in F^A$. Then we may find $v_n \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $(\mathcal{E}_1^A)^{\frac{1}{2}}$-$\lim_{n\to\infty} v_n = v$. Analogous to the above calculations $(D_\sigma v_n)_{n\in\mathbb{N}}$ is a $L^2_{d,k}$-Cauchy-sequence. Hence, we define $D_\sigma v := \lim_{n\to\infty} D_\sigma v_n$. Summarized it holds

$$\begin{aligned}
\tilde{\mathcal{E}}^A(u,v) = \lim_{n\to\infty} \tilde{\mathcal{E}}^A(u,v_n) &\underset{(4.3)}{=} \lim_{n\to\infty} \int_{\mathbb{R}^d} \langle D_\sigma u(x), D_\sigma v_n(x)\rangle \, m(dx) \\
&= \int_{\mathbb{R}^d} \langle D_\sigma u(x), D_\sigma v(x)\rangle \, m(dx).
\end{aligned}$$

Therefore,

$$\tilde{\mathcal{E}}^A(u,v) = \int_{\mathbb{R}^d} \langle D_\sigma u(x), D_\sigma v(x)\rangle \, m(dx) \quad \text{for all } v, u \in F^A.$$

Left to show is the assertion $D_\sigma u = D_\sigma u \tau\sigma$. Let $D_\sigma u \in V$. Then there exists $(D_\sigma u_n)_{n\in\mathbb{N}}$ such that $D_\sigma u_n \in V_0$, $L^2_{d,k}$-$\lim_{n\to\infty} D_\sigma u_n = D_\sigma u$ and $u_n \in C_0^\infty(\mathbb{R}^d)$. Hence, we get

$$\begin{aligned}
& \|D_\sigma u - D_\sigma u\tau\sigma\|_{L^2_{d,k}} \\
\leq\; & \|D_\sigma u - D_\sigma u_n\|_{L^2_{d,k}} + \underbrace{\|D_\sigma u_n - D_\sigma u_n\tau\sigma\|_{L^2_{d,k}}}_{\underset{\text{Lemma 4.2}}{=} 0, \text{ since } u_n \in \mathcal{C}_0^\infty(\mathbb{R}^d)} \\
& + \|D_\sigma u_n\tau\sigma - D_\sigma u\tau\sigma\|_{L^2_{d,k}} \\
=\; & \|D_\sigma u - D_\sigma u_n\|_{L^2_{d,k}} + \|(D_\sigma u_n - D_\sigma u)\tau\sigma\|_{L^2_{d,k}} \\
\leq\; & \|D_\sigma u - D_\sigma u_n\|_{L^2_{d,k}} + \left(\int_{\mathbb{R}^d} |D_\sigma u_n - D_\sigma u|^2 \underbrace{\|\tau\sigma\|^2}_{\leq 1} \, dm \right)^{\frac{1}{2}} \\
\underset{n\to\infty}{\to}\; & 0.
\end{aligned}$$

(ii) Let $u \in \hat{F}^A$. Then we have by (i)

$$\tilde{\mathcal{E}}^A(u_t, \varphi) = \int_{\mathbb{R}^d} \langle D_\sigma u_t(x), D_\sigma \varphi(x) \rangle \, m(dx) \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d) \text{ and a.e. } t.$$

Further we deduce analogous to Lemma 2.12 the existence of functions $u^n \in \mathcal{C}_0^\infty([0,T] \times \mathbb{R}^d), n \in \mathbb{N}$, such that $\mathcal{A}_1^A(u^n - u) := \int_0^T \mathcal{E}_1^A(u^n - u)\, dt \to 0$. Hence, we define $\psi := \lim_{n\to\infty} \nabla u^n \sigma$ in $L^2([0,T] \times \mathbb{R}^d; \mathbb{R}^k)$ and $\phi := \psi\tau$. Then we calculate by using Lemma 4.2

$$D_\sigma u^n = \nabla u^n \sigma = \nabla u^n \sigma\tau\sigma \underset{(\star)}{\to} \psi\tau\sigma = \phi\sigma \text{ in } L^2([0,T] \times \mathbb{R}^d; \mathbb{R}^k).$$

$$\begin{aligned}
(\star) \qquad & \lim_{n\to\infty} \|(\nabla u^n \sigma - \psi)\tau\sigma\|_{L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^k)} \\
= \quad & \lim_{n\to\infty} \int_0^T \int_{R^d} |(\nabla u^n \sigma - \psi)\tau\sigma|^2 \, dm\, dt \\
\leq \quad & \lim_{n\to\infty} \int_0^T \int_{R^d} |\nabla u^n \sigma - \psi|^2 \underbrace{\|\tau\sigma\|^2}_{\leq 1} \, dm\, dt \\
\leq \quad & \lim_{n\to\infty} \|\nabla u^n \sigma - \psi \;\|_{L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^k)} = 0
\end{aligned}$$

Therefore, $|\phi\sigma| \in L^2([0,T] \times \mathbb{R}^d)$. Since

$$\lim_{n\to\infty} \|\nabla u^n \sigma - \phi\sigma\|_{L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^k)} = 0,$$

we can find a subsequence $(n_k)_{k\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ and a zeroset $\Lambda_1 \subset [0,T]$ such that

$$\lim_{k\to\infty} \|\nabla u_t^{n_k} \sigma - \phi_t \sigma\|^2_{L^2_{d,k}} = 0 \quad \text{for all } t \in [0,T] \backslash \Lambda_1.$$

Since $\lim_{n\to\infty} \mathcal{A}_1^A(u^n - u) = 0$, we also have $\lim_{k\to\infty} \mathcal{A}_1^A(u^{n_k} - u) = 0$ and therefore, we may find a subsequence $(n_{k_l})_{l\in\mathbb{N}}$ of $(n_k)_{k\in\mathbb{N}}$ and a zeroset $\Lambda_2 \subset [0,T]$ such that

$$\lim_{l\to\infty} \tilde{\mathcal{E}_1}^A (u_t^{n_{k_l}} - u_t) = 0 \quad \text{for all } t \in [0,T] \backslash \Lambda_2.$$

Define $\Lambda := \Lambda_1 \cup \Lambda_2$ and fix $t \in [0,T]\backslash\Lambda$. Clearly,

$$\tilde{\mathcal{E}}^A(u_t^{n_{k_l}}, \varphi) = \int_{\mathbb{R}^d} \langle \nabla u_t^{n_{k_l}} \sigma(x), D_\sigma \varphi(x) \rangle \, m(dx) \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d).$$

Since

$$\lim_{l\to\infty} |\tilde{\mathcal{E}}^A(u_t^{n_{k_l}} - u_t, \varphi)| \leq \lim_{l\to\infty} \left(\tilde{\mathcal{E}}^A(u_t^{n_{k_l}} - u_t)^{\frac{1}{2}} \tilde{\mathcal{E}}^A(\varphi)^{\frac{1}{2}} \right) \quad = \quad 0$$

and

$$\begin{aligned}
& \lim_{l\to\infty} \left| \int_{\mathbb{R}^d} \langle \nabla u_t^{n_{k_l}} \sigma - \phi_t \sigma, D_\sigma \varphi \rangle \, dm \right| \\
\leq \quad & \lim_{l\to\infty} \left(\int_{\mathbb{R}^d} |\nabla u_t^{n_{k_l}} \sigma - \phi_t \sigma|^2 \, dm \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} |D_\sigma \varphi|^2 \, dm \right)^{\frac{1}{2}} \\
= \quad & 0,
\end{aligned}$$

we deduce the following equation for u_t

$$\tilde{\mathcal{E}}^A(u_t,\varphi) = \int_{\mathbb{R}^d} \langle \phi_t\sigma, D_\sigma\varphi\rangle\, dm \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d).$$

Therefore, the assertion $D_\sigma u_t = \phi_t\sigma$ follows by uniqueness.

(iii) Define $v := \lim_{n\to\infty} D_\sigma u^n$ in $L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^k)$. So we may find a zeroset $\Lambda_1 \subset [0,T]$ and a subsequence $(n_k)_{k\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ such that it holds for every $t \in [0,T]\backslash\Lambda_1$

$$\lim_{k\to\infty} \|v_t - D_\sigma u_t^{n_k}\|_{L^2_{d,k}} = 0. \tag{4.5}$$

Since $u^n \to u$ in $L^2((0,T)\times\mathbb{R}^d)$, we have $u^{n_k} \to u$ in $L^2((0,T)\times\mathbb{R}^d)$ and can find a subsequence $(n_{k_l})_{l\in\mathbb{N}}$ of $(n_k)_{k\in\mathbb{N}}$ and a zeroset $\Lambda_2 \subset [0,T]$ such that for every $t \in [0,T]\backslash\Lambda_2$

$$\lim_{l\to\infty} \|u_t^{n_{k_l}} - u_t\|_2 = 0. \tag{4.6}$$

Fix the set $\Lambda := \Lambda_1 \cup \Lambda_2$ and denote the infinitesimal generator associated to $\tilde{\mathcal{E}^A}$ by $\tilde{L}^A$. Moreover, fix a $t \in [0,T]\backslash\Lambda$ and let $\varphi \in \mathcal{D}(\tilde{L}^A) \subset F^A$. Then

$$\begin{aligned}
(v_t, D_\sigma\varphi) &\underset{(4.5)}{=} \lim_{l\to\infty} (D_\sigma u_t^{n_{k_l}}, D_\sigma\varphi) = \lim_{l\to\infty} \int_{R^d} \langle D_\sigma u_t^{n_{k_l}}, D_\sigma\varphi\rangle\, dm \\
&= \lim_{l\to\infty} \tilde{\mathcal{E}}^A(u_t^{n_{k_l}},\varphi) = -\lim_{l\to\infty} (u_t^{n_{k_l}}, \tilde{L}^A\varphi) \underset{(4.6)}{=} -(u_t, \tilde{L}^A\varphi) \\
&= \tilde{\mathcal{E}}^A(u_t,\varphi) = (D_\sigma u_t, D_\sigma\varphi).
\end{aligned}$$

Hence, it holds

$$0 = (v_t - D_\sigma u_t, D_\sigma\varphi) \quad \text{for all } \varphi \in \mathcal{D}(\tilde{L}^A).$$

If we can show that $\overline{\{D_\sigma\varphi : \varphi \in \mathcal{D}(\tilde{L}^A)\}}^{L^2_{d,k}} = V$, the assertion $v_t = D_\sigma u_t$ will follow. Note that by [MR92, I. Theorem 2.13] it holds that $\mathcal{D}(\tilde{L}^A)$ is dense in F^A. First we will show

$$\hat{V} := \{D_\sigma\varphi : \varphi \in F^A\} \subset V.$$

Let $\varphi \in F^A$. Then there exists $\varphi_n \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $\varphi_n \to \varphi$ w.r.t. $(\tilde{\mathcal{E}}_1{}^A)^{\frac{1}{2}}$. Hence, $D_\sigma\varphi_n \to D_\sigma\varphi$ in $L^2_{d,k}$. Since $V = \overline{\{D_\sigma\varphi : \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)\}}^{L^2_{d,k}}$, we get $D_\sigma\varphi \in V$ and consequently

$$\{D_\sigma\varphi : \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)\} \subset \{D_\sigma\varphi : \varphi \in F^A\} \subset V.$$

Now it is obvious that $\hat{V}$ is dense in V. Next let us show that

$$\tilde{V} := \{D_\sigma\varphi : \varphi \in \mathcal{D}(\tilde{L}^A)\} \subset \{D_\sigma\varphi : \varphi \in F^A\} \subset \overline{\{D_\sigma\varphi : \varphi \in \mathcal{D}(\tilde{L}^A)\}}^{L^2_{d,k}}.$$

Let $\varphi \in F^A$. Then there exists $\varphi_n \in \mathcal{D}(\tilde{L}^A)$ such that $\varphi_n \to \varphi$ w.r.t. $(\tilde{\mathcal{E}}_1{}^A)^{\frac{1}{2}}$. Hence, we conclude that the limit of $D_\sigma\varphi_n$ exists in $L^2_{d,k}$. Summarized we get $\tilde{V} \underset{\text{densely}}{\subset} V$ and therefore the assertion follows. □

4.2 Solution of the Nonlinear Equation

In this nonlinear framework we use the same definition of a solution as in [BPS05]. The proposition in the next section shows the existence of a unique solution under Lipschitz conditions. In Section 4.4 the case of more general monotonicity conditions is treated.

Definition 4.5. [*Solution of the nonlinear equation*]
A solution of equation (4.1) *is a system* $u = (u^1, u^2, \dots, u^l)$ *of* l *elements in* $\hat{F}$, *for which we denote by* $D_\sigma u_t$ *the* $\mathbb{R}^l \otimes \mathbb{R}^k$*-matrix whose rows are* $D_\sigma u_t^i, i = 1, \dots, l$, *which has the property that each function* $f^i(\cdot, \cdot, u, D_\sigma u)$ *belongs to* $L^1([0,T]; L^2)$, *and such that the function* u^i *satisfies the following weak sense equation associated to* $(\phi^i, f^i(\cdot, \cdot, u, D_\sigma u))$ *for all* $\varphi \in \mathcal{C}_T$*:*

$$\int_0^T [(u_t^i, \partial_t \varphi_t) + \mathcal{E}(u_t^i, \varphi_t)]\, dt \tag{4.7}$$
$$= \int_0^T (f_t^i(u_t, D_\sigma u_t), \varphi_t)\, dt + (\phi^i, \varphi_T) - (u_0^i, \varphi_0).$$

Definition 4.6. [*mild equation*]
For every $i \in \{1, \dots, l\}$ *we define the mild equation*

$$u^i(t,x) = P_{T-t}\phi^i(x) + \int_t^T P_{s-t} f^i(s, \cdot, u_s, D_\sigma u_s)(x)\, ds, \ m\text{-a.e.}. \tag{4.8}$$

We say that u *solves the mild equation, if every* u^i *solves the mild equation.*

Lemma 4.7. u *is a solution of the nonlinear equation* (4.1)*, if and only if it solves the mild equation* (4.8)*.*

Proof. The assertion follows by Proposition 3.8 (cf. [BPS05, p.33]). □

Notation. For $u, v \in F^l$ we define $\mathcal{E}(u,v) := \sum_{i=1}^l \mathcal{E}(u^i, v^i)$ and $\mathcal{E}^A(u,v) := \sum_{i=1}^l \mathcal{E}^A(u^i, v^i)$. We denote by $L^2([0,T] \times \mathbb{R}^d; \mathbb{R}^l)$ the function space $L^2([0,T] \times \mathbb{R}^d, dt \times dm; \mathbb{R}^l)$ and by $L^2_{d,l}$ the function space $L^2(\mathbb{R}^d, m; \mathbb{R}^l)$.

4.3 The Case of Lipschitz Conditions

We follow [BPS05, Proposition 3.1].

Proposition 4.8. *Consider a measurable function*

$$f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^l \times \mathbb{R}^l \otimes \mathbb{R}^k \to \mathbb{R}^l$$

such that

$$|f(t,x,y,z) - f(t,x,y',z')| \leq C(|y - y'| + |z - z'|) \tag{4.9}$$

with t, x, y, y', z, z' *arbitrary and* $C \in \mathbb{R}_+$ *constant. Let* $f^0 \in L^2([0,T] \times \mathbb{R}^d; \mathbb{R}^l)$ *and* $\phi \in L^2_{d,l}$*. Then the equation* (4.1) *admits a unique solution* $u \in \hat{F}^l$*, which satisfies the following estimate:*

$$\|u\|_T^2 \leq \frac{1}{2 - K_A} e^{T(1+2C+C^2+C_A)} \Big(\|\phi\|_2^2 + \|f^0\|^2_{L^2([0,T]\times\mathbb{R}^d)} \Big).$$

Proof. By relation (4.9) we have

$$\begin{aligned} |f(\cdot,\cdot,u,D_\sigma u)| &\leq |f(\cdot,\cdot,u,D_\sigma u) - f(\cdot,\cdot,0,0)| + |f(\cdot,\cdot,0,0)| \\ &\leq C(|u| + |D_\sigma u|) + |f^0|. \end{aligned}$$

Note that if $u \in \hat{F}^l$, then it follows by Proposition 4.4(ii) that $|D_\sigma u|$ is an element of $L^2([0,T]\times\mathbb{R}^d)$. Since it holds $f^0 \in L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^l)$, we get in this situation $f(\cdot,\cdot,u,D_\sigma u) \in L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^l)$.

Now let us define the operator $A : \hat{F}^l \to \hat{F}^l$ by

$$(Au)^i(t,x) = P_{T-t}\phi^i(x) + \int_t^T P_{s-t}f^i(s,\cdot,u_s,D_\sigma u_s)(x)\,ds, \qquad i = 1,\dots,l.$$

Then we know by Proposition 3.8 that $Au \in \hat{F}^l$.

Next we will show that if T is sufficiently small, then A is a contraction with respect to $\|\cdot\|_T$. Afterwards the existence and uniqueness of a solution will follow by a recurrence procedure.

In the following we write $f^i_{u,s} := f^i(s,\cdot,u_s,D_\sigma u_s)$. Let us start with an estimate for $\mathcal{E}(Au_t - Av_t)$.

$$\begin{aligned}
&[\mathcal{E}(Au_t - Av_t)]^{\frac12} \\
\underset{\text{def.}}{=}\; & \left[\sum_{i=1}^l \tilde{\mathcal{E}}\left(\int_t^T P_{s-t}(f^i_{u,s} - f^i_{v,s})\,ds\right)\right]^{\frac12} \\
\underset{\text{Lemma 2.9}}{=}\; & \left[\sum_{i=1}^l \int_t^T\int_t^T \left(\tilde{\mathcal{E}}\left(P_{s-t}(f^i_{u,s} - f^i_{v,s}), P_{r-t}(f^i_{u,r} - f^i_{v,r})\right)\right) ds\,dr\right]^{\frac12} \\
\leq\; & \left[\sum_{i=1}^l \int_t^T\int_t^T \left(\tilde{\mathcal{E}}\left(P_{s-t}(f^i_{u,s} - f^i_{v,s})\right)^{\frac12}\tilde{\mathcal{E}}\left(P_{r-t}(f^i_{u,r} - f^i_{v,r})\right)^{\frac12}\right) dsdr\right]^{\frac12} \\
=\; & \left[\sum_{i=1}^l\left(\underbrace{\int_t^T \mathcal{E}\left(P_{s-t}(f^i_{u,s} - f^i_{v,s})\right)^{\frac12}\,ds}_{\geq 0}\right)^2\right]^{\frac12} \\
\leq\; & \int_t^T\sum_{i=1}^l \mathcal{E}\left(P_{s-t}(f^i_{u,s} - f^i_{v,s})\right)^{\frac12}\,ds \\
\underset{\text{Lemma 1.9}}{\leq}\; & \sqrt{\tilde{C}}\int_t^T \|f_{u,s} - f_{v,s}\|_2\frac{ds}{\sqrt{s-t}}
\end{aligned}$$

Further by using equation (4.9) and

$$\begin{aligned}
(\star) \qquad \int_0^s\sqrt{\frac{(T-t)}{(s-t)}}\,dt &= \int_0^s\sqrt{\frac{(T-s+s-t)}{(s-t)}}\,dt = \int_0^s\sqrt{\frac{(T-s+t)}{t}}\,dt \\
&\leq \int_0^T\sqrt{\frac{T}{t}+1}\,dt \leq \sqrt{2T}\int_0^T\frac{1}{\sqrt{t}}\,dt \leq 2\sqrt{T}\sqrt{2T} \\
&\leq T2\sqrt{2}
\end{aligned}$$

it follows that

$$\begin{aligned}
\int_0^T \mathcal{E}(Au_t - Av_t)\,dt &\leq \tilde{C}\int_0^T \left(\int_t^T \|f_{u,s}-f_{v,s}\|_2 \frac{ds}{\sqrt{s-t}}\right)^2 dt \\
&\leq \tilde{C}\int_0^T \left(\int_t^T \|f_{u,s}-f_{v,s}\|_2^2 \frac{ds}{\sqrt{s-t}} \int_t^T \frac{ds}{\sqrt{s-t}}\right) dt \\
&= 2\tilde{C}\int_0^T \left(\int_t^T \|f_{u,s}-f_{v,s}\|_2^2 \frac{ds}{\sqrt{s-t}}\sqrt{T-t}\right) dt \\
&= 2\tilde{C}\int_0^T \left(\|f_{u,s}-f_{v,s}\|_2^2 \int_0^s \sqrt{\frac{(T-t)}{(s-t)}}\,dt\right) ds \\
&\underset{(\star)}{\leq} 4\sqrt{2}\tilde{C}T\int_0^T \|f_{u,s}-f_{v,s}\|_2^2\,ds \\
&\leq 8\sqrt{2}\tilde{C}C^2T\int_0^T (\|u_s-v_s\|_2^2 + \|D_\sigma u_s - D_\sigma v_s\|_2^2)\,ds \\
&= 8\sqrt{2}\tilde{C}C^2T\int_0^T (\|u_s-v_s\|_2^2 + \mathcal{E}^A(u_s-v_s))\,ds \\
&\underset{(A2)}{\leq} TK_1\|u-v\|_T^2,
\end{aligned}$$

where K_1 is a constant, which depends on C_A, K_A, C, T and $\tilde{C}$. Moreover, we have

$$\begin{aligned}
\|Au_t - Av_t\|_2^2 &= \sum_{i=1}^l \left\|\int_t^T P_{s-t}(f_{u,s}^i - f_{v,s}^i)\,ds\right\|_2^2 \\
&\leq T\int_0^T \sum_{i=1}^l \|f_{u,s}^i - f_{v,s}^i\|_2^2\,ds \\
&\leq T\int_0^T \|f_{u,s}-f_{v,s}\|_2^2\,ds \\
&\leq TK_2\|u-v\|_T^2,
\end{aligned}$$

where K_2 is a constant, which depends on C_A, K_A, C and T. Finally, we obtain

$$\|Au - Av\|_T^2 \leq KT\|u-v\|_T^2$$

where K is a constant, which depends on C_A, K_A, the Lipschitz constant C, T and the constant $\tilde{C}$ from Lemma 1.9.

Now let us define

$$\|u\|_{[T_a,T_b]} := \left(\sup_{t\in[T_a,T_b]} \|u_t\|_2^2 + \int_{T_a}^{T_b} \mathcal{E}(u_t)\,dt\right)^{\frac{1}{2}}$$

where $0 \leq T_a \leq T_b \leq T$. Fix T_1 sufficiently small such that $KT_1 < 1$. Then the following relation holds:

$$\|Au - Av\|_{[0,T_1]}^2 < \|u-v\|_{[0,T_1]}^2.$$

Banach's fixed point theorem yields

$$\exists! u_1 \in \hat{F}_{[0,T_1]} : Au_1 = u_1$$

where $\hat{F}_{[T_a,T_b]} := \mathcal{C}([T_a,T_b];L^2) \cap L^2((T_a,T_b);F)$ for $T_a \in [0,T]$ and $T_b \in [T_a,T]$. Hence, u_1 satisfies the weak equation over the interval $[0,T_1]$ and also $[T_1-\varepsilon,T_1]$, where ε such that $T_1 \geq \varepsilon > 0$.

Analogous to the above calculations we get for $\tilde{T}_1 := T_1 - \varepsilon$, where $\varepsilon > 0$ fixed, small enough and K is as above

$$\|Au_t - Av_t\|^2_{\tilde{T}_1} \quad \leq \quad K(T-\tilde{T}_1)\|u-v\|^2_{[\tilde{T}_1,T]}.$$

Now we choose T such that $T_1 = T - \tilde{T}_1$ and therefore $T_2 := T_1 + \tilde{T}_1$ such that $K(T_2 - \tilde{T}_1) < 1$. By using Banach's fixed point theorem again we conclude

$$\exists! u_2 \in \hat{F}_{[\tilde{T}_1,T_2]} : Au_2 = u_2.$$

Hence, u_1 and u_2 satisfy the weak equation over the interval $[T_1-\varepsilon,T_1]$. By the uniqueness of u_1 and u_2 it follows that $u_1(t) = u_2(t)$ for almost every $t \in [T_1-\varepsilon,T_1]$. Therefore, we can construct a solution over the interval $[0,T_2]$.

Clearly there exists $n \in \mathbb{N}$ such that $T < n(T_1-\varepsilon)$. Hence, the construction is done after n steps. Finally, the uniqueness of the fixed points implies the existence of a unique solution over the interval $[0,T]$.

In order to obtain the estimate in the statement, let us start by writing

$$\begin{aligned}
\left|\int_t^T (f_{u,s} - f_s^0, u_s)\,ds\right| &\underset{(4.9)}{\leq} \int_t^T C(|u_s| + |D_\sigma u_s|, |u_s|)\,ds \\
&\leq C\int_t^T \|u_s\|_2^2\,ds + C\int_t^T \|D_\sigma u_s\|_2\|u_s\|_2\,ds.
\end{aligned}$$

Hence, we conclude

$$\begin{aligned}
&\left|\int_t^T (f_{u,s}, u_s)\,ds\right| \\
\leq\ & \int_t^T |(f_s^0,u_s)|\,ds + C\int_t^T \|u_s\|_2^2\,ds + C\int_t^T \|D_\sigma u_s\|_2\|u_s\|_2\,ds \\
\leq\ & \frac{1}{2}\int_t^T \|f_s^0\|_2^2\,ds + \left(\frac{1}{2} + C + \frac{1}{2}C^2\right)\int_t^T \|u_s\|_2^2\,ds + \frac{1}{2}\int_t^T \mathcal{E}^A(u_s)\,ds.
\end{aligned}$$

By relation (3.5) of Proposition 3.8 it follows that

$$\begin{aligned}
&\|u_t\|_2^2 + 2\int_t^T \mathcal{E}(u_s)\,ds \\
\underset{(3.5)}{=}\ & 2\int_t^T (f_{u,s},u_s)\,ds + \|\phi\|_2^2 \\
\leq\ & \|\phi\|_2^2 + \int_t^T \|f_s^0\|_2^2\,ds + (1+2C+C^2+C_A)\int_t^T \|u_s\|_2^2\,ds \\
& + \int_t^T K_A\,\mathcal{E}(u_s)\,ds.
\end{aligned}$$

Gronwall's lemma yields

$$\|u_t\|_2^2 + (2 - K_A) \int_t^T \mathcal{E}(u_s)\, ds \le e^{T(1+2C+C^2+C_A)} \left(\|\phi\|_2^2 + \int_0^T \|f_s^0\|_2^2\, ds \right)$$

and hence we get

$$\|u\|_T^2 \le \frac{1}{2 - K_A} e^{T(1+2C+C^2+C_A)} \left(\|\phi\|_2^2 + \int_0^T \|f_s^0\|_2^2\, ds \right).$$

□

4.4 The Case of Monotonicity Conditions

4.4.1 The Monotonicity Conditions

Let $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^l \times \mathbb{R}^l \otimes \mathbb{R}^k \to \mathbb{R}^l$ be a measurable function and $\phi \in L^2(\mathbb{R}^d, m; \mathbb{R}^l)$ be the final condition of (4.1). In this section we show the existence of a unique solution under monotonicity conditions on f, cf. [BPS05, p.35]. We impose the following conditions:

(H1) [Lipschitz condition in z]
There exists a fixed constant $C > 0$ such that for t, x, y, z, z' arbitrary

$$|f(t,x,y,z) - f(t,x,y,z')| \le C|z - z'|.$$

(H2) [Monotonicity condition in y]
For t, x, y, y', z arbitrary, there exists a fixed constant $\mu \in \mathbb{R}$ such that

$$\langle y - y', f(t,x,y,z) - f(t,x,y',z) \rangle \le \mu |y - y'|^2.$$

(H3) [Continuity condition in y]
For t, x and z fixed, the map

$$y \mapsto f(t,x,y,z)$$

is continuous.

(H4)
For each $r > 0$

$$f'^{,r} \in L^1([0,T]; L^2).$$

(H5)

$$\|\phi\|_\infty < \infty, \|f^0\|_\infty < \infty, |\phi| \in L^2, |f^0| \in L^1([0,T]; L^2)$$

Lemma 4.9. *(i) If $\rho > 0$, the last two conditions in (H5) are ensured by the boundness of ϕ and f^0. (cf. [BPS05, page 35])*
(ii) If f is Lipschitz continuous in y with Lipschitz constant K, then $(H2)$ is

fulfilled with constant K.
(iii) The conditions (H1),(H4) and (H5) imply that, if $u \in \hat{F}$ is bounded, then $|f(u, D_\sigma u)| \in L^1([0,T]; L^2)$. (cf. [BPS05, page 35])
(iv) We can assume in (H2) without loss of generality that $\mu = 0$. (cf. [BPS05, page 36])

Proof. (i) By Lemma 2.1 we have $m(\mathbb{R}^d) < \infty$. Hence, the assertion follows.
(ii) $\langle y - y', f(t,x,y) - f(t,x,y')\rangle \le |y-y'||f(t,x,y) - f(t,x,y')| \le K|y-y'|^2$.
(iii) We have by (H1)

$$\begin{aligned} & |f(u, D_\sigma u)| - |f'(u)| - |f^0| \le |f(u, D_\sigma u) - f'(u) - f^0| \le C|D_\sigma u| \\ \Rightarrow \quad & |f(u, D_\sigma u)| \le C|D_\sigma u| + |f'(u)| + |f^0|. \end{aligned}$$

Since $u \in \hat{F}$ is bounded, we deduce by (H4) and (H5) that

$$|f(u, D_\sigma u)| \in L^1([0,T]; L^2).$$

This can be proved as follows:

$$\begin{array}{lcl} u \in \hat{F} & \underset{(4.4)(ii)}{\Rightarrow} & |D_\sigma u| \in L^2([0,T] \times \mathbb{R}^d) \underset{(\star)}{\Rightarrow} |D_\sigma u| \in L^1([0,T]; L^2). \\ (H5) & \Rightarrow & |f^0| \in L^1([0,T]; L^2), \\ (H4) & \Rightarrow & f^{\prime,r} \in L^1([0,T]; L^2). \end{array}$$

Since u is bounded, there exists $\tilde{r}$ such that $|u| \le \tilde{r}$. Hence, we get $|f'(u)| \le |f^{\prime,\tilde{r}}|$ and therefore $f'(u) \in L^1([0,T]; L^2)$. It remains to show $(\star)$

$$\begin{aligned} \int_0^T \|D_\sigma u\|_2 \, dt &= \int_0^T \left(\int_{\mathbb{R}^d} |D_\sigma u|^2 \, dm \right)^{\frac{1}{2}} dt \\ &\le T^{\frac{1}{2}} \left(\int_0^T \int_{\mathbb{R}^d} |D_\sigma u|^2 \, dm \, dt \right)^{\frac{1}{2}} \\ &< \infty. \end{aligned}$$

(iv) First of all we will show that the function $(t,x) \mapsto \psi(t,x) := \varphi(t,x)\exp(\mu t)$ is an element of C_T for all $\varphi \in \mathcal{C}_T$ and $\mu \in \mathbb{R}$. Let $\mu \in \mathbb{R}$ be fixed and $\varphi \in \mathcal{C}_T$. We will verify the properties (i)-(iv) of Lemma 2.4 for the function ψ.

- Fix $t \in [0,T]$. Then $\exp(\mu t)$ is constant and hence, it is obvious that $\exp(\mu t)\varphi_t \in F$ for almost every t.

-

$$\begin{aligned} \int_0^T \mathcal{E}(\exp(\mu t)\varphi_t) \, dt &= \int_0^T (\exp(\mu t))^2 \mathcal{E}(\varphi_t) \, dt \\ &\le \sup_{t \in [0,T]} (\exp(\mu t))^2 \int_0^T \mathcal{E}(\varphi_t) \, dt \\ &< \infty \end{aligned}$$

- Since we have

$$
\begin{aligned}
& \lim_{h\to 0}\left\|\varphi_t \frac{\exp((t+h)\mu)-\exp(t\mu)}{h} - \varphi_t\mu\exp(t\mu)\right\|_2 \\
\leq\ & \|\varphi_t\|_2 \lim_{h\to 0}\left(\frac{\exp((t+h)\mu)-\exp(t\mu)}{h} - \mu\exp(t\mu)\right) \\
=\ & 0
\end{aligned}
$$

and

$$
\begin{aligned}
& \left\|\frac{\varphi_{t+h}-\varphi_t}{h}\exp((t+h)\mu) - \partial_t\varphi_t\exp(t\mu)\right\|_2 \\
=\ & \exp(t\mu)\left\|\frac{\varphi_{t+h}-\varphi_t}{h}\exp(h\mu) - \partial_t\varphi_t\right\|_2 \\
\leq\ & \exp(t\mu)\Bigg(\left\|\frac{\varphi_{t+h}-\varphi_t}{h}\exp(h\mu) - \partial_t\varphi_t\exp(h\mu)\right\|_2 \\
& + \|\partial_t\varphi_t\exp(h\mu) - \partial_t\varphi_t\|_2\Bigg) \\
\leq\ & \exp(t\mu)\left(\underbrace{|\exp(h\mu)|}_{\to 1}\underbrace{\left\|\frac{\varphi_{t+h}-\varphi_t}{h} - \partial_t\varphi_t\right\|_2}_{\to 0,\text{ since }\varphi\in\mathcal{C}_T} + \underbrace{(\exp(h\mu)-1)}_{\to 0}\underbrace{\|\partial_t\varphi_t\|_2}_{\in\mathbb{R}}\right) \\
\underset{h\to 0}{\longrightarrow}\ & 0,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
& \left\|\frac{\varphi_{t+h}\exp((t+h)\mu)-\varphi_t\exp(t\mu)}{h} - \big(\varphi_t\mu\exp(t\mu) + \partial_t\varphi_t\exp(t\mu)\big)\right\|_2 \\
=\ & \Bigg\|\frac{\varphi_{t+h}-\varphi_t}{h}\exp((t+h)\mu) + \varphi_t\frac{\exp((t+h)\mu)-\exp(t\mu)}{h} \\
& -\big(\varphi_t\mu\exp(\mu t) + \partial_t\varphi_t\exp(t\mu)\big)\Bigg\|_2 \\
\leq\ & \left\|\varphi_t\frac{\exp((t+h)\mu)-\exp(t\mu)}{h} - \varphi_t\mu\exp(t\mu)\right\|_2 \\
& + \left\|\frac{\varphi_{t+h}-\varphi_t}{h}\exp((t+h)\mu) - \partial_t\varphi_t\exp(t\mu)\right\|_2 \\
\underset{h\to 0}{\longrightarrow}\ & 0.
\end{aligned}
$$

- By the above calculation it holds

$$\partial_t(\exp(t\mu)\varphi_t) = \partial_t\varphi_t\exp(t\mu) + \varphi_t\mu\exp(t\mu) \text{ in } L^2.$$

Hence, we have to show

$$
\begin{aligned}
& \|\partial_t(\exp(t\mu)\varphi_t) - \partial_t(\exp((t+h)\mu)\varphi_{t+h})\|_2 \\
\leq\ & \|\partial_t\varphi_t\exp(t\mu) - \partial_t\varphi_{t+h}\exp((t+h)\mu)\|_2 \\
& +\|\varphi_t\mu\exp(t\mu) - \varphi_{t+h}\mu\exp((t+h)\mu)\|_2 \\
\overset{!}{\underset{h\to 0}{\longrightarrow}}\ & 0.
\end{aligned}
$$

This can be seen as follows:

$$
\begin{aligned}
& \|\partial_t\varphi_t \cdot \exp(t\mu) - \partial_t\varphi_{t+h} \cdot \exp((t+h)\mu)\|_2 \\
= \; & \exp(t\mu)\|\partial_t\varphi_t - \partial_t\varphi_{t+h} + \partial_t\varphi_{t+h} - \partial_t\varphi_{t+h}\exp(h\mu))\|_2 \\
\leq \; & \exp(t\mu)\|\partial_t\varphi_t - \partial_t\varphi_{t+h}\|_2 + \exp(t\mu)(1-\exp(h\mu))\|\partial_t\varphi_{t+h}\|_2 \\
\underset{h\to 0}{\longrightarrow} \; & 0,
\end{aligned}
$$

$$
\begin{aligned}
& \|\varphi_t\mu\exp(t\mu) - \varphi_{t+h}\mu\exp((t+h)\mu)\|_2 \\
\leq \; & \mu\exp(t\mu)\|\varphi_t - \varphi_{t+h} + \varphi_{t+h} - \varphi_{t+h}\exp(h\mu)\|_2 \\
\leq \; & \mu\exp(t\mu)\|\varphi_t - \varphi_{t+h}\|_2 + \mu\exp(t\mu)\|\varphi_{t+h} - \varphi_{t+h}\exp(h\mu)\|_2 \\
\leq \; & \mu\exp(t\mu)\|\varphi_t - \varphi_{t+h}\|_2 + \mu\exp(t\mu)(1-\exp(h\mu))\|\varphi_{t+h}\|_2 \\
\underset{h\to 0}{\longrightarrow} \; & 0.
\end{aligned}
$$

Finally, Lemma 2.4 yields:

$$(t,x) \mapsto \varphi_t(x)\exp(\mu t) \in \mathcal{C}_T \quad \text{for all } \mu \in \mathbb{R}, \varphi \in \mathcal{C}_T.$$

Now let us make the change $u_t^\star = \exp(\mu t)u_t$ for the solution and the changes $\phi^\star = \exp(\mu T)\phi$ and $f_t^\star(y,z) = \exp(\mu t)f_t(\exp(-\mu t)y, \exp(-\mu t)z) - \mu y$ for the data. Next we will prove that u is a solution associated to the data (ϕ, f), if and only if $u^\star$ is a solution associated to the data $(\phi^\star, f^\star)$. Let us start by writing equation (4.7) for u

$$
\begin{aligned}
& \int_0^T (f_t^i(u_t, D_\sigma u_t), \varphi_t)\,dt + (u_T^i, \varphi_T) - (u_0^i, \varphi_0) \\
& = \int_0^T \mathcal{E}(u_t^i, \varphi_t) + (u_t^i, \partial_t\varphi_t))\,dt.
\end{aligned}
$$

By the above calculations this equation is equivalent to

$$
\begin{aligned}
& \int_0^T (f_t^i(u_t, D_\sigma u_t), \exp(\mu t)\varphi_t)\,dt + (u_T^i, \exp(\mu T)\varphi_T) - (u_0^i, \exp(\mu\cdot 0)\varphi_0) \\
& = \int_0^T \mathcal{E}(u_t^i, \exp(\mu t)\varphi_t) + (u_t^i, \partial_t(\exp(\mu t)\varphi_t))\,dt.
\end{aligned}
$$

Moreover, this is equivalent to

$$
\begin{aligned}
& \int_0^T (\exp(\mu t)f_t^i(u_t, D_\sigma u_t) - \mu\exp(\mu t)u_t^i, \varphi_t)\,dt + (u_T^i, \exp(\mu T)\varphi_T) \\
& = \int_0^T \mathcal{E}(u_t^i, \exp(\mu t)\varphi_t) + (u_t^i, \exp(\mu t)\partial_t\varphi_t)\,dt + (u_0^i, \exp(\mu\cdot 0)\varphi_0)
\end{aligned}
$$

and hence also to

$$
\begin{aligned}
& \int_0^T (f_t^{i,\star}(u_t\exp(\mu t), D_\sigma u_t\exp(\mu t)), \varphi_t)\,dt + (u_T^i\exp(\mu T), \varphi_T) \\
& = \int_0^T \mathcal{E}(u_t^i\exp(\mu t), \varphi_t) + (u_t^i\exp(\mu t), \partial_t\varphi_t)\,dt + (u_0^i\exp(\mu\cdot 0), \varphi_0).
\end{aligned}
$$

By substituting $u_t^\star = u_t \exp(\mu t)$ this equation is the weak equation for $u^\star$

$$\int_0^T (f_t^{i,\star}(u_t^\star, D_\sigma u_t^\star), \varphi_t)\, dt + (u_T^{i,\star}, \varphi_T)$$
$$= \int_0^T \mathcal{E}(u_t^{i,\star}, \varphi_t) + (u_t^{i,\star}, \partial_t \varphi_t)\, dt + (u_0^{i,\star}, \varphi_0).$$

Left to show is that the function $f^\star$ satisfies the conditions (H1)-(H5). It is obvious that (H1), (H3)-(H5) are not altered by the above transformation. Therefore, let us prove that $f^\star$ satisfies (H2) with $\mu = 0$

$$\begin{aligned}
& \langle y - \tilde{y}, f^\star(t,x,y,z) - f^\star(t,x,\tilde{y},z)\rangle \\
= \;& \langle y - \tilde{y}, \mu\tilde{y} + \exp(\mu t) f(t,x,\exp(-\mu t)y, \exp(-\mu t)z)\rangle \\
& -\langle y - \tilde{y}, \mu y + \exp(\mu t) f(t,x,\exp(-\mu t)\tilde{y}, \exp(-\mu t)z)\rangle \\
= \;& \langle y - \tilde{y}, \mu\tilde{y} - \mu y\rangle \\
& +(\exp(\mu t))^2 \langle \exp(-\mu t)y - \exp(-\mu t)\tilde{y}, f(t,x,\exp(-\mu t)y, \exp(-\mu t)z)\rangle \\
& -(\exp(\mu t))^2 \langle \exp(-\mu t)y - \exp(-\mu t)\tilde{y}, f(t,x,\exp(-\mu t)\tilde{y}, \exp(-\mu t)z)\rangle \\
\underset{(H2)}{\leq} \;& -|y-\tilde{y}|^2\mu + \underbrace{\mu(\exp(\mu t))^2 |\exp(-\mu t)y - \exp(-\mu t)\tilde{y}|^2}_{=\mu|y-\tilde{y}|^2} \\
= \;& 0.
\end{aligned}$$

Thus, by making the transformation $f \to f^\star$, we may assume that $\mu = 0$. □

4.4.2 Estimates for the Solution

In this section we prove two important estimates for a solution of (4.1). These are essential tools in the proof of the uniqueness and existence theorem in Section 4.4.3. The $\|\cdot\|_T$-estimate will be proved under a weaker form of condition (H2) with $\mu = 0$ denoted by (H2').

$$(H2') \qquad \langle y, f'(t,x,y)\rangle \leq 0.$$

Lemma 4.10. *Let f satisfy the conditions (H1), (H2') and (H5). Then there exists a constant $K \in \mathbb{R}_+$, which depends on K_A, C_A, T, μ and C such that for a solution u the following relation holds:*

$$\|u\|_T^2 \leq K \left(\|\phi\|_2^2 + \int_0^T \|f_t^0\|_2^2\, dt \right). \tag{4.10}$$

Proof. (cf. idea of the proof of [BPS02, Lemma 3.8])

Since u is a solution of (4.1), we have by Proposition 3.8

$$\|u_t\|_2^2 + 2\int_t^T \mathcal{E}(u_s)\, ds = 2\int_t^T (f_s, u_s)\, ds + \|u_T\|_2^2.$$

The conditions (H1) and (H2') yield

$$\begin{aligned}\langle f_s(u_s, D_\sigma u_s), u_s\rangle &= \langle f_s(u_s, D_\sigma u_s) - f_s(u_s,0) + f_s(u_s,0) - f_s^0 + f_s^0, u_s\rangle \\ &= \langle f_s(u_s, D_\sigma u_s) - f_s(u_s,0) + f_s'(u_s) + f_s^0, u_s\rangle \\ &\leq |f_s(u_s, D_\sigma u_s) - f_s(u_s,0)||u_s| + \underbrace{\langle f_s'(u_s), u_s\rangle}_{\leq 0 \text{ by (H2')}} + |f_s^0||u_s| \\ &\underset{(H1)}{\leq} (C|D_\sigma u_s| + |f_s^0|)|u_s|.\end{aligned}$$

Hence, it follows

$$\begin{aligned} & \|u_t\|_2^2 + 2\int_t^T \mathcal{E}(u_s)\,ds \\ \leq\ & 2\int_t^T (C|D_\sigma u_s| + |f_s^0|, |u_s|)\,ds + \|u_T\|_2^2 \\ \leq\ & \underbrace{\int_t^T \|D_\sigma u_s\|_2^2\,ds}_{\leq \int_t^T (K_A\mathcal{E}(u_s)+C_A\|u_s\|_2^2)\,ds} + \int_t^T \|f_s^0\|_2^2\,ds + (C^2+1)\int_t^T \|u_s\|_2^2\,ds + \|u_T\|_2^2.\end{aligned}$$

Gronwalls' lemma yields

$$\|u\|_T^2 \leq \underbrace{\frac{1}{2-K_A}\exp(T(1+C^2+C_A))}_{=:K}\left(\|\phi\|_2^2 + \int_0^T \|f_s^0\|_2^2\,ds\right).$$

□

The aim of the following is to give an upper estimate for a solution of the nonlinear equation. This will be done under additional conditions in Lemma 4.18. Let us start by presenting two useful approximation lemmas.

Lemma 4.11. *Let $f \in L^1([0,T];L^2)$, $\phi \in L^2$ and u be a weak solution of (3.1) associated to the data (ϕ, f). Then there exists $f_n \in \mathcal{C}^1([0,T];L^2)$ and $\phi_n \in \mathcal{D}(L)$ such that*

(i) $u_{n,t} := P_{T-t}\phi_n + \int_t^T P_{s-t} f_{n,s}\,ds$ *is a weak solution for (ϕ_n, f_n),*

(ii) $\lim_{n\to\infty} \int_t^T \|f_{n,s} - f_s\|_2\,ds = 0,$

(iii) $\lim_{n\to\infty} \|\phi_n - \phi\|_2 = 0,$

(iv) $\lim_{n\to\infty} \|u_n - u\|_T = 0.$

Proof. To verify (ii) let $f \in L^1([0,T];L^2)$. Then there exists $f_n \in C^1([0,T];L^2)$ such that $\lim_{n\to\infty} \int_0^T \|f_{t,n} - f_t\|_2\,dt = 0$. The existence of such f_n follows by the well known fact $\mathcal{C}([0,T];L^2) \underset{\text{dense}}{\subset} L^1([0,T];L^2)$ and arguments analogous to the approximation in the proof of Lemma 2.10. In order to show assertion (iii), let $\phi \in L^2$. Define $\phi_\lambda := \lambda G_\lambda \phi$. Then $\phi_\lambda \in \mathcal{D}(L)$ (cf.[MR92, I. Proposition

1.5]) and $\phi_\lambda \underset{\lambda\to\infty}{\to} \phi$ in L^2 by the strong continuity of G_λ. Now the assertion (i) follows directly by Proposition 3.8 and (iv) follows by (ii),(iii) and equation (3.6). □

Lemma 4.12. *Let $f \in \mathcal{C}^1([0,T]; L^2)$, $\phi \in \mathcal{D}(L)$ and u be a weak solution of* (3.1) *associated to the data (ϕ, f). Then there exists $f_n \in \mathcal{C}^1([0,T]; L^2)$ bounded and $\phi_n \in \mathcal{D}(L)$ bounded such that*

$$
\begin{aligned}
&(i) && u_{n,t} := P_{T-t}\phi_n + \int_t^T P_{s-t} f_{n,s}\, ds \ \text{ is a weak solution for } (\phi_n, f_n),\\
&(ii) && \lim_{n\to\infty} \int_t^T \|f_{n,s} - f_s\|_2\, ds = 0,\\
&(iii) && \lim_{n\to\infty} \|\phi_n - \phi\|_2 = 0,\\
&(iv) && \lim_{n\to\infty} \|u_n - u\|_T = 0.
\end{aligned}
$$

Proof. To verify (ii) let $f \in C^1([0,T]; L^2)$. Then there exists $f_n \in C^1([0,T]; L^2)$ bounded such that $\lim_{n\to\infty} \sup_{t\in[0,T]} \|f_n - f\|_2 = 0$. The existence of such f_n can be shown analogously to the approximation in the proof of Lemma 2.7. In order to show (iii), let $\phi \in \mathcal{D}(L)$, then we can find $\psi \in L^2$ such that $G_1\psi = \phi$ (cf.[MR92, proof of I. Proposition 1.5]). Define $\phi_n := G_1(\psi \wedge n \vee -n)$. Since $G_1(L^2(\mathbb{R}^d, m)) = \mathcal{D}(L)$, we have $\phi_n \in \mathcal{D}(L)$ (cf.[MR92, proof of I. Proposition 1.5]). With the same argumentation as in the proof of Lemma 1.11 it follows that ϕ_n is bounded. Moreover, by [MR92, I. Remark 2.9 (i)] we can deduce $\phi_n \underset{n\to\infty}{\to} \phi$ in $(F, \tilde{\mathcal{E}}_1^{\frac{1}{2}})$. The assertions (i) and (iv) follow by the same argumentation as (i) and (iv) in the above lemma. □

From now on we assume the following additional conditions:

$$
\begin{aligned}
&\text{(A3)} && d_i = 0 \text{ for } i = 1, \ldots, d,\\
&\text{(A4)} && c \in L^\infty(\mathbb{R}^d; \mathbb{R}_+).
\end{aligned}
$$

By $(A3)$ the bilinear form (2.1) has the following representation for $u, v \in \mathcal{C}_0^\infty(\mathbb{R}^d)$:

$$
\begin{aligned}
\mathcal{E}(u,v) \quad := \quad & \sum_{i,j=1}^d \int_{\mathbb{R}^d} a^{i,j}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j}\, m(dx) \\
& + \sum_{i=1}^d \int_{\mathbb{R}^d} \frac{\partial u(x)}{\partial x_i} v(x) b_i(x)\, m(dx) \\
& + \int_{\mathbb{R}^d} u(x) v(x) c(x)\ m(dx).
\end{aligned} \tag{4.11}
$$

Analogous to Chapter 2 we construct $F, \hat{F}$ and C_T w.r.t. the norm associated to $\mathcal{E}$ in (4.11). In the next proposition we will prove the assertion of [BPS05, Proposition 2.10] for our framework in the case of nice functions. The main arguments are the same, but note that in contrast to the symmetric case of

[BPS05], we do not have an explicit form of the bilinear form (4.11) for $u, v \in F$. Hence, the proof is more technical than in the symmetric case. Moreover, the assertion (i) in our framework contains a term depending on c. For the proof we need the following lemma, which can be easily verified:

Lemma 4.13. *If $A \in \mathbb{R}^l \otimes \mathbb{R}^k$ and $y \in R^l$, then one has*

$$tr(AA^\star)|y|^2 \geq \langle y, AA^\star y\rangle.$$

Proposition 4.14. *Let $u = (u^1, \cdots, u^l)$ be a vector valued function where each component is a weak solution of the linear equation (3.1) associated to certain data $f^i \in \mathcal{C}^1([0,T]; L^2)$ bounded and $\phi^i \in \mathcal{D}(L)$ bounded for $i = 1, \cdots, l$. Denote by ϕ, f the vectors $\phi = (\phi^1, \cdots, \phi^l)$, $f = (f^1, \cdots, f^l)$ and by $D_\sigma u$ the matrix whose rows consist of the row vectors $D_\sigma u^i$. Then the following relations hold m-almost everywhere*

$$(i) \qquad |u_t|^2 + 2\int_t^T P_{s-t}\left(|D_\sigma u_s|^2 + \frac{1}{2}c|u_s|^2\right) ds = P_{T-t}|\phi|^2 + 2\int_t^T P_{s-t}\langle u_s, f_s\rangle\, ds,$$

$$(ii) \qquad |u_t| \leq P_{T-t}|\phi| + \int_t^T P_{s-t}\langle \hat{u}_s, f_s\rangle\, ds.$$

Here we write $\hat{x} = x/|x|$, for $x \in \mathbb{R}^l$, $x \neq 0$ and $\hat{x} = 0$, if $x = 0$.

Proof. By Proposition 3.6 it holds $u \in b\mathcal{C}_T$.
(i) First we prove the relation in the case $l = 1$. If we can check that u^2 verifies the equation

$$(\partial_t + L)u^2 + 2uf - 2|D_\sigma u|^2 - cu^2 = 0, \qquad u_T^2 = \phi^2 \tag{4.12}$$

in the weak sense with test functions of $b\mathcal{C}_T$, the assertion will follow by Proposition 3.12. We need the following two relations:

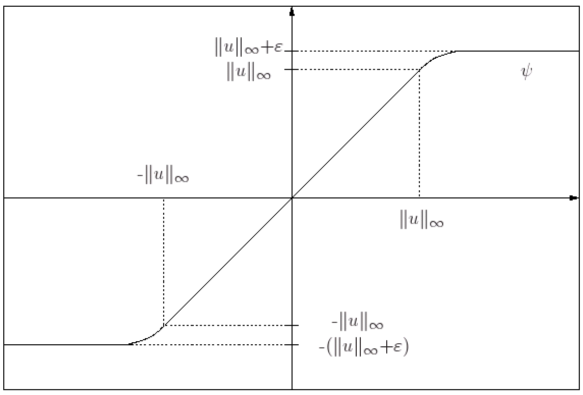
- $$\begin{aligned} &\int_0^T (u_t^2, \partial_t\varphi_t)\, dt \qquad (4.13)\\ =\ &\int_0^T \int_{\mathbb{R}^d} (\partial_t(u_t^2\varphi_t))\, dm\, dt - \int_t^T (\partial_t u_t^2, \varphi_t)\, dt \\ =\ &\int_0^T (\partial_t(u_t\varphi_t), u_t)\, dt + \int_0^T (\partial_t u_t, u_t\varphi_t)\, dt - \int_0^T (\partial_t u_t^2, \varphi_t) dt \\ =\ &\int_0^T (\partial_t(u_t\varphi_t), u_t)\, dt + \int_0^T (\partial_t u_t, u_t\varphi_t)\, dt + \int_0^T (\partial_t\varphi_t, u_t^2)\, dt \\ &\quad - \int_0^T \int_{\mathbb{R}^d} (\partial_t(u_t^2\varphi_t))\, dm\, dt \\ =\ &2\int_0^T (u_t, \partial_t(u_t\varphi_t))\, dt + (u_0^2, \varphi_0) - (u_T^2, \varphi_T), \end{aligned}$$

- $$\mathcal{E}(u_t^2, \varphi_t) = 2\mathcal{E}(u_t, u_t\varphi_t) - (2|D_\sigma u_t|^2 + cu_t^2, \varphi_t). \tag{4.14}$$

[Proof of equation 4.14]
First note that for $v, w \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned}
\mathcal{E}^A(v^2, w) &= \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a^{i,j} \frac{\partial v^2}{\partial x_i} \frac{\partial w}{\partial x_j} \, m(dx) \\
&= \sum_{i,j=1}^{d} 2 \int_{\mathbb{R}^d} a^{i,j} v \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} \, m(dx) \\
&= \sum_{i,j=1}^{d} 2 \int_{\mathbb{R}^d} a^{i,j} \frac{\partial v}{\partial x_i} \frac{\partial (vw)}{\partial x_j} \, m(dx) \\
&\quad -2 \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a^{i,j} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} w \, m(dx) \\
&= 2\mathcal{E}^A(v, vw) - 2(|D_\sigma v|^2, w), \\
\mathcal{E}^B(v^2, w) &= \sum_{i=1}^{d} \int_{\mathbb{R}^d} \frac{\partial v^2}{\partial x_i} w b_i \, m(dx) \\
&= 2 \sum_{i=1}^{d} \int_{\mathbb{R}^d} \frac{\partial v}{\partial x_i} v w b_i \, m(dx) \\
&= 2\mathcal{E}^B(v, vw), \\
\mathcal{E}^C(v^2, w) &= \int_{\mathbb{R}^d} c v^2 w \, m(dx) \\
&= 2 \int_{\mathbb{R}^d} c v v w \, m(dx) - \int_{\mathbb{R}^d} c v^2 w \, m(dx) \\
&= 2\mathcal{E}^C(v, vw) - (cv^2, w).
\end{aligned}$$

Now let us approximate u_t^2 and φ_t by $\mathcal{C}_0^\infty(\mathbb{R})$ functions. For simplicity of notation we will write u instead of u_t and φ instead of φ_t in the following calculations. Take $u_n \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $u_n \to u$ in $\| \cdot \|_{\tilde{\mathcal{E}}_1^{\frac{1}{2}}}$. Let $1 > \varepsilon > 0$ and ψ be a smooth function on $\mathbb{R}$ with bounded derivative such that $\|\psi\|_\infty \leq \|u\|_\infty + \varepsilon$ and $\psi(x) = x$ for $|x| \leq \|u\|_\infty$.



Now define $\psi_n := \psi(u_n)$. Then we deduce by Theorem 1.13

$$\mathcal{E}(\psi_n) \leq \sup_{n\in\mathbb{N}} \|\psi'\|_\infty^2 \mathcal{E}(u_n) < \infty.$$

Since $(|\psi_n|^2)_{n\in\mathbb{N}}$ is uniformly integrable and there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ such that $\lim_{k\to\infty} \psi(u_{n_k}(x)) = u(x)$ m-a.e., it follows by [Bau92, Korollar 21.5]

$$\lim_{k\to\infty} \|\psi_{n_k} - u\|_2 = 0.$$

By Lemma 1.19 we obtain that there exists a subsequence of $(\psi_{n_k})_{k\in\mathbb{N}}$ such that for its Cesáro mean, denoted by w_n, it holds:

$$w_n \underset{n\to\infty}{\to} u \text{ in } (F, \tilde{\mathcal{E}}_1^{\frac{1}{2}}).$$

Further we have

$$\sup_{n\in\mathbb{N}} \|w_n\|_\infty \leq \|u\|_\infty + 1 < \infty.$$

Hence, it follows

$$\|w_n^2 - u^2\|_2 \leq \|w_n\|_\infty \|w_n - u\|_2 + \|u\|_\infty \|w_n - u\|_2 \underset{n\to\infty}{\to} 0$$

and by Corollary 1.15

$$\sup_{n\in\mathbb{N}} \mathcal{E}(w_n^2) \leq 4 \sup_{n\in\mathbb{N}} (\|w_n\|_\infty^2 \mathcal{E}(w_n)) < \infty.$$

Now we use Lemma 1.19 to deduce

$$\lim_{n\to\infty} \mathcal{E}(w_n^2, \varphi_t) = \mathcal{E}(u^2, \varphi_t) \text{ for all } \varphi \in \mathcal{C}_T.$$

With the same arguments as above we construct a sequence $(\varphi_k)_{k\in\mathbb{N}}$ of $C_0^\infty(\mathbb{R}^d)$ functions such that

$$\sup_{k\in\mathbb{N}} \|\varphi_k\|_\infty < \infty \text{ and } \lim_{k\to\infty} \mathcal{E}_1(\varphi_k - \varphi) = 0.$$

It is easy to see that there exists a subsequence $(k_m)_{m\in\mathbb{N}}$ of $(k)_{k\in\mathbb{N}}$ such that $\varphi_{k_m} \to \varphi$ m-a.e.. Let us set $\varphi_m := \varphi_{k_m}$. Now it is a simple matter to check that

$$\mathcal{E}(u^2, \varphi) = \lim_{m\to\infty} \lim_{n\to\infty} \mathcal{E}(w_n^2, \varphi_m).$$

Next we will approximate the right hand side of (4.14). Let us start by writing

$$\begin{aligned} & \mathcal{E}(w_n, w_n\varphi_m) - \mathcal{E}(u, u\varphi) \\ = \; & \mathcal{E}(w_n - u, w_n\varphi_m) - \mathcal{E}(u, u\varphi - w_n\varphi_m) \\ = \; & \mathcal{E}(w_n - u, w_n\varphi_m) - \mathcal{E}(u, u\varphi - u\varphi_m) - \mathcal{E}(u, u\varphi_m - w_n\varphi_m). \quad (4.15) \end{aligned}$$

Since

$$
\begin{aligned}
& \lim_{n\to\infty} \mathcal{E}(w_n - u, w_n\varphi_m) \\
\leq \quad & \lim_{n\to\infty} \Big(K_{\mathcal{E}}\mathcal{E}_1(w_n-u)^{\frac{1}{2}}\mathcal{E}_1(w_n\varphi_m)^{\frac{1}{2}} + \|w_n-u\|_2\|w_n\varphi_m\|_2\Big) \\
= \quad & \lim_{n\to\infty} \Big[K_{\mathcal{E}}\mathcal{E}_1(w_n-u)^{\frac{1}{2}} \\
& \quad \Big(\mathcal{E}(w_n\varphi_m) + \|w_n\varphi_m\|_2\Big)^{\frac{1}{2}} + \|w_n-u\|_2\|w_n\varphi_m\|_2\Big] \\
\underset{\text{Corollary 1.15}}{\leq} \quad & \lim_{n\to\infty} \Big[K_{\mathcal{E}}\mathcal{E}_1(w_n-u)^{\frac{1}{2}}\Big(\big(\underbrace{2\|w_n\|_\infty^2}_{\text{bdd. in } n}\sup_{m\in\mathbb{N}}\mathcal{E}(\varphi_m) \\
& +2\sup_{m\in\mathbb{N}}\|\varphi_m\|_\infty^2 \underbrace{\mathcal{E}(w_n)}_{\text{bdd. in } n}\big) + \sup_{m\in\mathbb{N}}\|\varphi_m\|_\infty \underbrace{\|w_n\|_2}_{\text{bdd. in } n}\Big)^{\frac{1}{2}} \\
& +\|w_n-u\|_2\sup_{m\in\mathbb{N}}\|\varphi_m\|_\infty \underbrace{\|w_n\|_2}_{\text{bdd. in } n}\Big] \\
= \quad & 0,
\end{aligned}
$$

it follows that

$$\lim_{m\to\infty}\lim_{n\to\infty}\mathcal{E}(w_n-u, w_n\varphi_m) = 0.$$

Let us examine the second term of (4.15). Since we have on the one hand

$$\|u(\varphi-\varphi_m)\|_2 \leq \|u\|_\infty\|\varphi-\varphi_m\|_2 \underset{n\to\infty}{\to} 0$$

and on the other

$$
\begin{aligned}
& \mathcal{E}(u(\varphi-\varphi_m))^{\frac{1}{2}} \\
\underset{\text{Corollary 1.15}}{\leq} \quad & \|u\|_\infty\mathcal{E}(\varphi-\varphi_m)^{\frac{1}{2}} + \|\varphi-\varphi_m\|_\infty\mathcal{E}(u)^{\frac{1}{2}} \\
\leq \quad & \sup_{m\in\mathbb{N}}\|u\|_\infty\mathcal{E}(\varphi-\varphi_m)^{\frac{1}{2}} + \Big(\|\varphi\|_\infty + \sup_{m\in\mathbb{N}}\|\varphi_m\|_\infty\Big)\mathcal{E}(u)^{\frac{1}{2}} \\
< \quad & \infty,
\end{aligned}
$$

it follows by Lemma 1.19

$$\lim_{m\to 0}\mathcal{E}(u, u(\varphi-\varphi_m)) = 0.$$

The convergence of the last term of (4.15) can be shown analogously to the second one. Hence, we get

$$\lim_{m\to\infty}\lim_{n\to\infty}\mathcal{E}(w_n, w_n\varphi_m) = \mathcal{E}(u, u\varphi).$$

Next we have to show

$$\lim_{m\to\infty}\lim_{n\to\infty}(cw_n^2, \varphi_m) = (cu^2, \varphi).$$

The convergence follows from

$$\lim_{m\to\infty}|(cw_n^2, \varphi_m-\varphi)| \leq \sup_{x\in\mathbb{R}^d}|c(x)|\sup_{n\in\mathbb{N}}\|w_n^2\|_2\lim_{m\to\infty}\|\varphi_m-\varphi\|_2 = 0$$

and

$$\begin{aligned}
& |(c(w_n^2 - u^2), \varphi)| \\
\leq\ & \sup_{x \in \mathbb{R}^d} |c(x)| \|\varphi\|_2 \|w_n^2 - w_n u + w_n u - u^2\|_2 \\
\leq\ & \sup_{x \in \mathbb{R}^d} |c(x)| \|\varphi\|_2 \left(\|w_n\|_\infty \|w_n - u\|_2 + \|u\|_\infty \|w_n - u\|_2 \right) \\
\leq\ & \underbrace{(\sup_{x \in \mathbb{R}^d} |c(x)|}_{\underset{(A4)}{\leq} \text{ constant}} \underbrace{\|\varphi\|_2 (\|w_n\|_\infty + \|u\|_\infty))}_{\leq \text{ constant}} \|w_n - u\|_2 \\
\underset{n\to\infty}{\longrightarrow}\ & 0.
\end{aligned}$$

At last we have to verify

$$\lim_{m\to\infty} \lim_{n\to\infty} (|D_\sigma w_n|^2, \varphi_m) = (|D_\sigma u|^2, \varphi).$$

This follows from

$$\begin{aligned}
& \lim_{m\to\infty} \lim_{n\to\infty} |(|D_\sigma w_n|^2, \varphi_m) - (|D_\sigma u|^2, \varphi)| \\
\leq\ & \lim_{m\to\infty} \lim_{n\to\infty} \Big(|(|D_\sigma w_n|^2 - |D_\sigma u|^2, \varphi_m)| + |(|D_\sigma u|^2, \varphi_m - \varphi)| \Big)
\end{aligned}$$

by the following two calculations:

$$\begin{aligned}
& |(|D_\sigma w_n|^2 - |D_\sigma u|^2, \varphi_m)| \\
\leq\ & \sup_{m\in\mathbb{N}} \|\varphi_m\|_\infty \int_{\mathbb{R}^d} \big| |D_\sigma w_n|^2 - |D_\sigma u|^2 \big| \, dm \\
=\ & \sup_{m\in\mathbb{N}} \|\varphi_m\|_\infty \int_{\mathbb{R}^d} \big| (|D_\sigma w_n| - |D_\sigma u|)(|D_\sigma w_n| + |D_\sigma u|) \big| \, dm \\
\leq\ & \sup_{m\in\mathbb{N}} \|\varphi_m\|_\infty \left(\int_{\mathbb{R}^d} \big| |D_\sigma w_n| - |D_\sigma u| \big|^2 \, dm \right)^{\frac{1}{2}} \\
& \cdot \left(\int_{\mathbb{R}^d} \big| |D_\sigma w_n| + |D_\sigma u| \big|^2 \, dm \right)^{\frac{1}{2}} \\
\leq\ & \sup_{m\in\mathbb{N}} \|\varphi_m\|_\infty \mathcal{E}^A(w_n - u)^{\frac{1}{2}} \underbrace{\left(\sqrt{2}\mathcal{E}^A(w_n)^{\frac{1}{2}} + \sqrt{2}\mathcal{E}^A(u)^{\frac{1}{2}} \right)}_{\text{bdd. in } n} \\
\underset{n\to\infty}{\longrightarrow}\ & 0,
\end{aligned}$$

$$\lim_{m\to\infty} |(|D_\sigma u|^2, \underbrace{\varphi_m - \varphi}_{\leq \sup_{m\in\mathbb{N}} \|\varphi_m\|_\infty + \|\varphi\|_\infty})| = \int_{\mathbb{R}^d} |D_\sigma u|^2 \lim_{m\to\infty} |(\varphi_m - \varphi)| \, dm = 0.$$

Summarized we get for $\varphi \in b\mathcal{C}_T$, $u \in b\mathcal{C}_T$ and almost every t

$$\begin{aligned}
\mathcal{E}(u_t^2, \varphi_t) &= \lim_{m\to\infty} \lim_{n\to\infty} \mathcal{E}(w_{t,n}^2, \varphi_{t,m}) \\
&= \lim_{m\to\infty} \lim_{n\to\infty} \Big(2\mathcal{E}(w_{t,n}, w_{t,n}\varphi_{t,m}) - (2|D_\sigma w_{t,n}|^2 + c|w_{t,n}|^2, \varphi_{t,m}) \Big) \\
&= 2\mathcal{E}(u_t, u_t\varphi_t) - (2|D_\sigma u_t|^2 + c u_t^2, \varphi_t).
\end{aligned}$$

Since u is a weak solution of (3.1), we have

$$\begin{aligned}&\int_0^T (u_t, \partial_t(u_t\varphi_t))\,dt - (u_T, u_T\varphi_T) + (u_0, u_0\varphi_0) - \int_0^T (f_t, u_t\varphi_t)\,dt\\ = &\ -\int_0^T \mathcal{E}(u_t, u_t\varphi_t)\,dt.\end{aligned}$$

By (4.13) we obtain

$$\begin{aligned}&\frac{1}{2}\int_0^T (u_t^2, \partial_t\varphi_t)\,dt + \frac{1}{2}(u_0^2, \varphi_0) - \frac{1}{2}(u_T^2, \varphi_T) - \int_0^T (f_t, u_t\varphi_t)\,dt\\ = &\ -\int_0^T \mathcal{E}(u_t, u_t\varphi_t)\,dt.\end{aligned}$$

Moreover, by (4.14) it follows that

$$\begin{aligned}&\frac{1}{2}\int_0^T (u_t^2, \partial_t\varphi_t)\,dt + \frac{1}{2}(u_0^2, \varphi_0) - \frac{1}{2}(u_T^2, \varphi_T) - \int_0^T (f_t, u_t\varphi_t)\,dt\\ = &\ \int_0^T \left[-\frac{1}{2}\mathcal{E}(u_t^2, \varphi_t) - \left(|D_\sigma u_t|^2 + \frac{1}{2}c|u_t|^2, \varphi_t\right)\right]\,dt.\end{aligned}$$

This equation is equivalent to the weak form of equation (4.12)

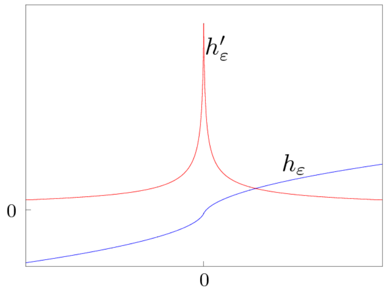
$$\begin{aligned}&\int_0^T (u_t^2, \partial_t\varphi_t)\,dt + (u_0^2, \varphi_0) - (u_T^2, \varphi_T) + \int_0^T \mathcal{E}(u_t^2, \varphi_t)\,dt\\ = &\ 2\int_0^T (f_t u_t, \varphi_t)\,dt - \int_0^T (2|D_\sigma u_t|^2 + c|u_t|^2, \varphi_t)\,dt.\end{aligned}$$

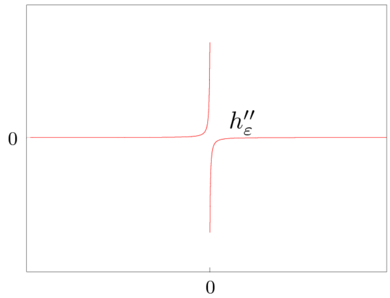
Hence, by Proposition 3.12 the relation (i) holds in the case $l = 1$. To deduce this relation in the case $l > 1$ it suffices to add the relations corresponding to the components $|u_t^i|^2, i = 1\cdots, l$.

$$\begin{aligned}&|u_t|^2 + 2\int_t^T P_{s-t}(|D_\sigma u_s|^2 + \frac{1}{2}c|u_s|^2)\,ds\\ = &\ \sum_{i=1}^l \left(|u_t^i|^2 + 2\int_t^T P_{s-t}(|D_\sigma u_s^i|^2 + \frac{1}{2}c|u_s^i|^2)\,ds\right)\\ = &\ \sum_{i=1}^l \left(P_{T-t}|\phi^i|^2 + 2\int_t^T P_{s-t}(u_s^i, f_s)\,ds\right)\\ = &\ P_{T-t}|\phi|^2 + 2\int_t^T P_{s-t}\langle u_s, f_s\rangle\,ds\end{aligned}$$

(ii) Let us define for $\varepsilon > 0$

$$h_\varepsilon(t) := \begin{cases}\sqrt{t+\varepsilon} - \sqrt{\varepsilon} & \text{for } t \geq 0\\ -(\sqrt{|t|+\varepsilon} - \sqrt{\varepsilon}) & \text{for } t < 0.\end{cases}$$





Easily we see that

$$|h_\varepsilon(t) - h_\varepsilon(s)| \leq K_\varepsilon |t - s| \text{ and } h_\varepsilon(0) = 0$$

where $K_\varepsilon := \sup_{s\in\mathbb{R}} |h'_\varepsilon(s)|$. Our first aim is to verify the following equation for $\varphi \in b\mathcal{C}_T$ and almost every t:

$$\begin{aligned}\mathcal{E}(h_\varepsilon(|u_t|^2), \varphi_t) &= \mathcal{E}(|u_t|^2, h'_\varepsilon(|u_t|^2)\varphi_t) - (h''_\varepsilon(|u_t|^2)|D_\sigma(|u_t|^2)|^2, \varphi_t) \\ &\quad + (c(h_\varepsilon(|u_t|^2) - |u_t|^2 h'_\varepsilon(|u_t|^2)), \varphi_t).\end{aligned}$$

Let us start by approximating φ_t and $|u_t|^2$. For simplicity of notation we will write u instead of u_t and φ instead of φ_t in the following calculations. Analogous to the proof of step (i) we construct $(\varphi_n)_{n\in\mathbb{N}}, \varphi_n \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $\sup_{n\in\mathbb{N}} \|\varphi_n\|_\infty < \infty$ and $\varphi_n \to \varphi$ in $\tilde{\mathcal{E}}_1^{\frac{1}{2}}$. Moreover, u^i can be approximated by $(u_n^i)_{n\in\mathbb{N}}, u_n^i \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $\sup_{n\in\mathbb{N}} \|u_n^i\|_\infty < \infty$, $u_n^i \to u^i$ m-a.e. and $u_n^i \to u^i$ in $\tilde{\mathcal{E}}_1^{\frac{1}{2}}$. Since $\lim_{n\to\infty} \||u_n|^2 - |u|^2\|_2 = 0$ and

$$\mathcal{E}(|u_n|^2) \leq \sum_{i=1}^d \mathcal{E}((u_n^i)^2) \underset{\text{Corollary 1.15}}{\leq} 4\sum_{i=1}^d \sup_{n\in\mathbb{N}} \|u_n^i\|_\infty^2 \mathcal{E}(u_n^i) < \infty,$$

we obtain by Lemma 1.19 that there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ such that the Cesáro mean $w_j := \frac{1}{j}\sum_{k=1}^j |u_{n_k}|^2$ converges to $|u|^2$ in $\tilde{\mathcal{E}}_1^{\frac{1}{2}}$. Hence, there exists a subsequence $(j_m)_{m\in\mathbb{N}}$ of $(j)_{j\in\mathbb{N}}$ such that $w_{j_m} \to |u|^2$ m-a.e..

From now on we fix the sequence $(w_n)_{n\in\mathbb{N}} := (w_{j_n})_{n\in\mathbb{N}}$. It is easy to see that $w_n \geq 0, \sup_{n\in\mathbb{N}} \|w_n\|_\infty < \infty$ and $w_n \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. Note that for w_n and φ_m

the following equations hold:

$$\begin{aligned}
\mathcal{E}^A(h_\varepsilon(w_n),\varphi_m) &= \int_{\mathbb{R}^d}\sum_{i,j=1}^d a^{i,j}\frac{\partial h_\varepsilon(w_n)}{\partial x_i}\frac{\partial\varphi_m}{\partial x_j}\,m(dx)\\
&= \int_{\mathbb{R}^d}\sum_{i,j=1}^d a^{i,j}h'_\varepsilon(w_n)\frac{\partial w_n}{\partial x_i}\frac{\partial\varphi_m}{\partial x_j}\,m(dx)\\
&= \int_{\mathbb{R}^d}\sum_{i,j=1}^d a^{i,j}\frac{\partial w_n}{\partial x_i}\frac{\partial(h'_\varepsilon(w_n)\varphi_m)}{\partial x_j}\,m(dx)\\
&\quad -\int_{\mathbb{R}^d}h''_\varepsilon(w_n)\sum_{i,j=1}^d a^{i,j}\frac{\partial w_n}{\partial x_i}\frac{\partial w_n}{\partial x_j}\varphi_m\,m(dx)\\
&= \int_{\mathbb{R}^d}\sum_{i,j=1}^d a^{i,j}\frac{\partial w_n}{\partial x_i}\frac{\partial(h'_\varepsilon(w_n)\varphi_m)}{\partial x_j}\,m(dx)\\
&\quad -(h''_\varepsilon(w_n)|D_\sigma(w_n)|^2,\varphi_m)\\
&= \mathcal{E}^A(w_n,h'_\varepsilon(w_n)\varphi)-(h''_\varepsilon(w_n)|D_\sigma(w_n)|^2,\varphi_m),
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}^B(h_\varepsilon(w_n),\varphi_m) &= \int_{\mathbb{R}^d}\sum_{i=1}^d\frac{\partial(h_\varepsilon(w_n))}{\partial x_i}\varphi_m b_i\,m(dx)\\
&= \int_{\mathbb{R}^d}\sum_{i=1}^d\frac{\partial w_n}{\partial x_i}h'_\varepsilon(w_n)\varphi_m b_i\,m(dx)\\
&= \mathcal{E}^B(w_n,h'_\varepsilon(w_n)\varphi_m),
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}^C(h_\varepsilon(w_n),\varphi_m) &= \int_{\mathbb{R}^d}c\,\varphi_m(h_\varepsilon(w_n)-w_nh'_\varepsilon(w_n))\,m(dx)\\
&\quad +\int_{\mathbb{R}^d}cw_nh'_\varepsilon(w_n)\varphi_m\,m(dx)\\
&= \mathcal{E}^C(w_n,h'_\varepsilon(w_n)\varphi_m)+(c(h_\varepsilon(w_n)-w_nh'_\varepsilon(w_n)),\varphi_m).
\end{aligned}$$

Our next aim is to verify

$$\lim_{n\to\infty}\lim_{m\to\infty}\mathcal{E}(h_\varepsilon(w_n),\varphi_m)=\mathcal{E}(h_\varepsilon(|u|^2),\varphi) \tag{4.16}$$

and

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\Big[\mathcal{E}(w_n,h'_\varepsilon(w_n)\varphi_m)-(h''_\varepsilon(w_n)|D_\sigma(w_n)|^2,\varphi_m) \qquad (4.17)\\
&\qquad\qquad +(c(h_\varepsilon(w_n)-w_nh'_\varepsilon(w_n)),\varphi_m)\Big]\\
&= \mathcal{E}(|u|^2,h'_\varepsilon(|u|^2)\varphi)-(h''_\varepsilon(|u|^2)|D_\sigma(|u|^2)|^2,\varphi)\\
&\qquad\qquad +(c(h_\varepsilon(|u|^2)-|u|^2h'_\varepsilon(|u|^2)),\varphi).
\end{aligned}$$

Let us start with equation (4.16). Since we have by Theorem 1.13 $\mathcal{E}(h_\varepsilon(w_n))\le K_\varepsilon^2\mathcal{E}(w_n)$ for all $n\in\mathbb{N}$ and it holds

$$\lim_{n\to\infty}\|h_\varepsilon(w_n)-h_\varepsilon(|u|^2)\|_2\le K_\varepsilon\lim_{n\to\infty}\|w_n-|u|^2\|_2=0,$$

we obtain by Lemma 1.19

$$\lim_{n\to\infty} \mathcal{E}(h_\varepsilon(w_n), \varphi) = \mathcal{E}(h_\varepsilon(|u|^2), \varphi) \text{ for all } \varphi \in \mathcal{C}_T.$$

Further we deduce for all $n \in \mathbb{N}$

$$\mathcal{E}(h_\varepsilon(w_n), \varphi) = \lim_{m\to\infty} \mathcal{E}(h_\varepsilon(w_n), \varphi_m)$$

by the following calculation:

$$\begin{aligned}
& \lim_{m\to\infty} |\mathcal{E}(h_\varepsilon(w_n), \varphi_m - \varphi)| \\
\leq\ & \lim_{m\to\infty} \Big(K_{\mathcal{E}} \mathcal{E}_1(h_\varepsilon(w_n))^{\frac{1}{2}} \mathcal{E}_1(\varphi_m - \varphi)^{\frac{1}{2}} + \|h_\varepsilon(w_n)\|_2 \|\varphi_m - \varphi\|_2 \Big) \\
\leq\ & \lim_{m\to\infty} \left(K_{\mathcal{E}} K_\varepsilon \sup_{n\in\mathbb{N}} \mathcal{E}_1(w_n)^{\frac{1}{2}} \mathcal{E}_1(\varphi_m - \varphi)^{\frac{1}{2}} + K_\varepsilon \sup_{n\in\mathbb{N}} \|w_n\|_2 \|\varphi_m - \varphi\|_2 \right) \\
=\ & 0.
\end{aligned}$$

Hence, the first equation is shown. Next we have to verify equation (4.17). Let us start by showing the convergence in m. Since we have

$$\lim_{m\to\infty} \|h'_\varepsilon(w_n)(\varphi_m - \varphi)\|_2 \leq K_\varepsilon \lim_{m\to\infty} \|\varphi_m - \varphi\|_2 = 0,$$

the convergence of the first term will follow by Lemma 1.19, if we can show

$$\sup_{m\in\mathbb{N}} \mathcal{E}(h'_\varepsilon(w_n)\varphi_m) < \infty. \tag{4.18}$$

Let us define $v := h'_\varepsilon(w_n)\varphi_m$, $v_1 := K_\varepsilon \varphi_m$ and $v_2 := \tilde{K}_\varepsilon \|\varphi_m\|_\infty w_n$ where $\tilde{K}_\varepsilon := \sup_{s\in\mathbb{R}} |h''_\varepsilon(s)|$. Then we have m-a.e.

$$|v| = |h'_\varepsilon(w_n)\varphi_m| \leq K_\varepsilon |\varphi_m| \leq K_\varepsilon |\varphi_m| + \tilde{K}_\varepsilon \|\varphi_m\|_\infty |w_n| = |v_1| + |v_2|$$

and

$$\begin{aligned}
& |v(x) - v(y)| \\
=\ & |h'_\varepsilon(w_n)(x)\varphi_m(x) - h'_\varepsilon(w_n)(y)\varphi_m(y)| \\
\leq\ & |h'_\varepsilon(w_n)(x)\varphi_m(x) - h'_\varepsilon(w_n)(x)\varphi_m(y)| \\
& + |\varphi_m(y)||h'_\varepsilon(w_n)(x) - h'_\varepsilon(w_n)(y)| \\
\leq\ & K_\varepsilon |\varphi_m(x) - \varphi_m(y)| + \|\varphi_m\|_\infty \tilde{K}_\varepsilon |w_n(x) - w_n(y)| \\
=\ & |v_1(x) - v_1(y)| + |v_2(x) - v_2(y)|.
\end{aligned}$$

Hence, by Corollary 1.14 it follows that

$$\begin{aligned}
\mathcal{E}(h'_\varepsilon(w_n)\varphi_m)^{\frac{1}{2}} &= \mathcal{E}(v)^{\frac{1}{2}} \\
&\leq \mathcal{E}(v_1)^{\frac{1}{2}} + \mathcal{E}(v_2)^{\frac{1}{2}} \\
&= K_\varepsilon \mathcal{E}(\varphi_m)^{\frac{1}{2}} + \tilde{K}_\varepsilon \|\varphi_m\|_\infty \mathcal{E}(w_n)^{\frac{1}{2}} \\
&\leq K_\varepsilon \sup_{m\in\mathbb{N}} \mathcal{E}(\varphi_m)^{\frac{1}{2}} + \tilde{K}_\varepsilon \sup_{m\in\mathbb{N}} \|\varphi_m\|_\infty \sup_{n\in\mathbb{N}} \mathcal{E}(w_n)^{\frac{1}{2}} \\
&< \infty.
\end{aligned}$$

Therefore, we can apply Lemma 1.19 and deduce for all $n \in \mathbb{N}$

$$\lim_{m\to\infty} \mathcal{E}(w_n, h'_\varepsilon(w_n)\varphi_m) = \mathcal{E}(w_n, h'_\varepsilon(w_n)\varphi).$$

The remaining terms of equation (4.17) converge in m by the following arguments for fixed n:

$$\begin{aligned} & \left|\Big(h''_\varepsilon(w_n)|D_\sigma(w_n)|^2, \varphi_m - \varphi\Big)\right| \\ = & \left|\Big(h''_\varepsilon(w_n)|(\nabla w_n)\sigma|^2, \varphi_m - \varphi\Big)\right| \\ \le & \sup_{x\in\mathbb{R}^d} |(\nabla w_n)\sigma|\tilde{K}_\varepsilon \|(\nabla w_n)\sigma\|_2 \|\varphi_m - \varphi\|_2 \\ \underset{m\to\infty}{\to} & \; 0, \end{aligned}$$

$$\lim_{m\to\infty} |(ch_\varepsilon(w_n), \varphi_m - \varphi)| \le \sup_{x\in\mathbb{R}^d} |c(x)| \|h_\varepsilon(w_n)\|_2 \lim_{m\to\infty} \|\varphi_m - \varphi\|_2 = 0$$

and

$$\lim_{m\to\infty} |(w_n h'_\varepsilon(w_n), \varphi_m - \varphi)| \le K_\varepsilon \|w_n\|_2 \lim_{m\to\infty} \|\varphi_m - \varphi\|_2 = 0.$$

Next we have to pass to the limit in n

$$\begin{aligned} (a) \quad & \lim_{n\to\infty} \mathcal{E}(w_n, h'_\varepsilon(w_n)\varphi) = \mathcal{E}(|u|^2, h'_\varepsilon(|u|^2)\varphi), \\ (b) \quad & \lim_{n\to\infty} (h''_\varepsilon(w_n)|D_\sigma(w_n)|^2, \varphi) = (h''_\varepsilon(|u|^2)|D_\sigma(|u|^2)|^2, \varphi), \\ (c) \quad & \lim_{n\to\infty} (c(h_\varepsilon(w_n)), \varphi) = (c(h_\varepsilon(|u|^2)), \varphi), \\ (d) \quad & \lim_{n\to\infty} (c(w_n h'_\varepsilon(w_n)), \varphi) = (c(|u|^2 h'_\varepsilon(|u|^2)), \varphi). \end{aligned}$$

(a) First of all let us note that analogous to equation (4.18) we can show that

$$\sup_{n\in\mathbb{N}} \mathcal{E}(h'_\varepsilon(w_n)\varphi) < \infty.$$

Since we have

$$\lim_{n\to\infty} \|(h'_\varepsilon(w_n) - h'_\varepsilon(|u|^2))\varphi\|_2 \le \sup_{x\in\mathbb{R}^d} |\varphi|\tilde{K}_\varepsilon \lim_{n\to\infty} \|w_n - |u|^2\|_2 = 0,$$

it follows by Lemma 1.19

$$\lim_{n\to\infty} \mathcal{E}(|u|^2, (h'_\varepsilon(w_n) - h'_\varepsilon(|u|^2))\varphi) = 0.$$

Now we deduce the assertion (a)

$$\begin{aligned} & |\mathcal{E}(w_n, h'_\varepsilon(w_n)\varphi) - \mathcal{E}(|u|^2, h'_\varepsilon(|u|^2)\varphi)| \\ \le \; & |K_\mathcal{E} \mathcal{E}_1(w_n - |u|^2)^{\frac{1}{2}} \mathcal{E}_1(h'_\varepsilon(w_n)\varphi)^{\frac{1}{2}}| + \|w_n - |u|^2\|_2 \|h'_\varepsilon(w_n)\varphi\|_2 \\ & + |\mathcal{E}(|u|^2, (h'_\varepsilon(w_n) - h'_\varepsilon(|u|^2))\varphi)| \\ \underset{n\to\infty}{\to} & \; 0. \end{aligned}$$

(b) Since $w_n \to |u|^2$ m-a.e., we calculate

$$\begin{aligned}
& \lim_{n\to\infty} \left| \Big((h''_\varepsilon(w_n) - h''_\varepsilon(|u|^2))|D_\sigma(|u|^2)|^2, \varphi \Big) \right| \\
\leq\ & \sup_{s\in\mathbb{R}} |h'''_\varepsilon(s)| \lim_{n\to\infty} \Big(\underbrace{|w_n - |u|^2|}_{\leq \sup_{n\in\mathbb{N}} \|w_n\|_\infty + \|u\|^2_\infty} |D_\sigma(|u|^2)|^2, |\varphi| \Big) \\
\leq\ & \sup_{s\in\mathbb{R}} |h'''_\varepsilon(s)| \sup_{x\in\mathbb{R}^d} |\varphi(x)| \int_{\mathbb{R}^d} \lim_{n\to\infty} |w_n - |u|^2||D_\sigma(|u|^2)|^2 \, dm \\
=\ & 0
\end{aligned}$$

and

$$\begin{aligned}
& |(h''_\varepsilon(w_n) \left(|D_\sigma(w_n)|^2 - |D_\sigma(|u|^2)|^2 \right), \varphi)| \\
\leq\ & \tilde{K}_\varepsilon \sup_{x\in\mathbb{R}^d} |\varphi(x)| \int_{\mathbb{R}^d} \left(|D_\sigma(w_n)|^2 - |D_\sigma(|u|^2)|^2 \right) \, dm \\
\leq\ & \tilde{K}_\varepsilon \sup_{x\in\mathbb{R}^d} |\varphi(x)| \left(\int_{\mathbb{R}^d} \Big| |D_\sigma w_n| - |D_\sigma |u|^2| \Big|^2 \, dm \right)^{\frac{1}{2}} \\
& \cdot \left(\int_{\mathbb{R}^d} \Big| |D_\sigma w_n| + |D_\sigma |u|^2| \Big|^2 \, dm \right)^{\frac{1}{2}} \\
\leq\ & \tilde{K}_\varepsilon \sup_{x\in\mathbb{R}^d} |\varphi(x)| \mathcal{E}^A(w_n - |u|^2)^{\frac{1}{2}} \left(\sqrt{2}\mathcal{E}^A(w_n)^{\frac{1}{2}} + \sqrt{2}\mathcal{E}^A(|u|^2)^{\frac{1}{2}} \right) \\
\underset{n\to\infty}{\longrightarrow}\ & 0.
\end{aligned}$$

(c)

$$\begin{aligned}
& \lim_{n\to\infty} |(c(h_\varepsilon(w_n) - h_\varepsilon(|u|^2)), \varphi)| \\
\leq\ & \sup_{x\in\mathbb{R}^d} |c(x)| \|\varphi\|_2 K_\varepsilon \lim_{n\to\infty} \|w_n - |u|^2\|_2 \\
=\ & 0
\end{aligned}$$

(d)

$$\begin{aligned}
& \lim_{n\to\infty} |(c((w_n - |u|^2)h'_\varepsilon(w_n), \varphi)| \\
\leq\ & \sup_{x\in\mathbb{R}^d} |c(x)| \|\varphi\|_2 K_\varepsilon \lim_{n\to\infty} \|w_n - |u|^2\|_2 \\
=\ & 0
\end{aligned}$$

$$\begin{aligned}
& \lim_{n\to\infty} |(c|u|^2, (h'_\varepsilon(w_n) - h'_\varepsilon(|u|^2))\varphi)| \\
\leq\ & \sup_{x\in\mathbb{R}^d} |c(x)| \tilde{K}_\varepsilon \|\varphi\|_\infty \|u\|_\infty \|u\|_2 \lim_{n\to\infty} \|w_n - |u|^2\|_2 \\
=\ & 0
\end{aligned}$$

Finally, we obtain equation (4.17). Summarized we have shown that

$$\begin{aligned}
0 \ =\ & \int_0^T \Big(- \mathcal{E}(h_\varepsilon(|u_t|^2), \varphi_t) + \mathcal{E}(|u_t|^2, h'_\varepsilon(|u_t|^2)\varphi_t) \\
& - (h''_\varepsilon(|u_t|^2)|D_\sigma(|u_t|^2)|^2, \varphi_t) + (c(h_\varepsilon(|u_t|^2) - |u_t|^2 h'_\varepsilon(|u_t|^2)), \varphi_t) \Big) \, dt.
\end{aligned}$$

Since we have the identity (cf. Lemma 2.6)

$$\partial_t(h_\varepsilon(|u|^2)) = h'_\varepsilon(|u|^2)\partial_t(|u|^2),$$

$h_\varepsilon(|u|^2)$ is a weak solution of

$$\begin{aligned} & (\partial_t + L)h_\varepsilon(|u|^2) \\ = \; & h'_\varepsilon(|u|^2)(\partial_t + L)|u|^2 + h''_\varepsilon(|u|^2)|D_\sigma(|u|^2)|^2 - c(h_\varepsilon(|u|^2) - |u|^2 h'_\varepsilon(|u|^2)). \end{aligned} \tag{4.19}$$

Further by the product rule for generalized gradients it holds

$$\begin{aligned} |D_\sigma(|u|^2)|^2 &= 4\left|\sum_{i=1}^{l} u^i D_\sigma(u^i)\right|^2 \\ &= 4\left|u^1 D_\sigma(u^1) + \cdots + u^l D_\sigma(u^l)\right|^2 \\ &= 4\left|u^\star D_\sigma(u)\right|^2 \\ &= 4\langle (u^\star D_\sigma(u))^\star, (u^\star D_\sigma(u))^\star\rangle \\ &= 4\langle u, D_\sigma(u)(u^\star D_\sigma(u))^\star\rangle \\ &= 4\langle u, D_\sigma u(D_\sigma u)^\star u\rangle. \end{aligned} \tag{4.20}$$

Now we deduce

$$\begin{aligned} & (\partial_t + L)h_\varepsilon(|u|^2) + c(h_\varepsilon(|u|^2) - |u|^2 h'_\varepsilon(|u|^2)) \\ \underset{(4.19)}{=} \; & h'_\varepsilon(|u|^2)(\partial_t + L)|u|^2 + h''_\varepsilon(|u|^2)|D_\sigma(|u|^2)|^2 \\ \underset{(4.20),(i)}{=} \; & h'_\varepsilon(|u|^2)(-2\langle u, f\rangle + 2|D_\sigma u|^2 + c|u|^2) \\ & +4h''_\varepsilon(|u|^2)\langle u, D_\sigma u(D_\sigma u)^\star u\rangle \\ = \; & \frac{-\langle u, f\rangle + |D_\sigma u|^2}{(|u|^2+\varepsilon)^{\frac{1}{2}}} - \frac{|u|^2\langle \hat{u}, D_\sigma u(D_\sigma u)^\star \hat{u}\rangle}{(|u|^2+\varepsilon)^{\frac{3}{2}}} + c|u|^2 h'_\varepsilon(|u|^2) \\ = \; & \frac{-\langle u, f\rangle}{(|u|^2+\varepsilon)^{\frac{1}{2}}} \\ & + \frac{\varepsilon(|D_\sigma u|^2) + |u|^2(|D_\sigma u|^2 - \langle \hat{u}, D_\sigma u(D_\sigma u)^\star \hat{u}\rangle)}{(|u|^2+\varepsilon)^{\frac{3}{2}}} + c|u|^2 h'_\varepsilon(|u|^2) \\ \underset{\text{Lemma 4.13}}{\geq} \; & \frac{-\langle u, f\rangle}{(|u|^2+\varepsilon)^{\frac{1}{2}}} + c|u|^2 h'_\varepsilon(|u|^2). \end{aligned}$$

Hence, we conclude

$$(\partial_t + L)h_\varepsilon(|u|^2) + \frac{\langle u, f\rangle}{(|u|^2+\varepsilon)^{\frac{1}{2}}} + c\left(h_\varepsilon(|u|^2) - 2|u|^2 h'_\varepsilon(|u|^2)\right) \geq 0.$$

Since $c\left(h_\varepsilon(|u|^2) - 2|u|^2 h'_\varepsilon(|u|^2)\right) \leq 0$, we get by Proposition 3.12

$$h_\varepsilon(|u_t|^2) \;\; \leq \;\; P_{T-t}h_\varepsilon(|\phi|^2) + \int_t^T P_{s-t}\frac{\langle u_s, f_s\rangle}{(|u_s|^2+\varepsilon)^{\frac{1}{2}}}\,ds.$$

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence such that $\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$. Now we show the existence of a subsequence of $(\varepsilon_n)_{n\in\mathbb{N}}$ such that passing to the limit in the above relation will yield the assertion. More precisely we have to show that there exists a subsequence such that the following equations hold m-a.e.:

$$(a) \qquad \lim_{n\to\infty} h_{\varepsilon n}(|u_t|^2) = |u_t|,$$

$$(b) \qquad \lim_{n\to\infty} P_{T-t} h_{\varepsilon n}(|\phi|^2) = P_{T-t}|\phi|^2,$$

$$(c) \qquad \lim_{n\to\infty} \int_t^T P_{s-t} \frac{\langle u_s, f_s\rangle}{(|u_s|^2+\varepsilon_n)^{\frac{1}{2}}}\, ds = \int_t^T P_{s-t}\langle \hat{u}_s, f_s\rangle\, ds.$$

Before starting the calculations we note that $h_{\varepsilon n}(|x|^2) \leq |x|$ for all $x \in \mathbb{R}$.

(a)

$$\lim_{n\to\infty} h_{\varepsilon n}(|u_t|^2) \underset{\text{def.}}{=} \lim_{n\to\infty}\left(\sqrt{|u_t|^2+\varepsilon_n} - \sqrt{\varepsilon_n}\right) = |u_t|$$

(b) Since we have

$$\begin{aligned}
\lim_{n\to\infty} \|P_{T-t}(h_{\varepsilon n}(|\phi|^2) - |\phi|)\|_2 &\leq \lim_{n\to\infty} \|\underbrace{h_{\varepsilon n}(|\phi|^2) - |\phi|}_{\leq 2|\phi|}\|_2 \\
&= \|\lim_{n\to\infty} h_{\varepsilon n}(|\phi|^2) - |\phi|\|_2 \\
&= 0,
\end{aligned}$$

there exists a subsequence such that (b) holds m-a.e..
(c) First note that we have for $|u_s| > 0$

$$\lim_{n\to\infty}\left|\frac{\langle u_s, f_s\rangle}{(|u_s|^2+\varepsilon_n)^{\frac{1}{2}}} - \frac{\langle u_s, f_s\rangle}{|u_s|}\right| = 0.$$

Hence, we calculate

$$\begin{aligned}
&\lim_{n\to\infty}\left\|\int_t^T \left(P_{s-t}\frac{\langle u_s, f_s\rangle}{(|u_s|^2+\varepsilon_n)^{\frac{1}{2}}} - P_{s-t}\frac{\langle u_s, f_s\rangle}{|u_s|}\right) ds\right\|_2 \\
\leq\ &\lim_{n\to\infty}\int_t^T \left\|\underbrace{\frac{\langle u_s, f_s\rangle}{(|u_s|^2+\varepsilon_n)^{\frac{1}{2}}} - \frac{\langle u_s, f_s\rangle}{|u_s|}}_{\leq 2|f_s|}\right\|_2 ds \\
=\ &\int_t^T \left\|\lim_{n\to\infty}\left(\frac{\langle u_s, f_s\rangle}{(|u_s|^2+\varepsilon_n)^{\frac{1}{2}}} - \frac{\langle u_s, f_s\rangle}{|u_s|}\right)\right\|_2 ds \\
=\ &0
\end{aligned}$$

and deduce that a subsequence of the sequence in (b) exists such that (c) holds m-a.e..

□

The next corollary is a version of the above proposition for general data.

Corollary 4.15. *Let $u = (u^1, \cdots, u^l)$ be a vector valued function where each component is a weak solution of the linear equation (3.1) associated to certain data $f^i \in L^1([0,T]; L^2)$ and $\phi^i \in L^2$ for $i = 1, \cdots, l$. Denote by ϕ, f the vectors $\phi = (\phi^1, \cdots, \phi^l)$, $f = (f^1, \cdots, f^l)$, and by $D_\sigma u$ the matrix whose rows consist of the row vectors $D_\sigma u^i$. Then the following relations hold m-almost everywhere*

$$\begin{aligned}(i) \qquad & |u_t|^2 + 2\int_t^T P_{s-t}\left(|D_\sigma u_s|^2 + \frac{1}{2}c|u_s|^2\right) ds \\ & \qquad = P_{T-t}|\phi|^2 + 2\int_t^T P_{s-t}\langle u_s, f_s\rangle\, ds, \\ (ii) \qquad & |u_t| \le P_{T-t}|\phi| + \int_t^T P_{s-t}\langle \hat{u}_s, f_s\rangle\, ds.\end{aligned}$$

Proof. (i) Analogous to the proof of the above proposition it is enough to verify the assertion (i) for $l = 1$. Let $\phi \in \mathcal{D}(L)$ and $f \in \mathcal{C}^1([0,T]; L^2)$. Then by Lemma 4.12 there exists $\phi_n \in \mathcal{D}(L)$ bounded and $f_n \in \mathcal{C}^1([0,T]; L^2)$ bounded such that

$$\begin{aligned}(a) \qquad & u_{n,t} := P_{T-t}\phi_n + \int_t^T P_{s-t} f_{n,s}\, ds \quad \text{is a weak solution,} \\ (b) \qquad & \lim_{n\to\infty} \int_t^T \|f_{n,s} - f_s\|_2\, ds = 0, \\ (c) \qquad & \lim_{n\to\infty} \|\phi_n - \phi\|_2 = 0, \\ (d) \qquad & \lim_{n\to\infty} \|u_n - u\|_T = 0.\end{aligned}$$

By the above proposition it holds

$$\begin{aligned} & |u_{n,t}|^2 + 2\int_t^T P_{s-t}\left(|D_\sigma u_{n,s}|^2 + \frac{1}{2}c|u_{n,s}|^2\right) ds \\ = \quad & P_{T-t}|\phi|^2 + 2\int_t^T P_{s-t}(u_{n,s} f_{n,s})\, ds.\end{aligned} \tag{4.21}$$

Hence, we have to pass to the limit.

Since we have by (d) $\|u_{n,t} - u_t\|_2 \to 0$ for all $t \in [0,T]$, it follows for a subsequence $(n_1)_{n_1\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ that $||u_{n_1,t}| - |u_t|| \le |u_{n_1,t} - u_t| \to 0$ m-a.e. and hence $|u_{n_1,t}|^2 \to |u_t|^2$ m-a.e..

Fix $t \in [0,T]$. By (c) it follows that $\|P_{T-t}|\phi_{n_1}| - P_{T-t}|\phi|\|_2 \to 0$ and hence there exists a subsequence $(n_2)_{n_2\in\mathbb{N}}$ such that $P_{T-t}|\phi_{n_2}| \to P_{T-t}|\phi|$ m-a.e.. By (b) and (d) we obtain

$$\begin{aligned} & \left\|\int_t^T P_{s-t}\left((u_{n_2,s} f_{n_2,s}) - (u_s f_s)\right) ds\right\|_1 \\ \le \quad & \int_t^T \left(\|u_{n_2,s}\|_2 \|f_{n_2,s} - f_s\|_2 + \|f_s\|_2 \|u_{n_2,s} - u_s\|_2\right) ds \\ \le \quad & \sup_{s\in[0,T]} \|u_{n_2,s}\|_2 \int_t^T \|f_{n_2,s} - f_s\|_2\, ds + \sup_{s\in[0,T]} \|u_{n_2,s} - u_s\|_2 \int_t^T \|f_s\|_2\, ds \\ \to \quad & 0.\end{aligned}$$

Hence, there exists a subsequence $(n_3)_{n_3\in\mathbb{N}}$ such that we get

$$\lim_{n_3\to\infty}\int_t^T P_{s-t}(u_{n_3,s}f_{n_3,s})\,ds = \int_t^T P_{s-t}(u_s f_s)\,ds \quad m\text{-a.e..}$$

Again by (d), we calculate

$$\begin{aligned}
& \int_t^T \left\| |D_\sigma u_{n_3,s}|^2 - |D_\sigma u_s|^2 \right\|_1 \, ds \\
\leq\ & \int_t^T \|D_\sigma u_{n_3,s}\|_2 \| |D_\sigma u_{n_3,s}| - |D_\sigma u_s| \|_2 \, ds \\
& + \int_t^T \|D_\sigma u_s\|_2 \| |D_\sigma u_{n_3,s}| - |D_\sigma u_s| \|_2 \, ds \\
\leq\ & \left(\int_t^T \|D_\sigma u_{n_3,s}\|_2^2\,ds\right)^{\frac{1}{2}} \left(\int_t^T \|D_\sigma u_{n_3,s} - D_\sigma u_s\|_2^2\,ds\right)^{\frac{1}{2}} \\
& + \left(\int_t^T \|D_\sigma u_s\|_2^2\,ds\right)^{\frac{1}{2}} \left(\int_t^T \|D_\sigma u_{n_3,s} - D_\sigma u_s\|_2^2\,ds\right)^{\frac{1}{2}} \\
\leq\ & \underbrace{\left(\left(\int_t^T \|D_\sigma u_{n_3,s}\|_2^2\,ds\right)^{\frac{1}{2}} + \left(\int_t^T \|D_\sigma u_s\|_2^2\,ds\right)^{\frac{1}{2}}\right)}_{\leq \tilde{K} \text{ constant}} \\
& \cdot \left(\int_t^T \|D_\sigma u_{n_3,s} - D_\sigma u_s\|_2^2\,ds\right)^{\frac{1}{2}} \\
\leq\ & \tilde{K}\left(\int_t^T K_A \mathcal{E}(u_{n_3,s} - u_s) + C_A \|u_{n_3,s} - u_s\|_2^2\,ds\right)^{\frac{1}{2}} \\
\to\ & 0
\end{aligned}$$

and obtain for a subsequence $(n_4)_{n_4\in\mathbb{N}}$ m-a.e.

$$\lim_{n_4\to\infty}\int_t^T P_{s-t}(|D_\sigma u_{n_4,s}|^2)\,ds = \int_t^T P_{s-t}(|D_\sigma u_s|^2)\,ds.$$

Since

$$\begin{aligned}
& \int_t^T \|c|u_{n_4,s}|^2 - c|u_s|^2\|_1\,ds \\
\leq\ & \sup_{x\in\mathbb{R}^d}|c(x)| \int_t^T (\|u_{n_4,s}\|_2 + \|u_s\|_2)\|u_{n_4,s} - u_s\|_2\,ds \\
\to\ & 0,
\end{aligned}$$

there exists a subsequence $(n_5)_{n_5\in\mathbb{N}}$ such that m-a.e.

$$\lim_{n_5\to\infty}\int_t^T P_{s-t}(c|u_{n_5,s}|^2)\,ds = \int_t^T P_{s-t}(c|u_s|^2)\,ds.$$

Summarized we can find a subsequence $(n_5)_{n_5\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ such that by passing to the limit in equation (4.21), the assertion (i) holds m-a.e. for $\phi \in \mathcal{D}(L)$ and $f \in \mathcal{C}^1([0,T];L^2)$. Absolutely analogous to these calculations we can show that by using Lemma 4.11 the assertion (i) holds for $f \in L^1([0,T];L^2)$ and $\phi \in L^2$.

(ii) Let $\phi^i \in \mathcal{D}(L)$ and $f^i \in \mathcal{C}^1([0,T];L^2)$. Then there exist by Lemma 4.12 approximating functions $\phi^i_n \in \mathcal{D}(L)$ bounded and $f^i_n \in \mathcal{C}^1([0,T];L^2)$ bounded such that

$$(a) \qquad u^i_{n,t} = P_{T-t}\phi^i_n + \int_t^T P_{s-t} f^i_{n,s}\, ds \quad \text{is a weak solution,}$$

$$(b) \qquad \lim_{n\to\infty} \int_t^T \|f^i_{n,s} - f^i_s\|_2\, ds = 0,$$

$$(c) \qquad \lim_{n\to\infty} \|\phi^i_n - \phi^i\|_2 = 0,$$

$$(d) \qquad \lim_{n\to\infty} \|u^i_n - u^i\|_T = 0.$$

By the above proposition it holds:

$$|u_{n,t}| \le P_{T-t}|\phi| + \int_t^T P_{s-t}\langle \hat{u}_{n,s}, f_{n,s}\rangle\, ds. \tag{4.22}$$

It is easy to see that analogous to the above calculations the first two terms of this equation converge m-a.e. along a subsequence. Hence, we will only examine the last term.

$$\begin{aligned}
& \lim_{n\to\infty} \left\| \int_t^T P_{s-t}\langle \hat{u}_{n,s}, f_{n,s}\rangle - P_{s-t}\langle \hat{u}_s, f_s\rangle\, ds \right\|_2 \\
\le\ & \lim_{n\to\infty} \int_t^T \left\| \langle \hat{u}_{n,s}, f_{n,s}\rangle - \langle \hat{u}_s, f_s\rangle \right\|_2 ds \\
\le\ & \lim_{n\to\infty} \int_t^T \| \underbrace{|\hat{u}_{n,s}|}_{=1} |f_{n,s} - f_s| \|_2 + \| |\hat{u}_{n,s} - \hat{u}_s| |f_s| \|_2\, ds \\
\le\ & \underbrace{\lim_{n\to\infty} \int_t^T \|f_{n,s} - f_s\|_2\, ds}_{=0} + \lim_{n\to\infty} \int_t^T \| \underbrace{|\hat{u}_{n,s} - \hat{u}_s|}_{\le 2} |f_s| \|_2\, ds \\
\le\ & \int_t^T \| \lim_{n\to\infty} |\hat{u}_{n,s} - \hat{u}_s| |f_s| \|_2\, ds \\
=\ & 0
\end{aligned}$$

Here we have chosen by (d) a subsequence such that

$$\lim_{n\to\infty} |u^i_{n,s} - u^i_s| = 0 \quad \text{for i=1,...,l} \quad \text{a.e..}$$

Summarized we can find a subsequence and a zero set such that we can pass to the limit in equation (4.22) and get (ii) for $\phi^i \in \mathcal{D}(L)$ and $f^i \in \mathcal{C}^1([0,T];L^2)$. Absolutely analogous to these calculations we can show that by Lemma 4.11 the assertion (ii) holds for $f^i \in L^1([0,T];L^2)$ and $\phi^i \in L^2$. □

The next lemma presents two useful relations. We follow [BPS05, Lemma 2.12]. Our proof of the second estimate bases on the Markov process associated to $(T_t)_{t\geq 0}$.

Lemma 4.16. *If $f, g \in L^1([0,T]; L^2)$ and $\phi \in L^2$, then the following relations hold m-a.e.:*

$$(i) \qquad \int_t^T P_{s-t}(f_s P_{T-s}\phi)\, ds \leq \frac{1}{2}\left[\frac{1}{2}P_{T-t}\phi^2 + \int_t^T \int_s^T P_{s-t}\left(f_s P_{r-s} f_r\right)\, dr\, ds\right],$$

$$\begin{aligned}(ii) \qquad & \int_t^T \int_s^T P_{s-t}[(f_s+g_s)P_{r-s}(f_r+g_r)]\, dr\, ds \\ & \leq 2\int_t^T \int_s^T P_{s-t}(f_s P_{r-s} f_r)\, dr\, ds + 2\int_t^T \int_s^T P_{s-t}(g_s P_{r-s} g_r)\, dr\, ds.\end{aligned}$$

Remark 4.17. *In the case of the above lemma it holds for every $\varepsilon > 0$*

$$\int_t^T P_{s-t}(f_s P_{T-s}\phi)\, ds \leq \frac{1}{2}\left[\frac{1}{2}\varepsilon^2 P_{T-t}\phi^2 + \frac{1}{\varepsilon^2}\int_t^T \int_s^T P_{s-t}\left(f_s P_{r-s} f_r\right)\, dr\, ds\right].$$

Proof of Lemma 4.16. (i) Let us define

$$h_t = P_{T-t}\phi, \qquad v_t = \int_t^T P_{s-t} f_s\, ds.$$

By relation (i) from Corollary 4.15 we deduce

$$h_t^2 + 2\int_t^T P_{s-t}\left(|D_\sigma h_s|^2 + \frac{1}{2}c|h_s|^2\right) ds = P_{T-t}\phi^2, \tag{4.23}$$

$$\begin{aligned} v_t^2 + 2\int_t^T & P_{s-t}\left(|D_\sigma v_s|^2 + \frac{1}{2}c|v_s|^2\right) ds \\ &= 2\int_t^T P_{s-t}\left(f_s \int_s^T P_{r-s} f_r\, dr\right) ds, \end{aligned} \tag{4.24}$$

$$\begin{aligned} h_t v_t + 2\int_t^T & P_{s-t}\left(\langle D_\sigma h_s, D_\sigma v_s\rangle + \frac{1}{2}c(h_s v_s)\right) ds \\ &= 2\int_t^T P_{s-t}\left(f_s P_{T-s}\phi\right) ds. \end{aligned} \tag{4.25}$$

The equation (4.23) follows from Corollary 4.15 by setting $u_t = h_t$ and (4.24) follows by setting $u_t = v_t$. Since $u_t = h_t + v_t$, the equation (4.25) follows from Corollary 4.15

$$\begin{aligned} & |h_t + v_t|^2 + 2\int_t^T P_{s-t}\left(|D_\sigma(h_s+v_s)|^2 + \frac{1}{2}c|h_s+v_s|^2\right) ds \\ &= P_{T-t}|\phi|^2 + 2\int_t^T P_{s-t}((h_s+v_s)f_s)\, ds \end{aligned}$$

by subtracting the equations (4.23) and (4.24). The relation (i) is a consequence of the preceding relations.

$$
\begin{aligned}
& \int_t^T P_{s-t}(f_s P_{T-s}\phi)\, ds \\
\underset{(4.25)}{=} \quad & \frac{1}{2} h_t v_t + \int_t^T P_{s-t}\langle D_\sigma h_s, D_\sigma v_s\rangle\, ds + \int_t^T P_{s-t}\left(\frac{1}{2} c(h_s v_s)\right) ds \\
\underset{\substack{c \geq 0 \\ \text{Lemma1.10}}}{\leq} \quad & \frac{1}{2}\left[\frac{1}{2}(h_t^2 + v_t^2) + \int_t^T P_{s-t}(|D_\sigma h_s|^2 + |D_\sigma v_s|^2)\, ds \right. \\
& \left. + \int_t^T P_{s-t}\left(\frac{1}{2} c(h_s^2 + v_s^2)\right) ds \right] \\
= \quad & \frac{1}{2}\left[\frac{1}{2} h_t^2 + \int_t^T P_{s-t}|D_\sigma h_s|^2\, ds + \int_t^T P_{s-t}\left(\frac{1}{2} c h_s^2\right) ds \right. \\
& \left. + \frac{1}{2} v_t^2 + \int_t^T P_{s-t}|D_\sigma v_s|^2\, ds + \int_t^T P_{s-t}\left(\frac{1}{2} c v_s^2\right) ds \right] \\
\underset{(4.23),(4.24)}{=} \quad & \frac{1}{2}\left[\frac{1}{2} P_{T-t}\phi^2 + \int_t^T P_{s-t}\left(f_s \int_s^T P_{r-t} f_r\, dr\right) ds\right] \\
= \quad & \frac{1}{2}\left[\frac{1}{2} P_{T-t}\phi^2 + \int_t^T \int_s^T P_{s-t}\left(f_s P_{r-s} f_r\right) dr\, ds\right]
\end{aligned}
$$

(ii) First let us prove that the symmetric bilinear form

$$
Q(f,g) := \int_t^T \int_s^T P_{s-t}(f_s P_{r-s} g_r)\, dr\, ds + \int_t^T \int_s^T P_{s-t}(g_s P_{r-s} f_r)\, dr\, ds
$$

is non-negative.

$$
\begin{aligned}
& \int_t^T \int_s^T P_{s-t}(f_s P_{r-s} f_r)(x)\, dr\, ds \\
= \quad & \int_t^T \int_s^T P_{s-t}(f_s(x) E_x[f_r(X_{r-s})])\, dr\, ds \\
= \quad & \int_t^T \int_s^T E_x\left[f_s(X_{s-t}) E_{X_{s-t}}[f_r(X_{r-s})]\right] dr\, ds \\
= \quad & \int_t^T \int_s^T E_x\left[f_s(X_{s-t}) E_x[f_r(X_{r-t})|\mathcal{F}_{s-t}]\right] dr\, ds \\
= \quad & \int_t^T \int_s^T E_x\left[f_s(X_{s-t}) f_r(X_{r-t})\right] dr\, ds \\
= \quad & \frac{1}{2} E_x\left[\int_t^T f_s(X_{s-t}) ds \int_t^T f_r(X_{r-t}) dr\right] \\
\geq \quad & 0
\end{aligned}
$$

Then it is easy to see that

$$
\begin{aligned}
& \int_t^T \int_s^T P_{s-t}[(f_s+g_s)P_{r-s}(f_r+g_r)]\,dr\,ds \\
= {} & \int_t^T \int_s^T P_{s-t}(f_s P_{r-s} g_r)\,dr\,ds + \int_t^T \int_s^T P_{s-t}(g_s P_{r-s} f_r)\,dr\,ds \\
& + \int_t^T \int_s^T P_{s-t}(f_s P_{r-s} f_r)\,dr\,ds + \int_t^T \int_s^T P_{s-t}(g_s P_{r-s} g_r)\,dr\,ds \\
= {} & Q(f,g) \\
& + \int_t^T \int_s^T P_{s-t}(f_s P_{r-s} f_r)\,dr\,ds + \int_t^T \int_s^T P_{s-t}(g_s P_{r-s} g_r)\,dr\,ds \\
\leq {} & \frac{1}{2}(Q(f,f)+Q(g,g)) \\
& + \int_t^T \int_s^T P_{s-t}(f_s P_{r-s} f_r)\,dr\,ds + \int_t^T \int_s^T P_{s-t}(g_s P_{r-s} g_r)\,dr\,ds \\
= {} & 2\int_t^T \int_s^T P_{s-t}(f_s P_{r-s} f_r)\,dr\,ds + 2\int_t^T \int_s^T P_{s-t}(g_s P_{r-s} g_r)\,dr\,ds.
\end{aligned}
$$

□

Finally, the next lemma presents an upper estimate for a solution u. We follow the idea of [BPS05, Lemma 3.3]. Note that in the framework of [BPS05] $\mathcal{E}$ is conservative and hence it is possible to deduce an estimate for $\|u\|_T$ from the first equation of the following lemma. Since generally our Dirichlet form is not conservative, we have proved a $\|\cdot\|_T$ estimate directly in Lemma 4.10.

Lemma 4.18. *Assume that u is a solution of (4.1) such that the conditions (H1) and (H2') hold. Then there exists a constant $\tilde{K}$, which depends on C, μ and T such that*

$$
\begin{aligned}
(i) \qquad & |u_t|^2 + 2\int_t^T P_{s-t}\left(|D_\sigma u_s|^2 + \frac{1}{2}c|u_s|^2\right) ds \\
\leq {} & \tilde{K}\left(P_{T-t}|\phi|^2 + \int_t^T \int_s^T P_{s-t}(|f_s^0| P_{r-s}|f_r^0|)\,dr\,ds\right).
\end{aligned}
$$

Moreover, there exists a constant K, which depends on C,μ and T such that

$$(ii) \qquad \|u\|_\infty \leq K(\|\phi\|_\infty + \|f^0\|_\infty).$$

Proof. Writing relation (i) from Corollary 4.15 for the solution u we get

$$
\begin{aligned}
& |u_t|^2 + 2\int_t^T P_{s-t}\left(|D_\sigma u_s|^2 + \frac{1}{2}c|u_s|^2\right) ds \qquad (4.26) \\
= {} & P_{T-t}|\phi|^2 + 2\int_t^T P_{s-t}\langle u_s, f_s(u_s, D_\sigma u_s)\rangle\,ds.
\end{aligned}
$$

By (H1) and (H2') we deduce

$$\begin{aligned}\langle u_s, f_s(u_s, D_\sigma u_s)\rangle &\underset{(H2')}{\leq} \langle u_s, f_s(u_s, D_\sigma u_s) - f'(s,x,u_s,0)\rangle \\ &= \langle u_s, f_s(u_s, D_\sigma u_s) - f(u_s,0) + f_s^0\rangle \\ &= \langle u_s, f_s(u_s, D_\sigma u_s) - f_s(u_s,0)\rangle + \langle u_s, f_s^0\rangle \\ &\underset{(H1)}{\leq} |u_s|(C|D_\sigma u_s| + |f_s^0|)\end{aligned}$$

and therefore by Corollary 4.15 (ii)

$$|u_s| \leq P_{T-s}|\phi| + \int_s^T P_{r-s}(C|D_\sigma u_r| + |f_r^0|)\, dr.$$

Finally, we get

$$\begin{aligned}&\int_t^T P_{s-t}\langle u_s, f_s(u_s, D_\sigma u_s)\rangle\, ds \qquad (4.27)\\ \leq\ &\int_t^T P_{s-t}\left[\left(P_{T-s}|\phi| + \int_s^T P_{r-s}(C|D_\sigma u_r| + |f_r^0|)\, dr\right)(C|D_\sigma u_s| + |f_s^0|)\right] ds.\end{aligned}$$

Note that it holds

$$\begin{aligned}&\int_t^T \int_s^T P_{s-t}\left(|D_\sigma u_s| P_{r-s}|D_\sigma u_r|\right) dr\, ds \\ \leq\ &\int_t^T \int_s^T \frac{1}{2}\left(P_{s-t}\left(|D_\sigma u_s|^2 + (P_{r-s}|D_\sigma u_r|)^2\right)\right) dr\, ds \\ \leq\ &\int_t^T \int_s^T \frac{1}{2}\left(P_{s-t}\left(|D_\sigma u_s|^2 + P_{r-s}|D_\sigma u_r|^2\right)\right) dr\, ds \\ =\ &\int_t^T \int_s^T \frac{1}{2}\left(P_{s-t}|D_\sigma u_s|^2 + P_{s-t}P_{r-s}|D_\sigma u_r|^2\right) dr\, ds \\ =\ &\int_t^T \int_s^T \frac{1}{2}\left(P_{s-t}|D_\sigma u_s|^2\right) dr\, ds + \int_t^T \int_s^T \frac{1}{2}\left(P_{r-t}|D_\sigma u_r|^2\right) dr\, ds \\ \leq\ &(T-t)\int_t^T P_{s-t}|D_\sigma u_s|^2\, ds.\end{aligned}$$

We use the relations (i) and (ii) from Lemma 4.16 and the equations (4.26) and (4.27) to obtain

$$
\begin{aligned}
& |u_t|^2 + 2\int_t^T P_{s-t}\left(|D_\sigma u_s|^2 + \frac{1}{2}c|u_s|^2\right) ds \\
\le\ & P_{T-t}|\phi|^2 + 2\Big(\int_t^T P_{s-t}\Big[\Big(P_{T-s}|\phi| \\
& \qquad + \int_s^T P_{r-s}(C|D_\sigma u_r| + |f_r^0|)\, dr\Big)(C|D_\sigma u_s| + |f_s^0|)\Big]\, ds\Big) \\
=\ & P_{T-t}|\phi|^2 + 2\left[\underbrace{\int_t^T P_{s-t}(P_{T-s}|\phi| C|D_\sigma u_s|)\, ds}_{\le \frac{1}{2}(\frac{1}{2}P_{T-t}|\phi|^2 + C^2\int_t^T\int_s^T P_{s-t}(|D_\sigma u_s| P_{r-s}|D_\sigma u_r|)\, dr\, ds)}\right] \\
& +2\left[\underbrace{\int_t^T P_{s-t}(P_{T-s}|\phi||f_s^0|)\, ds}_{\le \frac{1}{2}(\frac{1}{2}P_{T-t}|\phi|^2 + \int_t^T\int_s^T P_{s-t}(|f_s^0| P_{r-s}|f_r^0|)\, dr\, ds)}\right] \\
& +2\left[\underbrace{\int_t^T P_{s-t}\left[\int_s^T P_{r-s}(C|D_\sigma u_r| + |f_r^0|)\, dr \cdot (|f_s^0| + C|D_\sigma u_s|)\right] ds}_{=\int_t^T\int_s^T P_{s-t}[P_{r-s}(C|D_\sigma u_r| + |f_r^0|)\cdot(|f_s^0| + C|D_\sigma u_s|)]\, dr\, ds}\right] \\
\underset{(i)}{\le}\ & 2P_{T-t}|\phi|^2 + C^2\int_t^T\int_s^T P_{s-t}(|D_\sigma u_s| P_{r-s}|D_\sigma u_r|)\, dr\, ds \\
& + \int_t^T\int_s^T P_{s-t}(|f_s^0| P_{r-s}|f_r^0|)\, dr\, ds \\
& +2\int_t^T\int_s^T P_{s-t}\left[P_{r-s}(C|D_\sigma u_r| + |f_r^0|)\cdot(|f_s^0| + C|D_\sigma u_s|)\right] dr\, ds \\
\underset{(ii)}{\le}\ & 2P_{T-t}|\phi|^2 + 5C^2\int_t^T\int_s^T P_{s-t}(|D_\sigma u_s| P_{r-s}|D_\sigma u_r|)\, dr\, ds \\
& +5\int_t^T\int_s^T P_{s-t}(|f_s^0| P_{r-s}|f_r^0|)\, dr\, ds.
\end{aligned}
$$

Summarized we get

$$
\begin{aligned}
& |u_t|^2 + 2\int_t^T P_{s-t}\left(|D_\sigma u_s|^2 + \frac{1}{2}c|u_s|^2\right) ds \\
\le\ & 2P_{T-t}|\phi|^2 + 5C^2(T-t)\int_t^T P_{s-t}|D_\sigma u_s|^2\, ds \\
& +5\int_t^T\int_s^T P_{s-t}(|f_s^0| P_{r-s}|f_r^0|)\, dr\, ds.
\end{aligned}
$$

Hence, the first estimate of this lemma holds on the interval $[T-\varepsilon, T]$ where $\varepsilon > 0$ such that $C^2\varepsilon 5 = 1$. It is easy to see that we can deduce by iteration the estimate over the interval $[0,T]$.

Left to show is the upper estimate for u. We obtain from the first estimate:

$$\begin{aligned}
|u_t|^2 &\le \sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}\tilde{K}\left(P_{T-t}|\phi|^2 + \underbrace{\int_t^T\int_s^T P_{s-t}(|f_s^0|P_{r-s}|f_r^0|)\,dr\,ds}_{\le (T-t)\int_t^T P_{s-t}|f_s^0|^2\,ds}\right)\\
&\le \sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}\left(\tilde{K}P_{T-t}|\phi|^2 + \tilde{K}(T-t)\int_t^T P_{s-t}|f_s^0|^2\,ds\right)\\
&\le \sup_{t\in[0,T]}\left(\tilde{K}\|\phi^2\|_\infty + \tilde{K}(T-t)\int_t^T \sup_{x\in\mathbb{R}^d}|P_{s-t}|f_s^0|^2|\,ds\right)\\
&\le \tilde{K}\|\phi^2\|_\infty + \tilde{K}T\int_0^T \sup_{x\in\mathbb{R}^d}||f_s^0|^2|\,ds\\
&\le \underbrace{\max(\tilde{K},\tilde{K}T^2)}_{:=K^2}(\|\phi^2\|_\infty + \|f^0\|_\infty^2)\\
&\le K^2(\|\phi\|_\infty^2 + \|f^0\|_\infty^2).
\end{aligned}$$

Hence, we have

$$|u_t| \le K\left(\sqrt{\|\phi\|_\infty^2 + \|f^0\|_\infty^2}\right) \le K\left(\|\phi\|_\infty + \|f^0\|_\infty\right).$$

Finally,

$$\|u\|_\infty \le K\left(\|\phi\|_\infty + \|f^0\|_\infty\right).$$

□

4.4.3 The Existence and Uniqueness Theorem

We consider the conditions (A1)-(A4). The next theorem follows the lines of arguments of [BPS05, Theorem 3.2] in the case $\rho > 0$.

Theorem 4.19. *Under the conditions (H1)-(H5) and $\rho > 0$ there exists a unique solution of equation* (4.1)*. It satisfies the following estimates with constants K_1 and K_2*

$$\begin{aligned}
\|u\|_T^2 &\le K_1\left(\|\phi\|_2 + \int_0^T \|f_t^0\|_2\,dt\right),\\
\|u\|_\infty &\le K_2\left(\|\phi\|_\infty + \|f^0\|_\infty\right)
\end{aligned}$$

where K_1 depends only on the constants C,μ,T,C_A,K_A and K_2 only on C,μ,T.

Proof. [**Uniqueness**]
Let u_1 and u_2 be two solutions of equation (4.1). By using (3.5) for the difference

$u_1 - u_2$ we get

$$\begin{aligned}
& \|u_{1,t} - u_{2,t}\|_2^2 + 2\int_t^T \mathcal{E}(u_{1,t} - u_{2,t})\,ds \\
\underset{(3.5)}{=} \quad & 2\int_t^T (f(s,\cdot,u_{1,s},D_\sigma u_{1,s}) - f(s,\cdot,u_{2,s},D_\sigma u_{2,s}), u_{1,s} - u_{2,s})\,ds \\
\underset{(H1),(H2)}{\leq} \quad & 2\int_t^T C(|D_\sigma u_{1,s} - D_\sigma u_{2,s}|, |u_{1,s} - u_{2,s}|)\,ds \\
\leq \quad & C^2 \int_t^T \|u_{1,s} - u_{2,s}\|_2^2\,ds + \int_t^T \underbrace{\mathcal{E}^A(u_{1,s} - u_{2,s})}_{\leq K_A \mathcal{E}(u_{1,s}-u_{2,s}) + C_A \|u_{1,s}-u_{2,s}\|_2^2}\,ds
\end{aligned}$$

and therefore

$$\begin{aligned}
& \|u_{1,t} - u_{2,t}\|_2^2 \\
\leq \quad & (C^2 + C_A)\int_t^T \|u_{1,s} - u_{2,s}\|_2\,ds + \underbrace{(K_A - 2)\int_t^T \mathcal{E}(u_{1,s} - u_{2,s})\,ds}_{\leq 0} \\
\leq \quad & (C^2 + C_A)\int_t^T \|u_{1,s} - u_{2,s}\|_2\,ds.
\end{aligned}$$

By Gronwall's lemma it follows that

$$\|u_{1,t} - u_{2,t}\|_2^2 \leq 0 \cdot \exp((C^2 + C_A)T).$$

This implies $\|u_{1,t} - u_{2,t}\|_2 = 0$ for all $t \in [0,T]$ and hence $u_1 = u_2$.

[**Existence**]
The existence will be proved in four steps.

[**Step 1:**]
We suppose the existence of $r \in \mathbb{R}$ such that

$$r \geq 1 + K(\|\phi\|_\infty + \|f^0\|_\infty + \|f'^{,1}\|_\infty)$$

where K is the constant appearing in Lemma 4.18(ii), and such that f is uniformly bounded on the set

$$A_r = [0,T] \times \mathbb{R}^d \times \{|y| \leq r\} \times \mathbb{R}^l \otimes \mathbb{R}^k.$$

We define

$$M := \sup\{|f(t,x,y,z)| : (t,x,y,z) \in A_r\} < \infty.$$

Next we will regular f with respect to the variable y. Let us define

$$f_n(t,x,y,z) := n^l \int_{\mathbb{R}^l} f(t,x,y',z)\varphi(n(y-y'))\,dy'$$

where $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ of support contained in $\{|y| \leq 1\}$ such that $\int_{\mathbb{R}^l} \varphi\,dm = 1$. Then it holds (cf. [Kan03, Theorem II.1.2])

$$f = \lim_{n\to\infty} f_n.$$

We set

$$h_n(t,x,y,z) := f_n\Big(t,x,\underbrace{\frac{r-1}{|y|\vee(r-1)}}_{\leq 1}y,z\Big)$$

and denote $\tilde{y_h} := \frac{r-1}{|y|\vee(r-1)}y$. The derivatives $\partial_{y_i} f_n$ satisfy (cf. [Alt06, 2.12 Faltung $\langle 4\rangle$])

$$\begin{aligned}\partial_{y_i} f_n(t,x,y,z) &= \int_{\mathbb{R}^l} f(t,x,y',z)n^l\partial_{y_i}\varphi(n(y-y'))\,dy' \\ &= \int_{\mathbb{R}^l} f(t,x,y-\frac{y'}{n},z)\partial_{y_i}\varphi(y')\,dy'.\end{aligned}$$

Thus, we deduce for $|y| \leq r-1$

$$\partial_{y_i} f_n(t,x,y,z) \quad\leq\quad M\int_{\mathbb{R}^l}\partial_{y_i}\varphi(y')\,dy'.$$

This shows that the partial derivatives $\partial_{y_i} f_n$ are uniformly bounded on A_{r-1} for each n. Since $f_n = h_n$ on A_{r-1}, the partial derivatives $\partial_{y_i} h_n$ are also uniformly bounded on A_{r-1}. Hence, the functions h_n satisfy the Lipschitz condition with respect to y and z. Thus, by Proposition 4.8 each h_n determines a solution $u_n \in \hat{F}^l$ with data (ϕ, h_n).

Now we will show that f_n satisfies $(H1)$ and $(H2')$:

$$\begin{aligned}&|f_n(t,x,y,z) - f_n(t,x,y,\tilde{z})| \\ =\;& n^l\int_{\mathbb{R}^l}\left[f(t,x,y',z) - f(t,x,y',\tilde{z})\right]\varphi(n(y-y'))\,dy' \\ \underset{(H1)}{\leq}\;& C|z-\tilde{z}|n^l\int_{\mathbb{R}^l}\varphi(n(y-y'))\,dy' \\ =\;& C|z-\tilde{z}|,\end{aligned}$$

$$\begin{aligned}&\langle y, f_n'(t,x,y)\rangle \\ =\;& \langle y, f_n(t,x,y,0) - f_n(t,x,0,0)\rangle \\ =\;& \langle y, n^l\int_{\mathbb{R}^l} f(t,x,y',0)\varphi(n(y-y'))\,dy' - n^l\int_{\mathbb{R}^l} f(t,x,y',0)\varphi(n(-y'))dy'\rangle \\ =\;& \langle y, \int_{\mathbb{R}^l}\Big(f(t,x,y-\frac{y'}{n},0) - f(t,x,0-\frac{y'}{n},0)\Big)\varphi(y')\,dy'\rangle \\ =\;& \int_{\mathbb{R}^l}\Big\langle\Big(y-\frac{y'}{n}\Big) - \Big(-\frac{y'}{n}\Big), \Big(f(t,x,y-\frac{y'}{n},0) - f(t,x,-\frac{y'}{n},0)\Big)\Big\rangle\varphi(y')\,dy' \\ \underset{(H2)}{\leq}\;& 0.\end{aligned}$$

Therefore, it is easy to see that h_n also satisfies $(H1)$ and $(H2')$ with the same constants ($C > 0$ and $\mu = 0$):

$$|h_n(t,x,y,z) - h_n(t,x,y,z')| \quad=\quad |f_n(t,x,\tilde{y_h},z) - f_n(t,x,\tilde{y_h},z')| \leq C|z-z'|,$$

$$\begin{aligned}\langle y, h_n'(t,x,y)\rangle &= \frac{|y|\vee(r-1)}{r-1}\langle \tilde{y}_h, h_n'(t,x,y)\rangle\\ &= \frac{|y|\vee(r-1)}{r-1}\langle \tilde{y}_h, f_n'(t,x,\tilde{y}_h)\rangle\\ &\leq 0.\end{aligned}$$

Since

$$\begin{aligned}|h_n(t,x,0,0)| &= |f_n(t,x,0,0)|\\ &\leq n^l\int_{\mathbb{R}^l}|f(t,x,y',0)-f^0(t,x)+f^0(t,x)||\varphi(n(-y'))|\,dy'\\ &\leq |f^0(t,x)|+f^{\prime,1}(t,x),\end{aligned}$$

we deduce by Lemma 4.18 that $\|u_n\|_\infty \leq r-1$ and by Lemma 4.10 that there exists a constant K_T such that $\|u_n\|_T \leq K_T$. This can be derived as follows: Let us denote the constant that appears in Lemma 4.18(ii) by K_1 and the constant from Lemma 4.10 by K_2.

$$\begin{aligned}\|u_n\|_\infty &\leq K_1(\|\phi\|_\infty+\|h_n^0\|_\infty)\\ &\leq K_1(\|\phi\|_\infty+\|f^0\|_\infty+\|f^{\prime,1}(t,x)\|_\infty)\leq r-1,\\ \|u_n\|_T^2 &\leq K_2\left(\|\phi\|_2+\int_0^T\|h_n^0\|_2\,dt\right)\\ &\leq K_2\left(\|\phi\|_2+\int_0^T(\|f^0\|_2+\|f^{\prime,1}\|_2)\,dt\right)=:K_T^2\underset{(H4),(H5)}{<}\infty\end{aligned}$$

Since by definition $h_n=f_n$ on A_{r-1}, it follows that u_n satisfies (4.1) in the weak sense with data (ϕ,f_n).

For $b>0$ we define

$$d_{n,b}(t,x):=\sup_{|y|\leq r-1,|z|\leq b}|f(t,x,y,z)-f_n(t,x,y,z)|.$$

It holds for $|y|\leq r-1$

$$\begin{aligned}f_n(t,x,y,z) &= n^l\int_{\mathbb{R}^l}f(t,x,y',z)\varphi(n(y-y'))\,dy'\\ &\leq Mn^l\int_{\{|y'|\leq r\}}\varphi(n(y-y'))\,dy'=M.\end{aligned}$$

Hence, we deduce that $|d_{n,b}(t,x)|\leq 2M$. Moreover, since we have y-continuity and uniform z-continuity of f, we obtain that for fixed t,x and b the family of functions $\{f(t,x,\cdot,z)||z|\leq b\}$ is equicontinuous, and hence by Arzela Ascoli's theorem compact in $\mathcal{C}(\{|y|\leq r-1\})$. Since the fact that convolution operators approach the identity uniformly on compact sets (cf. [Kan03, Theorem II.1.2.(2)]), we get

$$\lim_{n\to\infty}d_{n,b}(t,x)=0.$$

Therefore, by Lebesgue's theorem it follows that $\lim_{n\to\infty} d_{n,b} = 0$ in $L^2(\,dt\times m)$. Moreover, it holds for $u \in \hat{F}^l, |u| \le r-1$

$$\begin{aligned} |f(u, D_\sigma u) - f_n(u, D_\sigma u)| &\le \mathbb{1}_{|D_\sigma u|\le b} d_{n,b} + 2M\mathbb{1}_{|D_\sigma u|>b} \\ &\le d_{n,b} + \frac{2M}{b}|D_\sigma u|. \end{aligned} \tag{4.28}$$

Next we will show that $(u_n)_{n\in\mathbb{N}}$ is a $\|\cdot\|_T$-Cauchy-sequence. Let us start by writing relation (3.5) for the difference $u_l - u_n$.

$$\begin{aligned}
&\|u_{l,t} - u_{n,t}\|_2^2 + 2\int_t^T \mathcal{E}(u_{l,s} - u_{n,s})\,ds \\
\underset{(3.5)}{=}\; & 2\int_t^T (f_l(s,\cdot,u_{l,s},D_\sigma u_{l,s}) - f_n(s,\cdot,u_{n,s},D_\sigma u_{n,s}), u_{l,s} - u_{n,s})\,ds \\
\le\; & 2\int_t^T (|f_l(u_{l,s},D_\sigma u_{l,s}) - f(u_{l,s},D_\sigma u_{l,s})|, |u_{l,s} - u_{n,s}|)\,ds \\
& +2\int_t^T (|f(u_{n,s},D_\sigma u_{n,s}) - f_n(u_{n,s},D_\sigma u_{n,s})|, |u_{l,s} - u_{n,s}|)\,ds \\
& +2\int_t^T (|f(u_{l,s},D_\sigma u_{l,s}) - f(u_{l,s},D_\sigma u_{n,s})|, |u_{l,s} - u_{n,s}|)\,ds \\
& +2\int_t^T (f(u_{l,s},D_\sigma u_{n,s}) - f(u_{n,s},D_\sigma u_{n,s}), u_{l,s} - u_{n,s})\,ds \\
\underset{(H2)}{\le}\; & 2\int_t^T (|f_l(u_{l,s},D_\sigma u_{l,s}) - f(u_{l,s},D_\sigma u_{l,s})|, |u_{l,s} - u_{n,s}|)\,ds \\
& +2\int_t^T (|f(u_{n,s},D_\sigma u_{n,s}) - f_n(u_{n,s},D_\sigma u_{n,s})|, |u_{l,s} - u_{n,s}|)\,ds \\
& +2\int_t^T (|f(u_{l,s},D_\sigma u_{l,s}) - f(u_{l,s},D_\sigma u_{n,s})|, |u_{l,s} - u_{n,s}|)\,ds \\
\underset{(H1)}{\le}\; & 2\int_t^T (|f_l(u_{l,s},D_\sigma u_{l,s}) - f(u_{l,s},D_\sigma u_{l,s})|, |u_{l,s} - u_{n,s}|)\,ds \\
& +2\int_t^T (|f(u_{n,s},D_\sigma u_{n,s}) - f_n(u_{n,s},D_\sigma u_{n,s})|, |u_{l,s} - u_{n,s}|)\,ds \\
& +2\int_t^T C(|D_\sigma u_{l,s} - D_\sigma u_{n,s}|, |u_{l,s} - u_{n,s}|)\,ds \\
\underset{(4.28)}{\le}\; & 2\int_t^T (d_{l,b}(s,\cdot) + d_{n,b}(s,\cdot), |u_{l,s} - u_{n,s}|)\;ds \\
& +2\int_t^T \left(\frac{2M}{b}(|D_\sigma u_{l,s}| + |D_\sigma u_{n,s}|), |u_{l,s} - u_{n,s}|\right)\,ds \\
& +2\int_t^T C(|D_\sigma u_{l,s} - D_\sigma u_{n,s}|, |u_{l,s} - u_{n,s}|)\,ds
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \int_t^T \|d_{l,b}(s,\cdot)\|_2^2\,ds + \int_t^T \|d_{n,b}(s,\cdot)\|_2^2\,ds \\
& + \frac{1}{b^2}\int_t^T (\|D_\sigma u_{l,s}\|_2^2 + \|D_\sigma u_{n,s}\|_2^2)\,ds \\
& + (1+4M^2+C^2)\int_t^T \|u_{l,s}-u_{n,s}\|_2^2\,ds \\
& + \int_t^T \|D_\sigma u_{l,s} - D_\sigma u_{n,s}\|_2^2\,ds
\end{aligned}
$$

Since we have $\|u_n\|_T \leq K_T$ for all n, we deduce

$$\int_0^T \|D_\sigma u_{l,s}\|_2^2\,ds < K_T$$

where the constant K_T is independent of l and b. Thus, for b, l, n large enough, we get for an arbitrary $\varepsilon > 0$

$$\|u_{l,t}-u_{n,t}\|_2^2 + \int_t^T \mathcal{E}(u_{l,s}-u_{n,s})\,ds \leq \frac{\varepsilon}{2-K_A} + \tilde{K}\int_t^T \|u_{l,s}-u_{n,s}\|_2^2\,ds$$

where $\tilde{K}$ depends on C, M , C_A , μ and K_A. It is easy to see that Gronwall's lemma implies that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence. Let us define the $\|\cdot\|_T$-limit $u := \lim_{n\to\infty} u_n$ and take a subsequence $(n_k)_{k\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ such that $u_{n_k} \to u$ a.e..

Now we show that u is a solution of (4.1) associated to (ϕ, f). Since $u_{n_k} \to u$ a.e., it follows

$$f(\cdot,\cdot,u_{n_k},D_\sigma u) \to f(\cdot,\cdot,u,D_\sigma u) \text{ in } L^2(\,dt\times m)$$

by the following calculation:

$$
\begin{aligned}
& \lim_{k\to\infty}\int_t^T\int_{\mathbb{R}^l} \underbrace{|f(\cdot,\cdot,u_{n_k},D_\sigma u) - f(\cdot,\cdot,u,D_\sigma u)|^2}_{\leq 4M^2}\,dm\,dt \\
\underset{Lebesgue}{=}\ & \int_t^T\int_{\mathbb{R}^l} \lim_{k\to\infty} |f(\cdot,\cdot,u_{n_k},D_\sigma u) - f(\cdot,\cdot,u,D_\sigma u)|^2\,dm\,dt \\
\underset{(H3)}{=}\ & 0.
\end{aligned}
$$

Since $\|u_{n_k} - u\|_T \to 0$, we obtain by $(A2)$ that

$$\|D_\sigma u - D_\sigma u_{n_k}\|_{L^2(\,dt\times m)} \to 0.$$

Then by (H1) it follows that

$$
\begin{aligned}
& \lim_{k\to\infty} \|f(\cdot,\cdot,u_{n_k},D_\sigma u) - f(\cdot,\cdot,u_{n_k},D_\sigma u_{n_k})\|_{L^2(\,dt\times m)} \\
\underset{(H1)}{\leq}\ & \lim_{k\to\infty} C\|D_\sigma u - D_\sigma u_{n_k}\|_{L^2(\,dt\times m)} \\
=\ & 0.
\end{aligned}
$$

By using (4.28) and passing to the limit first in k and then in b we get

$$\begin{aligned}
& \|f(\cdot,\cdot,u_{n_k},D_\sigma u_{n_k}) - f_{n_k}(\cdot,\cdot,u_{n_k},D_\sigma u_{n_k})\|_{L^2(\,dt\times m)} \\
\leq\ & \left\| d_{n_k,b} + \frac{2M}{b}|D_\sigma u_{n_k}| \right\|_{L^2(\,dt\times m)} \\
\leq\ & \|d_{n_k,b}\|_{L^2(\,dt\times m)} + \frac{2M}{b}\|D_\sigma u_{n_k}\|_{L^2(\,dt\times m)} \\
\leq\ & \|d_{n_k,b}\|_{L^2(\,dt\times m)} + \frac{2M}{b}\sqrt{K_T} \\
\to\ & 0.
\end{aligned}$$

Finally, we conclude

$$\begin{aligned}
& \lim_{k\to\infty} \|f_{n_k}(u_{n_k},D_\sigma u_{n_k}) - f(u,D_\sigma u)\|_{L^2(\,dt\times m)} \\
\leq\ & \lim_{k\to\infty} \|f_{n_k}(u_{n_k},D_\sigma u_{n_k}) - f(u_{n_k},D_\sigma u_{n_k})\|_{L^2(\,dt\times m)} \\
& + \lim_{k\to\infty} \|f(u_{n_k},D_\sigma u_{n_k}) - f(u_{n_k},D_\sigma u)\|_{L^2(\,dt\times m)} \\
& + \lim_{k\to\infty} \|f(u_{n_k},D_\sigma u) - f(u,D_\sigma u)\|_{L^2(\,dt\times m)} \\
=\ & 0.
\end{aligned}$$

By passing to the limit in the weak equation associated to u_{n_k} with data (ϕ, f_{n_k}), it follows that u is the solution associated to (ϕ, f).

[**Step 2:**]
In this step we will prove the assertion under the assumption that there exists some constant r such that $f'^{,r}$ is uniformly bounded and

$$r \geq 1 + K(\|\phi\|_\infty + \|f^0\|_\infty + \|f'^{,1}\|_\infty)$$

where K is the constant appearing in Lemma 4.18(ii). Let us define

$$f_n(t,x,y,z) := f\Big(t,x,y,\underbrace{\frac{n}{|z|\vee n}}_{\leq 1} z\Big), \qquad n \in \mathbb{N}\backslash\{0\}.$$

Since it holds for $|y| \leq r$

$$\begin{aligned}
|f_n| &= \left| f\left(t,x,y,\frac{n}{|z|\vee n} z\right) + f(t,x,y,0) - f(t,x,y,0) - f^0 + f^0 \right| \\
&\underset{(H1)}{\leq} Cn + \|f'^{,r}\|_\infty + \|f^0\|_\infty,
\end{aligned}$$

f_n is bounded on A_r by $Cn + \|f'^{,r}\|_\infty + \|f^0\|_\infty$. It is easy to see that each of the functions f_n satisfies the same conditions as f. Hence, we apply step 1 and obtain the existence of a solution u_n associated to the data (ϕ, f_n). By Lemma 4.18 we get

$$\|u_n\|_\infty \leq K(\|\phi\|_\infty + \|\underbrace{f_n^0}_{=f^0}\|_\infty) \leq r - 1$$

and by Lemma 4.10

$$\|u_n\|_T^2 \leq K\left(\|\phi\|_2 + \int_0^T \|f_t^0\|_2\,dt\right) \leq K_T$$

where $K_T \in \mathbb{R}_+$ fix. The conditions (H1) and (H2) yield

$$\begin{aligned}
& |(f_l(u_l, D_\sigma u_l) - f_n(u_n, D_\sigma u_n), u_l - u_n)| \qquad (4.29)\\
\underset{(H2)}{\leq}\ & |(f_l(u_l, D_\sigma u_l) - f_l(u_l, D_\sigma u_n) + f_l(u_n, D_\sigma u_n) - f_n(u_n, D_\sigma u_n), u_l - u_n)|\\
\underset{(H1)}{\leq}\ & C(|D_\sigma u_l - D_\sigma u_n|, |u_l - u_n|)\\
& +|(f_l(u_n, D_\sigma u_n) - f_n(u_n, D_\sigma u_n), u_l - u_n)|.
\end{aligned}$$

By the relations for $n \leq l$

$$f_n(t,x,y,z)\mathbb{1}_{|z|\leq n} \underset{\text{def.}}{=} f(t,x,y,z)\mathbb{1}_{|z|\leq n}$$

and

$$\begin{aligned}
& |f_l(t,x,y,z) - f_n(t,x,y,z)|\mathbb{1}_{|z|\geq n}\\
=\ & |f_l(t,x,y,z) - f(t,x,y,0) + f(t,x,y,0) - f_n(t,x,y,z)|\mathbb{1}_{|z|\geq n}\\
\leq\ & 2C|z|\mathbb{1}_{|z|\geq n}
\end{aligned}$$

we conclude that

$$|(f_l(u_n, D_\sigma u_n) - f_n(u_n, D_\sigma u_n), u_l - u_n)| \leq |\left(2C|D_\sigma u_n|\mathbb{1}_{\{|D_\sigma u_n|\geq n\}}, |u_l - u_n|\right)|.$$

Next we will show that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

$$\begin{aligned}
& \|u_l - u_n\|_2^2 + 2\int_t^T \mathcal{E}(u_l - u_n)\,ds\\
\underset{(3.5)}{=}\ & 2\int_t^T (f_l(u_l, D_\sigma u_l) - f_n(u_n, D_\sigma u_n), u_l - u_n)\,ds\\
\underset{(4.29)}{\leq}\ & 2\int_t^T C(|D_\sigma u_l - D_\sigma u_n|, |u_l - u_n|)\,ds\\
& +2\int_t^T \underbrace{|(f_l(u_n, D_\sigma u_n) - f_n(u_n, D_\sigma u_n), u_l - u_n)|}_{\leq(2C|u_l - u_n|, ||D_\sigma u_n|\mathbb{1}_{\{|D_\sigma u_n|\geq n\}}|)}\,ds
\end{aligned}$$

$$
\begin{aligned}
\underset{\|u_n\|_\infty \leq r-1}{\leq} \quad & C^2 \int_t^T \|u_l - u_n\|_2^2 \, ds + \int_t^T \mathcal{E}^A(u_l - u_n) \, ds \\
& + 8C(r-1) \int_t^T \int_{\mathbb{R}^d} |D_\sigma u_n| \mathbb{1}_{\{|D_\sigma u_n| \geq n\}} \, dm \, ds \\
\underset{(A2)}{\leq} \quad & (C^2 + C_A) \int_t^T \|u_l - u_n\|_2^2 \, ds + K_A \int_t^T \mathcal{E}(u_l - u_n) \, ds \\
& + 8C(r-1) \int_t^T \left[\left(\int_{\mathbb{R}^d} \mathbb{1}^2{}_{|D_\sigma u_n| \geq n} \, dm \right)^{\frac{1}{2}} \right. \\
& \quad \left. \cdot \left(\int_{\mathbb{R}^d} |D_\sigma u_n|^2 \, dm \right)^{\frac{1}{2}} \right] ds \\
\leq \quad & (C^2 + C_A) \int_t^T \|u_l - u_n\|_2^2 \, ds + K_A \int_t^T \mathcal{E}(u_l - u_n) \, ds \\
& + 8C(r-1) \left(\int_t^T \int_{\mathbb{R}^d} \mathbb{1}^2{}_{|D_\sigma u_n| \geq n} \, dm \, ds \right)^{\frac{1}{2}} \\
& \quad \cdot \left(\int_t^T \int_{\mathbb{R}^d} |D_\sigma u_n|^2 \, dm \, ds \right)^{\frac{1}{2}} \\
= \quad & (C^2 + C_A) \int_t^T \|u_l - u_n\|_2^2 \, ds + K_A \int_t^T \mathcal{E}(u_l - u_n) \, ds \\
& + 8C(r-1) \left(\int_t^T \|\mathbb{1}_{\{|D_\sigma u_n| \geq n\}}\|_2^2 \, ds \right)^{\frac{1}{2}} \left(\int_t^T \|D_\sigma u_n\|_2^2 \, ds \right)^{\frac{1}{2}}
\end{aligned}
$$

Since $\|u_n\|_T^2 \leq K_T$ for all n, we have $\int_0^T \|D_\sigma u_{s,n}\|_2^2 \, ds \leq K_T$ independent of n. Hence,

$$
n^2 \int_t^T \|\mathbb{1}_{\{|D_\sigma u_n| \geq n\}}\|_2^2 \, ds \quad \leq \quad \int_t^T \||D_\sigma u_n| \mathbb{1}_{\{|D_\sigma u_n| \geq n\}}\|_2^2 \, ds \leq K_T.
$$

Therefore, we conclude for n big enough

$$
\|u_l - u_n\|^2 + (2 - K_A) \int_t^T \mathcal{E}(u_l - u_n) \, ds \leq (C^2 + C_A) \int_t^T \|u_l - u_n\|_2^2 \, ds + \varepsilon.
$$

By Gronwalls' lemma it is easy to see that $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. Hence, the $\| \cdot \|_T$-limit $u := \lim_{n \to \infty} u_n$ is well defined. Now we can find a subsequence $(n_k)_{k \in \mathbb{N}}$ of $(n)_{n \in \mathbb{N}}$ such that

$$
(u_{n_k}, D_\sigma u_{n_k}) \underset{k \to \infty}{\longrightarrow} (u, D_\sigma u) \quad \text{a.e.}
$$

and conclude a.e.

$$
\begin{aligned}
& |f_{n_k}(u_{n_k}, D_\sigma u_{n_k}) - f(u, D_\sigma u)| \\
= \quad & \left| f\left(u_{n_k}, \frac{n_k}{|D_\sigma u_{n_k}| \vee n_k} D_\sigma u_{n_k}\right) - f(u, D_\sigma u)\right| \\
\leq \quad & \left| f\left(u_{n_k}, \frac{n_k}{|D_\sigma u_n| \vee n_k} D_\sigma u_{n_k}\right) - f(u_{n_k}, D_\sigma u)\right| \\
& + |f(u_{n_k}, D_\sigma u) - f(u, D_\sigma u)| \\
\leq \quad & C \left| \frac{n_k}{|D_\sigma u_{n_k}| \vee n_k} D_\sigma u_{n_k} - D_\sigma u\right| + |f(u_{n_k}, D_\sigma u) - f(u, D_\sigma u)| \\
\underset{k\to\infty}{\longrightarrow} \quad & 0.
\end{aligned}
$$

Since

$$
\begin{aligned}
& |f(u, D_\sigma u) - f_{n_k}(u_{n_k}, D_\sigma u_{n_k})| \\
= \quad & |f(u, D_\sigma u) - f(u, 0) - f_n(u_{n_k}, D_\sigma u_{n_k}) + f_{n_k}(u_{n_k}, 0)| \\
& + | - f_{n_k}(u_{n_k}, 0) + f(u, 0) + f^0 - f^0| \\
\leq \quad & |f(u, D_\sigma u) - f(u, 0)| + |f_{n_k}(u_{n_k}, D_\sigma u_{n_k}) - f_{n_k}(u_{n_k}, 0)| \\
& + |f_{n_k}(u_{n_k}, 0) - f^0| + |f(u, 0) - f^0| \\
\leq \quad & C(|D_\sigma u| + |D_\sigma u_{n_k}|) + 2f'^{,r},
\end{aligned}
$$

by Lebesgue's theorem it follows that

$$\lim_{k\to\infty} f_{n_k}(u_{n_k}, D_\sigma u_{n_k}) = f(u, D_\sigma u)$$

in L^1. By passing to the limit in the weak equation we conclude that u is a solution of (4.1) associated to the data (ϕ, f).

[**Step 3:**]
Now we only suppose that $f'^{,1}$ is bounded. Hence, we can choose a constant r such that

$$r \geq 1 + K(\|\phi\|_\infty + \|f^0\|_\infty + \|f'^{,1}\|_\infty)$$

where K is the constant that appears in Lemma 4.18(ii). Let us define

$$f_n := \frac{n}{f'^{,r} \vee n}(f - f^0) + f^0.$$

Easily we see that the functions f_n have the same properties as f. We present for example $(H1)$ in the case $f'^{,r}(t,x) \geq n$:

$$
\begin{aligned}
& |f_n(t,x,y,z) - f_n(t,x,y,z')| \\
\underset{\text{def.}}{=} \quad & \Big| \frac{n}{f'^{,r}(t,x) \vee n}(f(t,x,y,z) - f^0(t,x)) + f^0(t,x) \\
& - \left(\frac{n}{f'^{,r}(t,x) \vee n}(f(t,x,y,z') - f^0(t,x)) + f^0(t,x)\right) \Big| \\
= \quad & \left| \frac{n}{f'^{,r}(t,x) \vee n}(f(t,x,y,z) - f(t,x,y,z'))\right| \\
\underset{(H1)}{\leq} \quad & \underbrace{\frac{n}{f'^{,r}(t,x) \vee n}}_{\leq 1} C|z - z'|.
\end{aligned}
$$

Since $f_n(t,x,y,z) = f(t,x,y,z)$ for $f^{\prime,r} \le n$, we deduce

$$\lim_{n\to\infty} f_n = f.$$

Let us introduce the following notation:

$$f_n^{\prime,r}(t,x) := \sup_{|y|\le r} |f_n^{\prime}(t,x,y)| \text{ and } f_n^{\prime}(t,x,y) := f_n(t,x,y,0) - f^0(t,x).$$

Note that we have

$$f_n(t,x,0,0) = \frac{n}{f^{\prime,r}\vee n}(f(t,x,0,0) - f^0(t,x)) + f^0(t,x) = f^0(t,x).$$

Next we will show that

$$|f_n^{\prime,r}| \le n \wedge |f^{\prime,r}|.$$

If $f^{\prime,r} \le n$, then we have

$$|f_n^{\prime,r}| = \sup_{|y|\le r} |f_n(t,x,y,0) - f^0(t,x)| = f^{\prime,r}$$

and if $f^{\prime,r} > n$, then

$$\begin{aligned} |f_n^{\prime,r}| &= \sup_{|y|\le r} |f_n(t,x,y,0) - f^0(t,x)| \\ &\le \sup_{|y|\le r} \left| \left(\frac{n}{f(t,x,y) - f^0(t,x)} (f(t,x,y) - f^0(t,x)) \right) \right| \\ &< n. \end{aligned}$$

Hence, $f_n^{\prime}$ is uniformly bounded. By the preceding step we obtain that there exists a solution u_n associated to the data (ϕ, f_n) such that by Lemma 4.10 and 4.18 it holds

$$\|u_n\|_\infty \le r-1 \text{ and } \|u_n\|_T \le const.$$

Next we prove the convergence of u_n. We have for $n \le l$

$$\begin{aligned} & |f_l - f_n| \\ = \;& |f - f^0| \left| \frac{l}{f^{\prime,r}\vee l} - \frac{n}{f^{\prime,r}\vee n} \right| \\ = \;& |f(t,x,y,z) - f(t,x,y,0) + f(t,x,y,0) - f(t,x,0,0)| \left| \frac{l}{f^{\prime,r}\vee l} - \frac{n}{f^{\prime,r}\vee n} \right| \\ \le \;& (C|z| + |f^{\prime}|) \left| \frac{l}{f^{\prime,r}\vee l} - \frac{n}{f^{\prime,r}\vee n} \right| \\ \le \;& (C|z| + |f^{\prime}|) \mathbb{1}_{\{f^{\prime,r} > n\}}. \end{aligned}$$

Hence it holds

$$\begin{aligned} & \int_t^T |(f_l(u_n, D_\sigma u_n) - f_n(u_n, D_\sigma u_n), u_l - u_n)|\, ds \qquad (4.30) \\ \underset{\|u\|_\infty \le r-1}{\le} \;& 2(r-1) \int_t^T \int_{\{f^{\prime,r}>n\}} (C|D_\sigma u_n| + f^{\prime,r})\, dm\, ds. \end{aligned}$$

To show the convergence of u_n we start as in the preceding step:

$$\begin{aligned}
&\|u_{l,t}-u_{n,t}\|_2^2+2\int_t^T \mathcal{E}(u_{l,s}-u_{n,s})\,ds \\
= \quad & 2\int_t^T (f_l(s,\cdot,u_{l,s},D_\sigma u_{l,s})-f_n(s,\cdot,u_{n,s},D_\sigma u_{n,s}),u_{l,s}-u_{n,s})\,ds. \\
\underset{(4.29)}{\leq} \quad & 2\int_t^T C(|D_\sigma u_l-D_\sigma u_n|,|u_l-u_n|)\,ds \\
& +2\int_t^T |(f_l(u_n,D_\sigma u_n)-f_n(u_n,D_\sigma u_n),u_l-u_n)|\,ds \\
\underset{(4.30)}{\leq} \quad & \int_t^T \|D_\sigma u_l-D_\sigma u_n\|_2^2\,ds+C^2\int_t^T \|u_l-u_n\|_2^2\,ds \\
& +4(r-1)\int_t^T\int_{\{f^{\prime,r}>n\}}(C|D_\sigma u_n|+f^{\prime,r})\,dm\ ds.
\end{aligned}$$

Note that we have on the one side

$$\lim_{n\to\infty}\int_t^T\int_{\{f^{\prime,r}>n\}} f^{\prime,r}\,dm\,ds=0$$

and on the other one

$$\int_t^T\int_{\{f^{\prime,r}>n\}}|D_\sigma u_n|\,dm\,dt \quad\leq\quad \|1\!\!1_{\{f^{\prime,r}>n\}}\|_{L^2(dt\times m)}\|D_\sigma u_n\|_{L^2(dt\times m)}\to 0.$$

Fix $\varepsilon>0$. Then for n big enough it follows by (A2) that

$$\|u_{l,t}-u_{n,t}\|_2^2+(2-K_A)\int_t^T \mathcal{E}(u_{l,s}-u_{n,s})\,ds\leq\varepsilon+(C^2+C_A)\int_t^T\|u_l-u_n\|_2^2\,ds.$$

By Gronwall's lemma we deduce that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Hence, u_n converges to a limit u, which solves the equation (4.1) with data (ϕ,f).

[**Step 4:**]
Now we prove the theorem without additional conditions. Let us define

$$f_n:=\frac{n}{f^{\prime,1}\vee n}(f-f^0)+f^0.$$

Then it holds

$$\lim_{n\to\infty} f_n=f \text{ where } f_n=f \text{ for } f^{\prime,1}\leq n.$$

Moreover, we define

$$f_n^{\prime,1}=\sup_{|y|\leq 1}|f_n^\prime(t,x,y)| \text{ and } f_n^\prime(t,x,y)=f_n(t,x,y,0)-f^0(t,x).$$

Analogous to the calculations in step 3 it follows that $f^{\prime,1}\leq n\Rightarrow|f_n^{\prime,1}|=|f^{\prime,1}|$ and $f^{\prime,1}>n\Rightarrow|f_n^{\prime,1}|<n$. Therefore, we deduce

$$|f_n^{\prime,1}|\leq n\wedge|f^{\prime,1}|.$$

Since $f_n^{\prime,1}$ is uniformly bounded, we apply step 3. Thus, we get a solution u_n for the data (ϕ,f_n). The convergence of u_n can be shown analogous to step 3. □

Notation. The bilinear form (2.1) has the following representation:

$$\mathcal{E}^\rho(u,v) := \sum_{i,j=1}^d \int_{\mathbb{R}^d} a^{i,j}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}\,m(dx) + \sum_{i=1}^d \int_{\mathbb{R}^d} \frac{\partial u(x)}{\partial x_i} v(x) b_i(x)\,m(dx) + \int_{\mathbb{R}^d} c(x)u(x)v(x)\,m(dx). \tag{4.31}$$

where $u, v \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. We denote the function spaces, which are associated to (4.31) by $F_\rho, \hat{F}_\rho$ and C_T^ρ. In the case $\rho = 0$ we drop ρ in the notation, i.e. $\mathcal{E}^0 = \mathcal{E}$. Further we denote by $(\ ,\ )_\rho$ the inner product on $L^2(\mathbb{R}^d, m)$.

To treat the general case ($\rho = 0$) we need to modify condition ($A1$):

$$(A1)' \qquad A = \tilde{A} \text{ and } A \text{ is bounded}$$

and additionally assume

- ($A5$) $\exists\,\sigma^{-1}$ such that $\sigma\sigma^{-1} = \mathbb{1}$ and $|\sigma^{-1}(x)| < \infty$ uniformly,
- ($A6$) $-\nabla \cdot b \geq 0$,
- ($A7$) $b \in L^2(\mathbb{R}^d, dx)$,
- ($A8$) $\mathcal{E}(u) < \infty \Rightarrow u \in F$.

By (A5) and (A7) the Dirichlet form has the following representation for $u, v \in F_\rho$, $\rho \geq 0$:

$$\mathcal{E}^\rho(u,v) = \int_{\mathbb{R}^d} \langle D_\sigma u, D_\sigma v\rangle\,dm + \int_{\mathbb{R}^d} cuv\,dm + \int_{\mathbb{R}^d} \langle (D_\sigma u)\sigma^{-1}, b\rangle v\,dm.$$

Furthermore, by condition ($A6$) there exists a measure μ_b such that for $u \in F_\rho$, $\rho \geq 0$ it holds

$$\mathcal{E}^{B,\rho}(u,u) = \frac{1}{2}\int_{\mathbb{R}^d} u^2 \exp(-\rho\theta)\,d\mu_b.$$

Note that condition ($A5$) is an assumption on the coefficients $a^{i,j}$, usually called global strict ellipticity. More precisely condition ($A5$) is equivalent to the non-degeneracy of L.

Lemma 4.20. *Let $\rho > 0$. Then it holds*

$$\mathcal{E}^\rho(u,\varphi) = \mathcal{E}(u, \varphi\exp(-\theta\rho)) + (M_\rho u, \varphi)_\rho$$

for $u \in F_\rho, \varphi \in bF_\rho$, where

$$M_\rho u = \rho\langle D_\sigma\theta, D_\sigma u\rangle.$$

Proof. It is easy to see that the only term, which is not trivial, is

$$\begin{aligned}\mathcal{E}^{A,\rho}(u,\varphi) &= \int_{\mathbb{R}^d} \langle D_\sigma u, D_\sigma\varphi\rangle\,dm \\ &\stackrel{!}{=} \int_{\mathbb{R}^d} \langle D_\sigma u, D_\sigma(\varphi\exp(-\rho\theta)) - D_\sigma(\exp(-\rho\theta))\varphi\rangle\,dx.\end{aligned}$$

Since

$$\left|\int_{\mathbb{R}^d} \langle D_\sigma u, D_\sigma \varphi - D_\sigma(\varphi \exp(-\rho\theta)) \exp(\rho\theta) + D_\sigma(\exp(-\rho\theta)) \exp(\rho\theta)\varphi \rangle \, dm\right|$$
$$\leq \|D_\sigma u\|_{2,\rho} \|D_\sigma \varphi - D_\sigma(\varphi \exp(-\rho\theta)) \exp(\rho\theta) + D_\sigma(\exp(-\rho\theta)) \exp(\rho\theta)\varphi\|_{2,\rho},$$

it is enough to show that the last term is zero. Take $\varphi_n \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $\varphi_n \to \varphi$ in $L^2(\mathbb{R}^d, m)$ and $D_\sigma \varphi_n \to D_\sigma \varphi$ in $L^2(\mathbb{R}^d, m)$. Then we obtain in $L^2(\mathbb{R}^d, m)$

$$\begin{aligned} & \lim_{n\to\infty} \Big(D_\sigma(\varphi_n \exp(-\rho\theta)) \exp(\rho\theta) \Big) \\ = & \lim_{n\to\infty} \Big(D_\sigma \varphi_n + D_\sigma(\exp(-\rho\theta)) \exp(\rho\theta)\varphi_n \Big) \\ = & \; D_\sigma \varphi - \rho D_\sigma(\theta)\varphi \end{aligned}$$

and the assertion follows. □

Theorem 4.21. *Under the conditions (H1)-(H5) and $\rho = 0$ there exists a unique solution of equation* (4.1). *It satisfies the following estimates with constants K_1 and K_2*

$$\begin{aligned} \|u\|_T^2 & \leq K_1 \left(\|\phi\|_2 + \int_0^T \|f_t^0\|_2 \, dt \right), \\ \|u\|_\infty & \leq K_2 \left(\|\phi\|_\infty + \|f^0\|_\infty \right) \end{aligned}$$

where K_1 depends only on the constants C, μ, T, C_A, K_A and K_2 only on C, μ, T.

Proof. W.l.o.g. we consider the case of a single equation ($l = 1$ in Definition 4.5). We set for $\rho > 0$

$$f^\rho(t, x, y, z) := f(t, x, y, z) + \rho \sum_{l=1}^k \sum_{i=1}^d \sigma_l^i(x) \partial_i \theta(x) z_l(x)$$

and define

$$(\partial_t + L_\rho)u + f^\rho(u, D_\sigma u) = 0, \qquad u_T = \phi. \tag{4.32}$$

The associated weak equation has the form

$$\begin{aligned} & \int_0^T \mathcal{E}^\rho(u_t, \varphi_t) + (u_t, \partial_t \varphi_t)_\rho \, dt && (4.33) \\ = & \int_0^T (f_t^\rho, \varphi_t)_\rho \, dt + (u_T, \varphi_T)_\rho - (u_0, \varphi_0)_\rho, && \forall \varphi \in \mathcal{C}_T^\rho. \end{aligned}$$

[Step 1] We easily see that f^ρ satisfies the conditions (H2)-(H5). Hence, we show (H1):

$$\begin{aligned} & |f^\rho(t, x, y, z) - f^\rho(t, x, y, \tilde{z})| \\ \leq & \; |f(t, x, y, z) - f(t, x, y, \tilde{z})| + \left| \rho \sum_{l=1}^k \sum_{i=1}^d \sigma_l^i \partial_i \theta (z_l - \tilde{z}_l) \right| \\ \leq & \left(C + \left| \rho \sum_{l=1}^k \sum_{i=1}^d \sigma_l^i \partial_i \theta \right| \right) |z - \tilde{z}|. \end{aligned}$$

Thus, the existence of a unique solution of (4.32) is obtained in $\hat{F}_\rho$ for $\rho > 0$ by the above theorem.

[Step 2] In this step we prove the existence of a unique function, which satisfies the weak equation (4.7) in a weaker sense for $\rho = 0$ (cf. equation (4.34)). Moreover, we prove that $u^\rho = u^{\tilde{\rho}}$ for all $\rho, \tilde{\rho} > 0$, where u^ρ resp. $u^{\tilde{\rho}}$ satisfies the weak equation (4.33) associated to ρ resp. $\tilde{\rho}$.

Fix $\rho > 0$ and define $f_n \in C_0^\infty(\mathbb{R}^d)$ such that $f_n(x) = 1$ for $x \in B_n(0)$, $f_n(x) = 0$ for $x \in B_{2n}^C(0)$, $\frac{\partial f_n(x)}{\partial x_i}$ are uniformly bounded and $\left|\frac{\partial f_n(x)}{\partial x_i}\right| \to 0$, where $B_r(0)$ denotes the $\mathbb{R}^d$ dimensional ball with radius r centered at 0.

Let us show $\varphi \in b\mathcal{C}_T \Rightarrow (\varphi f_n \exp(\theta\rho)) \in b\mathcal{C}_T^\rho$:

- $\|f_n \exp(\theta\rho)(\varphi_{t+h} - \varphi_t)\|_{2,\rho} \leq const\, \|\varphi_{t+h} - \varphi_t\|_{2,\rho} \to 0$
- $\|f_n \exp(\theta\rho)\left(\frac{\varphi_{t+h} - \varphi_t}{h} - \partial_t\varphi_t\right)\|_{2,\rho} \to 0$
- $\|f_n \exp(\theta\rho)(\partial_t\varphi_{t+h} - \partial_t\varphi_t)\|_{2,\rho} \to 0$
- $$\begin{aligned}&\int_0^T \mathcal{E}^\rho(\varphi_t f_n \exp(\theta\rho))\, dt \\ &\leq \int_0^T 2\|\varphi_t\|_\infty^2 \underbrace{\mathcal{E}^\rho(f_n \exp(\theta\rho))}_{<\infty,\text{ since } f_n\in C_0^\infty(\mathbb{R}^d), \exp(\theta\rho)\in C^1(\mathbb{R}^d)} dt \\ &\quad + \underbrace{\int_0^T 2\|f_n \exp(\theta\rho)\|_\infty^2 \mathcal{E}^\rho(\varphi_t)\, dt}_{<\infty,\text{ since } \varphi\in b\mathcal{C}_T^\rho} \\ &< \infty,\end{aligned}$$

Let u^ρ be a solution of (4.32), then we have for all $\varphi \in b\mathcal{C}_T^\rho$

$$\int_0^T \mathcal{E}^\rho(u_t^\rho, \varphi_t) + (u_t^\rho, \partial_t\varphi_t)_\rho\, dt = \int_0^T (f_t^\rho, \varphi_t)_\rho\, dt + (u_T^\rho, \varphi_T)_\rho - (u_0^\rho, \varphi_0)_\rho.$$

Since $(\tilde{\varphi} f_n \exp(\theta\rho)) \in b\mathcal{C}_T^\rho$ for all $\tilde{\varphi} \in b\mathcal{C}_T$, it holds:

$$\begin{aligned}&\int_0^T \mathcal{E}^\rho(u_t^\rho, \tilde{\varphi}_t f_n \exp(\theta\rho)) + (u_t^\rho, \partial_t\tilde{\varphi}_t f_n \exp(\theta\rho))_\rho\, dt \\ =\ &\int_0^T (f_t^\rho, \tilde{\varphi}_t f_n \exp(\theta\rho))_\rho\, dt + (u_T^\rho, \tilde{\varphi}_T f_n \exp(\theta\rho))_\rho - (u_0^\rho, \tilde{\varphi}_0 f_n \exp(\theta\rho))_\rho.\end{aligned}$$

Moreover, the above equation is equivalent to

$$\begin{aligned}&\int_0^T \mathcal{E}^\rho(u_t^\rho, \tilde{\varphi}_t f_n \exp(\theta\rho)) + (u_t^\rho, \partial_t f_n \tilde{\varphi}_t)\, dt \\ =\ &\int_0^T (f_t^\rho, f_n \tilde{\varphi}_t)\, dt + (u_T^\rho, f_n \tilde{\varphi}_T) - (u_0^\rho, f_n \tilde{\varphi}_0)\end{aligned}$$

and also

$$\begin{aligned}&\int_0^T \mathcal{E}(u_t^\rho, f_n\tilde{\varphi}_t) + (\rho\langle D_\sigma\theta, D_\sigma u_t^\rho\rangle, f_n\tilde{\varphi}_t) + (u_t^\rho, \partial_t\tilde{\varphi}_t f_n)\,dt\\ = &\int_0^T (f_t + \rho\sum_{l=1}^k\sum_{i=1}^d \sigma_l^i\partial_i\theta(D_\sigma u_t^\rho)_l, f_n\tilde{\varphi}_t)\,dt + (u_T^\rho, f_n\tilde{\varphi}_T) - (u_0^\rho, f_n\tilde{\varphi}_0).\end{aligned}$$

Finally, this yields to a weaker form of (4.7)

$$\begin{aligned}&\int_0^T \mathcal{E}(u_t^\rho, f_n\tilde{\varphi}_t) + (u_t^\rho, \partial_t\tilde{\varphi}_t f_n)\,dt\\ = &\int_0^T (f_t, f_n\tilde{\varphi}_t)\,dt + (u_T^\rho, f_n\tilde{\varphi}_T) - (u_0^\rho, f_n\tilde{\varphi}_0).\end{aligned} \tag{4.34}$$

Note that $(f_n\tilde{\varphi}) \in b\mathcal{C}_T$ for all $\tilde{\varphi} \in b\mathcal{C}_T$.

Now let $u \in \hat{F}_{\tilde{\rho}}$ be a function, which satisfies (4.34) for all $\varphi \in b\mathcal{C}_T$ and f_n as above for a fixed $\tilde{\rho} \geq 0$. Fix $\rho \geq \tilde{\rho}$ and take $\tilde{\varphi} \in b\mathcal{C}_T^\rho$.

Let us show $(\tilde{\varphi}\exp(-\theta\rho)) \in b\mathcal{C}_T$:

- $$\int_{\mathbb{R}^d} |\tilde{\varphi}_{t+h} - \tilde{\varphi}_t|^2 \exp(-\theta\rho)\,dm \leq \|\tilde{\varphi}_{t+h} - \tilde{\varphi}_t\|_{2,\rho} \to 0$$
- $$\int_{\mathbb{R}^d} \left|\left(\frac{\tilde{\varphi}_{t+h} - \tilde{\varphi}_t}{h} - \partial_t\tilde{\varphi}_t\right)\right|^2 \exp(-\theta\rho)\,dm \to 0$$
- $$\int_{\mathbb{R}^d} |\partial_t\tilde{\varphi}_{t+h} - \partial_t\tilde{\varphi}_t|^2 \exp(-\theta\rho)\,dm \to 0$$
- $$\begin{aligned}&\int_0^T \mathcal{E}(\tilde{\varphi}_t\exp(-\theta\rho))\,dt\\ &\leq \int_0^T\int_{\mathbb{R}^d} 2|D_\sigma\tilde{\varphi}_t|^2\exp(-\theta\rho)^2 + 2|D_\sigma\exp(-\theta\rho)|^2\tilde{\varphi}_t^2\,dx\,dt\\ &\quad + \int_0^T\int_{\mathbb{R}^d} (\tilde{\varphi}_t\exp(-\theta\rho))^2\,d\mu_b\,dt\\ &\quad + \int_0^T\int_{\mathbb{R}^d} c(\tilde{\varphi}_t\exp(-\theta\rho))^2\,dx\,dt\\ &< \infty.\end{aligned}$$

By the same arguments as above, we conclude that u satisfies the weak equation (4.33) for ρ with test functions $\tilde{\varphi}f_n$ where $\tilde{\varphi} \in b\mathcal{C}_T^\rho$ and f_n as above. By passing to the limit in this equation we conclude that u is a solution for every $\rho > \tilde{\rho}$. Note that in this equation we can pass n to the limit, since we have $u \in \hat{F}_\rho$. The only not trivial convergence is

$$\int_0^T \mathcal{E}^\rho(u_t, \tilde{\varphi}_t f_n)\,dt \to \int_0^T \mathcal{E}^\rho(u_t, \tilde{\varphi}_t)\,dt.$$

Let us examine this term:

$$
\begin{aligned}
&\left|\int_0^T \mathcal{E}^\rho(u_t, \tilde{\varphi}_t(f_n-1))\,dt\right| \\
\leq\ & \int_0^T K_{\mathcal{E}}\mathcal{E}_1^\rho((u_t))^{\frac{1}{2}}\mathcal{E}_1^\rho(\tilde{\varphi}_t(f_n-1))^{\frac{1}{2}} + |(u_t, \tilde{\varphi}_t(f_n-1))|\,dt \\
\leq\ & K_{\mathcal{E}}\left(\int_0^T \mathcal{E}_1^\rho(u_t)\,dt\right)^{\frac{1}{2}}\left(\int_0^T \mathcal{E}_1^\rho(\tilde{\varphi}_t(f_n-1))\,dt\right)^{\frac{1}{2}} \\
&+\int_0^T |(u_t, \tilde{\varphi}_t(f_n-1))_\rho|\,dt.
\end{aligned}
$$

Easily we see that the last term converges to zero. Hence, we examine only the first one:

$$
\begin{aligned}
&\int_0^T \mathcal{E}^\rho(\tilde{\varphi}_t(f_n-1))\,dt \\
\leq\ & \int_0^T\int_{\mathbb{R}^d} 2|D_\sigma\tilde{\varphi}_t|^2(f_n-1)^2 + 2|D_\sigma(f_n-1)|^2|\tilde{\varphi}_t|^2\,dm\,dt \\
&+\int_0^T\int_{\mathbb{R}^d}(\tilde{\varphi}_t(f_n-1))^2\exp(-\theta\rho)\,d\mu_b\,dt \\
&+\int_0^T\int_{\mathbb{R}^d} c(\tilde{\varphi}_t(f_n-1))^2\,dm\,dt \\
\underset{n\to\infty}{\longrightarrow}\ & 0.
\end{aligned}
$$

Now fix $\rho_1 > 0$. Then there exists a solution u^{ρ_1} of the weak equation (4.33) associated to ρ_1 (cf. step 1). Now we can conclude that u^{ρ_1} also satisfies the equation (4.34) with test functions of the form $\tilde{\varphi} f_n$ where $\tilde{\varphi} \in \mathcal{C}_T$ and f_n as above. Moreover, we obtain by the above argumentation that u^{ρ_1} satisfies the weak equation (4.33) for all $\rho > \rho_1$ with test functions $\tilde{\varphi} \in b\mathcal{C}_T^\rho$. By step 1 there exists a unique solution u^ρ of (4.32) for every $\rho > 0$. Hence, by uniqueness it follows that $u^{\rho_1} = u^\rho$ for all $\rho > \rho_1$.

Finally, we can deduce that a solution $u^{\tilde{\rho}}$ of (4.32) associated to $\tilde{\rho}$, is a solution of (4.32) for all $\rho > 0$.

[Step 3] Let $u^{\tilde{\rho}}$ be a solution of (4.32). Then by (2.) $u^{\tilde{\rho}}$ is a solution of (4.32) for every $\rho > 0$. Moreover, by Theorem 4.19 it holds:

$$
\|u^{\tilde{\rho}}\|_{T,\rho}^2 \leq K\left(\|\phi\|_{2,\rho} + \int_0^T \|f_t^0\|_{2,\rho}\,dt\right) \text{ for all } \rho > 0.
$$

Letting $\rho \to 0$, the estimate passes to the limit and we get

$$
\begin{aligned}
\limsup_{\rho\to 0}\|u^{\tilde{\rho}}\|_{T,\rho}^2 &\leq \lim_{\rho\to 0} K\left(\|\phi\|_{2,\rho} + \int_0^T \|f_t^0\|_{2,\rho}\,dt\right) \\
&= K\left(\|\phi\|_2 + \int_0^T \|f_t^0\|_2\,dt\right).
\end{aligned}
$$

Next we have to verify that

$$\|u^{\tilde\rho}\|_T = \lim_{\rho\to 0} \|u^{\tilde\rho}\|_{T,\rho}.$$

Clearly it holds

$$\limsup_{\rho\to 0} \|u^{\tilde\rho}\|_{T,\rho} \le \|u^{\tilde\rho}\|_T.$$

Since

$$\liminf_{\rho\to 0} \sup_{t\in[0,T]} \|u^{\tilde\rho}\|_{2,\rho} \ge \sup_{t\in[0,T]} \|u^{\tilde\rho}\|_2$$

and

$$\begin{aligned}\int_0^T \mathcal{E}^0(u_t^{\tilde\rho})\,dt &= \int_0^T \int_{\mathbb{R}^d} \Big(|D_\sigma u_t^{\tilde\rho}|^2 + c(u_t^{\tilde\rho})^2\Big) \lim_{\rho\to 0} \exp(-\rho\theta)\,dx\,dt \\ &\quad + \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} (u_t^{\tilde\rho})^2 \lim_{\rho\to 0} \exp(-\rho\theta)\,d\mu_b\,dt \\ &\le \liminf_{\rho\to 0} \int_0^T \mathcal{E}^\rho(u_t^{\tilde\rho})\,dt,\end{aligned}$$

it follows $\|u^{\tilde\rho}\|_T = \lim_{\rho\to 0}\|u^{\tilde\rho}\|_{T,\rho}$. Easily we see that the second estimate holds:

$$\|u^{\tilde\rho}\|_\infty \le K(\|\phi\|_\infty + \|f^0\|_\infty).$$

[Step 4] In this step we show that a solution $u^{\tilde\rho} \in \hat{F}^{\tilde\rho}$ of (4.32) for $\tilde\rho > 0$ is an element of $\hat{F}$. Note that since $\mathcal{E}(u_t^{\tilde\rho}) < \infty$ for almost every t (cf. step 3), it follows by assumption (A8) that $u_t^{\tilde\rho} \in F$ for almost every t. Moreover, we have already verified that $\int_0^T \mathcal{E}_1(u_t^{\tilde\rho})\,dt < \infty$. Hence, it is left to show that $u_t^{\tilde\rho} \in \mathcal{C}([0,T];L^2)$.

Since $u^{\tilde\rho}$ is a solution of (4.32) for all $\rho > 0$ (cf. step 2), we can deduce by equation (3.5) that

$$\begin{aligned}\Big|\|u_t^{\tilde\rho}\|_{2,\rho}^2 - \|u_{t+h}^{\tilde\rho}\|_{2,\rho}^2\Big| &\le 2\left[\left|\int_t^{t+h} (u_s^{\tilde\rho}, f_s^\rho)_\rho\,ds\right| + \left|\int_t^{t+h} \mathcal{E}^\rho(u_s^{\tilde\rho})\,ds\right|\right] \\ &\le 2\left[\left|k_1\int_t^{t+h} \|f_s^\rho\|_{2,\rho}\,ds\right| + \left|\int_t^{t+h} \mathcal{E}^\rho(u_s^{\tilde\rho})\,ds\right|\right] \\ &\le 2\left[\left|k_1\int_t^{t+h} \|f_s\|_{2,\rho}\,ds + \int_t^{t+h} k_2\rho\mathcal{E}^{A,\rho}(u_s^{\tilde\rho})^{\frac{1}{2}}\,ds\right|\right. \\ &\quad \left. + \left|\int_t^{t+h} \mathcal{E}^\rho(u_s^{\tilde\rho})\,ds\right|\right]\end{aligned}$$

where k_1, k_2 are constants. Let us make an additional assumption on the function f: $\int_0^T \|f_s\|_2\,ds < \infty$. We point out that, if $u^{\tilde\rho} \in \hat{F}$, this assumption is always fulfilled (cf. Lemma 4.9). Analogous to the arguments of step 3 we can show that

- $$\lim_{\rho\to 0} \|u_t^{\tilde\rho}\|_{2,\rho}^2 = \|u_t^{\tilde\rho}\|_2^2,$$
- $$\lim_{\rho\to 0} \int_t^{t+h} \mathcal{E}^\rho(u_s^{\tilde\rho})\, ds = \int_t^{t+h} \mathcal{E}(u_s^{\tilde\rho})\, ds,$$
- $$\lim_{\rho\to 0} \int_t^{t+h} \rho\mathcal{E}^{A,\rho}(u_s^{\tilde\rho})^{\frac{1}{2}}\, ds = 0.$$

Hence, we only examine the term, which depends on f

$$\begin{aligned}
\limsup_{\rho\to 0} \int_t^{t+h} \|f_s\|_{2,\rho}\, ds &\leq \int_t^{t+h} \|f_s\|_2\, ds \\
&= \int_t^{t+h} \|f_s \lim_{\rho\to 0} \exp\Big(-\frac{\theta\rho}{2}\Big)\|_2\, ds \\
&\leq \liminf_{\rho\to 0} \int_t^{t+h} \|f_s\|_{2,\rho}\, ds.
\end{aligned}$$

Summarized it holds:

$$\Big| \|u_t^{\tilde\rho}\|_2^2 - \|u_{t+h}^{\tilde\rho}\|_2^2 \Big| \quad\leq\quad 2\left[\left| \int_t^{t+h} \|f_s\|_2\, ds \right| + \left| \int_t^{t+h} \mathcal{E}(u_s^{\tilde\rho})\, ds \right| \right].$$

By passing h to zero it follows

$$\lim_{h\to 0} \Big| \|u_t^{\tilde\rho}\|_2^2 - \|u_{t+h}^{\tilde\rho}\|_2^2 \Big| = 0.$$

Now fix a sequence $h_n \underset{n\to\infty}{\longrightarrow} 0$. Clearly, since $u^{\tilde\rho} \in \hat F^\rho$ for $\rho > 0$, the following convergence holds for every subsequence $(n_k)_{k\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$:

$$\|u_{t+h_{n_k}}^{\tilde\rho} - u_t^{\tilde\rho}\|_{2,\rho} \to 0.$$

Therefore, there exists a subsequence such that $u_{t+h_{n_{k_l}}}^{\tilde\rho} \to u_t^{\tilde\rho}$ for m-almost every x. Hence, $u_{t+h_{n_{k_l}}}^{\tilde\rho} \to u_t^{\tilde\rho}$ for dx-almost every x. Consequently, it follows that $u_{t+h_n}^{\tilde\rho} \to u_t^{\tilde\rho}$ in measure (cf. [Bau92, Korollar 20.8]). Now we can obtain by [Bau92, Satz 21.7] that $u_{t+h_n}^{\tilde\rho} \to u_t^{\tilde\rho}$ in $L^2(\mathbb{R}^d, dx)$. Since this reasoning holds for every sequence $h_n \to 0$ as $n \to \infty$,

$$u^{\tilde\rho} \in \mathcal{C}([0,T], L^2).$$

[Step 5] Let $u \in \hat F_\rho$ be a solution of (4.32) for $\rho > 0$. The existence follows by step 1. In step 2 we have shown that u satisfies (4.34) for $\varphi \in b\mathcal{C}_T$ and $f_n \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ as above.

$$\begin{aligned}
&\int_0^T \mathcal{E}(u_t, f_n\varphi_t) + (u_t, \partial_t\varphi_t f_n)\, dt \\
=\ &\int_0^T (f_t, f_n\varphi_t)\, dt + (u_T, f_n\varphi_T) - (u_0, f_n\varphi_0)
\end{aligned}$$

Since by step 4 it holds $u \in \hat F$, we can pass n to the limit in this equation. Now we see that u satisfies the weak equation (4.7) for $\rho = 0$. Thus, u is a solution of (4.1). □

The next proposition is a comparison result. We follow [BPS05, Proposition 3.4].

Proposition 4.22. *Let* $f^i : [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^k \to \mathbb{R}$, $\rho \geq 0$ *and* $u^i \in \hat{F}^\rho, i = 1,2$ *be such that* $f^i(u^i, D_\sigma u^i) \in L^1([0,T]; L^2(\mathbb{R}^d, m))$. *Assume that* f^1 *satisfies the conditions (H1) and (H2) and that the following inequality holds*

$$f^1(u^2, D_\sigma u^2) \leq f^2(u^2, D_\sigma u^2).$$

If u^i *is a solution of the equation (4.1) with data* $(\phi^i, f^i), i = 1,2$ *such that* $\phi^1 \leq \phi^2$, *then one has*

$$u^1 \leq u^2.$$

Proof. Let us define

$$v := u^1 - u^2, \psi := \phi^1 - \phi^2 \text{ and } g = f^1(\cdot,\cdot,u^1, D_\sigma u^1) - f^2(\cdot,\cdot,u^2, D_\sigma u^2).$$

Then v is a solution of equation (3.1) associated to the data (ψ, g). Moreover, it holds $v_T^+ = 0$. Hence, we can apply Lemma 3.10

$$\|v_t^+\|_{2,\rho}^2 + 2\int_t^T \mathcal{E}^\rho(v_s^+)\, ds \leq 2\int_t^T (g_s, v_s^+)_\rho\, ds.$$

By (H1), (H2) and the condition $f^1(u^2, D_\sigma u^2) \leq f^2(u^2, D_\sigma u^2)$ we deduce

$$\begin{aligned} gv^+ &= (f^1(\cdot,\cdot,u^1, D_\sigma u^1) - f^1(\cdot,\cdot,u^2, D_\sigma u^1))v^+ \\ &\quad + (f^1(\cdot,\cdot,u^2, D_\sigma u^1) - f^1(\cdot,\cdot,u^2, D_\sigma u^2))v^+ \\ &\quad + (f^1(\cdot,\cdot,u^2, D_\sigma u^2) - f^2(\cdot,\cdot,u^2, D_\sigma u^2))v^+ \\ &\leq \mu(v^+)^2 + C|D_\sigma v^+|v^+. \end{aligned}$$

Now it follows

$$\begin{aligned} & \|v_t^+\|_{2,\rho}^2 + 2\int_t^T \mathcal{E}^\rho(v_s^+)\, ds \\ \leq\ & 2\int_t^T \int_{\mathbb{R}^d} \left(\mu(v^+)^2 + C|D_\sigma v^+|v^+\right)\, dm\, ds \\ =\ & 2\mu\int_t^T \int_{\mathbb{R}^d} (v^+)^2\, dm\, ds + 2\int_t^T \int_{\mathbb{R}^d} C|D_\sigma v^+|v^+\, dm\, ds \\ \leq\ & (2\mu + C^2)\int_t^T \|v_s^+\|_{2,\rho}^2\, ds + \int_t^T \mathcal{E}^{\rho,A}(v_s^+)\, ds \\ \leq\ & (2\mu + C^2 + C_A)\int_t^T \|v_s^+\|_{2,\rho}^2\, ds + \int_t^T K_A \mathcal{E}^\rho(v_s^+)\, ds. \end{aligned}$$

By applying Gronwall's lemma we get $v^+ = 0$. Hence, $u^1 \leq u^2$. ☐

Appendix A

The Bochner Integral

In this section we outline a useful proposition for the Bochner integral. For more details we refer to [Coh94, Appendix E], [PR07, Appendix A] and [Yos71, V.5. Bochner's Integral].

Proposition A.1. *Let* $(B, \|\cdot\|_B)$, $(Y, \|\cdot\|_Y)$ *be Banach spaces,* $f \in L^1([0,T]; B)$ *and* $P \in L(B,Y)$, *where* $L(B,Y)$ *is the set of all bounded linear operators* $P : B \to Y$. *Then we have*

$$(i) \quad \left\| \int_0^T f \, dt \right\|_B \leq \int_0^T \|f\|_B \, dt \ \textit{(Bochner inequality)}$$

$$(ii) \quad \int_0^T P \circ f \, dt = P \left(\int_0^T f \, dt \right).$$

Proof. See [Yos71, V.5. Bochner's Integral]. □

Appendix B

Backward Gronwall's Inequality

In this section we present a backward version of Gronwall's lemma. It is a simplified version of [SY08, Lemma 3.1].

Lemma B.1. *Let* $u : [0, T] \to \mathbb{R}$ *be a integrable function and* $\beta, \alpha \in \mathbb{R}_+$*. If it holds for* $0 \le t \le T$

$$u(t) \le \alpha + \beta \int_t^T u(r)\, dr,$$

then

$$u(t) \le \alpha + \beta\alpha \int_t^T e^{\beta(r-t)}\, dr$$

and

$$u(t) \le \alpha e^{\beta T}.$$

Proof.

$$\begin{aligned}
& \frac{d}{dt}\left(\exp(-(T-t)\beta) \int_t^T u(r)\, dr\right) \\
= \quad & \beta \exp(-(T-t)\beta) \int_t^T u(r)\, dr - \exp(-(T-t)\beta) u(t) \\
= \quad & \exp(-(T-t)\beta) \left(\beta \int_t^T u(r)\, dr - u(t)\right) \\
\ge \quad & -\alpha \exp(-(T-t)\beta)
\end{aligned}$$

Hence, by integration we deduce

$$\int_t^T u(s)\, ds \le \exp((T-t)\beta)\alpha \int_t^T \exp(-(T-r)\beta)\, dr.$$

Finally, we deduce

$$\begin{aligned} u(t) &\leq \alpha + \beta \exp((T-t)\beta)\alpha \int_t^T \exp(-(T-r)\beta)\, dr \\ &= \alpha + \beta\alpha \int_t^T \exp((r-t)\beta)\, dr \\ &\leq \alpha \exp(\beta T). \end{aligned}$$

□

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