Stochastic Evolution Equations with Lévy Noise and Applications to the Heath-Jarrow-Morton Model

Diplomarbeit
vorgelegt von Kristian Knäble
im Studiengang Wirtschaftsmathematik

Fakultäten für
Mathematik und Wirtschaftswissenschaften
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Chapter 0

Introduction

In this diploma thesis we will solve a stochastic partial differential equation with Lipschitz coefficients where the noise is a Hilbert space-valued Lévy process. Moreover we will apply our results to the Heath-Jarrow-Morton interest rate model from mathematical finance.

At first we want to present the framework in which we treat the problem: We consider the following type of stochastic differential equation on a separable (infinite dimensional) Hilbert space H

$$\begin{cases} df(t) = (Af(t) + a(t, f(t)))\ dt + \sigma(t, f(t))\ dL(t),\ t \in [0, T] \\ f(0) = \xi \end{cases} \tag{1}$$

where $L(t)$, $t \in [0, T]$, is a Lévy process taking values in a separable Hilbert space G. A is the (possibly unbounded) generator of a strongly continuous semigroup $S(t)$, $t \geq 0$, of linear operators on H. The drift coefficient $a : [0, T] \times \Omega \times H \to H$ and the noise coefficient $\sigma : [0, T] \times \Omega \times H \to L_2(G, H)$(:=space of Hilbert-Schmidt operators from G into H) are measurable mappings. $\xi \in L^2(\Omega, \mathcal{F}, P; H)$ is a random initial value. We impose the following condition on L

$$\int_{\{\|x\| \geq 1\}} \|x\|^2\ \nu(dx) < \infty, \tag{2}$$

where ν is the corresponding Lévy measure governing the jumps of the Lévy process. Thereby $L(t)$ is in $L^2(\Omega, \mathcal{F}, P; G)$ at any time $0 \leq t \leq T$.
A mild solution of problem (1) is a predictable process $f(t)$, $t \in [0, T]$, such that

$$f(t) = S(t)\xi + \int_0^t S(t-s)a(s, f(s))\, ds + \int_0^t S(t-s)\sigma(s, f(s))\, dL(s) \quad P\text{-a.s.}$$

As far as we know there are no results on the equation (1) on infinite-dimensional Hilbert spaces with state-dependent noise coefficient and a general Lévy process as integrator. We will prove the existence of a unique mild solution to this equation under Lipschitz conditions on a and σ, which is the central result of this diploma thesis.

We start with some comments on the history of the problem, while at the same time trying to motivate our proceedings.

In order to give sense to the mild solution we have to define the stochastic integral with respect to a Hilbert space-valued Lévy process. We apply the Lévy-Itô decomposition from Albeverio and Rüdiger [AlRü 05]. Then the definition of the integral w.r.t. the Brownian motion part of the Lévy process is taken from [DaPrZa 92] (cf. *Appendix A*). The integral w.r.t. to the jump part is constructed as a stochastic integral w.r.t. a Hilbert space-valued martingale measure following Applebaum [App]. A martingale measure (called martingale-valued measure in [App]) is a mapping $M : [0, t] \times S \times \Omega \to G$ where $[0, t]$ is a time interval, S a Lusin topological space and $(\Omega, \mathcal{F}, P)$ a probability space. Basically, it is a Hilbert space-valued martingale in the time component and a σ-finite measure in the S-component. In the real-valued case ($G = \mathbb{R}$) this concept was introduced by Walsh [Wal 86] in order to treat stochastic partial differential equations. Our main example is the Lévy martingale measure formed by the jump part of the Lévy process which admits many desirable properties.

Métivier [Met 77] defines the stochastic integral for a wide class of càdlàg semimartingales as integrators. In *Appendix B* we give a detailed review of his construction in the case of square-integrable martingales. One can as well use this approach to define the stochastic integrals, but the construction suggested by Applebaum [App] turns out to be much more useful when we want to derive existence (and uniqueness) results for equation (1). In both cases the integrals are defined as L^2-limits of the integrals of simple processes approximating the integrands in a suitable L^2-space.

A different approach is to define the stochastic integral as a limit in distribution. This was carried out by Chojnowska-Michalik [C-M 87] for deterministic integrands and Lévy processes as integrators.

In the case that L is a Brownian motion equation (1) was examined by Da Prato and Zabczyk [DaPrZa 92] and solved via a fixed-point argument. The first to consider a Hilbert space-valued Lévy process as integrator was Chojnowska-Michalik [C-M 87]. For $a \equiv 0$ and σ the identity she constructed (based on her integration theory) what she termed a mild solution, but what is in fact even a weak one. Applebaum [App] obtained the same result for a constant noise coefficient using different methods including martingale measure integration. This was later generalized by S. Stolze [Sto 05] to allow for a Lipschitz non-linearity. C. Knoche [Kno 03] proved the existence of a mild solution for a state-dependent noise coefficient and L a compensated Poisson random measure.

We will now give an overview of the contents of the different chapters and point out our contributions. Additional information may be found at the beginning of every chapter.

In *Chapter 1* we will introduce Lévy processes on separable Hilbert spaces. Refering to Albeverio and Rüdiger [AlRü 05] we define the Poisson random measure and the compensated Poisson random measure corresponding to a Lévy process. We also quote the main result from their paper: the Lévy-Itô decomposition in separable Hilbert spaces (shown by them for separable Banach spaces). Next we give the definition of a Hilbert space-valued martingale measure taken from [App]. As our main example we use a slight modification of the Lévy martingale measure discussed in [Sto 05]. Finally we introduce the stochastic integral with respect to a special class of martingale measures (the Levy one among them). Basically we carry out the construction from [App] with some complements from [Sto 05]. We add a slight generalization by considering as integrands mappings which take values in a certain class of linear operators from one Hilbert space into another (maybe different) one.

Chapter 2 is devoted to the study of equation (1). As a fundamentally new result we show the existence of a mild solution using the methods of [DaPrZa 92] and some complements to their approach worked out by K. Frieler and C. Knoche [FriKno 01]. Hereby condition (2) on the Lévy process allows us to treat the jump term of the stochastic integral with basically the same techniques as the Brownian motion term. The solution is then found as a fixed-point of the contraction $f \mapsto \gamma(f)$ defined by

$$\gamma(f)(t) = S(t)\xi + \int_0^t S(t-s)a(s, f(s))\, ds + \int_0^t S(t-s)\sigma(s, f(s))\, dL(s).$$

on a suitable Banach space of processes. Existence and uniqueness are a consequence of Banach's fixed-point theorem. This main result of the diploma thesis is stated in Theorem 2.1, while its proof covers most of Chapter 2.

In recent years Lévy processes have played an important role in finance. While traditionally Brownian motion is used as a source of randomness, many newer models use Lévy processes to allow for jumps (which may be interpreted as external shocks) and get a better fit to empirical data. For an overview of the numerous applications of Lévy processes to finance consult the book of Schoutens [Sch 03] and the references therein.

We will concentrate on the Heath-Jarrow-Morton interest rate model in *Chapter 3*. It was introduced by Heath, Jarrow and Morton [HJM 92] to model the dynamics of bond prices (enabling them to price bond options) via the evolution of forward interest rates as Itô processes. They stated the famous HJM drift condition which guarantees an arbitrage-free movement of the bond prices. By a change of parametrization Musiela [Mus 93] transformed the model to the framework of stochastic evolution equations on an infinite-dimensional function space. Filipović [Fil 01] gives a rigorous treatment of this approach and generalizes the model to allow for a state-dependent volatility structure and an infinite-dimensional driving Brownian motion. We will present a summary of this development and then make the transition to Lévy noise. The equation for the forward rates then reads

$$\begin{cases} df_t = (\frac{\partial}{\partial x} f_t + \alpha(t, f_t))\ dt + \sigma(t, f_t)\ dL(t) \\ f_0 = f(0, \cdot) \in H \end{cases}$$

where H is a suitable Hilbert space of real-valued functions on $[0, \infty]$. Among the first to consider non-Gaussian noise are Björk et al. [BDKR 97] who add a compensated as well as a non-compensated Poisson random measure part. Eberlein and Raible [EbeRai 99] suggest a model with a (finite-dimensional) Lévy process, while Raible [Rai 00] presents strong empirical evidence for the use of Lévy noise instead of pure Gaussian one. Finally, Jakubowski and Zabczyk [JaZa 04] and Özkan and Schmidt [ÖzkSch 05] work with HJM models driven by infinite-dimensional Lévy processes and develop the corresponding HJM-type drift conditions. We will extend these models by considering state-dependent volatility coefficients which give more flexibility in modeling. Here the results from Chapter 2 are essential in deriving the existence of an HJM model with Lévy noise in Proposition 3.2.

I wish to thank Prof. Dr. Michael Röckner who led me to the study of stochastic differential equations with Lévy noise. His lectures which I have attended since my first semester were an excellent guide to modern mathematics, especially stochastic calculus. Moreover, I am grateful for support and helpful proposals in connection with this diploma thesis.

I would also like to thank Dr. Walter Hoh for giving a report on this thesis. Special thanks go to my brother, Florian Knäble.

Chapter 1

Lévy Processes and Stochastic Integration

Lévy processes on a separable Hilbert space G are introduced in section 1.1 as stochastic processes with independent and stationary increments. To a Lévy process L we can assign the *Poisson random measure* $N(t, dx)$ which is for any $A \in \mathcal{B}(G - \{0\})$ with $0 \notin \bar{A}$ given by

$$N(t, A) := |\{0 < s \leq t : \Delta L(s) \in A\}| = \sum_{0<s\leq t} 1_A(\Delta L(s))$$

with $\Delta L(s)$ the "jump" of the process L at time s. Centralization then gives the *compensated Poisson random measure* $\tilde{N}(t, dx)$. From [AlRü 05] we take the *Lévy-Itô decomposition* which states that any Lévy process can be written as the sum of a deterministic drift, a Brownian motion, an integral with respect to the compensated Poisson random measure and an integral with respect to the Poisson random measure. We show that the last term may be expressed in the first and third one, if the expected number of "big jumps" decreases sufficiently quickly. In section 1.2 we give the definition of *martingale measures* in the Hilbert space case developed in [App]. In the real-valued case this concept is due to [Wal 86]. Basically, a martingale measure is a G-valued set function depending on a time component, a Borel set and a random component, which for a fixed set is a martingale and locally a measure in the set component. Our main example is the *Lévy martingale measure* M given by

$$M(t, A) := \int_{A-\{0\}} x\, \tilde{N}(t, dx),\ t \geq 0,$$

for $A \in \mathcal{B}(G - \{0\})$ with $0 \notin \bar{A}$. It is a slight modification of the one mentioned in [App] and discussed in [Sto 05]. M is a *nuclear* martingale measure, i.e. its martingale component has a covariance operator which is positive, self-adjoint and of trace class. Since the third part of the Lévy-Itô decomposition is described by M, we want to construct the stochastic integral with respect to nuclear martingale measures. This is done in section 1.3 following the approach of [App], later carried out in detail in [Sto 05]. (There the integral is called strong stochastic integral.) We make a slight generalization by considering integrands that take values in the linear operators from one Hilbert space into another one. As usual the stochastic integral is first defined for simple functions via an isometry. These simple functions are dense in a space called $\mathcal{N}^2(T)$ to which the integral can be extended by L^2-limits.

1.1 Lévy processes in Hilbert spaces

Let be $(G, (\ ,\ )_G)$ a separable Hilbert space; $(\Omega, \mathcal{F}, P)$ a complete probability space with $(\mathcal{F}_t)$, $t \geq 0$, a right-continuous filtration on $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}_0$ contains all P-nullsets.

Definition 1.1 *Fix $T > 0$. A subset $A \subset [0,T] \times \Omega$ of the form $A =]s,t] \times F$ where $F \in \mathcal{F}_s$, $0 \leq s < t \leq T$, or $\{0\} \times F$, $F \in \mathcal{F}_0$, is called* predictable rectangle. *The family of predictable rectangles is denoted by $\mathcal{R}_T$.*

Let be $\mathcal{P}_T = \sigma(\mathcal{R}_T)$, the σ-algebra generated by $\mathcal{R}_T$. $\mathcal{P}_T$ is called the σ-algebra of the predictable sets; *a stochastic process X measurable with respect to $\mathcal{P}_T$ is called* predictable.

For E a Banach space $\mathcal{B}(E)$ denotes the Borel σ-algebra *on E, i.e. the σ-algebra generated by all open subsets of E.*

Definition 1.2 *A G-valued stochastic process L adapted to $(\mathcal{F}_t)$, $t \geq 0$, is a* Lévy process *if*

- *$L(0) = 0$*
- *L has increments independent of the past, i.e. $L(t) - L(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t < \infty$*
- *L has stationary increments, i.e. $L(t) - L(s)$ has the same distribution as $L(t-s)$ for all $0 \leq s < t < \infty$*
- *L is stochastically continuous, i.e. for all $t \geq 0$ and $\epsilon > 0$ holds*

$$\lim_{s \to t} P(\|L(s) - L(t)\|_G > \epsilon) = 0$$

- *L has strongly càdlàg paths, i.e. the paths of L are right-continuous and always have left limits w.r.t. the strong topology.*

Examples of Lévy processes are Q-Brownian motion (see App. A), or (for $G = \mathbb{R}$) the Poisson process and standard Brownian motion.

A σ-finite measure ν on $G - \{0\}$ is called a *Lévy measure* if

$$\int_{G-\{0\}} (\|x\|^2 \wedge 1)\, \nu(dx) < \infty.$$

(An alternative convention is to define the Lévy measure on the whole of G via the assignment $\nu(\{0\}) = 0$.)

Let be $\Delta L(s) := L(s) - L(s-),\ s > 0$ the "jump" of L at time t. We say that a Lévy process has *bounded jumps* if there exists a constant $J > 0$ with

$$\sup_{t \geq 0} \|\Delta L(t)\| < J.$$

Define for $A \in \mathcal{B}(G - \{0\})$ with $0 \notin \bar{A}$ and $t > 0$

$$N(t, A) := |\{0 < s \leq t : \Delta L(s) \in A\}| = \sum_{0<s\leq t} 1_A(\Delta L(s)). \tag{1.1}$$

N admits the following properties:

Proposition 1.1 *1. For $A \in \mathcal{B}(G - \{0\})$ with $0 \notin \bar{A}$ fixed the process $(N(t, A)),\ t \geq 0$, is a Poisson process.*

2. For $t \geq 0$ and $\omega \in \Omega$ fixed $N(t, \cdot)(\omega)$ is a set function from $\{A \in \mathcal{B}(G - \{0\}) : 0 \notin \bar{A}\}$ to $\mathbb{R}_+ \cup \{+\infty\}$. For P-a.a. $\omega \in \Omega$ there exists a unique σ-finite measure on $\mathcal{B}(G - \{0\})$ extending this set function. We denote this measure by $N(t, dx)$.

3. Set $\tilde{\nu}(A) := E[N(1, A)]$ for $A \in \mathcal{B}(G - \{0\})$ with $0 \notin \bar{A}$. Then $\tilde{\nu}$ has a unique extension to a σ-finite measure ν on $\mathcal{B}(G - \{0\})$. Moreover, ν is a Lévy measure.

Proof.

1. (cf. [AlRü 05] Thm. 2.7, even for G a separable Banach space)
2. (cf. [AlRü 05] Thm. 2.13, Cor. 2.14)
3. (cf. [AlRü 05] Thm. 2.17, Cor. 2.18)

We follow the convention of [AlRü 05] and call N a *Poisson random measure*.

If $\tilde{N}$ denotes the compensated Poisson random measure, i.e. $\tilde{N}(dt, dx) := N(dt, dx) - dt \otimes \nu(dx)$, we have the following result:

Proposition 1.2 *Let $f \in L^2(G, \mathcal{B}(G), \nu; G)$. Then for any $t \geq 0$ and $A \in \mathcal{B}(G - \{0\})$ the integral*

$$\int_A f(x)\,\tilde{N}(t,dx)$$

exists and

$$E[\|\int_A f(x)\,\tilde{N}(t,dx)\|^2] = t\int_A \|f(x)\|^2\,\nu(dx) < \infty.$$

Proof. (cf. [AlRü 05] Thm. 3.25)

Now we can formulate a very important representation for Lévy processes which is given by the following theorem:

Theorem 1.1 (Lévy-Itô decomposition) *For any G-valued Lévy process $L = (L(t))_{t\geq 0}$ there exist $b \in G$, a Brownian motion $(B_Q(t))_{t\geq 0}$ with covariance operator Q, independent of $N(\cdot, A)$ for any $A \in \mathcal{B}(G - \{0\})$ with $0 \notin \bar{A}$, such that:*

$$L(t) = tb + B_Q(t) + \int_{\{\|x\|<1\}} x\ \tilde{N}(t,dx) + \int_{\{\|x\|\geq 1\}} x\ N(t,dx).$$

Here the Poisson random measure N and the Lévy measure ν are defined as in Prop. 1.1.
The triple (b, Q, ν) is called the characteristics *of the process L.*

Proof. (cf. [AlRü 05] Thm. 4.1)
The first integral is well-defined, since $id_{\{\|x\|<1\}} \in L^2(G, \mathcal{B}(G), \nu; G)$ (cf. Prop. 1.2). Moreover, seen as a process in t it is a square-integrable martingale (see [Sto 05] Lem. 2.4.8).

The càdlàg property of the Lévy process guarantees that for a given $\omega \in \Omega$ on any finite interval $[0,T]$ there are only finitely many jumps with norm ≥ 1. Otherwise we could find an accumulation point $\bar{t} \in [0,T]$ where $t \mapsto L(t)(\omega)$ would not have a left limit. Hence we can write the second integral just as a random finite sum in G:

$$\int_{\{\|x\|\geq 1\}} x\ N(t,dx) = \sum_{0<s\leq t} \Delta L(s) 1_{\{\|x\|\geq 1\}}(\Delta L(s)).$$

Thus this term is of finite variation on any $[0,T]$.

Remark 1.1 *If we denote the distribution of $L(t)$ by μ_t, then μ_t is infinitely divisible. The famous* Lévy-Khintchine formula *gives us the characteristic function of $L(t)$ as $E[\exp(i(u, L(t))_G] = \exp(-t\phi(u))$, where*

$$\phi(u) := -i(u,b)_G + \frac{1}{2}(Qu,u)_G - \int_G \exp(i(u,x)_G) - 1 - i(u,x)_G 1_{\{\|x\|<1\}}(x)\,\nu(dx))$$

with (b, Q, ν) *the characteristics of* L *from Theorem 1.1.*

Remark 1.2 $(L(t))_{t\in[0,T]}$ *can be written in the form* $L(t) = M(t)+V(t)$, $t \in [0,T]$*, where* M *is a square-integrable martingale and* V *is a càdlàg process with bounded variation. Here*

$$M(t) := B_Q(t) + \int_{\{\|x\|<1\}} x\ \tilde{N}(t, dx) \text{ and}$$

$$V(t) := tb + \int_{\{\|x\|\geq 1\}} x\ N(t, dx).$$

Hence L *is a semimartingale of the type discussed at the end of Appendix B. So we could use the construction derived there to define the stochastic integral of operator-valued stochastic processes w.r.t. to a general Lévy process. Unfortunately, the isometry developed in App. B is too general to be of much help in Chapter 2 where we want to study stochastic equations with Lévy noise. Therefore we will introduce a construction that makes use of the special structure of the* $\tilde{N}(t, dx)$*-term.*

We say that a Lévy process fulfills condition (F) if for the corresponding Lévy measure holds

$$\int_{\{\|x\|\geq 1\}} \|x\|^2\, \nu(dx) < \infty. \tag{1.2}$$

Note that via the definition of Lévy measure (F) actually yields $\int_G \|x\|^2\, \nu(dx) < \infty$. Bounded jumps of L are sufficient for (F), since then we have

$$\int_{\{\|x\|\geq 1\}} \|x\|^2\, \nu(dx) \leq J^2 \int_{\{\|x\|\geq 1\}} 1\, \nu(dx) < \infty.$$

Lemma 1.1 *If* L *is a Lévy process with characteristics* (b, Q, ν) *fulfilling (F), the Lévy-Itô decomposition of* L *can be written in the following way:*

$$L(t) = tm + B_Q(t) + \int_G x\ \tilde{N}(t, dx) \tag{1.3}$$

where $m := b + \int_{\{\|x\|\geq 1\}} x\, \nu(dx)$.

Proof. We have

$$\int_{\{\|x\|\geq 1\}} x\ N(t, dx) = \int_{\{\|x\|\geq 1\}} x\ \tilde{N}(t, dx) + t \int_{\{\|x\|\geq 1\}} x\, \nu(dx).$$

The integral w.r.t. the compensated Poisson measure exists, since thanks to (F) $id_{\{\|x\|\geq 1\}} \in L^2(G, \mathcal{B}(G), \nu; G)$ (see again Prop. 1.2). (F) also ensures that m has finite norm:

$$\|m\| \leq \|b\| + \|\int_{\{\|x\|\geq 1\}} x\, \nu(dx)\| \leq \|b\| + \int_{\{\|x\|\geq 1\}} \|x\|^2\, \nu(dx) < \infty.$$

□

1.2 Martingale measures

Define $S := G,\ \Sigma := \mathcal{B}(S),\ \mathcal{A}_0 := \{A \in \Sigma : 0 \notin \bar{A}\},\ \mathcal{A} := \mathcal{A}_0 \cup \{A \cup \{0\} : A \in \mathcal{A}_0\},\ S_n := \{x \in S : \frac{1}{n} \leq \|x\|\},\ \Sigma_n := \mathcal{B}(S_n)$, then $S = \bigcup_{n\in\mathbb{N}} S_n$.

Definition 1.3 *A martingale measure is a set function $M : \mathbb{R}_+ \times \mathcal{A} \times \Omega \to G$ with the following properties:*
$M(0, A) = M(t, \emptyset) = 0$ a.s. for all $A \in \mathcal{A},\ t \geq 0$. For $t > 0$ $M(t, \cdot)$ is

1. *finitely additive, i.e. $M(t, A \cup B) = M(t, A) + M(t, B)$ a.s. for all $A, B \in \mathcal{A}$ disjoint*
2. *σ-finite, i.e. $\sup\{E[\|M(t, A)\|_G^2 | A \in \Sigma_n] < \infty\}$ for all $n \in \mathbb{N}$*
3. *countably additive on each Σ_n, $n \in \mathbb{N}$, i.e. for any sequence decreasing to the the empty set $(A_j) \subset \Sigma_n$ we have $\lim_{j\to\infty} E[\|M(t, A_j)\|_G^2 = 0$*

For each $A \in \mathcal{A}$ the process $(M(t, A))_{t\geq 0}$ is a strongly càdlàg square-integrable martingale. Finally the zero set in 1. is independent of t.

A martingale measure M is called *orthogonal* if for any disjoint $A, B \in \mathcal{A}$ and any orthonormal base (e_n) of G the process

$$((M(t, A), e_n)_G \cdot (M(t, B), e_m)_G)_{t\geq 0}$$

is a (real-valued) martingale for all $m, n \in \mathbb{N}$. In particular the process $((M(t, A), M(t, B))_G)_{t\geq 0}$ is a martingale.

M has *independent increments* if $M((s, t], A)$ is independent of $\mathcal{F}_s$ for all $A \in \mathcal{A},\ 0 \leq s < t < \infty$. Here $M((s, t], A) := M(t, A) - M(s, A)$.

Let $T = (T_A, A \in \mathcal{A})$ be a family of bounded non-negative self-adjoint operators on G. T is a *positive-operator valued (POV) measure* on (S, Σ) if

1. $T_\emptyset = 0$
2. $T_{A\cup B} = T_A + T_B$ for all $A, B \in \mathcal{A}$ disjoint.

T is *trace class* if every $T_A, A \in A$, is trace class.

T is called *decomposable* if there exist a σ-finite measure μ on (S, Σ) and a family $(T_x, x \in S)$ of bounded non-negative self-adjoint operators on G s.t. $x \mapsto T_x y$ is measurable for all $y \in G$ and

$$T_A y = \int_A T_x y \, \mu(dx)$$

for all $A \in \mathcal{A}$ and $y \in G$.

M is *nuclear* with (T, ρ) if for all $0 \leq s < t < \infty,\ A \in \mathcal{A},\ x, y \in G$

$$E[(M((s,t],A),x)_G(M((s,t],A),y)_G] = (x, T_A y)_G \rho((s,t])$$

where $T = (T_A, A \in \mathcal{A})$ is a POV measure which is trace class and ρ is a Radon measure on $(0, \infty)$. If T is decomposable we call M *decomposable*.

Our key example (and motivation) for these concepts is as follows:

Theorem 1.2 *Consider a Lévy process fulfilling (F) with Lévy-Itô decomposition (1.3). Then M defined by*

$$M(t,A) := \int_{A-\{0\}} x\, \tilde{N}(t,dx),\ t \geq 0,\ A \in \mathcal{A},$$

is an orthogonal martingale measure with independent increments. We call it a Lévy martingale measure

Proof. (cf. [Sto 05] Thm. 2.5.2)

Proposition 1.3 *The Lévy martingale measure M is nuclear with (T, dt) where dt denotes Lebesgue measure on $\mathbb{R}_+$ and $T = \{T_A;\ A \in \mathcal{A}\}$ with*

$$T_A y = \int_{A-\{0\}} (x,y)_G x\, \nu(dx).$$

In particular T is decomposable with ν and

$$T_x = (x, \cdot)_G x.$$

Proof. (cf. [Sto 05] Prop. 2.5.4)

In [Sto 05] these results are in fact proved for $S = \{\|x\| < 1\}$ instead of $S = G$. But the crucial point is to have the inequality $\int_S \|x\|^2\, \nu(dx) < \infty$, which we obtain from condition (F)! Of course, for $S = \{\|x\| < 1\}$ this is automatically fulfilled for any Lévy process.

For later use we calculate

$$\begin{aligned} \|T_x^{\frac{1}{2}}\|^2_{L_2(G)} &= tr(T_x) = \sum_{n\in\mathbb{N}} (T_x e_n, e_n) \qquad (1.4)\\ &= \sum_{n\in\mathbb{N}} ((x,e_n)x, e_n) = \sum_{n\in\mathbb{N}} (x,e_n)^2 = \|x\|^2_G \end{aligned}$$

where (e_n), $n \in \mathbb{N}$, is an orthonormal basis of G.

1.3 Stochastic integrals

Our aim is to define the stochastic integral $\int_0^t \int_S R(s,x)\, M(ds,dx)$ for an operator-valued process R. M is an orthogonal martingale measure with independent increments, nuclear and decomposable. Keeping in mind the Lévy martingale measure we take $S = G$ and $\rho = dt$ Lebesgue measure. However, note that the construction works for S a Lusin topological space (i.e. a continuous one-to-one image of a Polish space) and ρ a Radon measure on $(0,\infty)$.

We consider again separable Hilbert spaces G and H with orthonormal bases (e_n), $n \in \mathbb{N}$, resp. (f_n), $n \in \mathbb{N}$. Let $L(G,H)$ denote the space of all linear bounded operators from G to H with operator norm $\|R\| := \sup_{\|g\|_G \leq 1} \|Rg\|_H$. Then $(L(G,H), \|\cdot\|)$ is a Banach space (see [ReSi 80] Thm. III.2.). Since the *norm topology* generated by $\|\cdot\|$ is too strong for our purposes, we consider the *strong topology* on $L(G,H)$ instead:

$$R_n \xrightarrow{s} R \ \text{ iff } \ R_n g \to Rg \text{ for all } g \in G.$$

The corresponding Borel σ-algebra $\mathcal{L}$ is generated by sets of the form

$$\{R \in L(G,H); Rg \in A\} \text{ with } g \in G, A \in \mathcal{B}(H).$$

(cf. [DaPrZa 96], p.24ff.).
Moreover we need the space $L_2(G,H)$ of *Hilbert-Schmidt operators*. An operator R is Hilbert-Schmidt if $tr(R^*R) < \infty$. The space $L_2(G,H)$ with inner product $(R_1,R_2)_{L_2} := tr(R_1^* R_2)$ and induced norm $\|R\|_{L_2} = tr(R^*R)^{\frac{1}{2}}$ is a separable Hilbert space and a two-sided $L(G,H)$-ideal, i.e. for $R \in L_2(G,H)$, $C_1 \in L(G)$, $C_2 \in L(H)$ we have

$$\|C_2 R C_1\|_{L_2} \leq \|C_2\| \|C_1\| \|R\|_{L_2}$$

(cf. [Wei 80] p.138). $L_2(G,H)$ is a strongly measurable subset of $L(G,H)$ (cf. [DaPrZa 96], p.25).

Definition 1.4 *Let be $\mathcal{N}^2(T) = \mathcal{N}^2(T;\nu,dt)$ the space of all mappings X on $[0,\tilde{T}] \times S \times \Omega$ taking values in the linear (possibly unbounded) operators from G into H, such that*

1. *For any $g \in G$ the H-valued mapping $(t,x) \mapsto X(t,x)g$ is $\mathcal{P}_{\tilde{T}} \otimes \mathcal{B}(S)$-measurable.*

2. *For any $(t,x,\omega) \in [0,\tilde{T}] \times S \times \Omega$ $X(t,x)(\omega) \circ T_x^{\frac{1}{2}}$ is a Hilbert-Schmidt operator and we have*

$$\|X\|_{\mathcal{N}^2(T)} := (E[\int_0^{\tilde{T}} \int_S \|X(s,x) T_x^{\frac{1}{2}}\|_{L_2}^2 \, \nu(dx)\, ds])^{\frac{1}{2}} < \infty.$$

([App] and [Sto 05] treat the case $G = H$.)

Lemma 1.2 *(cf. [Sto 05], Lemma 3.1.1)*
The mapping

$$(X,Y) \mapsto E[\int_0^{\tilde{T}} \int_S tr(X(t,x)T_x Y(t,x)^*) \ \nu(dx)\, dt]$$

is an inner product in $\mathcal{N}^2(T)$ and with respect to this inner product $\mathcal{N}^2(T)$ is a Hilbert space.

Proof. Inner product is obvious. Now let (R_n) be a Cauchy sequence in $\mathcal{N}^2(T)$. By the Riesz-Fischer Theorem $(R_n T_x^{\frac{1}{2}})$ converges to some S in the space $L^2(([0,\tilde{T}] \times \Omega \times S, \mathcal{P}_{\tilde{T}} \otimes \mathcal{B}(S), dt \otimes P \otimes \nu); (L_2(G,H), \mathcal{B}(L_2(G,H)))$. Hence there exists a subsequence (R_{n_k}) such that

$$\lim_{k\to\infty} R_{n_k}(t,\omega,x)T_x^{\frac{1}{2}} = S(t,\omega,x) \ \text{ in } \|\cdot\|_{L_2(G,H)} \ dt \otimes P \otimes \nu\text{-a.s.}$$

For (t,ω,x) fixed choose an orthonormal basis (e_m) of G with each e_m either in $ker(T_x^{\frac{1}{2}})$ or in its orthogonal complement $(ker(T_x^{\frac{1}{2}}))^\perp$. Next we define

$$R(t,\omega,x)g := \begin{cases} S(t,\omega,x)(T_x^{\frac{1}{2}})^{-1}g & \text{if } g \in T_x^{\frac{1}{2}}(G) \\ 0 & \text{if } g \in (T_x^{\frac{1}{2}}(G))^\perp \end{cases},$$

where

$$(T_x^{\frac{1}{2}})^{-1} : T_x^{\frac{1}{2}}((ker(T_x^{\frac{1}{2}}))^\perp) = T_x^{\frac{1}{2}}(G) \to (ker(T_x^{\frac{1}{2}}))^\perp$$

is the pseudo-inverse of $T_x^{\frac{1}{2}}$. To finish the proof we show that $(R_{n_k}(t,\omega,x)T_x^{\frac{1}{2}})$ converges to $R(t,\omega,x)T_x^{\frac{1}{2}}$ in $\|\cdot\|_{L_2(G,H)}$ $dt \otimes P \otimes \nu$-a.s. which implies convergence of (R_n) to R in $\mathcal{N}^2(T)$:

$$\begin{aligned}
& \|(R_{n_k}(t,\omega,x) - R(t,\omega,x))T_x^{\frac{1}{2}}\|^2_{L_2(G,H)} \\
= & \sum_{m=1}^{\infty} \|(R_{n_k}(t,\omega,x) - R(t,\omega,x))T_x^{\frac{1}{2}} e_m\|_H^2 \\
= & \sum_{m=1}^{\infty} \|R_{n_k}(t,\omega,x)T_x^{\frac{1}{2}} e_m - S(t,\omega,x)(T_x^{\frac{1}{2}})^{-1} T_x^{\frac{1}{2}} e_m\|_H^2 \\
= & \sum_{e_m \in ker(T_x^{\frac{1}{2}}))^\perp} \|R_{n_k}(t,\omega,x)T_x^{\frac{1}{2}} e_m - S(t,\omega,x)e_m\|_H^2 \\
\leq & \sum_{m=1}^{\infty} \|(R_{n_k}(t,\omega,x)T_x^{\frac{1}{2}} - S(t,\omega,x))e_m\|_H^2
\end{aligned}$$

$$= \|R_{n_k}(t,\omega,x)T_x^{\frac{1}{2}} - S(t,\omega,x)\|^2_{L_2(G,H)}.$$

□

As usual the construction of the integral is started by considering simple functions. We denote by $\mathcal{S}^2(T) := \mathcal{S}^2(T;\nu,dt)$ the subspace of all $R \in \mathcal{N}^2(T)$ which have the following form:

$$R = \sum_{i=0}^{M}\sum_{j=0}^{N} R_{ij} 1_{(t_i,t_{i+1}]} 1_{A_j}, \tag{1.5}$$

where $M, N \in \mathbb{N}$ and $0 = t_0 < t_1 < \ldots < t_{M+1} = \tilde{T}$. The $A_0, \ldots, A_{N+1} \in \mathcal{A}$ are disjoint sets (having finite ν-measure!) and each R_{ij} is an $\mathcal{F}_{t_i}/\mathcal{L}$-measurable random variable with values in $L(G,H)$ (equivalently $R_{ij}g$ is $\mathcal{F}_{t_i}$-measurable for any $g \in G$).

Lemma 1.3 *(cf. [Sto 05], Lemma 3.1.2)*
The subspace $\mathcal{S}^2(T)$ is dense in $\mathcal{N}^2(T)$.

Proof. We have to show $\mathcal{S}^2(T)^\perp = \{0\}$ (where $\mathcal{S}^2(T)^\perp$ is the orthogonal complement of $\mathcal{S}^2(T)$ in $\mathcal{N}^2(T)$). Consider for $k, l \in \mathbb{N}$ the operators $S_{kl} \in L(G,H)$ and $U_{lk} \in L(H,G)$ defined by

$$S_{kl}e_n = \begin{cases} f_l & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases},$$

$$U_{lk}f_m = \begin{cases} e_k & \text{if } m = l \\ 0 & \text{if } m \neq l \end{cases}.$$

It is easy to see that the adjoint of S_{kl} is $S^*_{kl} = U_{lk}$. Note that

$$tr(S_{kl}T_xU_{lk}) \leq tr(T_x)$$

and hence the mapping with constant value S_{kl} is an element of $\mathcal{N}^2(T)$. Consider the simple function $S \in \mathcal{S}^2(T)$ defined by

$$S(s,\omega,x) = 1_B(s)1_F(\omega)1_A(x)S_{kl},$$

where $B = (t_1, t_2]$ with $t_1, t_2 \in [0, \tilde{T}]$, $A \in \mathcal{A}$, $F \in \mathcal{F}_{t_1}$. Then for arbitrary $R \in \mathcal{S}^2(T)^\perp$ we have $(R,S)_{\mathcal{N}^2(T)} = 0$. Hence

$$\begin{aligned} & E[\int_0^{\tilde{T}}\int_S tr(R(t,x)T_xS(t,x)^*)\ \nu(dx)\,dt] \\ = {} & E[1_F\int_B\int_A\sum_{n=1}^{\infty}(R(t,x)T_xU_{lk}f_n, f_n)_H\ \nu(dx)\,dt] \\ = {} & E[1_F\int_B\int_A(R(t,x)T_xe_k, f_l)_H\ \nu(dx)\,dt] = 0. \end{aligned}$$

Now define a signed measure μ on $\mathcal{P}_{\tilde{T}} \otimes \mathcal{B}(S)$ by setting

$$\mu(G) := \int_G (R(t,x)T_x e_k, f_l)_H \; dt\, dP\, \nu(dx).$$

For any $G \in \mathcal{P}_{\tilde{T}} \otimes \mathcal{B}(S)$ of the type $B \times F \times A$ we have $\mu(G) = 0$. Since the system of such sets is closed against intersections and generates the σ-algebra $\mathcal{P}_{\tilde{T}} \otimes \mathcal{B}(S)$ we can conclude that $\mu = 0$ on $\mathcal{P}_{\tilde{T}} \otimes \mathcal{B}(S)$. Thus $(R(t,x)T_x e_k, f_l)_H = 0 \; dt \otimes P \otimes \nu$-a.e. for any $k, l \in \mathbb{N}$ and therefore $R(t,x)T_x = 0 \; dt \otimes P \otimes \nu$-a.e. But then

$$\begin{aligned} \|R\|^2_{\mathcal{N}^2(T)} &= E[\int_0^{\tilde{T}} \int_S \|R(t,x)T_x^{\frac{1}{2}}\|^2_{L_2} \, \nu(dx)\, dt] \\ &= E[\int_0^{\tilde{T}} \int_S tr(R(t,x)T_x R(t,x)^*) \, \nu(dx)\, dt] = 0 \end{aligned}$$

and we obtain $\mathcal{S}^2(T)^\perp = \{0\}$.

□

For $t \in [0, \tilde{T}]$ and every simple function $R \in \mathcal{S}^2(T)$ (cf. (1.5)) we define

$$J_t(R) := \sum_{i=0}^{M} \sum_{j=0}^{N} R_{ij} M((t \wedge t_i, t \wedge t_{i+1}], A_j). \tag{1.6}$$

It is easy to see that $J_t(R)$ does not depend on the representation of R.

Proposition 1.4 *(cf. [Sto 05], Prop. 3.1.3/[App], p.11/12)*
J_t, given by (1.6) for every $R \in \mathcal{S}^2(T)$, can be extended to an isometry from $\mathcal{N}^2(T;t)$ to $L^2(\Omega, \mathcal{F}, P; H)$.

Proof. Let $R \in \mathcal{S}^2(T)$ be a simple function as in (1.5). Since

$$E[\|J_t(R)\|^2] = E[\| \sum_{i=0}^{M} \sum_{j=0}^{N} R_{ij} M((t \wedge t_i, t \wedge t_{i+1}], A_j)\|^2],$$

we study the individual terms under the sum. By the martingale property of $M(\cdot, A_l)$ we obtain for $i < k$ such that $t_i, t_k < t$:

$$\begin{aligned} & E[(R_{ij} M((t_i, t \wedge t_{i+1}], A_j), R_{kl} M((t_k, t \wedge t_{k+1}], A_l))_H] \\ = \; & E[(R^*_{kl} R_{ij} M((t_i, t \wedge t_{i+1}], A_j), E[M((t_k, t \wedge t_{k+1}], A_l)|\mathcal{F}_{t_k}])_G] \\ = \; & 0. \end{aligned}$$

Let be $j \neq l$ and $N_{ij} := M((t_i, t \wedge t_{i+1}], A_j)$. Then $(N_{ij}, e_n)_G (e_m, N_{il})_G$ is a martingale for every $n, m \in \mathbb{N}$, because M is orthogonal. Moreover

$$\begin{aligned}
& E[(R_{ij} M(t_i, t \wedge t_{i+1}], A_j), R_{kl} M((t_k, t \wedge t_{k+1}], A_l))_H] \\
= \; & E[\sum_{n=1}^{\infty} (R_{ij} N_{ij}, f_n)_H (f_n, R_{il} N_{il})_H] \\
= \; & E[\sum_{n=1}^{\infty} (N_{ij}, R_{ij}^* f_n)_G (R_{il}^* f_n, N_{il})_G] \\
= \; & \sum_{n,m,r=1}^{\infty} E[(N_{ij}, e_m)_G (e_m, R_{ij}^* f_n)_G (R_{il}^* f_n, e_r)_G (e_r, N_{il})_G] \\
= \; & \sum_{n,m,r=1}^{\infty} E[e_m, R_{ij}^* f_n)_G (R_{il}^* f_n, e_r)_G E[(N_{ij}, e_m)_G (e_r, N_{il})_G | \mathcal{F}_{t_i}]] \\
= \; & 0.
\end{aligned}$$

Since M has independent increments and is nuclear we can conclude

$$\begin{aligned}
& E[\|R_{ij} M(t_i, t \wedge t_{i+1}], A_j)\|^2] \\
= \; & \sum_{n,m,r=1}^{\infty} E[(e_m, R_{ij}^* f_n)_G (R_{ij}^* f_n, e_r)_G] E[(N_{ij}, e_m)_G (e_r, N_{ij})_G] \\
= \; & \sum_{n,m,r=1}^{\infty} E[(e_m, R_{ij}^* f_n)_G (R_{ij}^* f_n, e_r)_G] (e_r, T_{A_j} e_m)_G ((t \wedge t_{i+1}) - t_i) \\
= \; & \sum_{m=1}^{\infty} E[(R_{ij} e_m, R_{ij} T_{A_j} e_m)_H] ((t \wedge t_{i+1}) - t_i) \\
= \; & E[tr(R_{ij}^* R_{ij} T_{A_j})] ((t \wedge t_{i+1}) - t_i).
\end{aligned}$$

And for any $A \in \mathcal{A}$ and some operator $Q \in L(G)$ we have

$$\begin{aligned}
tr(Q T_A) &= \sum_{n=1}^{\infty} (e_n, Q T_A e_n)_G \\
&= \sum_{n=1}^{\infty} (Q^* e_n, \int_A T_x e_n \, \nu(dx))_G = \int_A tr(Q T_x) \, \nu(dx).
\end{aligned}$$

Finally using the calculations from above we get

$$\begin{aligned}
E[\|J_t(R)\|^2] &= E[\|\sum_{i=0}^{M}\sum_{j=0}^{N} R_{ij}M(t\wedge t_i, t\wedge t_{i+1}], A_j)\|^2] \\
&= \sum_{i=0}^{M}\sum_{j=0}^{N} E[\|R_{ij}M(t\wedge t_i, t\wedge t_{i+1}], A_j)\|^2] \\
&= \sum_{i=0}^{M}\sum_{j=0}^{N} E[tr(R_{ij}^* R_{ij} T_{A_j})]((t\wedge t_{i+1}) - (t\wedge t_i)) \\
&= \sum_{i=0}^{M}\sum_{j=0}^{N} E[\int_{A_j} tr(R_{ij}^* R_{ij} T_x)\nu(dx)]((t\wedge t_{i+1}) - (t\wedge t_i)) \\
&= E[\int_0^{\tilde{T}}\int_S \|1_{(0,t]}(s)R(s,x)T_x^{\frac{1}{2}}\|^2_{L_2(G,H)}\nu(dx)ds] \\
&= \|R_{|(0,t]\times\Omega\times S}\|^2_{\mathcal{N}^2(T;t)}.
\end{aligned}$$

For general $R \in \mathcal{N}^2(T)$ we approximate R by a sequence $(R_n) \subset \mathcal{S}^2(T)$ (cf. Lemma 1.3). Hence

$$\lim_{n,m\to\infty} E[\|J_t(R_n) - J_t(R_m)\|^2] = \lim_{n,m\to\infty} \|(R_n - R_m)_{|(0,t]\times\Omega\times S}\|^2_{\mathcal{N}^2(T;t)} = 0.$$

Thus $(J_t(R_n))$ is a Cauchy sequence in the Hilbert space $L^2(\Omega, \mathcal{F}, P; H)$ and we can define $J_t(R)$ as the L^2-limit of $J_t(R_n)$.

□

For any $R \in \mathcal{N}^2(T)$ we define the *(strong) stochastic integral* of R with respect to the orthogonal and nuclear martingale measure M by

$$\int_0^t\int_S R(s,x)M(ds,dx) := J_t(R)$$

for $t \in [0, \tilde{T}]$.

Proposition 1.5 *(cf. [Sto 05], Thm. 3.1.5)*
The process $(\int_0^t\int_S R(s,x)\, M(dx,ds))_{t\geq 0}$ is an H-valued strongly càdlàg square-integrable martingale. Furthermore,

$$\begin{aligned}
& E[\int_0^t\int_S \|R(s,x)T_x^{\frac{1}{2}}\|^2_{L_2(G,H)}\,\nu(dx)\,ds] \qquad (1.7)\\
= \; & E[\|\int_0^t\int_S R(s,x)\,M(dx,ds)\|^2_H],
\end{aligned}$$

where $t \in [0, \tilde{T}]$.

Proof. Consider a simple function $R \in \mathcal{S}^2(T)$ given by (1.5). Take $r \leq t$ and set $i_0 := \max\{i : t_i \leq r\}$. Without loss of generality we can assume $t_M < t$ and obtain

$$\begin{aligned}
& E[\int_0^t \int_S R(s,x) M(ds,dx) | \mathcal{F}_r] \\
= \;& \sum_{i=0}^{M} \sum_{j=0}^{N} E[R_{ij} M((t_i, t \wedge t_{i+1}], A_j) | \mathcal{F}_r] \\
= \;& \sum_{i=0}^{M} \sum_{j=0}^{N} \sum_{n=1}^{\infty} E[(R_{ij} M((t_i, t \wedge t_{i+1}], A_j), f_n)_H f_n | \mathcal{F}_r] \\
= \;& \sum_{i=0}^{M} \sum_{j=0}^{N} \sum_{n=1}^{\infty} E[(M((t_i, t \wedge t_{i+1}], A_j), R^*_{ij} f_n)_G | \mathcal{F}_r] f_n \\
= \;& \sum_{j=0}^{N} \sum_{i=0}^{i_0-1} \sum_{n=1}^{\infty} (M((t_i, t_{i+1}], A_j), R^*_{ij} f_n)_G f_n \\
+ \;& \sum_{j=0}^{N} \sum_{n=1}^{\infty} (E[M((t_{i_0}, t \wedge t_{i_0+1}], A_j) | \mathcal{F}_r], R^*_{i_0 j} f_n)_G f_n \\
+ \;& \sum_{i=i_0+1}^{M} \sum_{j=0}^{N} \sum_{n=1}^{\infty} E[(E[(M((t_i, t \wedge t_{i+1}], A_j) | \mathcal{F}_{t_i}], R^*_{ij} f_n)_G | \mathcal{F}_r] f_n \\
= \;& \sum_{j=0}^{N} (\sum_{i=0}^{i_0-1} R_{ij} M((t_i, t_{i+1}], A_j) + R_{i_0 j} M((t_{i_0}, r], A_j)) \\
= \;& \int_0^r \int_S R(s,x) \, M(dx, ds).
\end{aligned}$$

Thus we have proved the martingale property for simple functions. As before the result can be extended to any $R \in \mathcal{N}^2(T)$ via the isometry from Proposition 1.4: R can be written as the limit of some sequence $(R_n) \subset \mathcal{S}^2(T)$ in $\mathcal{N}^2(T)$. Hence the corresponding stochastic integral can be expressed as an L^2-limit of martingales which makes it a martingale again.
Proposition 3 in [Kun 70] states that every square-integrable martingale is automatically strongly càdlàg P-a.s.

$\square$

Chapter 2

Stochastic Equations with Lévy Noise

As before let be $(G, (\ ,\ )_G)$ and $(H, (\ ,\ )_H)$ separable Hilbert spaces. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with $(\mathcal{F}_t)$, $t \geq 0$, a right-continuous filtration on $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}_0$ contains all P-nullsets.
We fix a G-valued Lévy process L with characteristics (b, Q, ν). For $T > 0$ we consider the following type of stochastic equation with Lévy noise and state space H

$$\begin{cases} df(t) = (Af(t) + a(t, f(t)))\ dt + \sigma(t, f(t))\ dL(t),\ t \in [0, T] \\ f(0) = \xi \end{cases} \tag{2.1}$$

where

- $\xi \in L^2(\Omega, \mathcal{F}_0, P, H)$ is a given (possibly stochastic) initial condition
- $A : D(A) \to H$ is the infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0}$ of linear operators on H
- a is a measurable function from $([0, T] \times \Omega \times H, \mathcal{P}_T \otimes \mathcal{B}(H))$ into $(H, \mathcal{B}(H))$
- σ is a measurable function from $([0, T] \times \Omega \times H, \mathcal{P}_T \otimes \mathcal{B}(H))$ into $(L_2(G, H), \mathcal{B}(L_2(G, H))$.

It is well-known (see [Paz 83]) that there exist constants $\omega \geq 0$ and $M \geq 1$ such that $\|S(t)\|_{L(H)} \leq Me^{\omega t}$, $t \geq 0$. Hence we find

$$M_T := \sup_{t\in[0,T]} \|S(t)\|_{L(H)} < \infty.$$

Definition 2.1 *An H-valued predictable process $f(t),\ t \in [0,T]$, is called a mild solution of equation (2.1) if*

$$f(t) = S(t)\xi + \int_0^t S(t-s)a(s,f(s))\,ds + \int_0^t S(t-s)\sigma(s,f(s))\,dL(s)$$

P-a.s. for all $t \in [0,T]$.

In this chapter we show existence and uniqueness of the mild solution to problem (2.1) under Lipschitz conditions on a and σ by a fixed-point argument. In [DaPrZa 92] this is done for L an infinite dimensional Brownian motion. In order to use the isometry from Chapter 1, we have to impose the following condition on the Lévy measure of L

$$\int_{\{\|x\|\geq 1\}} \|x\|^2\,\nu(dx) < \infty.$$

[App] deals with the case that L is an H-valued Lévy process, $a(\cdot) \equiv 0$ and $\sigma(\cdot) \equiv C \in L(H)$. In [Sto 05] this is generalized to allow for a Lipschitz drift. There the jumps of L are required to fulfill (see p.70)

$$\sup_{t\in[0,T]} \|\Delta L(t)\| \in L^{2+\epsilon}(\Omega, \mathcal{F}, P) \text{ for some } \epsilon > 0. \tag{2.2}$$

Lemma 2.1 *(2.2) implies $\int_{\{\|x\|\geq 1\}} \|x\|^2\,\nu(dx) < \infty$.*

Proof. Set $\tilde{J} := \sup_{t\in[0,T]} \|\Delta L(t)\|$. Then using the definition of $\nu(dx)$ from Prop. 1.1 and Fubini's theorem we get

$$\begin{aligned}
\int_{\{\|x\|\geq 1\}} \|x\|^2\,\nu(dx) &= \int_{\{\|x\|\geq 1\}} \|x\|^2\,E[N(1,dx)] \\
&= E[\int_{\{\|x\|\geq 1\}} \|x\|^2\,N(1,dx)] \\
&= E[\sum_{0<s\leq 1} 1_{\{\|x\|\geq 1\}}(\Delta L(s))\,\|\Delta L(s)\|^2] \\
&\leq E[\tilde{J}^2 \cdot N(1,\{\|x\|\geq 1\})] < \infty.
\end{aligned}$$

Here the last inequality follows from Hölder's inequality, because $\tilde{J}^2 \in L^{1+\frac{\epsilon}{2}}(P)$ for some $\epsilon > 0$ by (2.2), and $N(1,\{\|x\|\geq 1\})$ is Poisson-distributed with intensity $\nu(\{\|x\|\geq 1\})$ and thus in all $L^p(P),\ 1 \leq p < \infty$.

$\square$

Moreover, in Appendix C we present an example of a Lévy process fulfilling our assumption, but not the one stated in [Sto 05]. Hence our condition is a strictly weaker one! Note that bounded jumps of L are sufficient for (2.2).

2.1 Existence of the mild solution

To prove the existence (and uniqueness) of a mild solution on $[0,T]$ we make the following **Assumptions**:

- $a : [0,T] \times \Omega \times H \to H$ is Lipschitz continuous in the third variable, i.e. there exists a constant $Lip_a > 0$ such that

  $$\|a(s,\omega,h_1) - a(s,\omega,h_2)\|_H \leq Lip_a \|h_1 - h_2\|_H$$

  for all $h_1, h_2 \in H$, $s \in [0,T]$, $\omega \in \Omega$.

- $\sigma : [0,T] \times \Omega \times H \to L_2(G,H)$ is Lipschitz continuous in the third variable, i.e. there exists a constant $Lip_\sigma > 0$ such that

  $$\|\sigma(s,\omega,h_1) - \sigma(s,\omega,h_2)\|_{L_2} \leq Lip_\sigma \|h_1 - h_2\|_H$$

  for all $h_1, h_2 \in H$, $s \in [0,T]$, $\omega \in \Omega$.

- There is a constant $C > 0$ with

  $$\sup_{(s,\omega)} \|a(s,\omega,0)\|_H \leq C \quad \text{and} \quad \sup_{(s,\omega)} \|\sigma(s,\omega,0)\|_{L_2} \leq C.$$

- L fulfills the condition (F), i.e.

  $$\int_{\{\|x\| \geq 1\}} \|x\|^2 \, \nu(dx) < \infty. \tag{2.3}$$

  We set $\int_G \|x\|^2 \, \nu(dx) =: C_\nu < \infty$.

Remark 2.1 *(Linear growth)*
The Lipschitz constant Lip_a can be chosen in such a way that

$$\|a(s,\omega,h)\|_H \leq Lip_a(1 + \|h\|_H)$$

for all $h \in H$, $s \in [0,T]$, $\omega \in \Omega$. The same applies to Lip_σ respectively.

Proof. For all $h \in H$

$$\begin{aligned} & \|a(s,\omega,h)\|_H \leq \|a(s,\omega,h) - a(s,\omega,0)\| + \|a(s,\omega,0)\| \\ \leq \quad & Lip_a \|h\| + \sup_{(s,\omega)} \|a(s,\omega,0)\| \leq (Lip_a \vee C\|) \, (1 + \|h\|). \end{aligned}$$

And of course we still have for all $h_1, h_2 \in H$

$$\|a(s,\omega,h_1) - a(s,\omega,h_2)\|_H \leq (Lip_a \vee \; C\|) \, \|h_1 - h_2\|_H.$$

The same argument works for Lip_σ.

Remark 2.2 *The assumption for L gives us (cf. equation (1.4)):*

$$\int_G \|T_x^{\frac{1}{2}}\|_{L_2}^2 \,\nu(dx) = \int_G \|x\|^2 \,\nu(dx) = C_\nu < \infty.$$

Next we define the space where we want to find the mild solution of the above stochastic differential equation. A process $Y : [0,T] \times \Omega \to H$ is called *H-predictable*, if it is $\mathcal{P}_T/\mathcal{B}(H)$-measurable. We set

$$\mathcal{H}^2(T,H) := \{Y(t), t \in [0,T] | Y \text{ is an } H\text{-predictable process s.t.}$$
$$\sup_{t\in[0,T]} E[\|Y(t)\|^2] < \infty\}$$

and for $Y \in \mathcal{H}^2(T,H)$

$$\|Y\|_{\mathcal{H}^2} := \sup_{t\in[0,T]} (E[\|Y(t)\|^2])^{\frac{1}{2}}.$$

Then $(\mathcal{H}^2(T,H), \|\;\|_{\mathcal{H}^2})$ is a Banach space.

Theorem 2.1 *Assume that a, σ and L fulfill the conditions stated above. Then for every initial condition $\xi \in L^2(\Omega, \mathcal{F}_0, P, H)$ there exists a unique mild solution $f(t)$, $t \in [0,T]$, of equation (2.1). Moreover, the solution is continuous as a mapping from $[0,T]$ to $L^2(\Omega, \mathcal{F}, P; H)$.*

The proof of the theorem uses the following lemmas.

Lemma 2.2 *If f is a predictable H-valued process and σ and $S(t)$, $t \geq 0$, are as above, then the mapping*

$$(s,\omega) \mapsto 1_{(0,t]}(s) S(t-s)\sigma(f(s,\omega))$$

is $\mathcal{P}_T/\mathcal{B}(L_2(G,H))$-measurable for all $t \in [0,T]$.

Proof. (cf. [FrKn 01], Lemma 3.6, p.69)

Lemma 2.3 *Let Φ be a process on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,T]})$ with values in a Banach space E. If Φ is adapted to $(\mathcal{F}_t)_{t\in[0,T]})$, and stochastically continuous then there exists a predictable version of Φ.*

Proof. ([DaPrZa 92], Proposition 3.6 (ii), p.76)

Lemma 2.4 *Let Φ be a predictable H-valued process which is P-a.s. Bochner integrable. Then the process given by*

$$(\int_0^t S(t-s)\Phi(s)\,ds)_{t\in[0,T]}$$

is P-a.s. continuous and adapted to $(\mathcal{F}_t)_{t\in[0,T]}$. This especially implies that it is predictable.

Proof. ([FrKn 01], Lemma 3.9., p.70)

Lemma 2.5 *Let $(x_{n,m})_{m\in\mathbb{N}}$, $n \in \mathbb{N}$, be sequences of real numbers such that for each $n \in \mathbb{N}$ there exists $x_n \in \mathbb{R}$ with*

$$x_{n,m} \longrightarrow x_n \text{ as } m \to \infty.$$

If there exists a further sequence $(y_n)_{n\in\mathbb{N}}$ such that $|x_{n,m}| \leq y_n\ \forall\, m \in \mathbb{N}$ and $\sum_{n\in\mathbb{N}} y_n < \infty$ then

$$\sum_{n\in\mathbb{N}} x_{n,m} \longrightarrow \sum_{n\in\mathbb{N}} x_n \text{ as } m \to \infty.$$

Proof. The claim follows by Lebesgue's dominated convergence theorem with respect to the measure $\mu := \sum_{n\in\mathbb{N}} \delta_n$.

□

Lemma 2.6 *Let $(\Omega, \mathcal{F})$ be a measurable space. Let E be a metric space with metric d and $f : \Omega \to E$ strongly measurable. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of E-valued simple functions (i.e. f_n is $\mathcal{F}/\mathcal{B}(E)$-measurable and takes only finitely many values) such that for arbitrary $\omega \in \Omega$ the sequence $d(f(\omega), f_n(\omega))$, $n \in \mathbb{N}$, is monotonely decreasing to zero.*

Proof. ([DaPrZa 92], Lemma 1.1)

Proof of Theorem 2.1
Let $t \in [0,T]$, $\xi \in L^2(\Omega, \mathcal{F}_0, P, H)$ and $f \in \mathcal{H}^2(T, H)$. We define

$$\gamma(f)(t) := S(t)\xi + \int_0^t S(t-s)a(s, f(s))\,ds + \int_0^t S(t-s)\sigma(s, f(s))\,dL(s).$$

Thus by Definition 2.1 a mild solution of problem (2.1) with initial condition $\xi \in L^2(\Omega, \mathcal{F}_0, P, H)$ is an H-predictable process such that $\gamma(f)(t) = f(t)$ P-a.s. for all $t \in [0,T]$. Hence we have to look for a fixed-point of γ, i.e. an f such that $\gamma(f) = f$ in $\mathcal{H}^2(T, H)$.
Therefore we show that γ is a well-defined mapping from $\mathcal{H}^2(T,H)$ to $\mathcal{H}^2(T,H)$ which also is a strict contraction. That means there exists $C < 1$ such that for all $f_1, f_2 \in \mathcal{H}^2(T,H)$

$$\|\gamma(f_1) - \gamma(f_2)\|_{\mathcal{H}^2} \leq C\|f_1 - f_2\|_{\mathcal{H}^2}.$$

Then we get the existence and uniqueness of the mild solution $f \in \mathcal{H}^2(T,H)$ with initial condition $\xi \in L^2(\Omega, \mathcal{F}_0, P, H)$ by Banach's fixed-point theorem.

Step 1. The mapping $\gamma : \mathcal{H}^2(T,H) \to \mathcal{H}^2(T,H)$ is well-defined.

Let $\xi \in L^2(\Omega, \mathcal{F}_0, P, H)$ and $f \in \mathcal{H}^2(T,H)$. Then $(S(t)\xi)_{t\in[0,T]} \in \mathcal{H}^2(T,H)$ (cf. [FrKn 01], Proof of Thm. 3.2, Step 2, p.74).

The process $a(s, f(s))$, $t \in [0,T]$, is P-a.s. Bochner integrable because

$$\begin{aligned} E[\int_0^t \|a(s,f(s))\|\,ds)] &\leq \int_0^t E[Lip_a(1 + \|f(s)\|)]\,ds)] \\ &\leq T\,Lip_a(1 + \|f\|_{\mathcal{H}^2(T,H)}) < \infty. \end{aligned}$$

Moreover it is predictable and hence by Lemma 2.4 the process

$$\int_0^t S(t-s)a(s,f(s))\,ds,\ 0 \leq t \leq T,$$

is well-defined and admits a predictable version. It is in $\mathcal{H}^2(T,H)$, since

$$\begin{aligned} &\sup_{t\in[0,T]} (E[\|\int_0^t S(t-s)a(s,f(s))\,ds\|^2])^{\frac{1}{2}} \\ \leq\ & M_T T^{\frac{1}{2}} \sup_{t\in[0,T]} (E[\int_0^t \|a(s,f(s))\|_H^2\,ds])^{\frac{1}{2}} \\ \leq\ & M_T T^{\frac{1}{2}} Lip_a \sup_{t\in[0,T]} (\int_0^t E[(1 + \|f(s)\|)^2]\,ds)^{\frac{1}{2}} \\ \leq\ & M_T T^{\frac{1}{2}} Lip_a \sup_{t\in[0,T]} (\int_0^t E[2(1 + \|f(s)\|^2)]\,ds)^{\frac{1}{2}} \\ \leq\ & M_T T^{\frac{1}{2}} Lip_a \sqrt{2} \sup_{t\in[0,T]} (\int_0^t 1 + E[\|f(s)\|^2]\,ds)^{\frac{1}{2}} \\ \leq\ & M_T T\,Lip_a \sqrt{2}(1 + \|f\|_{\mathcal{H}^2}) < \infty. \end{aligned}$$

Hence it only remains to prove that the process

$$(\int_0^t S(t-s)\sigma(s,f(s))\,dL(s))_{t\in[0,T]}$$

is well-defined and admits a version which is an element of $\mathcal{H}^2(T,H)$. We use Lemma 1.3 to decompose the Lévy process into

$$L(t) = tm + B_Q(t) + \int_G x\ \tilde{N}(t,dx).$$

Then we define the stochastic integral as

$$\begin{aligned} \int_0^t S(t-s)\sigma(s,f(s))\,dL(s) := & \int_0^t S(t-s)\sigma(s,f(s))m\,ds && (2.4)\\ + & \int_0^t S(t-s)\sigma(s,f(s))\,dB_Q(s) && (2.5)\\ + & \int_0^t\int_G S(t-s)\sigma(s,f(s))x\,\tilde{N}(dt,dx) && (2.6) \end{aligned}$$

and show the required properties for each summand. First we prove that the processes are well-defined, then that they have finite $\mathcal{H}^2(T,H)$-norm, and finally that we can find a predictable version.

Claim 1: The integrals are well-defined.
1. σ is $\mathcal{P}_T/\mathcal{B}(L_2(G,H))$-measurable. Hence σm is $\mathcal{P}_T/\mathcal{B}(H)$-measurable for all $m \in G$. Moreover

$$E[\int_0^T \|\sigma(s,f(s))m\|_H\,ds] \le T\,Lip_\sigma \|m\|(1+\|f\|_{\mathcal{H}^2(T,H)}) < \infty.$$

Therefore σm is predictable and P-a.s. Bochner integrable. By Lemma 2.4 we get that

$$(\int_0^t S(t-s)\sigma(s,f(s))m\,ds)_{t\in[0,T]}$$

exists and has a predictable version.

2. The stochastic integrals $\int_0^t S(t-s)\sigma(s,f(s))\,dB_Q(s)$, $t \in [0,T]$, are well-defined because the processes $1_{(0,t]}(s)S(t-s)\sigma(s,f(s))$, $s \in [0,t]$, are in $\mathcal{N}_B^2(0,T)$ (cf. App. A) for all $t \in [0,T]$:
(i) The mapping

$$(s,\omega) \to 1_{(0,t]}(s)S(t-s)\sigma(s,\omega,f(s,\omega))$$

is $\mathcal{P}_T/\mathcal{B}(L_2(G,H))$-measurable by Lemma 2.2.
(ii) With respect to the norm we have

$$\begin{aligned}
& E[\int_0^t \|S(t-s)\sigma(s,f(s))Q^{\frac{1}{2}}\|_{L_2}^2\, ds] \\
\leq\ & M_T^2 Lip_\sigma^2 tr(Q) \int_0^t E[(1+\|f(s)\|)^2]\, ds \\
\leq\ & M_T^2 T Lip_\sigma^2 tr(Q) 2(1+\|f\|_{\mathcal{H}^2}^2) < \infty.
\end{aligned}$$

We know from Prop. A.3 that the stochastic integral is again a martingale and hence is $(\mathcal{F}_t)$-adapted.

3. Define a Lévy martingale measure by

$$M(t,A) := \int_{A-\{0\}} x\ \tilde{N}(t,dx),\ \ A \in \mathcal{A}.$$

Then we now from Chapter 2 that M is an orthogonal martingale measure with independent increments which is nuclear and decomposable (see Thm. 1.2, Prop. 1.3). Thus we can define the stochastic integrals with respect to a Lévy martingale measure by

$$\int_0^t \int_G S(t-s)\sigma(s,f(s))x\, \tilde{N}(dt,dx)) \ \ := \ \ \int_0^t \int_G S(t-s)\sigma(s,f(s))\, M(ds,dx).$$

They are well-defined because the processes $1_{(0,t]}(s)S(t-s)\sigma(s,f(s)),\ s \in [0,t]$, are in $\mathcal{N}^2(T;\nu,dt)$ (cf. Def. **??**) for all $t \in [0,T]$:
(i) The mapping

$$(s,\omega) \to 1_{(0,t]}(s)S(t-s)\sigma(s,\omega,f(s,\omega))$$

is $\mathcal{P}_T/\mathcal{B}(L_2(G,H))$-measurable by Lemma 2.2. Hence

$$(s,\omega) \to 1_{(0,t]}(s)S(t-s)\sigma(s,\omega,f(s,\omega))g$$

is $\mathcal{P}_T/\mathcal{B}(H))$-measurable for all $g \in G$. (Note that σ does not depend on $x \in G$!)
(ii) For the norm we obtain with $T_x = (x,\cdot)_G x$

$$\begin{aligned}
& E[\int_0^t \int_S \|S(t-s)\sigma(s,f(s))T_x^{\frac{1}{2}}\|_{L_2}^2\, \nu(dx)ds] \\
\leq\ & \int_0^t E[\|S(t-s)\|_{L(H)}^2 \|\sigma(s,f(s))\|_{L_2}^2] \int_G \|T_x^{\frac{1}{2}}\|_{L_2(G)}^2\, \nu(dx)\, ds \\
\leq\ & M_T^2 Lip_\sigma^2 \int_0^t E[(1+\|f(s)\|)^2] C_\nu\, ds \\
\leq\ & M_T^2 T Lip_\sigma^2 2C_\nu (1+\|f\|_{\mathcal{H}^2}^2) < \infty.
\end{aligned}$$

Again the stochastic integral is a martingale and therefore $(\mathcal{F}_t)$-adapted (cf. Prop. 1.5).

Claim 2: The three expressions (2.4), (2.5)and (2.6) have finite $\mathcal{H}^2(T,H)$-norm.

1. Similar to the calculations for the term involving $a(\cdot)$ we can conclude:

$$\begin{aligned}
&\sup_{t\in[0,T]}(E[\|\int_0^t S(t-s)\sigma(s,f(s))m\,ds\|^2])^{\frac{1}{2}}\\
\leq\;& M_T T^{\frac{1}{2}}\|m\|\sup_{t\in[0,T]}(E[\int_0^t \|\sigma(s,f(s))\|_{L_2}^2\,ds])^{\frac{1}{2}}\\
\leq\;& M_T T^{\frac{1}{2}}\|m\|Lip_\sigma\sup_{t\in[0,T]}(\int_0^t E[(1+\|f(s)\|)^2]\,ds)^{\frac{1}{2}}\\
\leq\;& M_T T^{\frac{1}{2}}\|m\|Lip_\sigma\sqrt{2}\sup_{t\in[0,T]}(\int_0^t 1+E[\|f(s)\|^2]\,ds)^{\frac{1}{2}}\\
\leq\;& M_T T\|m\|Lip_\sigma\sqrt{2}(1+\|f\|_{\mathcal{H}^2})<\infty.
\end{aligned}$$

2. For (2.5) we get:

$$\begin{aligned}
&\sup_{t\in[0,T]}(E[\|\int_0^t S(t-s)\sigma(s,f(s))\,dB_Q(s)\|^2])^{\frac{1}{2}}\\
=\;& \sup_{t\in[0,T]}(E[\int_0^t \|S(t-s)\sigma(s,f(s))Q^{\frac{1}{2}}\|_{L_2}^2\,ds])^{\frac{1}{2}}\\
\leq\;& M_T Lip_\sigma tr(Q)^{\frac{1}{2}}\sup_{t\in[0,T]}(\int_0^t E[(1+\|f(s)\|)^2]\,ds)^{\frac{1}{2}}\\
\leq\;& M_T T^{\frac{1}{2}} Lip_\sigma tr(Q)^{\frac{1}{2}}\sqrt{2}(1+\|f\|_{\mathcal{H}^2})<\infty.
\end{aligned}$$

3. Finally for (2.6) we obtain:

$$\begin{aligned}
&\sup_{t\in[0,T]}(E[\|\int_0^t\int_G S(t-s)\sigma(s,f(s))\,M(ds,dx)\|^2])^{\frac{1}{2}}\\
=\;& \sup_{t\in[0,T]}(E[\int_0^t\int_G \|S(t-s)\sigma(s,f(s))T_x^{\frac{1}{2}}\|_{L_2}^2\,\nu(dx)ds])^{\frac{1}{2}}\\
\leq\;& \sup_{t\in[0,T]}(\int_0^t E[\|S(t-s)\|^2\|\sigma(s,f(s))\|_{L_2}^2]\int_G \|T_x^{\frac{1}{2}}\|_{L_2(G)}^2\,\nu(dx)\,ds)^{\frac{1}{2}}\\
\leq\;& M_T Lip_\sigma\sup_{t\in[0,T]}(\int_0^t E[(1+\|f(s)\|)^2]C_\nu\,ds)^{\frac{1}{2}}\\
\leq\;& M_T T^{\frac{1}{2}} Lip_\sigma\sqrt{2}\,C_\nu^{\frac{1}{2}}(1+\|f\|_{\mathcal{H}^2})<\infty.
\end{aligned}$$

Claim 3: For each process there is a predictable version.

To prove this claim we will use Lemma 2.3. Hence we have to show that the processes are adapted and stochastically continuous.

1. The existence of a predictable version was already proved in Claim 1, 1.

2. The following argument goes back to [FriKno 01] and [Kno 03].
As seen in the proof of Claim 1, 2.

$$Z(t) := \int_0^t S(t-s)\sigma(s, f(s))\, dB_Q(s),\ t \in [0,T],$$

is $(\mathcal{F}_t)$-adapted. In addition we show that it is continuous in the mean square and therefore stochastically continuous:

For $\alpha > 1$ the process $Z^\alpha(t) := \int_0^{t/\alpha} S(t-s)\sigma(s, f(s))\, dB_Q(s),\ t \in [0,T]$, is mean-square continuous.

To prove this claim we first use the semigroup property and get that

$$\begin{aligned} &\int_0^{t/\alpha} S(t-s)\sigma(s, f(s))\, dB_Q(s) \\ = &\int_0^{t/\alpha} S(t-\alpha s)S((\alpha-1)s)\sigma(s, f(s))\, dB_Q(s),\ t \in [0,T], \end{aligned}$$

where we set $\Phi^\alpha(s) := 1_{(0,T]}(s)S((\alpha-1)s)\sigma(s, f(s)),\ s \in [0,T]$.
Then it is clear that $\Phi^\alpha(t),\ t \in [0,T]$, is an element of $\mathcal{N}_B^2(0,T)$. Hence we have to show now that the process

$$\tilde{Z}(t) := \int_0^{t/\alpha} S(t-\alpha s)\tilde{\Phi}(s)\, dB_Q(s),\ t \in [0,T],$$

is continuous for each $\alpha > 1$ and $\tilde{\Phi} \in \mathcal{N}_B^2(0,T)$.

(a) In the first step let $\tilde{\Phi}$ be a simple process of the form

$$\tilde{\Phi} = \sum_{i=1}^m u_i 1_{A_i} \tag{2.7}$$

where $m \in \mathbb{N}$, $u_i \in L_2(G,H)$ and $A_i \in \mathcal{P}_T$, $1 \le i \le m$. We take arbitrary $0 \le r < t \le T$ and get

$$
\begin{aligned}
& (E[\| \int_0^{t/\alpha} S(t-\alpha s)\tilde{\Phi}(s)\, dB_Q(s) - \int_0^{r/\alpha} S(r-\alpha s)\tilde{\Phi}(s)\, dB_Q(s)\|^2])^{\frac{1}{2}} \\
\le\ & (E[\| \int_0^{r/\alpha} [S(t-\alpha s) - S(r-\alpha s)]\tilde{\Phi}(s)\, dB_Q(s)\|^2])^{\frac{1}{2}} \\
+\ & (E[\| \int_{r/\alpha}^{t/\alpha} S(t-\alpha s)\tilde{\Phi}(s)\, dB_Q(s)\|^2])^{\frac{1}{2}} \\
\le\ & (E[\int_0^{r/\alpha} \|[S(t-\alpha s) - S(r-\alpha s)]\tilde{\Phi}(s)Q^{\frac{1}{2}}\|_{L_2}^2\, ds])^{\frac{1}{2}} \\
+\ & (E[\int_{r/\alpha}^{t/\alpha} \|S(t-\alpha s)\tilde{\Phi}(s)Q^{\frac{1}{2}}\|_{L_2}^2\, ds])^{\frac{1}{2}} \\
\le\ & \sum_{i=1}^m (E[\int_0^{r/\alpha} 1_{A_i}(s,\cdot)\|[S(t-\alpha s) - S(r-\alpha s)]u_i Q^{\frac{1}{2}}\|_{L_2}^2\, ds])^{\frac{1}{2}} \\
+\ & \sum_{i=1}^m (E[\int_{r/\alpha}^{t/\alpha} 1_{A_i}(s,\cdot)\|S(t-\alpha s)u_i Q^{\frac{1}{2}}\|_{L_2}^2\, ds])^{\frac{1}{2}} \\
\le\ & tr(Q)^{\frac{1}{2}} \cdot \sum_{i=1}^m (\int_0^{r/\alpha} \|[S(t-\alpha s) - S(r-\alpha s)]u_i\|_{L_2}^2\, ds)^{\frac{1}{2}} \qquad (2.8) \\
+\ & tr(Q)^{\frac{1}{2}} \cdot \sum_{i=1}^m (\int_{r/\alpha}^{t/\alpha} \|S(t-\alpha s)u_i\|_{L_2}^2\, ds)^{\frac{1}{2}}. \qquad (2.9)
\end{aligned}
$$

The second summand (2.9) converges to zero because for any $1 \le i \le m$

$$\int_{r/\alpha}^{t/\alpha} \|S(t-\alpha s)u_i\|_{L_2}^2\, ds \le \frac{t-r}{\alpha} M_T^2 \|u_i\|_{L_2}^2 \longrightarrow 0 \quad \text{as}\, r \uparrow t \,\text{or}\, t \downarrow r.$$

The same is true for the first summand (2.8) for the following reason:
Let be e_n, $n \in \mathbb{N}$, an orthonormal basis of G. Then for any $s \in [0,T]$ and $1 \le i \le m$ we have that

$$
\begin{aligned}
& 1_{[0,r/\alpha)}(s)\|[S(t-\alpha s) - S(r-\alpha s)]u_i\|_{L_2}^2 \\
=\ & \sum_{n \in \mathbb{N}} 1_{[0,r/\alpha)}(s)\|[S(t-\alpha s) - S(r-\alpha s)]u_i e_n\|_H^2
\end{aligned}
$$

where

$$1_{[0,r/\alpha)}(s)\|[S(t-\alpha s) - S(r-\alpha s)]u_i e_n\|^2 \longrightarrow 0 \quad \text{as}\, r \uparrow t \,\text{or}\, t \downarrow r$$

by the strong continuity of the semigroup $S(t)$. Combined with the inequality

$$1_{[0,r/\alpha)}(s)\|[S(t-\alpha s)-S(r-\alpha s)]u_i e_n\|^2 \leq 4M_T^2\|u_i e_n\|^2$$

for all $n \in \mathbb{N}$, $1 \leq i \leq m$, we get the pointwise convergence

$$1_{[0,r/\alpha)}(s)\|[S(t-\alpha s)-S(r-\alpha s)]u_i\|_{L_2}^2 \longrightarrow 0 \quad \text{as}\, r \uparrow t \,\text{or}\, t \downarrow r$$

by Lemma 2.5. Finally, the following integrable upper bound

$$1_{[0,r/\alpha)}(s)\|[S(t-\alpha s)-S(r-\alpha s)]u_i\|_{L_2}^2 \leq 4M_T^2\|u_i\|_{L_2}^2 \in L^1([0,T],dx)$$

for all $s \in [0,T]$, $0 \leq r < t \leq T$, allows us to use Lebesgue's dominated convergence theorem. Thus we obtain the convergence of the integrals

$$\int_0^{r/\alpha} \|[S(t-\alpha s)-S(r-\alpha s)]u_i\|_{L_2}^2\, ds,\ 1 \leq i \leq m,$$

we were looking for.
Hence we have proved the continuity of

$$\int_0^{t/\alpha} S(t-\alpha s)\tilde{\Phi}(s)\, dB_Q(s),\ t \in [0,T],$$

in the case that Φ is a simple process.

(b) Now consider an arbitrary $\tilde{\Phi}$ from $\mathcal{N}_B^2(0,T)$:
There exists a sequence $(\tilde{\Phi}_n)_{n\in\mathbb{N}}$ of simple processes considered in (a) such that (see Lemma 2.6)

$$E[\int_0^T \|\Phi(s)-\Phi_n(s)\|_{L_2}^2\, ds] \xrightarrow{n\to\infty} 0.$$

By step (a) we know that for each $n \in \mathbb{N}$

$$\tilde{Z}^n(t) := \int_0^{t/\alpha} S(t-\alpha s)\tilde{\Phi}_n(s)\, dB_Q(s),\ t \in [0,T],$$

is continuous. To show the continuity of

$$\tilde{Z}(t) = \int_0^{t/\alpha} S(t-\alpha s)\tilde{\Phi}(s)\, dB_Q(s),\ t \in [0,T],$$

we prove that $\tilde{Z}^n$ converges to $\tilde{Z}$ uniformly in $t \in [0,T]$:

$$\begin{aligned}
& \sup_{t\leq T} E[\|\tilde{Z}^n(t) - \tilde{Z}(t)\|_H^2] \\
= & \sup_{t\leq T} E[\|\int_0^{t/\alpha} S(t-\alpha s)(\tilde{\Phi}_n(s) - \tilde{\Phi}(s))\, dB_Q(s)\|^2] \\
= & \sup_{t\leq T} E[\int_0^{t/\alpha} \|S(t-\alpha s)(\tilde{\Phi}_n(s) - \tilde{\Phi}(s))Q^{\frac{1}{2}}\|_{L_2}^2\, ds] \\
\leq & M_T^2 tr(Q) E[\int_0^T \|(\tilde{\Phi}_n(s) - \tilde{\Phi}(s))\|_{L_2}^2\, ds] \\
& \xrightarrow{n\to\infty} 0.
\end{aligned}$$

Taking $\tilde{\Phi} = \Phi^\alpha$ we thus obtain the continuity of

$$Z^\alpha(t) = \int_0^{t/\alpha} S(t-\alpha s)\Phi^\alpha(s)\, dB_Q(s) = \int_0^{t/\alpha} S(t-s)\sigma(s, f(s))\, dB_Q(s).$$

for $\alpha > 1$. With this result we can prove the assertion we are interested in.

(c) To establish the mean-square continuity of

$$Z(t) = \int_0^t S(t-s)\sigma(s, f(s))\, dB_Q(s),\ t \in [0,T],$$

we proceed as in (b) and show that Z^{α_n} converges to Z uniformly in $t \in [0,T]$ for $(\alpha_n)_{n\in\mathbb{N}}$ any sequence of real numbers such that $\alpha_n \downarrow 1$ as $n \to \infty$:

$$\begin{aligned}
& \sup_{t\leq T} E[\|Z^{\alpha_n}(t) - Z(t)\|_H^2] \\
= & \sup_{t\leq T} E[\|\int_0^{t/\alpha_n} S(t-s)\sigma(s, f(s))\, dB_Q(s) - \int_0^t S(t-s)\sigma(s, f(s))\, dB_Q(s)\|^2] \\
= & \sup_{t\leq T} E[\|\int_0^T 1_{(\frac{t}{\alpha_n}, t]}(s) S(t-s)\sigma(s, f(s))\, dB_Q(s)\|^2] \\
\leq & M_T^2 tr(Q) Lip_\sigma^2 \sup_{t\leq T} E[\int_0^T 1_{(\frac{t}{\alpha_n}, t]}(s)(1 + \|f(s)\|)^2\, ds] \\
\leq & M_T^2 tr(Q) Lip_\sigma^2 2(1 + \|f\|_{\mathcal{H}^2}) \sup_{t\leq T}(t - \frac{t}{\alpha_n}) \\
\leq & M_T^2 tr(Q) Lip_\sigma^2 2(1 + \|f\|_{\mathcal{H}^2}) T \frac{\alpha_n - 1}{\alpha_n} \\
& \xrightarrow{n\to\infty} 0.
\end{aligned}$$

Thus Z is adapted and continuous in the mean square, and the application of Lemma 2.3 finally gives the existence of a predictable version.

3. The proof is basically the same as the previous one: The fact that

$$Y(t) := \int_0^t \int_G S(t-s)\sigma(s, f(s))\, M(ds, dx),\ t \in [0, T],$$

is $(\mathcal{F}_t)$-adapted again follows from the martingale property as already mentioned in the proof of Claim 1, 3.
For $\alpha > 1$ we define $Y^\alpha(t) := \int_0^{t/\alpha} \int_G S(t-s)\sigma(s, f(s))\, M(ds, dx),\ t \in [0, T]$.
Remember that $\Phi^\alpha(s) = 1_{(0,T]}(s) S((\alpha - 1)s)\sigma(s, f(s)),\ s \in [0, T]$.
Then $\Phi^\alpha(t),\ t \in [0, T]$, is also an element of $\mathcal{N}^2(T; \nu, dt)$, and we show as in 2. that the process

$$\tilde{Y}(t) := \int_0^{t/\alpha} \int_G S(t - \alpha s)\tilde{\Phi}(s)\, M(ds, dx),\ t \in [0, T],$$

is mean-square continuous for each $\alpha > 1$ and $\tilde{\Phi} \in \mathcal{N}^2(T; \nu, dt)$.

(a) To show continuity in the case that $\tilde{\Phi}$ is a simple process of the form defined above in (2.7) we use the following estimate (analogously to that in 2.):

$$\begin{aligned}
& (E[\| \int_0^{t/\alpha} \int_G S(t - \alpha s)\tilde{\Phi}(s)\, M(ds, dx) \\
- & \int_0^{r/\alpha} \int_G S(r - \alpha s)\tilde{\Phi}(s)\, M(ds, dx)\|^2])^{\frac{1}{2}} \\
\leq & (E[\| \int_0^{r/\alpha} \int_G [S(t - \alpha s) - S(r - \alpha s)]\tilde{\Phi}(s)\, M(ds, dx)\|^2])^{\frac{1}{2}} \\
+ & (E[\| \int_{r/\alpha}^{t/\alpha} \int_G S(t - \alpha s)\tilde{\Phi}(s)\, M(ds, dx)\|^2])^{\frac{1}{2}} \\
\leq & (E[\int_0^{r/\alpha} \int_G \|[S(t - \alpha s) - S(r - \alpha s)]\tilde{\Phi}(s) T_x^{\frac{1}{2}}\|_{L_2}^2\, \nu(dx)\, ds])^{\frac{1}{2}} \\
+ & (E[\int_{r/\alpha}^{t/\alpha} \int_G \|S(t - \alpha s)\tilde{\Phi}(s) T_x^{\frac{1}{2}}\|_{L_2}^2\, \nu(dx)\, ds])^{\frac{1}{2}} \\
\leq & C_\nu^{\frac{1}{2}} \cdot \sum_{i=1}^m (\int_0^{r/\alpha} \|[S(t - \alpha s) - S(r - \alpha s)]u_i\|_{L_2}^2\, ds)^{\frac{1}{2}} \\
+ & C_\nu^{\frac{1}{2}} \cdot \sum_{i=1}^m (\int_{r/\alpha}^{t/\alpha} \|S(t - \alpha s)u_i\|_{L_2}^2\, ds)^{\frac{1}{2}}.
\end{aligned}$$

The two summands converge to zero by the same arguments as in 2. and thus we have proved that

$$\int_0^{t/\alpha} \int_G S(t-s)\tilde{\Phi}(s)\, M(ds,dx),\ t \in [0,T],$$

is continuous for $\tilde{\Phi}$ a simple process.

(b) For arbitrary $\tilde{\Phi}$ from $\mathcal{N}^2(T;\nu,dt)$, which does **not** depend on $x \in G$, we again have the sequence of approximating simple processes $(\tilde{\Phi}_n)_{n\in\mathbb{N}}$ (see Lemma 2.6). We proceed exactly as in the previous proof and show that

$$\tilde{Y}^n(t) := \int_0^{t/\alpha} \int_S S(t-\alpha s)\tilde{\Phi}_n(s)\, M(ds,dx),\ t \in [0,T],$$

converges uniformly in $t \in [0,T]$ to

$$\tilde{Y}(t) = \int_0^{t/\alpha} \int_S S(t-\alpha s)\tilde{\Phi}(s)\, M(ds,dx),\ t \in [0,T].$$

Hence:

$$\begin{aligned}
& \sup_{t\le T} E[\|\tilde{Y}^n(t)-\tilde{Y}(t)\|_H^2] \\
= \ & \sup_{t\le T} E[\|\int_0^{t/\alpha}\int_S S(t-\alpha s)(\tilde{\Phi}_n(s)-\tilde{\Phi}(s))\, M(ds,dx)\|^2] \\
= \ & \sup_{t\le T} E[\int_0^{t/\alpha}\int_S \|S(t-\alpha s)(\tilde{\Phi}_n(s)-\tilde{\Phi}(s))T_x^{\frac{1}{2}}\|_{L_2}^2\, \nu(dx)ds] \\
\le \ & M_T^2 C_\nu E[\int_0^T \|(\tilde{\Phi}_n(s)-\tilde{\Phi}(s))\|_{L_2}^2\, ds] \\
& \xrightarrow{n\to\infty} 0.
\end{aligned}$$

Taking $\tilde{\Phi} = \Phi^\alpha$ we get for any $\alpha > 1$ the continuity of

$$Y^\alpha(t) = \int_0^{t/\alpha}\int_S S(t-\alpha s)\Phi^\alpha(s)\, M(ds,dx) = \int_0^{t/\alpha}\int_S S(t-s)\sigma(s,f(s))\, M(ds,dx).$$

(c) Finally, to prove the mean-square continuity of

$$Y(t) = \int_0^t \int_S S(t-s)\sigma(s,f(s))\, M(ds,dx),\ t \in [0,T],$$

we proceed as before and show the uniform convergence of Y^{α_n} to Y where $(\alpha_n)_{n\in\mathbb{N}}$ is any sequence of real numbers with $\alpha_n \downarrow 1$ as $n \to \infty$:

$$\begin{aligned}
& \sup_{t\le T} E[\|Y^{\alpha_n}(t) - Y(t)\|_H^2] \\
= \; & \sup_{t\le T} E[\|\int_0^T \int_S 1_{(\frac{t}{\alpha_n},t]}(s) S(t-s)\sigma(s,f(s))\, M(ds,dx)\|^2] \\
\le \; & M_T^2 C_\nu Lip_\sigma^2 \sup_{t\le T} E[\int_0^T 1_{(\frac{t}{\alpha_n},t]}(s)(1+\|f(s)\|)^2\, ds] \\
\le \; & M_T^2 C_\nu Lip_\sigma^2 2(1+\|f\|_{\mathcal{H}^2}) \sup_{t\le T}(t - \frac{t}{\alpha_n}) \\
\le \; & M_T^2 C_\nu Lip_\sigma^2 2(1+\|f\|_{\mathcal{H}^2}) T \frac{\alpha_n - 1}{\alpha_n} \\
& \xrightarrow{n\to\infty} 0.
\end{aligned}$$

So Y is adapted and continuous in the mean square. Thus Lemma 2.3 again yields the existence of a predictable version.

Step 2. The mapping $\gamma : \mathcal{H}^2(T,H) \to \mathcal{H}^2(T,H)$ is a strict contraction.

Let $f_1, f_2 \in \mathcal{H}^2(T,H)$, $\xi \in L^2(\Omega, \mathcal{F}_0, P, H)$. Then we get

$$\begin{aligned}
& \|\gamma(f_1) - \gamma(f_2)\|_{\mathcal{H}^2} = \|(\int_0^t S(t-s)(a(f_1(s)) - a(f_2(s)))\, ds \\
+ \; & \int_0^t S(t-s)(\sigma(f_1(s)) - \sigma(f_2(s)))\, dL(s))_{t\in[0,T]}\|_{\mathcal{H}^2} \\
\le \; & \sup_{t\in[0,T]} (E[\|\int_0^t S(t-s)(a(f_1(s)) - a(f_2(s)))\, ds\|^2])^{\frac{1}{2}} \\
+ \; & \sup_{t\in[0,T]} (E[\|\int_0^t S(t-s)(\sigma(f_1(s)) - \sigma(f_2(s)))\, dL(s)\|^2])^{\frac{1}{2}}.
\end{aligned}$$

The first summand can be estimated by

$$\begin{aligned}
\le \; & T^{\frac{1}{2}} \sup_{t\in[0,T]} (E[\int_0^t \|S(t-s)(a(f_1(s)) - a(f_2(s)))\|^2\, ds])^{\frac{1}{2}} \\
\le \; & M_T Lip_a T^{\frac{1}{2}} \sup_{t\in[0,T]} (\int_0^t E[\|(f_1(s)) - f_2(s))\|^2]\, ds)^{\frac{1}{2}} \\
\le \; & M_T Lip_a T \|f_1 - f_2\|_{\mathcal{H}^2}.
\end{aligned}$$

As in **Step 1** we decompose our Lévy process according to Lemma 1.3 and obtain the following upper bound for the second summand:

$$
\begin{aligned}
& \sup_{t\in[0,T]} (E[\| \int_0^t S(t-s)(\sigma(f_1(s)) - \sigma(f_2(s)))m\, ds\|^2])^{\frac{1}{2}} && (2.10)\\
+ \quad & \sup_{t\in[0,T]} (E[\| \int_0^t S(t-s)(\sigma(f_1(s)) - \sigma(f_2(s)))\, dB_Q(s)\|^2])^{\frac{1}{2}} && (2.11)\\
+ \quad & \sup_{t\in[0,T]} (E[\| \int_0^t \int_G S(t-s)(\sigma(f_1(s)) - \sigma(f_2(s)))x\tilde{N}(dt,dx))\|^2])^{\frac{1}{2}} && (2.12)
\end{aligned}
$$

1. It is easy to see that (2.10)

$$
\begin{aligned}
&\leq T^{\frac{1}{2}} \sup_{t\in[0,T]} (E[\int_0^t \|S(t-s)(\sigma(f_1(s)) - \sigma(f_2(s)))m\|^2\, ds])^{\frac{1}{2}}\\
&\leq M_T Lip_\sigma \|m\| T^{\frac{1}{2}} \sup_{t\in[0,T]} (\int_0^t E[\|f_1(s) - f_2(s)\|^2]\, ds)^{\frac{1}{2}}\\
&\leq M_T Lip_\sigma \|m\| T \|f_1 - f_2\|_{\mathcal{H}^2}.
\end{aligned}
$$

2. We get that (2.11)

$$
\begin{aligned}
&= \sup_{t\in[0,T]} (E[\int_0^t \|S(t-s)(\sigma(f_1(s)) - \sigma(f_2(s)))Q^{\frac{1}{2}}\|_{L_2}^2\, ds])^{\frac{1}{2}}\\
&\leq M_T Lip_\sigma tr(Q)^{\frac{1}{2}} \sup_{t\in[0,T]} (E[\int_0^t \|f_1(s) - f_2(s)\|^2\, ds])^{\frac{1}{2}}\\
&\leq M_T Lip_\sigma tr(Q)^{\frac{1}{2}} T^{\frac{1}{2}} \|f_1 - f_2\|_{\mathcal{H}^2}.
\end{aligned}
$$

3. Finally we conclude that (2.12)

$$
\begin{aligned}
&= \sup_{t\in[0,T]} (E[\| \int_0^t \int_G S(t-s)(\sigma(f_1(s)) - \sigma(f_2(s)))\, M(ds,dx))\|^2])^{\frac{1}{2}}\\
&= \sup_{t\in[0,T]} (E[\int_0^t \int_G \|S(t-s)(\sigma(f_1(s)) - \sigma(f_2(s)))T_x^{\frac{1}{2}}\|_{L_2}^2\, \nu(dx)ds])^{\frac{1}{2}}\\
&\leq \sup_{t\in[0,T]} (E[\int_0^t \|S(t-s)\|_{L(H)}^2 \|\sigma(f_1(s)) - \sigma(f_2(s))\|_{L_2}^2 \int_G \|T_x^{\frac{1}{2}}\|_{L_2}^2\, \nu(dx)ds])^{\frac{1}{2}}\\
&\leq M_T Lip_\sigma \sup_{t\in[0,T]} (E[\int_0^t \|f_1(s) - f_2(s)\|^2 C_\nu\, ds])^{\frac{1}{2}}\\
&\leq M_T Lip_\sigma T^{\frac{1}{2}} C_\nu^{\frac{1}{2}} \|f_1 - f_2\|_{\mathcal{H}^2}.
\end{aligned}
$$

Hence by taking $T = T_1$ sufficiently small we can find $C < 1$ such that

$$\|\gamma(f_1) - \gamma(f_2)\|_{\mathcal{H}^2(T_1,H)} \leq C\|f_1 - f_2\|_{\mathcal{H}^2(T_1,H)}$$

for all $\xi \in L^2(\Omega, \mathcal{F}_0, P, H)$, $f_1, f_2 \in \mathcal{H}^2(T_1, H)$. Thus we have established the existence of a unique mild solution on $[0, T_1]$.

For general T we start with the unique mild solution f on $[0, T_1]$. Then we solve again for the new initial condition $f(T_1)$. Since the constants involved, $M_T, Lip_a, Lip_\sigma, C_\nu, tr(Q), \|m\|$, only depend on T (if at all) we can proceed exactly as before and get a unique mild solution on $[T_1, 2T_1]$. To be precise, set $\tilde{\mathcal{F}}_t := \mathcal{F}_{t+T_1}$, $\tilde{a}(t,\cdot) := a(t + T_1, \cdot)$, $\tilde{\sigma}(t,\cdot) := \sigma(t + T_1, \cdot)$ and $\tilde{L}(t) := L(t + T_1) - L(T_1)$, $t \in [0, T - T_1]$. We consider the equation

$$\begin{cases} d\tilde{f}(t) = (A\tilde{f}(t) + \tilde{a}(t, \tilde{f}(t)))\, dt + \tilde{\sigma}(t, \tilde{f}(t))\, d\tilde{L}(t) \\ \tilde{f}(0) = f(T_1) \in L^2(\Omega, \tilde{\mathcal{F}}_0, P, H). \end{cases}$$

Guaranteed a unique mild solution $\tilde{f}$ on $[0, T_1 \wedge (T - T_1)]$ we define

$$f(t) := \begin{cases} f(t) & , t \in [0, T_1] \\ \tilde{f}(t - T_1) & , t \in (T_1, 2T_1 \wedge T], \end{cases}$$

thus extending the solution to $[0, 2T_1 \wedge T]$
We continue this procedure until we have constructed the solution f on the whole interval $[0, T]$.

Step 3. The mild solution $f : [0, T] \to L^2(\Omega, \mathcal{F}, P; H)$ is continuous.

For $0 \leq s \leq t \leq T$ we can conclude

$$E[\|S(t)\xi - S(s)\xi\|^2] \leq E[\|S(t) - S(s)\|^2_{L(H)}\|\xi\|^2] \leq 4M_T^2 E[\|\xi\|^2] < \infty.$$

Due to the strong continuity of $S(t)$, $t \geq 0$, we get the pointwise convergence

$$\|[S(t) - S(s)]\xi\|_H \longrightarrow 0 \quad \text{as}\, s \uparrow t\, \text{or}\, t \downarrow s.$$

Hence the application of Lebesgue's dominated convergence theorem yields the L^2-continuity of $(S(t)\xi)_{t\in[0,T]}$.

The proof for $(\int_0^t S(t-s)a(s,f(s))\,ds)_{t\in[0,T]}$ makes use of the same argument: We take arbitrary $0 \le r < t \le T$ and get

$$\begin{aligned}
& (E[\| \int_0^t S(t-s)a(s,f(s))\,ds - \int_0^r S(r-s)a(s,f(s))\,ds\|^2])^{\frac{1}{2}} \\
\le \; & (E[\| \int_0^r [S(t-s) - S(r-s)]a(s,f(s))\,ds\|^2])^{\frac{1}{2}} \qquad (2.13) \\
+ \; & (E[\| \int_r^t S(t-s)a(s,f(s))\,ds\|^2])^{\frac{1}{2}}. \qquad (2.14)
\end{aligned}$$

Again by using Lebesgue's theorem we show that (2.13) tends to zero:

$$\begin{aligned}
& E[\| \int_0^r [S(t-s) - S(r-s)]a(s,f(s))\,ds\|^2] \\
\le \; & rE[\int_0^r \|[S(t-s) - S(r-s)]a(s,f(s))\|^2\,ds] \\
\le \; & 4M_T^2 r^2 Lip_a^2 2(1 + \|f\|_{\mathcal{H}^2}^2) < \infty.
\end{aligned}$$

Moreover by the strong continuity of $S(t)$, $t \ge 0$, we have for all (s,ω)

$$\|[S(t-s) - S(r-s)]a(s,f(s))\|_H \longrightarrow 0 \quad \text{as}\, r \uparrow t \,\text{or}\, t \downarrow r.$$

And (2.14) converges to zero because

$$\begin{aligned}
E[\| \int_r^t S(t-s)a(s,f(s))\,ds\|^2] & \le (t-r)E[\int_r^t \|S(t-s)a(s,f(s))\|^2\,ds] \\
& \le (t-r)^2 M_T^2 Lip_a^2 2(1 + \|f\|_{\mathcal{H}^2}^2) \\
& \longrightarrow 0 \quad \text{as}\, r \uparrow t \,\text{or}\, t \downarrow r.
\end{aligned}$$

The continuity of $(\int_0^t S(t-s)\sigma(s,f(s))\,dL(s))_{t\in[0,T]}$ follows (cf. (2.4) ff.) from the continuity of $\int_0^t S(t-s)\sigma(s,f(s))m\,ds$, $\int_0^t S(t-s)\sigma(s,f(s))\,dB_Q(s)$ and $\int_0^t \int_G S(t-s)\sigma(s,f(s))\,M(dt,dx)$.
For the second and third term this property has already been shown in **Step 1**, **Claim 3**, 2. & 3. The proof for the first term is completely analogous to that for $(\int_0^t S(t-s)a(s,f(s))\,ds)_{t\in[0,T]}$ and so we are finally done.

$\square$

Chapter 3

Application to Heath-Jarrow-Morton Interest Rate Models

In this final chapter we want to apply our results from Chapter 2 to modeling bond prices via Heath-Jarrow-Morton interest rate models.
Section 3.1 states some basic concepts and definitions from general interest rate theory. In section 3.2 we introduce the classical *Heath-Jarrow-Morton model* put forward in [HJM 92]. It attempts to capture the movement of bond prices by modeling *forward rates* ("expected future interest rates") with stochastic equations driven by Brownian motion. Similar to [Fil 01] we show how to reformulate the original approach in the framework of stochastic evolution equations. Motivated by empirical findings we then switch from Gaussian noise to the more general Lévy noise in the underlying equation in section 3.3. Following [ÖzkSch 05] we develop an HJM-type condition for the drift coefficient to guarantee the absence of arbitrage in our model. Finally, we use our existence and uniqueness theorem from section 2.1 to prove the existence of an HJM model with Lévy noise (conditional upon an assumption on the drift).

3.1 General interest rate theory

As before let $(\Omega, \mathcal{F}, P)$ be a complete probability space with $(\mathcal{F}_t)$, $t \geq 0$, a right-continuous filtration on $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}_0$ contains all P-nullsets.We fix a time horizon $\bar{T} > 0$.

A *zero-coupon bond (ZCB) of maturity* T is a financial security paying to its holder one unit of cash at the prespecified date T in the future. I.e.

the bond's *face value* is one unit of currency (normalized for convenience).

We assume throughout the whole chapter that the bonds have **no** risk of default.

By $B(t,T)$ we denote the price of a zero-coupon bond of maturity T at any instant $t \leq T$. $B(t,T) \leq 1$, because the interest earned on this bond appears as a discount to the face value. Clearly $B(T,T) = 1$ for any $T \leq \bar{T}$. We assume that $B(\cdot,T)$ follows a strictly positive and adapted process on $(\Omega, \mathcal{F}, P)$.

Since the interest received depends on the time to maturity, interest rates are not a one-dimensional object. Thus modeling them requires a vector- or function-valued process. We assume the existence of a complete set of zero-coupon bonds for all maturities $T \in [0, \bar{T}]$. (In reality bonds with a finite number of maturities between 0 and at least 30 years are traded.)

The *term structure of interest rates* (at time t) is the set of *yields-to-maturity* $(Y(t,T))$, $t < T \leq \bar{T}$, where

$$Y(t,T) = -\frac{1}{T-t} \ln B(t,T), \ t < T \leq \bar{T}.$$

This is derived from the discount equation (using continuous compounding)

$$B(t,T) = \exp(-Y(t,T)(T-t)), \ t < T \leq \bar{T}.$$

$(Y(t,T))$, $t \leq T \leq \bar{T}$, is known as the *yield curve* at time t.

The Short Rate

The *short rate* $r(t) = Y(t,t) := \lim_{T \downarrow t} Y(t,T)$ is the rate for instantaneous borrowing or lending at date t. Since the short rate can fluctuate over time, we consider the process $(r(t))$, $t \geq 0$. (Note that the short rate is a theoretical construction which cannot be directly observed in real life. It may be approximated by the overnight, one-week or one-month interest rate.)

The *money-market account* is one unit of cash invested in the short rate and continuously "rolled over", i.e. instantaneously reinvested. At time t its value is

$$D(t) = \exp(\int_0^t r(s)\, ds).$$

Originally, the short rate was modeled as a (one-dimensional) stochastic process to calculate prices of bonds and bond options. For an account of the most popular models see e.g. [Shr 04], 6.5 and 10.2. The primary shortcoming of these so-called *short rate models* is that they cannot capture complicated yield curve behavior as changes in slope or curvature.

3.2 The classical Heath-Jarrow-Morton model

Instead of using only the short rate as a state variable, Heath, Jarrow and Morton (HJM) proposed in their seminal paper ([HJM 92]) to use the entire forward rate curve as the (infinite-dimensional) state variable. In the HJM model an entire forward curve evolves simultaneously. Moreover, the HJM model uses all the information available in the initial term structure of interest rates.

Forward Interest Rates
We call $f(t,T)$ the *forward (interest) rate* at date $t \leq T$ for instantaneous riskfree borrowing or lending at date T. One should think of $f(t,T)$ as the interest rate over the infinitesimal time interval $[T, T+dt]$ as seen from time t. Hence the short rate is given by $r(t) = f(t,t)$.

If we specify a family of forward rates $f(t,T),\ 0 \leq t \leq T \leq \bar{T}$, then we can express the bond price as

$$B(t,T) = \exp(-\int_t^T f(t,u)\,du).$$

If we assume that the family of bond prices $B(t,T)$ is sufficiently smooth with respect to the maturity T, we may formally define

$$f(t,T) := -\frac{\partial \ln B(t,T)}{\partial T},\ 0 \leq t \leq T \leq \bar{T}. \tag{3.1}$$

Thus we can calculate zero-coupon bond prices from forward rates and vice versa; the two concepts contain equivalent information. (However, note that forward rates are a mathematical idealization, not directly observable.)

The Heath-Jarrow-Morton Model
Assume that $f(0,T),\ 0 \leq T \leq \bar{T}$, is known at time 0. We call this the *initial forward rate curve*. In the HJM model the forward rate at later times t for investing at still later times T is given by

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)\,ds + \int_0^t \sigma(s,T)\cdot dW(s).$$

Or written in differential form:

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\cdot dW(t),\ 0 \leq t \leq T. \tag{3.2}$$

Here the variable T is held constant. W is a d-dimensional standard Brownian motion. The coefficient functions $\alpha : \Delta^2 \times \Omega \to \mathbb{R}$ and $\sigma : \Delta^2 \times \Omega \to \mathbb{R}^d$ are adapted processes in the t variable for each fixed T, where $\Delta^2 := \{(t,T) \in \mathbb{R}^2 : 0 \le t \le T \le \bar{T}\}$ is a triangle set.

The dynamics of the forward rates can be used to gain that of the bond prices (cf. [Shr 04], 10.3 for details), given by

$$dB(t,T) = B(t,T)\,(m(t,T)\,dt + \sum_{j=1}^{d} \sigma_j^*(t,T)\,dW^j(t))$$

where

$$\begin{aligned} m(t,T) &= f(t,t) - \int_t^T \alpha(t,u)\,du + \frac{1}{2}\sum_{j=1}^{d}(\int_t^T \sigma_j(t,u)\,du)^2, \\ \sigma_j^*(t,T) &= -\int_t^T \sigma_j(t,u)\,du. \end{aligned}$$

From (3.2) we get the dynamics of the short rate under this model as

$$r(t) = f(0,t) + \int_0^t \alpha(s,t)\,ds + \int_0^t \sigma(s,t)\cdot dW(s).$$

The HJM model includes zero-coupon bonds with maturity T for each $T \in [0,\bar{T}]$. To rule out the possibility of arbitrage by trading in these bonds we have to guarantee that each discounted bond price process

$$D(t)^{-1}B(t,T) = \exp(-\int_0^t r(s)\,ds)B(t,T),\ 0 \le t \le T$$

is a (local) martingale ("First fundamental theorem of asset pricing"; cf. [DelSch 94]). We use a standard approach and work under a *risk-neutral setting*, i.e. we assume that risk-adjustments have already been made with the measure P and thus prices can be derived as if all traders were risk-neutral. Then we give conditions on α ensuring that the price processes are martingales under P. Such P is then called a *risk-neutral measure* or *martingale measure* (We will prefer the former name, because the latter has nothing to do with our notion of martingale measure from section 1.2).

This leads to the *HJM no-arbitrage condition* (cf. [HJM 92] Prop.3 (18)) relating forward drifts and volatilities:

$$\alpha(t,T) = \sum_{j=1}^{d} \sigma_j(t,T) \int_t^T \sigma_j(t,u)\,du,\ 0 \le t \le T \le \bar{T}. \tag{3.3}$$

In terms of bond prices this results in replacing the drift coefficient $m(t,T)$ by the short rate $r(t)$. Thus under a risk-neutral setting in the HJM model the bond price dynamics are given by

$$dB(t,T) = B(t,T)\,(r(t)\,dt + \sum_{j=1}^{d} \sigma_j^*(t,T)\,dW^j(t)). \tag{3.4}$$

The solution of (3.4) can be written as

$$B(t,T) = B(0,T)D(t)\,\exp(\sum_{j=1}^{d}(\int_0^t \sigma_j^*(s,T)\,dW^j(s) - \frac{1}{2}\int_0^t \sigma_j^*(s,T)^2\,ds)). \tag{3.5}$$

The HJM model was later extended to allow for an infinite number of driving Brownian motions, i.e. an infinite number of factors influencing the forward rate movement. For example, Filipovic ([Fil 01]) considered the approach

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)\,ds + \sum_{j\in\mathbb{N}} \int_0^t \sigma^j(s,T)\,d\beta^j(s) \tag{3.6}$$

where again T is held constant, $0 \leq t \leq T$, $\sigma^j : \Delta^2 \times \Omega \to \mathbb{R}$, $j \in \mathbb{N}$, and the β^j, $j \in \mathbb{N}$, form a sequence of independent standard Brownian motions. Then for (g_j), $j \in \mathbb{N}$, the standard orthonormal basis in ℓ^2, the series $B_Q := \sum_{j\in\mathbb{N}} \beta^j g_j$ defines a Q-Brownian motion in the weighted sequence Hilbert space ℓ^2_λ. Here $\ell^2_\lambda := \{(v_j)_{j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} | \sum_{j\in\mathbb{N}} \lambda_j v_j^2 < \infty\}$ where $\lambda = (\lambda_j)_{j\in\mathbb{N}}$ is a sequence of strictly positive numbers with $\sum_{j\in\mathbb{N}} \lambda_j < \infty$ (cf. [Fil 01], Prop. 2.1.1.).

Thus in the HJM framework the dynamics of the forward rate are given by a system of infinitely many stochastic differential equations indexed by T.

Musiela Parametrization

Naturally this led to the idea to treat the whole system as *one* infinite dimensional process. I.e. transforming (3.6) into a Hilbert space-valued stochastic evolution equation, thus entering the field of stochastic partial differential equations.

To avoid problems with a varying state space depending on t we switch to a parametrization proposed by Musiela (see [Mus 93]): For $x \geq 0$ set $T = x + t$. Denote by $(S(t))_{t\geq 0}$ the semigroup of right shifts, i.e.

$$S(t)g(x) := g(x+t),\ t \geq 0,\ g : \mathbb{R}_+ \to \mathbb{R}.$$

Applying these definitions we can rewrite (3.6) as

$$\begin{aligned} f(t, x+t) &= S(t) f(0, x) + \int_0^t S(t-s) \alpha(s, x+s)\, ds \\ &+ \sum_{j \in \mathbb{N}} \int_0^t S(t-s) \sigma^j(s, x+s)\, d\beta^j(s). \end{aligned} \tag{3.7}$$

Now we consider $f_t(\cdot) := f(t, \cdot + t),\ t \geq 0$, as a stochastic process with state space H, where H is a suitable separable Hilbert space of functions from $\mathbb{R}_+$ to $\mathbb{R}$. We call H the *space of forward curves*.

Possible choices of H are thoroughly discussed in [Fil 01]. He proposed a family of weighted Sobolev spaces (H_w) defined as follows:

Definition 3.1 *(State space for the HJM model)*

$$H_w := \{f : \mathbb{R}_+ \to \mathbb{R} : f \textit{ is absolutely continuous}, \int_0^\infty f'(x)^2 w(x)\, dx < \infty\},$$

where $w : \mathbb{R}_+ \to [1, \infty)$ *is a non-decreasing* C^1*-function s.t.* $\int_0^\infty w(x)^{-\frac{1}{3}}\, dx < \infty$ *and* f' *denotes the weak derivative of* f.

Then for a fixed weight function w the space H_w is a separable Hilbert space w.r.t. the inner product

$$(f, g)_{H_w} := f(0) g(0) + \int_0^\infty f'(x) g'(x) w(x)\, dx, \quad f, g \in H_w.$$

Moreover, the shift semigroup $(S(t))_{t \geq 0}$ is strongly continuous on H_w with infinitesimal generator $A = \frac{\partial}{\partial x}$. There exists a constant C such that for any $f \in H_w$ we have $\|f\|_{L^\infty(\mathbb{R}_+)} \leq C \|f\|_{H_w}$. (cf. [Fil 01], Thm. 5.1.1.)

Introducing the notation $\alpha_t := \alpha(t, \cdot + t)$, $\sigma_t^j := \sigma^j(t, \cdot + t)$ and $f_0 := f(0, \cdot)$ equation (3.7) transforms into

$$f_t = S(t) f_0 + \int_0^t S(t-s) \alpha_s\, ds + \sum_{j \in \mathbb{N}} \int_0^t S(t-s) \sigma_s^j\, d\beta^j(s)$$

Thus f looks like a a mild solution to the stochastic evolution equation

$$\begin{cases} df_t = (\frac{\partial}{\partial x} f_t + \alpha_t)\ dt + \sum_{j \in \mathbb{N}} \sigma_t^j\ d\beta^j(t) \\ f_0 = f(0, \cdot) \end{cases} \tag{3.8}$$

Finally allowing state-dependent coefficients (we could have done this before) we arrive at the following stochastic equation for the forward rates:

$$\begin{cases} df_t = (\frac{\partial}{\partial x} f_t + \alpha(t, f_t))\ dt + \sigma(t, f_t)\ dB_Q(t) \\ f_0 = f(0, \cdot) \end{cases} \tag{3.9}$$

where

- $f(0,\cdot) \in H$ is the (deterministic) initial forward curve
- α is a measurable function from $([0,T] \times \Omega \times H, \mathcal{P}_T \otimes \mathcal{B}(H))$ into $(H, \mathcal{B}(H))$
- σ is a measurable function from $([0,T] \times \Omega \times H, \mathcal{P}_T \otimes \mathcal{B}(H))$ into $(L_2(G,H), \mathcal{B}(L_2(G,H))$
- B_Q is the Q-Brownian motion on $G := \ell^2_\lambda$.

Hence (3.9) is just a special case of our setup from Chapter 2, with $A = \frac{\partial}{\partial x}$ and $L = B_Q$.

3.3 HJM models with Lévy noise

The *log return* between times t and $t + \Delta t$ on a zero-coupon bond maturing at T is defined as $\ln(B(t+\Delta t, T)) - \ln(B(t,T))$. It can be shown that log bond returns resulting from Gaussian HJM models approximately follow a Normal distribution.

But as in the case of stock prices, empirical studies (see [Rai 00] for a detailed study concerning German government bonds) show that this normality assumption is not really true in reality. Log returns of bonds calculated from historical data turn out to follow a leptokurtic distribution. I.e. very small and very large price movements occur more often than predicted by a Gaussian law. Hence it seems reasonable to replace the Normal distribution by a more flexible one to obtain a more realistic model. That means switching to the much wider class of Lévy processes, with the driving Brownian motion from the classical HJM model as just one prominent example.

Instead of starting with the driving SDE Eberlein and Raible [EbRai 99] suggest to use the explicit bond price formula in the Gaussian framework (3.5) and replace the Brownian motion with a Lévy process. They study the one-dimensional case, while in [Rai 00] the multi-dimensional setting with d independent Lévy processes $(L^j(t))$, $j = 1, \dots d$, is considered. The derived bond price process is of the form

$$B(t,T) = B(0,T)D(t)\,\frac{\exp(\sum_{j=1}^d \int_0^t \sigma_j^*(s,T)\,dL^j(s))}{\exp(\sum_{j=1}^d \int_0^t \theta_j(\sigma_j^*(s,T))\,ds)} \tag{3.10}$$

where $\theta_j(u) := \ln(E[\exp(uL^j(1))])$ is the logarithm of the moment-generating function of the jth Lévy process at time 1. In the classical HJM model we choose $L^j = W^j$, $j = 1, \dots d$ and $\theta_j(u) = u^2/2$ and get back formula (3.5).

It is shown that the approach (3.10) also yields the martingale property for each discounted bond price process

$$D(t)^{-1}B(t,T) = \exp(-\int_0^t r(s)ds)B(t,T),\ 0 \leq t \leq T,$$

thus ruling out arbitrage. ([Rai 00], Col.7.10, with integrability conditions on L and σ.)

HJM equation with Lévy noise

We will explore another approach, i.e. we begin with the driving SDE and replace the Q-Brownian motion in (3.9) by an infinite-dimensional Lévy process.

Concerning the two different approaches we refer to [App 04], p.273, l.18-24, where a similar topic is discussed in the context of modeling a stock price by a simple one-dimensional linear stochastic equation:

> The use of Lévy processes in finance is at a relatively early stage of development and there seems to be some disagreement in the literature as to whether it is best to employ a stochastic exponential to model stock prices, (...), or to use *geometric Lévy motion*, $S(t) = e^{X(t)}$ (the reader can check that these are, more or less, equivalent when X is Gaussian). Indications are that the former is of greater theoretical interest while the latter may be more realistic in practical models.

Starting with the dynamics leads to the equation

$$\begin{cases} df_t = (\frac{\partial}{\partial x} f_t + \alpha(t, f_t))\ dt + \sigma(t, f_t)\ dL(t) \\ f_0 = f(0, \cdot) \end{cases}. \tag{3.11}$$

Then a mild solution to this equation has to satisfy

$$f_t = S(t)f_0 + \int_0^t S(t-s)\alpha(s, f_s)\, ds + \int_0^t S(t-s)\sigma(s, f_s)\, dL(s). \tag{3.12}$$

Working with the Musiela parametrization the short rate is given by $r(t) = f_t(0)$ and the discounting process by $D(t)^{-1} = \exp(-\int_0^t f_s(0)\, ds)$.

We take $H = G = H_w$ for a fixed weight function w. (For the definition and properties of the state space H_w see Def. 3.1.) We will loosely follow the setting of [ÖzkSch 05], section 3, and assume that our Lévy process L is a martingale, more precisely that L fulfills condition (F) and can be written as

$$L(t) = B_Q(t) + \int_H x\, \tilde{N}(t, dx). \tag{3.13}$$

Such L have second moments in the following sense

$$\begin{aligned} E[\|L(t)\|^2] &= E[\|B_Q(t) + \int_H x\,\tilde{N}(t,dx)\|^2] \\ &\leq 2(E[\|B_Q(t)\|^2] + E[\|\int_H x\,\tilde{N}(t,dx)\|^2]) \\ &= 2(t\ tr(Q) + t\int_H \|x\|^2\,\nu(dx)) < \infty,\ t\in[0,\bar{T}]. \end{aligned}$$

Here the last equality results from Prop. A.2 and Prop. 1.2. For Lévy processes with second moments we have the following useful representation

Proposition 3.1 *([ÖzkSch 05], Prop. 2.1.)*
Consider a Lévy process $(L(t))_{t\in[0,\bar{T}]}$ with values in H and $E[\|L(t)\|^2] < \infty$ for all $t \in [0,\bar{T}]$. Then for (e_k), $k \in \mathbb{N}$, an arbitrary orthonormal basis of H we have the following decomposition

$$L(t) = \sum_{k=1}^{\infty}(L(t), e_k)_H e_k,$$

where the series converges in L^2. Moreover, the process $(l_k(t))_{t\in[0,\bar{T}]}$ defined by $l_k(t) := (L(t), e_k)_H$ is a real-valued Lévy process for any $k \in \mathbb{N}$.

Proof. Consider $t \in [0,\bar{T}]$ fixed. Then by the Bessel inequality we have for any $m \in \mathbb{N}$

$$\begin{aligned} E[\|\sum_{k=1}^{m}(L(t),e_k)_H e_k\|^2] &= \sum_{k,j=1}^{m} E[(L(t),e_k)(L(t),e_j)](e_k,e_j) \\ &= \sum_{k=1}^{m} E[(L(t),e_k)^2] \leq E[\|L(t)\|^2]. \end{aligned}$$

Hence the series converges in L^2. The fact that the processes l_k are real-valued Lévy processes is clear by the definition.

□

To ensure the existence of exponential moments for L needed later, we impose the following condition on ν

$$\int_{\|x\|\geq 1} \exp((c,x)_H)\,\nu(dx) < \infty,\ \forall\, c \in H. \qquad (3.14)$$

Unlike [ÖzkSch 05] we allow the coefficients to explicitly depend on the state f_t. As before σ is a measurable function from $[0,T]\times\Omega\times H$ into $L_2(H)$ and α is a measurable function from $[0,T]\times\Omega\times H$ into H. We require σ to be Lipschitz continuous in the third variable and uniformly bounded. The process $\alpha(\cdot,f.)$ is assumed to be P-a.s. Bochner integrable on $[0,\bar{T}]$.

To shorten notation we define $\alpha^*(s,f_s)(T) := \int_0^{T-s}\alpha(s,f_s)(u)\,du$ and also $\sigma_k^*(s,f_s)(T) := \int_0^{T-s}[\sigma(s,f_s)e_k](u)\,du$.

As outlined in section 3.2, to guarantee the absence of arbitrage we have to make sure that all discounted bond prices follow local martingales under the risk-neutral measure P. This is done in the next theorem which states a HJM-type condition relating forward drifts and volatilities (compare formula (3.3))

Theorem 3.1 *(cf. [ÖzkSch 05], Thm. 3.1.)*
All discounted bond prices are local martingales, if for all $0\le t\le T\le\bar{T}$ the following condition holds P-a.s.:

$$\begin{aligned} 0 &= -\alpha^*(t,f_t)(T)+\frac{1}{2}\sum_{k=1}^{\infty}\lambda_k[\sigma_k^*(t,f_t)(T)]^2 \\ &+ \int_H [\exp(\int_0^{T-t}[\sigma(t,f_t)x](u)\,du)-1-\int_0^{T-t}[\sigma(t,f_t)x](u)\,du]\ \nu(dx). \end{aligned} \tag{3.15}$$

In the proof we make use of the following Itô formula for Lévy processes obtained from [Kun 70]:

Theorem 3.2 *Let $(L(t))_{t\in[0,\bar{T}]}$ be a Lévy process with values in the separable Hilbert space H. Moreover, let $(\sigma(t))_{t\in[0,\bar{T}]}$ be a predictable $L(H)$-valued process which is bounded and set $X(t) := \int_0^t\sigma(s)\,dL(s)$ for all $t\in[0,\bar{T}]$. Denote by λ_k and e_k, $k\in\mathbb{N}$, the eigenvalues resp. eigenvectors of Q. For an open subset $A\subset H$ and a twice differentiable function $F:A\to H$ with uniformly continuous second derivatives on bounded subsets of H it holds, that*

$$\begin{aligned} F(X(t)) &= F(X(0))+\int_0^t DF(X(s-))\,dX(s) \\ &+ \frac{1}{2}\int_0^t\sum_{k=1}^{\infty}\lambda_k D^2F(X(s-))\,(\sigma(s)e_k,\sigma(s)e_k)\,ds \\ &+ \sum_{s\le t}[\Delta(F(X))(s)-DF(X(s-))\Delta X(s)]. \end{aligned}$$

Note that the second derivative $D^2F(\cdot)$ is a bilinear mapping, and we just write $D^2F(\cdot)\,(g,h)$ for this mapping evaluated at g and h.

Proof of Thm. 3.1. Define

$$y(t,T) := -\int_0^{T-t} f_t(u)\,du.$$

Then $B(t,T) = \exp(y(t,T))$ and we first derive the dynamics of the process $(y(t,T))_{t\geq 0}$. From (3.12) we get

$$\begin{aligned} y(t,T) &= -\int_0^{T-t} [f_0(u+t) + \int_0^t \alpha(s,f_s)(u+t-s)\,ds \\ &+ (\int_0^t S(t-s)\sigma(s,f_s)\,dL(s))(u)]\,du. \end{aligned} \tag{3.16}$$

Setting $t=0$ yields

$$\begin{aligned} -\int_0^{T-t} f_0(u+t)\,du &= y(0,T) + \int_0^T f_0(u)\,du - \int_0^{T-t} f_0(u+t)\,du \\ &= y(0,T) + \int_0^t f_0(u)\,du. \end{aligned} \tag{3.17}$$

Since we have to consider the discounted bond prices, it is convenient to have the short rate explicitly appear in the dynamics of y. Again from (3.12) we conclude

$$\begin{aligned} \int_0^t f_u(0)\,du &= \int_0^t f_0(u)\,du + \int_0^t \int_0^u \alpha(s,f_s)(u-s)\,ds\,du \\ &+ \int_0^t (\int_0^u S(u-s)\sigma(s,f_s)\,dL(s))(0)\,du. \end{aligned} \tag{3.18}$$

Inserting (3.17) and (3.18) into (3.16) we obtain

$$\begin{aligned} y(t,T) &= y(0,T) + \int_0^t f_u(0)\,du \\ &- \int_0^t \int_0^u \alpha(s,f_s)(u-s)\,ds\,du - \int_0^T \int_0^t \alpha(s,f_s)(u-s)\,ds\,du \\ &- \int_0^{T-t} (\int_0^t S(t-s)\sigma(s,f_s)\,dL(s))(u)\,du \\ &- \int_0^t (\int_0^u S(u-s)\sigma(s,f_s)\,dL(s))(0)\,du. \end{aligned} \tag{3.19}$$

Now we can interchange the order of the α-integrals by using Fubini's theorem. This gives us the following expression for the sum of the α-integrals

$$-\int_0^t \int_s^T \alpha(s,f_s)(u-s)\,du\,ds.$$

We apply the decomposition of L from Prop. 3.1 to get

$$\begin{aligned}(\int_0^t S(t-s)\sigma(s,f_s)\,dL(s))(u) &= \sum_{k=1}^{\infty}\int_0^t [\sigma(s,f_s)e_k](u+t-s)\,dl_k(s) \\ &= \sum_{k=1}^{\infty}\int_0^t \sigma_k(s,f_s)(u+t-s)\,dl_k(s),\end{aligned}$$

where $\sigma_k(s,f_s)(u) := [\sigma(s,f_s)e_k](u)$. Note that σ, and thus σ_k for any $k \in \mathbb{N}$, is uniformly bounded.

This property also allows us to use the stochastic Fubini theorem (cf. [Sto 05], Thm. 3.3.4; [DaPrZa 92], Thm. 4.18) to interchange the order of integration in the last two terms of (3.19). Hence we obtain

$$\begin{aligned}&\int_0^{T-t}(\int_0^t S(t-s)\sigma(s,f_s)\,dL(s))(u)\,du \\ =\;& \int_0^{T-t}\sum_{k=1}^{\infty}\int_0^t \sigma_k(s,f_s)(u+t-s)\,dl_k(s)\,du \\ =\;& \sum_{k=1}^{\infty}\int_0^t\int_t^T \sigma_k(s,f_s)(u-s)\,du\,dl_k(s)\end{aligned}$$

as well as

$$\int_0^t(\int_0^u S(u-s)\sigma(s,f_s)\,dL(s))(0)\,du = \sum_{k=1}^{\infty}\int_0^t\int_s^t \sigma_k(s,f_s)(u-s)\,du\,dl_k(s).$$

Combining these calculations leads to the following formulation of y:

$$\begin{aligned}y(t,T) &= y(0,T) + \int_0^t f_u(0)\,du - \int_0^t\int_s^T \alpha(s,f_s)(u-s)\,ds \\ &- \sum_{k=1}^{\infty}\int_0^t\int_s^T \sigma_k(s,f_s)(u-s)\,du\,dl_k(s).\end{aligned}$$

Finally we use the abbreviations $\alpha^*(s,f_s)(T)$ and $\sigma_k^*(s,f_s)(T)$ introduced above and obtain the dynamics of y:

$$\begin{aligned}y(t,T) &= y(0,T) + \int_0^t f_u(0)\,du - \int_0^t \alpha^*(s,f_s)(T)\,du\,ds \\ &- \sum_{k=1}^{\infty}\int_0^t \sigma_k^*(s,f_s)(T)\,dl_k(s). \qquad (3.20)\end{aligned}$$

Again, note that the σ_k^* are bounded. In order to apply the Itô formula from 3.2 we need a more functional analytic representation of the equation for y. So we define $\Phi : [0, \bar{T}] \times \Omega \to L(H)$ by

$$[\Phi(s)g](\cdot) := \int_s^{\cdot} [\sigma(s, f_s)g](u - s)\, ds, \quad g \in H.$$

Then

$$\begin{aligned}
\int_0^t [\Phi(s)\, dL(s)](T) &= \sum_k \int_0^t [\Phi(s)e_k](T)\, dl_k(s) \\
&= \sum_k \int_0^t \int_s^T [\sigma(s, f_s)e_k](u - s)\, ds\, dl_k(s) \\
&= \sum_k \int_0^t \sigma_k^*(s, f_s)(T)\, du\, dl_k(s).
\end{aligned}$$

Setting $m(s, f_s)(\cdot) := f_s(0) - \alpha^*(s, f_s)(\cdot)$ we get

$$y(t) = y(0) + \int_0^t m(s, f_s)\, ds - \int_0^t \Phi(s)\, dL(s). \tag{3.21}$$

Since $B(t, T) = \exp(y(t, T))$, we define

$$F : H \to H, \quad g(\cdot) \mapsto \exp(g(\cdot)),$$

where $\exp(g(\cdot))$ is the function h such that $h(x) = \exp(g(x))$, $x \geq 0$. Then $B(t, \cdot) = F(y(t, \cdot))$. For two real-valued functions functions g, h we set $(g \times h)(\cdot) := g(\cdot)h(\cdot)$. It is easy to show that $DF(\cdot) = F(\cdot) \times id$ and $D^2F(\cdot) = F(\cdot) \times id \times id$. Hence the application of Theorem 3.2 yields

$$\begin{aligned}
B(t) &= B(0) + \int_0^t DF(y(s-))\, (m(s, f_s)\, ds - \Phi(s)\, dL(s)) \\
&+ \frac{1}{2} \int_0^t \sum_{k=1}^{\infty} \lambda_k D^2F(y(s-))\, (\Phi(s)e_k, \Phi(s)e_k)\, ds \\
&+ \sum_{s \leq t} [F(y(s)) - F(y(s-)) - DF(y(s-))\, \Phi(s)\Delta L(s)].
\end{aligned}$$

Inserting the derivatives of F we conclude

$$\begin{aligned}
B(t) &= B(0) + \int_0^t B(s-) \times (m(s, f_s)\, ds - \int_0^t B(s-) \times \Phi(s)\, dL(s)) \\
&+ \frac{1}{2} \int_0^t \sum_{k=1}^{\infty} \lambda_k B(s-) \times (\Phi(s)e_k) \times (\Phi(s)e_k)\, ds \\
&+ \sum_{s \leq t} [\Delta B(s) - B(s-) \times (\Phi(s)\Delta L(s))].
\end{aligned}$$

By evaluating $B(t,T)$ at maturity T we can deduce

$$\begin{aligned} B(t,T) &= B(0,T) + \int_0^t B(s-,T)[f_s(0) - \alpha^*(s,f_s)(T)]\,ds \\ &- \sum_k \int_0^t B(s-,T)\sigma_k^*(s,f_s)(T)\,dl_k(s) \\ &+ \frac{1}{2}\int_0^t B(s-,T)\sum_{k=1}^{\infty}\lambda_k[\sigma_k^*(s,f_s)(T)]^2\,ds \\ &+ \sum_{s\leq t}[\Delta B(s,T) - B(s-,T)(\Phi(s)\Delta L(s))(T)]. \end{aligned}$$

Since $B(s) = F(y(s))$ we get $B(s)/B(s-) = \exp(\Phi(s)\Delta L(s))$ and thus obtain

$$\begin{aligned} \Delta B(s,T) &= B(s-,T)(\frac{B(s,T)}{B(s-,T)} - 1) \\ &= B(s-,T)(\exp([\Phi(s)\Delta L(s)](T)) - 1). \end{aligned}$$

This leads to

$$\begin{aligned} &\sum_{s\leq t}[\Delta B(s,T) - B(s-,T)(\Phi(s)\Delta L(s))(T)] \\ = &\sum_{s\leq t} B(s-,T)[\exp((\Phi(s)\Delta L(s))(T)) - 1 - (\Phi(s)\Delta L(s))(T)]. \end{aligned}$$

And this expression can also be written as

$$\int_0^t \int_H B(s-,T)[\exp((\Phi(s)x)(T)) - 1 - (\Phi(s)x)(T)]\,N(ds,dx).$$

Since $D(t)^{-1}$ is real-valued and of finite variation, applying Itô's product rule yields

$$d[D(t)^{-1}B(t,T)] = (-f_{t-}(0))D(t-)^{-1}B(t-,T)\,dt + D(t-)^{-1}\,dB(t,T)$$

and therefore the discounted bond price process fulfills:

$$\begin{aligned}
& D(t)^{-1}B(t,T) && (3.22)\\
= \; & D(0)^{-1}B(0,T) - \int_0^t D(s)^{-1}B(s-,T)\alpha^*(s,f_s)(T)\,ds \\
- \; & \sum_k \int_0^t D(s)^{-1}B(s-,T)\sigma_k^*(s,f_s)(T)\,dl_k(s) \\
+ \; & \frac{1}{2}\int_0^t D(s)^{-1}B(s-,T)\sum_{k=1}^{\infty}\lambda_k[\sigma_k^*(s,f_s)(T)]^2\,ds \\
+ \; & \int_0^t\int_H D(s)^{-1}B(s-,T)[\exp((\Phi(s)x)(T)) - 1 - (\Phi(s)x)(T)]\,\tilde{N}(ds,dx) \\
+ \; & \int_0^t\int_H D(s)^{-1}B(s-,T)[\exp((\Phi(s)x)(T)) - 1 - (\Phi(s)x)(T)]\,\nu(dx)\,ds.
\end{aligned}$$

The term $D(0)^{-1}B(0,T)$ is just a constant. The Lévy processes l_k are martingales by assumption. The same holds for $\int_H x\,\tilde{N}(\cdot,dx)$. Hence the stochastic integrals w.r.t. them are local martingales. Remembering, that $(\Phi(s)x)(T) = \int_0^{T-s}[\sigma(s,f_s)x](u)\,du$, we conclude.

□

Note that if the forward rates are positive (a suitable property indeed), the discounted bond prices will also be true martingales as they are bounded by 1.

Writing out the abbreviations (3.15) reads

$$\begin{aligned}
0 \; = \; & -\int_0^{T-t}\alpha(t,f_t)(u)\,du + \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k\Big(\int_0^{T-t}[\sigma(t,f_t)e_k](u)\,du\Big)^2 \\
+ \; & \int_H [\exp\Big(\int_0^{T-t}[\sigma(t,f_t)x](u)\,du\Big) - 1 - \int_0^{T-t}[\sigma(t,f_t)x](u)\,du]\;\nu(dx).
\end{aligned}$$

Taking derivatives on both sides we get

$$\begin{aligned}
\alpha(t,f_t)(T-t) \; = \; & \sum_{k=1}^{\infty}\lambda_k[\sigma(t,f_t)e_k](T-t)\int_0^{T-t}[\sigma(t,f_t)e_k](u)\,du && (3.23)\\
+ \; & \int_H [\sigma(t,f_t)x](T-t)\Big(\exp\Big(\int_0^{T-t}[\sigma(t,f_t)x](u)\,du\Big) - 1\Big)\;\nu(dx).
\end{aligned}$$

Here interchanging differentiation and summation is justified by [Fil 01] Lem. 4.3.2. (uniform convergence on compacts); interchanging differentiation and integration by Lebesgue's dominated convergence theorem and (3.14).

We can now state the following (conditional) proposition on the existence of an HJM model with Lévy noise and state-dependent coefficients:

Proposition 3.2 *Let be* $\sigma : ([0,T]\times\Omega\times H, \mathcal{P}_T\otimes\mathcal{B}(H)) \to (L_2(H), \mathcal{B}(L_2(H)))$ *measurable, uniformly bounded and Lipschitz continuous in the third variable. Moreover let L be a Lévy process satisfying (3.13) and (3.14). Define the drift coefficient* α *by formula (3.23).*
If then $\alpha : ([0,T]\times\Omega\times H, \mathcal{P}_T\otimes\mathcal{B}(H)) \to (H, \mathcal{B}(H))$ *is measurable and Lipschitz continuous in the third variable with* $\sup_{(s,\omega)}\|a(s,\omega,0)\|_H \leq C$*, we get an HJM model with Lévy noise. I.e., for any initial forward curve* $f_0 \in H$ *equation (3.11) has a unique mild solution, describing the forward rate dynamics and thus the arbitrage-free movement of the discounted bond prices.*

Proof. Existence (and uniqueness) of the mild solution is just an application of Theorem 2.1. Absence of arbitrage is due to the special form of α derived in Theorem 3.1.

$\square$

Of course, it would be desirable to give explicit (additional) conditions on σ and L which would ensure that α has the required properties.

In the case of $L = B_Q$ (hence $\nu \equiv 0$) this is done in [Fil 01]:
Define H^0 as a closed subspace of H by

$$H^0 := H^0_w := \{f \in H_w : f(\infty) = 0\}.$$

Assume that B_Q takes values in H^0. If σ is a measurable, Lipschitz continuous (in f) and uniformly bounded mapping from $([0,T]\times\Omega\times H, \mathcal{P}_T\otimes\mathcal{B}(H))$ into $(L_2(H^0), \mathcal{B}(L_2(H^0)))$, α is a mapping from $([0,T]\times\Omega\times H, \mathcal{P}_T\otimes\mathcal{B}(H))$ into $(H, \mathcal{B}(H))$ with the same properties.
Proof. (cf. [Fil 01] Lem. 5.2.1., Lem. 5.2.2. ii))

In the case of general L finding explicit conditions for the moment remains an open problem. While the first term of α is covered by Filipovic's results, there seems to be no straightforward approach for the second one. One might think of strengthening the requirements on the state space or a special form of σ. Another idea to get the Lipschitz continuity of α could be to assume the Frechét differentiability of σ and then find conditions under which this would give the Frechet differentiability of α with bounded derivative.

Appendix A

Q-Brownian Motion and Stochastic Integration

Let be $(G, (\ ,\ )_G)$ and $(H, (\ ,\ )_H)$ separable Hilbert spaces; $(\Omega, \mathcal{F}, P)$ a complete probability space with $(\mathcal{F}_t)$, $t \geq 0$, a right-continuous filtration on $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}_0$ contains all P-nullsets.

Definition A.1 *An operator $T \in L(G, H)$ is called* nuclear *if there exist sequences $(a_j)_{j\in\mathbb{N}}$ in H and $(b_j)_{j\in\mathbb{N}}$ in G such that*

$$Tx = \sum_{j\in\mathbb{N}} a_j (b_j, x)_G \quad \textit{for all } x \in G$$

and

$$\sum_{j\in\mathbb{N}} \|a_j\|_H \|b_j\|_G < \infty.$$

The space of nuclear operators from G to H is denoted by $L_1(G, H)$.

Proposition A.1 *The space $L_1(G, H)$ endowed with the norm*

$$\|T\|_{L_1(G,H)} := \inf\{\sum_j \|a_j\|_H \|b_j\|_G \ : \ Tx = \sum_{j\in\mathbb{N}} a_j (b_j, x)_G \quad \textit{for all } x \in G\}$$

is a Banach space.

Proof. ([MeVo 92], 16.25 Cor.)

Definition A.2 *For $T \in L(G)$ and (e_n), $n \in \mathbb{N}$, an orthonormal basis of G we define the* trace *of T as*

$$tr(T) := \sum_{n\in\mathbb{N}} (Te_n, e_n)_G$$

if the series converges.

This expression is well-defined independently of the choice of (e_n), $n \in \mathbb{N}$ (cf. [ReSi 80, Thm. VI.18]). Moreover, for $T \in L_1(G)$ we have that $|tr(T)| \leq \|T\|_{L_1(G)}$.

If T^* is the adjoint operator of T, then T^*T is non-negative. By the square-root lemma ([ReSi 80], Thm. VI.9) for every non-negative bounded linear operator Q on G the square-root $Q^{\frac{1}{2}}$ exists, i.e. a unique non-negative bounded linear operator with $Q^{\frac{1}{2}}Q^{\frac{1}{2}} = Q$. Hence for any $T \in L(G)$ we can define $|T| := (T^*T)^{\frac{1}{2}}$. T is called *trace class* if $tr|T| < \infty$. We have that $L_1(G)$ is the space of trace class operators and $tr|T| = \|T\|_{L_1(G)}$.

Definition A.3 *A G-valued stochastic process B_Q adapted to $(\mathcal{F}_t)$, $t \geq 0$, is a Q-Brownian motion if*

- $B_Q(0) = 0$
- *B_Q has increments independent of the past, i.e. $B_Q(t) - B_Q(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t < \infty$*
- *B_Q has stationary Gaussian increments, i.e. $P \circ (B_Q(t) - B_Q(s))^{-1} = \mathcal{N}(0, (t-s)Q)$ for all $0 \leq s < t < \infty$*
- *B_Q has P-a.s. continuous trajectories*

Here $\mathcal{N}(0, Q)$ denotes a Gaussian probability measure on G with mean 0 and covariance operator Q. Q is a non-negative, symmetric trace class operator (cf. [FriKno 01], Section 1.1).

Definition A.4 *Let M be a stochastic process with values in a separable Banach space E. The process M is called an $\mathcal{F}_t$-martingale, if*

- *$E[\|M(t)\|] < \infty$ for all $0 \leq t < \infty$*
- *M(t) is $\mathcal{F}_t$-measurable for all $0 \leq t < \infty$*
- *$E[M(t)|\mathcal{F}_s] = M(s)$ P-a.s. for all $0 \leq s \leq t < \infty$*

Proposition A.2 *A Q-Brownian motion $B_Q(t)$, $t \in [0, T]$, is a continuous square-integrable martingale. Moreover, $E[\|B_Q(t)\|^2] = t \cdot tr(Q) < \infty$.*

Proof. ([FriKno 01], Prop. 1.20 & Prop. 1.3).

Definition A.5 *Set $G_0 := Q^{\frac{1}{2}}(G)$. Let be $\mathcal{N}_B^2(0, T)$ the space of all mappings X on $[0, T] \times \Omega$ taking values in $L_2(G_0, H)$, such that*

1. *X is predictable, i.e. $\mathcal{P}_T/\mathcal{B}(L_2(G_0,H))$-measurable.*
2. *For any $(t,\omega) \in [0,T] \times \Omega$ $X(t,\omega) \circ Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator and we have*

$$\|X\|_T := (E[\int_0^T \|X(s)Q^{\frac{1}{2}}\|_{L_2}^2\, ds])^{\frac{1}{2}} < \infty.$$

Proposition A.3 *For $X \in \mathcal{N}_B^2(0,T)$ the stochastic integrals*

$$\int_0^t X(s)\, dB_Q(s),\ t \in [0,T],$$

are well-defined and we have

$$E[\|\int_0^t X(s)\, dB_Q(s)\|^2] = E[\int_0^t \|X(s)Q^{\frac{1}{2}}\|_{L_2}^2\, ds],\ t \in [0,T]. \tag{A.1}$$

Moreover the process $(\int_0^t X(s)\, dB_Q(s))_{t\in[0,T]}$ is a continuous square-integrable martingale with respect to $\mathcal{F}_t$, $t \in [0,T]$.

These results are taken from Section 1.3 of [FriKno 01] which is based on [DaPrZa 92]. There a detailed construction of the stochastic integral with respect to a Q-Brownian motion is carried out (including more general integrands and further properties of the integral).

Appendix B

The Stochastic Integral in Hilbert Spaces with respect to General Martingales

Let be $(\Omega, \mathcal{F}, P)$ a complete probability space. Let $(\mathcal{F}_t)$, $t \geq 0$, be a right-continuous filtration on $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}_0$ contains all P-nullsets.
$(H, (\ ,\ )_H)$ and $(G, (\ ,\ )_G)$ are (infinite-dimensional) separable Hilbert spaces.
Fix $T > 0$. A G-valued martingale $M = (M_t)_{t\in[0,T]}$ is called *square-integrable martingale*, if for any $t \in [0,T]$ holds $E(\|M_t\|^2) < \infty$.
Following Métivier [Met 77], our aim is to define the stochastic integral "$\int X\, dM$" for M a square-integrable martingale and X from a wide class of stochastic processes taking values in the linear (possibly unbounded) operators from G to H.

First we recall some basic facts about tensor products in Hilbert spaces used in the following sections (cf. [Tre 67]):

B.1 Tensor products in Hilbert spaces

Let $(H, (\ ,\ )_H)$ and $(G, (\ ,\ )_G)$ be separable Hilbert spaces.

Definition B.1 *The algebraic tensor product $H \otimes G$ of H and G is defined as the (smallest) vector space such that*

1. *there exists a bilinear mapping π from $H \times G := \{(h,g) \mid h \in H,\ g \in G)\}$ into $H \otimes G$*
2. *any bilinear mapping $b : H \times G \to K$, K any Hilbert space, can be*

written in the form $b = u_b \circ \pi$ where u_b is a uniquely defined linear map from $H \otimes G$ into K (depending on b).

For existence and uniqueness (up to isomorphism) of the tensor product see e.g. ([Tre 67], Theorem 39.1, p.404).

Proposition B.1 *On $H \otimes G$ there exists a unique norm such that for any continuous b the corresponding map u_b is also continuous, and moreover $\|b\|_{B(H\times G,K)} = \|u_b\|_{L(H\otimes G,K)}$.*

Proof. ([Tre 67], Proposition 43.4, p.438 & Proposition 43.12(b), p.443).

Definition B.2 *The norm from B.1 is called* trace norm *and is denoted by $\|\cdot\|_1$. The completion of the space $H \otimes G$ w.r.t. $\|\cdot\|_1$ is called the* projective tensor product *of the spaces H and G, denoted by $H\hat{\otimes}_1 G$.*

For $K = \mathbb{R}$ and $b = (\ ,\ )_G$ the linear mapping u_b in the factorization $b = u_b \circ \pi$ is called the *trace* on $G\hat{\otimes}_1 G$ and we write $u_b = tr$. Thus tr is the unique linear continuous extension of the mapping

$$g \otimes g' \ \mapsto \ (g, g')_G.$$

On $H \otimes G$ we can introduce an inner product which is the unique linear continuous extension of the mapping

$$< h \otimes g, h' \otimes g' > \mapsto \ (h, h')_H \cdot (g, g')_G.$$

We can assign to $h\otimes g$ the linear mapping $(h, \cdot)_H g$. Thereby $H\otimes G$ is uniquely embedded into $L(H, G)$. For $H\hat{\otimes}_1 G$ we have the following characterization:

Proposition B.2 *$H\hat{\otimes}_1 G$ is (canonically) isomorphic to the space of nuclear (or trace class) operators from H to G, i.e. $L_1(H, G)$.*

Proof. ([Tre 67], p.495 ff.).

To the element $Q \in H\hat{\otimes}_1 G$ we assign the operator $\tilde{Q} \in L_1(H, G)$ given by the following equation:

$$(\tilde{Q}h, g)_G =< Q, h \otimes g >, \ h \in H, \ g \in G.$$

Then $\|Q\|_1 = \|\tilde{Q}\|_{L_1(H,G)}$.

Proposition B.3 *$H\hat{\otimes}_1 G$ has the properties of the dual space of a separable Banach space.*

Proof. ([Tre 67], Theorem 48.5', p.498).

For $u \in L(G,H)$ we denote by $u \otimes u$ the unique linear continuous extension of the mapping

$$u \otimes u(g \otimes g') \mapsto u(g) \otimes u(g')$$

from $G \otimes G$ to $H \otimes H$.

B.2 Doleans measures

Definition B.3 *A subset $A \subset [0,T] \times \Omega$ of the form $A =]s,t] \times F$ where $F \in \mathcal{F}_s$, $0 \le s < t \le T$, or $\{0\} \times F$, $F \in \mathcal{F}_0$, is called* predictable rectangle. *The family of predictable rectangles is denoted by $\mathcal{R}_T$.*

Let be $\mathcal{P}_T = \sigma(\mathcal{R}_T)$, the σ-algebra generated by $\mathcal{R}_T$. $\mathcal{P}_T$ is called the σ-algebra of the predictable sets; *a stochastic process X measurable with respect to $\mathcal{P}_T$ is called* predictable.

Definition B.4 *(cf. [Met 77], 2.3/2.4, p.6/7)*
Let be $(Z_t)_{t\in[0,T]}$ a real-valued process adapted to $(\mathcal{F}_t)_{t\ge 0}$ with $E(|Z_t|) < \infty$ for all $t \in [0,T]$. The real-valued function λ_Z is defined on $\mathcal{R}_T$ by setting

$$\lambda_Z(]s,t] \times F) = E(1_F \cdot (Z_t - Z_s)) \ , \ \lambda_Z(\{0\} \times F) = 0.$$

λ_Z is additive and therefore can be extended to a content on $\mathcal{A}_T$, the ring generated by $\mathcal{R}_T$.

If λ_Z has a σ-additive extension from $\mathcal{R}_T$ to $\mathcal{P}_T$, also denoted by λ_Z, this σ-additive measure on $([0,T] \times \Omega, \mathcal{P}_T)$ is called the Doleans measure of the *process Z.*

Remark B.1 *It is easy to show that the process Z is a martingale (sub-martingale / supermartingale), if and only if λ_Z is identically zero (positive / negative).*

Proposition B.4
1. *If M_t, $t \in [0,T]$, is a G-valued $\mathcal{F}_t$-martingale then $\|M_t\|^2$, $t \in [0,T]$, is a positive real-valued $\mathcal{F}_t$-submartingale.*
2. *If additionally M has right-continuous paths, then the process $\|M_t\|^2$, $t \in [0,T]$, admits a Doleans measure.*

Proof.
1. ([DaPrZa 92], Proposition 3.7, p.78)
2. ([Met 77], Prop. 2.6, p.–I.9– & Prop. 20.1, p.–III.20–)

□

The positive Doleans measure of $\|M_t\|^2$ from Proposition B.4 will be denoted by μ_M. From [Kun 70] we know that every square-integrable martingale admits a strongly càdlàg version. In the following we only consider this càdlàg version.

The measure $\mathbf{a_M}$(cf. [Met 77], p.-I.83-)
For a square-integrable martingale M we have $\|M_t \otimes M_t\|_1 = \|M_t\|_G^2$. Hence the Banach space-valued random variable $M_t \otimes M_t$ (taking values in $G\hat{\otimes}_1 G$) is integrable. For any predictable rectangle $]s,t] \times F$ we define

$$a_M(]s,t] \times F) := E(1_F \cdot (M_t - M_s)^{\otimes 2}) \tag{B.1}$$

The equation

$$\begin{aligned} E(1_F \cdot (M_t - M_s)^{\otimes 2}) &= E(1_F \cdot (M_t^{\otimes 2} - M_s^{\otimes 2})) \\ - \quad E(1_F \cdot M_s \otimes (M_t - M_s)) &- E(1_F \cdot (M_t - M_s) \otimes M_s) \end{aligned}$$

combined with the martingale property gives

$$a_M(]s,t] \times F) = E(1_F \cdot (M_t^{\otimes 2} - M_s^{\otimes 2})).$$

Therefore it is clear that a_M is additive and that it is possible to extend a_M to an additive function on $\mathcal{A}_T$, the ring generated by $\mathcal{R}_T$.

For the proof of the next proposition we need the following abstract theorem from the theory of linear operators:

Theorem B.1 *Let (S, Σ, μ) be a σ-finite positive measure space and let T be a continuous linear map of $L^1(S, \Sigma, \mu)$ into the dual space B^* of a separable Banach space B. Then there is a μ-essentially unique function $b^*(\cdot)$ on S to B^* such that $b^*(\cdot)b$ is μ-essentially bounded for each $b \in B$ and*

$$(Tf)b = \int_S b^*(s)bf(s)\,\mu(ds), \quad f \in L^1(S, \Sigma, \mu),\ b \in B. \tag{B.2}$$

Moreover, $\|T\| = \operatorname{ess\,sup}_{s\in S} \|b^*(s)\|$. Conversely, if $b^*(\cdot)$ is any function on S to B^* such that $b^*\cdot)b$ is measurable for each $b \in B$, and such that

$$\operatorname*{ess\,sup}_{s\in S} \|b^*(s)\| = M < \infty,$$

then equation (B.2) defines a continuous linear map T of $L^1(S, \Sigma, \mu)$ into B^* whose norm is M.

Proof. ([DunSch 57], VI.8.6, 6Theorem, p.508).

Proposition B.5 *(cf. [Met 77], p.-I.84-)*
There exists a unique (up to μ_M-equivalence) predictable process Q_M with values in $G\hat{\otimes}_1 G$ such that

$$\forall \varphi \in L^1([0,T]\times\Omega,\mathcal{P}_T,\mu_M): \quad \int \varphi \; da_M \;=\; \int \varphi \; Q_M \; d\mu_M.$$

Proof.
From (B.1) follows

$$\begin{aligned} & \|a_M(]s,t]\times F)\|_1 \;\le\; E(1_F\cdot\|(M_t-M_s)^{\otimes 2}\|_1) \\ =\; & E(1_F\cdot\|M_t-M_s\|_G^2) \;=\; E(1_F\cdot(\|M_t\|_G^2-\|M_s\|_G^2)) \\ =\; & \mu_M(]s,t]\times F) \end{aligned} \tag{B.3}$$

For any real-valued process X of the form

$$X=\sum_{i=1}^n \gamma_i\cdot 1_{]s_i,t_i]\times F_i}, \quad]s_i,t_i]\times F_i\in\mathcal{R}_T \;\; \forall\, i,$$

we set

$$\int X \; da_M := \sum_{i=1}^n \gamma_i\cdot a_M(]s_i,t_i]\times F_i).$$

By inequality (B.3) we get

$$\begin{aligned} & \|\int X \; da_M\|_1 \;\le\; \sum_{i=1}^n \|\gamma_i\cdot 1_{]s_i,t_i]\times F_i}\|_1 \\ =\; & \sum_{i=1}^n |\gamma_i|\cdot\mu_M(]s_i,t_i]\times F_i) \;=\; \int |X| \; d\mu_M \;=\; \|X\|_{L^1(\mu_M)} \end{aligned}$$

Therefore the mapping $X\mapsto\int X\;da_M$ has a unique linear, continuous extension to a contraction from $L^1([0,T]\times\Omega,\mathcal{P}_T,\mu_M)$ into $G\hat{\otimes}_1 G$; also denoted by $X\mapsto\int X\;da_M$.
By Prop. B.3 $G\hat{\otimes}_1 G$ is the dual space of a separable Banach space. Hence the application of the first part of Theorem B.1 completes the proof.

□

Remark B.2 *The equation $a_M(A):=\int 1_A\;da_M$, $A\in\mathcal{P}_T$, defines a σ-additive measure on $\mathcal{P}_T$.*
The process Q_M turns out as the "density function" of the $G\hat{\otimes}_1 G$-valued measure a_M w.r.t. the real-valued measure μ_M, and the measure a_M as the Doleans measure of the $G\hat{\otimes}_1 G$-valued process $M\otimes M$.

Q_M can be chosen in such a way that for any $(t,\omega) \in [0,T] \times \Omega$ $Q(t,\omega)$ is a positive element of $G\hat{\otimes}_1 G$. I.e., $< Q_M, g \otimes g > \geq 0$ for all $g \in G$:
For any $g \in G$ and $A =]s,t] \times F \in \mathcal{R}_T$ holds

$$\begin{aligned} < a_M(A), g \otimes g > &= < E(1_F \cdot (M_t - M_s)^{\otimes 2}), g \otimes g > \\ &= E(1_F (M_t - M_s, g)_G^2) \geq 0. \end{aligned}$$

Hence the real-valued measure $A \mapsto < a_M(A), g \otimes g >$ is positive, and because of

$$< a_M(A), g \otimes g > = \int_A < Q(t,\omega), g \otimes g > \, d\mu_M(t,\omega)$$

the function $< Q(t,\omega), g \otimes g >$ is μ_M-a.e. positive. Since G is separable, it is possible to choose a version of Q_M which fulfills $< Q_M(t,\omega), g \otimes g > \geq 0$ for all $g \in G$ and all $(t,\omega) \in [0,T] \times \Omega$. (cf. [Met 77], 11.7, p.-I.85-)

To the process Q_M we assign the process $\tilde{Q}_M$ taking values in $L_1(G)$ and given by the following equation:

$$(\tilde{Q}_M g, g')_G = < Q_M, g \otimes g' >, \;\; g, g' \in G.$$

Since $Q_M(\cdot)$ is positive, the same holds for $\tilde{Q}_M(\cdot)$ and therefore $\tilde{Q}_M^{\frac{1}{2}}(\cdot)$ is well-defined. In fact, $\tilde{Q}_M(\cdot)$ is also symmetric and thus $\tilde{Q}_M^{\frac{1}{2}}(\cdot)$ a Hilbert-Schmidt operator.

B.3 Stochastic integrals with respect to general martingales

Similar to the Brownian motion or the martingale measure case (see section 1.3), in the first step the stochastic integral is defined for so-called simple processes. In the second step the construction is then extended (via an L^2-isometry) to the closure of such processes in a suitable Hilbert space.

Definition B.5 *$\mathcal{M}_T^2(G)$ is the vector space of square-integrable càdlàg G-martingales with the inner product $(M,N) := E((M_T, N_T)_G)$. $\mathcal{M}_T^2(G)$ is a Hilbert space isomorphic to $L^2(\Omega, \mathcal{F}_T, P, G)$.*

Definition B.6 *(cf. [Met 77], 32.1, p.-V.7-)*
Let be $L^*(G, H; \mathcal{P}_T, M)$ the space of processes X taking values in the linear (possibly unbounded) operators from G into H, which have the following properties

1. *The domain of* $X(t,\omega)$ *contains* $\tilde{Q}_M^{\frac{1}{2}}(t,\omega)(G) \subset G,\ (t,\omega) \in [0,T] \times \Omega$.

2. *For any* $g \in G$ *the* H*-valued process* $X \circ \tilde{Q}_M^{\frac{1}{2}}(g)$ *is predictable.*

3. *For any* $(t,\omega) \in [0,T] \times \Omega$ $X(t,\omega) \circ \tilde{Q}_M^{\frac{1}{2}}(t,\omega)$*is a Hilbert-Schmidt operator and we have*

$$\int_{[0,T]\times\Omega} \|X \circ \tilde{Q}_M^{\frac{1}{2}}\|_{L_2}^2 \; d\mu_M \; < \; \infty.$$

Proposition B.6 *([Met 77], 32.2, p.-V.8-)*
For any $X, Y \in L^*(G,H;\mathcal{P}_T,M)$ *the process* $tr(X \circ \tilde{Q}_M \circ Y^*)$ *is predictable and* μ_M*-integrable. The mapping*

$$(X,Y) \mapsto \int_{[0,T]\times\Omega} tr(X \circ \tilde{Q}_M \circ Y^*) \; d\mu_M$$

is an inner product on $L^*(G,H;\mathcal{P}_T,M)$ *and with respect to this inner product* $L^*(G,H;\mathcal{P}_T,M)$ *is complete; i.e. a Hilbert space.*

Proof.
Claim 1. $tr(X \circ \tilde{Q}_M \circ Y^*)$ is a predictable real-valued process.
Because of the polarization identity

$$\begin{aligned} & tr(X \circ \tilde{Q}_M \circ Y^*) = tr(Y \circ \tilde{Q}_M \circ X^*) \\ = \; & \frac{1}{4}\big(tr((Y+X) \circ \tilde{Q}_M \circ (Y^*+X^*)) - tr((Y-X) \circ \tilde{Q}_M \circ (Y^*-X^*))\big) \end{aligned}$$

it is enough to show that for any $X \in L^*(G,H;\mathcal{P}_T,M)$ $tr(X \circ \tilde{Q}_M \circ X^*)$ is a predictable process.
Let be (e_n) an orthonormal basis of G. By

$$tr(X \circ \tilde{Q}_M \circ X^*) = \|X \circ \tilde{Q}_M^{\frac{1}{2}}\|_{L_2}^2 = \sum_n \|X \circ \tilde{Q}_M^{\frac{1}{2}}(e_n)\|_H^2$$

and property 2. we can conclude that the process $tr(X \circ \tilde{Q}_M \circ X^*)$ is indeed predictable.
Claim 2. The mapping given above defines an inner product on $L^*(G,H;\mathcal{P}_T,M)$.
We have

$$tr(X \circ \tilde{Q}_M \circ Y^*) \leq \|X \circ \tilde{Q}_M^{\frac{1}{2}}\|_{L_2} \cdot \|Y \circ \tilde{Q}_M^{\frac{1}{2}}\|_{L_2}.$$

Applying Hölder's inequality and property 3. gives

$$\int_{[0,T]\times\Omega} tr(X \circ \tilde{Q}_M \circ Y^*) \; d\mu_M \leq \int_{[0,T]\times\Omega} \|X \circ \tilde{Q}_M^{\frac{1}{2}}\|_{L_2} \cdot \|Y \circ \tilde{Q}_M^{\frac{1}{2}}\|_{L_2} \; d\mu_M$$

$$\leq (\int_{[0,T]\times\Omega} \|X \circ \tilde{Q}_M^{\frac{1}{2}}\|_{L_2}^2 \, d\mu_M)^{\frac{1}{2}} \cdot (\int_{[0,T]\times\Omega} \|Y \circ \tilde{Q}_M^{\frac{1}{2}}\|_{L_2}^2 \, d\mu_M)^{\frac{1}{2}} \ < \ \infty.$$

Hence the mapping defines a positive definite, symmetric, continuous bilinear form on $L^*(G, H; \mathcal{P}_T, M)$.

Claim 3. $L^*(G, H; \mathcal{P}_T, M)$ is complete for this inner product.

Let be (X_n) a Cauchy sequence in $L^*(G, H; \mathcal{P}_T, M)$, i.e.

$$\lim_{n,m\to\infty} \int_{[0,T]\times\Omega} \|(X_n - X_m) \circ \tilde{Q}_M^{\frac{1}{2}}\|_{L_2}^2 \, d\mu_M = 0.$$

By the Riesz-Fischer Theorem $(X_n \circ \tilde{Q}_M^{\frac{1}{2}})$ converges to some Y in the space $L^2([0,T] \times \Omega, \mathcal{P}_T, \mu_M; L_2(G, H))$. Hence there exists a subsequence $(X_{n_k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} X_{n_k} \circ \tilde{Q}_M^{\frac{1}{2}}(t,\omega) = Y(t,\omega) \quad \mu_M\text{-a.e.}$$

Since $\tilde{Q}_M^{\frac{1}{2}}(t,\omega)f = 0$ implies $Y(t,\omega)f = 0$, $Y(t,\omega)$ can be written in the form (cf. the proof of Lemma 1.2)

$$Y(t,\omega) = X(t,\omega) \circ \tilde{Q}_M^{\frac{1}{2}}$$

where $X(t,\omega)$ is a linear mapping from $\tilde{Q}_M^{\frac{1}{2}}(G)$ into H. Obviously X has the required properties 1.–3., hence belongs to $L^*(G, H; \mathcal{P}_T, M)$.

□

Now we introduce the space of simple processes:

Definition B.7 *([Met 77], 32.3, p.-V.10-)*
$\mathcal{E}(G, H)$ denotes the vector space of processes of the form

$$X = \sum_{i=1}^{n} 1_{A_i} \cdot u_i$$

where $A_i \in \mathcal{R}_T$ and $u_i \in L(G, H)$ for any i. Such processes are called simple *processes.*
$\tilde{L}^2(G, H; \mathcal{P}_T, M)$ is defined as the closure of $\mathcal{E}(G, H)$in $L^*(G, H; \mathcal{P}_T, M)$.

Unfortunately we get no explicit characterization of $\tilde{L}^2(G, H; \mathcal{P}_T, M)$ and have to content ourselves with the following result:

Proposition B.7 *([Met 77], 32.4, p.-V.10-)*
The space $\tilde{L}^2(G, H; \mathcal{P}_T, M)$ contains all processes X with the following properties

1. $\forall\ (t,\omega) \in [0,T] \times \Omega$ *we have* $X(t,\omega) \in L(G,H)$.

2. $\forall\ g \in G$ *the* H*-valued process* Xg *is predictable.*

3. $\int_{[0,T]\times\Omega} tr(X \circ \tilde{Q}_M \circ X^*)\ d\mu_M\ <\ \infty$.

Proof.
Claim 1. It is enough to show that any process X with properties 1.–3. and $\sup_{t,\omega} \|X(t,\omega)\| \leq K$ for some constant K belongs to $\tilde{L}^2(G,H;\mathcal{P}_T,M)$.
Let be X a process fulfilling 1.–3., $(g_n)_{n\in\mathbb{N}}$ a countable dense subset of the unit ball in G. Then we have

$$\|X(s,\omega)\| = \sup_n \|X(t,\omega)g_n\|_H$$

and therefore the process $\|X\|$ is predictable. Thus the process $1_{\{\|X\|\leq n\}}X$ has properties 1.–3. and is, of course, bounded by n.
For any $(s,\omega) \in [0,T] \times \Omega$ we get

$$\lim_{n\to\infty} \|1_{\{\|X\|\leq n\}}X(s,\omega) - X(s,\omega)\| = 0.$$

Hence we have for all $(s,\omega) \in [0,T] \times \Omega$:

$$\lim_{n\to\infty} \|(1_{\{\|X\|\leq n\}}X(s,\omega) - X(s,\omega)) \circ \tilde{Q}_M^{\frac{1}{2}}(s,\omega)\|_{L_2} = 0.$$

Since

$$\|(1_{\{\|X\|\leq n\}}X(s,\omega) - X(s,\omega)) \circ \tilde{Q}_M^{\frac{1}{2}}(s,\omega)\|_{L_2}^2 =$$

$$1_{\{\|X\|\geq n\}}\|X(s,\omega) \circ \tilde{Q}_M^{\frac{1}{2}}(s,\omega)\|_{L_2}^2\ \leq\ \|X(s,\omega) \circ \tilde{Q}_M^{\frac{1}{2}}(s,\omega)\|_{L_2}^2$$

we can conclude by Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} \int_{[0,T]\times\Omega} \|(1_{\{\|X\|\leq n\}}X(s,\omega) - X(s,\omega)) \circ \tilde{Q}_M^{\frac{1}{2}}(s,\omega)\|_{L_2}^2\ d\mu_M = 0.$$

Claim 2. Any process X with properties 1.–3. and $\|X\| \leq K$ for some K is in $\tilde{L}^2(G,H;\mathcal{P}_T,M)$.
Let be Y a mapping from $[0,T] \times \Omega$ into the Banach space $L(G,H)$ with $\|Y\| \leq K$ which is strongly measurable with respect to $\mathcal{P}_T$. Then by Lemma 2.6 there exists a sequence (Y_n) in $\mathcal{E}(G,H)$ converging to $Y(t,\omega)$ in $L(G,H)$ for all $(t,\omega) \in [0,T] \times \Omega$. For such a sequence we have

$$\lim_{n\to\infty} \int_{[0,T]\times\Omega} \|(Y - Y_n) \circ \tilde{Q}_M^{\frac{1}{2}}\|_{L_2}^2\ d\mu_M = 0.$$

Now consider a process X from Claim 2. We show that such an X can be approximated in $L^*(G,H;\mathcal{P}_T,M)$ by a sequence of strongly measurable processes of the type discussed above. Then it is clear that X belongs to $\tilde{L}^2(G,H;\mathcal{P}_T,M)$.
Let be (f_i) resp. (e_i) an orthonormal basis of H resp. G. Denote by π_H^n resp. π_G^n the orthogonal projection from H resp. G onto the subspace generated by $\{f_1,\dots,f_n\}$ resp. $\{e_1,\dots,e_n\}$. Set $X_n := \pi_H^n \circ X \circ \pi_G^n$. Then for any n we have that $X_n([0,T]\times\Omega)$ is separable and that $X_n g$ is predictable for all $g \in G$. This gives us the strong measurability of X_n.
For any i we get:

$$\lim_{n\to\infty} \|(\pi_H^n \circ X \circ \pi_G^n - X)\circ \tilde{Q}_M^{\frac{1}{2}} e_i\|_H^2 \;=\; 0 \tag{B.4}$$

$$\|(\pi_H^n \circ X \circ \pi_G^n - X)\circ \tilde{Q}_M^{\frac{1}{2}} e_i\|_H^2 \;\le\; 4K^2 \|\tilde{Q}_M^{\frac{1}{2}} e_i\|_G^2 \tag{B.5}$$

$$\sum_{i=1}^{\infty} \|\tilde{Q}_M^{\frac{1}{2}} e_i\|_G^2 \;=\; \|\tilde{Q}_M^{\frac{1}{2}}\|_{L_2}^2 \;<\; \infty. \tag{B.6}$$

It follows for all (t,ω) by (B.4) – (B.6) that

$$\lim_{n\to\infty} \|(X_n - X)\circ \tilde{Q}_M^{\frac{1}{2}}\|_{L_2}^2 = \lim_{n\to\infty} \sum_i \|(\pi_H^n \circ X \circ \pi_G^n - X)\circ \tilde{Q}_M^{\frac{1}{2}} e_i\|_H^2 \;=\; 0$$

with

$$\|(X_n - X)\circ \tilde{Q}_M^{\frac{1}{2}}\|_{L_2}^2 \;\le\; 4\|X \circ \tilde{Q}_M^{\frac{1}{2}}\|_{L_2}^2.$$

Hence again by Lebesgue's dominated convergence theorem we can finally conclude

$$\lim_{n\to\infty} \int_{[0,T]\times\Omega} \|(X - X_n)\circ \tilde{Q}_M^{\frac{1}{2}}\|_{L_2}^2 \, d\mu_M = 0$$

and the proposition is proved.

□

Remark B.3 *([Met 77], 32.5, p.-V.12-, presenting a counterexample)*
The space of processes which have properties 1.–3. of Proposition B.7 is generally not closed in $\tilde{L}^2(G,H;\mathcal{P}_T,M)$.

Finally, we define the stochastic integral for simple processes and prove the isometry:

Proposition B.8 *([Met 77], 32.6, p.-V.14-)*
Let be $M \in \mathcal{M}_T^2(G)$ and $X \in \tilde{L}^2(G, H; \mathcal{P}_T, M)$. Then there exists a unique isometric mapping from $\tilde{L}^2(G, H; \mathcal{P}_T, M)$ into $\mathcal{M}_T^2(H)$ such that the process $(1_F \cdot (u(M_{s\wedge t}) - u(M_{r\wedge t})))_{t\in[0,T]}$ is the image of the process $X = 1_{]r,s]\times F} \cdot u$ for any $0 \le r < s \le T$, $F \in \mathcal{F}_r$, $u \in L(G, H)$.

Proof.
Claim 1. $Z = (\sum_{i=1}^n (1_{F_i} \cdot (u_i(M_{s_i\wedge t}) - u_i(M_{r_i\wedge t})))_{t\in[0,T]}$ is an $(\mathcal{F}_t)$-martingale.
Adapted is clear from the construction, and integrability follows from Claim 2. Hence we only have to show the martingale property.
For $0 \le s < t \le T$ consider an arbitrary set $A \in \mathcal{F}_s$. Then we obtain by Prop. E.11 from [Coh 80], $F_i \in \mathcal{F}_{r_i}$ $\forall i$, and the martingale property of M:

$$\begin{aligned}
\int_A Z(t)\ dP &= \int_A \sum_{i=1}^n (1_{F_i} \cdot (u_i(M_{s_i\wedge t}) - u_i(M_{r_i\wedge t})))\ dP \\
= & \sum_{i=1}^n \int_{A\cap F_i} u_i(M_{s_i\wedge t}) - u_i(M_{r_i\wedge t})\ dP \\
= & \sum_{i=1}^n u_i(\int_{A\cap F_i} M_{s_i\wedge t} - M_{r_i\wedge t}\ dP) \\
= & \sum_{i=1}^n u_i(\int_{A\cap F_i} M_{s_i\wedge s} - M_{r_i\wedge s}\ dP) \\
= & \int_A \sum_{i=1}^n (1_{F_i} \cdot (u_i(M_{s_i\wedge s}) - u_i(M_{r_i\wedge s})))\ dP = \int_A Z(s)\ dP.
\end{aligned}$$

Claim 2. The mapping

$$X = \sum_{i=1}^n 1_{]r_i,s_i]\times F_i} \cdot u_i \ \mapsto \ \sum_{i=1}^n (1_{F_i} \cdot (u_i(M_{s_i\wedge t}) - u_i(M_{r_i\wedge t})))_{t\ge 0}$$

is an isometry from $\mathcal{E}(G, H)$ into $\mathcal{M}_T^2(H)$.
W.l.o.g. we can assume that the sets $]r_i, s_i] \times F_i$ are mutually disjoint (otherwise we could take a finer partition). Hence the following chain of equations

gives the assertion:

$$
\begin{aligned}
& E(\| \sum_{i=1}^{n} 1_{F_i} \cdot (u_i(M_{s_i}) - u_i(M_{r_i})) \|_H^2) \\
= \;& E(\sum_{i=1}^{n} \| 1_{F_i} \cdot u_i(M_{s_i} - M_{r_i}) \|_H^2) \\
= \;& E(\sum_{i=1}^{n} 1_{F_i} \cdot tr[u_i \otimes u_i (M_{s_i} - M_{r_i})^{\otimes 2}]) \\
= \;& tr[\sum_{i=1}^{n} u_i \otimes u_i \; E(1_{F_i} \cdot (M_{s_i} - M_{r_i})^{\otimes 2})] \\
= \;& \sum_{i=1}^{n} tr[u_i \otimes u_i \int_{]r_i, s_i] \times F_i} Q_M \; d\mu_M] \\
= \;& \sum_{i=1}^{n} \int_{]r_i, s_i] \times F_i} tr[u_i \circ \tilde{Q}_M \circ u_i^*] \; d\mu_M \\
= \;& \int_{[0,T] \times \Omega} tr[X \circ \tilde{Q}_M \circ X^*] \; d\mu_M = \|X\|_{L^*(G,H;\mathcal{P}_T,M)}.
\end{aligned}
$$

By the isometry and because $\mathcal{M}_T^2(H)$ is complete everything can be extended to the closure of $\mathcal{E}(G,H)$, i.e. $\tilde{L}^2(G,H;\mathcal{P}_T,M)$. Hence the Proposition is proved.

□

Definition B.8 *(cf. [Met 77], 32.7, p.-V.15-)*
Let be $M \in \mathcal{M}_T^2(G)$ and $X \in \tilde{L}^2(G,H;\mathcal{P}_T,M)$. The image of X under the isometry of Proposition B.8 is called the L^2-stochastic integral *of X w.r.t. M, and is denoted by $\int X \; dM$.*
The value of the martingale $\int X \; dM$ at time $0 \le t \le T$ is denoted by $\int_0^t X \; dM$.

Example B.1 (Q-Brownian motion) *For M a Q-Brownian motion the isometry from Prop. B.8*

$$
E[\| \int_0^t X(s) \, dM(s) \|^2] = \int_{[0,t] \times \Omega} \| X(s,\omega) \tilde{Q}_M^{\frac{1}{2}}(s,\omega) \|_{L_2}^2 \, d\mu_M(s,\omega), \;\; t \in [0,T],
$$

simply reads (compare formula (A.1))

$$
E[\| \int_0^t X(s) \, dB_Q(s) \|^2] = E[\int_0^t \| X(s) Q^{\frac{1}{2}} \|_{L_2}^2 \, ds], \;\; t \in [0,T].
$$

In this case the "covariance structure" of the martingale is very easy with $\tilde{Q}(s,\omega) \equiv \frac{Q}{tr(Q)}$ *and* $\mu_M = tr(Q)\, dt \otimes P$ *where* dt *denotes Lebesgue measure on* $[0,T]$.

Remark B.4 *Consider a semimartingale* $(Z_t)_{t\in[0,T]}$ *of the form* $Z = M + V$, *where* $M \in \mathcal{M}_T^2(G)$ *and* $(V_t)_{t\in[0,T]}$ *is a* G*-valued,* $(\mathcal{F}_t)$*-adapted process with bounded variation and càdlàg paths.*
For $X \in \tilde{L}^2(G, H; \mathcal{P}_T, M)$ *we can define*

$$\int_0^t X(s)\, dZ(s) := \int_0^t X(s)\, dM(s) + \int_0^t X(s)\, dV(s),\ \ t \in [0,T]$$

where the first integral on the right-hand side is an L^2*-stochastic integral constructed above and the second one is understood pathwise for any* $\omega \in \Omega$ *as a Riemann-Stieltjes integral.*

Appendix C

Example from Chapter 2

We give an example of a (real-valued) Lévy process which fulfills condition (F), but does not meet the requirement from [Sto 05], (2.2). That means

$$\int_{\{\|x\|\geq 1\}} \|x\|^2 \, \nu(dx) < \infty,$$

while

$$\sup_{t\in[0,T]} \|\Delta L(t)\| \notin L^{2+\epsilon}(\Omega, \mathcal{F}, P) \text{ for any } \epsilon > 0.$$

Let be $P_\lambda(t),\ t \geq 0$, a Poisson process with parameter λ and $\tilde{P}_\lambda(t) := P_\lambda(t) - \lambda t,\ t \geq 0$, the corresponding compensated Poisson process. Then $P(P_\lambda(1) = 0) = \exp(-\lambda)$ and $P(P_\lambda(1) > 0) = 1 - \exp(-\lambda)$.

Set $\nu(k) := \frac{1}{k^3 (\ln k)^2}$ for $k \geq 2$. Clearly $\sum_{k=2}^\infty \nu(k) < \infty$. If for independent processes $\tilde{P}_{\nu(k)}$, $k \geq 2$, we define $L(t) := \sum_{k=2}^\infty k \, \tilde{P}_{\nu(k)}(t)$, then L is a real-valued Lévy process with corresponding Lévy measure $\nu = \sum_{k=2}^\infty \nu(k) \, 1_{\{k\}}$ and

$$\int_{\{\|x\|\geq 1\}} \|x\|^2 \, \nu(dx) = \sum_{k=2}^\infty k^2 \, \nu(k) = \sum_{k=2}^\infty \frac{1}{k \, (\ln k)^2} < \infty.$$

The convergence of the sum is a consequence of

$$\lim_{N\to\infty} \int_2^N \frac{1}{x \, (\ln x)^2} \, dx = \lim_{N\to\infty} \int_{\ln 2}^{\ln N} \frac{1}{y^2} \, dy = \lim_{N\to\infty} \left(-\frac{1}{\ln N} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2} \, .$$

For simplicity we consider $T = 1$. Then we get for any $\epsilon > 0$ with $C > 0$ a varying constant

$$
\begin{aligned}
E[\sup_{t\in[0,1]} \|\Delta L(t)\|^{2+\epsilon}] &= \sum_{k=2}^{\infty} k^{2+\epsilon}\, P(\sup_{t\in[0,1]} \|\Delta L(t)\| = k) \\
&= \sum_{k=2}^{\infty} k^{2+\epsilon}\, (1-\exp(-\nu(k))) \prod_{n>k} \exp(-\nu(n)) \\
&= \sum_{k=2}^{\infty} k^{2+\epsilon}\, (1-\exp(-\nu(k))) \exp(-\sum_{n>k} \nu(n)) \\
&\geq C\sum_{k=2}^{\infty} k^{2+\epsilon}\, (1-\exp(-\nu(k))) \\
&= C\sum_{k=2}^{\infty} k^{2+\epsilon}\, (1-\sum_{l=0}^{\infty} \frac{(-\nu(k))^l}{l!}) \\
&\geq C\sum_{k=2}^{\infty} k^{2+\epsilon}\, (\nu(k)-\nu(k)^2) \\
&\geq C\sum_{k=2}^{\infty} \frac{k^{\epsilon}}{k\,(\ln k)^2} \\
&\geq C\sum_{k=2}^{K(\epsilon)} \frac{k^{\epsilon}}{k\,(\ln k)^2} + C\sum_{k=K(\epsilon)+1}^{\infty} \frac{1}{k} = \infty.
\end{aligned}
$$

Hence we have shown that (F) does not imply the condition from [Sto 05].

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