Diplomarbeit

Two Mathematical Models for Stochastic Resonance in an Asymmetric Double-Well Potential

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1. Introduction

Life's necessities: Food,
water, shelter, . . . noise.

(from [Hän02])

Stochastic resonance (SR) is a quite prominent example for a convincingly simple model from an applied science (climatology), that not only has found lots of applications in other sciences but has also inspired mathematicians to adapt their methods and develop new results to faciliate a rigorous study of the model and its properties.

In this text we present, after a short review of the development of the model and some interesting applications, two important mathematical approaches to SR: The sample paths large deviations approach based on the Freidlin-Wentzell theory, which was the first rigorous mathematical approach to SR, and the pathwise approach developed by N. Berglund and B. Gentz. A summary of the mathematical results can be found in Section 1.2.

1.1. SR – The Physical Model and its Applications

1.1.1. Historical Note

In the eighties of the last century, scientists analyzing the earth mean temperature curve of the last 400,000 years (which had been calculated from the biochemical composition of antarctic ice cores) made the observation that the earth climate shows two stable states ("warm age", "ice age"), which alternate periodically every 100,000 years. The changes from warm to ice age and back are matching noticeably well with planetary cycles ("Milankovich cycles"), where the gravitational influence of other planets in the solar system makes the earth slightly deviate from its standard orbit around the sun. However, these slight fluctuations in the distance between earth and sun and the resulting fluctuations in the amount of incoming energy explain only a small fraction of the temperature difference between warm and ice age. This discrepancy was the motivation for the development of SR. The basic idea of the model is that "warm age" and "ice age" are stable states in the otherwise dynamic energy balance development of the earth, and that the quasi-random fast fluctuations of the earth mean temperature, which are caused by the day-to-day changes of the weather, together with the comparably slow fluctuations of incoming energy from the sun, which change the energy balance configuration, trigger the transition between these stable states.

A more detailed exposition of the so-called energy-balance-model and the role of SR therein can be found e.g. in [IP04], where also more references are provided. Among the pioneering publications are [BPSV83] and [Nic82].

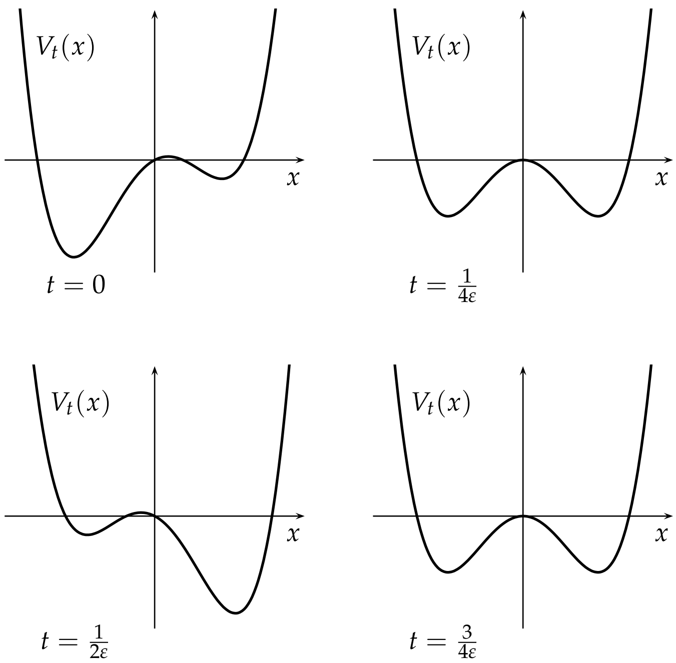

Figure 1.1.: Asymmetric double-well potential $V_t(x)$ for different values of t.

However, specialists are nowadays convinced that SR can not sufficiently describe the dynamics underlying the transitions between warm and ice ages, which is why we do not cover the topic in more detail.

However, other phenomena analyzed in climatology seem to follow an SR scheme: New observations and long-time climate simulations point out that SR might be an appropriate model to describe intermediate changes in the atlantic circulation during the last ice age, the so-called Dansgaard–Oeschger events (see [GR02]). It remains to be seen whether refined simulations can confirm this result.

Before we describe applications of SR beyond climatology let us describe the underlying physical model.[1]

1.1.2. The Physical Model

Consider the graph of the function $V_t : \mathbb{R} \to \mathbb{R}$ defined by

$$V_t(x) := \frac{1}{4} \cdot x^4 - \frac{1}{2} \cdot x^2 + \lambda_0 \cdot \cos(2\pi \cdot \varepsilon t) \cdot x\,, \tag{1.1.1}$$

where t parametrizes ‘time’ and $\lambda_0 > 0$ is a constant such that for any $t \in \mathbb{R}$ the graph of $V_t(x)$ has two separate local minima. This is called an asymmetric double well potential, and it changes continuously in time with period $\frac{1}{\varepsilon}$. See Figure 1.1 for the graph of $V_t(x)$ in the cases $t \in \left\{0, \frac{1}{4\varepsilon}, \frac{1}{2\varepsilon}, \frac{3}{4\varepsilon}\right\}$.

[1] Readers who are generally interested in stochastic models in climatology will easily find a wide array of research papers available. Valuable starting points are [vSvSM01] and the proceedings volumes [IvS01, IM02].

It is obvious that an unperturbed ('deterministic') particle in one of the potential wells will remain in that well for infinitely long time, independent of whether the well is deep or flat.

If on the other hand there is a strong stochastic perturbation acting on the particle, it will randomly move forth and back between the two wells, again independent of their depth.

The most interesting situation occurs when there is a moderate stochastic perturbation acting on the particle. The particle will then typically stay in the well it occupies for some time, until the random diffusion drives it over the potential barrier into the other well. It is intuitively clear that the exit from a flat well happens faster than the exit from a deep well (more precisely, we will show below that the time the process typically spends in a well grows exponentially fast with the depth of the well). Let us call the typical exit time from a flat well T_w and the exit time from a deep well T_W.

If now the motion speed of the potential is such that the flat well remains 'almost flat' for longer than T_w, a typical particle will leave the flat well. If in the same setting the period of time during which a well is deeper than the other is shorter than T_W, a typical particle will stay inside the deep well – until it becomes flat again. In this situation, the typical particle will always jump from the flat into the deep well, remain there until the deep well becomes (almost) flat, jump back, etc. This concurrence, where a slight modification of the basic situation (the potential) and a mild stochastic perturbation together faciliate a transition between different stable states, is called noise-induced synchronization or stochastic resonance (SR).

Actually, there are different models for SR (see also below) and a number of modifications (e.g. SR in a symmetric double-well potential, non-continuous simplifications of the potential up to a two-state system, SR-like behavior of a particle in a multiwell-potential (one- or multidimensional), etc.). However, in this text we will consider the model of a particle under stochastic perturbation in a continuously changing double-well potential, as described above.

1.1.3. Applications of SR

In this subsection we provide an overview of some interesting applications of SR beyond the original climatology results. The applications divide roughly into two groups, one of biological and one of physical topics.

Biology. There are a great number of research results concerning SR in biology. Most of them belong to neurology and behavior research. There are lots of scientific results showing that the presence of a certain amount of (electric and/or mechanical) noise makes neurons and sense organs more sensitive to input signals. This has behavioral consequences in so far as a better perception of the environment lets animals and humans react in a more appropriate way. Possible applications of research in this direction include a better understanding of neurological diseases and technical devices like gloves that raise the tactile sensitivity of the user by inducing electric or mechanic noise (see e.g. [Hän02] for a recent review and further references, and [KNE$^+$03] for a concrete application example in health care).

However, the underlying concept of SR is different from ours: The concept

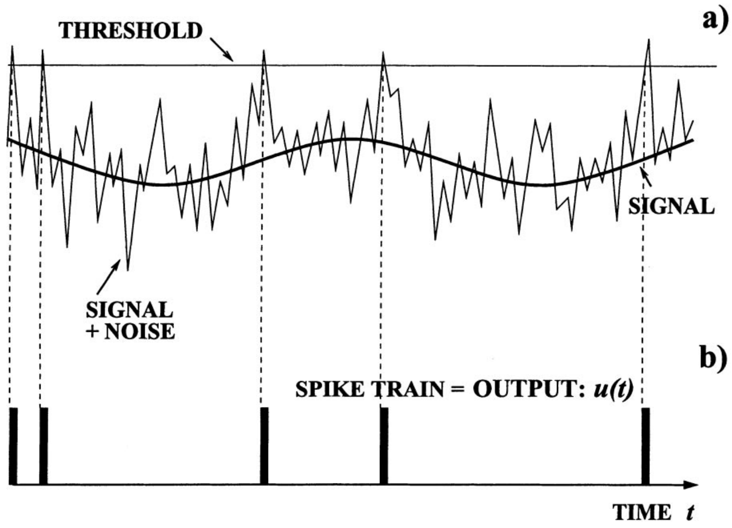

Figure 1.2.: A different concept: SR in neurons (Figure from [Hän02]).

of SR in most of the applications to biology is the following: A signal alone (e.g. the electric field emitted by moving zooplancton) is too weak to be recognized, because the respective receptor (e.g. a sense organ of a paddlefish) only reacts if incoming signals are above a certain threshold. But if the signal is complemented by noise (e.g. the emissions of hundreds of creatures), the combined effect overcomes the threshold and enables recognition of the signal (see Figure 1.2).

Even though most of the applications of SR in biology belong to the above-mentioned concept, there also are biological applications of SR as described in Subsection 1.1.2 above. For example, there is strong empirical evidence that neuronal growth, especially the transition between forward and backward motion of the leading edge of a neuronal growth cone, is subject to a stochastic resonance mechanism (see e.g. [BLK06]). In this case, the stable states are 'forward motion' and 'backward motion'.

Physics. After the invention of SR in climatology, it took some time until the topic gained momentum. An important result that established the concept of SR outside climatology was published by McNamara and Wiesenfeld (see [MW88], [MW89]), who also extended the underlying theory. They showed the following result:

Consider a bistable ring laser, the stable states being the two directions of oscillation (clockwise, counterclockwise). An optical modulation device is placed in the ring laser beam. It is possible to control the direction of oscillation with such a device (see [RSW87]).

Now, two control signals are applied to the modulation device: A sinusoidal signal with a frequency of e.g. 2 kHz (the 'slow modulation' of the SR model) and a high-frequency noise with variable intensity. In this situation it is possible to specify an amount of noise which significantly optimizes the response of the laser (i.e. the synchronization of transitions between the rotation directions to the periodic input signal) in comparison to the response without noise or with too much noise.

Another example of SR in laser physics has been shown in [GMR99]: A modulation of the control current plus a varied amount of noise causes the polarization of a laser beam to flip according to an SR pattern.

An explicit analog simulation of SR has been done using so-called optical tweezers. An optical tweezer is the real-world analogue to the tractor beam 'invented' in science fiction literature: A laser beam providing an attractive force of about a trillionth (10^{-12}) of a Newton on a particle. Based on this principle, it is possible to use two laser beams, each controlled by a filter, to create a force field that resembles a continuously changing double well potential. It has been shown in [SL92], [BSPB04] that the perturbations caused by heat (at room temperature) on a particle of 1 μm diameter in water, together with an appropriate modulation of the abovementioned configuration of optical traps, leads to transitions of the particle between the two traps which clearly follow an SR pattern.

In [CDM03], the authors provide empirical evidence that even geomagnetic polarity reversals follow an SR pattern, where the periodic slow modulation of the bistable geomagnetic dynamo due to planetary (e.g. Milankovich) cycles – directly or indirectly, possibly via climate changes, different spin, tidal effects or other – coincides with fast fluctuations in the liquid core of the earth and thus leads to periodic flips in the polarity of the earth's magnetic field.

The comparably simple basic idea of SR has led to a huge amount of further research, considering theoretical aspects and variations of the model as well as various applications: The classical review on SR [GHJM98], which alone cites more than 300 references, is by now (according to ISI Web of Science) cited in more than 1,200 scientific articles. More topics remain open: An interesting question for empirical research might e.g. be whether an SR-like mechanism can appropriately explain the development of liquidity (or underlying psychological factors) on the stock markets.

As final remark let us mention that, even though our focus is on SR with periodic slow modulation, it has been shown that SR works with aperiodic modulation as well (see e.g. [CCCI96]).

1.2. Mathematical Treatment of SR

In this section we summarize the main results of the following two chapters.

Freidlin's approach. Apparently, the first mathematically rigorous treatment of SR was published in the article [Fre00] by M. Freidlin. In this paper, Freidlin analyzes the quasi-deterministic behaviour of a stochastic particle in a multiwell potential landscape, based on the idea of a hierarchy of cycles of the stable states. For each of the cycles rotation rate, exit rate and main state are determined. This allows the identification of a (quasi-deterministic) metastable state for any chosen pair of initial point and timescale.

In the case of a stochastic particle in an asymmetric double-well potential $V_t : \mathbb{R} \to \mathbb{R}$, the underlying result boils down to the following: Let the potential V_t have two potential wells which are separated by a barrier. Let $0 < w < W$ be the respective depths of the flat and the deep potential well relative to the maximum of the barrier. Let T_w denote the exit time (i.e., time until transition) from the flat well and T_W the exit time from the deep well. Then T_w is of the order $\exp\left[\frac{2w}{\varepsilon}\right]$, and T_W is of the order $\exp\left[\frac{2W}{\varepsilon}\right]$. This fact has been commonly used in physics for a long time and is known as "Kramer's law" or "Arrhenius' law". A rigorous formulation and proof based on Freidlin-Wentzell theory of random perturbations of dynamic systems is presented in Chapter 3.

If in this situation the potential is periodically "flipped" such that the flat and the deep well are exchanged by each others after a period of time T with

$$T_w \ll T \ll T_W \ ,$$

a synchronization of the particle position with the position of the deep well (hence, SR) may be expected with high probability.

This one of is the simplest mathematically rigorous approaches to SR. However, it does not cover potentials that change continuously in time, which – at least in terms of applications – is the much more natural approach.

The approach of Berglund and Gentz. In [BG02b], N. Berglund and B. Gentz introduce a new sample-paths approach to SR. They consider two classes – symmetric and asymmetric – of continuously and periodically modulated double well potentials (in the asymmetric case, basically a generalization of V_t defined in (1.1.1)) and the behaviour of a stochastically perturbed particle therein. They show that there exists is a threshold for the noise intensity: Below this value, there is no transition and above this value, transition happens – both with probability exponentially close to 1. Furthermore, they precisely specify the parameters (basically, barrier height and lenght of modulation period) that control the behavior of the particle and they rigorously describe the dependence of the abovementioned threshold on these parameters.

However, in [BG02b] only the symmetric case is completely presented and proven. For the asymmetric case, only the results are presented and actually very few of them proven. In Chapter 4, we provide a complete exposition of the asymmetric case, with small optimizations of the results by Berglund and Gentz.

Let us only sketch the general structure of the approach: In a first step, the behavior of a deterministic (unperturbed) particle in the modulated potential is analyzed. After that the behavior stochastically perturbed particle is estimated with respect to the deterministic path. This is done in two parts, first for noise intensity below the threshold and then for noise above the threshold. The estimates in the stochastic case are based on the relation between the stochastic process describing the motion of the particle in the potential – which is modeled as the strong solution of a stochastic differential equation (SDE) – and the process that solves a linearized version of that SDE.

Since Freidlin's paper [Fre00], more mathematical research concerning SR has been done. An interesting alternative to the approaches presented here is the spectral power amplification approach that goes back to the work of McNamara and Wiesenfeld [MW89] and has recently been further developed e.g. by P. Imkeller and others (see e.g. [IP02, IP04]). The core idea of this approach is the following: Consider a randomly perturbed particle in an asymmetric, periodically changing double well potential with period Δ. Then the dependence of the spectral component of period Δ of the particle's path on the strength of the stochastic perturbation is mathematically obtained. It is possible to identify the amount of noise for which the Δ-component of the path becomes maximal in comparison to other (weaker or stronger) amounts of noise. This has been extensively analyzed for different variations of the SR model (time/space discretization, up to a two-state Markov process), including comparisons of the results for the model variations. One of the main results of this work is that intrawell fluctuations of the process may cause a dramatic deviation of the 'original' model's behavior from that of the reduced (Markov chain) model.

Other mathematical approaches to SR make e.g. use of refined multidimensional generalizations of diffusion exit results or of the Fokker-Planck equation characterizing the distribution of the stochastic particle in the double well potential.

The aims (and structure) of this thesis. In Chapter 3, we provide a self-contained, yet streamlined presentation of the large deviations approach to SR via diffusion exit results, understandable for anybody with basic knowledge in Stochastic Analysis. This chapter is based on the lecture notes [Gen03] by B. Gentz, which are, however, not very much detailed. Thus, even though the basic structure of the chapter follows that of [Gen03], most of the results are based on other sources – especially [DZ98] and [FW98] – to obtain a complete presentation of this approach to SR with all necessary results and proofs included.

In Chapter 4, we present the pathwise approach to SR in an asymmetric double-well potential. This chapter is based on the article [BG02b] by N. Berglund and B. Gentz. However, in this article only the case of a symmetric double-well potential is completely covered, whereas the case of an asymmetric potential (which all the abovementioned applications belong to) is presented in the fashion of an outline with many of the proofs missing. In our presentation, the asymmetric case is treated completely in all detail.

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2. Mathematical Preliminaries

2.1. Prerequisites

The reader is assumed to have at least basic knowledge in probability theory and stochastic analysis (stochastic processes, stochastic integration, stochastic differential equations), especially the theory of Brownian motion. Furthermore, some knowledge of functional analysis is necessary.

More than sufficient preparation in stochastics is provided by the lectures of Professor Röckner in probability theory (notes of these lectures (in German, see [Röc03]) have been typeset by the author of this thesis and can be found on his internet site, as well as that of Professor Röckner) and his introduction to stochastic analysis (notes of parts of this lecture (again in German, see [Röc04]) have been typeset by S. Stolze).

2.2. Basic Notions

For any suitable space X and any subset $A \subset X$ we denote by $\bar{A}$ the closure, by $\mathring{A}$ the interior and by A^c the complement of A in X.

Inclusions of set are denoted as follows: "$\subset$" is used in the sense of "$\subseteq$". Whenever we exclude the identity of the compared sets, we use "$\subsetneq$".

The infimum of a function over an empty set is defined to be ∞.

The derivative of a function φ with respect to time will often be denoted by $\dot{\varphi}$.

For a topological space X, $\mathcal{B}(X)$ will always denote the Borel σ-algebra, i.e. the σ-algebra generated by the topology. Throughout this text, we will always assume that probability spaces are completed; nevertheless, we will denote the (completed) Borel σ-algebra by $\mathcal{B}(X)$. Furthermore, in a topological space X the *neighborhood* N of a set $A \subset X$ is an open set $N \subset X$ such that $A \subset N$ (unless we explicitly note that A should be closed).

Let (X, d) be a metric space, $m \in X$, and $\delta > 0$ a real number. Then

$$B(m, \delta) := \{x \in X \mid d(x, m) < \delta\}$$

is the (open) ball of radius δ centered at m. The distance between a set $A \subset X$ and a point $x \in A^c$ is given by

$$d(x, A) := \inf_{z \in A} d(x, z) \ ,$$

and the (closed) δ-blowup of A by

$$A^\delta := \{y \in X \mid d(y, A) \leqslant \delta\} \ .$$

As usual, $\mathbb{R}$ denotes the set of real numbers and $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$.

By $\|x\|$ we denote the norm of any $x \in \mathbb{R}^d$, and by $|r|$ the absolute value of $r \in \mathbb{R}$.

The transpose of a matrix or vector M is denoted by M^T.

2.2.1. Function Spaces and Norms

We fix a time interval $[0, T] \subset \mathbb{R}_+$.

We denote by

$$\mathcal{C} := \mathcal{C}\big([0, T]; \mathbb{R}^d\big)$$

the set of all continuous functions from $[0, T]$ to $\mathbb{R}^d$, and by

$$\mathcal{C}_0 := \{\varphi \in \mathcal{C} \mid \varphi_0 = 0\}$$

the subset of all functions in $\mathcal{C}$ starting at 0. For any φ from $\mathcal{C}$ or $\mathcal{C}_0$ we define

$$\|\varphi\|_\infty := \|\varphi\|_{[0,T]} := \sup_{t \in [0,T]} \|\varphi_t\| \ .$$

By

$$\mathcal{L}^2 := \mathcal{L}^2\big([0, T]; \mathbb{R}^d\big)$$

we denote the set of all square-integrable functions from $[0, T]$ to $\mathbb{R}^d$. Furthermore, we set

$$H_1 := H_1\big([0, T]; \mathbb{R}^d\big) := \left\{ \int_0^\cdot f(s) \, \mathrm{d}s \,\middle|\, f \in \mathcal{L}^2 \right\}$$

for the set of all absolutely continuous functions from $[0, T]$ to $\mathbb{R}^d$ which have a square-integrable derivative and start at 0. For any $\varphi \in H_1$ we define

$$\|\varphi\|_{H_1} := \left(\int_0^T \|\dot{\varphi}_s\|^2 \, \mathrm{d}s \right)^{\frac{1}{2}} \ .$$

3. SR through Freidlin-Wentzell theory

Even though the theory presented in this chapter originally stems from [FW98], the structure of chapter is mainly based on the lecture notes "Random Perturbations of Dynamical Systems" by Barbara Gentz (cf. [Gen03]). Further frequently used references are [DZ98], [DS01] and [FW98].[1]

In this chapter we provide a complete exposition of the large deviations theory necessary to prove the diffusion exit results which form the basis of Freidlin's approach to SR.

We start with general large deviations theory and prove the classical theorem by Schilder on large deviations sample paths of Brownian motion. After that, we show how large deviations results can be generalized using the contraction principle and apply this to Schilder's result to obtain a large deviations principle for strong solutions of certain stochastic differential equations.

This lays the fundament for the second section of this chapter, where we prove the classical results on diffusion exit from a domain: The behavior of a stochastic particle in e.g. a potential well can be modeled by an SDE. The exit of the particle from the well (or a certain domain therein) is a large deviation from its expected behavior. Thus, the generalized large deviations principle from the first section is the tool we need to analyze the diffusion exit behavior of the particle. The results of this section justify the estimates for potential well transition quoted in the first part of Section 1.2. Hence, they lay the foundation to this simple mathematical approach to SR.

3.1. Large Deviations

Our main reason for studying large deviations is that we want to understand the behaviour of strong solutions to stochastic differential equations in $\mathbb{R}^d$ of the form

$$\begin{cases} \mathrm{d}x_t^\varepsilon &= b(x_t^\varepsilon)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}W_t \\ x_0^\varepsilon &= x\,, \end{cases}$$

where we assume that the noise intensity $\sqrt{\varepsilon}$ is small, and $(W_t)_{t\geqslant 0}$ is a d-dimensional Brownian motion[2].

[1] Every lemma/theorem/... carries a reference to the source for the original version it is based on. A proof only carries a separate reference if it is not from the same source as the assertion.

[2] We assume throughout this text that the Brownian motion is continuous.

Throughout this chapter we assume that b, σ are bounded and uniformly Lipschitz continuous – thus, a unique, strong solution exists.

It seems obvious that for sufficiently small ε the perturbed process x_t^ε should be close to the solution of the deterministic ordinary differential equation

$$\begin{cases} \mathrm{d}x_t &= b(x_t)\ \mathrm{d}t \\ x_0 &= x\ . \end{cases}$$

And indeed, if b is Lipschitz continuous with Lipschitz constant L_b, then we have that

$$\|x_t^\varepsilon - x_t\| \leqslant L_b \int_0^t \|x_s^\varepsilon - x_s\|\ \mathrm{d}s + \sqrt{\varepsilon} \cdot \|W_t\|\ .$$

Applying Gronwall's inequality this leads to the estimate

$$\sup_{t\in[0,T]} \|x_t^\varepsilon - x_t\| \leqslant \sqrt{\varepsilon} \cdot \sup_{t\in[0,T]} \|W_t\| \cdot \exp[L_b T]\ .$$

In other words, the behaviour of $\|x_t^\varepsilon - x_t\|$ for $t \in [0,T]$ can be estimated if we know the behaviour of the d-dimensional Brownian motion (W_t):

$$P\big[\sup_{t\in[0,T]} \|x_t^\varepsilon - x_t\| \geqslant \delta\big] \leqslant P\left[\sup_{t\in[0,T]} \|W_t\| \geqslant \frac{\delta}{\sqrt{\varepsilon}} \cdot \exp[-L_b T]\right]\ .$$

Hence, to measure the event that x_t^ε deviates away from x_t during $[0,T]$, we basically need to know the probability that W_t leaves a ball of some radius r before the time T. This we can estimate using Lemma 3.1.1 below:

$$P\big[\sup_{t\in[0,T]} \|x_t^\varepsilon - x_t\| \geqslant \delta\big] \leqslant 4d \cdot \exp\left[-\frac{\delta^2 \cdot \exp[-2L_b T]}{2d \cdot \varepsilon T}\right]\ .$$

Let us take a closer look at this estimate. As might have been expected, the probability of leaving a δ-neighborhood of the deterministic solution increases with T and/or ε, and decreases as δ grows. More precisely, for increasing δ and/or decreasing ε the probability of leaving the δ-neighborhood of x_t decays exponentially.

Hence, if we choose $A \in \mathcal{B}(\mathcal{C})$ such that no path $(\varphi_t) \in A$ remains inside the δ-neighborhood of the deterministic solution for all $t \in [0,T]$, then the path $x^\varepsilon := (x_t^\varepsilon)_{0\leqslant t\leqslant T}$ of the solution of the perturbed equation satisfies the inequality

$$P[x^\varepsilon \in A] \leqslant 4d \cdot \exp\left[-\frac{\delta^2 \cdot \exp[-2L_b T]}{2d \cdot \varepsilon T}\right] \xrightarrow{\varepsilon\to 0} 0\ . \tag{3.1.1}$$

The event $x^\varepsilon \in A$ for sets A as described above and $\varepsilon \to 0$ is what we call a *large deviation*: The expected behaviour would of course be $x^\varepsilon \notin A$ for any such A, because a typical path x^ε should "remain near the deterministic solution" for small enough $\varepsilon > 0$. In other words, when we look for large deviations of a stochastic process, we consider atypical behaviour of that process. Our first aim is to find the *rate* at which the probability in (3.1.1) tends to zero as $\varepsilon \to 0$, depending on the choice of A. To achieve this, we have to find a better estimate for the probability, which takes into account the choice of A.

However, in general it is not possible to obtain the exact rate for the decay of the probability, but only the exponential rate. To formulate this exponential estimate, we select special sets A, namely δ-neighborhoods of some $\varphi \in \mathcal{C}$ (with respect to the supremum norm $\|\cdot\|_{[0,T]} = \|\cdot\|_\infty$). Our new aim for this section is then to find a *rate function* $I : \mathcal{C} \to [0,\infty]$ such that

$$P\big[\|x^\varepsilon - \varphi\|_\infty < \delta\big] \approx \exp\left[-\frac{I(\varphi)}{\varepsilon}\right]$$

as $\varepsilon \to 0$.

Before we start into the theory of large deviations, let us prove the estimate for the Brownian motion in $\mathbb{R}^d$, which we have used above. We use the following notation:

$$W_t^\varepsilon := \sqrt{\varepsilon} \cdot W_t\ .$$

Lemma 3.1.1 (large deviations for W_t^ε)**.** *[DZ98, Lemma 5.2.1] For any (integer) dimension d and any set of positive constants $\tau, \varepsilon, \delta$, the following estimate holds:*

$$P\big[\sup_{t\in[0,\tau]} \|W_t^\varepsilon\| \geqslant \delta\big] \leqslant 4d \cdot \exp\left[-\frac{\delta^2}{2d\cdot\tau\varepsilon}\right]. \tag{3.1.2}$$

Proof. Let us first fix that for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and any $\alpha > 0$

$$\big\{x \in \mathbb{R}^d \mid \|x\|^2 \geqslant \alpha\big\} \subset \bigcup_{i=1}^d \left\{x \in \mathbb{R}^d \,\middle|\, |x_i|^2 \geqslant \frac{\alpha}{d}\right\}.$$

If we denote by $(W_t^{(1)})$ a Brownian motion in $\mathbb{R}$, we obtain the estimate

$$\begin{aligned} P\big[\sup_{t\in[0,\tau]} \|W_t^\varepsilon\| \geqslant \delta\big] &= P\left[\sup_{t\in[0,\tau]} \|W_t\|^2 \geqslant \frac{\delta^2}{\varepsilon}\right] \\ &\leqslant d \cdot P\left[\sup_{t\in[0,\tau]} (W_t^{(1)})^2 \geqslant \frac{\delta^2}{d\cdot\varepsilon}\right]. \end{aligned}$$

Since the laws of $W_t^{(1)}$ and $\sqrt{\tau}\cdot W_{\frac{t}{\tau}}^{(1)}$ are identical, this estimate and time scaling imply:

$$P\big[\sup_{t\in[0,\tau]} \|W_t^\varepsilon\| \geqslant \delta\big] \leqslant d \cdot P\left[\sup_{t\in[0,1]} |W_t^{(1)}| \geqslant \frac{\delta}{\sqrt{\tau\cdot d\varepsilon}}\right]. \tag{3.1.3}$$

Because the distribution of Brownian motion is symmetric, i.e. $W_t^{(1)}$ and $-W_t^{(1)}$ have the same law in $\mathcal{C}_0$, we see that

$$\begin{aligned} P\left[\sup_{t\in[0,1]} |W_t^{(1)}| \geqslant \eta\right] &\leqslant 2 \cdot P\big[\sup_{t\in[0,1]} W_t^{(1)}) \geqslant \eta\big] = 4 \cdot P\big[W_1^{(1)} \geqslant \eta\big] \\ &\leqslant 4 \cdot \exp\left[-\frac{\eta^2}{2}\right], \end{aligned}$$

where the equation is an application of the reflection principle. Combining the this with (3.1.3) completes the proof. $\square$

3.1.1. Basic Definitions

Situation 3.1.2 (basic general situation)**.** *Let X be a topological space, and $\mathcal{B}$ a σ-algebra over X such that $\mathcal{B} \supset \mathcal{B}(X)$.*[3]

We consider a family $\{\mu_\varepsilon\}_{\varepsilon>0}$ of probability measures on the measurable space $(X, \mathcal{B})$. We want to describe the limiting behaviour of the measures μ_ε as $\varepsilon \to 0$ by a *rate function* I. More precisely, we want to understand, whether the exponential bounds on the values that μ_ε assigns to sets from $\mathcal{B}$ can be asymptotically formulated in terms of such a rate function. Such an asymptotical formulation of exponential bounds we will call the *large deviations principle*.

Before our first definitions we recall that for any topological space X a function $f : X \to \bar{\mathbb{R}}$ is named *lower semi-continuous* if $\{x \in X \mid f(x) \leqslant \alpha\}$ is a closed set in X for any $\alpha \in \mathbb{R}$.

Definition 3.1.3 (rate function)**.** [DS01, p. 32–33] A lower semi-continuous function $I : X \to [0, \infty]$ is named a *rate function*. (In [FW98], rate functions are called *action functional*.)

If for all $\alpha \in \mathbb{R}$ the level sets

$$\Phi_I(\alpha) := \{x \in X \mid I(x) \leqslant \alpha\}$$

of a rate function I are compact subsets of X, we call I a *good* rate function (by lower semi-continuity of I, the level sets Φ_I are closed anyway).

Definition 3.1.4 (large deviations principle)**.** [DZ98, pp. 5, 7] We say that the family of measures $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies the *large deviations principle* with rate function I if, for all $A \in \mathcal{B}$, the following holds:

$$-\inf_{x\in\mathring{A}} I(x) \leqslant \liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(A) \leqslant \limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(A) \leqslant -\inf_{x\in\bar{A}} I(x)\ . \tag{3.1.4}$$

For better differentiation from the following notion of a *weak* large deviations principle, we will also call the principle defined above a *full* large deviations principle.

The collection $\{\mu_\varepsilon\}_{\varepsilon>0}$ is said to satisfy a *weak large deviations principle* with rate function I, if the upper bound in (3.1.4) holds for all compact sets A (instead of all $A \in \mathcal{B}$) and the lower bound holds for all $A \in \mathcal{B}$.

In general, the limit in (3.1.4) does not exist, i.e. the lim sup and lim inf are not equal. If, however, $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies (3.1.4) and we have for a set $A \in \mathcal{B}$ that

$$\inf_{x\in\mathring{A}} I(x) = \inf_{x\in\bar{A}} I(x)\ ,$$

then the limit

$$\lim_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(A) = -\inf_{x\in A} I(x)$$

does exist, and A is called an *I-continuity set*.

[3]This is the general setting considered in [DZ98]; cf. p. 4 there.

In the next remark, by giving a probability measure μ on $(X, \mathcal{B})$ the attribute *non-atomic*, we simply mean that $\mu(\{x\}) = 0$ shall hold for all $x \in X$.

Remark 3.1.5 (note on interior and closure of A in (3.1.4))**.** *[DZ98, p. 5] The use of interior and closure of A in the formulation of* (3.1.4) *is explicitly necessary if we assume $\{\mu_\varepsilon\}_{\varepsilon>0}$ to be non-atomic probability measures.*

We will show this for the lower bound: Assume that in (3.1.4) *the lower bound holds for A instead of $\mathring{A}$. Let $\{\mu_\varepsilon\}_{\varepsilon>0}$ be non-atomic probability measures. Then for any $x \in X$*

$$-I(x) = -\inf_{x \in \{x\}} I(x) \leqslant \liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(\{x\}) = -\infty \; ,$$

hence $I(x) = \infty$ for all $x \in X$. But, because of $\bar{X} = X$ and the upper bound in (3.1.4), *this implies*

$$0 = \limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(X) \leqslant -\inf_{x \in X} I(x) = -\infty \; ,$$

which is obviously wrong. Hence, our assumption, that in the formulation of a large deviations principle we may replace the interior of the considered set by the set itself, must have been wrong.

The following basic properties of good rate functions will reappear throughout the proofs of large deviations facts.

Remark 3.1.6 (good rate functions)**.** *[[DS01, Lemma 2.1.2] and [DZ98, Lemma 4.1.6]] Let I be a good rate function.*

(i) Since I has compact level sets $\{I \leqslant \alpha\}$, the infimum $\inf_{x \in A} I(x)$ is achieved over any non-empty closed set $A \subset X$.

(ii) Let $\{F_\delta\}_{\delta>0}$ be a nested family of closed sets, i.e., for any $\delta < \delta'$ we assume that $F_\delta \subset F_{\delta'}$. If we set $F_0 := \bigcap_{\delta>0} F_\delta$, then

$$\inf_{y \in F_0} I(y) = \lim_{\delta \to 0} \inf_{y \in F_\delta} I(y) \; .$$

(iii) **(on metric space X).** *Assume that (X, d) is a metric space. Then*

$$\inf_{y \in \bar{A}} I(y) = \lim_{\delta \to 0} \inf_{y \in A^\delta} I(y) \; ,$$

where A^δ is the (closed) δ-blowup of A.

Proof. Part (i) is trivial.

(ii) By construction, we have $F_0 \subset F_\delta$ for all $\delta > 0$. Thus, we only need to prove that for any $\eta > 0$

$$\lim_{\delta \to 0} \inf_{y \in F_\delta} I(y) =: \gamma \geqslant \inf_{y \in F_0} I(y) - \eta \; .$$

This is obvious if $\gamma = \infty$. Hence, we assume that $\gamma < \infty$, fix an $\eta > 0$ and set $\alpha := \gamma + \eta$. It remains to prove that $\alpha \geqslant \inf_{y \in F_0} I(y)$.

The sets $\{F_\delta \cap \Phi_I(\alpha)\}_{\delta>0}$ are non-empty (by definition of γ and α), nested and compact. Consequently, the set

$$F_0 \cap \Phi_I(\alpha) = \bigcap_{\delta>0} F_\delta \cap \Phi_I(\alpha)$$

is also non–empty, hence $\inf_{y\in F_0} I(y) \leqslant \alpha$.

(iii) The distance function $d(\,\cdot\,, A)$ is continuous. This implies that the sets $\{A^\delta\}_{\delta>0}$ are closed and nested. Furthermore, we have that

$$\bigcap_{\delta>0} A^\delta = \{y \in X \mid d(y, A) = 0\} = \bar{A} \ .$$

Now the assertion follows from (ii). □

The definition of a weak and a full large deviations principle raises the question how the two are related. While it is obvious that "full implies weak", the implication does not generally hold in the opposite direction:

Example 3.1.7 (weak $\not\Rightarrow$ full)**.** [Gen03, Example 2.9] Consider the Dirac measures $\mu_n := \delta_n$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The family $\{\mu_n\}_{n\in\mathbb{N}}$, setting $\varepsilon = \frac{1}{n}$, satisfies a weak large deviations principle with a good rate function: Let $F \in \mathcal{B}(\mathbb{R})$ be a compact set and n large enough. Then $\mu_n(F) = 0$. Hence, the upper bound in (3.1.4) holds for the rate function $I :\equiv \infty$. At the same time, this rate function makes the lower bound in (3.1.4) trivial for any $F \in \mathcal{B}(\mathbb{R})$.

If, on the other hand, we choose $F := [1, \infty[$, then we see that

$$\limsup_{n\to\infty} \frac{1}{n} \cdot \log \mu_n(F) = 0 > -\infty = -\inf_{x\in\bar{F}} I(x) \ ,$$

which contradicts the upper bound in (3.1.4).

Before the next remark we recall that a family $\{\mu_\varepsilon\}_{\varepsilon>0}$ of probability measures on X is called *exponentially tight*, if for any (arbitrarily big) $\alpha < \infty$ there exists a compact set K_α such that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(K_\alpha^c) < -\alpha \ .$$

I.e., it can be specified that as $\varepsilon \to 0$ the probability measures are "concentrated on K_α".

Remark 3.1.8 (conditions for "weak implies full")**.** *[DZ98, Lemma 1.2.18] Assume that $\{\mu_\varepsilon\}_{\varepsilon>0}$ is an exponentially tight collection of probability measures on $(X, \mathcal{B})$.*

If I is a rate function and $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies a weak large deviations principle with rate function I, then I is a good rate function and $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies a full large deviations principle with rate function I.

We do not need this result in the following. Thus, we state it without proof.

Finally, we state equivalent formulations for the upper and lower bounds in (3.1.4).

The motivation for this reformulation comes from the following observation: Since $\mu_\varepsilon(X) \equiv 1$ for all ε, the upper bound in (3.1.4) implies that for any rate function I governing a large deviations principle, the infimum over X has to be $\inf_{x\in X} I(x) = 0$. Especially, for any good rate function I we can find an $x \in X$ with $I(x) = 0$. If, on the other hand, $\inf_{x\in\bar{A}} I(x) = 0$ holds for a set $A \in \mathcal{B}$, this implies that the upper bound in (3.1.4) is automatically fulfilled for this set A. In the same way, $\inf_{x\in\mathring{A}} I(x) = \infty$ implies that the lower bound in (3.1.4) holds for the set A under consideration.

Lemma 3.1.9 (reformulation of bounds in (3.1.4))**.** *[Gen03, Lemma 2.10]*[4]

(i) The upper bound in (3.1.4) *is equivalent to the following statement:*
For all $\alpha < \infty$ and all $A \in \mathcal{B}$ such that $\bar{A} \subset \Phi_I(\alpha)^c$, the inequality

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(A) \leqslant -\alpha$$

holds.

(ii) The lower bound in (3.1.4) *is equivalent to the following statement:*
For all x with $I(x) < \infty$ and all $A \in \mathcal{B}$ such that $x \in \mathring{A}$, we have that

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(A) \geqslant -I(x) \ . \tag{3.1.5}$$

We won't need part (i) below. Hence, we only prove part (ii).

Proof. Assume that the lower bound in (3.1.4) holds. Choose x such that $I(x) < \infty$ and an $A \in \mathcal{B}$ such that $x \in \mathring{A}$. Then

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(A) \overset{(3.1.4)}{\geqslant} -\inf_{y\in\mathring{A}} I(y) \geqslant -I(x) \ .$$

For the converse implication, let $I : X \to \bar{\mathbb{R}}$ be a rate function and $A \in \mathcal{B}$ such that $\inf_{x\in\mathring{A}} I(x) < \infty$ (otherwise, the lower bound in (3.1.4) is trivial). Choose $x \in \mathring{A}$ such that $I(x) < \infty$ and let $\{\mu_\varepsilon\}_{\varepsilon>0}$ be a family of measures such that (3.1.5) is fulfilled. Then

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(A) \geqslant -I(x) \ .$$

This implies the lower bound in (3.1.4). □

3.1.2. Excursus: Logarithmic Equivalence

During[5] our later proofs, we will quite often make estimates based on the following observations:

Definition 3.1.10 (logarithmic equivalence)**.** Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences of strictly positive real numbers. If

$$\lim_{n\to\infty} \frac{1}{n} \cdot (\log a_n - \log b_n) = 0$$

holds, we say that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are *logarithmically equivalent*. We denote

$$a_n \simeq b_n \ .$$

[4] Without proof.
[5] This subsection is based on a note which I found in [dH00, Section I.1].

Remark 3.1.11 (log. estimate for sums). *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences of strictly positive real numbers. Then the following holds:*

$$a_n + b_n \simeq a_n \vee b_n \ . \tag{3.1.6}$$

Proof. For any $n \in \mathbb{N}$ we set $c_n := a_n \vee b_n$. Hence, (3.1.6) holds if and only if $a_n + b_n \simeq c_n$.

On the one hand, we have that

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{n} \cdot \big(\log(a_n + b_n) - \log c_n\big)\\ &\quad \leqslant \lim_{n\to\infty} \frac{1}{n} \cdot \big(\log(2c_n) - \log c_n\big) = \lim_{n\to\infty} \frac{1}{n} \cdot \log 2 = 0 \ .\end{aligned}$$

On the other hand, $a_n + b_n \geqslant c_n$ implies that for any $n \in \mathbb{N}$ the following holds:

$$\log(a_n + b_n) - \log c_n \geqslant 0 \ ,$$

and consequently

$$\lim_{n\to\infty} \frac{1}{n} \cdot \big(\log(a_n + b_n) - \log c_n\big) \geqslant 0 \ . \qquad \square$$

Remark 3.1.12 (a late explanation). *[OV05, p. 3] Let $\big\{(\Omega, \mathcal{A}, \mu_n)\big\}_{n\in\mathbb{N}}$ be a series of probability spaces, I an $\mathcal{A}$-measurable function and $A_1, A_2 \in \mathcal{A}$ disjoint sets, such that*

$$\lim_{n\to\infty} \frac{1}{n} \cdot \log \mu_n(A_i) = -I(A_i)$$

holds for $i = 1, 2$. Then, the above result implies that

$$\lim_{n\to\infty} \frac{1}{n} \cdot \log \mu_n(A_1 \cup A_2) = -\min\big\{I(A_1), I(A_2)\big\} \ .$$

A heuristic iteration, yielding

$$\text{“ } I(A) = \inf_{x\in A} I(x) \text{ ” },$$

explains why it makes sense to formulate the large deviations principle using the infimum of a rate function over the set under consideration.

3.1.3. Existence and Uniqueness Properties

If[6] the space X under consideration has a “coarse” topology, the information provided by a large deviations principle may be relatively poor: E.g., if the topology is given by $\{\emptyset, X\}$, then the large deviations principle on the space $\big(X, \mathcal{B}(X)\big)$ only implies that $\inf_{x\in X} I(x) = 0$, and nothing more. Hence, if we intend to prove uniqueness of the rate function, we have to make further assumptions on the topology: we will be able to show, that uniqueness of the rate function holds, if X is a regular Hausdorff space.

[6]This subsection is entirely based on [DZ98, Section 4.1].

Let us recall that a topological space X is called a *Hausdorff space*, if for every pair of points $x \neq y$ in X we can find disjoint open sets $A, B \subset X$, such that $x \in A$ and $y \in B$ (we say, x and y are separated by open neighborhoods).

Furthermore, a Hausdorff space X is named *regular*, if for any closed set $F \subset X$ and any point $x \in F^c$ we can find disjoint open sets $A, B \subset X$ such that $F \subset A$ and $x \in B$.

In the following remark we collect some facts from topology.

Remark 3.1.13 (about regular Hausdorff spaces)**.** *[DZ98, pp. 102–103] Let X be a regular Hausdorff space and $x \in X$.*

(i) *For any neighborhood A of x there exists a neighborhood B of x such that $\bar{B} \subset A$.*

(ii) **(metric spaces).** *Every metric space is a regular Hausdorff space. Furthermore, every real topological vector space with Hausdorff property is regular.*

(iii) **(lower semi-continuous functions).** *Any lower semi-continuous function $f : X \to \bar{\mathbb{R}}$ satisfies*

$$f(x) = \sup\big\{ \inf_{y \in A} f(y) \bigm| A \text{ is neighborhood of } x \big\} .$$

This implies that for every $y \in X$ and any (arbitrarily small) $\delta > 0$ we can find a neighborhood $G(y, \delta)$ of y, such that

$$\big(f(y) - \delta\big) \wedge \frac{1}{\delta} \leqslant \inf_{z \in G(y,\delta)} f(z) .$$

Now, (i) allows us to select a neighborhood $F(y, \delta)$ of y such that $\overline{F(y, \delta)} \subset G(y, \delta)$, and we obtain

$$\inf_{z \in \overline{F(y,\delta)}} f(z) \geqslant \inf_{z \in G(y,\delta)} f(z) \geqslant \big(f(y) - \delta\big) \wedge \frac{1}{\delta} .$$

In metric spaces, sets of the form $G(y, \delta)$ might be selected as balls $B(y, \tilde{\delta})$ with small enough radius $\tilde{\delta}$ (which does not have to be equal to δ!). We will come across this kind of construction during subsequent proofs.

Now we show the promised uniqueness result for rate functions.

Lemma 3.1.14 (uniqueness of rate function)**.** *[DZ98, Lemma 4.1.4] Let X be a regular Hausdorff space. A family $\{\mu_\varepsilon\}_{\varepsilon>0}$ of probability measures on X can not have more than one rate function associated with its large deviations principle.*

Proof. Assume that there are two rate functions I_1, I_2, such that $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies the large deviations principle with both of them. Without loss of generality, we assume that there exists an $x_0 \in X$ such that $I_1(x_0) > I_2(x_0)$.

Now, fix $\delta > 0$ and consider the open set A that fulfills

$$x_0 \in A \quad \text{and} \quad \inf_{y \in \bar{A}} I_1(y) \geqslant \big(I_1(x_0) - \delta\big) \wedge \frac{1}{\delta} .$$

Such a set A exists by part (iii) of the preceding remark. The assumed large deviations principle for $\{\mu_\varepsilon\}_{\varepsilon>0}$ implies that

$$-\inf_{y \in \bar{A}} I_1(y) \geqslant \limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(A) \geqslant \liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(A) \geqslant -\inf_{y \in A} I_2(y) ,$$

hence,

$$I_2(x_0) \geqslant \inf_{y \in A} I_2(y) \geqslant \inf_{y \in \bar{A}} I_1(y) \geqslant \big(I_1(x_0) - \delta\big) \wedge \frac{1}{\delta} \,.$$

Since δ can be selected arbitrarily small, this contradicts the assumption that $I_1(x_0) > I_2(x_0)$. □

Before we show an existence result, let us recall that in any topological space X, a subset $\mathcal{A}$ of the topology is named a *base* of the topology, if any open subset of X (i.e., any element of the topology) is a union of sets from $\mathcal{A}$.

Theorem 3.1.15 (existence of weak large deviations principle)**.** *[DZ98, Theorem 4.1.11] Let X be a topological space and $\mathcal{A}$ a base of the topology on X. Furthermore, let $\{\mu_\varepsilon\}_{\varepsilon>0}$ be a family of measures on $(X, \mathcal{B})$ and define, for any $A \in \mathcal{A}$,*

$$\mathcal{L}_A := -\liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(A) \,.$$

Finally we set, for any $x \in X$,

$$I(x) := \sup\big\{\mathcal{L}_A \;\big|\; A \in \mathcal{A} \text{ such that } x \in A\big\} \,. \tag{3.1.7}$$

If now, for all $x \in X$,

$$I(x) = \sup\big\{-\limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(A) \;\big|\; A \in \mathcal{A} \text{ such that } x \in A\big\} \tag{3.1.8}$$

holds, then the family $\{\mu_\varepsilon\}$ satisfies a weak large deviations principle with the rate function $I(x)$.

The identity (3.1.8) automatically holds if the limit $\lim_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(A)$ exists for all $A \in \mathcal{A}$ (not necessarily finite).

Proof. By definition, I is a nonnegative function, and for any $x \in X$ with $I(x) > \alpha$ for a constant α we can find an $A \in \mathcal{A}$ such that $\mathcal{L}_A > \alpha$. For such a set A we have that for any $y \in A$, again by the definition of I, $I(y) \geqslant \mathcal{L}_A > \alpha$. In other words, for any x with $I(x) > \alpha$ we can find a neighborhood $A \in \mathcal{A}$ such that $I(y) > \alpha$ for all $y \in A$. Hence, for any α the set $\big\{x \in X \;\big|\; I(x) > \alpha\big\}$ is open, and this implies that I as defined in (3.1.7) is a rate function.

Select an open set $G \subset X$. For any $x \in G$ we can find a set $A \in \mathcal{A}$ such that $x \in A \subset G$. Hence, we obtain that

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(G) \geqslant \liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(A) = -\mathcal{L}_A \geqslant -I(x) \,.$$

We have seen in Lemma 3.1.9(ii) that this is equivalent to the lower bound in a large deviations principle.

We note that the condition (3.1.8) was not necessary to prove the fact that I is a rate function, nor the lower bound. We need this condition only to prove the upper bound for compact sets:

Fix a constant $\delta > 0$ and a compact subset F of X and set

$$I^\delta(x) := \big(I(x) - \delta\big) \wedge \frac{1}{\delta} \,.$$

Assumption (3.1.8) implies that for every $x \in F$ there exists a set $A_x \in \mathcal{A}$ (which may depend on δ), such that

$$x \in A_x \quad \text{and} \quad -\limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(A_x) \geqslant I^\delta(x) . \tag{3.1.9}$$

The family $\{A_x\}_{x \in F}$ forms an open cover of F, and since F is compact, there exists a finite cover of F by a sub-family $A_{x_1}, \ldots, A_{x_m}$ of neighborhoods, such that (3.1.9) is fulfilled for any pair x_i, A_{x_i}. Obviously,

$$\mu_\varepsilon(F) \leqslant \sum_{i=1}^{m} \mu_\varepsilon(A_{x_i}) ,$$

and this implies that

$$\begin{aligned} \limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(F) &\leqslant \max_{i=1,\ldots,m} \limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(A_{x_i}) \leqslant - \min_{i=1,\ldots,m} I^\delta(x_i) \\ &\leqslant - \inf_{x \in F} I^\delta(x) . \end{aligned}$$

We complete the proof by taking the limit for $\delta \to 0$. □

In [DZ98, Section 4.2], further results can be found on the existence of large deviations principles as well as on properties of rate functions.

3.1.4. Sample-Path Large Deviations for Brownian Motion: Schilder's Theorem

In[7] this subsection we present the classical large deviations result for Brownian motion first proved by Schilder. We do not only do this for historical reasons: The results provided here will be of use later.

Let W_t be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with state space $\mathbb{R}^d$, where $W_0 = 0$. For any $\varepsilon > 0$ we define

$$W_t^\varepsilon := \sqrt{\varepsilon} \cdot W_t .$$

The following theorem states a large deviations principle for the distribution of this scaled Brownian motion as $\varepsilon \to 0$.

We fix a time $T \in \mathbb{R}_+$ and a dimension $d \in \mathbb{N}$ and recollect that $\mathcal{C}_0 := \{\varphi \in \mathcal{C}([0,T]; \mathbb{R}^d) \mid \varphi_0 = 0\}$.

Theorem 3.1.16 (Schilder, 1966)**.** *[[Gen03, Theorem 2.2], [Sch66]] The family $\{P \circ (W^\varepsilon)^{-1}\}_{\varepsilon > 0}$ of probability measures on $(\mathcal{C}_0, \mathcal{B}(\mathcal{C}_0))$ satisfies a large deviations principle with the good rate function*

$$I(\varphi) := I^{BM}_{[0,T],0}(\varphi) := \begin{cases} \frac{1}{2} \|\varphi\|_{H_1}^2 & \text{if } \varphi \in H_1 , \\ +\infty & \text{otherwise} . \end{cases} \tag{3.1.10}$$

[7] This subsection is largely based on [Gen03, Section 2.2].

In other words, the relation

$$-\inf_{\varphi\in\mathring{\Gamma}} I(\varphi) \leqslant \liminf_{\varepsilon\to 0} \varepsilon \log P[W^\varepsilon \in \Gamma] \leqslant \limsup_{\varepsilon\to 0} \varepsilon \log P[W^\varepsilon \in \Gamma] \leqslant -\inf_{\varphi\in\bar{\Gamma}} I(\varphi) \tag{3.1.11}$$

holds for all $\Gamma \in \mathcal{B}(\mathcal{C}_0)$, and the rate function I has compact level sets $\{I \leqslant \alpha\}$.

Remark 3.1.17 (observation). *[Gen03, Remark 2.3] Since the paths of a Brownian motion are almost surely of unbounded variation, we have that $W^\varepsilon \notin H_1$ almost surely. Hence, for all $\varepsilon > 0$ we get that $I(W^\varepsilon) = +\infty$ almost surely.*

As we have seen in the general case (cf. the notes preceding Lemma 3.1.9), $\inf_{\varphi\in\mathcal{C}_0} I(\varphi) = 0$. Since, in the case of Schilder's theorem, I is a good rate function, there exists a φ such that $I(\varphi) = 0$; e.g., $\varphi(t) :\equiv 0$. Now, (3.1.11) implies that any set containing this φ has maximal probability with respect to $P \circ (W^\varepsilon)^{-1}$ as $\varepsilon \to 0$. In other words, W^ε "concentrates near the zero function". The use of the large deviations principle is that it allows to estimate the probability of rare events, namely that W^ε is "far away from the zero function".

Especially, Schilder's theorem allows us to optimize the estimate shown in Lemma 3.1.1:

Example 3.1.18 (application of Schilder's theorem). [Gen03, Example 2.4] We want to specify the probability that $W^\varepsilon \in \mathcal{C}_0([0,T];\mathbb{R}^d)$ leaves a ball of radius δ around the origin,

$$B := B(0,\delta) := \left\{\varphi \in \mathcal{C}_0\big([0,T];\mathbb{R}^d\big) \;\middle|\; \|\varphi\|_\infty < \delta\right\},$$

for some $T > 0$.

Since the typical spreading of the Brownian motion scales with $\sqrt{t}$, we expect that W^ε remains inside $B(0,\delta)$ as long as $T \ll \frac{\delta^2}{\varepsilon}$.

Here, $\inf_{\varphi\in B^c} I(\varphi)$ is obtained for any φ of the form $\varphi_s = \frac{s}{T}\cdot x$ for an x with $\|x\| = \delta$:

$$\inf_{\varphi\in B^c} I(\varphi) = I\left(\frac{s}{T}\cdot x\right) = \frac{1}{2}\int_0^T \left(\frac{\delta}{T}\right)^2 \mathrm{d}s = \frac{\delta^2}{2T}.$$

Schilder's theorem implies that $P[W^\varepsilon \notin B]$ decays like $\exp\big[-\frac{\delta^2}{2\varepsilon T}\big]$, which is small for $\delta^2 \gg \varepsilon T$, as expected.

We prove Schilder's theorem in three steps. First, we show that the rate function I, as defined in (3.1.10), is a good rate function. Afterwards, we prove the upper and lower bounds in (3.1.11), thus completing the proof of Schilder's theorem.

Lemma 3.1.19 (compactness of level sets). *[FW98, Chap. 3, Lem. 2.1(b)] The level sets of I,*

$$\Phi_I(\alpha) := \{\varphi \in \mathcal{C}_0 \mid I(\varphi) \leqslant \alpha\}, \quad \alpha \in [0,\infty[,$$

are compact.

Proof. Let us first note that for all $\varphi \in \Phi_I(\alpha)$, $\alpha \in [0,\infty[$,

$$\int_0^T \|\dot{\varphi}_s\|^2 \,\mathrm{d}s = \|\varphi\|_{H_1}^2 \leqslant 2\alpha \;.$$

This implies that for all $t \in [0,T]$

$$\|\varphi_t\| = \left\|\varphi_0 + \int_0^t \dot{\varphi}_s \,\mathrm{d}s\right\| \leqslant \|\varphi_0\| + \sqrt{T\int_0^T \|\dot{\varphi}_s\|^2 \,\mathrm{d}s} \leqslant \|\varphi_0\| + \sqrt{T\cdot 2\alpha} \;.$$

Consequently, all $\varphi \in \Phi_I(\alpha)$ are uniformly bounded for any $\alpha \in [0,\infty[$.

Furthermore, for any $\varphi \in \Phi_I(\alpha)$ and all choices of t and h such that $\{t, t+h\} \subset [0,T]$,

$$\begin{aligned}\|\varphi_{t+h} - \varphi_t\| &\leqslant \int_t^{t+h} \|\dot{\varphi}_s\| \,\mathrm{d}s \leqslant \sqrt{h\int_t^{t+h} \|\dot{\varphi}_s\|^2 \,\mathrm{d}s} \\ &\leqslant \sqrt{h\int_0^T \|\dot{\varphi}_s\|^2 \,\mathrm{d}s} \leqslant \sqrt{h\cdot 2\alpha} \xrightarrow{h\to 0} 0 \;,\end{aligned}$$

i.e., all elements of $\Phi_I(\alpha)$ are equicontinuous for any $\alpha \in [0,\infty[$.

The compactness of $\Phi_I(\alpha) \subset \mathcal{C}_0$ follows from Arzela-Ascoli. □

For next step in the proof of Schilder's theorem, we prove a lower bound for the probability that W^ε remains in a ball. This bound depends only on the centre of the ball and on ε.

Lemma 3.1.20 (lower bound for (3.1.11))**.** *[Gen03, Lemma 2.5] For all $\delta > 0$, all $\gamma > 0$ and all $K > 0$ there exists an $\varepsilon_0 = \varepsilon_0(\delta,\gamma,K,T) > 0$ such that for all $\varepsilon \leqslant \varepsilon_0$ and all $\varphi \in \mathcal{C}_0$ with $I(\varphi) < K$ we have*

$$P\big[\|W^\varepsilon - \varphi\|_\infty < \delta\big] \geqslant \exp\left[-\frac{1}{\varepsilon}\cdot\big(I(\varphi)+\gamma\big)\right] \;.$$

Note that by definition of I in (3.1.10), the boundedness of $I(\varphi)$ automatically implies that $\varphi \in H_1$.

This lemma implies the lower bound in (3.1.11):

Proof of Schilder's theorem, part I. The fact that I is a good rate function has been established through Lemma 3.1.19.

To prove the lower bound in (3.1.11), we select an arbitrary open set $G \subset \mathcal{C}_0$. If $\inf_{\varphi\in G} I(\varphi) = \infty$, the lower bound is trivial. Hence we assume that $\inf_{\varphi\in G} I(\varphi) < \infty$. Since I is a good rate function, this allows us to choose a $\varphi \in G$ such that $I(\varphi) < \infty$. And because G is open, we can choose a radius $r_\varphi > 0$ such that the ball $B(\varphi, r_\varphi)$ is contained in G. Now, the above lemma implies that

$$\liminf_{\varepsilon\to 0} \varepsilon \log P[W^\varepsilon \in G] \geqslant \liminf_{\varepsilon\to 0} \varepsilon \log P\big[W^\varepsilon \in B(\varphi, r_\varphi)\big] \geqslant -I(\varphi) \;.$$

We conclude the proof by taking the infimum over all $\varphi \in G$. □

Proof of Lemma 3.1.20. We fix $\delta > 0$, $\gamma > 0$, $K > 0$ and $\varphi \in \mathcal{C}_0$ such that $I(\varphi) \leqslant K$ (hence, $\varphi \in H_1$). We consider

$$P\big[\|W^\varepsilon - \varphi\|_\infty < \delta\big] = P\left[\left\|W - \frac{\varphi}{\sqrt{\varepsilon}}\right\|_\infty < \frac{\delta}{\sqrt{\varepsilon}}\right].$$

Applying Girsanov's formula, we get that

$$\begin{aligned} &P\big[\|W^\varepsilon - \varphi\|_\infty < \delta\big] \\ &\quad = \exp\left[-\frac{1}{2\varepsilon}\int_0^T \|\dot{\varphi}_s\|^2 \,\mathrm{d}s\right] \\ &\qquad \cdot \int_{\{W \in B(0,\delta/\sqrt{\varepsilon})\}} \exp\left[-\frac{1}{\sqrt{\varepsilon}}\int_0^T \langle \dot{\varphi}_s, \mathrm{d}W_s\rangle\right] \mathrm{d}P\,. \end{aligned}$$

We do now split the domain of integration two parts: We set $C := \sqrt{I(\varphi)\cdot\frac{4}{\varepsilon}}$ and define

$$A_C := \left\{\omega \in \Omega \;\middle|\; -\frac{1}{\sqrt{\varepsilon}}\int_0^T \langle \dot{\varphi}_s, \mathrm{d}W_s\rangle \leqslant -C\right\}.$$

To obtain a precise lower bound, we want to base our estimate on those $\omega \in \Omega$ where the integrand is not too small, i.e. A_C^c. Thus we first show that A_C is 'small': Using Chebychev's inequality and our above choice of C we get that

$$\begin{aligned} P(A_C) &= \frac{1}{2}\cdot P\left[\left|\frac{1}{\sqrt{\varepsilon}}\int_0^T \langle \dot{\varphi}_s, \mathrm{d}W_s\rangle\right| \geqslant C\right] \leqslant \frac{1}{2\varepsilon C^2}\cdot \mathbb{E}\left[\left(\int_0^T \langle \dot{\varphi}_s, \mathrm{d}W_s\rangle\right)^2\right] \\ &\leqslant \frac{1}{2\varepsilon C^2}\int_0^T \|\dot{\varphi}_s\|^2 \,\mathrm{d}s = \frac{1}{\varepsilon C^2}\cdot I(\varphi) = \frac{1}{4}\,. \end{aligned}$$

On the other hand we have that

$$\begin{aligned} &P\Big(\big\{\|W^\varepsilon - \varphi\|_\infty < \delta\big\} \cap A_C^c\Big) \\ &\quad \geqslant \exp\left[-\frac{I(\varphi)}{\varepsilon}\right]\cdot \exp[-C]\cdot P\left(\left\{W \in B\left(0, \frac{\delta}{\sqrt{\varepsilon}}\right)\right\} \cap A_C^c\right) \\ &\quad \geqslant \exp\left[-\frac{I(\varphi)}{\varepsilon} - C\right]\cdot\left(P(A_C^c) - P\left[W \in B\left(0, \frac{\delta}{\sqrt{\varepsilon}}\right)^c\right]\right). \end{aligned}$$

Since $P(A_C^c) \geqslant \frac{3}{4}$ we can find a small enough $\varepsilon > 0$ such that

$$P(A_C^c) - P\left[W \in B\left(0, \frac{\delta}{\sqrt{\varepsilon}}\right)^c\right] \geqslant \frac{1}{2}\,.$$

By the definition of C, we finally get that for any small enough ε (say, $\varepsilon \leqslant \varepsilon_0(\delta, \gamma, K, T)$)

$$\begin{aligned} &P\big[\|W^\varepsilon - \varphi\|_\infty < \delta\big] \geqslant P\Big(\big\{\|W^\varepsilon - \varphi\|_\infty < \delta\big\} \cap A_C^c\Big) \\ &\quad \geqslant \exp\left[-\frac{I(\varphi)+\gamma}{\varepsilon}\right]. \end{aligned}$$

□

Now, let us again consider the level sets of I,

$$\Phi_I(\alpha) := \{\varphi \in \mathcal{C}_0 \mid I(\varphi) \leqslant \alpha\}, \quad \alpha \geqslant 0 .$$

These level sets are special neighborhoods of the function $0 \in \mathcal{C}_0$. In the following lemma we provide an upper bound for the probability that W^ε leaves a δ-neighborhood of $\Phi_I(\alpha)$.

Lemma 3.1.21 (upper bound for (3.1.11))**.** *[Gen03, Lemma 2.6] For all $\delta > 0$, all $\gamma > 0$ and all $\alpha_0 > 0$ there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \leqslant \varepsilon_0$ and all $\alpha \leqslant \alpha_0$*

$$P\Big[\operatorname{dist}\big(W^\varepsilon, \Phi_I(\alpha)\big) \geqslant \delta\Big] \leqslant \exp\Big[-\frac{\alpha - \gamma}{\varepsilon}\Big] ,$$

where

$$\operatorname{dist}\big(\varphi, \Phi_I(\alpha)\big) := \min_{\psi \in \Phi_I(\alpha)} \|\varphi_t - \psi_t\|_\infty .$$

The statement of this lemma is equivalent to the upper bound in (3.1.11). We prove, however, only the one implication we actually need:

Proof of Schilder's theorem, part II. Choose an arbitrary closed set F. The result is trivial if $\inf_{\varphi\in F} I(\varphi) = 0$, hence we may assume that $\inf_{\varphi\in F} I(\varphi) > 0$ and choose a $\gamma > 0$ such that $\alpha := \inf_{\varphi\in F} I(\varphi) - \gamma > 0$. As I is a good rate function, the level set $\Phi_I(\alpha)$ is compact. By definition of α, its intersection $\Phi_I(\alpha) \cap F$ with the closed set F is empty, which implies $\delta := \operatorname{dist}\big(F, \Phi_I(\alpha)\big) > 0$. By Lemma 3.1.21 and the definition of α we get that

$$P[W^\varepsilon \in F] \leqslant P\Big[\operatorname{dist}\big(W^\varepsilon, \Phi_I(\alpha)\big) \geqslant \delta\Big] \leqslant \exp\Big[-\frac{\inf_{\varphi\in F} I(\varphi) - 2\gamma}{\varepsilon}\Big] ,$$

which completes the proof of Schilder's theorem. □

Finally, we prove Lemma 3.1.21. The main problem in the proof is the fact that $I(W^\varepsilon) = \infty$, hence we have to approximate the scaled Brownian motion by functions from H_1. We use random polygons to solve this problem.

Proof of Lemma 3.1.21. To construct an approximating random polygon $x^{n,\varepsilon}$ for W^ε, we divide the time intervall $[0,T]$ into parts of the identical length $\Delta > 0$. We will specify Δ later; for now we assume that $\frac{T}{\Delta} \in \mathbb{N}$. The approximating polygon $x^{n,\varepsilon}$ shall have the vertices

$$(0,0),\ \big(\Delta, x^\varepsilon_\Delta\big),\ \big(2\Delta, x^\varepsilon_{2\Delta}\big),\ \ldots,\ \big(T, x^\varepsilon_T\big) .$$

To prove the upper bound claimed in the lemma with the help of this approximation, we consider two events: Either $x^{n,\varepsilon}$ is a bad approximation of W^ε or $x^{n,\varepsilon}$ is so good an approximation that it leaves $\Phi_I(\alpha)$ whenever W^ε leaves the δ-neighborhood of the level set:

$$P\Big[\operatorname{dist}\big(W^\varepsilon, \Phi_I(\alpha)\big) \geqslant \delta\Big] \leqslant P\big[\|W^\varepsilon - x^{n,\varepsilon}\|_\infty \geqslant \delta\big] + P\big[I(x^{n,\varepsilon}) > \alpha\big] . \quad (3.1.12)$$

First, we prove an upper bound for the first summand in (3.1.12), which is the probability of $x^{n,\varepsilon}$ being a bad approximation. We use the fact that the distances $\|W_s^\varepsilon - x_s^{n,\varepsilon}\|$, considered on different time intervalls $\big[k\Delta, (k+1)\Delta\big[$, are identically distributed.

$$
\begin{aligned}
P\big[\|W^\varepsilon - x^{n,\varepsilon}\|_\infty \geqslant \delta\big] &= P\big[\sup_{0\leqslant s\leqslant T} \|W_s^\varepsilon - x_s^{n,\varepsilon}\| \geqslant \delta\big] \\
&\leqslant \quad \frac{T}{\Delta} \cdot P\big[\sup_{0\leqslant s\leqslant \Delta} \|W_s^\varepsilon - x_s^{n,\varepsilon}\| \geqslant \delta\big] \leqslant \frac{T}{\Delta} \cdot P\big[\sup_{0\leqslant s\leqslant \Delta} \|W_s^\varepsilon\| \geqslant \delta\big] \\
&\overset{\text{Lemma 3.1.1}}{\leqslant} \frac{4dT}{\Delta} \cdot \exp\left[-\frac{\delta^2}{2d\varepsilon\Delta}\right],
\end{aligned}
$$

and, choosing $\Delta := \frac{\delta^2}{2d\alpha_0}$, we get that for all $\varepsilon \leqslant \varepsilon_0 = \varepsilon_0(T, \delta, \gamma, \alpha_0)$

$$
\begin{aligned}
P\big[\|W^\varepsilon - x^{n,\varepsilon}\|_\infty \geqslant \delta\big] &\leqslant \frac{1}{2} \cdot \exp\left[-\frac{\alpha_0 - \varepsilon \log\left(\frac{4d^2T\alpha_0}{\delta^2}\right)}{\varepsilon}\right] \\
&\leqslant \frac{1}{2} \cdot \exp\left[-\frac{\alpha_0 - \gamma}{\varepsilon}\right].
\end{aligned}
$$

Next we estimate the second summand in (3.1.12) which specifies the probability that the approximation $x^{n,\varepsilon}$ leaves the level set. Since $x^{n,\varepsilon}$ is a polygon, we have that

$$
\begin{aligned}
I(x^{n,\varepsilon}) &= \frac{1}{2} \sum_{l=1}^{T/\Delta} \int_{(l-1)\Delta}^{l\Delta} \frac{\big\|\sqrt{\varepsilon} \cdot W_{l\Delta} - \sqrt{\varepsilon} \cdot W_{(l-1)\Delta}\big\|^2}{\Delta^2} \, \mathrm{d}s \\
&= \frac{\varepsilon}{2} \sum_{l=1}^{T/\Delta} \frac{\big\|W_{l\Delta} - W_{(l-1)\Delta}\big\|^2}{\Delta}.
\end{aligned}
$$

The sum on the right-hand side has the same distribution as the sum $\sum \xi_i^2$ over the squares of $\frac{dT}{\Delta}$ independent, one dimensional standard-normal random variables ξ_i, which can be estimated by Chebychev's inequality; hence, we achieve for any $\kappa \in \big]0, \frac{1}{2}\big[$, that

$$
\begin{aligned}
P\big[I(x^{n,\varepsilon}) > \alpha\big] &= P\left[\sum_{i=1}^{dT/\Delta} \xi_i^2 > \frac{2\alpha}{\varepsilon}\right] \leqslant \exp\left[-\frac{2\kappa\alpha}{\varepsilon}\right] \cdot \Big(\mathbb{E}\big[\exp[\kappa\xi_1^2]\big]\Big)^{\frac{dT}{\Delta}} \\
&\leqslant (1 - 2\kappa)^{-\frac{dT}{2\Delta}} \cdot \exp\left[-\frac{2\kappa\alpha}{\varepsilon}\right].
\end{aligned}
$$

Now we choose $\kappa = \frac{1}{2}\big(1 - \frac{\gamma}{2\alpha}\big)$. Then for any small enough ε

$$
P\big[I(x^{n,\varepsilon}) > \alpha\big] = \left(\frac{\gamma}{2\alpha}\right)^{-\frac{dT}{2\Delta}} \cdot \underbrace{\exp\left[-\frac{\alpha - \frac{\gamma}{2}}{\varepsilon}\right]}_{=\exp\left[-\frac{\gamma/2}{\varepsilon}\right] \cdot \exp\left[-\frac{\alpha-\gamma}{\varepsilon}\right]} \leqslant \frac{1}{2} \cdot \exp\left[-\frac{\alpha - \gamma}{\varepsilon}\right]
$$

and the lemma is proved. □

3.1.5. The Contraction Principle (continuous version)

Let us return to the general setting from Subsection 3.1.1.

As soon as we have established a large deviations principle with a good rate function for a family of probability measures $\{\mu_\varepsilon\}_{\varepsilon>0}$, the basic, continuous version of the contraction principle as proved below provides us with a large deviations principle for $\{\mu_\varepsilon \circ f^{-1}\}_{\varepsilon>0}$, where f is assumed to be a continuous mapping of spaces.

Theorem 3.1.22 (contraction principle – continuous version)**.** *[DZ98, Theorem 4.2.1] Let X, Y be topological spaces, $I : X \to [0,\infty]$ a good rate function and $f : X \to Y$ continuous. We define the function $I' : Y \to [0,\infty]$ by*

$$I'(y) := \inf\{ I(x) \mid x \in X \text{ such that } f(x) = y \} .$$

Then I' is a good rate function on Y.

If I governs a large deviations principle for $\{\mu_\varepsilon\}_{\varepsilon>0}$ on X, then I' governs a large deviations principle for the image measures $\{\mu_\varepsilon \circ f^{-1}\}_{\varepsilon>0}$ on Y.

Of course, this result may also be applied if X and Y are the same space but e.g. equipped with different topologies.

Proof. We first show that I' is a good rate function. The nonnegativity of I' is obvious by definition. Since I is a good rate function, we know that for all $y \in f(X)$ the infimum in the definition of I' is obtained for (at least) one $x \in X$. Hence, we get for the level sets $\Phi_{I'}(\alpha) := \{ y \in Y \mid I'(y) \leqslant \alpha \}$ that

$$\Phi_{I'}(\alpha) = \{ f(x) \mid x \in X \text{ s.th. } I(x) \leqslant \alpha \} = f\big(\Phi_I(\alpha)\big) ,$$

where $\Phi_I(\alpha) := \{ x \in X \mid I(x) \leqslant \alpha \}$ are the level sets of I. The compactness of the level sets of I in X implies the same for the level sets of I' in Y, which makes I' a good rate function.

If we can show that for all $A \in \mathcal{B}(Y)$,

$$\begin{aligned} -\inf_{y \in \mathring{A}} I'(y) &\leqslant \liminf_{\varepsilon \to 0} \varepsilon \log\big(\mu_\varepsilon \circ f^{-1}\big)(A) \\ &\leqslant \limsup_{\varepsilon \to 0} \varepsilon \log\big(\mu_\varepsilon \circ f^{-1}\big)(A) \leqslant -\inf_{y \in \bar{A}} I'(y) , \end{aligned} \tag{3.1.13}$$

the proof is complete.

By the definition of I', we know that for all $A \subset Y$,

$$\inf_{y \in A} I'(y) = \inf_{x \in f^{-1}(A)} I(x) .$$

To show the lower bound of (3.1.13), we have to prove that for all open sets $A \in \mathcal{B}(Y)$

$$-\inf_{x \in f^{-1}(A)} I(x) \leqslant \liminf_{\varepsilon \to 0} \varepsilon \log\big(\mu_\varepsilon \circ f^{-1}\big)(A) . \tag{3.1.14}$$

But since by continuity $f^{-1}(A)$ is open in X for any open $A \subset Y$, the LDP for $\{\mu_\varepsilon\}_{\varepsilon>0}$ implies that

$$-\inf_{x \in f^{-1}(A)} I(x) \leqslant \liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon\big(f^{-1}(A)\big) ,$$

which proves (3.1.14). The proof of (3.1.13) is completed by a similar argument for closed sets, utilizing the upper bound of the LDP for $\{\mu_\varepsilon\}_{\varepsilon>0}$. $\square$

Remark 3.1.23 (possible generalizations)**.** *[DZ98, p. 111]*

(i) *It can be proved, that this result holds even when $\mathcal{B}$ does not contain $\mathcal{B}(X)$.*

(ii) **(if I is not good).** *If in the preceding theorem the rate function I is not good, the large deviations principle for $\{\mu_\varepsilon \circ f^{-1}\}_{\varepsilon>0}$ on Y does still hold.*

However, it may happen in this case that I' as constructed above is no longer a rate function. For example, consider $X = Y = \mathbb{R}$, $I(x) \equiv 0$ and $f(x) = \exp[x]$: In this case, $I'(y) = 0$ for any $y > 0$, whereas for any $z \leqslant 0$ we have that $I'(z) = \infty$. This implies that e.g. $\Phi_{I'}(1) =]0, \infty[$ is an open set.

We want to generalize the contraction principle from continuous functions to functions which can be approximated in some sense by continuous ones. This will be done in Subsection 3.1.7; however, we first have to introduce some additional results.

3.1.6. Exponentially Good Approximations

Definition 3.1.24 (exponential equivalence of measures)**.** [DZ98, Def. 4.2.10] Let (Y, d) be a metric space and consider families $\{\mu_\varepsilon^{(1)}\}_{\varepsilon>0}$ and $\{\mu_\varepsilon^{(2)}\}_{\varepsilon>0}$ of probability measures on Y. The two families are called *exponentially equivalent*, if there exist a family $\big\{(\Omega, \mathcal{B}_\varepsilon, P_\varepsilon)\big\}_{\varepsilon>0}$ of probability spaces and two families $\{Z_\varepsilon^{(1)}\}_{\varepsilon>0}$ and $\{Z_\varepsilon^{(2)}\}_{\varepsilon>0}$ of Y-valued random variables with joint distributions $\{\bar{\mu}_\varepsilon\}_{\varepsilon>0}$ and marginal distributions $\{\mu_\varepsilon^{(1)}\}_{\varepsilon>0}$ and $\{\mu_\varepsilon^{(2)}\}_{\varepsilon>0}$, respectively, such that the following holds:

For any constant $\delta > 0$, we have that

$$\Big\{\omega \in \Omega \;\Big|\; \big(Z_\varepsilon^{(1)}(\omega), Z_\varepsilon^{(2)}(\omega)\big) \in \Gamma_\delta\Big\} \in \mathcal{B}_\varepsilon$$

and

$$\limsup_{\varepsilon\to 0} \varepsilon \log \bar{\mu}_\varepsilon(\Gamma_\delta) = -\infty ,$$

where we set

$$\Gamma_\delta := \big\{(\tilde{y}, y) \in Y \times Y \;\big|\; d(\tilde{y}, y) > \delta\big\} . \tag{3.1.15}$$

Families of random variables $\{Z_\varepsilon^{(1)}\}_{\varepsilon>0}$ and $\{Z_\varepsilon^{(2)}\}_{\varepsilon>0}$, which fulfill these conditions, are also named *exponentially equivalent*.

Remark 3.1.25 (if Y is separable)**.** *[DZ98, p. 114] If Y is a separable space, the required measurability automatically holds.*

The next result provides an idea, why the notion of exponentially equivalent families of probability measures is interesting for us.

Theorem 3.1.26 (large deviations principles for exp. equivalent measures)**.** *[DZ98, Theorem 4.2.13] If a family $\{\mu_\varepsilon^{(1)}\}_{\varepsilon>0}$ of probability measures on a metric space (Y, d) fulfills a large deviations principle with a good rate function I, and if we have a second family $\{\mu_\varepsilon^{(2)}\}_{\varepsilon>0}$ of probability measures which are exponentially equivalent to the aforementioned family, then both families of probability measures fulfill the same large deviations principle.*

The theorem is a consequence of Theorem 3.1.28; cf. the note following that theorem.

Definition 3.1.27 (exponentially good approximations)**.** [DZ98, Def. 4.2.14] Let (Y, d) be a metric space and Γ_δ defined as above. For each $\varepsilon > 0$ and any $m \in \mathbb{N}$, let $(\Omega, \mathcal{B}_\varepsilon, P_\varepsilon)$ be a probability space, and let the Y-valued random variables Z_ε and $Z_{\varepsilon,m}$ be distributed according to the joint distribution $\bar{\mu}_{\varepsilon,m}$, with marginal distributions μ_ε and $\mu_{\varepsilon,m}$, respectively.

The random variables $\{Z_{\varepsilon,m}\}_{\varepsilon>0,m\in\mathbb{N}}$ are called *exponentially good approximations* of $\{Z_\varepsilon\}_{\varepsilon>0}$ if, for every $\delta > 0$, we have that

$$\Big\{\omega \in \Omega \;\Big|\; \big(Z_\varepsilon(\omega), Z_{\varepsilon,m}(\omega)\big) \in \Gamma_\delta\Big\} \in \mathcal{B}_\varepsilon \quad \text{for all } m \in \mathbb{N}$$

and

$$\lim_{m\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \bar{\mu}_{\varepsilon,m}(\Gamma_\delta) = -\infty \,.$$

We call the families of measures $\{\mu_{\varepsilon,m}\}_{\varepsilon>0,m\in\mathbb{N}}$ *exponentially good approximations* of the family $\{\mu_\varepsilon\}_{\varepsilon>0}$ if we can construct a family $\big\{(\Omega, \mathcal{B}_\varepsilon, P_\varepsilon)\big\}_{\varepsilon>0}$ of probability spaces as above.

Theorem 3.1.28 (large deviations under exp. good approximation)**.** *[DZ98, Theorem 4.2.16] Let (Y, d) be a metric space, and suppose that for any $m \in \mathbb{N}$ the family $\{\mu_{\varepsilon,m}\}_{\varepsilon>0}$ of probability measures satisfies a large deviations principle with rate function I_m. Furthermore, assume that $\{\mu_{\varepsilon,m}\}_{\varepsilon>0,m\in\mathbb{N}}$ are exponentially good approximations of $\{\mu_\varepsilon\}_{\varepsilon>0}$. Then the following holds:*

(i) $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies a weak large deviations principle with the rate function

$$I(y) := \sup_{\delta>0} \liminf_{m\to\infty} \inf_{z\in B(y,\delta)} I_m(z) \,. \tag{3.1.16}$$

(ii) If I is a good rate function and for every closed set F in Y we have that

$$\inf_{y\in F} I(y) \leqslant \limsup_{m\to\infty} \inf_{y\in F} I_m(y) \,, \tag{3.1.17}$$

then $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies a full large deviations principle with rate function I.

If the rate functions I_m are good rate functions and independent of m, then the theorem implies that $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies a full large deviations principle with rate function $I = I_m$. In particular, this proves Theorem 3.1.26.

Proof. (i) Let us assume that $\{Z_{\varepsilon,m}\}_{\varepsilon>0,m\in\mathbb{N}}$ are exponentially good approximations of $\{Z_\varepsilon\}_{\varepsilon>0}$ with joint distributions $\{\bar{\mu}_{\varepsilon,m}\}_{\varepsilon>0,m\in\mathbb{N}}$ and marginal distributions $\{\mu_{\varepsilon,m}\}_{\varepsilon>0,m\in\mathbb{N}}$ and $\{\mu_\varepsilon\}_{\varepsilon>0}$, respectively. Let Γ_δ be defined as in (3.1.15).

By application of Theorem 3.1.15 (the existence theorem for a weak large deviations principle) to the (topological) base $\{B(y,\delta)\}_{y\in Y,\delta>0}$ of (Y,d), we obtain the claimed weak large deviations principle after proving that

$$\begin{aligned} I(y) &= -\inf_{\delta>0}\limsup_{\varepsilon\to 0}\varepsilon\log\mu_\varepsilon\big(B(y,\delta)\big) \\ &= -\inf_{\delta>0}\liminf_{\varepsilon\to 0}\varepsilon\log\mu_\varepsilon\big(B(y,\delta)\big)\ . \end{aligned} \tag{3.1.18}$$

To prove (3.1.18), we choose a pair $\delta>0$, $y\in Y$ and note that for any $m\in\mathbb{N}$ and $\varepsilon>0$, we have that

$$\big\{Z_{\varepsilon,m}\in B(y,\delta)\big\}\subset\big\{Z_\varepsilon\in B(y,2\delta)\big\}\cup\big\{(Z_\varepsilon,Z_{\varepsilon,m})\in\Gamma_\delta\big\}\ , \tag{3.1.19}$$

hence,

$$\mu_{\varepsilon,m}\big(B(y,\delta)\big)\leqslant\mu_\varepsilon\big(B(y,2\delta)\big)+\bar{\mu}_{\varepsilon,m}(\Gamma_\delta)\ . \tag{3.1.20}$$

The lower bounds in the large deviations principles for $\{\mu_{\varepsilon,m}\}_{\varepsilon>0,m\in\mathbb{N}}$ do now imply the following for all $m\in\mathbb{N}$:

$$\begin{aligned} -\inf_{z\in B(y,\delta)} I_m(z) &\leqslant \liminf_{\varepsilon\to 0}\varepsilon\log\mu_{\varepsilon,m}\big(B(y,\delta)\big) \\ &\leqslant \liminf_{\varepsilon\to 0}\varepsilon\log\Big(\mu_\varepsilon\big(B(y,2\delta)\big)+\bar{\mu}_{\varepsilon,m}(\Gamma_\delta)\Big) \\ &\leqslant \liminf_{\varepsilon\to 0}\varepsilon\log\mu_\varepsilon\big(B(y,2\delta)\big)\vee\limsup_{\varepsilon\to 0}\underbrace{\varepsilon\log\bar{\mu}_{\varepsilon,m}(\Gamma_\delta)}_{=:\xi_m}\ . \end{aligned}$$

By assumption, $\{\mu_{\varepsilon,m}\}_{\varepsilon>0,m\in\mathbb{N}}$ are exponentially good approximations of $\{\mu_\varepsilon\}_{\varepsilon>0}$; hence, $\xi_m\xrightarrow{m\to\infty}-\infty$ and we conclude that

$$\limsup_{m\to\infty}\big(-\inf_{z\in B(y,\delta)} I_m(z)\big)\leqslant\liminf_{\varepsilon\to 0}\varepsilon\log\mu_\varepsilon\big(B(y,2\delta)\big)\ . \tag{3.1.21}$$

Now we repeat this argument, exchanging the roles of $Z_{\varepsilon,m}$ and Z_ε; parallel to (3.1.19) we see that

$$\big\{Z_\varepsilon\in B(y,\delta)\big\}\subset\big\{Z_{\varepsilon,m}\in B(y,2\delta)\big\}\cup\big\{(Z_{\varepsilon,m},Z_\varepsilon)\in\Gamma_\delta\big\}\ ;$$

the analogue estimate to (3.1.20) is

$$\mu_\varepsilon\big(B(y,\delta)\big)\leqslant\mu_{\varepsilon,m}\big(B(y,2\delta)\big)+\bar{\mu}_{\varepsilon,m}(\Gamma_\delta)\ ,$$

which implies that for any $m\in\mathbb{N}$

$$\begin{aligned} &\limsup_{\varepsilon\to 0}\varepsilon\log\mu_\varepsilon\big(B(y,\delta)\big) \\ &\quad\leqslant\limsup_{\varepsilon\to 0}\varepsilon\log\Big(\mu_{\varepsilon,m}\big(B(y,2\delta)\big)+\bar{\mu}_{\varepsilon,m}(\Gamma_\delta)\Big) \\ &\quad\leqslant\limsup_{\varepsilon\to 0}\varepsilon\log\mu_{\varepsilon,m}\big(B(y,2\delta)\big)\vee\underbrace{\limsup_{\varepsilon\to 0}\varepsilon\log\bar{\mu}_{\varepsilon,m}(\Gamma_\delta)}_{\xrightarrow{m\to\infty}-\infty}\ . \end{aligned}$$

Thus, we obtain the following analogue to (3.1.21), this time using the upper bound of the large deviations principle for $\{\mu_{\varepsilon,m}\}_{\varepsilon>0,m\in\mathbb{N}}$:

$$\begin{aligned}
&\liminf_{m\to\infty}\Big(-\inf_{z\in\overline{B(y,2\delta)}} I_m(z)\Big) \\
&\quad\geqslant \liminf_{m\to\infty}\limsup_{\varepsilon\to 0}\varepsilon\log\mu_{\varepsilon,m}\big(B(y,2\delta)\big)\vee-\infty \\
&\quad\geqslant \limsup_{\varepsilon\to 0}\varepsilon\log\mu_{\varepsilon}\big(B(y,\delta)\big)\ .
\end{aligned}\tag{3.1.22}$$

Remember that for the justification of (3.1.21) and (3.1.22) we did not pose any assumptions on the selection of $y\in Y$ and $\delta>0$.

Because of $\overline{B(y,2\delta)}\subset B(y,3\delta)$ and the definition of I (cf. (3.1.16)), by taking the infimum over $\delta>0$ in (3.1.21) and (3.1.22) we obtain that

$$\begin{aligned}
\inf_{\delta>0}\liminf_{\varepsilon\to 0}\varepsilon\log\mu_{\varepsilon}\big(B(y,\delta)\big) &\geqslant \inf_{\delta>0}\limsup_{m\to\infty}\Big(-\inf_{z\in B(y,\delta)} I_m(z)\Big) \\
&= -\sup_{\delta>0}\liminf_{m\to\infty}\inf_{z\in B(y,\delta)} I_m(z) = -I(y)
\end{aligned}\tag{3.1.23}$$

and

$$\begin{aligned}
\inf_{\delta>0}\limsup_{\varepsilon\to 0}\varepsilon\log\mu_{\varepsilon}\big(B(y,\delta)\big) &\leqslant \inf_{\delta>0}\liminf_{m\to\infty}\Big(-\inf_{z\in B(y,\delta)} I_m(z)\Big) \\
&= -\sup_{\delta>0}\limsup_{m\to\infty}\inf_{z\in B(y,\delta)} I_m(z) \leqslant -I(y)\ .
\end{aligned}\tag{3.1.24}$$

Combining the estimates (3.1.23) and (3.1.24), we get that

$$\inf_{\delta>0}\limsup_{\varepsilon\to 0}\varepsilon\log\mu_{\varepsilon}\big(B(y,\delta)\big)\leqslant -I(y)\leqslant \inf_{\delta>0}\liminf_{\varepsilon\to 0}\varepsilon\log\mu_{\varepsilon}\big(B(y,\delta)\big)\ ,$$

which finally implies (3.1.18).

(ii) Fix $\delta>0$ and a closed set $F\subset Y$. Like above, we note that for any $m\in\mathbb{N}$ and all $\varepsilon>0$ we have that

$$\{Z_\varepsilon\in F\}\subset\{Z_{\varepsilon,m}\in F^\delta\}\cup\big\{(Z_\varepsilon,Z_{\varepsilon,m})\in\Gamma_\delta\big\}\ .$$

The large deviations principles for $\{\mu_{\varepsilon,m}\}_{\varepsilon>0,m\in\mathbb{N}}$ imply hat for any $m\in\mathbb{N}$ the following holds:

$$\begin{aligned}
\limsup_{\varepsilon\to 0}\varepsilon\log\mu_{\varepsilon}(F) &\leqslant \limsup_{\varepsilon\to 0}\log\big(\mu_{\varepsilon,m}(F^\delta)+\bar\mu_{\varepsilon,m}(\Gamma_\delta)\big) \\
&\leqslant \limsup_{\varepsilon\to 0}\varepsilon\log\mu_{\varepsilon,m}(F^\delta)\vee\limsup_{\varepsilon\to 0}\varepsilon\log\bar\mu_{\varepsilon,m}(\Gamma_\delta) \\
&\leqslant \Big(-\inf_{y\in F^\delta} I_m(y)\Big)\vee\limsup_{\varepsilon\to 0}\varepsilon\log\bar\mu_{\varepsilon,m}(\Gamma_\delta)\ .
\end{aligned}$$

Because we know that $\{Z_{\varepsilon,m}\}_{\varepsilon>0,m\in\mathbb{N}}$ are exponentially good approximations of $\{Z_\varepsilon\}_{\varepsilon>0}$, by taking $m\to\infty$ and applying condition (3.1.17) to the set F^δ (closed by definition!), we get that

$$\limsup_{\varepsilon\to 0}\varepsilon\log\mu_{\varepsilon}(F)\leqslant -\limsup_{m\to\infty}\inf_{y\in F^\delta} I_m(y)\leqslant -\inf_{y\in F^\delta} I(y)\ .$$

Using the property of I to be "good" and Remark 3.1.6(iii), we may take limits for $\delta \to 0$ and obtain the upper bound for the claimed full large deviations principle. $\square$

3.1.7. The Contraction Principle (extended)

Now we can extend the contraction principle for continuous functions, which we have proved above, to a more general class of functions.

The following result is a special case of the preceding theorem. Hence, the idea of the proof is to apply the preceding theorem.

Theorem 3.1.29 (contraction principle)**.** *[DZ98, Theorem 4.2.23] Let (Y, d) be a metric space and X a Hausdorff space. Let $\{\mu_\varepsilon\}_{\varepsilon>0}$ be a family of probability measures on X that satisfies the large deviations principle with a good rate function I. For any $m \in \mathbb{N}$, let $f_m : X \to Y$ be continuous mappings. If there exists a measurable map $f : X \to Y$ such that for any $\alpha < \infty$*

$$\limsup_{m\to\infty} \sup\Big\{ d\big(f_m(x), f(x)\big) \;\Big|\; x \in \Phi_I(\alpha) \Big\} = 0\,, \tag{3.1.25}$$

and if $\{\tilde{\mu}_\varepsilon\}_{\varepsilon>0}$ is a family of probability measures on Y for which $\{\mu_\varepsilon \circ f_m^{-1}\}_{\varepsilon>0, m\in\mathbb{N}}$ are exponentially good approximations, then $\{\tilde{\mu}_\varepsilon\}_{\varepsilon>0}$ satisfies a large deviations principle with the good rate function

$$I'(y) = \inf\big\{ I(x) \;\big|\; x \in X \textit{ such that } f(x) = y \big\}\,.$$

Proof. Since the functions $f_m : X \to Y$ are continuous by assumption, we may apply the contraction principle for continuous functions (Theorem 3.1.22), and obtain that for any $m \in \mathbb{N}$ the family of measures $\{\mu_\varepsilon \circ f_m\}_{\varepsilon>0}$ satisfies a large deviations principle on Y with the good rate function

$$I_m(y) := \inf\big\{ I(x) \;\big|\; x \in X \text{ such that } f_m(x) = y \big\}\,.$$

By (3.1.25), $f_m \xrightarrow{m\to\infty} f$ converges uniformly on any level set $\Phi_I(\alpha)$ if $\alpha < \infty$. Thus, by similar arguments as in the proof of Theorem 3.1.22, f is continuous on each of these level sets, I' is a good rate function on Y, and its level sets are $f\big(\Phi_I(\alpha)\big)$.

Now, fix a closed set $F \subset Y$ and define, for any $m \in \mathbb{N}$,

$$\gamma_m := \inf_{y\in F} I_m(y) = \inf_{x \in f_m^{-1}(F)} I(x)\,.$$

Assume that[8]

$$\gamma := \liminf_{m\to\infty} \gamma_m < \infty$$

and select a subsequence $(\gamma_{m_k})_{k\in\mathbb{N}}$ of $(\gamma_m)_{m\in\mathbb{N}}$, such that

$$\gamma_{m_k} \xrightarrow{k\to\infty} \gamma \quad \text{and} \quad \sup_{k\in\mathbb{N}} \gamma_{m_k} =: \beta < \infty\,.$$

[8] see below for the case $\gamma = \infty$

Since I is a good rate function and $f_m^{-1}(F)$ is a closed subset of X for any $m \in \mathbb{N}$, we can find a sequence $(x_k)_{k\in\mathbb{N}} \subset X$ such that $f_{m_k}(x_k) \in F$ and $I(x_k) = \gamma_{m_k} < \beta$ for all $k \in \mathbb{N}$. By the uniform convergence assumption (3.1.25), we know that for any $\delta > 0$ we can find a $K_0 \in \mathbb{N}$ such that for all $k \geqslant K_0$ we have $f(x_k) \in F^\delta$. This implies that, for all $\delta > 0$ and all $k \in \mathbb{N}$ large enough,

$$\inf_{y\in F^\delta} I'(y) \leqslant I'\big(f(x_k)\big) \leqslant I(x_k) = \gamma_{m_k} \; .$$

Hence, again for all $\delta > 0$,

$$\inf_{y\in F^\delta} I'(y) \leqslant \liminf_{k\to\infty} \inf_{y\in F} I_{m_k}(y)$$

(note: this holds trivially if $\gamma = \infty$).

Letting $\delta \to 0$, Remark 3.1.6(iii) shows that for every closed set F we have that

$$\inf_{y\in F} I'(y) \leqslant \liminf_{k\to\infty} \inf_{y\in F} I_{m_k}(y) \; . \tag{3.1.26}$$

This does, in particular, imply that the estimate

$$\inf_{y\in F} I'(y) \leqslant \limsup_{k\to\infty} \inf_{y\in F} I_{m_k}(y)$$

holds, which in turn implies that Assumption (3.1.17) from Theorem 3.1.28 holds.

If we now choose $F := \overline{B(y,\delta)}$, we obtain that

$$I'(y) \overset{\text{Rem. 3.1.13}}{=} \sup_{\delta>0} \inf_{z\in\overline{B(y,\delta)}} I'(z) \overset{(3.1.26)}{\leqslant} \sup_{\delta>0} \liminf_{k\to\infty} \inf_{z\in\overline{B(y,\delta)}} I_{m_k}(z) =: \bar{I}(y) \; .$$

Because $\bar{I}$ is the rate function defined in Theorem 3.1.28, we note that this proof is complete as soon as we can show that $\bar{I}(y) \leqslant I'(y)$ holds for all $y \in Y$ – because then Theorem 3.1.28 implies that $\{\tilde{\mu}_\varepsilon\}_{\varepsilon>0}$ satisfies a full large deviations principle with the rate function $I' = \bar{I}$.

To prove this estimate, we select an $y \in Y$ and assume without loss of generality, that $I'(y) =: \alpha < \infty$. But $I'(y) = \alpha$ implies that $y \in f\big(\Phi_I(\alpha)\big)$, i.e., there exists an $x \in \Phi_I(\alpha)$ such that $f(x) = y$. We set $y_m := f_m(x) \in f_m\big(\Phi_I(\alpha)\big)$, and thus obtain $I_m(y_m) \leqslant \alpha$ for all $m \in \mathbb{N}$. Now, condition (3.1.25) implies that $d(y, y_m) \xrightarrow{m\to\infty} 0$, hence,

$$\bar{I}(y) \leqslant \liminf_{m\to\infty} I_m(y_m) \leqslant \alpha \; ,$$

which proves the required estimate. □

3.1.8. Sample-Path Large Deviations for Strong Solutions of Stochastic Differential Equations: Freidlin-Wentzell Theory

Let us return to the original problem described in the beginning of this chapter. We want to understand the behaviour of the strong solution x^ε of a stochastic differential equation

$$\begin{cases} \mathrm{d}x_t^\varepsilon &= b(x_t^\varepsilon)\,\mathrm{d}t + \sqrt{\varepsilon}\cdot\sigma(x_t^\varepsilon)\,\mathrm{d}W_t\,, \quad t\in[0,T]\,, \quad T<\infty \\ x_0^\varepsilon &= x \end{cases} \tag{3.1.27}$$

in $\mathbb{R}^d$, where we assure the existence of a unique strong solution x^ε by assuming that b,σ are bounded and uniformly Lipschitz continuous.

Our aim in this subsection is to prove a large deviations principle and the corresponding rate function for x^ε, more precisely, for its distributions μ_ε as $\varepsilon\to 0$. We do this in two steps: First we describe a simple special case, and then we go into the details of the large deviations of x^ε.

Simple case: $\sigma\equiv 1$ and $x_0^\varepsilon = 0$

In this special case,[9] we can use the continuous version of the contraction principle to obtain a large deviations principle the distribution of x^ε from Schilder's theorem:

Let f be the unique solution in $\mathcal{C}_0$ of the integral equation

$$f(t) = \int_0^t b\big(f(s)\big)\,\mathrm{d}s + g(t)\,, \quad g\in\mathcal{C}_0,$$

and define

$$\begin{aligned} F:\ \ \mathcal{C}_0 &\ \to\ \mathcal{C}_0 \\ g &\ \mapsto\ F(g) := f\,. \end{aligned}$$

We note that $F\big(\sqrt{\varepsilon}\cdot W\big) = x^\varepsilon$.

To apply the contraction principle using F, we have to prove that F is continuous. We choose $g_1,g_2\in\mathcal{C}_0$ and denote $f_1 = F(g_1)$, $f_2 = F(g_2)$. Then we have that for all $t\in[0,T]$, $T<\infty$,

$$\|f_1-f_2\|_{[0,t]} \leqslant L_b\cdot\int_0^t \|f_1-f_2\|_{[0,s]}\,\mathrm{d}s + \|g_1-g_2\|_{[0,t]}\,,$$

where L_b is a Lipschitz constant such that $\big\|b(x)-b(y)\big\| \leqslant L_b\cdot\|x-y\|$. Gronwall's lemma now implies that

$$\|f_1-f_2\|_{[0,T]} \leqslant \exp[L_bT]\cdot\|g_1-g_2\|_{[0,T]}\,,$$

and hence the continuity of F.

By Schilder's theorem, $\big\{P\circ(W_t^\varepsilon)^{-1}\big\}_{\varepsilon>0}$, for $W_t^\varepsilon := \sqrt{\varepsilon}\cdot W_t$, satisfies a large deviations principle with the good rate function

$$I^{\mathrm{BM}}(\varphi) := I^{\mathrm{BM}}_{[0,T],0}(\varphi) := \begin{cases} \frac{1}{2}\cdot\|\varphi\|^2_{H_1} & \text{if } \varphi\in H_1 \\ +\infty & \text{otherwise}\,. \end{cases}$$

By application of the contraction principle (continuous version) we know that $x^\varepsilon = F(W^\varepsilon)$ satisfies a large deviations principle with the good rate function

$$\begin{aligned} I(f) &:= \inf\Big\{ I^{\mathrm{BM}}_{[0,T],0}(g) \ \Big|\ g\in\mathcal{C}_0 \text{ such that } F(g) = f\Big\} \\ &= \inf\bigg\{ \frac{1}{2}\cdot\|g\|^2_{H_1} \ \bigg|\ g\in\mathcal{C}_0 \text{ such that } F(g)=f\bigg\}\,. \end{aligned}$$

[9] This subsection is based on the first part of [Gen03, Section 2.5].

Remember that the H_1-norm of a function $g \in H_1 = H_1([0,T];\mathbb{R}^d)$ is defined by

$$\|g\|_{H_1} := \sqrt{\int_0^T \|\dot{g}(t)\|^2 \,\mathrm{d}t}\,.$$

Finally, we want to identify I. If $g \notin H_1$, then also $f = F(g) \notin H_1$. If $g \in H_1$, then f is a.s. differentiable with $\dot{f}(t) = b(f(t)) + \dot{g}(t)$, and $f(0) = 0$. Then there exists a constant $B > 0$ such that for all $t \in [0,T]$

$$\|\dot{f}(t)\| \leqslant B \int_0^t \|\dot{f}(s)\| \,\mathrm{d}s + \|b(0)\| + \|\dot{g}(t)\|\,.$$

Gronwall's lemma together with $g \in H_1$ implies that $f \in H_1$. Thus,

$$I(f) = \begin{cases} \frac{1}{2}\int_0^t \big\|\dot{f}(s) - b(f(s))\big\|^2 \,\mathrm{d}s & \text{if } g \in H_1 \ (\Rightarrow f \in H_1) \\ +\infty & \text{if } g \notin H_1 \ (\Rightarrow f \notin H_1)\,. \end{cases}$$

Less simple case

Let x^ε be the unique solution of (3.1.27).[10] We want to understand the large deviations behavior of this stochastic process. The first idea for a proof of a large deviations result in this situation would be to apply the same tools as we did above – construct some continuous transformation F and use the contraction principle. However, the map defined by x^ε on $\mathcal{C}$ does not necessarily have to be continuous: It can be shown that, if we replace the Brownian motion W_t by its polygonal approximation (hence, a continuous approximation), the solution of (3.1.27) differs in the limit from x^ε by a non–zero (so–called Wong–Zakai) correction term. The existence of this non–zero correction term contradicts the assumption of continuity. Hence, we may not use the continuous version of the contraction principle.

On the other hand, the mentioned correction term is of order ε, so we may expect that it will not influence large deviations results. Consequently, we guess that, even though we have just realized that the proof will not work as above, the rate function for this situation might in principle be the same as above:

$$I^{x^\varepsilon}_{[0,T],x}(f) := \inf\left\{ \frac{1}{2} \cdot \|g\|^2_{H_1} \;\middle|\; g \in H_1 \text{ such that} \right. \tag{3.1.28}$$
$$\left. f(t) = x + \int_0^t b(f(s)) \,\mathrm{d}s + \int_0^t \sigma(f(s))\, \dot{g}(s) \,\mathrm{d}s \right\}.$$

The following theorem confirms this guess:

Theorem 3.1.30 (Large deviations principle for $\{P \circ (x^\varepsilon)^{-1}\}_{\varepsilon>0}$)**.** *[DZ98, Theorem 5.6.7] Consider the stochastic differential equation* (3.1.27)*. Assume that b and σ are bounded and uniformly Lipschitz continuous, and x^ε the solution of* (3.1.27)*. Then the family $\{\tilde{\mu}_\varepsilon\}_{\varepsilon>0} := \{P \circ (x^\varepsilon)^{-1}\}_{\varepsilon>0}$ satisfies a large deviations principle with the good rate function $I^{x^\varepsilon}_{[0,T],x}$ as defined in* (3.1.28).

[10] This subsection is almost entirely based on [DZ98, Section 5.6]

Proof. Without loss of generality, we may assume that the initial conditon is $x = 0$, and we may choose $T = 1$. Hence, $\mathcal{C}_0$ is the support of $\tilde{\mu}_\varepsilon$.

Our proof is based on the construction of a series of processes $x^{\varepsilon,m}$ which are shown to approximate x^ε exponentially good and thus allow us to apply the contraction principle as formulated in Theorem 3.1.29.

For any $m \in \mathbb{N}$, let $x^{\varepsilon,m}$ be the solution of the stochastic differential equation

$$\begin{cases} \mathrm{d}x_t^{\varepsilon,m} &= b\big(x^{\varepsilon,m}_{\frac{\lfloor mt \rfloor}{m}}\big)\,\mathrm{d}t + \sqrt{\varepsilon}\cdot\sigma\big(x^{\varepsilon,m}_{\frac{\lfloor mt \rfloor}{m}}\big)\,\mathrm{d}W_t\,, \quad t \in [0,1], \\ x_0^{\varepsilon,m} &= 0\,, \end{cases} \tag{3.1.29}$$

where the drift and diffusion coefficients are by construction "frozen over $\big[\frac{k}{m}, \frac{k+1}{m}\big[$". Since for any $\varepsilon > 0$ and $m \in \mathbb{N}$, x^ε and $x^{\varepsilon,m}$ are strong solutions of (3.1.27) and (3.1.29), respectively, they are defined on the same probability space, and by Lemma 3.1.31 (below) we know that $\{x^{\varepsilon,m}\}_{\varepsilon>0,m\in\mathbb{N}}$ are exponentially good approximations of $\{x^\varepsilon\}_{\varepsilon>0}$.

Like in the "simple case" above, we define functions

$$\begin{aligned} F^m :\ \ \mathcal{C}_0 &\to \mathcal{C}_0 \\ g &\mapsto F^m(g) = h^m\,, \end{aligned}$$

where for any $t \in \big]\frac{k}{m}, \frac{k+1}{m}\big]$, $k = 0, \dots, m-1$, we set $h^m(0) := 0$ and

$$\begin{aligned} h^m(t) := h^m\left(\frac{k}{m}\right) + b\left(h^m\left(\frac{k}{m}\right)\right)\cdot\left[t - \frac{k}{m}\right] \\ + \sigma\left(h^m\left(\frac{k}{m}\right)\right)\cdot\left[g(t) - g\left(\frac{k}{m}\right)\right]. \end{aligned}$$

We want to use these functions as "approximating functions", applying Theorem 3.1.29 (note that $x^{\varepsilon,m} = F^m(W^\varepsilon)$). To do this, we first have to prove that for any $m \in \mathbb{N}$, F^m is continuous. Let $g_1, g_2 \in \mathcal{C}_0$, set $h_1^m = F^m(g_1)$, $h_2^m = F^m(g_2)$ and define

$$e(t) := \left\|h_1^m(t) - h_2^m(t)\right\|.$$

By the continuity and boundedness assumptions on b and σ, we have that for

$t \in \left]\frac{k}{m}, \frac{k+1}{m}\right]$ there exist $L_b, B > 0$ such that

$$\begin{aligned} e(t) \leqslant & \underbrace{\left\| h_1^m\left(\frac{k}{m}\right) - h_2^m\left(\frac{k}{m}\right) \right\|}_{=e(\frac{k}{m})} \\ & + \underbrace{\left\| b\left(h_1^m\left(\frac{k}{m}\right)\right) - b\left(h_2^m\left(\frac{k}{m}\right)\right) \right\|}_{\leqslant L_b \cdot e(\frac{k}{m})} \cdot \underbrace{\left[t - \frac{k}{m}\right]}_{\leqslant \frac{1}{m}} \\ & + \underbrace{\left\| \sigma\left(h_1^m\left(\frac{k}{m}\right)\right) - \sigma\left(h_2^m\left(\frac{k}{m}\right)\right) \right\|}_{\leqslant 2B} \\ & \quad \cdot \underbrace{\left\| g_1(t) - g_1\left(\frac{k}{m}\right) - g_2(t) + g_2\left(\frac{k}{m}\right) \right\|}_{\leqslant 2 \cdot \|g_1 - g_2\|_{[0,1]}} \\ \leqslant & \; C \cdot \left(e\left(\frac{k}{m}\right) + \|g_1 - g_2\|_{[0,1]} \right) \end{aligned}$$

where $C < \infty$ is bigger than

$$\max\left\{2,\ 2 \cdot \frac{L_b}{m},\ 4B\right\} ,$$

and hence,

$$\sup_{t \in \left]\frac{k}{m}, \frac{k+1}{m}\right]} e(t) \leqslant C \cdot \left(e\left(\frac{k}{m}\right) + \|g_1 - g_2\|_{[0,1]} \right) .$$

Knowing that $e(0) = 0$, we can iterate this bound for $k = 0, \ldots, m-1$ and thus obtain the continuity of F^m for all $m \in \mathbb{N}$.

Now, define

$$\begin{aligned} F : \quad H_1 & \rightarrow \mathcal{C}_0 \\ g & \mapsto F(g) = f , \end{aligned}$$

where f shall be the unique solution of the integral equation

$$f(t) = \int_0^t b\big(f(s)\big)\, \mathrm{d}s + \int_0^t \sigma\big(f(s)\big)\, \dot{g}(s)\, \mathrm{d}s , \quad t \in [0,1].$$

Again, existence and uniqueness of the solution are standard because of the a–priori conditions on b and σ. As soon as we prove that for any $\alpha < \infty$,

$$\lim_{m\to\infty} \sup\left\{ \left\|F^m(g) - F(g)\right\|_{[0,1]} \;\middle|\; g \text{ such that } \|g\|_{H_1} \leqslant \alpha \right\} = 0 , \qquad (3.1.30)$$

we know that F^m approximates F in the sense of (3.1.25). As we have noted above, by Lemma 3.1.31 we know that $\{\mu_{\varepsilon,m}\}_{\varepsilon>0,m\in\mathbb{N}}$ approximate $\{\tilde{\mu}_\varepsilon\}_{\varepsilon>0}$ exponentially good. Hence, the conditions for Theorem 3.1.29, the generalized version of the contraction principle, are fulfilled and we have proved the assertion of the theorem.

To prove (3.1.30), we fix $\alpha < \infty$ and $g \in H_1$ with $\|g\|_{H_1} \leqslant \alpha$. We write

$$h^m = F^m(g)\ , \quad f = F(g) \quad \text{and} \quad r(t) := \left\|f(t) - h^m(t)\right\|^2\ ,$$

and note that for all $t \in [0,1]$, h^m fulfills the integral equation

$$h^m(t) = \int_0^t b\left(h^m\left(\frac{\lfloor ms\rfloor}{m}\right)\right)\,\mathrm{d}s + \int_0^t \sigma\left(h^m\left(\frac{\lfloor ms\rfloor}{m}\right)\right)\dot{g}(s)\,\mathrm{d}s\ .$$

By the Cauchy-Schwartz inequality and the fact that $\frac{\lfloor ms\rfloor}{m} = \frac{\lfloor mt\rfloor}{m} = \text{const.}$ for all $s \in \left[\frac{\lfloor mt\rfloor}{m}, t\right[$, $t \in [0,1]$, we have that

$$\begin{aligned}
&\int_{\frac{\lfloor mt\rfloor}{m}}^t \sigma\left(h^m\left(\frac{\lfloor ms\rfloor}{m}\right)\right)\dot{g}(s)\,\mathrm{d}s \\
&\quad\leqslant \sqrt{\int_{\frac{\lfloor mt\rfloor}{m}}^t \sigma\left(h^m\left(\frac{\lfloor mt\rfloor}{m}\right)\right)^2 \mathrm{d}s \cdot \underbrace{\int_{\frac{\lfloor mt\rfloor}{m}}^t \dot{g}(s)^2\,\mathrm{d}s}_{\leqslant\|g\|_{H_1}^2\leqslant\alpha^2}} \\
&\quad\leqslant \sqrt{\left[t - \frac{\lfloor mt\rfloor}{m}\right]} \cdot \sigma\left(h^m\left(\frac{\lfloor mt\rfloor}{m}\right)\right)\cdot\alpha\ ,
\end{aligned}$$

and thus,

$$\begin{aligned}
&h^m(t) - h^m\left(\frac{\lfloor mt\rfloor}{m}\right) \\
&\quad= \int_{\frac{\lfloor mt\rfloor}{m}}^t b\left(h^m\left(\frac{\lfloor ms\rfloor}{m}\right)\right)\mathrm{d}s + \int_{\frac{\lfloor mt\rfloor}{m}}^t \sigma\left(h^m\left(\frac{\lfloor ms\rfloor}{m}\right)\right)\dot{g}(s)\,\mathrm{d}s \\
&\quad\leqslant \left[t - \frac{\lfloor mt\rfloor}{m}\right]\cdot b\left(h^m\left(\frac{\lfloor mt\rfloor}{m}\right)\right) \\
&\qquad+\sqrt{\left[t - \frac{\lfloor mt\rfloor}{m}\right]}\cdot\sigma\left(h^m\left(\frac{\lfloor mt\rfloor}{m}\right)\right)\cdot\alpha\ .
\end{aligned}$$

Because σ, b are bounded, we may find a constant δ_m for every $m \in \mathbb{N}$, which is independent of g, such that $\delta_m \xrightarrow{m\to\infty} 0$ and the following estimate holds for all $t \in [0,1]$:

$$h^m(t) - h^m\left(\frac{\lfloor mt\rfloor}{m}\right) \leqslant \delta_m(1+\alpha)\ . \tag{3.1.31}$$

By the Lipschitz continuity of b, σ we can find a constant $\tilde{C} < \infty$, independent of m and g, such that for all $t \in [0,1]$

$$\begin{aligned}
\sqrt{r(t)} &:= \|f(t) - h^m(t)\| \\
&= \Bigg\| \int_0^t \underbrace{b\big(f(s)\big) - b\left(h^m\left(\frac{\lfloor ms \rfloor}{m}\right)\right)}_{\leqslant \tilde{C}\cdot\|f(s)-h^m(\frac{\lfloor ms \rfloor}{m})\|} \, \mathrm{d}s \\
&\qquad + \int_0^t \underbrace{\left[\sigma\big(f(s)\big) - \sigma\left(h^m\left(\frac{\lfloor ms \rfloor}{m}\right)\right)\right]}_{\leqslant \tilde{C}\cdot\|f(s)-h^m(\frac{\lfloor ms \rfloor}{m})\|} \cdot \dot{g}(s) \, \mathrm{d}s \Bigg\| \\
&\leqslant \tilde{C} \sqrt{\int_0^t \left\| f(s) - h^m\left(\frac{\lfloor ms \rfloor}{m}\right)\right\|^2 \mathrm{d}s} \\
&\qquad + \tilde{C} \sqrt{\int_0^t \left\| f(s) - h^m\left(\frac{\lfloor ms \rfloor}{m}\right)\right\|^2 \mathrm{d}s \cdot \alpha^2} \\
&= (1+\alpha)\cdot \tilde{C} \sqrt{\int_0^t \left\| f(s) - h^m\left(\frac{\lfloor ms \rfloor}{m}\right)\right\|^2 \mathrm{d}s} \, .
\end{aligned}$$

This implies that $r(0) = 0$. By application of (3.1.31) we see that there are $K_1, K_2 < \infty$, depending on $\tilde{C}$ and α, such that for all $t \in [0,1]$

$$\begin{aligned}
r(t) = \|f(t) - h^m(t)\|^2 &\leqslant \underbrace{(1+\alpha)^2 \cdot \tilde{C}^2}_{=K_1} \int_0^t \left\| f(s) - h^m\left(\frac{\lfloor ms \rfloor}{m}\right)\right\|^2 \mathrm{d}s \\
&\leqslant K_1 \int_0^t \|f(s) - h^m(s)\|^2 \, \mathrm{d}s + \underbrace{K_1 \int_0^t \Bigg\| \underbrace{h^m(s) - h^m\left(\frac{\lfloor ms \rfloor}{m}\right)}_{\leqslant \delta_m(1+\alpha)} \Bigg\|^2 \mathrm{d}s}_{\leqslant \delta_m^2 \cdot \underbrace{K_1 \int_0^1 (1+\alpha)^2 \, \mathrm{d}s}_{=K_2}} \\
&\leqslant K_1 \int_0^t r(s) \, \mathrm{d}s + \delta_m^2 \cdot K_2 \, ,
\end{aligned}$$

hence $r(t) \leqslant K_2 \delta_m^2 \cdot \exp[K_1 t]$ by Gronwall's lemma. Finally, we obtain that

$$\|F(g) - F^m(g)\|_{[0,1]} \leqslant \sqrt{K_2} \cdot \delta_m \cdot \exp\left[\frac{K_1 \cdot 1}{2}\right] ,$$

which proves (3.1.30), because the selection of $g \in H_1$ was without further assumptions and because $\delta_m \xrightarrow{m\to\infty} 0$. □

Lemma 3.1.31 ($x^{\varepsilon,m}$ approximate x^ε exp. good)**.** *[DZ98, Lemma 5.6.9] In the situation of the above proof, the following holds for any $\delta > 0$:*

$$\lim_{m\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log P\big[\|x^{\varepsilon,m} - x^\varepsilon\|_{[0,1]} > \delta\big] = -\infty \,.$$

To prove this lemma, we need the following result; remember that, for a measurable space $(\Omega, \mathcal{F})$ and a filtration $\{\mathcal{F}_t\}_{t\in[0,\infty[}$, a function X mapping $[0,\infty[\, \times\, \Omega$ into a measurable space $(E, \mathcal{A})$ is named *progressively measurable*, if $X|_{[0,T]\times\Omega}$ is $\mathcal{B}_{[0,T]} \times \mathcal{F}_T$-measurable for every $T < \infty$.

Lemma 3.1.32. *[DZ98, Lemma 5.6.18] Let b_t and σ_t be progressively measurable processes, and consider the equation*

$$\mathrm{d}z_t = b_t\,\mathrm{d}t + \sqrt{\varepsilon}\cdot\sigma_t\,\mathrm{d}W_t \quad \textit{in } \mathbb{R}^d \,, \tag{3.1.32}$$

where z_0 is deterministic. Let $\tau_1 \in [0,1]$ be a stopping time with respect to the filtration of $\{W_t\}_{t\in[0,1]}$. Suppose that the coefficients of the diffusion matrix σ are uniformly bounded, and that for some constants M, B, ρ and any $t \in [0,\tau_1]$ we have that

$$\begin{aligned} \|\sigma_t\| &\leqslant M\cdot\big(\rho^2 + \|z_t\|^2\big)^{1/2} \\ \|b_t\| &\leqslant B\cdot\big(\rho^2 + \|z_t\|^2\big)^{1/2} \,. \end{aligned} \tag{3.1.33}$$

Then for any $\delta > 0$ and any $\varepsilon \leqslant 1$ the following estimate holds:

$$\varepsilon \log P\big[\sup_{t\in[0,\tau_1]} \|z_t\| \geqslant \delta\big] \leqslant K + \log\left(\frac{\rho^2 + \|z_0\|^2}{\rho^2+\delta^2}\right) ,$$

where $K := 2B + M^2(2+d)$.

Note that the notation in (3.1.32) is 'abbreviating': b and σ are dependent on variables in space and time. Cf. the proof of Lemma 3.1.31.

Proof. For any $y \in \mathbb{R}^d$ we define

$$\phi(y) := \big(\rho^2 + \|y\|^2\big)^{1/\varepsilon}$$

and set $u_t := \phi(z_t)$. By Itô's formula, u_t is the strong solution of the stochastic differential equation

$$\begin{aligned} \mathrm{d}u_t = {} & \big(\nabla\phi(z_t)\big)^T\,\mathrm{d}z_t + \frac{\varepsilon}{2}\cdot \mathrm{tr}\big(\sigma_t\sigma_t^T\, D^2\phi(z_t)\big)\,\mathrm{d}t \\ =: {} & g_t\,\mathrm{d}t + \tilde{\sigma}_t\,\mathrm{d}W_t \,, \end{aligned} \tag{3.1.34}$$

where $D^2\phi(y)$ denotes the matrix of second derivatives of $\phi(y)$. Because of

$$\nabla\phi(y) = \frac{1}{\varepsilon}\cdot\big(\rho^2+\|y\|^2\big)^{\frac{1}{\varepsilon}-1}\cdot 2y = \frac{2\phi(y)}{\varepsilon\cdot\big(\rho^2+\|y\|^2\big)}\cdot y$$

and (3.1.33), we have that, for any $t \in [0, \tau_1]$,

$$\left\| \left(\nabla\phi(z_t)\right)^T b_t \right\| \leqslant \frac{2\phi(z_t)\cdot B \cdot \|z_t\|}{\varepsilon\cdot\left(\rho^2+\|z_t\|^2\right)^{1/2}} = \frac{2B}{\varepsilon}\cdot\phi(z_t)\cdot\frac{\|z_t\|}{\left(\rho^2+\|z_t\|^2\right)^{1/2}}$$
$$\leqslant \frac{2B}{\varepsilon}\cdot u_t\ . \tag{3.1.35}$$

In a similar way we obtain that, for any $\varepsilon \leqslant 1$ and any $t \in [0, \tau_1]$,

$$\frac{\varepsilon}{2}\cdot \mathrm{tr}\left(\sigma_t\sigma_t^T\, D^2\phi(z_t)\right) \leqslant \frac{M^2(2+d)}{\varepsilon}\cdot u_t$$

(using that $(\sigma_t^{(i)})^2 \leqslant M^2\cdot\left(\rho^2+\|z_t\|^2\right)$ for all $i = 1, \ldots, d$ and estimating componentwise). With these estimates and (3.1.34), we get that, for any $t \in [0, \tau_1]$,

$$g_t \leqslant \frac{Ku_t}{\varepsilon}\ , \tag{3.1.36}$$

with K as defined above.

Now, fix $\delta > 0$ and define the stopping time

$$\tau_2 := \inf\left\{t \mid \|z_t\| \geqslant \delta\right\} \wedge \tau_1\ .$$

Since, similar to (3.1.35),

$$\|\tilde{\sigma}_t\| \leqslant \frac{2Mu_t}{\sqrt{\varepsilon}}$$

is uniformly bounded on $[0, \tau_2]$, we know that

$$u_t - \int_0^t g_s\, \mathrm{d}s$$

is a continuous martingale up to τ_2. Hence, we may apply Doob's optional sampling theorem and obtain that

$$\mathbb{E}[u_{t\wedge\tau_2}] = u_0 + \mathbb{E}\left[\int_0^{t\wedge\tau_2} g_s\, \mathrm{d}s\right]\ .$$

Combining this with (3.1.36) and the fact that u is non–negative by definition, we get that

$$\mathbb{E}[u_{t\wedge\tau_2}] \leqslant u_0 + \frac{K}{\varepsilon}\cdot\mathbb{E}\left[\int_0^{t\wedge\tau_2} u_s\, \mathrm{d}s\right] = u_0 + \frac{K}{\varepsilon}\cdot\mathbb{E}\left[\int_0^{t\wedge\tau_2} u_{s\wedge\tau_2}\, \mathrm{d}s\right]$$
$$\leqslant u_0 + \frac{K}{\varepsilon}\cdot\int_0^t \mathbb{E}[u_{s\wedge\tau_2}]\, \mathrm{d}s\ .$$

Gronwall's lemma implies now that

$$\mathbb{E}[u_{\tau_2}] = \mathbb{E}[u_{1\wedge\tau_2}] \leqslant u_0\cdot\exp\left[\frac{K}{\varepsilon}\right] = \phi(z_0)\cdot\exp\left[\frac{K}{\varepsilon}\right]\ .$$

We note that $\phi(y)$ is positive and monotonely increasing in $\|y\|$; hence, by Markov's inequality,

$$P\big[\|z_{\tau_2}\| \geqslant \delta\big] = P\big[\phi(z_{\tau_2}) \geqslant \phi(\delta)\big] \leqslant \frac{\mathbb{E}\big[\phi(z_{\tau_2})\big]}{\phi(\delta)} = \frac{\mathbb{E}[u_{\tau_2}]}{\phi(\delta)} \; .$$

Now we combine the two preceding inequalities and finally see that

$$\begin{aligned}\varepsilon \log P\big[\|z_{\tau_2}\| \geqslant \delta\big] &\leqslant \varepsilon \log\bigg(\frac{\mathbb{E}[u_{\tau_2}]}{\phi(\delta)}\bigg) \leqslant \varepsilon \log\bigg(\frac{\phi(z_0)\cdot \exp\big[\frac{K}{\varepsilon}\big]}{\phi(\delta)}\bigg) \\ &= K + \log\bigg(\frac{\phi(z_0)}{\phi(\delta)}\bigg) = K + \log\bigg(\frac{\rho^2 + \|z_0\|^2}{\rho^2+\delta^2}\bigg) \; .\end{aligned}$$

The lemma is proved, as we have that

$$\sup_{t\in[0,\tau_1]} \|z_t\| \geqslant \delta \quad \Leftrightarrow \quad \|z_{\tau_2}\| \geqslant \delta \; . \qquad \square$$

Proof of Lemma 3.1.31. We fix $\delta > 0$ and define, for any $\rho > 0$, the stopping time

$$\tau_1 := \inf\Big\{ t \;\Big|\; \|x_t^{\varepsilon,m} - x_{\frac{\lfloor mt\rfloor}{m}}^{\varepsilon,m}\| \geqslant \rho \Big\} \wedge 1 \; .$$

If we denote

$$z_t := x_t^{\varepsilon,m} - x_t^{\varepsilon} \; ,$$

we see that this process z_t fulfills the differential equation (3.1.32), with

$$z_0 = 0 \; , \quad b_t := b\big(x_{\frac{\lfloor mt\rfloor}{m}}^{\varepsilon,m}\big) - b(x_t^{\varepsilon}) \quad \text{and} \quad \sigma_t := \sigma\big(x_{\frac{\lfloor mt\rfloor}{m}}^{\varepsilon,m}\big) - \sigma(x_t^{\varepsilon}) \; .$$

Hence, as the assumptions on b and σ are fulfilled by any bounded and Lipschitz-continuous function, we may apply Lemma 3.1.32, which shows that for any $\delta > 0$ and any $\varepsilon \leqslant 1$ we can find a $K < \infty$ that is independent of $\varepsilon, \delta, \rho$ and m and makes the following estimate valid:

$$\varepsilon \log P\big[\sup_{t\in[0,\tau_1]} \|x_t^{\varepsilon,m} - x_t^{\varepsilon}\| \geqslant \delta\big] \leqslant K + \log\bigg(\frac{\rho^2}{\rho^2+\delta^2}\bigg) \; ,$$

which in turn implies that

$$\lim_{\rho\to 0} \sup_{m\geqslant 1} \limsup_{\varepsilon\to 0} \varepsilon \log P\big[\sup_{t\in[0,\tau_1]} \|x_t^{\varepsilon,m} - x_t^{\varepsilon}\| \geqslant \delta\big] = -\infty \; .$$

Since, by construction of τ_1, for any $\rho > 0$

$$\big\{\|x^{\varepsilon,m} - x^{\varepsilon}\|_{[0,1]} > \delta\big\} \subset \{\tau_1 < 1\} \cup \big\{\sup_{t\in[0,\tau_1]} \|x_t^{\varepsilon,m} - x_t^{\varepsilon}\| \geqslant \delta\big\} \; ,$$

the proof of the lemma is complete if we can show that

$$\lim_{m\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log P\Big[\sup_{t\in[0,1]} \big\|x_t^{\varepsilon,m} - x_{\frac{\lfloor mt\rfloor}{m}}^{\varepsilon,m}\big\| \geqslant \rho\Big] = -\infty \; . \qquad (3.1.37)$$

The boundedness of b and σ allows us to find a constant C such that for all $m \in \mathbb{N}$

$$\begin{aligned}\left\|x_t^{\varepsilon,m} - x^{\varepsilon,m}_{\frac{\lfloor mt \rfloor}{m}}\right\| &\leqslant \int_{\frac{\lfloor mt \rfloor}{m}}^t \left\|b\left(x^{\varepsilon,m}_{\frac{\lfloor mt \rfloor}{m}}\right)\right\| \,\mathrm{d}t + \sqrt{\varepsilon}\int_{\frac{\lfloor mt \rfloor}{m}}^t \left\|\sigma\left(x^{\varepsilon,m}_{\frac{\lfloor mt \rfloor}{m}}\right)\right\| \,\mathrm{d}W_t \\ &\leqslant C \cdot \left(\frac{1}{m} + \sqrt{\varepsilon} \cdot \max_{k=0,\dots,m-1} \sup_{s \in [0,\frac{1}{m}]} \left\|W_{s+\frac{k}{m}} - W_{\frac{k}{m}}\right\|\right) .\end{aligned}$$

This implies that for all $m > \frac{C}{\rho}$

$$\begin{aligned}P\left[\sup_{t\in[0,1]} \left\|x_t^{\varepsilon,m} - x^{\varepsilon,m}_{\frac{\lfloor mt \rfloor}{m}}\right\| \geqslant \rho\right] &\leqslant m \cdot P\left[\sup_{s\in[0,\frac{1}{m}]} \|W_s\| \geqslant \frac{\rho - \frac{C}{m}}{\sqrt{\varepsilon}\cdot C}\right] \\ &\leqslant 4d \cdot m \cdot \exp\left[-\frac{m\left(\rho - \frac{C}{m}\right)^2}{2d \cdot \varepsilon C^2}\right] ,\end{aligned}$$

where the last estimate is based on Lemma 3.1.1. This completes the proof. □

Finally, we extend the results of this section by the following theorem, which allows more general initial conditions, and the subsequent corollary, which we will need in the next section on diffusion exit from a domain.

Theorem 3.1.33. *[DZ98, Theorem 5.6.12] Consider the situation of Theorem 3.1.30. Let $x^{\varepsilon,y}$ denote the solution of the stochastic differential equation* (3.1.27) *for the initial condition $x_0^{\varepsilon,y} := y$. Then the following holds:*

(a) For any closed set $F \subset \mathcal{C}([0,1];\mathbb{R}^d)$,

$$\limsup_{\substack{\varepsilon\to 0,\\ y\to x}} \varepsilon \log P[x^{\varepsilon,y} \in F] \leqslant -\inf_{\varphi\in F} I^{x^\varepsilon}_{[0,1],x}(\varphi) . \tag{3.1.38}$$

(b) For any open set $G \subset \mathcal{C}([0,1];\mathbb{R}^d)$,

$$\liminf_{\substack{\varepsilon\to 0,\\ y\to x}} \varepsilon \log P[x^{\varepsilon,y} \in G] \geqslant -\inf_{\varphi\in G} I^{x^\varepsilon}_{[0,1],x}(\varphi) . \tag{3.1.39}$$

Corollary 3.1.34 (generalized large dev. principle)**.** *[DZ98, Corollary 5.6.15] Consider the situation of Theorem 3.1.30. Then for any compact set $K \subset \mathbb{R}^d$ and any closed set $F \subset \mathcal{C}([0,1];\mathbb{R}^d)$ we have that*

$$\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{y\in K} P[x^{\varepsilon,y} \in F] \leqslant -\inf_{\substack{\varphi\in F,\\ y\in K}} I^{x^\varepsilon}_{[0,1],y}(\varphi) . \tag{3.1.40}$$

Similarly, for any open set $G \subset \mathcal{C}([0,1];\mathbb{R}^d)$,

$$\liminf_{\varepsilon\to 0} \varepsilon \log \inf_{y\in K} P[x^{\varepsilon,y} \in G] \geqslant -\sup_{y\in K} \inf_{\varphi\in G} I^{x^\varepsilon}_{[0,1],y}(\varphi) . \tag{3.1.41}$$

Proof. Let $-I_K$ denote the right hand side of (3.1.40). Fix $\delta > 0$ and set

$$I_K^\delta := \min\left\{ I_K - \delta, \frac{1}{\delta} \right\} .$$

Then (3.1.38) implies that for every $x \in K$ we can find an $\varepsilon_x > 0$ such that for all $\varepsilon \leqslant \varepsilon_x$

$$\varepsilon \log \sup_{y \in B(x,\varepsilon_x)} P[x^{\varepsilon,y} \in F] \leqslant -I_K^\delta .$$

We can cover K by open balls $B(x_i, \varepsilon_{x_i})$ fulfilling this estimate, where $i \in I$ for some index set I. Choose $x_1, \dots, x_m \in K$ such that the compact set K is covered by the finite union $\bigcup_{i=1}^m B(x_i, \varepsilon_{x_i})$ of open balls. Hence, for $\varepsilon \leqslant \min_{i=1,\dots,n} \varepsilon_{x_i}$,

$$\varepsilon \log \sup_{y \in K} P[x^{\varepsilon,y} \in F] \leqslant -I_K^\delta .$$

Now take first $\varepsilon \to 0$ and then $\delta \to 0$; this implies (3.1.40).

(3.1.41) can be proved by a similar argument based on (3.1.39). □

Proof of Theorem 3.1.33. By Theorem 3.1.26 (on large deviations principles for exponentially equivalent families of measures), we only have to show that the family $\{x^{\varepsilon,x}\}_{\varepsilon>0}$ is exponentially equivalent to $\{x^{\varepsilon,x_\varepsilon}\}_{\varepsilon>0}$ whenever $x_\varepsilon \xrightarrow{\varepsilon\to 0} x$.

We fix a family $(x_\varepsilon)_{\varepsilon>0}$ such that $x_\varepsilon \xrightarrow{\varepsilon\to 0} x$ and denote

$$z_t := x_t^{\varepsilon,x_\varepsilon} - x_t^{\varepsilon,x} .$$

Then z_t fulfills (3.1.32), with

$$z_0 := x_\varepsilon - x , \quad \sigma_t := \sigma(x_t^{\varepsilon,x_\varepsilon}) - \sigma(x_t^{\varepsilon,x}) \quad \text{and} \quad b_t := b(x_t^{\varepsilon,x_\varepsilon}) - b(x_t^{\varepsilon,x}) .$$

The standard properties of b, σ imply that (3.1.33) holds for any $\rho > 0$ and $\tau_1 = 1$. Thus, applying Lemma 3.1.32, we obtain that for any $\delta > 0$ and any $\rho > 0$ we can find a $K < \infty$ which is independent of ε, δ and ρ, such that

$$\varepsilon \log P\big[\|x^{\varepsilon,x_\varepsilon} - x^{\varepsilon,x}\|_{[0,1]} \geqslant \delta\big] \leqslant K + \log\left(\frac{\rho^2 + \|x_\varepsilon - x\|^2}{\rho^2 + \delta^2}\right) .$$

Now, we take first $\rho \to 0$ and then $\varepsilon \to 0$, and get

$$\limsup_{\varepsilon\to 0} \varepsilon \log P\big[\|x^{\varepsilon,x_\varepsilon} - x^{\varepsilon,x}\|_{[0,1]} \geqslant \delta\big] \leqslant K + \limsup_{\varepsilon\to 0} \log\left(\frac{\|x_\varepsilon - x\|^2}{\delta^2}\right) .$$

This proves the exponential equivalence of $x^{\varepsilon,x_\varepsilon}$ and $x^{\varepsilon,x}$, and thus the theorem. □

3.2. Diffusion Exit from a Domain

In the last section[11] we were concerned with the question, in which way a dynamical system changes its behaviour under small stochastic perturbations, with a special focus on untypical behaviour – large deviations.

[11] Even though the basic skeleton of this section comes from [Gen03], virtually anything in here is based on [DZ98, Section 5.7] and [FW98, Chapter 3].

In this section we consider a special case of this problem, namely the situation when the system without stochastic perturbations has a stable equilibrium point and the property to drift towards that point (under certain conditions). We study such systems under small stochastic perturbations. Especially, we want to understand, when and where such systems under stochastic perturbations leave certain neighborhoods of the deterministic equilibrium state.

Situation 3.2.1 (deterministic equilibrium point)**.** *We consider the stochastic differential equation*

$$\begin{cases} \mathrm{d}x_t^\varepsilon &= b(x_t^\varepsilon)\,\mathrm{d}t + \sqrt{\varepsilon}\cdot\sigma(x_t^\varepsilon)\,\mathrm{d}W_t\,, \quad t\in[0,T], \\ x_0^\varepsilon &= x \end{cases} \tag{3.2.1}$$

in $\mathbb{R}^d$, and an open, bounded domain $D\subset\mathbb{R}^d$ with $0\in D$. We assume that b and σ are uniformly Lipschitz continuous on a neighborhood of $\bar{D}$, and that $B<\infty$ is large enough to bound

$$\sup_{x\in\bar{D}}\|b(x)\|\,, \quad \sup_{x\in\bar{D}}\|\sigma(x)\|$$

and the Lipschitz constants for b and σ.

We denote by $\{P_x\circ(x^\varepsilon)^{-1}\}_{\varepsilon>0}$ the family of distributions of the solution x^ε to (3.2.1) *with initial condition x, and by $\mathbb{E}_x$ the corresponding expectation.*

Furthermore, we assume that the ordinary differential equation

$$\mathrm{d}x_t = b(x_t)\,\mathrm{d}t \tag{3.2.2}$$

has a unique stable equilibrium point at $0\in D$, and that for all solutions $(x_t)_{t\geqslant 0}$ of (3.2.2) *with initial condition $x_0\in D$ the following holds:*

- $x_t\in D$ *for all* $t>0$,
- $\lim_{t\to\infty} x_t = 0$.

Finally, we need the $\mathcal{F}_t$-stopping time

$$\tau^\varepsilon := \inf\{t\geqslant 0 \mid x_t^\varepsilon\notin D\}\,,$$

where $\{\mathcal{F}_t\}_{t\geqslant 0}$ is the canonical right-continuous filtration[12] *generated by the Brownian motion $(W_t)_{t\geqslant 0}$.*

It seems obvious that in this situation the system (3.2.1) should stay inside D for small ε with very high probability, if $x_0\in D$. However, also in this situation large deviations occur and the stochastically perturbed system eventually leaves D even for small ε. Our aim in this section is to understand these large deviations from the expected behaviour, including exit times and the probabilities of exit through certain parts of ∂D.

Remark 3.2.2 (possible generalization)**.** *It is noted in [DZ98, p. 197, p. 201] that the results of this section (concerning diffusion exit from a domain) and the preceding subsection (concerning Freidlin-Wentzell theory) can be generalized to other forms of stochastic perturbations, as e.g. Poisson processes, or, more generally, Lévy noise.*

[12] I.e., let $\{\mathcal{F}_t^0\}_{t\geqslant 0}$ be the filtration generated by Brownian motion and then set $\mathcal{F}_t := \bigcap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}^0$ for any $t\geqslant 0$.

3.2.1. Quasi-Potentials

Definition 3.2.3 (cost function V, quasi-potential)**.** [DZ98, p. 198] Let us define the *cost function*

$$V(x,y;t) := \inf\Big\{ I^{x^\varepsilon}_{[0,t],x}(\varphi) \;\Big|\; \varphi \in \mathcal{C}([0,t];\mathbb{R}^d),\ \varphi_0 = x,\ \varphi_t = y \Big\} \tag{3.2.3}$$

$$= \inf\bigg\{ \frac{1}{2}\int_0^t \|u_s\|^2 \,\mathrm{d}s \;\bigg|\; u \in \mathcal{L}^2([0,t];\,\mathbb{R}^d) \quad \text{such that} \tag{3.2.4}$$

$$\varphi_s := x + \int_0^s b(\varphi_r)\,\mathrm{d}r + \int_0^s \sigma(\varphi_r)u_r\,\mathrm{d}r\,,$$

$$s \in [0,t]\,, \quad \text{fulfills} \quad \varphi_t = y \bigg\}\,.$$

Heuristically, this may be understood as the cost of forcing the stochastically perturbed system (3.2.1) to be at the point y at time t, when starting at x.

Furthermore, we define

$$V(x,y) := \inf_{t>0} V(x,y;t)\,.$$

Following the above heuristic idea for $V(x,y;t)$, this may be understood as the cost of forcing the system (3.2.1) with starting point x to reach y eventually.

If the deterministic dynamics of the system under consideration possess a unique stable equilibrium point at the origin (which is the case in our setting, as described in Situation 3.2.1), we fix the special case $y \mapsto V(0,y)$ and call it a *quasi-potential*.

This definition (and especially its independence of the initial condition x) is motivated by the following observation.

Remark 3.2.4 (role of initial condition of (3.2.1))**.** *[DZ98, p. 198] In our setting, as described in Situation 3.2.1, the typical behaviour of a solution x^ε with initial condition $x_0^\varepsilon \in D$ is to move towards the origin as $\varepsilon \to 0$, and to stay close to that point for exponentially long time (until a large deviation occurs).*

During this long time span, the probability of hitting a closed set $N \subset \partial D$ is always determined by $\inf_{z\in N} V(0,z)$. Furthermore, during any excursion from the stable point, the probability to return there is overwhelmingly higher than the probability to push on towards (and beyond) ∂D.

In other words, to determine the probability of an exit from the domain D, it is not the time spent away from the stable point which matters, but the a priori chance for a direct, fast exit from the origin to D^c due to a rare segment in the Brownian motion's path.

A rigorous justification for this remark will be given below (cf. the note following Lemma 3.2.18).

We do not generally assume that the drift coefficient b in the system (3.2.1) derives from a potential. However, this may be the case in certain situations. For those situations, we note the following interesting fact.

Lemma 3.2.5 (potential and quasi-potential)**.** *[adapted from [FW98, Chap. 4, Thm. 3.1]] Assume that $\sigma \equiv 1$ and that there exists a continuously differentiable potential U on $\bar{D}$ satisfying the following conditions:*

- $U(0) = 0$,
- $U(x) > 0$ *for all* $x \neq 0$,
- $\nabla U(x) \neq 0$ *for all* $x \neq 0$.

If now the drift term is of the form

$$b(x) = -\nabla U(x) ,$$

then the quasi-potential $V(0, y)$ *satisfies*

$$V(0, y) = 2 \cdot U(y) \quad \text{for all } y \in D_0 := \{y \in \bar{D} \mid U(y) \leqslant U_0 := \min_{z \in \partial D} U(z)\} .$$

If, in addition, U is twice continuously differentiable, then the rate function I has a unique extremal φ *on the set*

$$\Big\{\varphi \in \mathcal{C}\big(]-\infty, T]; \mathbb{R}^d\big) \;\Big|\; \lim_{s \searrow -\infty} \varphi_s = 0,\ \varphi_T = x\Big\} ,$$

and this extremal is the solution of the differential equation

$$\begin{cases} \mathrm{d}\varphi_s &= +\nabla U(\varphi_s)\ \mathrm{d}s , \quad s \in]-\infty, T], \\ \varphi_T &= x . \end{cases} \tag{3.2.5}$$

According to [FW98], this relation between a potential and the quasi-potential derived from it is the reason for the name "quasi-potential".

Proof. Let $\varphi : [T_1, T_2] \to \mathbb{R}^d$ be a path and let us first assume that $\varphi_s \in \bar{D}$ for all $s \in [T_1, T_2]$. Then the relation

$$U(\varphi_{T_2}) - U(\varphi_{T_1}) = \int_{T_1}^{T_2} \langle \nabla U(\varphi_s), \dot{\varphi}_s \rangle\ \mathrm{d}s$$

implies that (with the polarization identity)

$$\begin{aligned} I_{[T_1,T_2]}(\varphi) &= \frac{1}{2} \int_{T_1}^{T_2} \|\dot{\varphi}_s + \nabla U(\varphi_s)\|^2\ \mathrm{d}s \\ &= \frac{1}{2} \int_{T_1}^{T_2} \|\dot{\varphi}_s - \nabla U(\varphi_s)\|^2\ \mathrm{d}s + 2 \int_{T_1}^{T_2} \langle \nabla U(\varphi_s), \dot{\varphi}_s \rangle\ \mathrm{d}s \\ &\geqslant 2 \cdot \big(U(\varphi_{T_2}) - U(\varphi_{T_1})\big) . \end{aligned} \tag{3.2.6}$$

Hence, if we choose φ such that $\varphi_s \in \bar{D}$ for all $s \in [T_1, T_2]$, $\varphi_{T_1} = 0$ and $\varphi_{T_2} = x$ where $x \in D_0$, we see that

$$I_{[T_1,T_2],0}(\varphi) \geqslant 2U(x) . \tag{3.2.7}$$

If $\varphi_{T_1} = 0$ and $\varphi_{T_2} = x$, but there exist $s \in [T_1, T_2]$ such that $\varphi_s \notin \bar{D}$ (i.e., φ leaves $\bar{D}$ during $[T_1, T_2]$), there exists a time $\tau \in [T_1, T_2]$ such that $U(\varphi_\tau) = U_0$ and

$$I_{[T_1,T_2],0}(\varphi) \geqslant I_{[T_1,\tau],0}(\varphi) \overset{(3.2.7)}{\geqslant} 2U(\varphi_\tau) = 2U_0 \geqslant 2U(x) ,$$

which is the asserted lower bound for all φ such that $\varphi_{T_1} = 0$ and $\varphi_{T_2} = x$.

Now, let $\hat{\varphi}$ be a solution of (3.2.5) (we still do not assume that U is twice differentiable). Then $\lim_{s\to-\infty} \hat{\varphi}_s = 0$ and (3.2.6) becomes an equation, because $\left\|\dot{\hat{\varphi}}_s - \nabla U(\hat{\varphi}_s)\right\| = 0$ for all $s \in [T_1, T_2]$. Hence, similar to (3.2.6),

$$I_{[-\infty,T],0}(\hat{\varphi}) = 2\int_{-\infty}^{T} \left\langle \nabla U(\hat{\varphi}_s), \dot{\hat{\varphi}}_s \right\rangle \,\mathrm{d}s = 2U(x)\ ,$$

and we conclude that $\hat{\varphi}$ is an extremal of the rate function (because "$I \geqslant 2U$" has been shown above). By the definition of the quasi-potential, this implies that $V(0,x) = 2U(x)$, as claimed in the Lemma.

If $U \in \mathcal{C}^2$, the solution to (3.2.5) is unique. This implies the uniqueness of the extremal ("optimal") path. □

Remark 3.2.6. *[cf. [FW98, Chap. 4, Thm. 3.1]]*

- *The idea behind the restriction on the domain D_0, where the relation $V = 2U$ holds, is the following:*

  As before, we think of the quasi-potential as the representation of the "cost" of forcing the process to a certain point within D. The process will naturally leave D near a point $z \in \partial D$ with $U(z) = U_0$, where this is "as cheap as possible".

  It will not, on the other hand, approach points $y \in D$ where $U(y) > U_0$. In other words, the "cost" of forcing the process to points in $D \setminus D_0$ is simply "not interesting", the quasi-potential V "does not know these".

- *The second assertion of the lemma and the differential equation* (3.2.5) *reflect that the "cheapest way" for the process to reach some $x \in D$ is to "move directly towards the exit point", along the path which the deterministic process would take from x to the origin.*

Remark 3.2.7 (possible generalization)**.** *[Gen03, Lemma 3.6] The above lemma can be generalized for the situation where $b(x) = -\nabla U(x) + l(x)$, if l is a mapping $\bar{D} \to \mathbb{R}^d$ such that*

$$\left\langle l(x), \nabla U(x) \right\rangle \equiv 0\ .$$

If l is continuously differentiable, then the second assertion of the lemma can be extended to this generalized case, and the extremal φ is the solution of the differential equation

$$\begin{cases} \mathrm{d}\varphi_s &= +\nabla U(\varphi_s)\,\mathrm{d}s + l(\varphi_s)\,\mathrm{d}s\ , \quad s \in\,]-\infty, T], \\ \varphi_T &= x\ . \end{cases}$$

3.2.2. Classical Results

Situation 3.2.8 (additional assumptions)**.** *[[DZ98, p. 199] and [Gen03, Ass. 3.9–3.11]] We still remain in Situation 3.2.1 and state the following additional assumptions:*

(A1) We add to the assumption that the deterministic system has a unique, stable equilibrium point at $x^* = 0$ the requirement, that if $x_0 \in \partial D$, then $\lim_{t\to\infty} x_t = 0$.

(A2) $V_0 := \inf_{z \in \partial D} V(0, z) < \infty$.

(A3) There exist a constant $K > 0$ and a maximal radius $\varrho_0 > 0$, such that for all $\varrho \leqslant \varrho_0$ and all x_0, y with $\|x_0 - z\| + \|z - y\| \leqslant \varrho$ for a point $z \in \partial D \cup \{x^*\}$, there exist a "control" $u \in \mathcal{L}^2$ with $\|u\|_\infty < K$ and a time $T(\varrho)$, such that the path φ_t, defined by

$$\varphi_t := x_0 + \int_0^t b(\varphi_s)\,\mathrm{d}s + \int_0^t \sigma(\varphi_s) u_s\,\mathrm{d}s\,,$$

satisfies $\varphi_{T(\varrho)} = y$, and the time $T(\varrho)$ converges to 0 as $\varrho \to 0$.

Note that in (A3) neither x_0 nor y are required to be in D.

Remark 3.2.9 (about the assumptions). *[[DZ98, p. 199] and [Gen03, Rem. 3.12]]*

(A1): *The basic assumption on existence and uniqueness of a stable equilibrium point (as formulated in Situation 3.2.1) formalizes the setting that we are interested in: If the deterministic dynamics starting inside D were allowed to leave D, the conclusion that the perturbed dynamics did so, too, would be trivial. Hence, we concentrate on the case of the deterministic dynamics drifting towards $x^* = 0$.*

(A1) extends this assumptions on the dynamics to initial conditions $x_0 \in \partial D$ (whereas earlier only $x_0 \in D = \mathring{D}$ were included). By this stronger assumption, we exclude a characteristic boundary, i.e. ∂D may not fall together with the boundary between different domains of attraction of the deterministic dynamics.

For example, if the drift coefficient b is the derivative of a potential, the topologically closed domain $\bar{D}$ may not contain a local extremum between two potential wells.

This poses a problem for us, since our aim is to understand stochastic resonance, i.e. transitions between wells. Fortunately, we will be able to relax the extended assumption and thus allow characteristic boundaries; see Corollary 3.2.13.

(A2): *This assumption is obviously necessary to assure that the exit from D in finite time is possible at all.*

(A3): *The rather technical-looking assumption states that there exists a bounded control function u, such that the controlled process φ_t connects the initial condition x_0 and the point y within time $T(\varrho)$, and that this time $T(\varrho)$ becomes smaller the more x_0 and y move closer towards each other and some point z out of $x^* \cup \partial D$.*

Heuristically speaking, this assumption makes sure that leaving x^* or moving near (especially: crossing) ∂D is not "too expensive" in terms of the cost function that underlies the quasi-potential V.

It can be proved (cf. [DZ98, Exercise 5.7.29]) that if $a(x) = \sigma(x)\sigma(x)^T$ is positive definite for $x = x^* = 0$ and uniformly positive definite for $x \in \partial D$, this assumption is satisfied.

Assumption (A3) implies the following continuity property:

Lemma 3.2.10 (continuity of V near $x^* = 0$ and ∂D)**.** *[DZ98, Lemma 5.7.8] For any $\delta > 0$ there exists a radius $\varrho > 0$ small enough, such that the following inequalities hold:*

$$\sup\Big\{\inf_{t\in[0,1]} V(x,y;t) \;\Big|\; x,y \in B(0,\varrho) \subset D\Big\} < \delta\,,$$

$$\sup\Big\{\inf_{t\in[0,1]} V(x,y;t) \;\Big|\; x,y \in \mathbb{R}^d \quad s.th.\ \inf_{z\in\partial D}\big(\|y-z\| + \|x-z\|\big) \leqslant \varrho\Big\} < \delta\,.$$

Proof. The function φ as defined in (A3) fulfills

$$V(x,y;t) \leqslant I^{x^\varepsilon}_{[0,t],x}(\varphi) = \frac{1}{2}\int_0^t \|u_s\|^2\,\mathrm{d}s \leqslant \frac{1}{2}\cdot tK^2\,.$$

Assumption (A3) furthermore implies that $t := T(\varrho)$ converges to 0 as $\varrho \to 0$. Hence, by choosing ϱ small enough, the inequalities of the lemma are fulfilled. □

The following theorem is the main result of this subsection. It provides both an exponential growth rate for the exit time τ^ε and estimates for the exit location.

Theorem 3.2.11. *[DZ98, Theorem 5.7.11]*

(first exit time τ^ε). *For any initial condition $x \in D$ and any $\delta > 0$, the first exit time fulfills the estimate*

$$\lim_{\varepsilon\to 0} P_x\left[\exp\left[\frac{V_0-\delta}{\varepsilon}\right] < \tau^\varepsilon < \exp\left[\frac{V_0+\delta}{\varepsilon}\right]\right] = 1\,. \tag{3.2.8}$$

Furthermore, for any $x \in D$, we can estimate the expected value of τ^ε as follows:

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_x[\tau^\varepsilon] = V_0\,. \tag{3.2.9}$$

(first exit location). *For any closed set $N \subset \partial D$ such that $V_N := \inf_{z\in N} V(0,z) > V_0$, and all initial conditions $x \in D$, we have that*

$$\lim_{\varepsilon\to 0} P_x\big[x^\varepsilon_{\tau^\varepsilon} \in N\big] = 0\,. \tag{3.2.10}$$

In particular, if the quasi-potential V has a unique minimum z^* on ∂D, then the following holds for all initial conditions $x \in D$ and all $\delta > 0$:

$$\lim_{\varepsilon\to 0} P_x\big[\|x^\varepsilon_{\tau^\varepsilon} - z^*\| < \delta\big] = 1\,. \tag{3.2.11}$$

The result (3.2.9) about the asymptotic mean exit time has been predicted by physicists (in the case that the drift derives from a potential U) long before the mathematical theory actually provided the result. The assertion that the logarithm of the mean exit time behaves like $2U_0/\varepsilon$, where $U_0 := \min_{z\in\partial D} U(z)$, is known in this context as "Arrhenius' law" (the original reference for this is [Arr89]; about the relation between U and V see also Lemma 3.2.5).

Remark 3.2.12 (restriction). *[DZ98, p. 201] If $V|_{\partial D} : \partial D \to \mathbb{R}_+$ has more than one minimum, the exit occurs from a neighborhood of the set of minima. The weight among the minima can not be determined without further refinements of the underlying large deviations theory.*

Finally, we state the result that is fundamental for SR in an asymmetric double-well potential. The proof can be found at the end of this section.

Corollary 3.2.13 (characteristic boundary). *[DZ98, Corollary 5.7.16] The first part of Theorem 3.2.11 (concerning the first exit time) remains true without Assumption (A1), i.e. ∂D may fall together with a characteristic boundary with respect to the deterministic dynamics.*

To prove the theorem and the corollary, we need the following handful of lemmata.

Situation 3.2.14 (balls and spheres in D). *[DZ98, p. 198] Let us add to the assumptions stated in Situations 3.2.1 and 3.2.8, that from now on and for the rest of the subsection, whereever we talk about a ball $B(x, \varrho)$ or a sphere*

$$S(x, \varrho) := \partial B(x, \varrho)$$

with $x \in D$, we assume that the radius ϱ is small enough to ensure $B(x, \varrho) \subset D$ or $S(x, \varrho) \subset D$, respectively.

Furthermore, we define a new stopping time

$$\sigma_\varrho := \inf\{t \geqslant 0 \mid x_t^\varepsilon \in B(0, \varrho) \cup \partial D\} .$$

First we provide a uniform lower bound for the probability of an exit from D during a finite time T_0.

Lemma 3.2.15 (minimum exit probability during finite time). *[DZ98, Lemma 5.7.18] For any (arbitrarily small) $\eta > 0$ and any radius $\varrho > 0$ small enough for Lemma 3.2.10 to hold (with $\delta = \frac{\eta}{3}$), there exists a time $T_0 = T_0(\eta, \varrho) < \infty$ such that*

$$\liminf_{\varepsilon \to 0} \varepsilon \log \inf_{x \in B(0,\varrho)} P_x[\tau^\varepsilon \leqslant T_0] > -(V_0 + \eta) .$$

Proof. Fix $\eta > 0$ and let $\varrho > 0$ be as in the lemma.

The idea of this proof is to construct deterministic exiting paths ϕ^x for all $x \in B(0, \varrho)$ such that a certain neighborhood Ψ of the set of all such ϕ^x in the set of all (not necessarily deterministic) continuous paths fulfills the following properties:

- $\liminf_{\varepsilon \to 0} \varepsilon \log \inf_{x \in B(0,\varrho)} P_x[x^\varepsilon \in \Psi] > -(V_0 + \eta)$,
- $x^\varepsilon \in \Psi \quad \Rightarrow \quad \tau^\varepsilon \leqslant T_0$ for some $T_0 < \infty$.

We construct ϕ^x piecewise:

By the continuity of V near $x^* = 0$ (cf. Assumption (A3) and Lemma 3.2.10), we know that for small enough $\varrho > 0$ the following holds: For any $x \in B(0, \varrho)$ we can find a path ψ^x with

$$\psi_0^x = x \quad \text{and} \quad \psi_{t_x}^x = x^* = 0 ,$$

where $t_x \in [0,1]$, such that

$$I^{x^\varepsilon}_{[0,t_x],x}(\psi^x) \leqslant \frac{\eta}{3} \, .$$

Since V_0 is finite (cf. Assumption (A2)), there exists a continuous path ψ^0 with

$$\psi^0_{t_x} = 0 \quad \text{and} \quad \psi^0_{t_x+t_0} =: z \in \partial D \, ,$$

where t_0 is some finite time, such that

$$I^{x^\varepsilon}_{[t_x,t_x+t_0],0}(\psi^0) < V_0 + \frac{\eta}{3} \, .$$

Again by the continuity property of V, we can find a small radius $\kappa > 0$, such that for any point[13] $y \in S(z,\kappa) \cap (\bar{D})^c$ there exists a continuous path ψ^z with

$$\psi^z_{t_x+t_0} = z \quad \text{and} \quad \psi^z_{t_x+t_0+t_z} = y \, ,$$

where $t_z \in [0,1]$, such that

$$I^{x^\varepsilon}_{[t_x+t_0,t_x+t_0+t_z],z}(\psi^z) \leqslant \frac{\eta}{3} \, .$$

Since $\bar{D}$ is compact and $y \in (\bar{D})^c$, the Euclidean distance α between y and $\bar{D}$ is strictly positive. We remark that, by selection of t_z and t_x, we have that $t_x + t_z \leqslant 2$.

Now, we denote by ψ^y the continuous path with the following properties:

$$\begin{aligned} &\psi^y_{t_x+t_0+t_z} = y \, , \\ &\dot{\psi}^y_s = b(\psi^y_s) \quad \text{for all } s \in [t_x+t_0+t_z, t_0+2]. \end{aligned}$$

Hence, $I^{x^\varepsilon}_{[t_x+t_0+t_z,t_0+2],y}(\psi^y) = 0$. Note that we do not know, whether $\psi^y_{t_0+2}$ is an element of D or not.

Finally, we define the promised path

$$\begin{aligned} \phi^x : \quad [0,t_0+2] \quad &\to \quad \mathbb{R}^d \\ s \quad &\mapsto \quad \begin{cases} \psi^x_s & \text{if } s \in [0,t_x] \\ \psi^0_s & \text{if } s \in [t_x, t_x+t_0] \\ \psi^z_s & \text{if } s \in [t_x+t_0, t_x+t_0+t_z] \\ \psi^y_s & \text{if } s \in [t_x+t_0+t_z, t_0+2] \, . \end{cases} \end{aligned}$$

This path has, by construction, the following properties:

- $\phi^x_0 = x$,
- $\phi^x_s = y \in (\bar{D})^c$ for some $s \in [0, t_0+2]$,

[13] This is one of the rare occasions in this subsection where we do not assume $S(z,\kappa) \subset D$, in contrast to the general setting described in Situation 3.2.14.

- $I^{x^\varepsilon}_{[0,t_0+2],x}(\phi^x) < V_0 + \eta$.

Moreover, this construction, along with the properties mentioned above, is independent of the initial selection of $x \in B(0,\varrho)$.

For the final step of the proof we define the set of all continuous paths starting at some $x \in B(0,\varrho)$ and following ϕ^x outside $\bar{D}$:

$$\Psi := \bigcup_{x\in B(0,\varrho)} \left\{ \psi \in \mathcal{C}\big([0,t_0+2];\mathbb{R}^d\big) \;\middle|\; \|\psi - \phi^x\|_\infty < \frac{\alpha}{2} \right\}$$

(remember that the distance between y and $\bar{D}$ is $\alpha > 0$, hence, every $\psi \in \Psi$ actually leaves D!). If we set $T_0 := t_0 + 2$ (which is a finite time, since $t_0 < \infty$), we know that $x^\varepsilon \in \Psi$ implies $\tau^\varepsilon \leqslant T_0$. Thus, we obtain

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \varepsilon \log \inf_{x\in B(0,\varrho)} P_x[\tau^\varepsilon \leqslant T_0] \geqslant \liminf_{\varepsilon\to 0} \varepsilon \log \inf_{x\in B(0,\varrho)} P_x[x^\varepsilon \in \Psi] \\
&\overset{(*)}{\geqslant} - \sup_{x\in B(0,\varrho)} \inf_{\psi\in\Psi} I^{x^\varepsilon}_{[0,T_0],x}(\psi) \geqslant - \sup_{x\in B(0,\varrho)} I^{x^\varepsilon}_{[0,T_0],x}(\phi^x) > -(V_0+\eta) ,
\end{aligned}$$

where the inequality $(*)$ is justified by Corollary 3.1.34. □

The second helping fact is that the probability of x^ε remaining inside D arbitrarily long without visiting a small neighborhood of 0 is exponentially small.

Note that for the proof of this lemma we need Assumption (A1).

Lemma 3.2.16 (σ_ϱ can not become arbitrarily big)**.** *[DZ98, Lemma 5.7.19] For all $\varrho > 0$,*

$$\lim_{t\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \sup_{x\in D} P_x[\sigma_\varrho > t] = -\infty .$$

Proof. Let $\varrho > 0$. If $x^\varepsilon_0 = x \in B(0,\varrho)$, then $\sigma_\varrho = 0$ and the lemma trivially holds.

Hence, we choose x^ε such that $x^\varepsilon_0 = x \in D \setminus B(0,\varrho)$ and set

$$\Psi_t := \left\{ \phi \in \big([0,t];\mathbb{R}^d\big) \;\middle|\; \forall\, s \in [0,t] : \phi_s \in \overline{D \setminus B(0,\varrho)} \right\} .$$

Now,

$$\{\sigma_\varrho > t\} \subset \{x^\varepsilon \in \Psi_t\} . \tag{3.2.12}$$

By Corollary 3.1.34 we have that for all $t < \infty$

$$\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{x\in \overline{D\setminus B(0,\varrho)}} P_x[x^\varepsilon \in \Psi_t] \leqslant - \inf_{\psi\in\Psi_t} \underbrace{I^{x^\varepsilon}_{[0,t],\psi_0}(\psi)}_{=: I^{x^\varepsilon}_t(\psi)} .$$

We will now show that

$$\lim_{t\to\infty} \inf_{\psi\in\Psi_t} I^{x^\varepsilon}_t(\psi) = \infty . \tag{3.2.13}$$

This implies the assertion of the lemma because of the inclusion (3.2.12).

Choose a point $x \in \overline{D \setminus B(0,\varrho)}$ and denote by ϕ^x the trajectory of the deterministic system (3.2.2) with initial condition $\phi^x_0 = x$. By Assumption (A1) (and the general Situation 3.2.1), ϕ^x hits $S\big(0, \frac{\varrho}{3}\big)$ in finite time, denoted T_x. Since the deterministic drift coefficient b is assumed to be Lipschitz continuous, we can find an open neighborhood W_x of x such that for all $y \in W_x$ the path ϕ^y hits $S\big(0, \frac{2}{3}\varrho\big)$ before time T_x.

By iterating this argument for more points in $\overline{D \setminus B(0,\varrho)}$, we can construct enough open sets like W_x to cover $\overline{D \setminus B(0,\varrho)}$, each of these open sets related to a finite hitting time like T_x.

By compactness of $\overline{D \setminus B(0,\varrho)}$, we find a finite cover of $\overline{D \setminus B(0,\varrho)}$ by such open sets. Hence, there exists a finite time T such that for any $y \in \overline{D \setminus B(0,\varrho)}$ the path ϕ^y hits $S\big(0, \frac{2}{3}\varrho\big)$ before T.

Now, let us assume that (3.2.13) is wrong. Then we can find a constant $M < \infty$ such that for any $n \in \mathbb{N}$ there exists a continuous function $\psi^n \in \Psi_{nT}$ which fulfills $I^{x^\varepsilon}_{nT}(\psi^n) \leqslant M$. Thus, we may select functions $\psi^{n,k} \in \Psi_T$, $k \leqslant n$, so that

$$M \geqslant I^{x^\varepsilon}_{nT}(\psi^n) = \sum_{k=1}^{n} I^{x^\varepsilon}_{T}(\psi^{n,k}) \geqslant n \cdot \min_{k=1,\dots,n} I^{x^\varepsilon}_{T}(\psi^{n,k}) \; .$$

Since this works for any $n \in \mathbb{N}$, there exists a sequence $(\phi^n)_{n\in\mathbb{N}} \subset \Psi_T$ with $\lim_{n\to\infty} I^{x^\varepsilon}_{T}(\phi^n) = 0$. Now, remember that I^{x^ε} is a good rate function, hence

$$\big\{\phi \;\big|\; \phi_0 \in \bar{D} \quad \text{and} \quad I^{x^\varepsilon}_{[0,T],\phi_0} \leqslant 1\big\}$$

is a compact subset of $\mathcal{C}\big([0,T];\mathbb{R}^d\big)$. This implies that the sequence $(\phi^n)_{n\in\mathbb{N}}$ has a limit point $\psi^* \in \Psi_T$. By the general properties of the rate function I^{x^ε}, we have that $I^{x^\varepsilon}_{T}(\psi^*) = 0$. This implies that ψ^* has to be a trajectory of the deterministic system (3.2.2). But, being an element of Ψ_T, this path cannot hit $S\big(0, \frac{2}{3}\varrho\big)$ during $[0,T]$. This is a contradiction to the construction of T.

Hence, the assumption that (3.2.13) is wrong can not be true: (3.2.13) holds, and the proof is complete. □

Choose an initial condition $x^\varepsilon_0 = y \in S(0,2\varrho)$. The next lemma provides an upper bound for the probability that $x^\varepsilon_{\sigma_\varrho}$ meets some subset of ∂D instead of joining the small ball $B(0,\varrho)$.

Lemma 3.2.17 (probability of leaving D before visiting 0: upper bound)**.** *[DZ98, Lemma 5.7.21] For any closed set $N \subset \partial D$,*

$$\lim_{\varrho\to 0} \limsup_{\varepsilon\to 0} \varepsilon \log \sup_{y\in S(0,2\varrho)} P_y\big[x^\varepsilon_{\sigma_\varrho} \in N\big] \leqslant -\inf_{z\in N} V(0,z) \; .$$

Proof. We choose a closed set $N \subset \partial D$. Fix a constant $\delta > 0$ and define

$$V^\delta_N := \big(\inf_{z\in N} V(0,z) - \delta\big) \wedge \frac{1}{\delta} \; .$$

By the continuity property of the quasi-potential (cf. Assumption (A3) and the first assertion of Lemma 3.2.10), for any small enough $\varrho > 0$ the following holds:

$$\inf_{\substack{y\in S(0,2\varrho),\\ z\in N}} V(y,z) \geqslant \inf_{z\in N} V(0,z) - \underbrace{\sup_{y\in S(0,2\varrho)} V(0,y)}_{<\delta} \geqslant V_N^\delta \,. \tag{3.2.14}$$

By Lemma 3.2.16, we can select a time $T < \infty$ such that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{y\in S(0,2\varrho)} P_y[\sigma_\varrho > T] < \limsup_{\varepsilon\to 0} \varepsilon \log \sup_{x\in D} P_x[\sigma_\varrho > T] < -V_N^\delta \,.$$

Now, we collect those $\psi \in \mathcal{C}\big([0,T];\mathbb{R}^d\big)$ that hit N during $[0,T]$:

$$\Psi := \Big\{\psi \in \mathcal{C}\big([0,T];\mathbb{R}^d\big) \,\Big|\, \exists\, t\in[0,T] \quad \text{such that } \psi_t \in N\Big\} \,.$$

This allows us to extend (3.2.14):

$$\inf_{\substack{y\in S(0,2\varrho),\\ \psi\in\Psi}} I^{x^\varepsilon}_{[0,T],y}(\psi) \geqslant \inf_{\substack{y\in S(0,2\varrho),\\ z\in N}} V(y,z) \geqslant V_N^\delta \,,$$

and hence we get by Corollary 3.1.34, that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{y\in S(0,2\varrho)} P_y[x^\varepsilon \in \Psi] \leqslant -\inf_{\substack{y\in S(0,2\varrho),\\ \psi\in\Psi}} I^{x^\varepsilon}_{[0,T],y}(\psi) \leqslant -V_N^\delta \,.$$

Since

$$P_y\big[x^\varepsilon_{\sigma_\varrho} \in N\big] \leqslant P_y[\sigma_\varrho > T] + P_y[x^\varepsilon \in \Psi] \,,$$

we obtain that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{y\in S(0,2\varrho)} P_y\big[x^\varepsilon_{\sigma_\varrho} \in N\big] < -V_N^\delta \,.$$

Taking $\delta \to 0$ completes the proof of the lemma. □

The next lemma has a more general point of view in so far as it allows all initial conditions $x_0^\varepsilon \in D$ without asking for $S\big(0,\|x_0^\varepsilon\|\big)$ being a subset of D. It shows, that even in this situation x^ε can hardly avoid any small neighborhood of 0 as $\varepsilon \to 0$.

Lemma 3.2.18 (x^ε will visit 0 almost surely as $\varepsilon \to 0$)**.** *[DZ98, Lemma 5.7.22] For all $\varrho > 0$ and any initial condition $x \in D$,*

$$\lim_{\varepsilon\to 0} P_x\big[x^\varepsilon_{\sigma_\varrho} \in B(0,\varrho)\big] = 1 \,.$$

This result provides, by the way, the promised justification for Remark 3.2.4.

Proof. Choose $\varrho > 0$ such that $B(0,\varrho) \subset D$. If $x = x_0^\varepsilon \in B(0,\varrho)$, the assertion obviously holds. Hence, we select $x \in D \setminus B(0,\varrho)$. Let ϕ be the trajectory of the deterministic system (3.2.2) with $\phi_0 := x$, and set

$$T := \inf\left\{ t \geqslant 0 \,\middle|\, \phi_t \in S\left(0,\frac{\varrho}{2}\right)\right\} < \infty \,.$$

ϕ is a continuous path that does not hit the compact set ∂D, hence, the distance Δ between $\{\phi_t\}_{t\leqslant T}$ and ∂D is strictly positive. We set

$$\Delta' := \Delta \wedge \varrho$$

and let x^ε be the solution of (3.2.1) with initial condition $x_0^\varepsilon = x$. Then we have that

$$\text{if} \quad \sup_{t\in[0,T]} \|x_t^\varepsilon - \phi_t\| < \frac{\Delta'}{2}\,, \quad \text{then} \quad x^\varepsilon_{\sigma_\varrho} \in B(0,\varrho)\,. \tag{3.2.15}$$

The uniform Lipschitz continuity of b implies that

$$\|x_t^\varepsilon - \phi_t\| \leqslant B\int_0^t \|x_s^\varepsilon - \phi_s\|\,\mathrm{d}s + \sqrt{\varepsilon}\cdot\left\|\int_0^t \sigma(x_s^\varepsilon)\,\mathrm{d}W_s\right\|\,,$$

and by Gronwall's lemma we obtain that

$$\begin{aligned}\|x_t^\varepsilon - \phi_t\| &\leqslant \sqrt{\varepsilon}\cdot\left\|\int_0^t \sigma(x_s^\varepsilon)\,\mathrm{d}W_s\right\| \\ &\quad + B\int_0^t \sqrt{\varepsilon}\cdot\left\|\int_0^s \sigma(x_r^\varepsilon)\,\mathrm{d}W_r\right\|\cdot\exp\big[B\cdot(t-s)\big]\,\mathrm{d}s\,,\end{aligned}$$

which implies that

$$\sup_{t\in[0,T]} \|x_t^\varepsilon - \phi_t\| \leqslant \sqrt{\varepsilon}\cdot\exp[BT]\cdot\sup_{t\in[0,T]}\left\|\int_0^t \sigma(x_s^\varepsilon)\,\mathrm{d}W_s\right\|\,. \tag{3.2.16}$$

Finally, we obtain from (3.2.15) and (3.2.16) that by the maximal inequality and the Burkholder-Davis-Gundy inequality there exist constants $c^{(1)}, c^{(2)} \in\,]0,\infty[$, independent of z, such that

$$\begin{aligned}P_x\big[x^\varepsilon_{\sigma_\varrho} \in \partial D\big] &\leqslant P_x\left[\sup_{t\in[0,T]}\|x_t^\varepsilon - \phi_t\| \geqslant \frac{\Delta'}{2}\right] \\ &\leqslant P_x\left[\sup_{t\in[0,T]}\left\|\int_0^t \sigma(x_s^\varepsilon)\,\mathrm{d}W_s\right\| \geqslant \frac{\Delta'}{2\sqrt{\varepsilon}}\cdot\exp[-BT]\right] \\ &\leqslant \varepsilon c^{(1)}\cdot\mathbb{E}_x\left[\left(\sup_{t\in[0,T]}\left\|\int_0^t \sigma(x_s^\varepsilon)\,\mathrm{d}W_s\right\|\right)^2\right] \\ &\leqslant \varepsilon c^{(2)}\cdot\mathbb{E}_x\left[\int_0^T \operatorname{tr}\big(\sigma(x_s^\varepsilon)\sigma(x_s^\varepsilon)^T\big)\,\mathrm{d}s\right] \xrightarrow{\varepsilon\to 0} 0\,.\end{aligned}$$

□

Finally, we show that during a short time intervall x^ε will almost surely not leave its starting point too far behind.

Lemma 3.2.19 (x^ε is "slow"). *[DZ98, Lemma 5.7.23] For any $\varrho > 0$, any initial condition $x \in D$ and any constant $c > 0$ there exists a time $T(c,\varrho) < \infty$ such that*

$$\limsup_{\varepsilon\to 0}\varepsilon\log\sup_{x\in D} P_x\Big[\sup_{t\in[0,T(c,\varrho)]}\|x_t^\varepsilon - x\| \geqslant \varrho\Big] < -c\,.$$

Proof. First let us notice that

$$J_t := \int_0^t \sigma(x_s^\varepsilon)\, \mathrm{d}W_s$$

is a continuous and square-integrable martingale.

For any radius $\varrho > 0$ and all times $T \leqslant \frac{\varrho}{2B}$,

$$P_x\big[\sup_{t\in[0,T]} \|x_t^\varepsilon - x\| \geqslant \varrho\big] = P_x\left[\sup_{t\in[0,T]} \Bigg\| \underbrace{\int_0^t b(x_s^\varepsilon)\, \mathrm{d}s}_{\text{“sup”}\leqslant T\cdot B \leqslant \frac{\varrho}{2}} + \sqrt{\varepsilon}\cdot J_t \Bigg\| \geqslant \varrho\right]$$

$$\leqslant P_x\left[\sqrt{\varepsilon}\cdot \sup_{t\in[0,T]} \|J_t\| \geqslant \frac{\varrho}{2}\right] .$$

It suffices now to consider the corresponding one-dimensional problem (exchanging $\frac{\varrho}{2}$ by $\beta\varrho$, where β depends on the dimension). Hence, from here on let $\sigma(x_s^\varepsilon) =: \sigma_s$ be scalar (still bounded by B) and W_t a one-dimensional Brownian motion.

By time change there is a standard one-dimensional Brownian motion W'_t on the same probability space as W_t, such that almost surely $J_t = W'_{\tau(t)}$, where

$$\tau(t) := \int_0^t \sigma_\theta^2\, \mathrm{d}\theta .$$

We know that $\tau(t) \leqslant B^2 t$ almost surely, and τ is continuous and increasing. Hence,

$$P_x\big[\sqrt{\varepsilon} \sup_{t\in[0,T]} |J_t| \geqslant \beta\varrho\big] = P_x\Big[\sqrt{\varepsilon} \sup_{t\in[0,T]} \big|W'_{\tau(t)}\big| \geqslant \beta\varrho\Big]$$

$$\leqslant P_x\big[\sqrt{\varepsilon} \sup_{\tau\in[0,B^2T]} |W'_\tau| \geqslant \beta\varrho\big] \overset{(*)}{\leqslant} 4\exp\left[\frac{-\beta^2\varrho^2}{2\varepsilon B^2 T}\right] ,$$

where the inequality $(*)$ is justified by Lemma 3.1.1. Choosing

$$T = T(c,\varrho) < \min\left\{\frac{\varrho}{2B}, \frac{\beta^2\varrho^2}{2B^2c}\right\} ,$$

we obtain that

$$P_x\big[\sqrt{\varepsilon} \sup_{t\in[0,T]} |J_t| \geqslant \beta\varrho\big] \leqslant \min\left\{4\exp\left[-\frac{\beta^2\varrho}{\varepsilon B}\right], 4\exp\left[-\frac{c}{\varepsilon}\right]\right\} ,$$

which implies the assertion of the lemma. □

Now we can finally prove the main theorem. For the convenience of the reader we repeat the assertions made in the theorem.

Proof of Theorem 3.2.11. **(first exit time τ^ε).** The following is claimed: [14]

[14] The equation numbers here are the same as in the “original theorem”.

For any initial condition $x \in D$ and any $\delta > 0$, the first exit time fulfills the estimate

$$\lim_{\varepsilon \to 0} P_x \left[\exp\left[\frac{V_0 - \delta}{\varepsilon}\right] < \tau^\varepsilon < \exp\left[\frac{V_0 + \delta}{\varepsilon}\right] \right] = 1 \, . \quad ((3.2.8))$$

Furthermore, for any $x \in D$, we can estimate the expected value of τ^ε as follows:

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_x[\tau^\varepsilon] = V_0 \, . \qquad ((3.2.9))$$

We prove the equations by providing upper and lower bounds. Therefore, we fix an initial condition $x \in D$ and a constant $\delta > 0$. Without loss of generality, we assume $V_0 > 0$ and $\frac{\delta}{2} < V_0$.

Upper bound. [Gen03, Proof of Theorem 3.14] Set $\eta := \frac{\delta}{8}$ and choose a small radius $\varrho > 0$. Lemma 3.2.15 implies the existence of a time $T_0 = T_0(\eta, \varrho)$ and an $\varepsilon_0 > 0$, such that the estimate

$$P_x[\tau^\varepsilon \leqslant T_0] > \exp\left[-\frac{V_0 + 2\eta}{\varepsilon}\right]$$

holds uniformly for any $x \in B(0, \varrho)$ and all $\varepsilon \leqslant \varepsilon_0$. We may assume that $\varepsilon_0 < \eta$.

By Lemma 3.2.16, there exists a time $T_1 = T_1(\eta, \varrho)$, such that the estimate

$$P_x[\sigma_\varrho > T_1] < \exp\left[-\frac{\eta}{\varepsilon}\right]$$

holds uniformly for all $x \in D$ and all $\varepsilon \leqslant \varepsilon_0$ (where ε_0 is the same as above).

Now, we set $T := T_0 + T_1$. Then, for all $\varepsilon \leqslant \varepsilon_0$, the probability that x^ε leaves D before time T can be estimated as follows:

$$\begin{aligned} q := \inf_{x \in D} P_x[\tau^\varepsilon \leqslant T] &\geqslant \inf_{x \in D} P_x[\sigma_\varrho \leqslant T_1] \cdot \inf_{x \in B(0,\varrho)} P_x[\tau^\varepsilon \leqslant T_0] \qquad (3.2.17) \\ &\geqslant \left(1 - \exp\left[-\frac{\eta}{\varepsilon}\right]\right) \cdot \exp\left[-\frac{V_0 + 2\eta}{\varepsilon}\right] \\ &= \exp\left[-\frac{V_0 + 2\eta}{\varepsilon}\right] - \exp\left[-\frac{V_0 + 3\eta}{\varepsilon}\right] . \end{aligned}$$

By construction of ε_0 we have that $\varepsilon_0 < \eta$, hence for all $\varepsilon \leqslant \varepsilon_0$

$$\begin{aligned} & \exp\left[\frac{2\eta}{\varepsilon}\right] - \exp\left[\frac{\eta}{\varepsilon}\right] \geqslant 1 \\ \Leftrightarrow \quad & \exp\left[-\frac{V_0 + 2\eta}{\varepsilon}\right] - \exp\left[-\frac{V_0 + 3\eta}{\varepsilon}\right] \geqslant \exp\left[-\frac{V_0 + 4\eta}{\varepsilon}\right] . \end{aligned} \qquad (3.2.18)$$

A simple iteration using the strong Markov property shows that for any $k \in \mathbb{N}$

$$\sup_{x \in D} P_x[\tau^\varepsilon > kT] \leqslant (1-q)^k ,$$

and thus we obtain that

$$\begin{aligned} \sup_{x \in D} \mathbb{E}_x[\tau^\varepsilon] &\leqslant T\Big[1 + \sum_{k=1}^{\infty} \sup_{x \in D} P_x[\tau^\varepsilon > kT]\Big] \\ &\leqslant T \sum_{k=0}^{\infty} (1-q)^k = \frac{T}{q} \leqslant T \cdot \exp\left[\frac{V_0 + 4\eta}{\varepsilon}\right] , \end{aligned} \tag{3.2.19}$$

where the last inequality holds because of (3.2.17), (3.2.18). This proves the upper bound on the mean first-exit time, since

$$\begin{aligned} & \mathbb{E}_x[\tau^\varepsilon] \leqslant T \cdot \exp\left[\frac{V_0 + \frac{\delta}{2}}{\varepsilon}\right] \\ \Rightarrow \quad & \log \mathbb{E}_x[\tau^\varepsilon] \leqslant \log T + \frac{V_0 + \frac{\delta}{2}}{\varepsilon} \\ \Rightarrow \quad & \varepsilon \log \mathbb{E}_x[\tau^\varepsilon] \leqslant \varepsilon \log T + V_0 + \frac{\delta}{2} \end{aligned}$$

independent of the selection of $\delta > 0$, hence

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_x[\tau^\varepsilon] \leqslant V_0 .$$

Applying Markov's inequality and (3.2.19), we obtain

$$\begin{aligned} \sup_{x \in D} P_x\left[\tau^\varepsilon \geqslant \exp\left[\frac{V_0 + \delta}{\varepsilon}\right]\right] &\leqslant \exp\left[-\frac{V_0 + \delta}{\varepsilon}\right] \cdot \sup_{x \in D} \mathbb{E}_x[\tau^\varepsilon] \\ &\leqslant T \cdot \exp\left[\frac{V_0 + \frac{\delta}{2}}{\varepsilon}\right] \cdot \exp\left[-\frac{V_0 + \delta}{\varepsilon}\right] = T \cdot \exp\left[-\frac{\delta}{2\varepsilon}\right] \xrightarrow{\varepsilon \to 0} 0 , \end{aligned}$$

which proves that $\tau^\varepsilon < \exp\left[\frac{V_0+\delta}{\varepsilon}\right]$ P_x-a.s., as claimed in the assertion of the theorem.

Lower bound. [DZ98, Proof of Thm. 5.7.11] We assume that $\varrho > 0$ is small enough for $S(0, 2\varrho) \subset D$. We define sequences of stopping times $(\tau_m)_{m \in \mathbb{N}_0}$ and $(\theta_m)_{m \in \mathbb{N}_0}$ as follows:

$$\begin{aligned} \theta_0 \quad &:= 0 , \\ \tau_m \quad &:= \inf\{t \geqslant 0 \mid t \geqslant \theta_m ,\ x_t^\varepsilon \in B(0, \varrho) \cup \partial D\} , \\ \theta_{m+1} &:= \begin{cases} \infty & \text{if } x_{\tau_m} \in \partial D \\ \inf\{t \geqslant 0 \mid t > \tau_m ,\ x_t^\varepsilon \in S(0, 2\varrho)\} & \text{else} . \end{cases} \end{aligned}$$

We say that each time interval $[\tau_m, \tau_{m+1}]$ represents a significant excursion from $B(0, \varrho)$, and note that there must be an $m^* \in \mathbb{N}_0$ such that

$\tau^\varepsilon = \tau_{m^*}$ (note that $\tau^\varepsilon < \infty$ P_x-a.s. by the upper bound proved above). Furthermore, we note that

$$(z_m)_{m\in\mathbb{N}_0} := \big(x^\varepsilon_{\tau_m}\big)_{m\in\mathbb{N}_0}$$

is a Markov chain, since $(x^\varepsilon_t)_{t\geqslant 0}$ is a strong Markov process.

Consider $\delta > 0$, $V_0 > 0$ as chosen in the beginning of the proof. By Lemma 3.2.17, choosing $N = \partial D$ we obtain that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{y\in S(0,2\varrho)} P_y\big[x^\varepsilon_{\sigma_\varrho} \in \partial D\big] < -V_0 + \frac{\delta}{2}\,. \tag{3.2.20}$$

By Lemma 3.2.19, we can find a time $T_0 := T(V_0,\varrho)$, independent of ε, such that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{x\in D} P_x\big[\sup_{t\in[0,T_0]} \|x^\varepsilon_t - x\| \geqslant \varrho\big] < -V_0\,.$$

This implies that

$$\begin{aligned}\sup_{x\in D} P_x[\theta_m - \tau_{m-1} \leqslant T_0] &\leqslant \sup_{x\in D} P_x\big[\sup_{t\in[0,T_0]} \|x^\varepsilon_t - x\| \geqslant \varrho\big]\\ &\leqslant \exp\left[-\frac{V_0 - \frac{\delta}{2}}{\varepsilon}\right]\,.\end{aligned} \tag{3.2.21}$$

By construction of the stopping times θ_m, τ_m, we know that for any $m \geqslant 1$

$$\sup_{x\in D} P_x[\tau^\varepsilon = \tau_m] \leqslant \sup_{y\in S(0,2\varrho)} P_y\big[x^\varepsilon_{\sigma_\varrho} \in \partial D\big]\;;$$

hence, by (3.2.20) we can find an $\varepsilon_0 > 0$ such that for any $\varepsilon \leqslant \varepsilon_0$ and all $m \geqslant 1$

$$\sup_{x\in D} P_x[\tau^\varepsilon = \tau_m] \leqslant \exp\left[-\frac{V_0 - \frac{\delta}{2}}{\varepsilon}\right]\,. \tag{3.2.22}$$

We fix a $k \in \mathbb{N}_0$ and consider the event $\{\tau^\varepsilon \leqslant kT_0\}$. This event implies that either for $m \in \{0,\dots,k\}$ the event $\{\tau^\varepsilon = \tau_m\}$ occurs (these are disjoint for different m), or that at least one of the significant excursions $[\tau_m, \tau_{m+1}]$, for $m \in \{0,\dots,k-1\}$, has a length $\leqslant T_0$. Hence, we obtain that for any $x \in D$ and any $k \in \mathbb{N}_0$ the following holds:

$$\begin{aligned}P_x[\tau^\varepsilon \leqslant kT_0] &\leqslant \sum_{m=0}^{k} \Big(P_x[\tau^\varepsilon = \tau_m] + P_x\big[\min_{k=1,\dots,m} (\theta_m - \tau_{m-1}) \leqslant T_0\big]\Big)\\ &\leqslant P_x[\tau^\varepsilon = \tau_0] + 2k \cdot \exp\left[-\frac{V_0 - \frac{\delta}{2}}{\varepsilon}\right]\,.\end{aligned} \tag{3.2.23}$$

where we used (3.2.21) and (3.2.22) in the last step. We recall the identity

$$\{\tau^\varepsilon = \tau_0\} = \big\{x^\varepsilon_{\sigma_\varrho} \notin B(0,\varrho)\big\}$$

and choose

$$k := \left\lceil \frac{1}{T_0} \cdot \exp\left[\frac{V_0 - \delta}{\varepsilon}\right] + \frac{1}{T_0} \right\rceil .$$

Then the following estimate holds for any $x \in D$:

$$\begin{aligned}
&P_x\left[\tau^\varepsilon \leqslant \exp\left[\frac{V_0-\delta}{\varepsilon}\right]\right] \\
&\quad \leqslant P_x\left[\tau^\varepsilon \leqslant \exp\left[\frac{V_0-\delta}{\varepsilon}\right]+1\right] \leqslant P_x[\tau^\varepsilon \leqslant T_0 \cdot k] \\
&\quad \leqslant \underbrace{P_x\left[x^\varepsilon_{\sigma_\varrho} \notin B(0,\varrho)\right]}_{=:A} \\
&\qquad + \underbrace{2 \cdot \left(\frac{1}{T_0} \cdot \exp\left[\frac{V_0-\delta}{\varepsilon}\right] + \frac{1+T_0}{T_0}\right) \cdot \exp\left[-\frac{V_0 - \frac{\delta}{2}}{\varepsilon}\right]}_{=:B},
\end{aligned}$$

where we used (3.2.23) in the last step.

We will now show that $A, B \xrightarrow{\varepsilon \to 0} 0$, which implies that for any $x \in D$

$$\lim_{\varepsilon \to 0} P_x\left[\tau^\varepsilon > \exp\left[\frac{V_0-\delta}{\varepsilon}\right]\right] = 1$$

holds. This is the lower bound of (3.2.8), hence, (3.2.8) is completely proved.

By Lemma 3.2.18, $A \xrightarrow{\varepsilon \to 0} 0$ is trivial. For B we see that

$$\begin{aligned}
B &= \frac{2}{T_0} \cdot \exp\left[\frac{V_0 - \delta - V_0 + \frac{\delta}{2}}{\varepsilon}\right] + \frac{2+2T_0}{T_0} \cdot \exp\left[-\frac{V_0 - \frac{\delta}{2}}{\varepsilon}\right] \\
&= \underbrace{\frac{2}{T_0} \cdot \exp\left[\frac{-\delta}{2\varepsilon}\right]}_{\xrightarrow{\varepsilon \to 0} 0} + \underbrace{\frac{2+2T_0}{T_0} \cdot \exp\left[-\frac{V_0 - \frac{\delta}{2}}{\varepsilon}\right]}_{\xrightarrow{\varepsilon \to 0} 0},
\end{aligned}$$

where the convergence of the second summand relies on our initial assumption that $V_0 - \frac{\delta}{2} > 0$.

By Markov's inequality we know that for any $x \in D$

$$P_x\left[\tau^\varepsilon > \exp\left[\frac{V_0-\delta}{\varepsilon}\right]\right] \leqslant \exp\left[-\frac{V_0-\delta}{\varepsilon}\right] \cdot \mathbb{E}_x[\tau^\varepsilon]$$

holds; hence, the lower bound for $\mathbb{E}_x[\tau^\varepsilon]$ is also proved.

(first exit location). What we want to prove:

For any closed set $N \subset \partial D$ such that $V_N := \inf_{z \in N} V(0,z) > V_0$, and all initial conditions $x \in D$, we have that

$$\lim_{\varepsilon \to 0} P_x\left[x^\varepsilon_{\tau^\varepsilon} \in N\right] = 0 . \qquad ((3.2.10))$$

In particular, if the quasi-potential V has a unique minimum z^* on ∂D, then the following holds for all initial conditions $x \in D$ and all $\delta > 0$:

$$\lim_{\varepsilon \to 0} P_x\left[\|x^\varepsilon_{\tau^\varepsilon} - z^*\| < \delta\right] = 1 . \qquad ((3.2.11))$$

We fix a closed set $N \subset \partial D$, such that $V_N > V_0$. (If $V_N = \infty$, we use an arbitrarily large finite constant throughout the proof instead.)

The proof uses the same idea as the proof of the lower bound for τ^ε. We fix a constant $\eta > 0$ such that $\eta < \frac{V_N - V_0}{3}$, and choose $\varrho, \varepsilon_0 > 0$ small enough to show with Lemma 3.2.17, that for any $\varepsilon \leqslant \varepsilon_0$

$$\sup_{y \in S(0,2\varrho)} P_y\left[x^\varepsilon_{\sigma_\varrho} \in N\right] \leqslant \exp\left[-\frac{V_N - \eta}{\varepsilon}\right] \qquad (3.2.24)$$

holds. By Lemma 3.2.19, we know that there is a time $T_0 := T(V_N - \eta, \varrho)$, such that

$$\limsup_{\varepsilon \to 0} \varepsilon \log \sup_{x \in D} P_x\Big[\sup_{t \in [0,T_0]} \|x^\varepsilon_t - x\| \geqslant \varrho\Big] < -(V_N - \eta) .$$

This implies that for every $\varepsilon \leqslant \varepsilon_0$ (if necessary, we reduce ε_0 to a smaller value > 0), $(\tau_k)_{k \in \mathbb{N}_0}$ and $(\theta_k)_{k \in \mathbb{N}_0}$ as defined before, and any $k \in \mathbb{N}$, we have that

$$\begin{aligned} \sup_{x \in D} P_x[\tau_k \leqslant kT_0] &\leqslant k \cdot \sup_{x \in D} P_x\Big[\sup_{t \in [0,T_0]} \|x^\varepsilon_t - x\| \geqslant \varrho\Big] \\ &\leqslant k \cdot \exp\left[-\frac{V_N - \eta}{\varepsilon}\right] . \end{aligned} \qquad (3.2.25)$$

Like before we set $z_m := x^\varepsilon_{\tau_m}$ for all $m \in \mathbb{N}_0$. By decomposition of the event $\{x^\varepsilon_{\tau^\varepsilon} \in N\}$ we obtain for any $y \in B(0,\varrho)$ (for which $\tau^\varepsilon > \tau_0 = 0$) and any $k \in \mathbb{N}$, that

$$\begin{aligned} &P_y[x^\varepsilon_{\tau^\varepsilon} \in N] \\ &\quad\leqslant P_y[\tau^\varepsilon > \tau_k] + \sum_{m=1}^{k} P_y[\tau^\varepsilon > \tau_{m-1}] \cdot P_y[z_m \in N \mid \tau^\varepsilon > \tau_{m-1}] \\ &\quad\leqslant P_y[\tau^\varepsilon > kT_0] + P_y[\tau_k \leqslant kT_0] \\ &\qquad + \sum_{m=1}^{k} P_y[\tau^\varepsilon > \tau_{m-1}] \cdot \mathbb{E}_y\left[P_{x^\varepsilon_{\theta_m}}\left[x^\varepsilon_{\sigma_\varrho} \in N\right] \,\Big|\, \tau^\varepsilon > \tau_{m-1}\right] \\ &\quad\leqslant P_y[\tau^\varepsilon > kT_0] + P_y[\tau_k \leqslant kT_0] + k \cdot \sup_{x \in S(0,2\varrho)} P_x\left[x^\varepsilon_{\sigma_\varrho} \in N\right] \\ &\quad\leqslant P_y[\tau^\varepsilon > kT_0] + 2k \cdot \exp\left[-\frac{V_N - \eta}{\varepsilon}\right] , \end{aligned}$$

where we have used (3.2.24) and (3.2.25) in the last estimate. We have seen before (in the first part, proof of the upper bound for $\mathbb{E}_x[\tau^\varepsilon]$), that the following estimate holds for a finite time T and all $\varepsilon \leqslant \varepsilon_0$ (if necessary, we reduce $\varepsilon_0 > 0$ again):

$$\sup_{x \in D} \mathbb{E}_x[\tau^\varepsilon] \leqslant T \cdot \exp\left[\frac{V_0 + \eta}{\varepsilon}\right] .$$

By Markov's inequality, we thus have that

$$P_y[\tau^\varepsilon > kT_0] \leqslant \frac{1}{kT_0} \cdot \mathbb{E}_x[\tau^\varepsilon] \leqslant \frac{T}{kT_0} \cdot \exp\left[\frac{V_0 + \eta}{\varepsilon}\right] .$$

If we choose

$$k := \left\lceil \exp\left[\frac{V_0 + 2\eta}{\varepsilon}\right] \right\rceil ,$$

we finally obtain (3.2.10) because of the estimate

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \sup_{y \in B(0,\varrho)} P_y[x^\varepsilon_{\tau^\varepsilon} \in N] \\ &\leqslant \limsup_{\varepsilon \to 0} \Bigg(\underbrace{\frac{T}{kT_0} \cdot \exp\left[\frac{V_0 + \eta}{\varepsilon}\right]}_{\substack{= \frac{T}{T_0} \cdot \exp[-\frac{\eta}{\varepsilon}] \\ \xrightarrow{\varepsilon \to 0} 0}} + \underbrace{2k \cdot \exp\left[-\frac{V_N - \eta}{\varepsilon}\right]}_{\xrightarrow{\varepsilon \to 0} 0} \Bigg) = 0 . \end{aligned}$$

(3.2.11) is a special case of (3.2.10), if we consider

$$N := \left\{ z \in \partial D \mid \|z - z^*\| \geqslant \delta \right\} . \qquad \square$$

Proof of Corollary 3.2.13. For any $\beta > 0$ set

$$D^{-\beta} := \left\{ x \in D \mid \|x - z\| > \beta \ \ \forall\, z \in \partial D \right\} .$$

Since $D^{-\beta}$ are open with $\overline{D^{-\beta}} \subset D$, Assumption (A1) holds for $D^{-\beta}$ for any $\beta > 0$. Furthermore, if $\beta > 0$ is chosen small enough, Assumption (A3) also holds for $D^{-\beta}$. Thus, Theorem 3.2.11 holds for $D^{-\beta}$ for small enough $\beta > 0$.

The exit times $\tau^{\varepsilon,\beta}$ for $D^{-\beta}$ decrease monotonically over β. Hence, taking $\beta \to 0$ we obtain the lower bound for τ^ε because of the continuity property of the quasi-potential resulting from (A3).

To check the upper bound, we proceed as in the proof of Theorem 3.2.11. We only have to check (parallel to (3.2.17), (3.2.18)) that

$$\inf_{x \in D} P_x[\tau^\varepsilon \leqslant T] \geqslant \exp\left[-\frac{V_0 + 4\eta}{\varepsilon}\right]$$

still holds for all $\varepsilon \leqslant \varepsilon_0$. Obviously, this estimate holds for $\tau^{\varepsilon,\beta}$. Thus, we have

$$\inf_{x \in D} P_x[\tau^\varepsilon \leqslant T + 1] \geqslant \inf_{x \in D} P_x[\tau^{\varepsilon,\beta} \leqslant T] \cdot \inf_{x \in D \setminus D^{-\beta}} P_x[\tau^\varepsilon \leqslant 1] .$$

It remains to show that

$$\liminf_{\varepsilon \to 0} \varepsilon \log \inf_{x \in D \setminus D^{-\beta}} P_x[\tau^\varepsilon \leqslant 1] > -\eta \ . \tag{3.2.26}$$

This completes the proof of the corollary, because (3.2.17) is the only argument in the proof of the first part of Theorem 3.2.11 that relies on Assumption (A1). However, using the continuity property of V near ∂D implied by (A3), (3.2.26) follows from the same construction as Lemma 3.2.15. □

4. The Pathwise Approach of Berglund and Gentz

In this chapter we construct and analyze a pathwise mathematical model for SR. Basically, we consider the behavior of a particle in a one-dimensional, continuously changing double-well potential V as described in Section 1.1.2 (resp., a slightly generalized class of potentials). The method is is to analyze a stochastic differential equation modeling the behavior of that particle, where the drift term is a derivative of the potential V.

Our approach is divided into three steps: First we consider the deterministic case, i.e., a system with time-dependent drift and without stochastic perturbations. Then we prove an upper bound for the probability that the path of the stochastically perturbed system leaves a certain neighborhood of the corresponding deterministic path during a finite time t. We show that, for exponentially long times, the perturbed system behaves nearly the same as the deterministic system, if the stochastic perturbation is not too strong. In the third step we consider the case of a diffusion coefficient σ which is strong enough to ensure that, under certain assumptions, we see a transition between the wells happen with high probability whenever the potential reaches a certain state (i.e., when the well containing the particle is most shallow). This is the main mechanism behind our mathematical model of stochastic resonance, or, *noise-induced synchronization*.

This chapter is almost entirely based on the article [BG02b]. In this article, N. Berglund and B. Gentz analyze the behaviour of systems described by one-dimensional stochastic differential equations of the type

$$\mathrm{d}x_s = -\frac{\mathrm{d}}{\mathrm{d}x} V(x_s, s)\,\mathrm{d}s + \sigma\,\mathrm{d}W_s\,. \tag{4.0.1}$$

- W_s is a Brownian motion.
- $V(x,t)$ is a potential changing continuously in time, being $\frac{1}{\varepsilon}$-periodic in t, and showing two local minima for each point t in time, which are separated by a barrier.
- ε and σ, the frequency and the noise intensity, as well as the height of the potential barrier, are ("moderately small") parameters.

  The relation between these parameters is shown to control the transition probability.

In their article, Berglund and Gentz cover two related settings of the above type, namely the case of a symmetric potential, where the two wells decrease and increase simultaneously, and the case of an asymmetric potential, where the two wells achieve their respective maximum and minimum depth at different times (cf. the example in Section 1.1.2). The treatment of the two cases differs insofar as the symmetric case is provided completely with all proofs, whereas in the asymmetric case most of the proofs are left out, with a few hints on how the proofs may be carried over from the symmetric to the asymmetric situation.

The main result of [BG02b] is that in the situation with stochastic perturbation there exists a threshold σ^*, depending on ε and the barrier height, such that for $\sigma > \sigma^*$ a transition happens with probability close to 1, and for $\sigma < \sigma^*$ there is no transition with probability exponentially close to 1. Especially, there is no gradual change between "transition" and "no transition".

In this chapter, we provide a complete treatment of the asymmetric case and slightly optimize some of the estimates of [BG02b].

All citations of theorems, remarks, proofs, etc. are marked as such. Whereever we present a statement or proof that is adapted from the treatment of the symmetric case in [BG02b], we refer to the original statement and mark the reference with "sc" ("symmetric case").

Remark 4.0.20 (possible generalizations)**.** *[Gen03, p. 49] Several restrictions applied below are not necessary for the theory to hold. We will restrict ourselves to the one-dimensional case, even though the treatment of the multidimensional case is possible. We will only consider drift coefficients that derive from a potential, even though this limitation is not necessary, either. And, finally, we will always assume that the diffusion coefficient is constant, even though time-dependent diffusion coefficients could in principle be treated.*

4.1. Preliminaries

We consider the abovementioned stochastic differential equation, replacing the time s by the *slow time* $t := \varepsilon s$ (this technical trick provides us with a periodically changing system with period 1). By this substitution, considering the rescaling properties of the Brownian motion, we obtain

$$\begin{cases} \mathrm{d}x_t &= \frac{1}{\varepsilon} \cdot f(x_t, t)\, \mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\, \mathrm{d}W_t \\ x_{t_0} &= x_0 \,, \end{cases} \tag{4.1.1}$$

where we assume the following conditions to be fulfilled:

- f is the force that is the derivative of the given potential V; it is always assumed to fulfill Lipschitz and boundedness conditions, which secure existence and uniqueness of a strong solution $(x_t)_{t \geqslant t_0}$ to (4.1.1).

  It will turn out that this is no restriction, as we are only interested in the behaviour of the system in a neighborhood of the equilibrium branches, and we will pose assumptions on f from which the solvability conditions over this domain follow immediately.

- $(W_t)_{t \geqslant t_0}$ is a standard Wiener process on a probability space $(\Omega, \mathcal{F}, P)$.

- x_0 is the initial condition; it is always assumed to be square-integrable with respect to P and independent of W.

- t_0 is the starting time (not necessarily 0).

Since under these conditions x always has a continuous version, we assume that the paths $t \mapsto x_t(\omega)$ are continuous for P-a.e. $\omega \in \Omega$.

Our aim is to understand the "jumping between wells" of x; technically speaking, to analyze first-exit times of x from space-time sets: Let $\mathcal{A} \subset \mathbb{R} \times [t_0, t_1]$ be a Borel measurable space-time set. We assume that the space-time starting point (x_0, t_0) is in $\mathcal{A}$ and define the first-exit time of $\big((x_t, t)\big)_{t \geqslant t_0}$ from $\mathcal{A}$ by

$$\tau_{\mathcal{A}}(\omega) := \begin{cases} \infty & \text{for all } \omega \in \Omega \text{ s.th.} \\ & \big(x_t(\omega), t\big) \in \mathcal{A} \\ & \text{for all } t \in [t_0, t_1] \\ \inf\big\{t \in [t_0, t_1] \bigm| \big(x_t(\omega), t\big) \notin \mathcal{A}\big\} & \text{else.} \end{cases}$$

We will also call $\tau_{\mathcal{A}}$ the first-exit time of x from $\mathcal{A}$.

Example 4.1.1 (typical space-time set)**.** Let $g_1, g_2 : [t_0, t_1] \to \mathbb{R}$ be continuous functions such that $g_1 < g_2$. In this chapter, we will typically consider sets of the form

$$\mathcal{A} := \big\{(x, t) \in \mathbb{R} \times [t_0, t_1] \bigm| g_1(t) < x < g_2(t)\big\} .$$

In this case, $\tau_{\mathcal{A}}$ is a stopping time with respect to the canonical filtration on $(\Omega, \mathcal{F}, P)$ generated by $(x_t)_{t \geqslant t_0}$.

We introduce some additional notation:

Let $\varepsilon > 0$ be a small parameter, $t \in [t_0, t_1]$, and $\phi, \psi : (t, \varepsilon) \mapsto \mathbb{R}$ two functions. We write $\phi(t, \varepsilon) \asymp \psi(t, \varepsilon)$ if there exist constants $c_+, c_- > 0$, independent of t and ε, such that for all $t \in [t_0, t_1]$ and all sufficiently small ε we have that

$$c_- \cdot \phi(t, \varepsilon) \leqslant \psi(t, \varepsilon) \leqslant c_+ \cdot \phi(t, \varepsilon) .$$

We remark that $a \asymp b$ especially implies that a and b have the same sign. When working with estimates of this type, we always denote the greater of the two positive constants with subscript "$+$" and the smaller with subscript "$-$".

By $\mathbb{P}^{t_0, x_0}$ we denote probabilities with respect to the law of $(x_t)_{t \geqslant t_0}$, after starting in x_0 at time t_0, and by $\mathbb{E}^{t_0, x_0}$ we denote the corresponding expectation.

Remark 4.1.2 (about the following results)**.** *Most estimates provided below hold for small enough ε only, and often only for P-a.e. $\omega \in \Omega$.*

4.1.1. Concerning f

Until now we have posed only very broad assumptions on the force f that provides the deterministic drift part of the stochastic differential equation (4.1.1). We will now show a typical example for such a function, before we provide a precise definition of the class of functions f which we consider during this chapter.

Example 4.1.3 (typical force term)**.** Let $\lambda(t) := -(\lambda_c - a_0) \cdot \cos(2\pi t)$ and define

$$f(x, t) = -x^3 + x + \lambda(t) .$$

Since f shall be the derivative of a potential with two separated wells, we must ensure that f has three vanishing points. The number of these points depends on λ: If $\lambda = 0$, there are three vanishing points at $1, 0, -1$. If $|\lambda|$ is "big", there is only one vanishing point, as the local maximum and the local minimum value of f have the same sign. If $|\lambda| = \frac{2}{3\sqrt{3}} =: \lambda_c$, then either the local maximum or the local minimum of f is a vanishing point ("double root"). Hence, to ensure that f has three vanishing points, we must take care that for any $t \in [t_0, t_1]$ the inequality $\bigl|\lambda(t)\bigr| < \lambda_c$ holds. To achieve this, we require that $a_0 \in]0, \lambda_c[$.

For intuitive understanding, the reader should always keep in mind that the force f derived from the potential V is *not* $\frac{\mathrm{d}}{\mathrm{d}x} V$ but $-\frac{\mathrm{d}}{\mathrm{d}x} V$.

Situation 4.1.4 (class of functions f)**.** *We consider a class of functions $f : \mathbb{R}^2 \to \mathbb{R}$ which shall satisfy the following assumptions:*

smoothness. *$f \in \mathcal{C}^3(\mathcal{M}; \mathbb{R})$, where $\mathcal{M} := [-d, d] \times \mathbb{R}$ for a constant $d > 0$.*

periodicity. *For all $(x, t) \in \mathcal{M}$ we assume that $f(x, t + 1) = f(x, t)$.*

equilibrium branches. *There exist continuous functions*

$$x_-^*(t) < x_u^*(t) < x_+^*(t) \quad \forall t \,,$$

mapping $\mathbb{R} \to [-d, d]$ such that for any $(x, t) \in \mathcal{M}$ the equation $f(x, t) = 0$ holds if and only if $x \in \bigl\{x_-^*(t),\, x_u^*(t),\, x_+^*(t)\bigr\}$. These functions are called equilibrium branches *of f.*

We claim that these zeroes of f are isolated in the following sense: For any $\delta > 0$ there exists a constant $\rho > 0$ such that for all x with

$$\bigl|x - x_-^*(t)\bigr| \geqslant \delta \,, \quad \bigl|x - x_+^*(t)\bigr| \geqslant \delta \,, \quad \text{and} \quad \bigl|x - x_u^*(t)\bigr| \geqslant \delta$$

we have that $\bigl|f(x, t)\bigr| \geqslant \rho$.

stability. *The equilibrium branches x_+^*, x_-^* are stable, whereas the equilibrium branch x_u^* is unstable. In other words, for all $t \in \mathbb{R}$ we assume that*

$$\begin{aligned} a_-^*(t) &:= \partial_x f\bigl(x_-^*(t), t\bigr) < 0 \,, \\ a_+^*(t) &:= \partial_x f\bigl(x_+^*(t), t\bigr) < 0 \,, \\ a_u^*(t) &:= \partial_x f\bigl(x_u^*(t), t\bigr) > 0 \,. \end{aligned} \tag{4.1.2}$$

Especially, this implies that there exist constants $a_+, a_-, a_u > 0$, such that for any $t \in \mathbb{R}$

$$\begin{aligned} a_-^*(t) &< -a_- \\ a_+^*(t) &< -a_+ \\ a_u^*(t) &> a_u \,. \end{aligned}$$

(Intuitively speaking, for any t the potential V with derivative f is strictly convex near the stable equilibrium branches and strictly concave near the unstable equilibrium branch. This does explicitly exclude bifurcations of or from the equilibrium branches.)

behaviour near $t = 0$**.** *We want that* $x_+^*(t)$ *and* $x_u^*(t)$ *come close at integer times* t*. To achieve this, we assume that for* $t = 0$ *we have an "avoided (saddle-node) bifurcation": We assume the existence of a (fixed) point* $x_c \in \left]x_u^*(0), x_+^*(0)\right[$*, such that*

$$\begin{aligned}
&\partial_{xx} f(x_c, 0) < 0 \,, \\
&\partial_x f(x_c, t) \quad = \mathcal{O}(t^2) \,, \\
&f(x_c, t) \qquad = a_0 + a_1 t^2 + \mathcal{O}(t^3) \,,
\end{aligned} \tag{4.1.3}$$

where $a_1 > 0$ *and* $\partial_{xx} f(x_c, 0)$ *are both fixed and of order* 1*, while* $a_0 = a_0(\varepsilon) = \mathcal{O}_\varepsilon(1)$ *is a small positive parameter.*

(Intuitively speaking, x_c *is a local maximum of* $x \mapsto f(x, 0)$*. At* $t = 0$*, the curve* $t \mapsto f(x_c, t)$ *achieves a minimum value* $a_0 > 0$ *of order* ε*.)*

Remark 4.1.5. *These assumptions imply, together with the isolation and stability assumptions for the equilibrium branches, that* $x_+^*(t)$ *reaches a local minimum near* $t = 0$*, at a time* $t_+^* = \mathcal{O}(a_0)$*, and* $x_u^*(t)$ *reaches a local maximum near* $t = 0$*, at a time* $t_u^* = \mathcal{O}(a_0)$*.*

We summarize our assumptions as follows: For a small enough constant time $T \in \left]0, \frac{1}{2}\right[$ *(such that the derivatives of* $x_+^*(t)$ *and* $x_u^*(t)$ *vanish once during* $[-T, T]$*), the equilibrium branches of* f *and the linearizations of* f *near these branches satisfy:*

$$\begin{aligned}
x_+^*(t) - x_c &\asymp \begin{cases} \sqrt{a_0} & \text{if } |t| \in [0, \sqrt{a_0}] \\ |t| & \text{if } |t| \in [\sqrt{a_0}, T] \,, \end{cases} \\
a_+^*(t) &\asymp \begin{cases} -\sqrt{a_0} & \text{if } |t| \in [0, \sqrt{a_0}] \\ -|t| & \text{if } |t| \in [\sqrt{a_0}, T] \,, \end{cases} \\
x_u^*(t) - x_c &\asymp \begin{cases} -\sqrt{a_0} & \text{if } |t| \in [0, \sqrt{a_0}] \\ -|t| & \text{if } |t| \in [\sqrt{a_0}, T] \,, \end{cases} \\
a_u^*(t) &\asymp \begin{cases} \sqrt{a_0} & \text{if } |t| \in [0, \sqrt{a_0}] \\ |t| & \text{if } |t| \in [\sqrt{a_0}, T] \,, \end{cases} \\
x_-^*(t) - x_c &\asymp -1 \quad \text{if } |t| \in [0, T] \,, \\
a_-^*(t) &\asymp -1 \quad \text{if } |t| \in [0, T] \,.
\end{aligned} \tag{4.1.4}$$

We tighten our assumptions as follows: We assume that x *is rescaled such that* $\partial_{xx} f(x_c, 0) = -2$*. This will provide us nice Taylor estimates for* f *further below.*

behaviour near $t = t_c$**.** *We want that* $x_-^*(t)$ *and* $x_u^*(t)$ *come close at a time* $t_c \in \left]T, 1 - T\right[$*. We achieve this by making similar assumptions as above that shall hold at a point* $(x_c', t_c) \in \mathcal{M}$*.*

Note that while we assume that $t_c \in [0, 1] \setminus \big([-T, T] \cup [1 - T, 1 + T]\big)$*, we do not claim that*

$$[t_c - T, t_c + T] \cap \big([-T, T] \cup [1 - T, 1 + T]\big)$$

is empty: even though t_c is outside the T-neighborhood of any integer time, the T-neighborhoods of t_c and the integer times are not required to be disjoint.

behaviour between the close encounters. *We want to exclude the possibility of more close encounters (and almost-bifurcations). Hence, we assume that for any $t \in \,]T, t_c - T[$ and for any $t \in \,]t_c + T, 1 - T[$ the distances $\left|x_+^*(t) - x_u^*(t)\right|$ and $\left|x_u^*(t) - x_-^*(t)\right|$ as well as the derivatives in* (4.1.2) *are bounded away from zero.*

The assumption on the behaviour near $t = t_c$ follows immediately from the assumptions on the behaviour near $t = 0$ if we assume that $f\big(x, t + \frac{1}{2}\big) = -f(-x, t)$ holds for all $(x, t) \in \mathcal{M}$. This condition is fulfilled in Example 4.1.3.

The following remark lists some direct consequences of the above assumptions.

Remark 4.1.6 (Taylor estimates). *By Taylor's formula (and thanks to the rescaling we did to ensure that $\partial_{xx} f(x_c, 0) = -2$) we obtain that*

$$\partial_x f(x_c + \tilde{x}, t) = \partial_x f(x_c, t) + \tilde{x} \cdot \big(-2 + r_1(\tilde{x}, t)\big) \ , \tag{4.1.5}$$

where $r_1(0,0) = 0$, and by the smoothness assumptions on f we know that $r_1 \in \mathcal{C}^1$. Our assumption that $\partial_x f(x_c, t) = \mathcal{O}(t^2)$ implies together with (4.1.5) *that $\partial_x f(x, t)$ vanishes on a curve $\bar{x}(t) = x_c + \mathcal{O}(t^2)$.*

As $\bar{x}(0) = x_c$, hence $\partial_{xx} f\big(\bar{x}(0), 0\big) = -2$, we further see that

$$f\big(\bar{x}(t) + z, t\big) = f\big(\bar{x}(t), t\big) + z^2 \cdot \big(-1 + r_0(z, t)\big) \ ,$$

where $r_0 \in \mathcal{C}^1$ with $r_0(0,0) = 0$, and that

$$f\big(\bar{x}(t), t\big) = f(x_c, t) + \mathcal{O}(t^4) = a_0 + a_1 \cdot t^2 + \mathcal{O}(t^3)$$

for all $t \in [-T, T]$.

4.2. The Deterministic Case

As announced above, we first analyze the behaviour of the deterministic system corresponding to (4.1.1). Our aim is, to come back to the intuitive description from above, to understand the motion of a particle in the potential V under the assumption that it starts near the bottom of one of the potential wells and that this well remains well-separated from the other well throughout the time intervall under consideration.

Situation 4.2.1 (deterministic situation). *We analyze the behavior of the system*

$$\varepsilon \cdot \frac{\mathrm{d}}{\mathrm{d}t} x_t^{\mathrm{det}} = f(x_t^{\mathrm{det}}, t) \ , \tag{4.2.1}$$

where we require that the assumptions from Situation 4.1.4 are still valid and the initial point x_{-T}^{det} (starting time is $-T$, not 0) satisfies the condition

$$x_{-T}^{\mathrm{det}} - x_+^*(-T) \asymp \varepsilon \ . \tag{4.2.2}$$

We also use the notation

$$\varepsilon \dot{x}_t^{\text{det}} = f(x_t^{\text{det}}, t)$$

instead of (4.2.1).

Before we state the main theorem of this section, let us first take a look at the behavior of the solution to (4.2.1) during $[-1+T, -T]$. By our stability assumption we might expect that, if the initial point x_{-1+T}^{det} is chosen appropriately, the path of the system approaches the equilibrium branch x_+^* exponentially fast to a distance of zero and then follows its path.

However, this is not the case. We will see that there is a so-called *slow solution* $\hat{x}^{\text{det}}$ to (4.2.1), which remains in a neighborhood of the order ε of the equilibrium branch, and that *this solution* is approached exponentially fast by any solution x^{det} of (4.2.1) if only x_{-1+T}^{det} is chosen close enough to $x_+^*(-1+T)$.

This result justifies our assumption (4.2.2) on the initial condition x_{-T}^{det}: If we start the system at a time $t_0 = -1+T$ in any initial point close enough to the equilibrium branch, then the following theorem implies that $\left|x_{-T}^{\text{det}} - x_+^*(-T)\right| \asymp \varepsilon$ if only $1-2T \gg \varepsilon$. Thus, assumption (4.2.2) is compatible with the general concept of periodicity of the setting.

Proposition 4.2.2 (deterministic system during $[-1+T, -T]$; Gradšteĭn, Tihonov)**.** *[adapted from [Gen03, Theorem 4.2]]*[1] *Consider* (4.2.1) *under the assumptions of Situation 4.1.4. There exist constants $\varepsilon_0, c_0, c_1 > 0$, which depend on f only, such that for any $\varepsilon \in \,]0, \varepsilon_0]$ the system* (4.2.1)*, starting at time $-1+T$, has a solution $\hat{x}^{\text{det}}$ such that for all $t \in [-1+T, -T]$*

$$\left|\hat{x}_t^{\text{det}} - x_+^*(t)\right| \leqslant c_1 \cdot \varepsilon \tag{4.2.3}$$

holds, and, if the initial condition x_{-1+T}^{det} satisfies $\left|x_{-1+T}^{\text{det}} - x_+^*(-1+T)\right| \leqslant c_0$ and $x_{-1+T}^{\text{det}} > x_u^*(-1+T)$, then the corresponding solution x^{det} of (4.2.1) *fulfills for any $t \in [-1+T, T]$ the estimate*

$$|x_t^{\text{det}} - \hat{x}_t^{\text{det}}| \leqslant |x_{-1+T}^{\text{det}} - \hat{x}_{-1+T}^{\text{det}}| \cdot \exp\left[-\frac{a_+ \cdot \big(t - (-1+T)\big)}{2\varepsilon}\right]. \tag{4.2.4}$$

Proof. We consider the deviation

$$y_t := x_t^{\text{det}} - x_+^*(t)$$

of an arbitrary solution of (4.2.1) from the equilibrium branch. To analyze this deviation, we first develop estimates for $\varepsilon \dot{y}_t$ and then, in a second step, prove the first assertion of the theorem by showing that $|y_t| \leqslant c_1 \varepsilon$ for a properly chosen initial condition y_{-1+t}; the so-constructed solution x^{det} is the "ideal solution" $\hat{x}^{\text{det}}$. In the third step of the proof, we prove (4.2.4), hence, that all solutions of (4.2.1) which start in a neighbourhood of $x_+^*(-1+T)$, approach each others exponentially fast.

We observe that $\varepsilon \dot{y}_t = f(x_t^{\text{det}}, t) - \varepsilon \dot{x}_+^*(t)$. Using the Taylor expansion

$$f(x_t^{\text{det}}, t) = \underbrace{f\big(x_+^*(t), t\big)}_{=0} + \underbrace{\partial_x f\big(x_+^*(t), t\big)}_{=a_+^*(t)} \cdot y_t + b(y_t, t), \tag{4.2.5}$$

[1] [Gen03] refers to [Gra53] and [Tih52] as original sources.

where $\left|b(y_t,t)\right| \leqslant M \cdot y_t^2$ for all $t \in [-1+t,-T]$, $|y_t| \leqslant d$, and a big enough constant $M > 0$, we may extend our above observation to obtain

$$\varepsilon \dot{y}_t = \underbrace{a_+^*(t)}_{\leqslant -a_+} \cdot y_t + \underbrace{b(y_t,t)}_{\leqslant M y_t^2} - \varepsilon \dot{x}_+^*(t) \ .$$

To complete the estimate of $\varepsilon \dot{y}_t$, we need to show the boundedness of $\left|\dot{x}_+^*(t)\right|$. But

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} f\big(x_+^*(t),t\big) = \partial_x f\big(x_+^*(t),t\big) \cdot \dot{x}_+^*(t) + \partial_t f\big(x_+^*(t),t\big)$$

implies that

$$\dot{x}_+^*(t) = -\frac{\partial_t f\big(x_+^*(t),t\big)}{a_+^*(t)} \ .$$

By our general assumptions in Situation 4.1.4, we know that the derivatives of f are bounded over $\mathcal{M}$ and that $a_+^*(t)$ is bounded away from zero for all t: $a_+^*(t) \leqslant -a_+ < 0$. Hence, the above result proves that we can find a constant $B > 0$ such that $\left|\dot{x}_+^*(t)\right| \leqslant B$ for all $t \in [-1+T,-T]$. Combining this with the estimate for $\varepsilon \dot{y}_t$, we see that

$$\begin{aligned} \varepsilon \dot{y}_t &\leqslant -a_+ y_t + M y_t^2 + \varepsilon B \quad \text{if } y_t \geqslant 0 \\ \varepsilon \dot{y}_t &\geqslant -a_+ y_t - M y_t^2 - \varepsilon B \quad \text{if } y_t \leqslant 0. \end{aligned} \tag{4.2.6}$$

For the second step, we assume that $y_t \geqslant 0$; the other case is similar. We define v_t by

$$\varepsilon \dot{v}_t = \underbrace{-a_+ v_t + M v_t^2 + \varepsilon B}_{=:g(v_t)} \ .$$

Simple arithmetic proves that $g(v_t) = 0$ if and only if

$$v_t = \underbrace{\frac{a_+}{2M} \pm \sqrt{\frac{a_+^2}{4M^2} - \frac{\varepsilon B}{M}}}_{=:v_\pm^*} \ .$$

Hence, $v_\pm^*$ are solutions of $\varepsilon \dot{v}_t = g(v_t)$ if ε is small enough to guarantee the positivity of the argument of the square root. By definition, v_t dominates y_t whenever $v_{-1+T} \geqslant y_{-1+T}$. Hence, if $0 \leqslant y_{-1+T} \leqslant v_-^*$, we conclude that for any $t \in [-1+T,-T]$

$$y_t \leqslant v_-^* = \frac{a_+}{2M} - \sqrt{\frac{a_+^2}{4M^2} - \frac{\varepsilon B}{M}} \ .$$

For $y_t \leqslant 0$, we can apply a similar argument and obtain alltogether that

$$|y_t| \leqslant \underbrace{\frac{a_+}{2M} - \sqrt{\frac{a_+^2}{4M^2} - \frac{\varepsilon B}{M}}}_{=:v^*} \ .$$

Hence, whenever the initial condition y_{-1+T} fulfills $|y_{-1+T}| \leqslant v^*$, the corresponding process $(y_t)_{t\in[-1+T,-T]}$ meets the claimed estimate $|y_t| \leqslant c_1\varepsilon$ for a constant $c_1 > 0$ and all $t \in [-1+T, -T]$.

Finally, we come to the third step, the proof of (4.2.4). Let $\hat{x}^{\text{det}}$ be a solution of (4.2.1) that fulfills the estimate (4.2.3) – we have just proved the existence of such a process. Let x^{det} be another solution with $\left|x^{\text{det}}_{-1+T} - x^*_+(-1+T)\right| \leqslant c_0$ for a constant $c_0 > 0$ and $x^{\text{det}}_{-1+T} > x^*_u(-1+T)$, and set

$$z_t := x^{\text{det}}_t - \hat{x}^{\text{det}}_t \,.$$

By definition,

$$\varepsilon \dot{z}_t = f(x^{\text{det}}_t, t) - f(\hat{x}^{\text{det}}_t, t) \,,$$

and by Taylor expansion for f (cf. (4.2.5)) we see that

$$\varepsilon \dot{z}_t = \underbrace{\partial_x f\big(x^*_+(t), t\big)}_{=a^*_+(t)} \cdot \underbrace{\big((x^{\text{det}}_t - x^*_+(t)) - (\hat{x}^{\text{det}}_t - x^*_+(t))\big)}_{=x^{\text{det}}_t - \hat{x}^{\text{det}}_t = z_t} + \hat{b}(y_t, t) \,,$$

where again $\left|\hat{b}(y_t, t)\right| \leqslant M z_t^2$ for all $t \in [-1+T, -T]$. Hence,

$$\varepsilon \dot{z}_t \leqslant -a_+ \cdot z_t + M z_t^2 \,,$$

and as long as $0 \leqslant z_{-1+T} \leqslant \frac{a_+}{2M}$ we may conclude that

$$\varepsilon \dot{z}_t \leqslant -\frac{a_+}{2} \cdot z_t \,.$$

Thus, as long as $0 \leqslant z_{-1+T} \leqslant \frac{a_+}{2M}$,

$$z_t \leqslant z_{-1+T} \cdot \exp\left[-a_+ \cdot \frac{t-(-1+T)}{2\varepsilon}\right] .$$

In the case $0 \geqslant z_{-1+T} \geqslant -\frac{a_+}{2M}$, a similar estimate holds; consequently, we have that

$$|z_t| \leqslant |z_{-1+T}| \cdot \exp\left[-a_+ \cdot \frac{t-(-1+T)}{2\varepsilon}\right] ,$$

and (4.2.4) is proven for $c_0 := \frac{a_+}{2M}$. □

From here on, we concentrate on the behaviour of the system during the time interval $[-T, T]$ and remark in passing that the behaviour during the interval $[t_c - T, t_c + T]$ is in principle the same ("modulo glide reflection").

We sum up the results of this section in the following theorem:

Theorem 4.2.3 (deterministic system during $[-T, T]$)**.** *[BG02b, Theorem 2.5] Consider the setting of Situation 4.2.1. The curves x^{det}_t and $x^*_+(t)$ cross exactly once during $[-T, T]$, at a time $\tilde{t}$ such that*

$$\tilde{t} - t^*_+ \asymp \frac{\varepsilon}{\sqrt{a_0}} \wedge \varepsilon^{1/2} \quad (> 0) \,.$$

There is a constant $c_0 > 0$ such that

$$x_t^{\det} - x_+^*(t) \asymp \begin{cases} \frac{\varepsilon}{|t|} & \text{if } t \in \left[-T, -c_0\left(\sqrt{a_0} \vee \varepsilon^{1/2}\right)\right] \\ -\frac{\varepsilon}{|t|} & \text{if } t \in \left[c_0\left(\sqrt{a_0} \vee \varepsilon^{1/2}\right), T\right] . \end{cases} \tag{4.2.7}$$

This implies that during these time intervals $x_t^{\det} - x_c \asymp |t|$.
Notation: *We set $t_0 := -c_0\left(\sqrt{a_0} \vee \varepsilon^{1/2}\right)$.*
If $t \in [t_0, -t_0]$, then

$$x_t^{\det} - x_c \asymp \begin{cases} \sqrt{a_0} & \text{if } a_0 \geqslant \varepsilon \\ \varepsilon^{1/2} & \text{if } \varepsilon \geqslant a_0. \end{cases} \tag{4.2.8}$$

For all $t \in [-T, T]$, the linearization of f at $x_t^{\det}$ can be estimated by

$$\underbrace{\partial_x f(x_t^{\det}, t)}_{=:\bar{a}(t)} \asymp -\left(|t| \vee \sqrt{a_0} \vee \sqrt{\varepsilon}\right) . \tag{4.2.9}$$

We note that (4.2.8) especially implies that $x_t^{\det}$ always remains greater than x_c. In other words, for any choice of a_0 and ε, the deterministic path will never cross the saddle.

The remaining part of this section is devoted to the proof of this theorem.

Remark 4.2.4 ($x^{\det}$ and x_+^* cross once at $\tilde{t}$)**.** *[sc: [BG02b, Remark 3.1]] During the time interval $[-T, T]$, the curve $x_t^{\det}$ crosses the equilibrium branch $x_+^*(t)$ precisely once at a time $\tilde{t} > t_+^*$.*

Basically, this is a consequence of the fact that $t \mapsto x_t^{\det}$ is strictly decreasing if $x_t^{\det} > x_+^*(t)$ and strictly increasing if $x_t^{\det} < x_+^*(t)$.

Proof. Let $\tilde{t}_1, \tilde{t}_2, \ldots$ be the crossing times of $x_t^{\det}$ and $x_+^*(t)$ during $[-T, T]$. Since we assume (see (4.2.2)) that $x_{-T}^{\det} - x_+^*(-T) > 0$, $x_t^{\det}$ must be decreasing during $[-T, \tilde{t}_1]$, increasing during $[\tilde{t}_1, \tilde{t}_2]$, and so on. Further, $x_+^*(t)$ is decreasing in a neighborhood of $-T$ by (4.1.4).

Because of $f\left(x_+^*(t), t\right) = 0$ for all t, the crossing can not happen as long as x_+^* is decreasing. Hence, there must be a time t slightly smaller than $\tilde{t}_1$, such that $x_+^*(t)$ is increasing. By the same argument, there must be a time t slightly smaller than $\tilde{t}_2$ (and bigger than $\tilde{t}_1$) such that $x_+^*(t)$ is decreasing. On the other hand, by our general assumptions and the definition of t_+^*, we know that $x_+^*(t)$ is decreasing during $[-T, t_+^*[$ and increasing during $]t_+^*, T]$. Hence, the crossing times must fulfill $\tilde{t}_1 > t_+^*$ and $\tilde{t}_2 > T$. This proves the claimed uniqueness of $\tilde{t}$ as the single crossing time during $[-T, T]$ and the relation between $\tilde{t}$ and t_+^*.

We will see below that such a crossing actually happens. We will also determine the order of $\tilde{t}$ (see Propositions 4.2.8, 4.2.9 and estimates (4.2.27), (4.2.36)). □

For the next steps, we set

$$y_t := x_t^{\det} - x_+^*(t)$$

and recover that by assumption, we know that $y_{-T} \asymp \varepsilon$.

The dynamics of y_t are characterized by

$$\begin{aligned}\varepsilon \cdot \frac{\mathrm{d}}{\mathrm{d}t}\, y_t &= f\big(x_t^{\det}, t\big) - \varepsilon \cdot \frac{\mathrm{d}}{\mathrm{d}t}\, x_+^*(t) \\ &= a_+^*(t) \cdot y_t + b_+^*(y_t, t) - \varepsilon \cdot \frac{\mathrm{d}}{\mathrm{d}t}\, x_+^*(t)\ . \end{aligned} \tag{4.2.10}$$

By assumption (4.1.4) we know that

$$a_+^*(t) \asymp \begin{cases} -\sqrt{a_0} & \text{if } |t| \leqslant \sqrt{a_0} \\ -|t| & \text{if } |t| \in \big[\sqrt{a_0}, T\big]\ , \end{cases} \tag{4.2.11}$$

and the smoothness properties of $f : \mathcal{M} \to \mathbb{R}$ imply that $\big|b_+^*(y_t, t)\big| \leqslant M \cdot y_t^2$ for a big enough $M > 0$ and all $t \in [-T, T]$. Again from (4.1.4) we obtain the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\, x_+^*(t) \asymp \begin{cases} -1 & \text{if } t \in [-T, -\sqrt{a_0}] \\ \frac{1}{\sqrt{a_0}} \cdot (t - t_+^*) & \text{if } t \in [-\sqrt{a_0}, \sqrt{a_0}] \\ 1 & \text{if } t \in \big[\sqrt{a_0}, T\big]\ . \end{cases} \tag{4.2.12}$$

Remark 4.2.5 (concerning estimate (4.2.12))**.** *There is a small complication concerning the application of this estimate over the interval* $[-\sqrt{a_0}, \sqrt{a_0}]$*.*

By definition of the notation "$\asymp$", the estimate

$$f(t) \asymp g(t)$$

says that there are constants $c_+ \geqslant c_- > 0$, such that for all t

$$c_- \cdot g(t) \leqslant f(t) \leqslant c_+ \cdot g(t)\ .$$

Also in this situation, the estimate states that there must be two constants $c_{\boxplus} \geqslant c_{\boxminus} > 0$ such that $\frac{\mathrm{d}}{\mathrm{d}t} x_+^*(t)$ is for any $t \in [-\sqrt{a_0}, \sqrt{a_0}]$ inside the strip confined by $g_{\boxminus}(t) := \frac{c_{\boxminus}}{\sqrt{a_0}} \cdot (t - t_+^*)$ and $g_{\boxplus}(t) := \frac{c_{\boxplus}}{\sqrt{a_0}} \cdot (t - t_+^*)$. But $t_+^* \in [-\sqrt{a_0}, \sqrt{a_0}]$ implies that

$$g_{\boxplus}(t) \left\{ \begin{matrix} \leqslant \\ \geqslant \end{matrix} \right\} g_{\boxminus}(t) \left\{ \begin{matrix} \textit{if } t \leqslant t_+^* \\ \textit{if } t \geqslant t_+^* \end{matrix} \right\}\ .$$

Hence, (4.2.12) *can only be understood as follows:*

There exist constants $c_{\boxplus} \geqslant c_{\boxminus} > 0$ such that

$$\frac{c_{\boxplus}}{\sqrt{a_0}} \cdot (t - t_+^*) \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\, x_+^*(t) \leqslant \frac{c_{\boxminus}}{\sqrt{a_0}} \cdot (t - t_+^*) \leqslant 0 \quad \textit{for all } t \leqslant t_+^*$$

$$0 \leqslant \frac{c_{\boxminus}}{\sqrt{a_0}} \cdot (t - t_+^*) \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\, x_+^*(t) \leqslant \frac{c_{\boxplus}}{\sqrt{a_0}} \cdot (t - t_+^*) \quad \textit{for all } t \geqslant t_+^*\ .$$

We will need the following, technical result several times:

	$\beta \geqslant \varepsilon^{1/2}$ $(> \varepsilon)$	$\beta \leqslant \varepsilon^{1/2}$	
①	$t \in [-T, -\beta]$	$t \in [-T, -\varepsilon^{1/2}]$	④
②	$t \in [-\beta, \beta]$	$t \in [-\varepsilon^{1/2}, \varepsilon^{1/2}]$	⑤
③	$t \in [\beta, T]$	$t \in [\varepsilon^{1/2}, T]$	⑥

Table 4.1.: The six cases for the proof of Lemma 4.2.6.

Lemma 4.2.6. *[BG02b, Lemma 4.1] Let $\beta = \beta(\varepsilon) \geqslant 0$ be some parameter. Let $\tilde{a}$ be an arbitrary continuous function such that $\tilde{a}(t) \asymp -(\beta \vee |t|)$ for all $t \in [-T, T]$. Furthermore, let $\chi_0 \asymp 1$ and define*

$$\tilde{\alpha}(t,s) := \int_s^t \tilde{a}(u)\,\mathrm{d}u\ .$$

Then,

$$\begin{aligned} &\chi_0 \cdot \exp\left[\frac{\tilde{\alpha}(t,-T)}{\varepsilon}\right] + \frac{1}{\varepsilon}\int_{-T}^{t} \exp\left[\frac{\tilde{\alpha}(t,s)}{\varepsilon}\right] \mathrm{d}s \\ &\quad \asymp \begin{cases} \frac{1}{\beta \vee \varepsilon^{1/2}} & \text{if } |t| \in [0, \beta \vee \varepsilon^{1/2}] \\ \frac{1}{|t|} & \text{if } |t| \in [\beta \vee \varepsilon^{1/2}, T]\ . \end{cases} \end{aligned} \tag{4.2.13}$$

Proof. [adapted from [BG02a, Lemma 4.2]] For this proof we make use of the flow property of solutions to differential equations; obviously, the left hand side of (4.2.13), which we refer to as $\chi(t)$ during the proof, is the solution of a differential equation. We consider six different cases, which are listed in Table 4.1. During the proof, we use the circled numbers, ①–⑥, to identify the cases.

Let us start with ①. By the assumptions of the Lemma, there exist constants $\chi_+, \chi_- > 0$ such that for all $t \in [-T, T]$

$$\chi_0 \begin{cases} \leqslant \chi_+ \\ \geqslant \chi_-\ , \end{cases}$$

and constants $c_+, c_- > 0$ such that for all $t \in [-T, -\beta]$

$$\tilde{a}(t) \begin{cases} \leqslant -c_- \cdot |t| \\ \geqslant -c_+ \cdot |t|\ , \end{cases}$$

which implies that for any s, t with $-T \leqslant s \leqslant t \leqslant -\beta$

$$\tilde{\alpha}(t,s) \begin{cases} \leqslant \frac{c_-}{2} \cdot (t^2 - s^2) \\ \geqslant \frac{c_+}{2} \cdot (t^2 - s^2)\ . \end{cases}$$

It should be noted that both $\tilde{a}$ and $\tilde{\alpha}$ are $\leqslant 0$ throughout the proof. We use this to estimate functions of the type $\exp[\tilde{\alpha}]$ from above by 1. Furthermore, we remark that we may always enlarge c_+, χ_+ or diminish c_-, χ_- without loss of generality. By integration by parts[2] we obtain the following:

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_{-T}^{t} \exp\left[\frac{c_\pm}{2\varepsilon}\cdot(t^2-s^2)\right] \mathrm{d}s \qquad (4.2.14)\\
&\quad= \left[-\frac{1}{c_\pm \cdot s}\cdot \exp\left[\frac{c_\pm}{2\varepsilon}\cdot(t^2-s^2)\right]\right]_{-T}^{t}\\
&\qquad - \int_{-T}^{t} \frac{1}{c_\pm\cdot s^2}\cdot \exp\left[\frac{c_\pm}{2\varepsilon}\cdot(t^2-s^2)\right] \mathrm{d}s\\
&\quad= \underbrace{-\frac{1}{c_\pm\cdot t}}_{=\frac{1}{c_\pm\cdot|t|}>0} - \frac{1}{c_\pm\cdot T}\cdot \exp\left[\frac{c_\pm}{2\varepsilon}\cdot(t^2-T^2)\right]\\
&\qquad - \int_{-T}^{t} \frac{1}{c_\pm\cdot s^2}\cdot \exp\left[\frac{c_\pm}{2\varepsilon}\cdot(t^2-s^2)\right] \mathrm{d}s\,.
\end{aligned}
$$

Now we see the the upper bound for ① as follows:

$$
\begin{aligned}
\chi(t) &\leqslant \chi_+\cdot \exp\left[\frac{c_-}{2\varepsilon}\cdot(t^2-T^2)\right] + \frac{1}{\varepsilon}\int_{-T}^{t} \exp\left[\frac{c_-}{2\varepsilon}\cdot(t^2-s^2)\right] \mathrm{d}s\\
&= \chi_+\cdot \exp\left[\frac{c_-}{2\varepsilon}\cdot(t^2-T^2)\right] + \frac{1}{c_-\cdot|t|} - \frac{1}{c_-\cdot T}\cdot \exp\left[\frac{c_-}{2\varepsilon}\cdot(t^2-T^2)\right]\\
&\qquad \underbrace{- \int_{-T}^{t} \frac{1}{c_-\cdot s^2}\cdot \exp\left[\frac{c_-}{2\varepsilon}\cdot(t^2-s^2)\right] \mathrm{d}s}_{>0}\\
&\leqslant \frac{1}{c_-\cdot|t|} - \underbrace{\left(\frac{1}{c_-\cdot T} - \chi_+\right)}_{\substack{>0\\ \text{if } c_- \text{ small enough}}}\cdot \exp\left[\frac{c_-}{2\varepsilon}\cdot(t^2-T^2)\right]\\
&\leqslant \frac{1}{c_-}\cdot\frac{1}{|t|}\,.
\end{aligned}
$$

This argument also proves the upper bound for ④.

---

[2] We use

$$\int_a^b u(s)v'(s)\,\mathrm{d}s = \left[u(s)v(s)\right]_a^b - \int_a^b u'(s)v(s)\,\mathrm{d}s$$

with

$$v(s) = \exp\left[\frac{c_\pm}{2\varepsilon}\cdot(t^2-s^2)\right] \quad\text{and}\quad u(s) = -\frac{1}{c_\pm\cdot s}\,.$$

Let us target the lower bound for ①. Using (4.2.14), we see that

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{-T}^{t} \exp\left[\frac{c_+}{2\varepsilon}\cdot(t^2-s^2)\right]\,\mathrm{d}s &\geqslant \frac{1}{\varepsilon}\int_{(-T\vee 2t)}^{t} \exp\left[\frac{c_+}{2\varepsilon}\cdot(t^2-s^2)\right]\,\mathrm{d}s \\
&\geqslant \frac{1}{c_+\cdot|t|} + \frac{1}{c_+\cdot(-T\vee 2t)}\cdot\exp\left[\frac{c_+}{2\varepsilon}\cdot\left(t^2-(-T\vee 2t)^2\right)\right] \\
&\quad \underbrace{-\int_{(-T\vee 2t)}^{t}\frac{1}{c_+\cdot s^2}\cdot 1\,\mathrm{d}s}_{=\frac{1}{c_+}\cdot\left(\frac{1}{|t|}+\frac{1}{-T\vee 2t}\right)} \\
&= \frac{-1}{c_+\cdot(-T\vee 2t)}\cdot\left(1-\exp\left[\frac{c_+}{2\varepsilon}\cdot\left(t^2-(-T\vee 2t)^2\right)\right]\right)
\end{aligned} \tag{4.2.15}$$

This implies that for all $t\in[-T,-\beta]$

$$\begin{aligned}
\chi(t) &\geqslant \chi_-\cdot\exp\left[\frac{c_+}{2\varepsilon}\cdot(t^2-T^2)\right] \\
&\qquad + \frac{-1}{c_+\cdot(-T\vee 2t)}\cdot\left(1-\exp\left[\frac{c_+}{2\varepsilon}\cdot\left(t^2-(-T\vee 2t)^2\right)\right]\right).
\end{aligned}$$

Now, if $t\in\left[-T,-\frac{T}{2}\right]$, i.e. $(-T\vee 2t)=-T$, we get that

$$\begin{aligned}
\chi(t) &\geqslant \chi_-\cdot\exp\left[\frac{c_+}{2\varepsilon}\cdot(t^2-T^2)\right] + \frac{1}{c_+\cdot T}\cdot\left(1-\exp\left[\frac{c_+}{2\varepsilon}\cdot(t^2-T^2)\right]\right) \\
&= \frac{1}{c_+\cdot T} + \exp\left[\frac{c_+}{2\varepsilon}\cdot(t^2-T^2)\right]\cdot\underbrace{\left(\chi_- - \frac{1}{c_+\cdot T}\right)}_{\substack{>0 \\ \text{if } c_+ \text{ big enough}}} \\
&\geqslant \frac{1}{c_+\cdot T} \geqslant \frac{1}{2c_+}\cdot\frac{1}{|t|}\,,
\end{aligned}$$

which proves the lower bound for ① and ④ for all $t\in\left[-T,-\frac{T}{2}\right]$

To complete the proof of the lower bound for ①, we consider the case $t\in\left[-\frac{T}{2},-\beta\right]$, where $(-T\vee 2t)=2t$. In this case, the lower bound is proved as follows:

$$\begin{aligned}
\chi(t) &\geqslant \underbrace{\chi_-\cdot\exp\left[\frac{c_+}{2\varepsilon}\cdot(t^2-T^2)\right]}_{>0} + \frac{-1}{2c_+\cdot t}\cdot\left(1-\exp\left[\frac{c_+}{2\varepsilon}\cdot(-3t^2)\right]\right) \\
&\geqslant \frac{1}{2c_+\cdot|t|}\cdot\left(1-\exp\left[-\frac{3}{2}\cdot c_+\right]\right)
\end{aligned}$$

where we used the following estimate:

$$\exp\left[\frac{c_+}{2\varepsilon}\cdot(-3t^2)\right] \leqslant \exp\left[\frac{c_+}{2\varepsilon}\cdot(-3\beta^2)\right] \leqslant \exp\left[\frac{c_+}{2\varepsilon}\cdot(-3\varepsilon)\right]. \tag{4.2.16}$$

To show the lower bound for ④ in the case $t \in [-\frac{T}{2}, -\varepsilon^{1/2}]$, we simply have to replace (4.2.16) by

$$\exp\left[\frac{c_+}{2\varepsilon} \cdot (-3t^2)\right] \leqslant \exp\left[\frac{c_+}{2\varepsilon} \cdot (-3\varepsilon)\right] .$$

Now, let us consider ②. For $t \in [-\beta, \beta]$, we have

$$\chi(t) = \chi(-\beta) \cdot \exp\left[\frac{\tilde{\alpha}(t, -\beta)}{\varepsilon}\right] + \frac{1}{\varepsilon}\int_{-\beta}^{t} \exp\left[\frac{\tilde{\alpha}(t,s)}{\varepsilon}\right] \mathrm{d}s \; .$$

By the assumptions of the lemma, we have the following estimates for $\tilde{a}(t)$ and $\tilde{\alpha}(t)$ for all $t \in [-\beta, \beta]$:

$$\tilde{a}(t) \quad \begin{cases} \leqslant -c_-\beta \\ \geqslant -c_+\beta \, , \end{cases} \tag{4.2.17}$$

$$\tilde{\alpha}(t,s) \begin{cases} \leqslant -c_-\beta \cdot (t-s) \\ \geqslant -c_+\beta \cdot (t-s) \; . \end{cases} \tag{4.2.18}$$

From the above results we conclude that

$$\chi(-\beta) \asymp \frac{1}{|\beta|} \quad \Rightarrow \quad \chi(-\beta) \begin{cases} \geqslant \chi_- \cdot \frac{1}{|\beta|} \\ \leqslant \chi_+ \cdot \frac{1}{|\beta|} \, , \end{cases}$$

adjusting χ_+, χ_-, if necessary. Let us note that

$$\frac{1}{\varepsilon}\int_{-\beta}^{t} \exp\left[-\frac{c_\pm\beta}{\varepsilon} \cdot (t-s)\right] \mathrm{d}s = \frac{1}{c_\pm\beta} - \frac{1}{c_\pm\beta} \cdot \exp\left[-\frac{c_\pm\beta}{\varepsilon} \cdot (t+\beta)\right] .$$

Hence, the lower bound is proved as follows:

$$\begin{aligned} \chi(t) &\geqslant \chi_- \cdot \frac{1}{|\beta|} \cdot \exp\left[-\frac{c_+\beta}{\varepsilon} \cdot (t+\beta)\right] + \frac{1}{\varepsilon}\int_{-\beta}^{t} \exp\left[-\frac{c_+\beta}{\varepsilon} \cdot (t-s)\right] \mathrm{d}s \\ &= \frac{\chi_-}{\beta} \cdot \exp\left[-\frac{c_+\beta}{\varepsilon} \cdot (t+\beta)\right] + \frac{1}{c_+\beta} - \frac{1}{c_+\beta} \cdot \exp\left[-\frac{c_+\beta}{\varepsilon} \cdot (t+\beta)\right] \\ &= \frac{1}{c_+\beta} + \exp\left[-\frac{c_+\beta}{\varepsilon} \cdot (t+\beta)\right] \cdot \underbrace{\left(\frac{\chi_-}{\beta} - \frac{1}{c_+\beta}\right)}_{\substack{>0 \\ \text{if } c_+ \text{ big enough}}} \\ &\geqslant \frac{1}{c_+} \cdot \frac{1}{|\beta|} \; . \end{aligned}$$

For the corresponding upper bound, we obtain for all $t \in [-\beta, \beta]$ that

$$\begin{aligned}\chi(t) &\leqslant \chi_+ \cdot \frac{1}{\beta} \cdot \underbrace{\exp\Big[-\frac{c_-\beta}{\varepsilon}\cdot(t+\beta)\Big]}_{\leqslant 1} \\ &\quad + \frac{1}{c_-\beta} - \underbrace{\frac{1}{c_-\beta}\cdot\exp\Big[-\frac{c_-\beta}{\varepsilon}\cdot(t+\beta)\Big]}_{>0} \\ &\leqslant \Big(\chi_+ + \frac{1}{c_-}\Big)\cdot\frac{1}{\beta}\,.\end{aligned}$$

The proof for ⑤ is very similar. Even for $\varepsilon^{1/2} \geqslant \beta$ we may use $\tilde{a}(t) \asymp -(\beta \vee |t|)$ to obtain for all $t \in [-\varepsilon^{1/2}, \varepsilon^{1/2}]$ the estimate

$$\tilde{a}(t) \begin{cases} \leqslant -c_-\beta \\ \geqslant -c_+\beta\,; \end{cases} \tag{4.2.19}$$

we only have to remark that the constants $c_+, c_- > 0$ used here may be different from those in (4.2.17). In the same way, (4.2.18) holds in this case, too, but with the constants c_+, c_- from (4.2.19). Since $t - s \leqslant 2\varepsilon^{1/2}$ for all $t, s \in [-\varepsilon^{1/2}, \varepsilon^{1/2}]$ and

$$\exp\Big[-\frac{c_+\beta}{\varepsilon}\cdot 2\varepsilon^{1/2}\Big] \geqslant \exp\Big[-\frac{c_+\varepsilon^{1/2}}{\varepsilon}\cdot 2\varepsilon^{1/2}\Big] = \exp[-2c_+]\,,$$

we have, using that $\chi\big(-\frac{1}{\varepsilon^{1/2}}\big) \asymp \frac{1}{|\varepsilon^{1/2}|}$,

$$\begin{aligned}\chi(t) &\geqslant \chi_- \cdot \frac{1}{\varepsilon^{1/2}} \cdot \exp\Big[-\frac{c_+\varepsilon^{1/2}}{\varepsilon}\cdot 2\varepsilon^{1/2}\Big] \\ &\quad + \underbrace{\frac{1}{\varepsilon}\int_{-\varepsilon^{1/2}}^{t}\exp\Big[-\frac{c_+\beta}{\varepsilon}\cdot 2\varepsilon^{1/2}\Big]\,\mathrm{d}s}_{\geqslant 0} \\ &\geqslant \frac{\chi_-}{\exp[2c_+]}\cdot\frac{1}{\varepsilon^{1/2}}\,.\end{aligned}$$

The corresponding upper bound, again for $t \in [-\varepsilon^{1/2}, \varepsilon^{1/2}]$, follows from

$$\chi(t) \leqslant \chi_+ \cdot \frac{1}{\varepsilon^{1/2}} \cdot 1 + \underbrace{\frac{1}{\varepsilon}\cdot\int_{-\varepsilon^{1/2}}^{t} 1\,\mathrm{d}s}_{\leqslant \frac{1}{\varepsilon}\cdot 2\varepsilon^{1/2} = \frac{2}{\varepsilon^{1/2}}} \leqslant (\chi_+ + 2)\cdot\frac{1}{\varepsilon^{1/2}}\,.$$

Now we come to ③. Similar to (but not identical with) ①, we have

$$\tilde{a}(t) \quad \begin{cases} \leqslant -c_- \cdot |t| \\ \geqslant -c_+ \cdot |t|\,, \end{cases} \tag{4.2.20}$$

$$\tilde{\alpha}(t,s) \begin{cases} \leqslant -\frac{c_-}{2}\cdot(t^2 - s^2) \\ \geqslant -\frac{c_+}{2}\cdot(t^2 - s^2)\,. \end{cases} \tag{4.2.21}$$

The same computations as in (4.2.14) show that

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_\beta^t \exp\left[-\frac{c_\pm}{2\varepsilon}\cdot(t^2-s^2)\right]\,\mathrm{d}s \qquad (4.2.22)\\
&\quad = \frac{1}{c_\pm \cdot t} - \frac{1}{c_\pm \beta}\cdot\exp\left[-\frac{c_\pm}{2\varepsilon}\cdot(t^2-\beta^2)\right]\\
&\quad\quad + \int_\beta^t \frac{1}{c_\pm\cdot s^2}\cdot\exp\left[-\frac{c_\pm}{2\varepsilon}\cdot(t^2-s^2)\right]\,\mathrm{d}s\,.
\end{aligned}
$$

Hence, we see that

$$
\begin{aligned}
\chi(t) &\leqslant \chi_+\cdot\frac{1}{\beta}\cdot\exp\left[-\frac{c_-}{2\varepsilon}\cdot(t^2-\beta^2)\right] + \frac{1}{\varepsilon}\int_\beta^t \exp\left[-\frac{c_-}{2\varepsilon}\cdot(t^2-s^2)\right]\,\mathrm{d}s\\
&= \chi_+\cdot\frac{1}{\beta}\cdot\underbrace{\exp\left[-\frac{c_-}{2\varepsilon}\cdot(t^2-\beta^2)\right]}_{\leqslant 1}\\
&\quad + \frac{1}{c_-\cdot t} - \frac{1}{c_-\beta}\cdot\exp\left[-\frac{c_-}{2\varepsilon}\cdot(t^2-\beta^2)\right]\\
&\quad + \underbrace{\int_\beta^t \frac{1}{c_-\cdot s^2}\cdot\exp\left[-\frac{c_-}{2\varepsilon}\cdot(t^2-s^2)\right]\,\mathrm{d}s}_{\substack{\leqslant \int_\beta^t \frac{1}{c_-\cdot s^2}\,\mathrm{d}s\\ = \frac{1}{c_-\beta}-\frac{1}{c_-\cdot t}}}\\
&\leqslant \chi_+\cdot\frac{1}{\beta} + \frac{1}{c_-\beta}\cdot\underbrace{\left(1-\exp\left[-\frac{c_-}{2\varepsilon}\cdot(t^2-\beta^2)\right]\right)}_{<1}\\
&< \left(\chi_+ + \frac{1}{c_-}\right)\cdot\frac{1}{\beta}\,.
\end{aligned}
$$

Thus, we can find a constant C, such that $\chi(t) \leqslant C\cdot\frac{1}{|t|}$ for all $t\in[\beta,T]$. By precisely the same arguments, replacing β with $\varepsilon^{1/2}$, the upper bound in ⑥ follows.

Next, we prove the lower bound in ③, again using (4.2.22):

$$\begin{aligned}
\chi(t) &\geqslant \chi_- \cdot \frac{1}{\beta} \cdot \exp\left[-\frac{c_+}{2\varepsilon} \cdot (t^2 - \beta^2)\right] + \frac{1}{\varepsilon} \int_\beta^t \exp\left[-\frac{c_+}{2\varepsilon} \cdot (t^2 - s^2)\right] \mathrm{d}s \\
&= \chi_- \cdot \frac{1}{\beta} \cdot \exp\left[-\frac{c_+}{2\varepsilon} \cdot (t^2 - \beta^2)\right] \\
&\quad + \frac{1}{c_+ \cdot t} - \frac{1}{c_+ \beta} \cdot \exp\left[-\frac{c_+}{2\varepsilon} \cdot (t^2 - \beta^2)\right] \\
&\quad \underbrace{+ \int_\beta^t \frac{1}{c_+ \cdot s^2} \cdot \exp\left[-\frac{c_+}{2\varepsilon} \cdot (t^2 - s^2)\right] \mathrm{d}s}_{>0} \\
&\geqslant \frac{1}{\beta} \cdot \exp\left[-\frac{c_+}{2\varepsilon} \cdot (t^2 - \beta^2)\right] \cdot \underbrace{\left(\chi_- - \frac{1}{c_+}\right)}_{\substack{>0 \\ \text{if } c_+ \text{ big enough}}} + \frac{1}{c_+ \cdot t} \\
&\geqslant \frac{1}{c_+} \cdot \frac{1}{|t|} \, .
\end{aligned}$$

The lower bound for ⑥ is proved by the same arguments. □

Proposition 4.2.7 (y_t during $[-T, -|t_0|]$). *[sc: [BG02b, Proposition 3.3]; see also p. 1462 therein] There exists a constant $c_0 > 0$, such that the solution of* (4.2.10) *with initial condition $y_{-T} \asymp \varepsilon$ satisfies*

$$y_t \asymp \frac{\varepsilon}{|t|} \tag{4.2.23}$$

for any $t \in [-T, t_0] = \left[-T, -c_0\left(\sqrt{a_0} \vee \varepsilon^{1/2}\right)\right]$.

Proof. Let us first assume that $c_0 \geqslant 1$; during the proof, we will add more restrictions on c_0. We first prove the upper bound, i.e., we show that there exists a constant $c_1 > \frac{1}{\varepsilon} y_{-T} \cdot |T|$ such that for all $t \in [-T, t_0]$

$$y_t \leqslant c_1 \cdot \frac{\varepsilon}{|t|} \, .$$

Let us note that by choice of c_1, y_{-T} meets this estimate.

We set

$$\tau := \inf\left\{ t \in [-T, T] \;\middle|\; y_t \notin \left]0, c_1 \cdot \frac{\varepsilon}{|t|}\right[\right\} .$$

If we can prove that $\tau \geqslant t_0$, the assertion follows.

By (4.1.4), we know that $a_+^*(t) \asymp -|t|$ during $[-T, -\sqrt{a_0}]$, hence, also for all $t \in [-T, t_0]$. This implies that we can find a constant $c_- > 0$ such that $a_+^*(t) \leqslant -c_-|t|$ for all $t \in [-T, t_0]$.

By (4.2.12) we know that for all $t \in [-T, t_0] \subset [-T, -\sqrt{a_0}]$

$$\frac{\mathrm{d}}{\mathrm{d}x} x_+^*(t) \asymp -1 \, .$$

Thus, there exists a constant $c_+ > 0$ such that for all $t \in [-T, t_0]$

$$-\frac{\mathrm{d}}{\mathrm{d}t}\, x_+^*(t) \leqslant c_+ \, .$$

We use these estimates to modify (4.2.10) and obtain that for all $t \in [-T, t_0]$

$$\varepsilon \cdot \frac{\mathrm{d}}{\mathrm{d}t}\, y_t \leqslant \underbrace{-c_-|t| \cdot y_t + M \cdot y_t^2}_{= -c_-|t| \cdot y_t \cdot \left(1 - \frac{M \cdot y_t}{c_-|t|}\right)} + \varepsilon c_+ \, .$$

By definition of τ, we have that $y_t \in \left]0, c_1 \cdot \frac{\varepsilon}{|t|}\right[$ for all $t \in [-T, \tau \wedge t_0]$. Furthermore, there is a constant $M' > 0$ such that, by definition of t_0, we have that $t \leqslant t_0 \leqslant -c_0 M'$, hence, $|t| \geqslant c_0 M'$ for all $t \in [-T, \tau \wedge t_0]$. This implies that for all such t

$$\varepsilon \cdot \frac{\mathrm{d}}{\mathrm{d}t}\, y_t \leqslant -c_-|t| \cdot y_t \cdot \left[1 - \underbrace{\frac{M c_1 \varepsilon}{c_-|t|^2}}_{\leqslant \frac{M c_1 \varepsilon}{c_- c_0^2 M'^2}}\right] + \varepsilon c_+ \, .$$

For any constant c_1, we can choose $c_0 = c_0(c_1)$ big enough for the term in square brackets to be larger than $\frac{1}{2}$. Hence, for a "big enough" c_0 in this sense,

$$\frac{\mathrm{d}}{\mathrm{d}t}\, y_t \leqslant -\frac{c_-|t|}{2\varepsilon} \cdot y_t + c_+ = \frac{c_-}{2\varepsilon} \cdot t \cdot y_t + c_+ \, ,$$

since all $t \in [-T, t_0]$ fulfill $-|t| = t$. By assumption, the initial condition to our deterministic model is $y_{-T} \asymp \varepsilon$.

From all this, we obtain (using Theorem A.0.1) that

$$y_t \leqslant y_{-T} \cdot \exp\left[\int_{-T}^{t} \frac{c_-}{2\varepsilon} \cdot s \,\mathrm{d}s\right] + \int_{-T}^{t} \exp\left[-\int_{s}^{t} \frac{c_-}{2\varepsilon} \cdot \tilde{s} \,\mathrm{d}\tilde{s}\right] \cdot c_+ \,\mathrm{d}s \, ,$$

hence,

$$\frac{1}{\varepsilon} \cdot y_t \leqslant \frac{y_{-T}}{\varepsilon} \cdot \exp\left[\frac{c_-}{4\varepsilon} \cdot (t^2 - T^2)\right] + \frac{c_+}{\varepsilon} \int_{-T}^{t} \exp\left[\frac{c_-}{4\varepsilon} \cdot (t^2 - s^2)\right] \mathrm{d}s \, .$$

We note that $\frac{y_{-T}}{\varepsilon} \asymp 1$ and $\frac{c_+}{\varepsilon} \asymp \frac{1}{\varepsilon}$. Hence, we may apply Lemma 4.2.6, where $\frac{c_-}{2} \cdot |t|$ takes the role of $\tilde{a}(t)$, and obtain that there exists a constant $c_2 > 0$, independent of c_1 and τ (note that by construction c_+, c_- are independent of c_1 and τ, too), such that for all $t \in [-T, \tau \wedge t_0]$

$$y_t \leqslant c_2 \cdot \frac{\varepsilon}{|t|} \, . \tag{4.2.24}$$

Hence, we may choose $c_1 > c_2$.

Assume that $\tau \leqslant t_0$. Then (4.2.24) implies that

$$|\tau| \cdot y_\tau \leqslant c_2 \varepsilon \, . \tag{4.2.25}$$

On the other hand, by continuity of the path $t \mapsto y_t$ and the definition of τ, we have

$$|\tau| \cdot y_\tau = c_1\varepsilon \quad \overset{c_1 > c_2}{\Rightarrow} \quad \tau^2 \cdot y_\tau > c_2\varepsilon \; ,$$

which contradicts (4.2.25). Thus, the assumption that $\tau \leqslant t_0$ must be wrong and the proof of the upper bound is thus completed.

The proof of the lower bound is in principle the same: We have to show the existence of a constant $c_3 \in \left]0, \frac{y_{-T}}{\varepsilon} \cdot |T|\right[$ such that for all $t \in [-T, t_0]$

$$y_t \geqslant c_3 \cdot \frac{\varepsilon}{|t|} \; .$$

Let us note that by definition of c_3, y_{-T} meets this estimate.

Like above, we use the general assumptions on equilibrium branches and linearizations of f during $[-T, \sqrt{a_0}]$:

- $a_+^*(t) \asymp -|t|$, i.e. there exists a constant $c_\oplus > 0$ big enough for

  $$a_+^*(t) \geqslant -c_\oplus |t| \quad \text{for all } t \in [-T, t_0].$$

- $\frac{\mathrm{d}}{\mathrm{d}t} x_+^*(t) \asymp -1$, i.e. there exists a constant $c_\ominus > 0$ small enough for

  $$-\frac{\mathrm{d}}{\mathrm{d}t}\, x_+^*(t) \geqslant c_\ominus \quad \text{for all } t \in [-T, t_0].$$

- There exists a constant $M > 0$ such that $b_+^*(y_t, t) \geqslant -M \cdot y_t^2$ for all $t \in [-T, t_0]$.

Hence, we get that

$$\varepsilon \cdot \frac{\mathrm{d}}{\mathrm{d}t}\, y_t \geqslant \underbrace{-c_\oplus |t| \cdot y_t - M \cdot y_t^2}_{-c_\oplus |t| \cdot y_t \cdot \left(1 + \frac{M \cdot y_t}{c_\oplus |t|}\right)} + \varepsilon c_\ominus \; .$$

By the first part of the proof, we already know that there is a constant c_1 such that

$$y_t \leqslant \frac{c_1 \varepsilon}{|t|} \quad \text{for all } t \in [-T, t_0] \; .$$

Hence, because again $|t| \geqslant c_0 M'$ for a constant $M' > 0$,

$$\begin{aligned}
\varepsilon \cdot \frac{\mathrm{d}}{\mathrm{d}t}\, y_t &\geqslant -c_+ |t| \cdot y_t \cdot \left(1 + \frac{M c_1 \varepsilon}{c_+ |t|^2}\right) + \varepsilon c_- \\
&\geqslant -c_+ |t| \cdot y_t \cdot \left(1 + \frac{M c_1 \varepsilon}{c_+ M'^2 c_0^2}\right) + \varepsilon c_- \; .
\end{aligned}$$

If we only choose c_0 big enough (with respect to c_1), the term in brackets is smaller than 2. Then we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\, y_t \geqslant \frac{-2c_+ |t|}{\varepsilon} \cdot y_t + c_- = \frac{2c_+}{\varepsilon} \cdot t \cdot y_t + c_- \; ,$$

as $-|t| = t$ for all $t \in [-T, t_0]$.

From here on, the procedure is as above. Since we do not have the restriction that estimates only hold up to a time τ, the contradiction argument is not needed here. □

From now on we assume that c_0 is large enough for Proposition 4.2.7 to hold.

We note that the proposition implies that for all $t \in [-T, t_0]$ we have that

$$x_t^{\mathrm{det}} \asymp x_+^*(t) ,$$

hence, again for $t \in [-T, t_0]$,

$$x_t^{\mathrm{det}} - x_c \asymp \sqrt{a_0} , \quad \text{and} \quad y_{t_0} \asymp \frac{\varepsilon}{\sqrt{a_0}} \wedge \varepsilon .$$

Now we analyze the development of y_t for $|t| \leqslant |t_0|$. We will distinguish between the cases $y_{t_0} \asymp \frac{\varepsilon}{\sqrt{a_0}}$ ("a_0 not too small") and $y_{t_0} \asymp \sqrt{\varepsilon}$.

Proposition 4.2.8 (y_t during $[t_0, -t_0]$ for $a_0 \geqslant \gamma_0\varepsilon$). *[sc: [BG02b, Prop. 3.4]] There exists a constant $\gamma_0 > 0$, depending only on f and y_{t_0}, such that in the case $a_0 \geqslant \varepsilon\gamma_0$ we have that for all $t \in [t_0, -t_0]$*

$$y_t = C_1(t) \cdot (t_+^* - t) + C_2(t) , \tag{4.2.26}$$

where

$$C_1(t) \asymp \frac{\varepsilon}{a_0} \quad \textit{and} \quad C_2(t) \asymp \frac{\varepsilon^2}{a_0^{3/2}} .$$

The estimate in (4.2.26) shows in particular that y_t vanishes at a time $\tilde{t}$ such that

$$\tilde{t} - t_+^* \asymp \frac{\varepsilon}{\sqrt{a_0}} , \tag{4.2.27}$$

if $a_0 \geqslant \gamma_0\varepsilon$. Furthermore, if $t_1 \asymp \sqrt{a_0}$, (4.2.26) together with the assumption that $t_+^* = \mathcal{O}(a_0)$ implies that $y_{t_1} \asymp -\frac{\varepsilon}{\sqrt{a_0}}$, hence, $y_{t_1} \asymp -\frac{\varepsilon}{|t_1|}$.

Proof. As in the proof of Proposition 4.2.7, we begin with the upper bound.

Our proof is again based on estimates for $a_+^*(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t} x_+^*(t)$, see (4.1.4) and (4.2.12), respectively. However, these estimates are based on a certain partition of the interval $[-T, T]$, namely

$$[-T, T] = [-T, -\sqrt{a_0}] \cup [-\sqrt{a_0}, \sqrt{a_0}] \cup [\sqrt{a_0}, T] .$$

To use them for $t \in [t_0, -t_0]$, we have to remark that it is possible to extend the estimates from $t \in [-\sqrt{a_0}, \sqrt{a_0}]$ to $[t_0, -t_0]$ – due to the fact that the corresponding original estimates for the neighborhood of $[-\sqrt{a_0}, \sqrt{a_0}]$ are linear – by enlarging/shrinking the respective constants for upper and lower bound. Hence, we obtain that there exists a constant $c_- > 0$ small enough for

$$a_+^*(t) \leqslant c_- \cdot (-\sqrt{a_0}) \quad \text{for all } t \in [t_0, -t_0],$$

and two constants, $c_{\boxplus} > 0$ big enough and $c_{\boxminus} > 0$ small enough (cf. Remark 4.2.5), for

$$\frac{\mathrm{d}}{\mathrm{d}t} x_+^*(t) \geqslant \begin{cases} \frac{c_{\boxplus}}{\sqrt{a_0}} \cdot (t - t_+^*) & \text{for all } t \in [t_0, t_+^*] \\ \frac{c_{\boxminus}}{\sqrt{a_0}} \cdot (t - t_+^*) & \text{for all } t \in [t_+^*, -t_0]. \end{cases}$$

Furthermore, there is a constant $M > 0$ big enough for

$$\left|b_+^*(y_t, t)\right| \leqslant M \cdot y_t^2 \, .$$

Let $c_1 > \frac{a_0}{\varepsilon} \cdot y_{t_0}$ be a constant and set

$$\tau := \inf\left\{ t \in [t_0, T] \;\middle|\; |y_t| \geqslant \frac{c_1 \varepsilon}{a_0} \right\} .$$

By definition of c_1, we have that $\tau > t_0$.

Using

$$\frac{\left|b_+^*(y_t, t)\right|}{|y_t|} \leqslant M \cdot |y_t| \leqslant M \cdot \frac{c_1 \varepsilon}{a_0} \quad \text{for } t \in [t_0, \tau \wedge -t_0]$$

we get that, if we choose γ_0 (hence, a_0) big enough,

$$a_+^*(t) + \frac{\left|b_+^*(y_t, t)\right|}{|y_t|} \leqslant -\frac{c_- \sqrt{a_0}}{2} \quad \text{for } t \in [t_0, \tau \wedge -t_0]. \tag{4.2.28}$$

Based on these estimates and setting

$$c_\Box(t) := \begin{cases} c_{\boxplus} & \text{if } t \leqslant t_+^* \\ c_{\boxminus} & \text{if } t \geqslant t_+^* \, , \end{cases}$$

we obtain the following modification of (4.2.10) for all $t \in [t_0, \tau \wedge -t_0]$:

$$\frac{\mathrm{d}}{\mathrm{d}t} y_t \leqslant -\frac{c_- \sqrt{a_0}}{2\varepsilon} \cdot y_t - \frac{c_\Box(t)}{\sqrt{a_0}} \cdot (t - t_+^*) \, ,$$

which implies (using Theorem A.0.1) that for all $t \in [t_0, \tau \wedge -t_0]$

$$\begin{aligned} y_t &\leqslant y_{t_0} \cdot \exp\left[\int_{t_0}^t -\frac{c_- \sqrt{a_0}}{2\varepsilon} \,\mathrm{d}s\right] \\ &\quad + \int_{t_0}^t \exp\left[\int_s^t -\frac{c_- \sqrt{a_0}}{2\varepsilon} \,\mathrm{d}\tilde{s}\right] \cdot \frac{-c_\Box(s)}{\sqrt{a_0}} \cdot (s - t_+^*) \,\mathrm{d}s \\ &= y_{t_0} \cdot \exp\left[-\frac{c_- \sqrt{a_0}}{2\varepsilon} \cdot (t - t_0)\right] \\ &\quad + \int_{t_0}^t \exp\left[-\frac{c_- \sqrt{a_0}}{2\varepsilon} \cdot (t - s)\right] \cdot \frac{-c_\Box(s)}{\sqrt{a_0}} \cdot (s - t_+^*) \,\mathrm{d}s \, . \end{aligned}$$

By integration by parts[3] we obtain that

$$\begin{aligned}
&\int_{t_0}^{t} \exp\left[-\frac{c_- \sqrt{a_0}}{2\varepsilon}\cdot(t-s)\right]\cdot\frac{c_\Box(s)}{\sqrt{a_0}}\cdot(t_+^* - s)\,\mathrm{d}s \\
&\quad = \left[\frac{c_\Box(s)}{\sqrt{a_0}}\cdot(t_+^*-s)\cdot\frac{2\varepsilon}{c_-\sqrt{a_0}}\cdot\exp\left[-\frac{c_-\sqrt{a_0}}{2\varepsilon}\cdot(t-s)\right]\right]_{t_0}^{t} \\
&\qquad - \int_{t_0}^{t} -\frac{2\varepsilon\cdot c_\Box(s)}{c_- a_0}\cdot\exp\left[-\frac{c_-\sqrt{a_0}}{2\varepsilon}\cdot(t-s)\right]\,\mathrm{d}s\,.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\int_{t_0}^{t} c_\Box(s)\cdot\exp\left[-\frac{c_-\sqrt{a_0}}{2\varepsilon}\cdot(t-s)\right]\,\mathrm{d}s \\
&\quad = c_\Box(t)\cdot\frac{2\varepsilon}{c_-\sqrt{a_0}} - c_\Box(t_0)\cdot\frac{2\varepsilon}{c_-\sqrt{a_0}}\cdot\exp\left[-\frac{c_-\sqrt{a_0}}{2\varepsilon}\cdot(t-t_0)\right]\,.
\end{aligned}$$

Hence, we have for all $t\in[t_0,\tau\wedge -t_0]$

$$\begin{aligned}
y_t \leqslant\; & y_{t_0}\cdot\exp\left[-\frac{c_-\sqrt{a_0}}{2\varepsilon}\cdot(t-t_0)\right] + \frac{2\varepsilon\cdot c_\boxplus}{c_- a_0}\cdot(t_+^*-t) \\
& - \frac{2\varepsilon\cdot c_\boxplus}{c_- a_0}\cdot(t_+^*-t_0)\cdot\exp\left[-\frac{c_-\sqrt{a_0}}{2\varepsilon}\cdot(t-t_0)\right] \\
& + \frac{4\varepsilon^2\cdot c_\boxplus}{c_-^2 a_0^{3/2}}\cdot\left(1-\exp\left[-\frac{c_-\sqrt{a_0}}{2\varepsilon}\cdot(t-t_0)\right]\right)\,,
\end{aligned}$$

where we use that $c_\Box(t_0) = c_\boxplus$, and estimate $c_\Box(t)$ from above by $c_\boxplus$ for any $t \neq t_0$;

$$\begin{aligned}
= \;& \frac{2\varepsilon\cdot c_\boxplus}{c_- a_0}\cdot(t_+^*-t) + \frac{4\varepsilon^2\cdot c_\boxplus}{c_-^2 a_0^{3/2}} \\
& + \exp\left[-\frac{c_-\sqrt{a_0}}{2\varepsilon}\cdot(t-t_0)\right]\cdot\underbrace{\left(y_{t_0} - \frac{2\varepsilon\cdot c_\boxplus}{c_- a_0}\cdot(t_+^*-t_0) - \frac{4\varepsilon^2\cdot c_\boxplus}{c_-^2 a_0^{3/2}}\right)}_{=:\eta(\varepsilon)}\,.
\end{aligned} \tag{4.2.29}$$

By the properties of y_{t_0} and a_0, $\eta(\varepsilon)$ is of the order $\mathcal{O}\left(\frac{\varepsilon}{\sqrt{a_0}}\right)$.

To further analyze the right hand side, we first assume that $\eta(\varepsilon) > 0$ and $t\in[t_0,\tau\wedge t_+^*]$.

[3] We use

$$\int_a^b u(s)v'(s)\,\mathrm{d}s = u(s)v(s)\big|_a^b - \int v(s)u'(s)\,\mathrm{d}s$$

with

$$v(s) = \exp\left[-\frac{c_-\sqrt{a_0}}{2\varepsilon}\cdot(t-s)\right]\cdot\frac{2\varepsilon}{c_-\sqrt{a_0}} \quad\text{and}\quad u(s) = \frac{c_\Box(s)}{\sqrt{a_0}}\cdot(t_+^*-s)\,.$$

By convexity of $x \mapsto \exp x$, we have that

$$\exp\left[-\frac{c_-\sqrt{a_0}}{2\varepsilon}\cdot(t-t_0)\right] \leqslant \frac{t_+^*-t}{t_+^*-t_0} + \exp\left[-\frac{c_-\sqrt{a_0}}{2\varepsilon}\cdot(t_+^*-t_0)\right]. \quad (4.2.30)$$

Since $t_0 \asymp -\sqrt{a_0}$, $t_+^* = \mathcal{O}(a_0)$, we obtain that

$$\begin{aligned} &\frac{2\varepsilon\cdot c_\boxplus}{c_- a_0} + \frac{1}{t_+^*-t_0}\cdot\eta(\varepsilon) \asymp \frac{\varepsilon}{a_0} \\ &\quad\Rightarrow\quad \exists\, \tilde{C}_1 > 0 : \frac{2\varepsilon\cdot c_\boxplus}{c_- a_0} + \frac{1}{t_+^*-t_0}\cdot\eta(\varepsilon) \leqslant \tilde{C}_1\cdot\frac{\varepsilon}{a_0}\,. \end{aligned} \quad (4.2.31)$$

Since $x\exp[-x] \xrightarrow{x\to\infty} 0$, we get

$$\begin{aligned} &\underbrace{\eta(\varepsilon)\cdot\exp\left[-\frac{c_-\sqrt{a_0}}{2\varepsilon}\cdot(t_+^*-t_0)\right]}_{\xrightarrow{\varepsilon\to 0} 0} + \frac{4\varepsilon^2\cdot c_\boxplus}{c_-^2\, a_0^{3/2}} \asymp \frac{\varepsilon^2}{a_0^{3/2}} \\ &\quad\Rightarrow\quad \exists\, \tilde{C}_2 > 0 : \eta(\varepsilon)\cdot\exp\left[-\frac{c_-\sqrt{a_0}}{2\varepsilon}\cdot(t_+^*-t_0)\right] + \frac{4\varepsilon^2\cdot c_\boxplus}{c_-^2\, a_0^{3/2}} \leqslant \tilde{C}_2\cdot\frac{\varepsilon^2}{a_0^{3/2}}\,. \end{aligned} \quad (4.2.32)$$

If we apply these estimates to the right hand side of (4.2.29), we obtain for all $t \in [t_0, \tau\wedge t_+^*]$:

$$\begin{aligned} y_t &\overset{(4.2.29)}{\leqslant} \frac{2\varepsilon\cdot c_\boxplus}{c_- a_0}\cdot(t_+^*-t) + \frac{4\varepsilon^2 c_\boxplus}{c_-^2\, a_0^{2/3}} + \exp\left[-\frac{c_-\sqrt{a_0}}{2\varepsilon}\cdot(t-t_0)\right]\cdot\eta(\varepsilon) \\ &\overset{(4.2.30)}{\leqslant} \frac{2\varepsilon\cdot c_\boxplus}{c_- a_0}\cdot(t_+^*-t) + \frac{4\varepsilon^2 c_\boxplus}{c_-^2\, a_0^{3/2}} \\ &\qquad + \left(\frac{t_+^*-t}{t_+^*-t_0} + \exp\left[-\frac{c_-\sqrt{a_0}}{2\varepsilon}\cdot(t_+^*-t_0)\right]\right)\cdot\eta(\varepsilon) \\ &\overset{(4.2.31)}{\leqslant} \tilde{C}_1\cdot\frac{\varepsilon}{a_0}\cdot(t_+^*-t) + \frac{4\varepsilon^2 c_\boxplus}{c_-^2\, a_0^{3/2}} + \exp\left[-\frac{c_-\sqrt{a_0}}{2\varepsilon}\cdot(t_+^*-t_0)\right]\cdot\eta(\varepsilon) \\ &\overset{(4.2.32)}{\leqslant} \tilde{C}_1\cdot\frac{\varepsilon}{a_0}\cdot(t_+^*-t) + \tilde{C}_2\cdot\frac{\varepsilon^2}{a_0^{3/2}}\,. \end{aligned} \quad (4.2.33)$$

Thus, the assertion is proved for all $t \in [t_0, t_+^*\wedge\tau]$, under the assumption that $\eta(\varepsilon) > 0$.

For $t \geqslant t_+^*$ (but still $t \leqslant \tau\wedge -t_0$, $\eta(\varepsilon) > 0$), we exchange (4.2.30) by

$$\exp\left[-\frac{c_-\sqrt{a_0}}{2\varepsilon}\cdot(t-t_0)\right] \leqslant \exp\left[-\frac{c_-\sqrt{a_0}}{2\varepsilon}\cdot(t_+^*-t_0)\right],$$

substitute (4.2.31) with

$$\frac{2\varepsilon\cdot c_\boxplus}{c_- a_0} \asymp \frac{\varepsilon}{a_0}\,,$$

and re-use (4.2.32) to obtain that (4.2.33) holds for any $t \in [t_0, \tau \wedge -t_0]$ as long as $\eta(\varepsilon) > 0$ (with $\tilde{C}_1$, $\tilde{C}_2$ enlarged, if necessary).

The case $\eta(\varepsilon) \leqslant 0$ is even easier: The upper estimates (4.2.31), (4.2.32) do still hold, hence (4.2.33) also does.

If $\tau \geqslant -t_0$, the proof is now complete. If $\tau < -t_0$, we simply have to enlarge c_1, such that the bound in the definition of τ 'catches' y_{-t_0}. This is possible, because the involved processes are all continuous. But to guarantee the validity of (4.2.28), we need that

$$M \cdot \frac{c_1 \varepsilon}{a_0} \leqslant \frac{c_-}{2} \cdot \sqrt{a_0}\,.$$

Consequently, whenever c_1 is enlarged by multiplication with a constant $k > 1$, we have to multiply a_0 with $k^{2/3}$. However, for any $k > 1$ we have that $k > k^{2/3}$, hence, such a 'simultaneous growth' of c_1 and a_0 will both grow the bound for y_t that defines τ and let (4.2.28) remain valid.

Thus, the upper bound is finally proved for all $t \in [t_0, -t_0]$.

We will only outline the proof of the lower bound, as it is basically the same. By similar arguments as above, there exist constants $c_+, c_\boxplus > 0$ big enough and $c_\boxminus > 0$ small enough, such that

$$\begin{aligned} a_+^*(t) &\geqslant c_+ \cdot (-\sqrt{a_0}) \quad \text{for all } t \in [t_0, -t_0]\,,\\ \frac{\mathrm{d}}{\mathrm{d}t} x_+^*(t) &\begin{cases} \leqslant \frac{c_\boxminus}{\sqrt{a_0}} \cdot (t - t_+^*) & \text{for } t \in [t_0, t_+^*] \\ \leqslant \frac{c_\boxplus}{\sqrt{a_0}} \cdot (t - t_+^*) & \text{for } t \in [t_+^*, -t_0]\,, \end{cases} \end{aligned}$$

and $M > 0$ big enough for

$$\begin{aligned} &\left|b_+^*(y_t, t)\right| \leqslant M \cdot y_t^2 \\ &\quad \Rightarrow \quad b_+^*(y_t) \geqslant -M \cdot y_t^2\,. \end{aligned}$$

Now, let c_1 and τ be as above. Then we have for $t \in [t_0, \tau \wedge -t_0]$

$$\frac{b_+^*(y_t, t)}{|y_t|} \geqslant -M \cdot |y_t| \geqslant -M \cdot \frac{c_1 \varepsilon}{a_0}\,,$$

hence, if γ_0 (consequently, a_0) is big enough,

$$a_+^*(t) + \frac{b_+^*(y_t, t)}{|y_t|} \geqslant -2c_+ \cdot \sqrt{a_0} \quad \text{for } t \in [t_0, \tau \wedge -t_0]\,.$$

By the same steps as above, this time applying $c_\square(t_0) = c_\boxminus$ and the estimate $c_\square(t) \geqslant c_\boxminus$ for all $t \in]t_0, -t_0]$, we arrive at the following estimate:

$$\begin{aligned} y_t \geqslant{}& \frac{\varepsilon c_\boxminus}{2c_+ a_0} \cdot (t_+^* - t) + \frac{\varepsilon^2 c_\boxminus}{2c_+^2 a_0^{3/2}} \\ &+ \exp\left[-\frac{2c_+\sqrt{a_0}}{\varepsilon} \cdot (t - t_0)\right] \cdot \underbrace{\left(y_{t_0} - \frac{\varepsilon c_\boxminus}{2c_+ a_0} \cdot (t_+^* - t_0) - \frac{\varepsilon^2 c_\boxminus}{4c_+^2 a_0^{3/2}}\right)}_{=:\eta(\varepsilon)}\,, \end{aligned}$$

which can be shown to fulfill the assertion in the same way as presented for the upper bound. The estimates used here are basically the same as above, we only have to replace (4.2.31) with the estimate

$$\frac{\varepsilon c_{\boxminus}}{2c_+ a_0} \geqslant \check{C}_1 \cdot \frac{\varepsilon}{a_0}$$

for a constant $\check{C}_1 > 0$, and (4.2.30) with the estimate

$$\exp\left[-\frac{2c_+\sqrt{a_0}}{\varepsilon} \cdot (t - t_0)\right] \geqslant \exp\left[-\frac{4c_+\sqrt{a_0}}{\varepsilon} \cdot |t_0|\right]$$

(which are both derived in a similar way as before) to obtain that

$$y_t \geqslant \check{C}_1 \cdot \frac{\varepsilon}{a_0} \cdot (t_+^* - t) + \check{C}_2 \cdot \frac{\varepsilon^2}{a_0^{3/2}} ,$$

where $\check{C}_2 > 0$ is a constant fulfilling a lower bound analogon to (4.2.32). □

From now on we assume that γ_0 is big enough for Proposition 4.2.8 to hold.

Proposition 4.2.9 (y_t during $[t_0, t_1]$ for $a_0 < \gamma_0\varepsilon$)**.** *[BG02b, Prop. 4.2] Assume that $a_0 < \gamma_0\varepsilon$. We choose an arbitrary $t_1 \asymp \sqrt{\varepsilon}$. Then for any $t \in [t_0, t_1]$*

$$x_t^{\mathrm{det}} - x_c \asymp \sqrt{\varepsilon} , \tag{4.2.34}$$

and x_t^{det} crosses $x_+^*(t)$ at a time $\tilde{t}$ such that $\tilde{t} \asymp \sqrt{\varepsilon}$.

Together with the preceding proposition, this result implies that there exists a time $t_1 \asymp (\sqrt{a_0} \vee \varepsilon^{1/2})$ such that

$$y_{t_1} = x_{t_1}^{\mathrm{det}} - x_c - \big(x_+^*(t_1) - x_c\big) \asymp -\frac{\varepsilon}{|t_1|} .$$

Proof. This Proposition is proved in the asymmetric version in [BG02b, Proposition 4.2]. □

For $t \in [t_1, T]$, we have $y_t \asymp -\frac{\varepsilon}{|t|}$. The proof is similar to that of Proposition 4.2.7. This estimate implies that, because of $x_+^*(t) - x_c \asymp t$ for $t \in [t_1, T]$, we also have $x_t^{\mathrm{det}} - x_c \asymp t$ for $t \in [t_1, T]$.

Alltogether, we have shown that for $t \in [-T, T]$

$$\left|x_t^{\mathrm{det}} - x_+^*(t)\right| \asymp \frac{\varepsilon}{|t|} \wedge \frac{\varepsilon}{\sqrt{a_0}} \wedge \sqrt{\varepsilon} , \tag{4.2.35}$$

and the vanishing time $\tilde{t}$ of y_t fulfills

$$\tilde{t} - t_+^* \asymp \frac{\varepsilon}{\sqrt{a_0}} \wedge \sqrt{\varepsilon} . \tag{4.2.36}$$

Proposition 4.2.10 (linearization of f at x_t^{det})**.** *[BG02b, Prop. 4.3] For all $t \in [-T, T]$ and any $a_0 = \mathcal{O}_\varepsilon(1)$,*

$$\bar{a}(t) := \partial_x f(x_t^{\mathrm{det}}, t) \asymp -\big(|t| \vee \sqrt{a_0} \vee \sqrt{\varepsilon}\big) . \tag{4.2.37}$$

Proof. [sc: [BG02b, Prop. 3.7]] From the assumptions on $x^*_+(t)$ for $t \in [-T, T]$ and the behavior of y_t during $[-T, T]$, as proved in the preceding propositions, we conclude that

$$\begin{aligned}\tilde{x}_t := x_t^{\mathrm{det}} - x_c &\asymp \begin{cases} |t| & \text{for } |t| \in [-t_0, T] \\ \sqrt{a_0} & \text{for } t \in [t_0, -t_0] \text{ and } a_0 \geqslant \varepsilon \\ \sqrt{\varepsilon} & \text{for } t \in [t_0, -t_0] \text{ and } a_0 \leqslant \varepsilon \end{cases} \\ &\asymp |t| \vee \sqrt{a_0} \vee \sqrt{\varepsilon} \,. \end{aligned} \tag{4.2.38}$$

By (4.1.5) we obtain that for all $t \in [-T, T]$

$$\partial_x f(x_t^{\mathrm{det}}, t) = \partial_x f(x_c + \tilde{x}_t, t) \asymp -\big(|t| \vee \sqrt{a_0} \vee \sqrt{\varepsilon}\big) \,,$$

since by assumption $T < \frac{1}{2}$. □

Finally, we introduce a further solution to (4.2.1), which we will have to use during the proofs of the non-deterministic results.

Corollary 4.2.11 ("unstable solution")**.** *[BG02b, Theorem 2.5 and Prop. 4.3] The deterministic equation* (4.2.1) *has a solution $\hat{x}_t^{\mathrm{det}}$ that "tracks the unstable equilibrium branch" $x^*_u(t)$. This solution satisfies analogous claims as x_t^{det} does with respect to $x^*_+(t)$: $\hat{x}_t^{\mathrm{det}}$ and $x^*_u(t)$ cross once (during $[-T, T]$) at a time $\hat{t}$ such that*

$$\hat{t} - t_0^* \asymp -(\tilde{t} - t_+^*) \,,$$

and (4.2.7), (4.2.8) *and* (4.2.9) *hold for $\hat{x}_t^{\mathrm{det}}$ and $x^*_u(t)$, but with opposite signs. Furthermore, we have that*

$$\hat{a}(t) := \partial_x f(\hat{x}_t^{\mathrm{det}}, t) \asymp |t| \vee \sqrt{a_0} \vee \sqrt{\varepsilon} \,.$$

Proof. The equation

$$\varepsilon \cdot \frac{\mathrm{d}}{\mathrm{d}s} z_s = -f(z_s, -s) \,, \tag{4.2.39}$$

with f as before, has a stable equilibrium branch $z_0^*(s) = x_u^*(-s)$. The same arguments as above prove the existence of a solution z_s to (4.2.39) tracking $z_0^*(s)$. By $z_s = x_{-s}$ we obtain the path of the claimed solution $\hat{x}_t^{\mathrm{det}}$. □

The following Lemma is a consequence of Lemma 4.2.6. It will be needed in the following section.

Lemma 4.2.12. *Like in Lemma 4.2.6 we set*

$$\bar{\alpha}(t, s) := \int_s^t \bar{a}(u) \,\mathrm{d}u \,.$$

Then we obtain that

$$\begin{aligned}\zeta(t) := \frac{1}{2 \cdot |\bar{a}(-T)|} \cdot \exp\left[\frac{2\bar{\alpha}(t, -T)}{\varepsilon}\right] + \frac{1}{\varepsilon} \int_{-T}^t \exp\left[\frac{2\bar{\alpha}(t, s)}{\varepsilon}\right] \mathrm{d}s \\ \asymp \frac{1}{|t| \vee \sqrt{a_0} \vee \sqrt{\varepsilon}} \,.\end{aligned}$$

4.3. The Stochastic Case

Let us return to the stochastic equation (4.1.1), which we repeat here for convenience of the reader:

$$\begin{cases} \mathrm{d}x_t &= \frac{1}{\varepsilon} \cdot f(x_t, t)\, \mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\, \mathrm{d}W_t \\ x_{t_0} &= x_0 \,, \end{cases} \tag{(4.1.1)}$$

with $\sigma > 0$ and f as described in Situation 4.1.4. We note that $x_t = x_t^\varepsilon$ depends on $\varepsilon > 0$, but we stick to the notation of [BG02b]

We assume that the initial condition for $t_0 = -T$ fulfills

$$x_{-T} - x_+^*(-T) \asymp \varepsilon \,. \tag{4.3.1}$$

It has been proved in [BG02a, Theorem 2.4] that, if the system (4.1.1) and the corresponding deterministic system (4.2.1) start at the same initial point $x_{-1+T} = x^{\mathrm{det}}_{-1+T}$, and if this initial point fulfills the conditions listed in the assumptions of Proposition 4.2.2, the relation (4.3.1) is fulfilled with very high probability, if σ is not too large.

Our aim is to describe the dynamics of the strong solution $(x_t)_{t\in[-T,T]}$ to (4.1.1) in comparison to the corresponding deterministic path x_t^{det}.

4.3.1. The Stable Case

In this subsection, we show that if σ is small enough, x_t remains close enough to x_t^{det} to have no transition during $[-T, T]$.

The main idea we follow is to describe the deviation of x from x^{det} by the variance of the solution of the linearization of (4.1.1) at x_t^{det}, and then to extend this result to the solution of the (original) nonlinear system. For technical reasons, we will replace the variance by $\zeta(t)$ as defined in Lemma 4.2.12, which is asymptotically the same as the variance itself.

Let $h > 0$ be a constant. We set

$$\begin{aligned} \mathcal{B}(h) &:= \left\{ (r,t) \in [-d,d] \times [-T,T] \;\middle|\; |r - x_t^{\mathrm{det}}| < h \cdot \sqrt{\zeta(t)} \right\} \\ \tau_{\mathcal{B}(h)} &:= \inf\left\{ t \in [-T,T] \;\middle|\; (x_t, t) \notin \mathcal{B}(h) \right\} \,. \end{aligned}$$

The main result of this subsection is the following:

Theorem 4.3.1 (motion near stable equilibrium branches)**.** *[BG02b, Thm. 2.6] There exists a constant $h_0 > 0$, depending only on f, such that the following holds:*

(i) If $t \in \left[-T, -(\sqrt{a_0} \vee \sqrt{\varepsilon})\right]$ and $h < h_0 \cdot |t|^{3/2}$, then

$$\begin{aligned} &\mathbb{P}^{-T,x_{-T}}[\tau_{\mathcal{B}(h)} < t] \\ &\qquad \leqslant C(t,\varepsilon) \cdot \exp\left[-\frac{h^2}{2\sigma^2} \cdot \left(1 - \mathcal{O}(\varepsilon) - \mathcal{O}\left(\frac{h}{|t|^{3/2}} \right) \right) \right] \,. \end{aligned} \tag{4.3.2}$$

(ii) If $t \in \left[-(\sqrt{a_0} \vee \sqrt{\varepsilon}), T\right]$ *and* $h < h_0 \cdot (a_0^{3/4} \vee \varepsilon^{3/4})$*, then*

$$\begin{aligned} &\mathbb{P}^{-T,x_{-T}}\left[\tau_{\mathcal{B}(h)} < t\right] \\ &\qquad \leqslant C(t,\varepsilon) \cdot \exp\left[-\frac{h^2}{2\sigma^2} \cdot \left(1 - \mathcal{O}(\varepsilon) - \mathcal{O}\left(\frac{h}{a_0^{3/4} \vee \varepsilon^{3/4}}\right)\right)\right] . \end{aligned} \tag{4.3.3}$$

In both cases,

$$C(t,\varepsilon) := \frac{1}{\varepsilon^2} \cdot \left|\bar{\alpha}(t,-T)\right| + 2 . \tag{4.3.4}$$

Remark 4.3.2. *[BG02b, pp. 1429, 1437]*

(i) If h *is significantly larger than* σ*, the exponential factors in* (4.3.2) *and* (4.3.3) *become very small, i.e. a transition is very unlikely.*

(ii) $C(t,\varepsilon)$ *is a correction factor that models the increase of* $\mathbb{P}^{-T,x_{-T}}\left[\tau_{\mathcal{B}(h)} < t\right]$ *over time. However, if* $\frac{h}{\sigma} > \mathcal{O}\left(|\log\varepsilon|\right)$*, its effect becomes negligible, because the exponential factor is already very small.*

The authors of [BG02b] note that they believe that $C(t,\varepsilon)$ *is not optimal.*

(iii) The theorem implies that the typical spreading of the paths of (4.1.1) *around* x^{det} *is of the order* $\sigma\sqrt{\zeta(t)}$*, which by Lemma 4.2.12 fulfills*

$$\sigma\sqrt{\zeta(t)} \asymp \frac{\sigma}{\sqrt{|t|} \vee a_0^{1/4} \vee \varepsilon^{1/4}} .$$

(iv) In the case that $\sigma \ll a_0^{3/4} \vee \varepsilon^{3/4}$*, it is possible within the assumptions of the theorem to choose* $h \gg \sigma$*. Then (cf. (i)) the probability* $\mathbb{P}^{-T,x_{-T}}\left[\tau_{\mathcal{B}(h)} < t\right]$ *becomes exponentially small. In other words,* $\sigma \ll a_0^{3/4} \vee \varepsilon^{3/4}$ *implies that a transition is very unlikely.*

(v) If σ *is not so small, we can still apply part (i) of the theorem to show that for* t *of order* $-\sigma^{2/3}$ *(or smaller) a transition is unlikely.*

The rest of this subsection is devoted to the proof of this theorem. We analyze the deviation of the solution x of (4.1.1) from the solution x^{det} of (4.2.1) (with identical initial conditions $x_{-T} = x^{\mathrm{det}}_{-T}$ as described before) using[4]

$$z_t := x_t - x_t^{\mathrm{det}} .$$

This process satisfies the stochastic equation

$$\begin{cases} \mathrm{d}z_t &= \frac{1}{\varepsilon} \cdot \left(\bar{a}(t) \cdot z_t + \bar{b}(z_t,t)\right) \mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} \, \mathrm{d}W_t \\ z_{-T} &= 0 . \end{cases} \tag{4.3.5}$$

[4]Because of consistency, we use a different notation than the authors of [BG02b]: z_t for the deviation of the perturbed from the deterministic solution, because y_t is already used for the deviation of the deterministic solution from the equilibrium branch, and $\bar{a}$ instead of $\tilde{a}$ for $\partial_x f(x_t^{\mathrm{det}}, t)$ because this notation is already established in Proposition 4.2.10

Here, as before, $\bar{a}(t) = \partial_x f(x_t^{\det}, t)$, and by Taylor's formula we obtain that there exists a constant $M > 0$ such that

$$\left|\bar{b}(z,t)\right| \leqslant M \cdot \left(x_t^{\det} + |z|\right) \cdot z^2 \tag{4.3.6}$$

for all $t \in [-T, T]$ and $x_t^{\det} + |z| \leqslant d$.

We consider the solution z^0 to the linearization of (4.3.5) at $x^{\det}$, namely

$$\begin{cases} \mathrm{d}z_t^0 &= \frac{1}{\varepsilon} \cdot \bar{a}(t) \cdot z_t^0 \,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} \,\mathrm{d}W_t \\ z_{-T}^0 &= 0 \,. \end{cases} \tag{4.3.7}$$

The solution z^0 to this equation is a Gaussian process with expectation zero and variance

$$v(t) = \frac{\sigma^2}{\varepsilon} \int_{-T}^{t} \exp\left[\frac{2\bar{\alpha}(t,s)}{\varepsilon}\right] \mathrm{d}s \,, \quad \text{where } \bar{\alpha}(t,s) = \int_s^t \bar{a}(u) \,\mathrm{d}u \,.$$

Instead of the variance itself we will use for our estimates the function $\zeta(t)$ defined in Lemma 4.2.12. For any $t \in [-T + \varepsilon, T]$ we have that both $\frac{v(t)}{\sigma^2}$ and $\zeta(t)$ can be estimated by

$$\asymp \frac{1}{|t| \vee \sqrt{a_0} \vee \sqrt{\varepsilon}}$$

(the fact that $\frac{v(t)}{\sigma^2} \approx \zeta(t)$ for $t > -T + \varepsilon$ follows from the definition of ζ and (4.2.37): For $t > -T + \varepsilon$, the negativity of $\bar{a}$ lets the first summand of the definition of ζ become negligibly small).

The advantage of ζ is that it is bounded away from zero, which avoids technical problems (see e.g. the formulation of the following proposition) and is more realistic in terms of the general periodicity of our model.

The next proposition shows that z^0 is likely to remain in a strip of width proportional to $\sqrt{\zeta(t)}$.

Proposition 4.3.3. *[sc: [BG02b, Proposition 3.8]] For all $t \in [-T, T]$ and any $h > 0$,*

$$\mathbb{P}^{-T,0}\left[\sup_{s \in [-T,t]} \frac{|z_s^0|}{|\zeta(s)|} \geqslant h\right] \leqslant C(t,\varepsilon) \cdot \exp\left[-\frac{h^2}{2\sigma^2} \cdot \left(1 - \mathcal{O}(\varepsilon)\right)\right],$$

with $C(t,\varepsilon)$ as defined in (4.3.4).

Proof. For $k \in \mathbb{N}$ let

$$-T = u_0 < u_1 < \cdots < u_K = T \tag{4.3.8}$$

be a partition of $[-T, t]$. By [BG02a, Lemma 3.2] we have

$$\mathbb{P}^{-T,0}\left[\sup_{s \in [-T,t]} \frac{|z_t^0|}{|\zeta(s)|} \geqslant h\right] \leqslant 2 \sum_{k=1}^{K} P_k \,, \tag{4.3.9}$$

where

$$P_k = \exp\left[-\frac{h^2}{2\sigma^2} \cdot \inf_{s \in [u_{k-1}, u_k]} \frac{\zeta(s)}{\zeta(u_k)} \cdot \exp\left[\frac{2\bar{\alpha}(u_k, s)}{\varepsilon}\right]\right].$$

Let the partition (4.3.8) be such that

$$\bar{\alpha}(u_k, u_{k-1}) = -2\varepsilon^2 \quad \text{for all } k = 1, \dots, K-1, \tag{4.3.10}$$

where

$$K = \left\lceil \frac{|\bar{\alpha}(t, -T)|}{2\varepsilon^2} \right\rceil .$$

By Proposition 4.2.10, we have $\bar{a}(s) < 0$ for all $s \in [-T, T]$, hence,

$$\begin{aligned}
\zeta'(s) &= \frac{1}{2 \cdot |\bar{a}(-T)|} \cdot \exp\left[\frac{2\bar{\alpha}(t, -T)}{\varepsilon}\right] \cdot \frac{2\bar{a}(t)}{\varepsilon} \\
&\quad + \frac{1}{\varepsilon} \cdot \left[\frac{2\bar{a}(t)}{\varepsilon} \cdot \int_{-T}^{t} \exp\left[\frac{2\bar{\alpha}(t,s)}{\varepsilon}\right] \mathrm{d}s + 1\right] \\
&= \frac{1}{\varepsilon} \cdot \big(2\bar{a}(t) \cdot \zeta(t) + 1\big) \leqslant \frac{1}{\varepsilon} ,
\end{aligned}$$

where we used that

$$\int_{-T}^{t} \exp\left[\frac{2\bar{\alpha}(t,s)}{\varepsilon}\right] \mathrm{d}s = \exp\left[\frac{2\bar{\alpha}(t,-T)}{\varepsilon}\right] \cdot \int_{-T}^{t} \exp\left[-\frac{2\bar{\alpha}(s,-T)}{\varepsilon}\right] \mathrm{d}s .$$

Thus, we obtain

$$\begin{aligned}
&\inf_{s \in [u_{k-1}, u_k]} \frac{\zeta(s)}{\zeta(u_k)} \geqslant \frac{1}{\zeta(u_k)} \cdot \inf_{s \in [u_{k-1}, u_k]} \left[\zeta(u_k) - \frac{1}{\varepsilon} \cdot (u_k - s)\right] \\
&= 1 - \frac{1}{\zeta(u_k)} \cdot \frac{1}{\varepsilon} \cdot (u_k - u_{k-1}) .
\end{aligned}$$

For all k such that $|u_k| \geqslant \sqrt{a_0} \vee \sqrt{\varepsilon}$, we apply (4.3.10) and (4.2.37) to see that there exist constants $c_-^{(1)}, c_-^{(2)}, \dots > 0$ small enough for

$$2\varepsilon^2 = -\bar{\alpha}(u_k, u_{k-1}) \geqslant c_-^{(1)} \int_{u_{k-1}}^{u_k} |s| \,\mathrm{d}s \geqslant c_-^{(2)} \cdot |u_k| \cdot (u_k - u_{k-1}) .$$

Furthermore, by Lemma 4.2.12 we know that $\zeta(u_k) \asymp \frac{1}{|u_k|}$, i.e. $\zeta(u_k) \geqslant \frac{c_-^{(3)}}{|u_k|}$. Thus,

$$\frac{u_k - u_{k-1}}{\zeta(u_k)} \leqslant \frac{|u_k|}{c_-^{(3)}} \cdot \frac{2\varepsilon^2}{c_-^{(2)} |u_k|} = \mathcal{O}(\varepsilon^2) . \tag{4.3.11}$$

For all other $k = 1, \dots, K-1$ (i.e., such that $|u_k| < \sqrt{a_0} \vee \sqrt{\varepsilon}$) we have

$$2\varepsilon^2 \geqslant c_-^{(4)} \cdot (\sqrt{a_0} \vee \sqrt{\varepsilon}) \cdot (u_k - u_{k-1})$$

and

$$\zeta(u_k) \geqslant \frac{c_-^{(5)}}{\sqrt{a_0} \vee \sqrt{\varepsilon}} ,$$

hence again

$$\frac{u_k - u_{k-1}}{\zeta(u_k)} \leqslant \frac{\sqrt{a_0} \vee \sqrt{\varepsilon}}{c_-^{(5)}} \cdot \frac{2\varepsilon^2}{c_-^{(4)} \cdot (\sqrt{a_0} \vee \sqrt{\varepsilon})} = \mathcal{O}(\varepsilon^2) \,. \tag{4.3.12}$$

Now we apply this to (4.3.9) and obtain, using the definition of K,

$$\begin{aligned}
&\mathbb{P}^{-T,0}\left[\sup_{s\in[-T,t]} \frac{|z_t^0|}{|\zeta(s)|} \geqslant h\right] \\
&\quad\leqslant 2\sum_{k=1}^{K} \exp\left[-\frac{h^2}{2\sigma^2} \cdot \big(1-\mathcal{O}(\varepsilon)\big) \cdot \exp\left[\frac{2\bar{\alpha}(u_k,s)}{\varepsilon}\right]\right] \\
&\quad\leqslant 2 \cdot \left(\frac{|\bar{\alpha}(t,-T)|}{2\varepsilon^2} + 1\right) \cdot \exp\left[-\frac{h^2}{2\sigma^2} \cdot \big(1-\mathcal{O}(\varepsilon)\big)\right] ,
\end{aligned}$$

because for $\varepsilon \ll 1$ we have $\big(1-\mathcal{O}(\varepsilon)\big)^2 = 1-\mathcal{O}(\varepsilon)$. □

In the preceding proposition we have proved that the solution to the linearized version of (4.3.5) behaves as claimed in the theorem. Now we extend this result to the original, nonlinear system.

We define the notations

$$\begin{aligned}
\Omega_t^0(h) &:= \left\{\omega \in \Omega \;\middle|\; |z_s^0| \leqslant h\sqrt{\zeta(s)} \;\forall\, s \in [-T,t]\right\} \\
\Omega_t(h) &:= \left\{\omega \in \Omega \;\middle|\; |z_s| \leqslant h\sqrt{\zeta(s)} \;\forall\, s \in [-T,t]\right\} .
\end{aligned}$$

Furthermore, for two events $A, B \in \mathcal{A}$ in a probability space $(\Omega, \mathcal{A}, P)$ we say $A \overset{\text{a.s.}}{\subset} B$ if P-a.e. $\omega \in A$ are elements of B.

Proposition 4.3.4 (extension to nonlinear system)**.** *[sc: [BG02b, Prop. 3.10]] There exists a constant ϱ, depending only on f, such that the following holds:*

(i) If $t \in \big[-T, -(\sqrt{a_0} \vee \sqrt{\varepsilon})\big]$ and $h < \frac{|t|^{3/2}}{\varrho}$, then

$$\Omega_t^0(h) \overset{a.s.}{\subset} \Omega_t\left(\left[1 + \varrho \cdot \frac{h}{|t|^{3/2}}\right] \cdot h\right) . \tag{4.3.13}$$

(ii) If $t \in \big[-(\sqrt{a_0} \vee \sqrt{\varepsilon}), T\big]$ and $h < \frac{1}{\varrho} \cdot (a_0^{3/4} \vee \varepsilon^{3/4})$, then

$$\Omega_t^0(h) \overset{a.s.}{\subset} \Omega_t\left(\left[1 + \varrho \cdot \frac{h}{a_0^{3/4} \vee \varepsilon^{3/4}}\right] \cdot h\right) . \tag{4.3.14}$$

Proof. For this proof we define the process $\tilde{z}_t := z_t - z_t^0$, which fulfills the stochastic equation

$$\begin{cases} \mathrm{d}\tilde{z}_t &= \frac{1}{\varepsilon} \cdot \bar{a}(t) \cdot \tilde{z}_t \,\mathrm{d}t + \frac{1}{\varepsilon} \cdot \bar{b}(z_t, t)\,\mathrm{d}t \\ \tilde{z}_{-T} &= 0 \,, \end{cases}$$

and thus fulfills

$$\tilde{z}_t = \frac{1}{\varepsilon} \cdot \int_{-T}^{s} \exp\left[\frac{\bar{\alpha}(s,r)}{\varepsilon}\right] \cdot \bar{b}(z_r, r)\, \mathrm{d}r \,. \tag{4.3.15}$$

Let us first assume that $t \in \left[-T, -(\sqrt{a_0} \vee \sqrt{\varepsilon})\right]$. Let $\varrho > 0$ be a constant (more specific restrictions on the selection of ϱ follow below) and

$$\delta := \frac{\varrho h}{|t|^{3/2}} \,,$$

which by the assumptions on h implies that $\delta < 1$. We define the first-exit time

$$\tau := \inf\left\{ s \in [-T, t] \;\middle|\; \frac{|\tilde{z}_s|}{\sqrt{\zeta(s)}} \geqslant \delta h \right\} \quad \in [-T, t] \cup \{\infty\} \tag{4.3.16}$$

and the set

$$A := \Omega_t^0(h) \cap \left\{ \omega \in \Omega \mid \tau(\omega) < \infty \right\} \,. \tag{4.3.17}$$

The aim of this proof is to show that $\mathbb{P}(A) = 0$ and hence $\tau(\omega) = \infty$ for $\mathbb{P}$-almost all $\omega \in \Omega_t^0(h)$. This implies that for $s \in [-T, t]$ and $\mathbb{P}$-almost all $\omega \in \Omega_t^0(h)$ we have

$$\left| z_s(\omega) \right| \leqslant (1 + \delta) \cdot h \sqrt{\zeta(s)} \,,$$

which proves (4.3.13) (and, consequently, (4.3.2)).

We choose an $\omega \in A$ and an $s \in \left[-T, \tau(\omega)\right]$. Then, by the definition of $\Omega_t^0(h)$ we have for all $r \in [-T, s]$

$$\left| z_r^0(\omega) \right| \leqslant h \sqrt{\zeta(r)}$$

and, by the definition of τ,

$$\left| z_r(\omega) \right| \leqslant (1 + \delta) \cdot h \sqrt{\zeta(r)} < 2h \sqrt{\zeta(r)} \,.$$

By Lemma 4.2.12, this implies that there exists a constant $c_+ > 0$ such that for all $r \in [-T, s]$

$$\left| z_r(\omega) \right| \leqslant 2h \cdot \frac{\sqrt{c_+}}{\sqrt{|r|}} \,.$$

Furthermore, we know by (4.2.38) that we can find constants $c_+^{(1)}, c_+^{(2)} > 0$ such that for all $r \in [-T, s]$ (note that $s < 0$!)

$$x_r^{\det} - x_c \leqslant c_+^{(1)} \cdot |r| \quad \Rightarrow \quad x_r^{\det} \leqslant c_+^{(2)} \cdot |r| \leqslant \sup\{c_+^{(2)}, c_+\} \cdot |r| \,.$$

From here on we use c_+ to denote the Supremum in the above estimate. Applying (4.3.6) we get that for all $r \in [-T, s]$

$$\begin{aligned}
\left| \bar{b}(z_r, r) \right| &\leqslant M \cdot \left(x_r^{\det} + |z_r| \right) \cdot z_r^2 \leqslant M \cdot \left(c_+ |r| + 2h \cdot \frac{\sqrt{c_+}}{\sqrt{|r|}} \right) \cdot 4h^2 \cdot \frac{c_+}{|r|} \\
&= 4Mh^2 c_+^2 \cdot \left(1 + \frac{2h}{\sqrt{c_+}|r|^{3/2}} \right) \leqslant 4Mh^2 c_+^2 \cdot \left(1 + \frac{2h}{\sqrt{c_+}|s|^{3/2}} \right) ,
\end{aligned}$$

where we used $|r| \geqslant |s|$ in the last step.
Using this estimate to (4.3.15), we see that for all $s \in \big[-T, \tau(\omega)\big]$

$$|\tilde{z}_s| \leqslant 4Mh^2c_+^2 \cdot \left(1 + \frac{2h}{\sqrt{c_+}|s|^{3/2}}\right) \cdot \frac{1}{\varepsilon} \int_{-T}^{s} \exp\left[\frac{\bar{\alpha}(s,r)}{\varepsilon}\right] \mathrm{d}r \,.$$

From Lemma 4.2.6 we conclude that there exists a constant $c_+^{(3)}$ such that

$$\int_{-T}^{s} \exp\left[\frac{\bar{\alpha}(s,r)}{\varepsilon}\right] \mathrm{d}r \leqslant \frac{\sup\{c_+^{(3)}, c_+\}}{|s|}$$

(from now on we write c_+ instead of the supremum), hence,

$$|\tilde{z}_s| \leqslant 4M \cdot \frac{h^2 c_+^3}{|s|} \cdot \left(1 + \frac{2h}{\sqrt{c_+}|s|^{3/2}}\right) .$$

By Lemma 4.2.12 we know that there exists a constant $c_- > 0$ such that $\zeta(s) \geqslant \frac{c_-}{|s|}$. Consequently, for all $s \in [-T, \tau(\omega)]$

$$\frac{|\tilde{z}_s|}{h\sqrt{\zeta(s)}} \leqslant 4M \cdot \frac{hc_+^3}{\sqrt{c_-}\sqrt{|s|}} \cdot \left(1 + \frac{2h}{\sqrt{c_+}|s|^{3/2}}\right) .$$

Now we set

$$\varrho = \frac{2}{\sqrt{c_+}} \vee 8M \cdot \frac{c_+^3}{\sqrt{c_-}} \,. \tag{4.3.18}$$

Because of $1 > |s| \geqslant |t| > 0$, we have

$$\delta = \varrho \cdot \frac{h}{|t|^{3/2}} \geqslant \varrho \cdot \frac{h}{|s|^{3/2}} > \varrho \cdot \frac{h}{|s|^{1/2}} \,.$$

Hence, we see that for all $s \in [-T, \tau(\omega)]$

$$\frac{|\tilde{z}_s|}{h\sqrt{\zeta(s)}} \leqslant \frac{\varrho h}{2\sqrt{|s|}} \cdot \left(1 + \frac{\varrho h}{|s|^{3/2}}\right) < \frac{\delta}{2} \cdot (1+\delta) \overset{\delta<1}{<} \delta \,.$$

This estimate implies that for almost all $\omega \in A$

$$|\tilde{z}_{\tau(\omega)}| < \delta h \sqrt{\zeta\big(\tau(\omega)\big)} \,.$$

At the same time, by continuity of $\tilde{z}$ and the definition of τ, we know that for all $\omega \in \{\tau < \infty\}$ we have

$$|\tilde{z}_{\tau(\omega)}| = \delta h \sqrt{\zeta\big(\tau(\omega)\big)} \,.$$

Hence, $\mathbb{P}(A) = 0$ as requested above. This completes the proof of (4.3.13).
The proof of (4.3.14) is very similar. The main difference stems from the fact that the estimates for ζ and $x^{\det}$ are a bit more complicated in this case:

We choose $t \in \big[-(\sqrt{a_0} \vee \sqrt{\varepsilon}), T\big]$ and re-use the constant $\varrho > 0$ as chosen in (4.3.18). We set

$$\delta := \frac{\varrho h}{a_0^{3/4} \vee \varepsilon^{3/4}}$$

and note that the assumptions on h yield $\delta < 1$. We define A as above and $\tau \in \big[-(\sqrt{a_0} \vee \sqrt{\varepsilon}), t\big]$ similar to (4.3.16) and select $\omega \in A$ and $s \in \big[-(\sqrt{a_0} \vee \sqrt{\varepsilon}), \tau(\omega)\big]$.

By definition of $\Omega_t^0(h)$ we have for any $r \in \big[-(\sqrt{a_0} \vee \sqrt{\varepsilon}), s\big]$ that $\big|z_r^0(\omega)\big| \leqslant h\sqrt{\zeta(r)}$, hence, by the definition of τ, that

$$\big|z_r(\omega)\big| \leqslant (1+\delta) \cdot h\sqrt{\zeta(r)} < 2h\sqrt{\zeta(r)} \,.$$

With similar arguments as before, we obtain from (4.2.38) that there exists a constant $c_+ > 0$ such that $x_r^{\det} \leqslant c_+ \cdot \big(|r| \vee \sqrt{a_0} \vee \sqrt{\varepsilon}\big)$. Together with (4.3.6) this implies that

$$\begin{aligned}
&\big|\bar{b}(z_r, r)\big| \\
&\quad\leqslant M \cdot \left(c_+ \cdot \big(|r| \vee \sqrt{a_0} \vee \sqrt{\varepsilon}\big) + 2h \cdot \frac{\sqrt{c_+}}{\sqrt{|r|} \vee a_0^{1/4} \vee \varepsilon^{1/4}} \right) \\
&\qquad \cdot \frac{4h^2 \cdot c_+}{|r| \vee \sqrt{a_0} \vee \sqrt{\varepsilon}} \\
&\quad\leqslant 4Mh^2 c_+^2 \cdot \left(1 + \frac{2h}{\sqrt{c_+} \cdot (a_0^{3/4} \vee \varepsilon^{3/4})} \right),
\end{aligned}$$

where we used that $|r|^{3/2} \vee a_0^{3/4} \vee \varepsilon^{3/4} \geqslant a_0^{3/4} \vee \varepsilon^{3/4}$. By (4.3.15) and applying Lemma 4.2.6 as above we see that

$$\begin{aligned}
|\tilde{z}_s| &\leqslant 4Mh^2 c_+^2 \cdot \left(1 + \frac{2h}{\sqrt{c_+} \cdot (a_0^{3/4} \vee \varepsilon^{3/4})} \right) \cdot \frac{1}{\varepsilon} \int_{-T}^{s} \exp\left[\frac{\bar{\alpha}(s,u)}{\varepsilon}\right] \mathrm{d}u \\
&\leqslant 4M \cdot \frac{h^2 c_+^3}{|s| \vee \sqrt{a_0} \vee \sqrt{\varepsilon}} \cdot \left(1 + \frac{2h}{\sqrt{c_+} \cdot (a_0^{3/4} \vee \varepsilon^{3/4})} \right).
\end{aligned}$$

By Lemma 4.2.12 we know that there exists a $c_- > 0$ such that

$$\zeta(s) \geqslant \frac{c_-}{|s| \vee \sqrt{a_0} \vee \sqrt{\varepsilon}} \,,$$

hence,

$$\begin{aligned}
\frac{|\tilde{z}_s|}{h\sqrt{\zeta(s)}} &\leqslant 4M \cdot \frac{hc_+^3}{\sqrt{c_-} \cdot \big(\sqrt{|s|} \vee a_0^{1/4} \vee \varepsilon^{1/4}\big)} \cdot \left(1 + \frac{2h}{\sqrt{c_+} \cdot (a_0^{3/4} \vee \varepsilon^{3/4})} \right) \\
&= \frac{\varrho}{2} \cdot \frac{h}{\sqrt{|s|} \vee a_0^{1/4} \vee \varepsilon^{1/4}} \cdot \left(1 + \varrho \cdot \frac{h}{a_0^{3/4} \vee \varepsilon^{3/4}} \right).
\end{aligned}$$

By

$$\delta = \frac{\varrho h}{a_0^{3/4} \vee \varepsilon^{3/4}} \geqslant \frac{\varrho h}{a_0^{1/4} \vee \varepsilon^{1/4}} \geqslant \frac{\varrho h}{|s|^{1/2} \vee a_0^{1/4} \vee \varepsilon^{1/4}} ,$$

we finally see that also in this case

$$\frac{|\tilde{z}_s|}{h\sqrt{\zeta(s)}} \leqslant \frac{\delta}{2} \cdot (1+\delta) \overset{\delta<1}{<} \delta .$$

The remaining argument is as before. □

These two propositions prove the result claimed in the theorem: Proposition 4.3.3 proves that solutions z^0 of the linearized SDE fulfill

$$\mathbb{P}^{-T,0}\left[\tau_{\mathcal{B}(h)}(z^0) < t\right] \leqslant C(t,\varepsilon) \cdot \exp\left[-\frac{h^2}{2\sigma^2} \cdot \left(1 - \mathcal{O}(\varepsilon)\right)\right] ,$$

and Proposition 4.3.4 extends this result by showing that there exists a constant $\varrho > 0$ such that

$$\left\{\tau_{\mathcal{B}\left([1+\varrho\frac{h}{|t|^{3/2}}]\cdot h\right)}(z) < t\right\} \overset{\text{a.s.}}{\subset} \left\{\tau_{\mathcal{B}(h)}(z^0) < t\right\}$$

for all $t \in \left[-T, -(\sqrt{a_0} \vee \sqrt{\varepsilon})\right]$ and $h < \frac{1}{\varrho} \cdot |t|^{3/2}$, and

$$\left\{\tau_{\mathcal{B}\left([1+\varrho\frac{h}{a_0^{3/4}\vee\varepsilon^{3/4}}]\cdot h\right)}(z) < t\right\} \overset{\text{a.s.}}{\subset} \left\{\tau_{\mathcal{B}(h)}(z^0) < t\right\}$$

for all $t \in \left[-(\sqrt{a_0} \vee \sqrt{\varepsilon}), T\right]$ and $h < \frac{1}{\varrho} \cdot (a_0^{3/4} \vee \varepsilon^{3/4})$.

4.3.2. Transition

In this subsection, we consider the situation that σ is big enough for a transition from the potential well at $x_+^*(t)$ over the saddle at $x_u^*(t)$ to the potential well at $x_-^*(t)$. Theorem 4.3.1 shows that 'big enough' means $\sigma \geqslant C \cdot (a_0^{3/4} \vee \varepsilon^{3/4})$ for a reasonable constant $C > 0$. Our aim is to find an upper bound for the probability of "no transition".

Situation 4.3.5. *We consider* (4.1.1)*, still under the assumptions of Situation 4.1.4. Furthermore, we assume that σ is not small with respect to $a_0^{3/4} \vee \varepsilon^{3/4}$.*

We assume that there exist constants $\delta_0, \delta_1, \delta_2 \in [-d, d]$ such that for all $t \in [-T, T]$

$$\delta_0 < \delta_1 < x_c < x_t^{\det} < \delta_2 \tag{4.3.19}$$

and

$$f(x,t) \asymp -1 \quad \text{for } x \in [\delta_0, \delta_1] \text{ and } t \in [-T, T] , \tag{4.3.20}$$

$$\partial_{xx} f(x,t) \leqslant 0 \quad \text{for } x \in [\delta_1, \delta_2] \text{ and } t \in [-T, T] . \tag{4.3.21}$$

This especially implies that we may complement (4.3.19) *to obtain*

$$x_-^*(t) < \delta_0 < \delta_1 < x_u^*(t) < x_c < x_t^{\det} < \delta_2 \quad \text{for all } t \in [-T, T] ,$$

and that we may assume that δ_0 is of order -1.

The basic idea for δ_0 is that the constant is 'near the potential bottom' during $[-T, T]$ in the sense that a deterministic path starting at δ_0 approaches a neighborhood of the order ε of $x^*_-(t)$ exponentially fast by Tihonov's Theorem.

In many cases, like in Example 4.1.3, δ_2 may be chosen arbitrarily large. In this example, a typical selection for δ_1 is the inflection point of $x \mapsto f(x,t)$.

We will now bound the nontransition-probability from above using the following heuristic idea:

$$\begin{aligned}
&\mathbb{P}[\text{no transition}] \\
&\quad \leqslant \mathbb{P}\left[\text{process escapes away from } x^*_u \text{ (beyond } x^{\det}_t + h\sqrt{\zeta(s)})\right] \qquad (4.3.22)\\
&\qquad + \mathbb{P}\left[\text{process crosses neither } \delta_1 \text{ nor } x^{\det}_t + h\sqrt{\zeta(s)}\right] \\
&\qquad + \mathbb{P}\left[\text{process starting at } \delta_1 \text{ does not reach } \delta_0\right] .
\end{aligned}$$

The following theorem collects the results of this subsection.

Theorem 4.3.6 (nontransition probability). *[BG02b, Theorem 2.7] Let $c_1, c_2 > 0$ be constants and assume that*

$$c_1^{3/2}\sigma \geqslant a_0^{3/4} \vee \varepsilon^{3/4} .$$

Choose times $-T \leqslant t_0 \leqslant t_1 \leqslant t \leqslant T$ such that

$$t_1 \in [-c_1\sigma^{2/3}, c_1\sigma^{2/3}] \quad \textit{and} \quad t \geqslant t_1 + c_2\varepsilon .$$

Let $h > 2\sigma$ be such that

$$x^{\det}_s + h\sqrt{\zeta(s)} < \delta_2 \quad \textit{for all } s \in [t_0, t_1].$$

Then, for any small enough c_1, any large enough c_2 and any $x_0 \in \left]\delta_1, x^{\det}_{t_0} + \frac{1}{2} \cdot h\sqrt{\zeta(t_0)}\right]$ we have that

$$\begin{aligned}
&\mathbb{P}^{t_0,x_0}\left[x_s > \delta_0 \ \ \forall\, s \in [t_0, t]\right] \\
&\quad \leqslant \frac{3}{2} \cdot \left(\frac{|\bar{\alpha}(t_1, t_0)|}{\varepsilon} + 1\right) \cdot \exp\left[-\frac{\kappa h^2}{\sigma^2}\right] \qquad (4.3.23)\\
&\qquad + \frac{3}{2} \cdot \exp\left[-\kappa \cdot \frac{1}{\log\frac{h}{\sigma} \vee |\log \sigma^{2/3}|} \cdot \frac{\hat{\alpha}(t_1, -c_1\sigma^{2/3})}{\varepsilon}\right] \\
&\qquad + \exp\left[-\frac{\kappa}{\sigma^2}\right] ,
\end{aligned}$$

where $\kappa > 0$ is a constant and

$$\hat{\alpha}(t,s) := \int_s^t \hat{a}(u)\, \mathrm{d}u .$$

We already know from the preceding subsection, that for σ smaller than $a_0^{3/4} \vee \varepsilon^{3/4}$ the transition probability becomes exponentially small. The above result allows us to extend this connection between the relation of the parameters and the probability of (non-)transition.

Remark 4.3.7 (optimal choice of parameters)**.** *[BG02b, p. 1438ff] Consider the situation of Theorem 4.3.6 and the force term f defined in Example 4.1.3. In this case, the assumptions from Situation 4.3.5 are fulfilled for any arbitrarily large $\delta_2 > 0$; we choose δ_2 large, $t_1 = -\frac{1}{2}c_1\sigma^{2/3}$ and $h \asymp (\delta_2 - x_c) \cdot \sigma^{1/3}$. Then,*

$$\begin{aligned}
\left|\bar{\alpha}(t_1, t_0)\right| &= \left|\int_{t_0}^{t_1} \bar{a}(r)\,\mathrm{d}r\right| \leqslant t_0^2 \\
\hat{\alpha}(t_1, -c_1\sigma^{2/3}) &= \int_{-c_1\sigma^{2/3}}^{-\frac{1}{2}c_1\sigma^{2/3}} \hat{a}(r)\,\mathrm{d}r \leqslant \text{const.} \cdot \sigma^{4/3}\ .
\end{aligned}$$

Thus, in this case

$$\begin{aligned}
&\mathbb{P}^{t_0,x_0}\left[x_s > \delta_0 \ \ \forall\, s \in [t_0, t]\right] \\
&\quad \leqslant \frac{t_0^2}{\varepsilon} \cdot \exp\left[-\mathcal{O}\left(\frac{(\delta_2 - x_c)^2}{\sigma^{4/3}}\right)\right] \\
&\qquad + \exp\left[-\frac{\text{const.}}{\log\frac{\delta_2 - x_c}{\sigma^{2/3}} \vee |\log\sigma|} \cdot \frac{\sigma^{4/3}}{\varepsilon}\right] + \exp\left[-\frac{\kappa}{\sigma^2}\right]\ .
\end{aligned}$$

From this we conclude that for $a_0^{3/4} \vee \varepsilon^{3/4} \ll \sigma \ll \left(\frac{1}{|\log\varepsilon|}\right)^{3/4}$, the probability of a transition becomes exponentially close to 1.

For large σ, e.g. $\sigma \geqslant \left(\frac{1}{|\log\varepsilon|}\right)^{3/4}$, the variance of the path becomes so great that it does not make sense any longer to speak of a transition probability.

Remark 4.3.8 (reaching $x_-^*(t)$ from δ_0)**.** *[BG02a, Theorem 2.4] shows that x is likely to reach a small neighborhood of x_-^* when starting at δ_0.*

In the following lemma we specify conditions under which z^0, the solution of the linearized stochastic differential equation (4.3.7), dominates z. The idea to consider linearized versions of the stochastic differential equations under consideration is again among our basic tools in this subsection.

Lemma 4.3.9. *[modified version of [BG02b, Lemma 4.4]] Let t_0 as in the theorem be an initial time. We consider the following processes on $[t_0, T]$:*

- (x_t^{det}) *The stable solution of the deterministic equation* (4.2.1) *with initial condition $x_{t_0}^{\text{det}} \in [\delta_1, \delta_2]$, such that $x_t^{\text{det}} \in [\delta_1, \delta_2]$ for all $t \in [t_0, T]$.*
- (x_t) *The solution of* (4.1.1) *with initial condition $x_{t_0} \in [x_{t_0}^{\text{det}}, \delta_2]$.*
- (z_t) *The difference $z_t := x_t - x_t^{\text{det}}$. The initial point is $z_{t_0} = x_{t_0} - x_{t_0}^{\text{det}} \geqslant 0$.*
- (z_t^0) *The solution of* (4.3.7) *with initial condition $z_{t_0}^0 \in [z_{t_0}, \delta_2 - x_{t_0}^{\text{det}}]$.*

Then we have $z_t^0 \geqslant z_t$ for any $t \in [t_0, T]$.

Proof. [adapted from [BG02b, Lemma 3.11]] We show that $\tilde{z}_t := z_t - z_t^0 \in \mathbb{R}_-$ for all $t \in [t_0, T]$. $\tilde{z}_{t_0} \leqslant 0$ is obvious.

The definition of δ_1, δ_2 and the assumptions on x^{det} imply that for all $t \in [t_0, T]$ and all $z \in \mathbb{R}$ with $x_t^{\text{det}} + z \in [-d, d]$ for all t we have

$$f(x_t^{\text{det}} + z, t) \leqslant f(x_t^{\text{det}}, t) + \partial_x f(x_t^{\text{det}}, t) \cdot z\ . \tag{4.3.24}$$

By definition, z_t fulfills

$$\mathrm{d}z_t = \frac{1}{\varepsilon} \cdot \left[f(x_t^{\det} + z_t, t) - f(x_t^{\det}, t) \right] \mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} \, \mathrm{d}W_t \, .$$

Hence, for all $t \in [t_0, T]$,

$$\begin{aligned} \tilde{z}_s &= \tilde{z}_{t_0} + \frac{1}{\varepsilon} \int_{t_0}^{s} \underbrace{f(x_r^{\det} + z_r, r) - f(x_r^{\det}, r)}_{\overset{(4.3.24)}{\leqslant} \partial_x f(x_r^{\det}, r) \cdot z_r} - \partial_x f(x_r^{\det}, r) \cdot z_r^0 \, \mathrm{d}r \\ &= \tilde{z}_{t_0} + \frac{1}{\varepsilon} \int_{t_0}^{s} \underbrace{\partial_x f(x_r^{\det}, r)}_{=\bar{a}(r)} \cdot \tilde{z}_r \, \mathrm{d}r \, . \end{aligned}$$

With Gronwall's inequality, we see that for all $s \in [t_0, T]$

$$\tilde{z}_s \leqslant \tilde{z}_{t_0} \cdot \exp\left[\frac{\bar{\alpha}(s, t_0)}{\varepsilon} \right] \overset{\tilde{z}_{t_0} \leqslant 0}{\leqslant} 0 \, .$$

□

We define the stopping time

$$\tau := \inf\left\{ s \in [t_0, t_1] \,\middle|\, x_s \leqslant \delta_1 \right\} \quad \in [t_0, t_1] \cup \{\infty\}$$

and estimate, following the idea of (4.3.22),

$$\begin{aligned} &\mathbb{P}^{t_0, x_{t_0}} \left[x_s > \delta_0 \;\; \forall \, s \in [t_0, t] \right] \\ &\quad \leqslant \mathbb{P}^{t_0, x_{t_0}} \left[\sup_{s \in [t_0, t_1]} \frac{x_s - x_s^{\det}}{\sqrt{\zeta(s)}} > h \right] \\ &\qquad + \mathbb{P}^{t_0, x_{t_0}} \left[x_s \in \left] \delta_1, x_s^{\det} + h\sqrt{\zeta(s)} \right] \;\; \forall \, s \in [t_0, t_1] \right] \\ &\qquad + \mathbb{E}^{t_0, x_{t_0}} \left[\mathbb{I}_{\{\tau \leqslant t_1\}} \cdot \mathbb{P}^{\tau, \delta_1} \left[x_s > \delta_0 \;\; \forall \, s \in [\tau, t] \right] \right] . \end{aligned} \tag{4.3.25}$$

The following propositions are dedicated to the development of upper bounds for each of the summands on the right hand side of (4.3.25). The first summand on the right hand side is similar to the probabilities we have estimated in the previous subsection, but here we need it for a larger selection of σ as before:

Proposition 4.3.10 (first summand in (4.3.25))**.** *[modified version of [BG02b, Proposition 4.5]] Let* $x_{t_0} \in \left[\delta_1, x_{t_0}^{\det} + \frac{1}{2} h \sqrt{\zeta(t_0)} \right]$*. Then*

$$\mathbb{P}^{t_0, x_{t_0}} \left[\sup_{s \in [t_0, t_1]} \frac{x_s - x_s^{\det}}{\sqrt{\zeta(s)}} > h \right] \leqslant \frac{3}{2} \cdot \left(\frac{\left| \bar{\alpha}(t_1, t_0) \right|}{\varepsilon} + 1 \right) \cdot \exp\left[-\frac{\kappa h^2}{\sigma^2} \right] ,$$

where $\kappa \geqslant 0$.

As before, we denote

$$\bar{\alpha}(t, s) := \int_s^t \bar{a}(r) \, \mathrm{d}r \, , \quad \bar{a}(s) := \partial_x f(x_s^{\det}, s)$$

(cf. Proposition 4.2.10 and Lemma 4.2.12).

Proof. Part 1. We define a partition $t_0 = u_0 < u_1 < \cdots < u_k = t_1$ of $[t_0, t_1]$ such that

$$\left|\bar{\alpha}(u_k, u_{k-1})\right| = \varepsilon \quad \text{for } 1 \leqslant k < K := \left\lceil \frac{\left|\bar{\alpha}(t_1, t_0)\right|}{\varepsilon} \right\rceil . \tag{4.3.26}$$

With the same arguments as in the proof of (4.3.11), (4.3.12), using (4.3.26) instead of (4.3.10), we obtain that

$$\frac{u_{k+1} - u_k}{\zeta(u_k)} = \mathcal{O}(\varepsilon) \quad \text{for all } k = 0, \dots, K-2. \tag{4.3.27}$$

We set

$$\varrho_k := \frac{1}{2} \cdot h\sqrt{\zeta(u_k)}$$

and

$$\begin{aligned} Q_k \quad &:= \sup_{z_{u_k} \leqslant \varrho_k} \Bigg(\mathbb{P}^{u_k, z_{u_k}} \Bigg[\sup_{s \in [u_k, u_{k+1}]} \frac{z_s}{\sqrt{\zeta(s)}} > h \Bigg] \\ &\qquad + \mathbb{P}^{u_k, z_{u_k}} \Bigg[\sup_{s \in [u_k, u_{k+1}]} \frac{z_s}{\sqrt{\zeta(s)}} \leqslant h \,,\ z_{u_{k+1}} > \varrho_{k+1} \Bigg] \Bigg) \\ &\quad \text{for } k = 0, \dots, K-2 \\ Q_{K-1} &:= \sup_{z_{u_{K-1}} \leqslant \varrho_{K-1}} \mathbb{P}^{u_{K-1}, \varrho_{K-1}} \Bigg[\sup_{s \in [u_{K-1}, u_K]} \frac{z_s}{\sqrt{\zeta(s)}} > h \Bigg] . \end{aligned}$$

Then,

$$\mathbb{P}^{x_0, t_0} \Bigg[\sup_{s \in [t_0, t_1]} \frac{x_s - x_s^{\det}}{\sqrt{\zeta(s)}} > h \Bigg] \leqslant \sum_{k=0}^{K-1} Q_k .$$

Part 2. Estimating Q_k. We define the process $\left(z_s^{(k)}\right)_{s \in [u_k, u_{k+1}]}$ for $k = 0, \dots, K-1$ by

$$z_s^{(k)} := \varrho_k \cdot \exp\left[\frac{\bar{\alpha}(s, u_k)}{\varepsilon}\right] + \frac{\sigma}{\sqrt{\varepsilon}} \int_{u_k}^{s} \exp\left[\frac{\bar{\alpha}(s, r)}{\varepsilon}\right] \mathrm{d}W_r^{(k)} , \tag{4.3.28}$$

where the Brownian motion $\left(W_s^{(k)}\right)_{s \in [u_k, u_{k+1}]}$ is defined by $W_s^{(k)} := W_s - W_{u_k}$. $z^{(k)}$ solves the linearized equation (4.3.7) with initial condition $z_{u_k}^{(k)} = \varrho_k$. The variance of $z_{u_{k+1}}^{(k)}$ is

$$v_{u_{k+1}}^{(k)} = \frac{\sigma^2}{\varepsilon} \int_{u_k}^{u_{k+1}} \exp\left[\frac{2\bar{\alpha}(u_{k+1}, r)}{\varepsilon}\right] \mathrm{d}r . \tag{4.3.29}$$

We define for any $k = 0, \dots, K-1$ the stopping time

$$\tau_k^+ := \inf\left\{ s \in [u_k, u_{k+1}] \;\middle|\; z_s^{(k)} = h\sqrt{\zeta(s)} \right\} .$$

By Lemma 4.3.9 we have that $z_s \leqslant z_s^{(k)}$ during $[t_0, T]$. Hence,

$$\begin{aligned}
Q_k &\leqslant \mathbb{P}^{u_k,\varrho_k}[\tau_k^+ < u_{k+1}] + \mathbb{P}^{u_k,\varrho_k}[z_{k+1}^{(k)} > \varrho_{k+1}] \quad \text{for } k = 0, \dots, K-2 \\
Q_{K-1} &\leqslant \mathbb{P}^{u_{K-1},\varrho_{K-1}}[\tau_{K-1}^+ < u_K] .
\end{aligned}$$

These estimates no longer depend on z, but on $z^{(k)}$.

We first estimate $\mathbb{P}^{u_k,\varrho_k}[\tau_k^+ < u_{k+1}]$ for $k = 0, \dots, K-1$:

$$\begin{aligned}
&\mathbb{P}^{u_k,\varrho_k}[\tau_k^+ < u_{k+1}] \\
&\quad = \mathbb{P}^{u_k,\varrho_k}\big[\exists\, s \in [u_k, u_{k+1}] : z_s^{(k)} \geqslant h\sqrt{\zeta(s)}\big] ;
\end{aligned}$$

(4.3.28) is invariant in $\mathbb{P}$ under $\sigma \mapsto -\sigma$, hence

$$\leqslant \mathbb{P}^{u_k,\varrho_k}\left[\exists\, s \in [u_k, u_{k+1}] : z_s^{(k)} \leqslant h\sqrt{\zeta(u_k)} \cdot \exp\left[\frac{\bar{\alpha}(s, u_k)}{\varepsilon}\right] - h\sqrt{\zeta(s)}\right] ;$$

the upper bound for $z^{(k)}$ is negative by definition of ζ, hence

$$\begin{aligned}
&\leqslant \mathbb{P}^{u_k,\varrho_k}\big[\exists\, s \in [u_k, u_{k+1}] : z_s^{(k)} \leqslant 0\big] \\
&= 2 \cdot \mathbb{P}^{u_k,\varrho_k}[z_{u_{k+1}}^{(k)} \leqslant 0] \\
&= \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{-\varrho_k \exp\left[\frac{\bar{\alpha}(u_{k+1}, u_k)}{\varepsilon}\right] (v_{u_{k+1}}^{(k)})^{-1/2}} \exp\left[-\frac{r^2}{2}\right] \mathrm{d}r \\
&\leqslant \exp\left[-\frac{1}{2} \cdot \frac{\varrho_k^2 \cdot \exp\left[\frac{2\bar{\alpha}(u_{k+1}, u_k)}{\varepsilon}\right]}{v_{u_{k+1}}^{(k)}}\right] ,
\end{aligned}$$

thus, by $\exp\left[\frac{2\bar{\alpha}(u_{k+1}, u_k)}{\varepsilon}\right] \geqslant \exp[-2]$ (by (4.3.26); "=" for $k = 0, \dots, K-2$), the definition of ϱ_k and $1 = \sigma^2/\sigma^2$:

$$\leqslant \exp\left[-\frac{h^2}{8\sigma^2} \cdot \frac{\zeta(u_k)}{v_{u_{k+1}}^{(k)}/\sigma^2} \cdot \exp[-2]\right] .$$

For the upper bound of $\mathbb{P}^{u_k,\varrho_k}[z_{u_{k+1}}^{(k)} > \varrho_{k+1}]$ (for $k < K-1$) we need again the definition of ϱ_k to see that

$$\begin{aligned}
\left(\varrho_{k+1} - \varrho_k \cdot \exp\left[\frac{\bar{\alpha}(u_{k+1}, u_k)}{\varepsilon}\right]\right)^2 &= \varrho_k^2 \cdot \left(\frac{\varrho_{k+1}}{\varrho_k} - \exp[-1]\right)^2 \\
&= \frac{h^2 \cdot \zeta(u_k)}{4} \cdot \left(\sqrt{\frac{\zeta(u_{k+1})}{\zeta(u_k)}} - \exp[-1]\right)^2 . \qquad (4.3.30)
\end{aligned}$$

By the definition of ζ, we have for $k < K-1$ that there exist constants[5] $c_-^{(1)}, c_-^{(2)} > 0$ such that

$$\begin{aligned}\zeta(u_{k+1}) &= \zeta(u_k)\cdot\exp\left[\frac{2\bar{\alpha}(u_{k+1},u_k)}{\varepsilon}\right] + \frac{1}{\varepsilon}\int_{u_k}^{u_{k+1}}\exp\left[\frac{2\bar{\alpha}(u_{k+1},s)}{\varepsilon}\right]\mathrm{d}s\\ &\geqslant \zeta(u_k)\cdot\exp[-2] + \exp[-2]\cdot c_-^{(1)}\cdot\inf_{s\in[u_k,u_{k+1}]}\zeta(s)\ ,\end{aligned}$$

where we used (4.3.27) in the last step; hence,

$$\frac{\zeta(u_{k+1})}{\zeta(u_k)} \geqslant \exp[-2] + \exp[-2]\cdot\underbrace{c_-^{(1)}\cdot\frac{\inf_s\zeta(s)}{\zeta(u_k)}}_{\overset{\text{Lemma 4.2.12}}{\geqslant}\ c_-^{(2)}}\ . \tag{4.3.31}$$

Similar to the above estimate we obtain that for all $k = 0,\dots,K-2$

$$\begin{aligned}&\mathbb{P}^{u_k,\varrho_k}[z_{u_{k+1}}^{(k)} > \varrho_{k+1}]\\ &\leqslant \frac{1}{2}\cdot\exp\left[-\frac{1}{2}\cdot\frac{\left(\varrho_{k+1}-\varrho_k\cdot\exp\left[\frac{\bar{\alpha}(u_{k+1},u_k)}{\varepsilon}\right]\right)^2}{v_{u_{k+1}}^{(k)}}\right]\\ &\overset{(4.3.30)}{=} \frac{1}{2}\cdot\exp\left[-\frac{h^2}{8\sigma^2}\cdot\frac{\zeta(u_k)}{v_{u_{k+1}}^{(k)}/\sigma^2}\cdot\left(\sqrt{\frac{\zeta(u_{k+1})}{\zeta(u_k)}}-\exp[-1]\right)^2\right]\\ &\overset{(4.3.31)}{\leqslant} \frac{1}{2}\cdot\exp\left[-\frac{h^2}{8\sigma^2}\cdot\frac{\zeta(u_k)}{v_{u_{k+1}}^{(k)}/\sigma^2}\cdot\left(\sqrt{\exp[-2]\cdot(1+c_-^{(2)})}-\exp[-1]\right)^2\right]\ .\end{aligned}$$

Hence we see that for all $k = 1,\dots,K-1$

$$Q_k \leqslant \frac{3}{2}\cdot\exp\left[-\frac{h^2}{\sigma^2}\cdot\kappa_k\right]\ ,$$

where

$$\kappa_k = \frac{1}{8}\cdot\frac{\zeta(u_k)}{v_{u_{k+1}}^{(k)}/\sigma^2}\cdot\underbrace{\left(\exp[-2]\wedge\left(\sqrt{\exp[-2]\cdot(1+c_-^{(2)})}-\exp[-1]\right)\right)}_{=:C>0}\ .$$

By (4.3.29) we have that for any $k = 0,\dots,K-1$ there exists a $\theta^{(k)} \in \big[\exp[-2],1\big]$ such that

$$v_{u_{k+1}}^{(k)} = \frac{\sigma^2}{\varepsilon}\cdot(u_{k+1}-u_k)\cdot\theta^{(k)}\ ,$$

hence, applying (4.3.27),

$$\kappa_k\cdot\frac{8\theta^{(k)}}{\varepsilon C} = \mathcal{O}(\varepsilon)\ .$$

[5]more precisely, these constants depend on k, but we choose the minimum of the respective constants over all k

Thus, we may conclude that κ_k is of order 1 for all $k = 0, \dots, K-1$. We set $\kappa := \inf_k \kappa_k$ (thus, $\kappa \geqslant 0$) and conclude that for all $k = 0, \dots, K-1$

$$Q_k \leqslant \frac{3}{2} \cdot \exp\left[-\frac{h^2}{\sigma^2} \cdot \kappa\right] .$$

Part 3. Re-collecting. We finally obtain that

$$\begin{aligned}
&\mathbb{P}^{t_0,x_0}\left[\sup_{s\in[t_0,t_1]} \frac{x_s - x_s^{\det}}{\sqrt{\zeta(s)}} > h\right] \\
&\leqslant \sum_{k=0}^{K-1} \frac{3}{2} \cdot \exp\left[-\frac{h^2}{\sigma^2} \cdot \kappa\right] = \frac{3}{2} \cdot \left\lceil \frac{|\bar{\alpha}(t_1,t_0)|}{\varepsilon} \right\rceil \cdot \exp\left[-\frac{h^2}{\sigma^2} \cdot \kappa\right] \\
&\leqslant \frac{3}{2} \cdot \left(\frac{|\bar{\alpha}(t_1,t_0)|}{\varepsilon} + 1\right) \cdot \exp\left[-\frac{h^2}{\sigma^2} \cdot \kappa\right] . \qquad \square
\end{aligned}$$

In the next proposition we prove an upper bound for the probability of the event that x stays near x_+^* (i.e. "no transition") even though σ is not small with respect to $a_0^{3/4} \vee \varepsilon^{3/4}$. We have already seen in the previous subsection that for $\sigma \ll a_0^{3/4} \vee \varepsilon^{3/4}$ a transition is very unlikely and that for bigger σ a transition before a time of order $-\sigma^{2/3}$ is very unlikely. Hence, we consider now the situation where σ is of order $a_0^{3/4} \vee \varepsilon^{3/4}$ or bigger, and we focus on the behaviour of x beginning with a time of order $-\sigma^{2/3}$.

Proposition 4.3.11 (second summand in (4.3.25)). *[BG02b][Prop. 4.6] There exist constants $c_1 > 0$ and $\bar{\kappa} > 0$ such that, if $c_1^{3/2}\sigma \geqslant a_0^{3/4} \vee \varepsilon^{3/4}$ and $h > 2\sigma$, then*

$$\begin{aligned}
&\mathbb{P}^{-c_1\sigma^{2/3},x_0}\left[x_s \in \left]\delta_1, x_s^{\det} + h\sqrt{\zeta(s)}\right] \quad \forall\, s \in [-c_1\sigma^{2/3}, t_1]\right] \\
&\leqslant \frac{3}{2} \cdot \exp\left[-\bar{\kappa} \cdot \frac{1}{\log\frac{h}{\sigma} \vee |\log \sigma^{2/3}|} \cdot \frac{\hat{\alpha}(t_1, -c_1\sigma^{2/3})}{\varepsilon}\right]
\end{aligned}$$

for all $t_1 \in [-c_1\sigma^{2/3}, c_1\sigma^{2/3}]$ and any initial point

$$x_0 \in \left[\delta_1, x^{\det}_{-c_1\sigma^{2/3}} + h\sqrt{\zeta(-c_1\sigma^{2/3})}\right] .$$

Proof. Part 1. Reformulating the problem. We set

$$\hat{z}_t := x_t - \hat{x}_t^{\det} ,$$

where (as in the previous section) $\hat{x}^{\det}$ is the solution of the deterministic equation (4.2.1) that tracks the unstable equilibrium branch x_u^* of f. The process $\hat{z}$ is described by the SDE

$$d\hat{z}_t = \frac{1}{\varepsilon} \cdot \left[\hat{a}(t) \cdot \hat{z}_t + \hat{b}(\hat{z}_t, t)\right] dt + \frac{\sigma}{\sqrt{\varepsilon}}\, dW_t , \tag{4.3.32}$$

where Corollary 4.2.11 and Lemma 4.2.12 show that

$$\hat{a}(t) \asymp |t| \vee \sqrt{a_0} \vee \sqrt{\varepsilon} \asymp \frac{1}{\zeta(t)} ,$$

and the definition of δ_1, δ_2 implies that for all $\hat{z}_t \in [\delta_1 - \hat{x}_t^{\det}, \delta_2 - \hat{x}_t^{\det}]$

$$\hat{b}(\hat{z}_t, t) \leqslant 0 .$$

We introduce a parameter

$$\varrho = \varrho\left(\frac{h}{\sigma}\right) \geqslant 1$$

(to be specified later) and define a partition

$$-c_1\sigma^{2/3} = u_0 < u_1 < \cdots < u_K = t_1$$

of $[-c_1\sigma^{2/3}, t_1]$ by requiring that for all $k = 1, \dots, K-1$

$$\hat{\alpha}(u_k, u_{k-1}) = \varrho\varepsilon ,$$

where we set

$$K := \left\lceil \frac{\hat{\alpha}(t_1, -c_1\sigma^{2/3})}{\varrho\varepsilon} \right\rceil .$$

Furthermore, we define

$$Q_k := \sup\Big\{ \mathbb{P}^{u_k, \hat{z}_{u_k}} \Big[\hat{x}_s^{\det} + \hat{z}_s \in \left]\delta_1, x_s^{\det} + h\sqrt{\zeta(s)}\right] \quad \forall\, s \in [u_k, u_{k+1}] \Big] \\ \Big|\; \hat{z}_{u_k} \text{ such that } \hat{x}_{u_k}^{\det} + \hat{z}_{u_k} \in \left]\delta_1, x_{u_k}^{\det} + h\sqrt{\zeta(u_k)}\right] \Big\}$$

for each $k = 0, \dots, K-1$. Then,

$$\mathbb{P}^{-c_1\sigma^{2/3}, x_0} \Big[x_s \in \left]\delta_1, x_s^{\det} + h\sqrt{\zeta(s)}\right] \quad \forall\, s \in [-c_1\sigma^{2/3}, t_1] \Big] \leqslant \prod_{k=0}^{K-1} Q_k .$$

Our aim is to find a ϱ such that Q_k is bounded away from 1 for all $k = 0, \dots, K-2$.

Part 2. Estimating Q_k. We select an arbitrary $k \in [0, \dots, K-2]$ and define a subdivision of our partition by introducing times $\tilde{u}_{k,1}, \tilde{u}_{k,2}$ such that

$$u_k < \tilde{u}_{k,1} < \tilde{u}_{k,2} < u_{k+1}$$

and

$$\hat{\alpha}(\tilde{u}_{k,1}, u_k) = \frac{1}{3} \cdot \varrho\varepsilon , \quad \hat{\alpha}(\tilde{u}_{k,2}, u_k) = \frac{2}{3} \cdot \varrho\varepsilon ,$$

and stopping times

$$\tau_{k,1} := \left\{ s \in [u_k, \tilde{u}_{k,1}] \;\middle|\; \hat{z}_s \leqslant x_s^{\det} - \hat{x}_s^{\det} \right\} ,$$
$$\tau_{k,2} := \left\{ s \in [u_k, \tilde{u}_{k,2}] \;\middle|\; \hat{z}_s \leqslant 0 \right\} .$$

We note that $\hat{z}_s \leqslant x_s^{\det} - \hat{x}_s^{\det}$ if and only if $x_s \leqslant x_s^{\det}$.

To estimate Q_k, we decompose the underlying event:

$$
\begin{aligned}
&\mathbb{P}^{u_k,\hat{z}_{u_k}}\Big[\hat{x}_s^{\det}+\hat{z}_s \in \,]\delta_1, x_s^{\det}+h\sqrt{\zeta(s)}]\quad \forall\, s\in[u_k,u_{k+1}]\Big] \\
&\quad\leqslant \mathbb{P}^{u_k,\hat{z}_{u_k}}\Big[\hat{x}_s^{\det}+\hat{z}_s \in \,]x_s^{\det}, x_s^{\det}+h\sqrt{\zeta(s)}]\quad \forall\, s\in[u_k,\tilde{u}_{k,1}]\Big] \qquad (4.3.33)\\
&\qquad + \mathbb{E}^{u_k,\hat{z}_{u_k}}\Big[\mathbb{I}_{\{\tau_{k,1}<\tilde{u}_{k,1}\}} \\
&\qquad\qquad \cdot \mathbb{P}^{\tau_{k,1},\hat{z}_{\tau_{k,1}}}\Big[\hat{x}_s^{\det}+\hat{z}_s \in \,]\delta_1, x_s^{\det}+h\sqrt{\zeta(s)}] \\
&\qquad\qquad\qquad \forall\, s\in[\tau_{k,1},u_{k+1}]\Big]\Big]\,.
\end{aligned}
$$

We start by computing an upper bound for the first summand on the r.h.s. That is, we show that x_t is likely to fulfill $x_t < x_t^{\det}$ before $t = \tilde{u}_{k,1}$. Let $z^{(k)}$ be the solution of the linearized equation (4.3.7) with initial condition $z_{u_k}^{(k)} = z_{u_k} = x_{u_k} - x_{u_k}^{\det}$ (cf. also Part 2 in the proof of Proposition 4.3.10). The variance $v_{\tilde{u}_{k,1}}^{(k)}$ of $z_{\tilde{u}_{k,1}}^{(k)}$ fulfills

$$
\begin{aligned}
v_{\tilde{u}_{k,1}}^{(k)} &= \frac{\sigma^2}{\varepsilon}\int_{u_k}^{\tilde{u}_{k,1}} \exp\left[\frac{2\bar{\alpha}(\tilde{u}_{k,1},s)}{\varepsilon}\right] \mathrm{d}s \\
&\geqslant \inf_{s\in[u_k,\tilde{u}_{k,1}]} \frac{-\varepsilon}{2\bar{a}(s)}\cdot\frac{\sigma^2}{\varepsilon}\int_{u_k}^{\tilde{u}_{k,1}} -\frac{2\bar{a}(s)}{\varepsilon}\cdot\exp\left[\frac{2\bar{\alpha}(\tilde{u}_{k,1},s)}{\varepsilon}\right] \mathrm{d}s\,.
\end{aligned}
$$

Via partial integration (notice that $\frac{\mathrm{d}}{\mathrm{d}s}\bar{\alpha}(\tilde{u}_{k,1},s) = -\bar{a}(u)$) we see that

$$
\begin{aligned}
&\int_{u_k}^{\tilde{u}_{k,1}} 1\cdot\left(-\frac{2\bar{a}(s)}{\varepsilon}\right)\cdot\exp\left[\frac{2\bar{\alpha}(\tilde{u}_{k,1},s)}{\varepsilon}\right]\mathrm{d}s \\
&\quad = \left[1\cdot\exp\left[\frac{2\bar{\alpha}(\tilde{u}_{k,1},s)}{\varepsilon}\right]\right]_{u_k}^{\tilde{u}_{k,1}} = 1-\exp\left[\frac{2\bar{\alpha}(\tilde{u}_{k,1},u_k)}{\varepsilon}\right] \\
&\quad \geqslant 1-\exp\left[\frac{2}{3L}\cdot\varrho\right],
\end{aligned}
$$

where we used that by Proposition 4.2.10 and Corollary 4.2.11 there exists a constant $L > 0$ such that we can bound $\bar{\alpha}$ away from zero as follows:

$$
\frac{1}{3}\cdot\varrho\varepsilon = \int_{u_k}^{\tilde{u}_{k,1}} \hat{a}(u)\,\mathrm{d}u \leqslant L\int_{u_k}^{\tilde{u}_{k,1}} -\bar{a}(u)\,\mathrm{d}u = L\cdot\left|\bar{\alpha}(\tilde{u}_{k,1},u_k)\right|,
$$

which implies

$$
\bar{\alpha}(\tilde{u}_{k,1},u_k) \leqslant -\frac{1}{3L}\cdot\varrho\varepsilon\,.
$$

Thus,

$$
v_{\tilde{u}_{k,1}}^{(k)} \geqslant \frac{\sigma^2}{2}\cdot\inf_{s\in[u_k,\tilde{u}_{k,1}]}\frac{1}{|\bar{a}(s)|}\cdot\left(1-\exp\left[-\frac{2}{3L}\cdot\varrho\right]\right).
$$

From this and the assumption that $z_{u_k}^{(k)} \leqslant h\sqrt{\zeta(u_k)}$ (cf. the definition of Q_k) we conclude that

$$
\begin{aligned}
& z_{u_k}^{(k)} \cdot \exp\left[\frac{\bar{\alpha}(\tilde{u}_{k,1}, u_k)}{\varepsilon}\right] \cdot \frac{1}{\big(v_{\tilde{u}_{k,1}}^{(k)}\big)^{1/2}} \\
& \leqslant h\sqrt{\zeta(u_k)} \cdot \exp\left[-\frac{1}{3L} \cdot \varrho\right] \\
& \quad \cdot \sup_{s \in [u_k, \tilde{u}_{k,1}]} \sqrt{|\bar{a}(s)|} \cdot \frac{\sqrt{2}}{\sigma} \cdot \left(1 - \exp\left[-\frac{2}{3L} \cdot \varrho\right]\right)^{-1/2} \\
& =: B_{k,1}(\varrho) > 0 \,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \mathbb{P}^{u_k, x_{u_k}}\left[\hat{x}_s^{\text{det}} + \hat{z}_s \in \left]x_s^{\text{det}}, x_s^{\text{det}} + h\sqrt{\zeta(s)}\right] \ \ \forall\, s \in [u_k, \tilde{u}_{k,1}]\right] \\
& \leqslant \mathbb{P}^{u_k, x_{u_k}}\left[z_s > 0 \ \ \forall\, s \in [u_k, \tilde{u}_{k,1}]\right] \\
& \leqslant \mathbb{P}^{u_k, x_{u_k}}\left[z_s^{(k)} > 0 \ \ \forall\, s \in [u_k, \tilde{u}_{k,1}]\right] \\
& = 1 - 2 \cdot \mathbb{P}^{u_k, x_{u_k}}[z_{\tilde{u}_{k,1}}^{(k)} \leqslant 0] \\
& = 1 - \frac{2}{\sqrt{2\pi}} \cdot \int_{-\infty}^{-z_{u_k}^{(k)} \cdot \exp\left[\frac{\bar{\alpha}(\tilde{u}_{k,1}, u_k)}{\varepsilon}\right] \cdot \frac{1}{(v_{\tilde{u}_{k,1}}^{(k)})^{1/2}}} \exp\left[-\frac{r^2}{2}\right] \mathrm{d}r \\
& \leqslant \frac{2}{\sqrt{2\pi}} \int_0^{B_{k,1}(\varrho)} \exp\left[-\frac{r^2}{2}\right] \mathrm{d}r \\
& \leqslant \frac{2}{\sqrt{2\pi}} \cdot B_{k,1}(\varrho) \,.
\end{aligned}
$$

Now we assume that $\tau_{k,1} < \tilde{u}_{k,1}$ and focus the second summand of (4.3.33). We see that

$$
\begin{aligned}
& \mathbb{P}^{\tau_{k,1}, x_{\tau_{k,1}}^{\text{det}}}\left[\hat{x}_s^{\text{det}} + \hat{z}_s \in \left]\delta_1, x_s^{\text{det}} + h\sqrt{\zeta(s)}\right] \ \ \forall\, s \in [\tau_{k,1}, u_{k+1}]\right] \\
& \leqslant \mathbb{P}^{\tau_{k,1}, x_{\tau_{k,1}}^{\text{det}}}\left[\hat{z}_s \in \left]0, x_s^{\text{det}} - \hat{x}_s^{\text{det}} + h\sqrt{\zeta(s)}\right] \ \ \forall\, s \in [\tau_{k,1}, \tilde{u}_{k,2}]\right] \\
& \quad + \mathbb{E}^{\tau_{k,1}, x_{\tau_{k,1}}^{\text{det}}}\Bigg[\mathbb{I}_{\{\tau_{k,2} < \tilde{u}_{k,2}\}} \\
& \qquad \cdot \mathbb{P}^{\tau_{k,2}, \hat{x}_{\tau_{k,2}}^{\text{det}}}\left[\hat{x}_s^{\text{det}} + \hat{z}_s \in \left]\delta_1, x_s^{\text{det}} + h\sqrt{\zeta(s)}\right] \right. \\
& \qquad\qquad \left. \forall\, s \in [\tau_{k,2}, u_{k+1}]\right]\Bigg] \,.
\end{aligned} \tag{4.3.34}
$$

In the next step we derive an upper bound for the first summand on the right hand side of (4.3.34). Therefore, we consider the process $\hat{z}^{(k)}$, which is described by the linearized version of (4.3.32) with initial condition $\hat{z}_{\tau_{k,1}}^{(k)} =$

$x^{\det}_{\tau_{k,1}} - \hat{x}^{\det}_{\tau_{k,1}}$. The variance $\hat{v}^{(k)}_{\tilde{u}_{k,2}}$ of $\hat{z}^{(k)}_{\tilde{u}_{k,2}}$ fulfills

$$\begin{aligned}
&\exp\left[-\frac{2\hat{\alpha}(\tilde{u}_{k,2},\tau_{k,1})}{\varepsilon}\right]\cdot \hat{v}^{(k)}_{\tilde{u}_{k,2}} = \frac{\sigma^2}{\varepsilon}\int_{\tau_{k,1}}^{\tilde{u}_{k,2}} \exp\left[-\frac{2\hat{\alpha}(s,\tau_{k,1})}{\varepsilon}\right]\,\mathrm{d}s \\
&\geqslant -\inf_{s\in[\tau_{k,1},\tilde{u}_{k,2}]}\frac{\varepsilon}{2\hat{a}(s)}\cdot\frac{\sigma^2}{\varepsilon}\int_{\tau_{k,1}}^{\tilde{u}_{k,2}}(-1)\cdot\frac{2\hat{a}(s)}{\varepsilon}\cdot\exp\left[-\frac{2\hat{\alpha}(s,\tau_{k,1})}{\varepsilon}\right]\,\mathrm{d}s\,.
\end{aligned}$$

By partial integration we obtain that

$$\begin{aligned}
&\int_{\tau_{k,1}}^{\tilde{u}_{k,2}} 1\cdot\left(-\frac{2\hat{a}(s)}{\varepsilon}\right)\cdot\exp\left[-\frac{2\hat{\alpha}(s,\tau_{k,1})}{\varepsilon}\right]\,\mathrm{d}s \\
&= \exp\left[-\frac{2\hat{\alpha}(\tilde{u}_{k,2},\tau_{k,1})}{\varepsilon}\right] - 1 \leqslant \exp\Big[\underbrace{-\frac{2\hat{\alpha}(\tilde{u}_{k,2},\tilde{u}_{k,1})}{\varepsilon}}_{=-\frac{2}{3}\varrho}\Big] - 1\,.
\end{aligned}$$

Thus we get

$$\hat{v}^{(k)}_{\tilde{u}_{k,2}} \geqslant \inf_{s\in[\tau_{k,1},\tilde{u}_{k,2}]}\frac{1}{\hat{a}(s)}\cdot\frac{\sigma^2}{2}\cdot\left(1-\exp\left[-\frac{2}{3}\cdot\varrho\right]\right)\cdot\exp\left[\frac{2\hat{\alpha}(\tilde{u}_{k,2},\tau_{k,1})}{\varepsilon}\right],$$

which implies

$$\begin{aligned}
&\hat{z}^{(k)}_{\tau_{k,1}}\cdot\exp\left[\frac{\hat{\alpha}(\tilde{u}_{k,2},\tau_{k,1})}{\varepsilon}\right]\cdot\left(\hat{v}^{(k)}_{\tilde{u}_{k,2}}\right)^{-1/2} \\
&\leqslant \left(x^{\det}_{\tau_{k,1}} - \hat{x}^{\det}_{\tau_{k,1}}\right)\cdot\frac{\sqrt{2}}{\sigma}\cdot\sup_{s\in[\tau_{k,1},\tilde{u}_{k,2}]}\sqrt{\hat{a}(s)}\cdot\left(1-\exp\left[-\frac{2}{3}\cdot\varrho\right]\right)^{-1/2} \\
&=: B_{k,2}(\varrho) > 0\,.
\end{aligned}$$

This leads to the estimate

$$\begin{aligned}
&\mathbb{P}^{\tau_{k,1},x^{\det}_{\tau_{k,1}}}\left[\hat{z}_s\in\left]0, x^{\det}_s - \hat{x}^{\det}_s + h\sqrt{\zeta(s)}\right]\ \ \forall\, s\in[\tau_{k,1},\tilde{u}_{k,2}]\right] \\
&\leqslant \mathbb{P}^{\tau_{k,1},x^{\det}_{\tau_{k,1}}}\left[\hat{z}^{(k)}_s \geqslant 0\ \ \forall\, s\in[\tau_{k,1},\tilde{u}_{k,2}]\right] \\
&= 1 - 2\mathbb{P}^{\tau_{k,1},x^{\det}_{\tau_{k,1}}}\left[\hat{z}^{(k)}_{\tilde{u}_{k,2}} < 0\right] \\
&= 1 - \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{-\hat{z}^{(k)}_{\tau_{k,1}}\cdot\exp\left[\frac{\hat{\alpha}(\tilde{u}_{k,2},\tau_{k,1})}{\varepsilon}\right]\cdot\frac{1}{(\hat{v}^{(k)}_{\tilde{u}_{k,2}})^{1/2}}}\exp\left[-\frac{r^2}{2}\right]\,\mathrm{d}r \\
&\leqslant \frac{2}{\sqrt{2\pi}}\int_0^{B_{k,2}(\varrho)}\exp\left[-\frac{r^2}{2}\right]\,\mathrm{d}r \\
&\leqslant \frac{2}{\sqrt{2\pi}}\cdot B_{k,2}(\varrho)\,.
\end{aligned}$$

We aim at the second summand in (4.3.34) and assume that $\tau_{k,2} < \tilde{u}_{k,2}$. We use again the process $\hat{z}^{(k)}$, but this time we assume the initial point $\hat{z}_{\tau^{(k)}_{k,2}} = 0$

at the initial time $\tau_{k,2}$. The variance $\hat{v}^{(k)}_{u_{k+1}}$ of this process can be estimated as follows:

$$\begin{aligned}
\exp\left[-\frac{2\hat{\alpha}(u_{k+1},\tau_{k,2})}{\varepsilon}\right]\cdot\hat{v}^{(k)}_{u_{k+1}} &= \frac{\sigma^2}{\varepsilon}\int_{\tau_{k,2}}^{u_{k+1}}\exp\left[-\frac{2\hat{\alpha}(s,\tau_{k,2})}{\varepsilon}\right]\mathrm{d}s\\
&\geqslant -\inf_{s\in[\tau_{k,2},u_{k+1}]}\frac{\varepsilon}{2\hat{a}(s)}\cdot\frac{\sigma^2}{\varepsilon}\int_{\tau_{k,2}}^{u_{k+1}}(-1)\cdot\frac{2\hat{a}(s)}{\varepsilon}\cdot\exp\left[-\frac{2\hat{\alpha}(s,\tau_{k,2})}{\varepsilon}\right]\mathrm{d}s\,.
\end{aligned}$$

Similar to the above estimate, we obtain again by partial integration

$$\int_{\tau_{k,2}}^{u_{k+1}}1\cdot\left(-\frac{2\hat{a}(s)}{\varepsilon}\right)\cdot\exp\left[-\frac{2\hat{\alpha}(s,\tau_{k,2})}{\varepsilon}\right]\mathrm{d}s\leqslant\exp\Bigg[\underbrace{-\frac{2\hat{\alpha}(u_{k+1},\tilde{u}_{k,2})}{\varepsilon}}_{=-\frac{2}{3}\varrho}\Bigg]-1$$

and see thus that

$$\hat{v}^{(k)}_{u_{k+1}}\geqslant\inf_{s\in[\tau_{k,2},u_{k+1}]}\frac{1}{\hat{a}(s)}\cdot\frac{\sigma^2}{2}\cdot\left(1-\exp\left[-\frac{2}{3}\cdot\varrho\right]\right)\cdot\exp\Bigg[\underbrace{\frac{2\hat{\alpha}(u_{k+1},\tau_{k,2})}{\varepsilon}}_{\geqslant\frac{2}{3}\varrho}\Bigg]\,,$$

which implies

$$\begin{aligned}
&\left(\hat{x}^{\det}_{u_{k+1}}-\delta_1\right)\cdot\frac{1}{\left(\hat{v}^{(k)}_{u_{k+1}}\right)^{1/2}}\\
&\leqslant\left(\hat{x}^{\det}_{u_{k+1}}-\delta_1\right)\\
&\quad\cdot\sup_{s\in[\tau_{k,2},u_{k+1}]}\sqrt{\hat{a}(s)}\cdot\frac{\sqrt{2}}{\sigma}\cdot\left(1-\exp\left[-\frac{2}{3}\cdot\varrho\right]\right)^{-1/2}\cdot\exp\left[-\frac{1}{3}\cdot\varrho\right]\\
&=: B_{k,3}(\varrho)>0\,.
\end{aligned}$$

Hence, we see that

$$\begin{aligned}
&\mathbb{P}^{\tau_{k,2},\hat{x}^{\det}_{\tau_{k,2}}}\left[\hat{x}^{\det}_s+\hat{z}_s\in\left]\delta_1,x^{\det}_s+h\sqrt{\zeta(s)}\right]\ \forall\,s\in[\tau_{k,2},u_{k+1}]\right]\\
&\leqslant\mathbb{P}^{\tau_{k,2},\hat{x}^{\det}_{\tau_{k,2}}}\left[\hat{z}^{(k)}_s>\delta_1-\hat{x}^{\det}_s\ \ \forall\,s\in[\tau_{k,2},u_{k+1}]\right]\\
&\leqslant\mathbb{P}^{\tau_{k,2},\hat{x}^{\det}_{\tau_{k,2}}}\left[\hat{z}^{(k)}_{u_{k+1}}>\delta_1-\hat{x}^{\det}_{u_{k+1}}\right]\\
&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-(\delta_1-\hat{x}^{\det}_{u_{k+1}})\cdot\frac{1}{(\hat{v}^{(k)}_{u_{k+1}})^{1/2}}}\exp\left[-\frac{r^2}{2}\right]\mathrm{d}r\\
&\leqslant\frac{1}{2}+\frac{1}{\sqrt{2\pi}}\int_0^{B_{k,3}(\varrho)}\exp\left[-\frac{r^2}{2}\right]\mathrm{d}r\\
&\leqslant\frac{1}{2}+\frac{1}{\sqrt{2\pi}}\cdot B_{k,3}(\varrho)\,.
\end{aligned}$$

Combining these results, we see that

$$Q_k\leqslant\frac{1}{2}+\frac{1}{\sqrt{2\pi}}\cdot\left(2B_{k,1}(\varrho)+2B_{k,2}(\varrho)+B_{k,3}(\varrho)\right)\,.$$

As $\varrho \geqslant 1$ and $L > 0$, we know that there exists a constant $C > 0$ such that

$$Q_k \leqslant \frac{1}{2} + \frac{C}{\sigma} \cdot \sup_{s \in [u_k, u_{k+1}]} \sqrt{\hat{a}(s)} \cdot \left(h\sqrt{\zeta(u_k)} \cdot \exp\left[-\frac{1}{3L} \cdot \varrho \right] + \sup_{s \in [u_k, u_{k+1}]} (x_s^{\det} - \hat{x}_s^{\det}) + (\hat{x}_{u_{k+1}}^{\det} - \delta_1) \cdot \exp\left[-\frac{1}{3} \cdot \varrho \right] \right) .$$

Remember that by assumption we consider $t_1 \in [-c_1\sigma^{2/3}, c_1\sigma^{2/3}]$. Thus, there exist constants $C_a, C_\zeta, C_{a\zeta}, C_x, C_\delta > 0$ such that for all $s \in [-c_1\sigma^{2/3}, c_1\sigma^{2/3}]$ and all $k = 0, \dots, K-2$

$$\begin{aligned}
\sqrt{\hat{a}(s)} \quad &\leqslant C_a \cdot c_1^{1/2}\sigma^{1/3} \\
\sqrt{\zeta(s)} \quad &\leqslant C_\zeta \cdot \frac{1}{a_0^{1/4} \vee \varepsilon^{1/4}} \\
\Rightarrow \quad \sqrt{\hat{a}(s)} \cdot \sqrt{\zeta(s)} &\leqslant C_a C_\zeta \cdot \frac{c_1^{1/2}\sigma^{1/3}}{a_0^{1/4} \vee \varepsilon^{1/4}} \leqslant C_{a\zeta} \\
x_s^{\det} - \hat{x}_s^{\det} &\leqslant C_x \cdot c_1 \sigma^{2/3} \\
\Rightarrow \quad \sqrt{\hat{a}(s)} \cdot (x_s^{\det} - \hat{x}_s^{\det}) &\leqslant C_a C_x \cdot c_1^{3/2}\sigma \\
\hat{x}_{u_{k+1}}^{\det} - \delta_1 \quad &\leqslant C_\delta \, .
\end{aligned}$$

We choose $C_1 := C_{a\zeta} \vee (C_a C_x) \vee C_\delta$ and obtain that for all $k = 0, \dots, K-2$

$$Q_k \leqslant \frac{1}{2} + C_1 \cdot \left(\frac{h}{\sigma} \cdot \exp\left[-\frac{1}{3L} \cdot \varrho \right] + c_1^{3/2} + \frac{\sqrt{c_1}}{\sigma^{2/3}} \cdot \exp\left[-\frac{1}{3} \cdot \varrho \right] \right)$$

Finally, we specify $c_1^{3/2} := \frac{1}{18C_1}$ and

$$\varrho := 1 \vee 3L \cdot \log\left(18C_1 \cdot \frac{h}{\sigma} \right) \vee 3 \cdot \log\left(18C_1 \cdot \frac{\sqrt{c_1}}{\sigma^{2/3}} \right) .$$

This implies that for any $k = 0, \dots, K-2$ we have $Q_k \leqslant \frac{2}{3}$. Using that $Q_{K-1} \leqslant 1$, we get that

$$\prod_{k=0}^{K-1} Q_k \leqslant \left(\frac{2}{3} \right)^{K-1} = \frac{3}{2} \cdot \frac{1}{\left(\frac{3}{2}\right)^K} \leqslant \frac{3}{2} \cdot \exp\left[-\log\left(\frac{3}{2} \right) \cdot \frac{\hat{\alpha}(t_1, -c_1\sigma^{2/3})}{\varrho\varepsilon} \right] .$$

By selection of ϱ, we have that there exists a constant $\bar{\kappa} > 0$ such that

$$\log\left(\frac{3}{2} \right) \cdot \frac{1}{\varrho} \geqslant \bar{\kappa} \cdot \frac{1}{\log \frac{h}{\sigma} \vee -\log \sigma^{2/3}} ,$$

which proves the assertion. □

Finally, we consider the last summand on the right hand side of (4.3.25). We only state the proposition, because it is completely proved in [BG02b].

Proposition 4.3.12 (third summand in (4.3.25))**.** *[BG02b, Prop. 4.7] Let* $\varrho \in]0, \delta_1 - \delta_0]$ *and* $f_0 > 0$ *be constants such that*

$$f(x,t) \leqslant -f_0 < 0 \quad \text{for all } x \in [\delta_0, \delta_1 + \varrho] \text{ and } t \in [-T, T] \, .$$

Then we have for all $t_0 \in [-T, T - c\varepsilon]$ *that*

$$\mathbb{P}^{t_0,\delta_1}\left[x_s > \delta_0 \ \ \forall\, s \in [t_0, t_0 + c\varepsilon]\right] \leqslant \exp\left[-\frac{\tilde{\kappa}}{\sigma^2}\right] ,$$

where $\tilde{\kappa} = \frac{f_0 \varrho^2}{4(\delta_1 - \delta_0)}$ *and* $c = (\delta_1 - \delta_0)\frac{2}{f_0}$

A. Appendix: Differential Inequalities

Estimates based on differential inequalities are among the basic tools used in the Chapter 4. Here we present the necessary results.

The following theorem is based on [Fle80, Theorem 2.6.1]; the version presented there is more general than we need it here.

Theorem A.0.1. *Consider $[a,b] \subset \mathbb{R}$, let $p, q : [a,b] \to \mathbb{R}$ be continuous, and let $\phi : [a,b] \to \mathbb{R}$ be a differentiable function such that for all $t \in [a,b]$ the following holds*

$$\phi'(t) \leqslant p(t)\phi(t) + q(t) \ . \tag{A.0.1}$$

Then ϕ does not exceed the solution of the linear equation $x' = p(t)x + q(t)$ with initial condition $\phi(a)$, i.e. for all $t \in [a,b]$ we have that

$$\phi(t) \leqslant \phi(a) \cdot \exp\left[\int_a^t p(v)\ \mathrm{d}v\right] + \int_a^t q(u) \cdot \exp\left[\int_u^t p(v)\ \mathrm{d}v\right] \mathrm{d}u \ . \tag{A.0.2}$$

Proof. We set

$$r(t) := \exp\left[-\int_a^t p(v)\ \mathrm{d}v\right] ,$$

hence, $r(t) > 0$ for all $t \in [a,b]$ and r is continuously differentiable for all such t. Then for all $t \in [a,b]$ the following holds:

$$\begin{aligned}\big(\phi(t) \cdot r(t)\big)' &= r(t) \cdot \phi'(t) + \phi(t) \cdot r'(t) = r(t) \cdot \phi'(t) - \phi(t) \cdot r(t)p(t) \\ &\leqslant r(t) \cdot \big(p(t)\phi(t) + q(t)\big) - \phi(t) \cdot r(t)p(t) = r(t) \cdot q(t) \ .\end{aligned}$$

Thus we obtain by the fundamental theorem of calculus that for all $t \in [a,b]$

$$\phi(t)r(t) - \phi(a) \leqslant \int_a^t q(u)r(u)\ \mathrm{d}u \ .$$

This proves the assertion. □

This result especially implies that for ϕ, p as above the relation

$$\phi'(t) \leqslant p(t)\phi(t)$$

together with the initial condition $\phi(a) \leqslant 0$ implies that $\phi(t) \leqslant 0$ for all $t \in [a,b]$ and all continuous functions p.

Corollary A.0.2. *In the situation of the theorem, we replace* (A.0.1) *by*

$$\phi'(t) \geqslant p(t)\phi(t) + q(t) \ .$$

Then the same proof as for the theorem shows that

$$\phi(t) \geqslant \phi(a) \cdot \exp\left[\int_a^t p(v)\ \mathrm{d}v\right] + \int_a^t q(u) \cdot \exp\left[\int_u^t p(v)\ \mathrm{d}v\right] \mathrm{d}u \ .$$

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