Feller-type Properties and Path Regularities of Markov Processes

Judith Maria Nefertari Dohmann

Diplomarbeit an der Fakultät für Mathematik
der Universität Bielefeld

November 2001

Contents

CHAPTER 1

Introduction

Let E be a polish space with the complete metric d. Let L be an operator on $L^p(E,\mu)$, that generates a Markovian C_0-semigroup $(T_t)_{t\geq 0}$ on $L^p(E,\mu)$. A natural question is the existence of a Markov process $(\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ whose transition semigroup $(p_t)_{t\geq 0}$ is related to $(T_t)_{t\geq 0}$ via $p_t f$ is a μ-version of $T_t f$ for all $f \in \mathcal{B}_b(E)$ and $t > 0$ such that one has

$$P_x[t \mapsto X_t \text{ is continuous/càdlàg}] = 1 \quad \forall x \in E.$$

If in the case of $p = 2$ there exists a quasi-regular (generalized) Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ that generates the above semigroup $(T_t)_{t\geq 0}$ then there exists (under a second condition in the case of generalized Dirichlet forms) a Markov process $(\tilde{\Omega}, \mathcal{H}, (Z_t)_{t\geq 0}, (Q_x)_{x\in E})$ with transition semigroup $(\tilde{p}_t)_{t\geq 0}$ such that $\tilde{p}_t f$ is a μ-version of $T_t f$ for every $f \in \mathcal{B}_b(E)$ and $t \geq 0$ and such that the trajectories are P_x-a.s. càdlàg (or continuous under certain conditions) for $\mathcal{E}$-quasi all $x \in E$. But no general conditions are known implying that this is true for *every* $x \in E$.

Now one could use one of the following criteria to get the desired regularity, but one has to check the conditions for every measure Q_x which might be too difficult, and one does not use our additional information, namely that the process has already "nice" paths Q_μ-a.s.

- Continuous paths
  1. Kolmogorov found the following criterion (see e.g. [**Ba91**, 39.3 Satz]) for continuity of the sample paths:
     Suppose there exist $\alpha > 0$, $\beta > 0$, $c > 0$ such that
     $$\forall s,t \in \mathbb{R}_+ : E[d(X_s, X_t)^\alpha] \leq c \cdot |s-t|^{1+\beta}.$$
     Then there exists a modification of this process with continuous sample paths.
  2. A more general criterion is the following from [**Ba74**, 63.5 Lemma]: Let $(\Omega, \mathcal{A}, (X_t)_{t\geq 0}, P)$ be a stochastic process on E. The following condition is necessary and sufficient, that every random variable X_t can be changed on a t-dependent null-set, so that all paths of the resulting process $(\Omega, \mathcal{A}, (\tilde{X}_t)_{t\geq 0}, P)$ are continuous: There exists a countable dense subset S of $\mathbb{R}_+$ such that:

(a) For all $\eta > 0$, $k > 0$ we have
$\lim_{\delta\to 0} P\Big[\bigcup_{\substack{|s-t|\leq\delta \\ s,t\in S\, s,t\leq k}} \{d(X_s, X_t) \geq \eta\}\Big] = 0.$

(b) For all $t \in \mathbb{R}_+$ there exists a sequence $(s_n)_{n\in\mathbb{N}} \subset S$ such that $s_n \xrightarrow[n\to\infty]{} t$ and $\lim_{n\to\infty} P[d(X_{s_n}, X_t) \geq \eta] = 0$.

- In order to get càdlàg paths sufficient conditions are the following:
  1. (See [**We81**, Theorem 9.3]) Define
  $$\alpha_\varepsilon(h) := \sup_{x\in X, t\leq h} p_t(x, \{y \in E | d(x,y) \geq \varepsilon\}).$$
  Suppose $\alpha_\varepsilon(h) \xrightarrow[h\to 0]{} 0$ for any $\varepsilon > 0$. Then there exists a Markov family with $(p_t)_{t\geq 0}$ as transition function whose paths are càdlàg.
  2. If E is in addition locally compact there exists for every Feller process (where Feller means that for the transition semigroup p_t one has that $p_t C_\infty(E) \subset C_\infty(E)$ where $C_\infty(E)$ denotes the continuous functions on E vanishing at ∞) a modification with càdlàg paths, see for example [**ReYo91**, III(2.7) Theorem].

In this Diploma thesis we provide as the main theorems (see 2.3 Theorem for the continuous case and see 3.1 Theorem for the càdlàg case) a method to construct a process that is continuous or càdlàg P_x-a.s. for every $x \in E$ using that there exists a process $(\tilde{\Omega}, \mathcal{H}, (Z_t)_{t\geq 0}, (Q_x)_{x\in E})$ that is continuous or càdlàg Q_μ-a.s., where where $Q_\mu := \int Q_x \mu(dx)$ and the measure μ from above has full support, and for whose transitions semigroup $(\tilde{p}_t)_{t\geq 0}$ we have that for every $f \in \mathcal{B}_b(E)$ $\tilde{p}_t f$ is a μ-version of $T_t f$.

More exactly we need that there exists a semigroup of kernels $(p_t)_{t\geq 0}$ that has the strong Feller property and that for every $f \in \mathcal{B}_b(E)$ $p_t f$ is a μ-version of $T_t f$, that there exists for every $x \in E$ a countable family $(f_n)_{n\in\mathbb{N}}$ of functions in $D(L) \cap C(E)$ that generates a subbase for the neighborhood system of x, that $\lim_{t\downarrow 0} p_t f_n^2(x) = f_n^2(x)$ for every $n \in \mathbb{N}$ and either that the associated resolvent of kernels $(R_\alpha)_{\alpha>0}$ maps functions from $\mathcal{B}(E) \cap L^p(E,\mu)$ to continuous functions or that the functions f_n have the form $f_n = R_{\lambda_n}\varphi_n$ with $\varphi_n \in \mathcal{B}_b(E)$ and $\lambda_n > 0$ (for the second variant see 2.12 Theorem or 3.7 Theorem respectively).

The proofs of all cases of the main theorem are very parallel, so here we will consider them only once. At first we construct via Kolmogorov's Theorem a process $(E^{\mathbb{R}_+}, \mathcal{F}, (Y_t)_{t\geq 0}, (P_x)_{x\in E})$, then we provide a method that allows us to show that for a set Λ with $\theta_t^{-1}(\Lambda) \supset \Lambda$ for some $t > 0$ and of probability one with respect to P_μ we have $P_x[\theta_t^{-1}(\Lambda)] = 1$ for every $x \in E$. After that we show that the measures Q_μ and P_μ have the same finite dimensional distributions.

The next step is to find such a set Λ that describes the paths that can be modified to be continuous or càdlàg. For the continuous case we take as Λ the set of all paths that are locally uniformly continuous on the set of dyadic numbers, as in the proof of Kolmogorov's criterion for continuity of sample paths as it is presented in [**Ba91**, 39.3 Satz].

In the càdlàg case we fix first a countable family $(g_k)_{k\in\mathbb{N}}$ of bounded functions that separates the points of E and we take Λ_0 as the set of all paths that have only a finite number of up-crossings when composed with any member of the abovementioned family of functions. Then we find an increasing family $(K_n)_{n\in\mathbb{N}}$ of compact sets such that $Q_\mu[\lim_{n\to\infty}\sigma_{E\setminus K_n}=\infty]=1$ and take Λ_1 as the set of all paths whose restriction to $[0,M]\cap D$ stay for any finite M in one of these sets, where D denotes the nonnegative dyadic numbers.

Both of these sets are shift-invariant for $t\in D$ and they have probability one with respect to P_μ since the corresponding sets of the Dirichlet-form process have probability one.

In the end we have to deal with time equal zero. But there we have only to show the right-continuity whose proof is identical for both cases. Since E is not necessarily locally compact we need another method to ensure the convergence in time zero. We use that $e^{-\alpha t}R_\alpha f(X_t)$ is a supermartingale with respect to P_x for every $x\in E$ and for every positive function $f\in\mathcal{B}(E)$. Then there exists the limit $\lim_{t\downarrow 0}f(X_t)$ almost surely. In order to show that the limit $\lim_{t\downarrow 0}X_t$ exists almost surely we take for every point x a countable family of functions generating a subbase of the neighborhood system of this point, such that we can identify the limits via $L^2(E^{\mathbb{R}_+},P_x)$-convergence; here we need that $\lim_{t\downarrow 0}p_t f_n^2(x)=f_n^2(x)$.

This thesis is organized in the following way: Chapter 1 is this introduction, in Chapter 2 we formulate and prove the main theorem for the continuous case, in Chapter 3 we do the same for the càdlàg case. In Chapter 4 we consider the below mentioned application. In Appendix A we include a short summary on (generalized) Dirichlet forms and Appendix B contains some complements which are needed for the proof of the main theorem.

In Chapter 4 as an application we consider the following operator on C_0^∞:

$$L_{A,b}=\sum_{i,j}a_{ij}\partial_i\partial_j+\sum_i b_i\partial_i.$$

We prove that under the following conditions there exists a solution to the martingale problem for this operator for every $x\in\mathbb{R}^d$:

(i) μ is a probability measure with $L_{A,b}^*\mu=0$ i.e.
$L_{A,b}\varphi$ is μ-integrable and $\int_{\mathbb{R}^d}L_{A,b}\varphi\,d\mu=0\quad\forall\varphi\in C_0^\infty(\mathbb{R}^d)$.

(ii) $A=(a_{ij})_{i,j}$ is continuous, symmetric, nonnegative and nondegenerate with $a_{ij}\in H_{loc}^{1,p}(\mathbb{R}^d)$

(iii) $b_i \in L^p_{loc}(\mathbb{R}^d, \lambda)$ where λ denotes the Lebesgue-measure on $\mathbb{R}^d$.
(iv) $(L_{A,b}, C_0^\infty(\mathbb{R}^d))$ is $L^1(\mathbb{R}^d, \mu)$-unique

Also applications to the infinite dimensional situation are possible, see for example [**DaRö01**].

Finally I would like to thank Prof. Michael Röckner who led me to Stochastic Analysis. I am grateful for his support during writing this Diploma thesis, and for the opportunity to get an insight into current research.

Furthermore, I want to thank Dr. Wilhelm Stannat for patiently answering my questions.

CHAPTER 2

Continuous Case

Throughout this chapter let E be a polish space with Borel σ-field $\mathcal{B}(E)$ and a probability measure μ on $(E,\mathcal{B}(E))$ with $\mathrm{supp}[\mu]=E$.

Denote by $\mathcal{U}(x)$ the set of all neighborhoods of a given point x in the corresponding topological space and by $\mathbb{R}_+$ the interval $[0,\infty)$.

Let $\mathcal{X}$ be an arbitrary space of functions on E. By $L^p(E,\mu)\cap\mathcal{X}$ or $\mathcal{X}\cap L^p(E,\mu)$ we denote the subspace of $\mathcal{X}$ that contains all functions f with $\int_E |f|^p d\mu<\infty$.

Let $A: L^p(E,\mu)\to X$ be an arbitrary mapping. Then this mapping induces a mapping which we will call for simplicity of notation again A from $\mathcal{B}\cap L^p(E,\mu)$ to X by $Af := A\bar{f}$ where $\bar{f}$ is the equivalence class of functions that contains f.

By $\mathbb{R}_+$ we denote the interval $[0,\infty)$ and $\lim_{t\downarrow s}$ denotes $\lim_{t\to s, t>s}$.

2.1. Definition. Let $p\in[1,\infty]$.

(i) A kernel K on $(E,\mathcal{B}(E))$ is said to have the *p-strong Feller property*, if for all $f\in L^p(E,\mu)\cap\mathcal{B}(E)$ we have $Kf\in C(E)$, where $C(E)$ denotes the set of all continuous functions on E.

(ii) A semigroup $(p_t)_{t\geq 0}$ of kernels is said to have the *p-strong Feller property*, if p_t has the p-strong Feller property for all $t>0$.

(iii) A resolvent $(R_\alpha)_{\alpha>0}$ of kernels is said to have the *p-strong Feller property*, if R_α has the p-strong Feller property for all $\alpha>0$.

2.2. Remark. Since $\mathrm{supp}[\mu]=E$ we have that the ∞-strong Feller property coincides with the usual strong Feller property.

2.3. Theorem. *Let $(L,D(L))$ be the infinitesimal generator of a sub-Markovian semigroup $(T_t)_{t\geq 0}$ on $L^p(E,\mu)$ and denote by $(G_\alpha)_{\alpha>0}$ the associated resolvent. Let $(p_t)_{t\geq 0}$ be a Markovian semigroup of kernels on $(E,\mathcal{B}(E))$ and $(R_\alpha)_{\alpha>0}$ the associated resolvent of kernels on $(E,\mathcal{B}(E))$ such that*

1. $\forall f\in\mathcal{B}_b(E)\ \forall t\geq 0$: *$p_t f$ is a μ-version of $T_t f$.*
2. $(p_t)_{t\geq 0}$ *has the ∞-strong Feller property.*
3. $(R_\alpha)_{\alpha>0}$ *has the p-strong Feller property.*
4. $\forall x\in E\ \exists (f_n)_{n\in\mathbb{N}}\subset D(L)\cap C(E)$, $p_t f_n^2(x) \underset{t\downarrow 0}{\longrightarrow} f_n^2(x)$, $\{f_n^{-1}(U) | U\in\mathcal{U}(f_n(x)), n\in\mathbb{N}\}$ *is a subbase of $\mathcal{U}(x)$.*

Suppose there exists a Markov process $\mathbf{M}_Z = (\Omega, \mathcal{H}, (Z_t)_{t\geq 0}, (Q_x)_{x\in E})$ *whose transition semigroup* $(\tilde{p}_t)_{t\geq 0}$ *has the property that* $\tilde{p}_t f$ *is a* μ-*version of* $T_t f$ *for all* $f \in \mathcal{B}_b(E)$ *with*

$$Q_\mu[t \mapsto Z_t \textit{ is continuous}] = 1. \tag{2.1}$$

Then there exists a right process $\mathbf{M}_X = (E^{\mathbb{R}_+}, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ *with transition semigroup* $(p_t)_{t\geq 0}$ *and*

$$P_x[t \mapsto X_t \textit{ is continuous}] = 1 \quad \forall x \in E.$$

2.4. REMARK. Note that since μ is a probability measure, we have that every $f \in \mathcal{B}_b(E)$ is p-integrable, hence we can apply T_t on $f \in \mathcal{B}_b(E)$ in 1.

PROOF OF 2.3 THEOREM. By Kolmogorov's theorem there exists a Markov process $\mathbf{M}_Y = (E^{\mathbb{R}_+}, \mathcal{F}, (Y_t)_{t\geq 0}, (P_x)_{x\in E})$ with transition semigroup $(p_t)_{t\geq 0}$, family of shift operators $(\theta_t)_{t\geq 0}$ and filtration $(\mathcal{F}_t)_{t\geq 0}$.

The idea is to show that there exists a set $\Lambda \subset E^{\mathbb{R}_+}$ with $P_x(\Lambda) = 1$ for all $x \in E$, such that all paths in this set can be modified to be continuous.

In order to derive statements about P_x from statements about P_μ we will use the following lemma.

2.5. LEMMA. *Let* $(E^{\mathbb{R}_+}, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ *be a Markov process with transition semigroup* $(p_t)_{t\geq 0}$ *which has the* ∞-*strong Feller property and shift operator* θ_t, μ *a probability measure on* E *with* $\operatorname{supp}[\mu] = E$. *Suppose* $\Lambda \subset E^{\mathbb{R}_+}$, $P_\mu[\Lambda] = 1$, $t \in (0, \infty)$ *such that* $\theta_t^{-1}(\Lambda) \supset \Lambda$. *Then we have*

$$P_x[\theta_t^{-1}(\Lambda)] = 1 \quad \forall\, x \in \operatorname{supp}[\mu].$$

PROOF. $P_\mu[\Lambda] = 1$ implies $P_x[\Lambda] = 1$ for μ-a.e. $x \in E$, and hence $P_x[\theta_t^{-1}(\Lambda)] = 1$ for μ-a.e. $x \in E$. By

$$\begin{aligned} P_x[\theta_t^{-1}(\Lambda)] &= E_{P_x}[E_{P_x}[1_\Lambda \circ \theta_t | \mathcal{F}_t]] \\ &= E_{P_x}[E_{X_t}[1_\Lambda]] \\ &= p_t(E.[1_\Lambda])(x) \end{aligned}$$

and the ∞-strong Feller property of $(p_t)_{t\geq 0}$ $x \mapsto P_x[\theta_t^{-1}(\Lambda_0)]$ is continuous, hence $P_x[\theta_t^{-1}(\Lambda_0)] = 1$ for all $x \in E$. □

P_μ and Q_μ have the same finite dimensional distributions by the following lemma.

2.6. LEMMA. *Let* ν *be a probability measure on* $(E, \mathcal{B}(E))$ *and let* $(\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ *and* $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{X}_t)_{t\geq 0}, (\tilde{P}_x)_{x\in E})$ E-*valued Markov processes with transition semigroups* $(p_t)_{t\geq 0}$ *and* $(\tilde{p}_t)_{t\geq 0}$ *respectively such that* $p_t f = \tilde{p}_t \tilde{f}$ ν-*almost surely for all* $f, \tilde{f} \in \mathcal{B}_b(E)$ *with* $f = \tilde{f}$ ν-*almost surely, then* P_ν *and* $\tilde{P}_\nu$ *have the same finite dimensional distributions.*

PROOF. We will show that for any n and any $t_1 \leq \cdots \leq t_n$ we have

$$\begin{aligned} &P_\nu[X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n] \\ &= \int_E p_{t_1}(1_{A_1} \cdot p_{t_2-t_1}(1_{A_2} \ldots p_{t_n-t_{n-1}} 1_{A_n}) \ldots)(x)\mu(dx) \\ &= \int_E \tilde{p}_{t_1}(1_{A_1} \cdot \tilde{p}_{t_2-t_1}(1_{A_2} \ldots \tilde{p}_{t_n-t_{n-1}} 1_{A_n}) \ldots)(x)\mu(dx) \\ &= \tilde{P}_\nu[\tilde{X}_{t_1} \in A_1, \ldots, \tilde{X}_{t_n} \in A_n]. \end{aligned}$$

The second equality holds by our assumption, the first and the third equalities are proved in the same way, so we will only prove the first.

To this end we will show, that $\forall f_1, \ldots, f_n \in \mathcal{B}(E)$

$$E_{P_\mu}[f_1(X_{t_1}) \ldots f_n(X_{t_n})] = \int_E p_{t_1}(f_1 p_{t_2-t_1}(\ldots p_{t_n-t_{n-1}} f_n) \ldots)(x)\mu(dx).$$

This will be done by induction.

For $n = 1$ the assertion holds by definition of p_t and P_μ. Assume that it is valid for n then we have for $n+1$:

$$\begin{aligned} &E_{P_\nu}[f_1(X_{t_1}) \ldots f_{n+1}(X_{t_{n+1}})] = \\ &= \int_E E_{P_x}[f_1(X_{t_1}) \ldots f_n(X_{t_n}) f_{n+1}(X_{t_{n+1}})]\nu(dx) \\ &= \int_E E_{P_x}[f_1(X_{t_1}) \ldots f_n(X_{t_n}) E_{P_x}[f_{n+1}(X_{t_{n+1}})|\mathcal{F}_{t_n}]]\nu(dx) \\ &= \int_E E_{P_x}[f_1(X_{t_1}) \ldots f_n(X_{t_n}) E_{P_{X_{t_n}}}[f_{n+1}(X_{t_{n+1}-t_n})]]\nu(dx) \\ &= \int_E E_{P_x}[f_1(X_{t_1}) \ldots f_n(X_{t_n}) p_{t_{n+1}-t_n} f_{n+1}(X_{t_n})]\nu(dx) \\ &= \int_E p_{t_1}(f_1 p_{t_2-t_1}(\ldots p_{t_n-t_{n-1}}(f_n p_{t_{n+1}-t_n} f_{n+1})) \ldots)\nu(dx). \end{aligned}$$

□

2.7. REMARK. Note that P_x and $\tilde{P}_x$ do not necessarily have the same finite dimensional distributions.

Now we have to find an appropriate set Λ which characterizes the paths which can be modified to be continuous. This set should be shift-invariant so that we can apply 2.5 Lemma. And in order to use $Q_\mu[t \mapsto Z_t \text{ is continuous}] = 1$ to get the same result for P_μ it should be describable as a union or intersection of sets which depend only on a finite number of times.

Define now $S_n := \{k \cdot 2^{-n} | k \in \mathbb{N}_0\}$, $n \in \mathbb{N}$ and denote the set of non-negative dyadic numbers by D. We have $D = \bigcup_{n \in \mathbb{N}} S_n$. Furthermore define

$$\begin{aligned}A_k^{(l)} :=&\big\{\omega \in E^{\mathbb{R}^+}\big|\exists n_0 \forall n \geq n_0 \forall s,t \in S_n \cap [0,l], |s-t| \leq 2^{-n_0}: \\ &d(Y_s(\omega), Y_t(\omega)) < 2^{-k}\big\},\\ B_k^{(l)} :=&\big\{\omega \in \Omega\big|\exists n_0 \forall n \geq n_0 \forall s,t \in S_n \cap [0,l], |s-t| \leq 2^{-n_0}: \\ &d(Z_s(\omega), Z_t(\omega)) < 2^{-k}\big\}.\end{aligned}$$

The functions in each set fulfill the ε-δ-condition for uniform continuity on $D \cap [0,l]$ for $\varepsilon = 2^{-k}$.

2.8. LEMMA. $\forall k, l \in \mathbb{N}$

$$P_\mu\big[A_k^{(l)}\big] = Q_\mu\big[B_k^{(l)}\big] = 1.$$

PROOF. The second identity follows by (2.1). The first one by

$$\begin{aligned}P_\mu[A_k^{(l)}] &= P_\mu\Big[\bigcup_{n_0\in\mathbb{N}}\bigcap_{n\geq n_0}\bigcap_{\substack{s,t\in S_n\cap[0,l]\\|s-t|\leq 2^{-n_0}}}\big\{d(Y_s,Y_t) < 2^{-k}\big\}\Big]\\ &= \sup_{n_0\in\mathbb{N}}\inf_{n\geq n_0} P_\mu\Big[\bigcap_{\substack{s,t\in S_n\cap[0,l]\\|s-t|\leq 2^{-n_0}}}\big\{d(Y_s,Y_t) < 2^{-k}\big\}\Big]\\ &= \sup_{n_0\in\mathbb{N}}\inf_{n\geq n_0} Q_\mu\Big[\bigcap_{\substack{s,t\in S_n\cap[0,l]\\|s-t|\leq 2^{-n_0}}}\big\{d(Z_s,Z_t) < 2^{-k}\big\}\Big]\\ &= Q_\mu\Big[\bigcup_{n_0\in\mathbb{N}}\bigcap_{n\geq n_0}\bigcap_{\substack{s,t\in S_n\cap[0,l]\\|s-t|\leq 2^{-n_0}}}\big\{d(Z_s,Z_t) < 2^{-k}\big\}\Big]\\ &= Q_\mu[B_k^{(l)}].\end{aligned}$$

□

Define now

$$\Lambda_0 := \bigcap_{k\in\mathbb{N}}\bigcap_{l\in\mathbb{N}} A_k^{(l)}.$$

By 2.8 Lemma we know that $P_\mu[\Lambda_0] = 1$. Define $\lceil x \rceil$ for $x \in \mathbb{R}$ as follows: $\lceil x \rceil := \min\{z \in \mathbb{Z}, z \geq x\}$. For $t \in D$, we have $\theta_t^{-1}(\Lambda_0) \supset \Lambda_0$ since

$$\begin{aligned}&\theta_t^{-1}(A_k^{(l)})\\ &\quad= \{\omega \in E^{\mathbb{R}_+}|\exists n_0\, \forall n \geq n_0\, \forall r,s \in S_n \cap [t, l+t], |r-s| \leq 2^{-n_0}: \\ &\qquad d(Y_r(\omega), Y_s(\omega)) < 2^{-k}\}\\ &\quad\supset \{\omega \in E^{\mathbb{R}_+}|\exists n_0\, \forall n \geq n_0\, \forall r,s \in S_n \cap [0, \lceil l+t \rceil], |r-s| \leq 2^{-n_0}: \\ &\qquad d(Y_r(\omega), Y_s(\omega)) < 2^{-k}\} = A_k^{(\lceil l+t\rceil)}\end{aligned}$$

and then we have

$$\theta_t^{-1}(\bigcap_l A_k^{(l)}) = \bigcap_l \theta_t^{-1}(A_k^{(l)}) \supset \bigcap_l A_k^{(\lceil l+t \rceil)} = \bigcap_l A_k^{(l)}.$$

Therefore by 2.5 Lemma we have that $P_x[\theta_t^{-1}(\Lambda_0)] = 1$ for all $x \in E$. Define

$$\Lambda_0' := \bigcap_{t \in D, t>0} \theta_t^{-1}(\Lambda_0). \tag{2.2}$$

2.9. REMARK. For $\omega \in \theta_t^{-1}(\Lambda_0)$ the function $Y(w) : (t, \infty) \cap D \to E$ is locally uniformly continuous on $(t, \infty) \cap D$ and thus can be extended to a continuous function on (t, ∞), since E is complete.

We have $P_x[\Lambda_0'] = 1$ for all $x \in E$ and for $\omega \in \Lambda_0'$ we have the following property:

For every $t > 0$ the function $Y_{\cdot+} : (t, \infty) \to E$, $s \mapsto \lim_{r \downarrow s, r \in D} Y_r(\omega)$ is continuous.

Up to now we only needed the assumptions 1. and 2., but we have the above property for all positive times t, but not for time zero, so we have to do additional work, where we will use the remaining assumptions.

Define

$$\Lambda_1 := \{\omega \in E^{\mathbb{R}_+} | \lim_{\substack{s \downarrow 0 \\ s \in D}} Y_t(\omega) \text{ exists}\}.$$

We have $P_x[\Lambda_1] = 1$ for all $x \in E$ by the following lemma.

2.10. LEMMA. *Let $(L, D(L))$ be the infinitesimal generator of a sub-Markovian semigroup $(T_t)_{t \geq 0}$ on $L^p(E, \mu)$ and denote by $(G_\alpha)_{\alpha > 0}$ the associated resolvent. Let $(E^{\mathbb{R}_+}, \mathcal{F}, (Y_t)_{t \geq 0}, (P_x)_{x \in E})$ be a normal Markov process whose transition semigroup $(p_t)_{t \geq 0}$ and resolvent of kernels $(R_\alpha)_{\alpha > 0}$ fulfill the following assumptions:*

(i) $\forall f \in \mathcal{B}_b$ $\forall t \geq 0$: *$p_t f$ is a μ-version of $T_t f$.*

(ii) $\forall x \in E$ $\exists (f_n)_{n \in \mathbb{N}} \subset D(L) \cap C(E)$: *$\{f_n^{-1}(U) | U \in \mathcal{U}(f_n(x)), n \in \mathbb{N}\}$ is a subbase of $\mathcal{U}(x)$, $p_t f_n^2(x) \underset{t \downarrow 0}{\longrightarrow} f_n^2(x)$.*

(iii) *$(R_\alpha)_{\alpha > 0}$ has the p-strong Feller property.*

Then we have for all $x \in E$ that $\lim_{t \downarrow 0} Y_t = x$ P_x-a.s., and in particular $\lim_{t \downarrow 0} Y_t = Y_0$ P_x-a.s.

PROOF. For every $f \in D(L) \cap C(E)$ there exists $u \in L^p(E, \mu)$ such that $G_1 u = \bar{f}$ where $\bar{f}$ denotes the equivalence class of functions in $L^p(E, \mu)$ containing f. Take now $v \in L^p(E, \mu) \cap \mathcal{B}$ such that u contains v. Then we have that $R_1 v = f$ μ-a.s. But since f and $R_1 v$ are continuous by assumptions (ii) and (iii) and since $\text{supp}[\mu] = E$ we have that $f(x) = R_1 v(x)$ for all $x \in E$.

Then we have

$$\int_0^\infty e^{-s}|p_s v(x)|ds \le \int_0^\infty e^{-s} p_s|v|(x)ds = R_1|v|(x) < \infty$$

and since $1_{[0,t]}e^{-s}p_s v_n(x) \underset{t\downarrow 0}{\longrightarrow} 0$ pointwise, we have that

$$\begin{aligned}
|p_t f(x) - f(x)| &= |p_t R_1 v(x) - R_1 v(x)| \\
&= \left|p_t \int_0^\infty e^{-s} p_s v(x)ds - \int_0^\infty e^{-s} p_s v(x)ds\right| \\
&= \left|\int_0^\infty e^{-s} p_{t+s} v(x)ds - \int_0^\infty e^{-s} p_s v(x)ds\right| \\
&= \left|e^t \int_t^\infty e^{-s} p_s v(x)ds - \int_0^\infty e^{-s} p_s v(x)ds\right| \\
&\le \left|\int_0^t e^{-s} p_s v(x)ds\right| + (e^t - 1)\left|\int_t^\infty e^{-s} p_s v(x)ds\right| \\
&\underset{t\downarrow 0}{\longrightarrow} 0.
\end{aligned}$$

Fix now $x \in E$ and $(f_n)_{n\in\mathbb{N}}$ as in assumption (ii) above. Then there exists by our considerations above for every $n \in \mathbb{N}$ a function $v_n \in L^p(E,\mu) \cap \mathcal{B}(E)$ such that $f_n(y) = R_1 v_n(y) = R_1 v_n^+(y) - R_1 v_n^-(y)$ for all $y \in E$.

Since $e^{-t}R_1 f(Y_t)$ is a supermartingale for every $f \ge 0$ we have that the following limits exist P_x-a.s.

$$\lim_{t\downarrow 0} e^{-t} R_1 v_n^+(Y_t(\omega)) \qquad\qquad \lim_{t\downarrow 0} e^{-t} R_1 v_n^-(Y_t(\omega))$$

and hence

$$\lim_{t\downarrow 0} f_n(Y_t(\omega)).$$

Because of

$$\begin{aligned}
E_{P_x}[(f_n(Y_t) - f_n(x))^2] &= p_t f_n^2(x) - 2f_n(x)p_t f_n(x) + f_n^2(x) \\
&\underset{t\to 0}{\longrightarrow} f_n^2(x) - 2f_n^2(x) + f_n^2(x) \\
&= 0
\end{aligned}$$

we have that $\lim_{t\downarrow 0} f_n(Y_t) = f_n(x)$ in $L^2(E^{\mathbb{R}_+}, P_x)$. Hence $\lim_{t\downarrow 0} f_n(Y_t) = f_n(x)$ P_x-a.s.

Since we have the countable index set $\mathbb{N}$ there exists a set $N \subset E^{\mathbb{R}_+}$ such that for all $\omega \notin N$ and all $n \in \mathbb{N}$ we have that $\lim_{t\downarrow 0} f_n(Y_t) = f_n(x)$.

Applying 2.11 Lemma below we get that $\lim_{t\downarrow 0} Y_t = x$ P_x-a.s. Since Y_t is normal by assumption we have $Y_0 = x$ P_x-a.s. □

2.11. LEMMA. *Let X, Y be topological spaces, $x \in X$, $f_i : X \to Y$ continuous in x, $\forall i \in I$, such that $\{f_i^{-1}(U) | i \in I, U \in \mathcal{U}(f_i(x))\}$ is a subbase of $\mathcal{U}(x)$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and assume that $f_i(x_n) \xrightarrow[n\to\infty]{} f_i(x)$ in Y for all $i \in I$. Then $x_n \xrightarrow[n\to\infty]{} x$.*

PROOF. Assume $x_n \not\to x$. Then there exists a neighborhood O of x and a subsequence $(x_{n_k})_k$ such that $x_{n_k} \notin O$ for all $k \in \mathbb{N}$. Since $\{f_i^{-1}(U) | i \in I, U \in \mathcal{U}(f_i(x))\}$ is a subbase of $\mathcal{U}(x)$ there exist $i_1, \dots i_m \in I$, $U_1 \in \mathcal{U}(f_{i_1}(x)), \dots, U_m \in \mathcal{U}(f_{i_m}(x))$ with $\bigcap_{j=1}^m f_{i_j}^{-1}(U_j) \subset O$. Therefore we have for all $k \in \mathbb{N}$ that $x_{n_k} \notin \bigcap_{j=1}^m f_{i_j}^{-1}(U_j)$. By the pigeon-hole principle there exists $j \in \{1, \dots, m\}$ and a subsequence $(x_{n_{k_l}})_l$ such that for all $l \in \mathbb{N}$ we have $x_{n_{k_l}} \notin f_{i_j}^{-1}(U_j)$. Hence we have $f_{i_j}(x_{n_{k_l}}) \notin U_j$ for all $l \in \mathbb{N}$ and then we have $f_{i_j}(x_n) \not\to f_{i_j}(x)$ which is in contradiction to our assumptions. □

Define

$$\Lambda := \Lambda_0' \cap \Lambda_1$$

and fix an arbitrary $x_0 \in E$. Now we will define the new process $\mathbf{M}_X = (E^{\mathbb{R}_+}, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ by:

$$X_t(\omega) = \begin{cases} \lim\limits_{\substack{s\downarrow t\\ s\in D}} Y_t(\omega) & ,\omega \in \Lambda \\ x_0 & ,\omega \notin \Lambda. \end{cases}$$

This process has continuous paths for every ω, because for $\omega \in \Lambda$ the function $t \mapsto Y_t(\omega)$ is locally uniformly continuous on $D \cap [t, \infty)$ for $t > 0$ and $\lim_{\substack{s\downarrow 0\\ s\in D\setminus\{0\}}} Y_s(\omega) = Y_0(\omega)$.

Finally we will show that $\mathbf{M}_X$ is a right process with transition semigroup $(p_t)_{t\geq 0}$. To this end we will show that $P_x[X_t = Y_t] = 1$ for all $x \in E$.

$$\begin{aligned} E_{P_x}[d(X_t, Y_t)] &= E_{P_x}[d(\lim_{\substack{s\downarrow t\\ s\in D}} Y_s, Y_t)] \\ &= E_{P_x}[E_{P_x}[d(\lim_{\substack{s\downarrow t\\ s\in D}} Y_s, Y_t)|\mathcal{F}_t]] \\ &= E_{P_x}[E_{P_x}[d(\lim_{\substack{s\downarrow t\\ s\in D}} Y_{s-t}, Y_0) \circ \theta_t|\mathcal{F}_t]] \\ &= E_{P_x}[E_{Y_t}[d(\lim_{\substack{s\downarrow t\\ s\in D}} Y_{s-t}, Y_0)]] \\ &= 0, \end{aligned}$$

since $Y_t(\omega) \in E$ and $\lim_{t\downarrow 0} Y_t = Y_0$ by 2.10 Lemma. So $d(X_t, Y_t) = 0$ P_x-a.s. hence $P_x[X_t = Y_t] = 1$ and in particular X_t has the transition semigroup $(p_t)_{t\geq 0}$. Then X_t has the Markov property w.r.t. the filtration $(\mathcal{F}_t^{0,X})_{t\geq 0}$ where $\mathcal{F}_t^{0,X} := \sigma(X_s | 0 \leq s \leq t)$ by [**Ba91**, 42.3 Satz] and since $\mathcal{N} := \bigcap_{\nu\in\mathcal{P}(E)} \{A \in \mathcal{F} | P_\nu[A] \in \{0, 1\}\}$ is a σ-algebra independent of every $\mathcal{F}_t^{0,X}$ w.r.t. every P_ν we have by [**BlGe68**, Exercise 2.14] (for a proof see B.2 Proposition in Appendix B) that X_t has the Markov property w.r.t. $(\mathcal{F}_t^X)_{t\geq 0}$, where $\mathcal{F}_t^X := \sigma(\mathcal{F}_t^{0,X}, \mathcal{N})$.

Since the paths are continuous it has the strong Markov property with respect to $(\mathcal{F}_{t+}^X)_{t\geq 0}$ by B.1 Proposition, and since it is normal by construction it is a right process. □

Another version of 2.3 Theorem is the following one, here there is no assumption on the resolvent, but then we need to know more about the family of functions $(f_n)_{n\in\mathbb{N}}$ that we needed to ensure the right-continuity for time zero.

2.12. THEOREM. *Let $(L, D(L))$ be the infinitesimal generator of a sub-Markovian semigroup $(T_t)_{t\geq 0}$ on $L^p(E,\mu)$ and denote by $(G_\alpha)_{\alpha>0}$ the associated resolvent. Let $(p_t)_{t\geq 0}$ be a Markovian semigroup of kernels on $(E, \mathcal{B}(E))$ and $(R_\alpha)_{\alpha>0}$ the associated resolvent of kernels on $(E, \mathcal{B}(E))$ such that*

1. $\forall f \in \mathcal{B}_b(E)\ \forall t \geq 0$: *$p_t f$ is a μ-version of $T_t f$*
2. *$(p_t)_{t\geq 0}$ has the ∞-strong Feller property.*

4′. $\forall x \in E\ \exists (f_n)_{n\in\mathbb{N}}, f_n = R_{\lambda_n}\varphi_n, \varphi_n \in \mathcal{B}_b(E), \lambda_n > 0$
$p_t f_n^2(x) \underset{t\downarrow 0}{\longrightarrow} f_n^2(x)$, *$\{f_n^{-1}(U) | U \in \mathcal{U}(f_n(x))\}$ is a subbase of $\mathcal{U}(x)$.*

Suppose there exists a Markov process $\mathbf{M}_Z = (\Omega, \mathcal{H}, (Z_t)_{t\geq 0}, (Q_x)_{x\in E})$ whose transition semigroup $(\tilde{p}_t)_{t\geq 0}$ has the property that $\tilde{p}_t f$ is a μ-version of $T_t f$ for all $f \in \mathcal{B}_b(E)$ with

$$Q_\mu[t \mapsto Z_t \text{ is continuous}] = 1. \tag{2.3}$$

Then there exists a right process $\mathbf{M}_X = (E^{\mathbb{R}_+}, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ with transition semigroup $(p_t)_{t\geq 0}$ and

$$P_x[t \mapsto X_t \text{ is continuous}] = 1 \quad \forall x \in E.$$

PROOF. The proof of this theorem is the same as for 2.3 Theorem with the following lemma instead of 2.10 Lemma. We can apply it, since we have $R_\alpha f = \int_0^\infty e^{-\alpha t} p_t f$, so the resolvent $(R_\alpha)_{\alpha>0}$ has the ∞-strong Feller property. □

2.13. LEMMA. *Let $(L, D(L))$ be the infinitesimal generator of a sub-Markovian semigroup $(T_t)_{t\geq 0}$ on $L^p(E,\mu)$ and denote by $(G_\alpha)_{\alpha>0}$ the associated resolvent. Let $(E^{\mathbb{R}_+}, \mathcal{F}, (Y_t)_{t\geq 0}, (P_x)_{x\in E})$ be a normal Markov process whose transition semigroup $(p_t)_{t\geq 0}$ and resolvent of kernels $(R_\alpha)_{\alpha>0}$ fulfill the following assumptions:*

(i) $\forall f \in \mathcal{B}_b$: *$p_t f$ is a μ-version of $T_t f$.*
(ii) $\forall x \in E\ \exists (f_n)_{n\in\mathbb{N}}, f_n = R_{\lambda_n}\varphi_n, \varphi_n \in \mathcal{B}_b(E), \lambda_n > 0$
$p_t f_n^2(x) \underset{t\downarrow 0}{\longrightarrow} f_n^2(x)$, *$\{f_n^{-1}(U) | U \in \mathcal{U}(f_n(x))\}$ is a subbase of $\mathcal{U}(x)$.*
(iii) *$(R_\alpha)_{\alpha>0}$ has the ∞-strong Feller property.*

Then we have for all $x \in E$ that $\lim_{t\downarrow 0} Y_t = x$ P_x-a.s., and in particular $\lim_{t\downarrow 0} Y_t = Y_0$ P_x-a.s.

PROOF. Fix now $x \in E$ and $(f_n)_{n\in\mathbb{N}}$ as above, then for any f_n we have that

$$\int_0^\infty e^{-\lambda_n s}|p_s\varphi_n(x)|ds \le \int_0^\infty e^{-\lambda_n s}p_s|\varphi_n|(x)ds = R_{\lambda_n}|\varphi_n|(x) < \infty.$$

and since $1_{[0,t]}e^{-\lambda_n s}p_s\varphi_n(x) \underset{t\downarrow 0}{\longrightarrow} 0$ pointwise, we have that

$$\begin{aligned}
&|p_t f_n(x) - f_n(x)| \\
&\quad= |p_t R_{\lambda_n}\varphi_n(x) - R_{\lambda_n}\varphi_n(x)| \\
&\quad= \left|p_t \int_0^\infty e^{-\lambda_n s}p_s\varphi_n(x)ds - \int_0^\infty e^{-\lambda_n s}p_s\varphi_n(x)ds\right| \\
&\quad= \left|\int_0^\infty e^{-\lambda_n s}p_{t+s}\varphi_n(x)ds - \int_0^\infty e^{-\lambda_n s}p_s\varphi_n(x)ds\right| \\
&\quad= \left|e^{\lambda_n t}\int_t^\infty e^{-\lambda_n s}p_s\varphi_n(x)ds - \int_0^\infty e^{-\lambda_n s}p_s\varphi_n(x)ds\right| \\
&\quad\le \left|\int_0^t e^{-\lambda_n s}p_s\varphi_n(x)ds\right| + (e^{\lambda_n t}-1)\left|\int_t^\infty e^{-\lambda_n s}p_s\varphi_n(x)ds\right| \\
&\quad\underset{t\downarrow 0}{\longrightarrow} 0.
\end{aligned}$$

Since $e^{-\alpha t}R_\alpha f(Y_t)$ is a supermartingale for every $f \ge 0$ and $\alpha > 0$ we have that the following limits exist P_x-a.s.

$$\lim_{t\downarrow 0} e^{-\lambda_n t}R_{\lambda_n}\varphi_n^+(Y_t(\omega)) \qquad \lim_{t\downarrow 0} e^{-\lambda_n t}R_{\lambda_n}\varphi_n^-(Y_t(\omega))$$

and hence

$$\lim_{t\downarrow 0} f_n(Y_t(\omega)).$$

Because of

$$\begin{aligned}
E_{P_x}[(f_n(Y_t) - f_n(x))^2] &= p_t f_n^2(x) - 2f_n(x)p_t f_n(x) + f_n^2(x) \\
&\underset{t\to 0}{\longrightarrow} f_n^2(x) - 2f_n^2(x) + f_n^2(x) \\
&= 0
\end{aligned}$$

we have that $\lim_{t\downarrow 0} f_n(Y_t) = f_n(x)$ in $L^2(E^{\mathbb{R}_+}, P_x)$. Hence we have $\lim_{t\downarrow 0} f_n(Y_t) = f_n(x)$ P_x-a.s.

Since we have the countable index set $\mathbb{N}$ there exists a set $N \subset E^{\mathbb{R}_+}$ such that for all $\omega \notin N$ and all $n \in \mathbb{N}$ we have that $\lim_{t\downarrow 0} f_n(Y_t) = f_n(x)$.

Applying 2.11 Lemma we get that $\lim_{t\downarrow 0} Y_t = x$ P_x-a.s. Since Y_t is normal by assumption we have $Y_0 = x$ P_x-a.s. $\square$

In order to apply the above theorems one needs the existence of a Markov process $\mathbf{M}_Z = (\Omega, \mathcal{H}, (Z_t)_{t\geq 0}, (Q_x)_{x\in E})$ with the property $Q_\mu[t \mapsto Z_t \text{ is continuous }] = 1$. In the special case of $p = 2$ the existence of such a process follows with the theory of Dirichlet forms (cf. Section A.1 in the appendix or **[MaRö92]** for an exact definition). In this case ($p = 2$) one has the following corollary to 2.3 Theorem.

2.14. COROLLARY. *Suppose there exists a quasi-regular Dirichlet form* $(\mathcal{E}, D(\mathcal{E}))$ *with the local property associated with* L *on* $L^2(E, \mu)$, *where* $\operatorname{supp}[\mu] = E$ *and a Markovian semigroup of kernels* $(p_t)_{t\geq 0}$, *satisfying assumptions 1.–4. or assumptions 1., 2. and 4′. Then there exists a right process* $(E^{\mathbb{R}_+}, \mathcal{F}, X_t, P_x)$ *that is properly associated with* $(\mathcal{E}, D(\mathcal{E}))$, *such that*

$$P_x[t \mapsto Z_t \text{ is continuous }] = 1 \qquad \forall x \in E.$$

PROOF. By **[MaRö92**, Theorem IV.3.5] there exists an μ-tight special standard process $(\Omega, \mathcal{H}, (Z_t)_{t\geq 0}, (Q_x)_{x\in E_\Delta})$ that is properly associated to $(\mathcal{E}, D(\mathcal{E}))$. Because the semigroup $(T_t)_{t\geq 0}$ is Markovian, the process is conservative and we have $\zeta = \infty$. By **[MaRö92**, Theorem V.1.5] we have $Q_x[t \mapsto Z_t \text{ is continuous}] = 1$ for $\mathcal{E}$-q.e. $x \in E$, in particular we have $Q_\mu[t \mapsto Z_t \text{ is continuous}] = 1$. Then we apply 2.3 Theorem or 2.12 Theorem respectively and get a right process $(E^{\mathbb{R}_+}, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ with transition semigroup $(p_t)_{t\geq 0}$, hence it is associated to $(\mathcal{E}, D(\mathcal{E}))$. By assumption 1. it is properly associated to $(\mathcal{E}, D(\mathcal{E}))$.

□

The last corollary in this chapter states that we can take the image of the process constructed in 2.3 Theorem or 2.12 Theorem on the space $C_E([0,\infty))$ of continuous functions from $[0,\infty)$ to E.

2.15. COROLLARY. *In the situation of 2.3 Theorem there exists a family of probability measures* $(\tilde{P}_x)_{x\in E}$ *on* $C_E([0,\infty))$ *such that the coordinate process* $(\tilde{X}_t)_{t\geq 0}$ *is a strong Markov process with transition semigroup* $(p_t)_{t\geq 0}$.

PROOF. 2.3 Theorem gives us a process on $E^{\mathbb{R}_+}$. Consider the following mapping $T : E^{\mathbb{R}_+} \to C_E([0,\infty))$, $\omega \mapsto T(\omega)$ with $T(\omega)(t) = X_t(\omega)$ and take as $\tilde{P}_x$ the image measures of P_x under T. Since X_t is a Markov process w.r.t. its natural filtration $(\mathcal{F}_t)_t$ and since $\mathcal{F}_t = \{T^{-1}(A) | A \in \tilde{\mathcal{F}}_t\}$ we can express conditional expectation under $\tilde{P}_x$ in terms of conditional expectations under P_x as follows: $E_{\tilde{P}_x}[\tilde{f}|\tilde{\mathcal{F}}_t] = E_{P_x}[\tilde{f} \circ T|\mathcal{F}_t]$ for any $\mathcal{F}_\infty$-measurable function $\tilde{f} : C_E([0,\infty)) \to \mathbb{R}$. Then we get the Markov property of $(\tilde{X}_t)$ by the Markov property of (X_t).

For the transition semigroup of this process we have for any $f \in \mathcal{B}_b(E)$ that $E_{\tilde{P}_x}[f(\tilde{X}_t)] = E_{\tilde{P}_x}[f \circ \tilde{X}_t] = E_{P_x}[f \circ \tilde{X}_t \circ T] = E_{P_x}[f \circ X_t] =$

$p_t f(x)$, hence it has the strong Feller property. Since the paths of the process $\tilde{X}_t$ are continuous, hence especially right continuous, this process is a strong Markov process by B.1 Proposition. $\square$

CHAPTER 3

Càdlàg Case

In this chapter we will prove the càdlàg case of the main theorem. The only difference is that the assumed process has càdlàg paths instead of continuous paths Q_μ-a.s., the constructed process will have also only càdlàg paths. All the other conditions remain the same.

3.1. THEOREM. *Let $(L, D(L))$ be the infinitesimal generator of a sub-Markovian semigroup $(T_t)_{t\geq 0}$ on $L^p(E, \mu)$ and denote by $(G_\alpha)_{\alpha>0}$ the associated resolvent. Let $(p_t)_{t\geq 0}$ be a Markovian semigroup of kernels on $(E, \mathcal{B}(E))$ and $(R_\alpha)_{\alpha>0}$ the associated resolvent of kernels on $(E, \mathcal{B}(E))$ such that*

1. $\forall f \in \mathcal{B}_b(E)\ \forall t \geq 0$: *$p_t f$ is a μ-version of $T_t f$.*
2. *$(p_t)_{t\geq 0}$ has the ∞-strong Feller property.*
3. *$(R_\alpha)_{\alpha>0}$ has the p-strong Feller property.*
4. $\forall x \in E\ \exists (f_n)_{n\in\mathbb{N}} \subset D(L) \cap C(E)$, $p_t f_n^2(x) \underset{t\downarrow 0}{\longrightarrow} f_n^2(x), \{f_n^{-1}(U) | U \in \mathcal{U}(f_n(x)), n \in \mathbb{N}\}$ *is a subbase of* $\mathcal{U}(x)$.

Suppose there exists a Markov process $\mathbf{M}_Z = (\Omega, \mathcal{H}, (Z_t)_{t\geq 0}, (Q_x)_{x\in E})$ whose transition semigroup $(\tilde{p}_t)_{t\geq 0}$ has the property that $\tilde{p}_t f$ is a μ-version of $T_t f$ for all $f \in \mathcal{B}_b(E)$ and $t > 0$ with

$$Q_\mu[t \mapsto Z_t \text{ is càdlàg}] = 1. \tag{3.1}$$

Then there exists a right process $\mathbf{M}_X = (E^{\mathbb{R}_+}, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ with transition semigroup $(p_t)_{t\geq 0}$ and

$$P_x[t \mapsto X_t \text{ is càdlàg}] = 1 \qquad \forall x \in E.$$

3.2. REMARK. Note that since μ is a probability measure, we have that every $f \in \mathcal{B}_b(E)$ is p-integrable, hence we can apply T_t on $f \in \mathcal{B}_b(E)$ in 1.

PROOF OF 3.1 THEOREM. By Kolmogorov's theorem there exists a Markov process $\mathbf{M}_Y = (E^{\mathbb{R}_+}, \mathcal{F}, (Y_t)_{t\geq 0}, (P_x)_{x\in E})$ with transition semigroup $(p_t)_{t\geq 0}$, family of shift operators $(\theta_t)_{t\geq 0}$ and filtration $(\mathcal{F}_t)_{t\geq 0}$.

In order to use 2.5 Lemma we have to find an appropriate set Λ which characterizes the paths which can be modified to be càdlàg. This set should be shift-invariant and describable as a union or intersection of sets which depend only on a finite number of times.

Denote by $U_M(\alpha, \beta, f)$ the number of up-crossings over the interval $[\alpha, \beta]$ by the function $f_{|M}$. Let $(g_n)_{n\in\mathbb{N}}$ be a family of bounded continuous functions which separates the points of E. Such a family exists, because E is separable (take e.g. $g_n(x) := d(x, x_n) \wedge 1$, where $\{x_n | n \in \mathbb{N}\}$ is a countable dense subset of E).

It is well-known (cf. e.g. [**Wi79**, (37.1) Lemma]) that for a function $f : \mathbb{R}_+ \to \mathbb{R}$ and a dense subset S of $\mathbb{R}_+$the following conditions are equivalent:

- the function $t \mapsto \lim_{\substack{s\to t\\ s>t}} f(s)$ is càdlàg,
- $\forall \alpha, \beta \in \mathbb{Q},\ \alpha < \beta\ \forall n \in \mathbb{N}$: $U_{S\cap[0,n]}(\alpha, \beta, f) < \infty$.

Define now for $k, n \in \mathbb{N},\ \alpha, \beta \in \mathbb{Q}$

$$A^{(n)}_{\alpha,\beta,k} := \{\omega \in E^{R_+} | U_{[0,n]\cap D}(\alpha, \beta, g_k \circ Y_\cdot(\omega)) = \infty\},$$
$$B^{(n)}_{\alpha,\beta,k} := \{\omega \in \Omega | U_{[0,n]\cap D}(\alpha, \beta, g_k \circ Z_\cdot(\omega)) = \infty\}$$

and

$$\Lambda_0 := E^{\mathbb{R}_+} \setminus \bigcup_{\substack{\alpha,\beta\in\mathbb{Q}\\ n,k\in\mathbb{N}}} A^{(n)}_{\alpha,\beta,k}\,.$$

By the following lemma we have that $P_\mu[\Lambda_0] = 1$, and since $\theta_t^{-1}(A^{(n)}_{\alpha,\beta,k}) = \{\omega \in E^{\mathbb{R}_+} | U_{[t,n+t]\cap D}(\alpha, \beta, g_k \circ Y(\omega)) = \infty\}$ $\subset \{\omega \in E^{\mathbb{R}_+} | U_{[0,n+t]\cap D}(\alpha, \beta, g_k \circ Y(\omega)) = \infty\}$ we have for $t \in D$ that $\theta_t^{-1}(\Lambda_0) \supset \Lambda_0$.

3.3. LEMMA. $\forall n, k \in \mathbb{N}\ \forall \alpha, \beta \in \mathbb{Q}$:

$$P_\mu\big[A^{(n)}_{\alpha,\beta,k}\big] = Q_\mu\big[B^{(n)}_{\alpha,\beta,k}\big] = 0.$$

PROOF. By Lemma 2.6 we have that P_μ and Q_μ have the same finite dimensional distributions. The second identity follows from (3.1). The first one by

$$\begin{aligned}
&P_\mu[A^{(n)}_{\alpha,\beta,k}]\\
&= P_\mu\Big[\bigcap_{l\in\mathbb{N}}\{\omega \in \mathbb{R}_+ | U_{[0,n]\cap D}(\alpha, \beta, g_k \circ Y_\cdot(\omega)) \geq l\}\Big]\\
&= \inf_{l\in\mathbb{N}} P_\mu\Big[\bigcup_{m\in\mathbb{N}} \bigcup_{\substack{\{t_1,s_1,\dots,t_l,s_l\}\\ \subset[0,n]\cap S_m\\ t_1<s_1<\dots<t_l<s_l}} \{g_k(Y_{t_i}) < \alpha, g_k(Y_{s_i}) > \beta, i = 1, \dots, l\}\Big]\\
&= \inf_{l\in\mathbb{N}} \sup_{m\in\mathbb{N}} P_\mu\Big[\bigcup_{\substack{\{t_1,s_1,\dots,t_l,s_l\}\\ \subset[0,n]\cap S_m\\ t_1<s_1<\dots<t_l<s_l}} \{g_k(Y_{t_i}) < \alpha, g_k(Y_{s_i}) > \beta, i = 1, \dots, l\}\Big]\\
&= \inf_{l\in\mathbb{N}} \sup_{m\in\mathbb{N}} Q_\mu\Big[\bigcup_{\substack{\{t_1,s_1,\dots,t_l,s_l\}\\ \subset[0,n]\cap S_m\\ t_1<s_1<\dots<t_l<s_l}} \{g_k(Z_{t_i}) < \alpha, g_k(Z_{s_i}) > \beta, i = 1, \dots, l\}\Big]
\end{aligned}$$

$$= \cdots = Q_\mu[B^{(n)}_{\alpha,\beta,k}].$$

□

Applying 2.5 Lemma we get that $P_x[\theta_t^{-1}(\Lambda_0)] = 1$ for all $x \in E$.

If $\omega \in \theta_t^{-1}(\Lambda_0)$ then the function $g_k \circ Y_{\cdot+}(\omega) : (t,\infty) \to \mathbb{R}$, $s \mapsto \lim_{r\downarrow s, r\in D} g_k \circ Y_r(\omega)$ can be defined for each k and is càdlàg. Define

$$\Lambda_0' := \bigcap_{t\in D, t>0} \theta_t^{-1}(\Lambda_0).$$

3.4. Remark. For $\omega \in \Lambda_0'$ we have the following property: For every $t > 0$ and $k \in \mathbb{N}$ the function $g_k \circ Y_{\cdot+}(\omega) : [t,\infty) \to \mathbb{R}$, $g_k \circ Y_{s+} := \lim_{r\downarrow s, r\in D} g_k \circ Y_r(\omega)$ is well-defined and càdlàg. We now have to conclude the same for the function $Y_{\cdot+}(\omega) : [t,\infty) \to E$.

To this end let $(K_n)_{n\in\mathbb{N}}$ be an increasing sequence of compact subsets of E such that

$$Q_\mu\big[\lim_{n\to\infty} \sigma_{E\setminus K_n} < \infty\big] = 0. \tag{3.2}$$

These sets exist, because the process $(Z_t)_{t\geq 0}$ is μ-tight by **[MaRö92**, Theorem IV.1.15] and the semigroup $(p_t)_{t\geq 0}$ is Markovian (hence lifetime $\zeta = \infty$). Define also

$$\Omega_1 := \{\omega \in \Omega | \forall M \in \mathbb{N} \exists n \in \mathbb{N} \forall t \in D \cap [0,M] : Z_t(\omega) \in K_n\},$$
$$\Lambda_1 := \{\omega \in E^{\mathbb{R}_+} | \forall M \in \mathbb{N} \exists n \in \mathbb{N} \forall t \in D \cap [0,M] : Y_t(\omega) \in K_n\}.$$

3.5. Lemma.

$$P_\mu[\Lambda_1] = Q_\mu[\Omega_1] = 1.$$

Proof. From (3.2) we deduce

$$\begin{aligned}
1 &= Q_\mu[\lim_{n\to\infty} \sigma_{E\setminus K_n} = \infty] \\
&= Q_\mu[\{\omega \in \Omega | \forall M \in \mathbb{N} \exists n \in \mathbb{N} : \sigma_{E\setminus K_n} \geq M\}] \\
&\leq Q_\mu[\{\omega \in \Omega | \forall M \in \mathbb{N} \exists n \in \mathbb{N} \forall t \in D \cap [0,M] : Z_t(\omega) \in K_n\}] \\
&= \inf_{M\in\mathbb{N}} \sup_{n\in\mathbb{N}} Q_\mu[\{\omega \in \Omega | \forall l \in \mathbb{N} \forall t \in S_l \cap [0,M] : Z_t(\omega) \in K_n\}] \\
&= \inf_{M\in\mathbb{N}} \sup_{n\in\mathbb{N}} \inf_{l\in\mathbb{N}} Q_\mu[\{\omega \in \Omega | \forall t \in S_l \cap [0,M] : Z_t(\omega) \in K_n\}] \\
&= \inf_{M\in\mathbb{N}} \sup_{n\in\mathbb{N}} \inf_{l\in\mathbb{N}} P_\mu[\{\omega \in E^{\mathbb{R}_+} | \forall t \in S_l \cap [0,M] : Y_t(\omega) \in K_n\}] \\
&= \dots \\
&= P_\mu[\{\omega \in E^{\mathbb{R}_+} | \forall M \in \mathbb{N} \exists n \in \mathbb{N} \forall t \in D \cap [0,M] : Y_t(\omega) \in K_n\}] \\
&\leq 1.
\end{aligned}$$

□

Again we have that $\theta_t(\Lambda_1) \subset \Lambda_1$ if $t \in D$. Define

$$\Lambda_2 := \Lambda_0 \cap \Lambda_1 \quad \text{and} \quad \Lambda_2' := \bigcap_{t \in D, t>0} \theta_t^{-1}(\Lambda_2).$$

If we define

$$\Lambda_1' := \bigcap_{t \in D, t>0} \theta_t^{-1}(\Lambda_1)$$

we have

$$\Lambda_2' = \Lambda_0' \cap \Lambda_1'.$$

We know that $P_x[\Lambda_2'] = 1$.

3.6. LEMMA. *For every $\omega \in \Lambda_2'$ we have the following property: For every $t > 0$ the function $Y_{\cdot+}(\omega) : [t, \infty) \to E$, $Y_{s+}(\omega) := \lim_{r \downarrow s, r \in D} Y_r(\omega)$ is well-defined and càdlàg.*

PROOF. Fix $\omega \in \Lambda_2'$ and $s > 0$. Then there exists $m \in \mathbb{N}$ such that $\{Y_r(\omega) | r \in (s, s+1] \cap D\} \subset K_m$. Since K_m is compact there exist $(s_n) \subset (s, s+1] \cap D$, $s_n \downarrow s$ and $x \in E$ such that $\lim_{n\to\infty} Y_{s_n}(\omega) = x$. Suppose $\lim_{r \downarrow s, r \in D} Y_r$ does not exist. Then there exists a neighborhood O of x and a sequence $(\tilde{s}_n)_{n \in \mathbb{N}} \subset D \cap (s, s+1]$ such that $Y_{\tilde{s}_n}(\omega) \notin O$ for all $n \in \mathbb{N}$. Since K_m is compact there exists a subsequence $(\tilde{s}_{n_k})_{k \in \mathbb{N}}$ and $y \in E$ with $\lim_{k\to\infty} Y_{s_{n_k}}(\omega) = y$. Then there exists $l \in \mathbb{N}$ with $g_l(x) \neq g_l(y)$ But then $\lim_{\substack{s \downarrow t \\ s \in D}} g_l(Y_s(\omega))$ does not exist. Hence $Y_{\cdot+}(\omega)$ is welldefined.

That it is càdlàg follows in a similar way: Since g_k is continuous we have that $g_k(Y_{s+}(\omega)) = g_k(\lim_{r \downarrow s, r \in D} Y_r(\omega)) = \lim_{r \downarrow s, r \in D} g_k(Y_r(\omega))$ hence the function $s \mapsto g_k(Y_{s+}(\omega))$ is càdlàg. But since $s \mapsto g_k(Y_{s+})$ is càdlàg we have that for every k the limits $\lim_{h \downarrow 0} g_k(Y_{(s+h)+}(\omega))$ and $\lim_{h \downarrow 0} g_k(Y_{(s-h)+}(\omega))$ exist. Then by a similar argument as in the first part we have that $\lim_{h \downarrow 0} Y_{(s+h)+}(\omega)$ and $\lim_{h \downarrow 0} Y_{(s-h)+}(\omega)$ exist. And since $(g_k)_{n \in \mathbb{N}}$ separates the points of E we have that $\lim_{h \downarrow 0} Y_{(s+h)+}(\omega) = Y_{s+}(\omega)$. □

Again we have to do additional work for time zero. The remainder of this proof is completely analog to the proof of 2.3 Theorem.

$$\Lambda_3 := \{\omega \in E^{\mathbb{R}_+} | \lim_{\substack{s \downarrow 0 \\ s \in D}} Y_t(\omega) \text{ exists}\}.$$

We have $P_x[\Lambda_1] = 1$ for all $x \in E$ by 2.10 Lemma.
Define

$$\Lambda := \Lambda_2' \cap \Lambda_3$$

and fix an arbitrary $x_0 \in E$.

Define the new process $\mathbf{M}_X = (E^{\mathbb{R}_+}, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in E}$ by:

$$X_t(\omega) = \begin{cases} \lim_{\substack{s\downarrow t \\ s\in D}} Y_t(\omega) & ,\omega \in \Lambda \\ x_0 & ,\omega \notin \Lambda. \end{cases} \tag{3.3}$$

This process has càdlàg paths for every ω.

It remains to show that $P_x[X_t = Y_t] = 1$ for all $x \in E$.

$$\begin{aligned} E_{P_x}[d(X_t, Y_t)] &= E_{P_x}[d(\lim_{\substack{s\downarrow t \\ s\in D}} Y_s, Y_t)] \\ &= E_{P_x}[E_{P_x}[d(\lim_{\substack{s\downarrow t \\ s\in D}} Y_s, Y_t)|\mathcal{F}_t]] \\ &= E_{P_x}[E_{P_x}[d(\lim_{\substack{s\downarrow t \\ s\in D}} Y_{s-t}, Y_0) \circ \theta_t|\mathcal{F}_t]] \\ &= E_{P_x}[E_{Y_t}[d(\lim_{\substack{s\downarrow t \\ s\in D}} Y_{s-t}, Y_0)]] \\ &= 0. \end{aligned}$$

So $d(X_t, Y_t) = 0$ P_x-a.s. hence $P_x[X_t = Y_t] = 1$ and in particular $(X_t)_{t\geq 0}$ has the transition semigroup $(p_t)_{t\geq 0}$. Then $(X_t)_{t\geq 0}$ has the Markov property w.r.t. the filtration $(\mathcal{F}_t^{0,X})_{t\geq 0}$ where $\mathcal{F}_t^{0,X} := \sigma(X_s|0 \leq s \leq t)$ by [**Ba91**, 42.3 Satz] and since $\mathcal{N} := \bigcap_{\nu\in\mathcal{P}(E)}\{A \in \mathcal{F}|P_\nu[A] \in \{0,1\}\}$ is a σ-algebra indepent of every $\mathcal{F}_t^{0,X}$ w.r.t. every P_ν we have by B.2 that X_t has the Markov property w.r.t. $(\mathcal{F}_t^X)_{t\geq 0}$ where $\mathcal{F}_t^X := \sigma(\mathcal{F}_t^{0,X}, \mathcal{N})$. Since the paths are càdlàg it has the strong Markov property with respect to $(\mathcal{F}_{t+}^X)_{t\geq 0}$ by Proposition B.1. And since it is normal by construction $\mathbf{M}_X$ is a right process. □

As in Chapter 2 there exists the following version of 3.1 Theorem, where there is no assumption on the resolvent, but a stronger assumption on the family of functions $(f_n)_{n\in\mathbb{N}}$ that we needed to ensure the rightcontinuity for time zero.

3.7. THEOREM. *Let $(L, D(L))$ be the infinitesimal generator of a sub-Markovian semigroup $(T_t)_{t\geq 0}$ on $L^p(E, \mu)$ and denote by $(G_\alpha)_{\alpha>0}$ the associated resolvent. Let $(p_t)_{t\geq 0}$ be a Markovian semigroup of kernels on $(E, \mathcal{B}(E))$ and $(R_\alpha)_{\alpha>0}$ the associated resolvent of kernels on $(E, \mathcal{B}(E))$ such that*

1. $\forall f \in \mathcal{B}_b(E)$: *$p_t f$ is a μ-version of $T_t f$.*
2. *$(p_t)_{t\geq 0}$ has the ∞-strong Feller property.*

4′. $\forall x \in E\ \exists (f_n)_{n\in\mathbb{N}}, f_n = R_{\lambda_n}\varphi_n, \varphi_n \in \mathcal{B}_b(E), \lambda_n > 0$
$p_t f_n^2(x) \underset{t\downarrow 0}{\longrightarrow} f_n^2(x)$, *$\{f_n^{-1}(U)|U \in \mathcal{U}(f_n(x))\}$ is a subbase of $\mathcal{U}(x)$.*

Suppose there exists a Markov process $\mathbf{M}_Z = (\Omega, \mathcal{H}, (Z_t)_{t\geq 0}, (Q_x)_{x\in E})$ *whose transition semigroup* $(\tilde{p}_t)_{t\geq 0}$ *has the property that* $\tilde{p}_t f$ *is a* μ*-version of* $T_t f$ *for all* $f \in \mathcal{B}_b(E)$ *with*

$$Q_\mu[t \mapsto Z_t \text{ is càdlàg}] = 1. \tag{3.4}$$

Then there exists a right process $\mathbf{M}_X = (E^{\mathbb{R}_+}, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ *with transition semigroup* $(p_t)_{t\geq 0}$ *and*

$$P_x[t \mapsto X_t \text{ is càdlàg}] = 1 \quad \forall x \in E.$$

PROOF. The proof of this theorem is the same as for 3.1 Theorem with 2.13 Lemma instead of 2.10 Lemma. We can apply it, since we have $R_\alpha f = \int_0^\infty e^{-\alpha t} p_t f$, so the resolvent $(R_\alpha)_{\alpha>0}$ has the ∞-strong Feller property. □

In order to apply the above theorems one needs the existence of a Markov process $\mathbf{M}_Z = (\Omega, \mathcal{H}, (Z_t)_{t\geq 0}, (Q_x)_{x\in E})$ with the property $Q_\mu[t \mapsto Z_t$ is càdlàg $] = 1$. In the special case of $p = 2$ the existence of such a process follows with the theory of generalized Dirichlet forms (cf. Section A.2 in the appendix or **[St99a]** for an exact definition). In this case ($p = 2$) one has the following corollary to Theorem 3.1.

3.8. COROLLARY. *Suppose there exists a quasiregular generalized Dirichletform on* $L^2(E, \mu)$ *fulfilling the following assumption:*

There exists a linear subspace $\mathcal{Y} \subset \mathcal{H} \cap L^\infty(E, m)$ *such that* $\mathcal{Y} \cap \mathcal{F}$ *is dense in* $\mathcal{F}$, $\lim_{\alpha\to\infty}(\alpha G_\alpha u - u)_E = 0$ *in* $\mathcal{H}$ *for all* $u \in \mathcal{Y}$ *and for the closure* $\bar{\mathcal{Y}}$ *of* $\mathcal{Y}$ *in* $L^\infty(E, \mu)$ *it follows that* $u \wedge \alpha \in \bar{\mathcal{Y}}$ *for* $u \in \mathcal{Y}$ *and* $\alpha \geq 0$.

Let $(T_t)_{t\geq 0}$ *be the Markovian semigroup associated to* $(\mathcal{E}, D(\mathcal{E}))$ *and a semigroup of kernels* $(p_t)_{t\geq 0}$*, which fulfills assumptions 1.–4. or assumptions 1., 2. and 4'.*

Then there exists a right process $(E^{\mathbb{R}_+}, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ *that is properly associated with* $(\mathcal{E}, D(\mathcal{E}))$*. And we have*

$$P_x[t \mapsto Z_t \text{ is càdlàg }] = 1 \qquad \forall x \in E.$$

PROOF. By **[St99a**, Theorem IV.2.2] there exists a μ-tight special standard process $(\Omega, \mathcal{G}, (Z_t)_{t\geq 0}, (Q_x)_{x\in E_\Delta})$ that is properly associated to $(\mathcal{E}, D(\mathcal{E}))$. Because the semigroup $(T_t)_{t\geq 0}$ is Markovian, the process is conservative and we have $\zeta = \infty$. Then we have $Q_x[t \mapsto Z_t$ is càdlàg$] = 1$ for $\mathcal{E}$-q.e. $x \in E_\Delta$ and in particular $Q_\mu[t \mapsto Z_t$ is càdlàg$] = 1$. Now we can apply 3.1 Theorem or 3.7 Theorem respectively and get a right process $(E^{\mathbb{R}_+}, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ with transition semigroup $(p_t)_{t\geq 0}$, hence it is associated to $(\mathcal{E}, D(\mathcal{E}))$. By assumption 1. it is properly associated to $(\mathcal{E}, D(\mathcal{E}))$. □

The last corollary in this chapter states that we can take the image of the process constructed in 2.3 Theorem or 2.12 Theorem on the space $D_E([0, \infty))$ of càdlàg functions from $[0, \infty)$ to E.

3.9. COROLLARY. *In the situation of Theorem 3.1 there exists a family of probability measures* $(\tilde{P}_x)_{x\in E}$ *on* $D([0,\infty),E)$ *such that the coordinate process is a strong Markov process.*

PROOF. Theorem 3.1 gives us a process on $E^{\mathbb{R}_+}$. Consider the following mapping $T : E^{\mathbb{R}_+} \to D_E([0,\infty))$, $\omega \mapsto T(\omega)$ with $T(\omega)(t) = X_t(\omega)$ and take as $\tilde{P}_x$ the image measures of P_x under T. Since X_t is a Markov process w.r.t. its natural filtration $(\mathcal{F}_t)_t$ and since $\mathcal{F}_t = \{T^{-1}(A)|A \in \tilde{\mathcal{F}}_t\}$ we can express conditional expectation under $\tilde{P}_x$ in terms of conditional expectations under P_x as follows: $E_{\tilde{P}_x}[\tilde{f}|\tilde{\mathcal{F}}_t] = E_{P_x}[\tilde{f} \circ T|\mathcal{F}_t]$ for any $\mathcal{F}_\infty$-measurable function $f : D_E([0,\infty)) \to \mathbb{R}$. Then we get the Markov property of $(\tilde{X}_t)_{t\geq 0}$ by the Markov property of $(X_t)_{t\geq 0}$.

For the transition semigroup of this process we have for any $f \in \mathcal{B}_b(E)$ that $E_{\tilde{P}_x}[f(\tilde{X}_t)] = E_{\tilde{P}_x}[f \circ \tilde{X}_t] = E_{P_x}[f \circ \tilde{X}_t \circ T] = E_{P_x}[f \circ X_t] = p_t f(x)$, hence it has the strong Feller property. Since the paths of the process $(\tilde{X}_t)_{t\geq 0}$ are càdlàg, hence rightcontinuous, this process is a strong Markov process by B.1 Proposition. $\square$

CHAPTER 4

Applications

In this section we will take $E = \mathbb{R}^d$, $d \geq 2$ and denote by λ the Lebesgue measure on $\mathbb{R}^d$. Let

$$L_{A,b} = \sum_{i,j} a_{ij} \partial_i \partial_j + \sum_i b_i \partial_i$$

be an operator on $C_0^\infty(\mathbb{R}^d)$.

We will need the notion of L^p-uniqueness, as it is introduced in [**St99b**, Definition 1.8].

4.1. Definition. Let $p \in [1, \infty)$ and (A, D) be a densely defined operator on $L^p(X, m)$. We say that (A, D) is $L^p(X, m)$-unique, if there is only one extension of (A, D) on $L^p(X, m)$ that generates a C_0-semigroup.

4.2. Theorem. *Let μ be a probability measure on $\mathbb{R}^d$ and $L_{A,b}^* \mu = 0$ i.e. $\int_{\mathbb{R}^d} L_{A,b} \varphi d\mu = 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d)$. Assume the following conditions hold for some $p > d + 2$:*

1. *$A = (a_{ij})_{i,j}$ is continuous, symmetric, nonnegative and nondegenerate with $a_{ij} \in H_{loc}^{1,p}(\mathbb{R}^d)$*
2. *$b_i \in L_{loc}^p(\mathbb{R}^d, \lambda)$*
3. *$(L_{A,b}, C_0^\infty(\mathbb{R}^d))$ is $L^1(\mathbb{R}^d, \mu)$-unique.*

Then there exists a family of probability measures $(P_x)_{x \in \mathbb{R}^d}$ such that $\mathbf{M} = (C_E([0, \infty)), \mathcal{F}, (X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d})$ is a conservative Markov process whose transition semigroup $(p_t)_{t \geq 0}$ has the following properties:

- *$p_t f$ is a μ-version of $\bar{T}_t f$ for all $f \in \mathcal{B}_b(E)$, $t \geq 0$ where $(\bar{T}_t)_{t \geq 0}$ is the C_0-semigroup on $L^1(E, \mu)$ generated by the unique extension of $L_{A,b}$ that generates a C_0-semigroup.*
- *$(p_t)_{t \geq 0}$ has the ∞-strong Feller property.*

4.3. Remark. Although L^1-uniqueness implies by its definition the existence of such an extension we will not need this part, due to the following proposition. We will only use L^1-uniqueness to ensure that the associated semigroup $(\bar{T}_t)_{t \geq 0}$ is Markovian.

4.4. Proposition. *Suppose in the situation of the above theorem that you have only assumptions 1. and 2. Then there exists a closed extension $\bar{L}_{A,b}$ of $L_{A,b}$ that generates a C_0-semigroup $(\bar{T}_t)_{t \geq 0}$ on $L^1(\mathbb{R}^d, \mu)$.*

Proof. To this end we want to apply [**St99b**, Theorem 1.5]. So we have to check the following conditions:

1. μ is absolutely continuous w.r.t. Lebesgue measure.
2. $\operatorname{supp}[\mu] = \mathbb{R}^d$.
3. $\frac{d\mu}{dx} = \varphi^2$ where $\varphi \in H^{1,2}_{loc}(\mathbb{R}^d)$.
4. $a_{ij} \in H^{1,2}_{loc}(\mathbb{R}^d, \mu)$,$1 \leq i, j \leq d$.
5. A is locally strict elliptic.
6. $|b|^2 \in L^1_{loc}(\mathbb{R}^d, \mu)$.

1. By [**BoKrRö**, Corollary 2.9] μ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$ and for its density ρ we have $\rho \in H^{1,p}_{loc}$ and ρ is locally Hölder continuous.

2. Since $b_i \in L^p_{loc}(\mathbb{R}^d)$ we can apply [**BoKrRö**, Corollary 2.10] and get that for every compact $V \subset U \subset \mathbb{R}^d$ where $\overline{U}$ is compact one has $\sup_V \varphi^2 \leq C \inf_V \varphi^2$. Take now $V_n = \overline{B_n(0)}$ and $U_n = B_{n+1}(0)$. Since μ is a probability measure there exists n_0 such that $\varphi^2_{|V_{n_0}} \neq 0$ but then $\varphi^2 > 0$ on V_n for all $n \geq n_0$ and since $\mathbb{R}^d = \bigcup_{n=1}^\infty V_n$ we have $\varphi^2 > 0$ on $\mathbb{R}^d$ and hence $\operatorname{supp}[\mu] = \mathbb{R}^d$.

3. Since μ is a probability measure by assumption and $\operatorname{supp}[\mu] = \mathbb{R}^d$ its density ρ is positive and admits a representation φ^2 where $\varphi \in H^{1,2}_{loc}$ and φ is again locally Hölder continuous since the function $x \mapsto \sqrt{x}$ is locally Lipschitz continuous on $(0, \infty)$.

4. We have for $V \subset \mathbb{R}^d$, V compact $\int_V \sum_{k=1}^d |\partial_k a_{ij}|^2 + |a_{ij}|^2 d\mu = \int_V \sum_{k=1}^d |\partial_k a_{ij}|^2 + |a_{ij}|^2 \varphi^2 dx = \sum_{k=1}^d \int_V |\partial_k a_{ij} \varphi|^2 dx + \int_V |a_{ij}\varphi|^2 dx \leq (\max\{|\varphi(x)| | x \in V\})^2 \left(\sum_{k=1}^d \int_V |\partial_k a_{ij}|^2 + \int_V |a_{ij}| \right) < \infty$ since φ is continuous and since $p > d+2 \geq 2$ we have that $H^{1,p}_{loc}(\mathbb{R}^d, dx) \subset H^{1,2}_{loc}(\mathbb{R}^d, dx)$.

5. We have to show that for all $V \subset \mathbb{R}^d$ compact there exists $c_V, C_V > 0$ such that for all $x \in V$, $h \in \mathbb{R}^d$ we have $c_V \|h\|^2 \leq \langle A(x)h, h \rangle \leq C_V \|h\|^2$.

It is sufficient to show the above inequality for $h \in \mathbb{R}^d$ with $\|h\| = 1$:

The mapping $(x, h) \mapsto \langle A(x)h, h \rangle$ is continuous, since the mapping $x \mapsto A(x)$ is continuous with respect to the operator norm on $(\mathbb{R}^d, \langle \cdot, \cdot \rangle)$ and since we have the following inequality:

$$\begin{aligned}
&|\langle A(x)u, u \rangle - \langle A(y)v, v \rangle| \\
&\leq |\langle A(x)u, u \rangle - \langle A(x)u, v \rangle| + |\langle A(x)u, v \rangle - \langle A(x)v, v \rangle| \\
&\quad + |\langle A(x)v, v \rangle - \langle A(y)v, v \rangle| \\
&= |\langle A(x)u, u - v \rangle| + |\langle u - v, A(x)v \rangle| + |\langle (A(x) - A(y))v, v \rangle| \\
&\leq (|A(x)u| + |A(x)v|)\|u - v\| + |(A(x) - A(y))v| \|v\| \\
&\leq (\|A(x)\| \|u\| + \|A(x)\| \|v\|)\|u - v\| + \|A(x) - A(y)\| \|v\|^2.
\end{aligned}$$

Since $V \times \{\|h\| = 1\}$ is compact there exists $(x_0, h_0), (x_1, h_1) \in V \times \{\|h\| = 1\}$ such that $c_V := \langle A(x_0)h_0, h_0 \rangle = \inf_{x,h \in V \times \{\|h\|=1\}} \langle A(x)h, h \rangle$

and $C_V := \langle A(x_1)h_1, h_1\rangle = \sup_{x,h\in V\times\{\|h\|=1\}}\langle A(x)h, h\rangle$ Then we get $C_V < \infty$ and $c_V > 0$ since A is nondegenerate.

6. We have to show that for every $V \subset \mathbb{R}^d$ compact we have $\int_V \|b\|^2\, d\mu < \infty$. This we get by calculating

$$\begin{aligned}\int_V \|b\|^2 d\mu &= \int_V \sum_{i=1}^d b_i^2 \varphi^2 dx \\ &\le \sup_{x\in V}|\varphi|^2 \sum_{i=1}^d \int_V b_i^2 dx \\ &= \int_V 1_{\{b_i<1\}} b_i^2\, dx + \int_V 1_{\{b_i\ge 1\}} b_i^2\, dx \\ &\le \int_V dx + \int_V |b_i|^p dx < \infty,\end{aligned}$$

since p>2 and $\int_V dx < \infty$.

Since all prerequisites are fulfilled, we can apply [**St99b**, Theorem 1.5] and get a closed extension $(\bar{L}_{A,b}, D(\bar{L}_{A,b}))$ of $(L_{A,b}, C_0^\infty(\mathbb{R}^d))$ which generates a sub-Markovian C_0-semigroup $(\bar{T}_t)$ of contractions on $L^1(\mathbb{R}^d, \mu)$. □

Proof of 4.2. The extension of $(L_{A,b}, C_0^\infty)$ on $L^1(\mathbb{R}^d, \mu)$ that generates a C_0-semigroup $(\bar{T}_t)_{t\ge 0}$ will be denoted by $(\bar{L}_{A,b}, D(\bar{L}_{A,b}))$.

We observe that a_{ij} is locally Hölder continuous.

Indeed: Let $V \subset \mathbb{R}^d$ compact. Then there exists $R > 0$ such that $B_R(0) \supset V$. There exists $\chi \in C_0^\infty$ such that $\chi_{|V} = 1$ and $\chi(x) = 0$ for all $x \notin B_{R+1}(0)$. Define $U = B_{R+2}(0)$. Then we have that $(a_{ij}\chi)_{|U} \in H^{1,p}(U)$. Since ∂U is a Lipschitz-boundary we have by the Sobolev embedding theorem that there exists a $(1-\frac{d}{p})$-Hölder-continuous function $f : U \to \mathbb{R}$ with Hölder-constant L such that $f = a_{ij}\chi$ m-almost surely on U. But since a_{ij} is already continuous and since $\chi_{|V} = 1$ we have that $f(x) = a_{ij}(x)$ for $x \in V$. Hence we have that $|a_{ij}(x) - a_{ij}(y)| \le L_V |x-y|^{1-\frac{d}{p}}$.

Now we can apply [**St99b**, Corollary 2.2] and get that μ is $\bar{T}_t$-invariant since $(L_{A,b}, C_0^\infty)$ is L^1-unique by assumption.

Then we have that $(\bar{T}_t)_{t\ge 0}$ is Markovian. Indeed, we have $\int(1 - \bar{T}_t 1)\, d\mu = 0$. But since $(\bar{T}_t)_{t\ge 0}$ is sub-Markovian, we have $\bar{T}_t 1 \le 1$ and hence $1 - \bar{T}_t 1 \ge 0$. But since $\operatorname{supp}[\mu] = \mathbb{R}^d$ we have $\bar{T}_t 1 = 1$.

$(\bar{T}_t)_{t\ge 0}$ induces a Markovian C_0-semigroup $(T_t)_{t\ge 0}$ on $L^2(\mathbb{R}^d, \mu)$, since $2 > 1$ (for a proof using the Riesz-Thorin interpolation theorem see e.g. [**Eb99**, Lemma 1.11]). The new semigroup is again Markovian since it coincides with the old semigroup on $L^1(\mathbb{R}^d, \mu) \cap L^2(\mathbb{R}^d, \mu)$.

Let L denote the generator of $(T_t)_{t\ge 0}$ and $(G_\alpha)_{\alpha>0}$ the associated resolvent. Then by [**St99b**, Theorem 3.5] there exists a Markov process $(\Omega, \mathcal{H}, (Z_t)_{t\ge 0}, (Q_x)_{x\in E})$ such that $E_{Q_\cdot}[\int_0^\infty e^{-\alpha t} f(X_t)\, dt]$ is an $\mathcal{E}^0$-q.c.

μ-version of $G_\alpha f$ for all $f \in \mathcal{B}_b(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, \mu)$, $\alpha > 0$ and hence $E_{Q_\cdot}[f(X_t)]$ is a μ-version of $T_t f$ for all $f \in \mathcal{B}_b(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, \mu)$ and so it is a μ-version of $\bar{T}_t f$. By [**St99b**, Theorem 3.6] we get $P_x[t \mapsto Z_t$ is continuous$] = 1$ $\mathcal{E}^0$-q.e., hence in particular μ-a.e. and so we get $Q_\mu[t \mapsto Z_t$ is continuous$] = 1$.

By [**BoKrRö**, Theorem 4.1] there exist unique probability kernels $p_t(\cdot, dy), t > 0$ on $\mathbb{R}^d$ such that $p_t(x, dy) = p(t, x, y)dy$ where $p(t, x, y)$ is a locally Hölder continuous function on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, and for every $f \in L^1(\mathbb{R}^d, \mu)$ the function $x \mapsto p_t f(x) := \int_{\mathbb{R}^d} f(y)p(t, x, y)dy$ is a μ-version of $T_t f$ such that $(t, x) \mapsto p_t f(x)$ is continuous on $(0, \infty) \times \mathbb{R}^d$, and so for $f \in \mathcal{B}_b$ it is a μ-version of $\bar{T}_t f$.

To apply Corollary 2.15 we have to check the following conditions:

1. $\mathbb{R}^d$ is a polish space.
2. $\forall x \in \mathbb{R}^d\ \exists (f_n)_{n\in\mathbb{N}} \in D(\bar{L}_{A,b}) \cap C(E)$:
   $\{f_n^{-1}(U) | U \in \mathcal{U}(f_n(x)), n \in \mathbb{N}\}$ is a subbase of $\mathcal{U}(x)$,
   $p_t f_n^2(x) \underset{t\downarrow 0}{\longrightarrow} f_n^2(x)$.
3. $\forall f \in \mathcal{B}_b(E) \cap L^p(E, \mu) : p_t f$ is a μ-version of $\bar{T}_t f$.
4. $(p_t)_{t\geq 0}$ has the ∞-strong Feller property.
5. $(R_\alpha)_{\alpha>0}$ has the p-strong Feller property,
   where $R_\alpha f = \int_0^\infty e^{-\alpha t} p_t f dt$.

1. This is fulfilled for $\mathbb{R}^d$.

2. Consider the function $\psi(x) := \begin{cases} \exp(-\frac{x^2}{1-x^2}), & |x| < 1 \\ 0, & |x| \geq 1. \end{cases}$

Then $\operatorname{supp} \psi(x) = \overline{B_1(0)}$, $\psi(0) = 1$, $\psi, \psi^2 \in C_0^\infty(\mathbb{R}^d)$, and $\{\psi(x) > 0\} = B_1(0)$. Define $\psi_{y,n}(x) = \psi((y - x)n)$. Then $\operatorname{supp} \psi_{y,n} = \overline{B_{\frac{1}{n}}(y0)}$ and $\{\psi_{y,n} > 0\} = B_{\frac{1}{n}}(y)$. Since $\psi_{y,n}, \psi_{y,n}^2 \in D(\hat{L}_{A,b})$ we have that $p_t \psi_{x,n}(x) \to \psi_{x,n}(x)$ and $p_t \psi_{x,n}^2(x) \to \psi_{x,n}(x)$. Thus the second condition is fulfilled.

3. This was shown above.

4. and **5.** This follows by [**BoKrRö**, Theorem 4.1]. We even have that $(t, x) \mapsto p_t f(x)$ is continuous on $(0, \infty) \times \mathbb{R}^d$ for every $f \in L^1(\mathbb{R}^d, \mu)$.

By Corollary 2.15 there exist probability measures P_x such that $(C([0, \infty)), \mathbb{R}^d), \mathcal{B}(C([0, \infty), \mathbb{R}^d), (X_t)_{t\geq 0}, (P_x)_{x\in\mathbb{R}^d})$ with $X_t(\omega) = \omega(t)$ is a Markov process with continuous paths starting at every point $x \in \mathbb{R}^d$. □

As a result of the above theorem we get the following corollary which states that there exists a solution for the martingale problem for $L_{A,b}$ for *every* starting point $x \in \mathbb{R}^d$.

4.5. COROLLARY. *Under the conditions of Theorem 4.2 exists for every starting point $x \in \mathbb{R}^d$ a probability measure P_x on $C_{\mathbb{R}^d}([0, \infty))$ that is a solution for the martingale problem for $L_{A,b}$.*

PROOF. As a result of Lemma 2.10 and the construction of the probability measures (P_x) we have $P_x[X_0 = x] = 1$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. We have to show that $M_t := \varphi(X_t) - \int_0^t (L_{A,b}\varphi)(X_u)\,du$ is a $(\mathcal{F}_t)$-martingale under P_x. Let $s > t$, then we have

$$\begin{aligned}
E_{P_x}[M_s|\mathcal{F}_t] &= E_{P_x}[\varphi(X_s) - \int_0^s (L_{A,b}\varphi)(X_u)\,du|\mathcal{F}_t] \\
&= E_{P_x}[M_t - \varphi(X_t) + \varphi(X_s) - \int_t^s (L_{A,b}\varphi)(X_u)du|\mathcal{F}_t] \\
&= M_t + E_{P_x}[\varphi(X_s) - \varphi(X_t) - \int_t^s (L_{A,b}\varphi)(X_u)du|\mathcal{F}_t] \\
&= M_t + E_{P_x}[(\varphi(X_{s-t}) - \varphi(X_0) - \int_t^s (L_{A,b}\varphi)(X_{u-t})du) \circ \theta_t|\mathcal{F}_t] \\
&= M_t + E_{P_{X_t}}[\varphi(X_{s-t}) - \varphi(X_0) - \int_t^s (L_{A,b}\varphi)(X_{u-t})du] \\
&= M_t + p_{s-t}\varphi(X_t) - \varphi(X_t) - \int_t^s E_{P_{X_t}}[L_{A,b}\varphi(X_{u-t})]du \\
&= M_t + p_{s-t}\varphi(X_t) - \varphi(X_t) - \int_t^s p_{u-t}L_{A,b}\varphi(X_t)du \\
&= M_t + p_{s-t}\varphi(X_t) - \varphi(X_t) - \int_t^s \frac{d}{du}p_{u-t}\varphi(X_t)du \\
&= M_t + p_{s-t}\varphi(X_t) - \varphi(X_t) - (p_{s-t}\varphi(X_t) - p_{t-t}\varphi(X_t)) = M_t.
\end{aligned}$$

□

APPENDIX A

Dirichlet Forms and Generalized Dirichlet Forms

1. Dirichlet Forms and the local property

In this section we will only cite the most important definitions and the main theorem. For a complete reference see [**MaRö92**].

A.1. Definition. Let $(E, \mathcal{B}, m)$ be a measure-space. A coercive closed form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E, m)$ is called a *Dirichlet form* if for all $u \in D(\mathcal{E})$ the following holds:

$$u^+ \wedge 1 \in D(\mathcal{E}) \quad \text{and} \quad \mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0$$
$$\text{and} \quad \mathcal{E}(u - u^+ \wedge 1, u + u^+ \wedge 1) \geq 0.$$

A.2. Remark. It is shown in [**MaRö92**, Theorem I.2.8.] that $\mathcal{E}$ uniquely determines a pair of strongly continuous contraction resolvents $(G_\alpha)_{\alpha>0}$, $(\hat{G}_\alpha)_{\alpha>0}$ on $L^2(E, m)$ such that $\mathcal{E}(G_\alpha f, u) + (G_\alpha f, u) = (f, u) = \mathcal{E}(u, \hat{G}_\alpha f) + (u, \hat{G}_\alpha f)$. Corresponding to these resolvents are two strongly continuous contraction semigroups $(T_t)_{t\geq 0}$ and $(\hat{T}_t)_{t\geq 0}$.

It follows from [**MaRö92**, Proposition I.4.3.] and [**MaRö92**, Theorem I.4.4.] that the resolvents and the semigroups are sub-Markovian.

In order to define quasi-regularity we have to introduce some potential theoretic notions.

A.3. Definition. Let $\alpha \in (0, \infty)$. An element $u \in L^2(E, m)$ is called *α-excessive* if $e^{-\alpha t} T_t u \leq u$ for all $t > 0$.

For a function h on E and an open subset U of E define $\mathcal{L}_{h,U} := \{w \in D(\mathcal{E}) | w \geq h \ m\text{-a.e. on } U\}$.

It is shown in [**MaRö92**, Proposition III.1.5.] and [**MaRö92**, Remark III.1.6.] that for h and U such that $\mathcal{L}_{h,U} \neq \emptyset$ there exists a 1-excessive $h_U \in \mathcal{L}_{h,U}$ such that $h_U \leq u$ for every $u \in \mathcal{L}_{h,U}$ 1-excessive.

Define for $F \subset E$, F closed

$$D(\mathcal{E})_F := \{u \in D(\mathcal{E}) | u = 0 \ m\text{-a.e. on } E \setminus F\}.$$

A.4. Definition. (i) An increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E is called an *$\mathcal{E}$-nest* if $\bigcup_{k\geq 1} D(\mathcal{E})_{F_k}$ is dense in $D(\mathcal{E})$ w.r.t. $\tilde{\mathcal{E}}_1^{1/2}$.

(ii) A subset $N \subset E$ is called *$\mathcal{E}$-exceptional* if $N \subset \bigcap_{k\geq 1} F_k^c$ for some $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$.

(iii) A property is said to hold $\mathcal{E}$-quasi-everywhere ($\mathcal{E}$-q.e.) if there exists an $\mathcal{E}$-exceptional set N such that the property holds on $E \setminus N$.

(iv) An $\mathcal{E}$-q.e. defined function f on E is called *$\mathcal{E}$-quasi-continuous* if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that $f_{|F_k}$ is continuous for every $k \in \mathbb{N}$.

A.5. DEFINITION. A Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E,m)$ is called *quasi-regular* if the following conditions hold:

(i) There exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact sets.

(ii) There exists an $\tilde{\mathcal{E}}_1^{1/2}$-dense subset of $D(\mathcal{E})$ whose elements have $\mathcal{E}$-quasi-continuous m-versions.

(iii) There exists $u_n \in D(\mathcal{E})$, $n \in \mathbb{N}$, having $\mathcal{E}$-quasi-continuous m-versions $\tilde{u}_n$, $n \in \mathbb{N}$, and an $\mathcal{E}$-exceptional set $N \subset E$ such that $\{\tilde{u}_n | n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

Finally we have the following theorem:

A.6. THEOREM. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular Dirichlet form on $L^2(E,m)$.*

Then there exists a pair $(\mathbf{M}, \hat{\mathbf{M}})$ of m-tight special standard processes which is properly associated with $(\mathcal{E}, D(\mathcal{E}))$, i.e. for the transition semigroup $(p_t)_{t\geq 0}$ and $(\hat{p}_t)_{t\geq 0}$ we have that $p_t f$ is an m-version of $T_t f$, $\hat{p}_t f$ is an m-version of $\hat{T}_t f$, and $p_t f$ and $\hat{p}_t f$ are $\mathcal{E}$-q.c. for all $t > 0$, $f \in \mathcal{B}_b(E) \cap L^2(E,m)$.

Now we have to define the local property.

A.7. DEFINITION. $(\mathcal{E}, D(\mathcal{E}))$ is said to have the *local property* if for all $u, v \in D(\mathcal{E})$ with $\operatorname{supp}[u], \operatorname{supp}[v]$ compact and $\operatorname{supp}[u] \cap \operatorname{supp}[v] = \emptyset$ we have $\mathcal{E}(u,v) = 0$.

The local property is closely related to the continuity of the sample paths as states the following theorem:

A.8. THEOREM. *Let $\mathbf{M} = (\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ be an m-tight special standard process with life time ζ associated with $(\mathcal{E}, D(\mathcal{E}))$.*

Then $(\mathcal{E}, D(\mathcal{E}))$ has the local property if and only if we have for $\mathcal{E}$-q.e. $x \in E$ that $P_x[t \mapsto X_t$ is continuous on $[0,\zeta)]$.

2. Generalized Dirichlet Forms

In this appendix only the most important definitions are cited. For a complete reference see **[St99a]**.

Let us first define generalized Dirichlet forms:

Let $(E, \mathcal{B}, m)$ be a σ-finite measure space, $(\mathcal{A}, \mathcal{V})$ be a coercive closed form on $\mathcal{H} := L^2(E,m)$ and $(\Lambda, D(\Lambda))$ be a linear operator on $\mathcal{H}$ such that:

(i) $(\Lambda, D(\Lambda))$ generates a C_0-semigroup of contractions $(U_t)_{t\geq 0}$ on $\mathcal{H}$.

(ii) $(U_t)_{t\geq 0}$ can be restricted to a C_0-semigroup on $\mathcal{V}$.

Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we obtain that $\mathcal{V} \hookrightarrow \mathcal{H} \cong \mathcal{H}' \hookrightarrow \mathcal{V}'$ densely and continuously. Let $(\Lambda, \mathcal{F})$ be the closure of $\Lambda : D(\Lambda) \cap \mathcal{V} \to \mathcal{V}'$ and $(\hat{\Lambda}, \hat{\mathcal{F}})$ be the dual operator. Then the bilinear form associated with $\mathcal{A}$ and Λ is given by

$$\mathcal{E}(u,v) := \begin{cases} \mathcal{A}(u,v) - \langle \Lambda u, v\rangle, & u \in \mathcal{F}, v \in \mathcal{V} \\ \mathcal{A}(u,v) - \langle \hat{\Lambda} v, u\rangle, & u \in \mathcal{V}, v \in \hat{\mathcal{F}}, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the dualization of $\mathcal{V}'$ with $\mathcal{V}$.

A.9. Remark. It follows from [**St99a**, I.3] that $\mathcal{E}$ uniquely determines a pair of C_0-resolvents of contractions $(G_\alpha)_{\alpha>0}$, $(\hat{G}_\alpha)_{\alpha>0}$ on $\mathcal{H}$ such that $G_\alpha(\mathcal{H}) \subset \mathcal{F}$, $\hat{G}_\alpha(\mathcal{H}) \subset \hat{\mathcal{F}}$ and

$$\mathcal{E}(G_\alpha f, g) + \alpha(G_\alpha f, g) = (f,g)_{\mathcal{H}} = \mathcal{E}(g, \hat{G}_\alpha f) + \alpha(\hat{G}_\alpha f, g).$$

A.10. Definition. A bilinear form $\mathcal{E}$ associated with $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda))$ is called a *generalized Dirichlet form* if the following condition is satisfied:

$$u \in \mathcal{F} \implies u^+ \wedge 1 \in \mathcal{V} \text{ and } \mathcal{E}(u, u - u^+ \wedge 1) \geq 0.$$

A.11. Remark. It has been shown in [**St99a**, Proposition I.4.6.] that the above condition is equivalent to the sub-Markov property of $(G_\alpha)_{\alpha>0}$ and the corresponding semigroup $(T_t)_{t\geq 0}$ respectively.

In order to define quasi-regularity for generalized Dirichlet forms let us first introduce some potential theoretic notions: For $f \in \mathcal{H}$ let $\mathcal{L}_f := \{g \in \mathcal{H} | g \geq f\}$.

A.12. Definition. Let $\alpha > 0$. An element $u \in \mathcal{H}$ is called α-*excessive* if $\beta G_{\beta+\alpha} u \leq u$ for all $\beta \geq 0$. Let $\mathcal{P}_\alpha$ denote the set of all α-excessive $u \in \mathcal{V}$.

If $f \in \mathcal{H}$ and $U \subset E$, U open such that $\mathcal{L}_{f\cdot 1_U} \cap \mathcal{F} \neq \emptyset$ there exists a 1-excessive element $f_U \in \mathcal{L}_{f\cdot 1_U}$ such that $f_U \leq u$ for all $u \in \mathcal{L}_{f\cdot 1_U}$, u 1-excessive (cf. [**St99a**, Proposition III.1.7.]). f_U is called the 1-reduced element f_U of f on U.

A.13. Definition. (i) An increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E is called an $\mathcal{E}$-*nest* if for every element $u \in \mathcal{F} \cap \mathcal{P}_1$ it follows that $\lim_{k\to\infty} u_{E\setminus F_k} = 0$ in $\mathcal{H}$.

(ii) A subset $N \subset E$ is called $\mathcal{E}$-*exceptional* if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that $N \subset \bigcap_{k=1}^\infty F_k^c$.

(iii) Finally, a function $f : E \to \mathbb{R}$ is called $\mathcal{E}$-*quasi-continuous* ($\mathcal{E}$-*q.c.*) if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that $f_{|F_k}$ is continuous for all k.

A.14. DEFINITION. A generalized Dirichlet form $\mathcal{E}$ associated with $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda))$ is called *quasi-regular* if the following conditions hold:

(i) There exists an $\mathcal{E}$-nest $(F_k)_{k\geq 1}$ consisting of compact sets.

(ii) There exists a dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$-q.c. m-versions.

(iii) There exist $u_n \in \mathcal{F}$, $n \in \mathbb{N}$ having $\mathcal{E}$-q.c. m-versions $\tilde{u}_n$, $n \in \mathbb{N}$ and an $\mathcal{E}$-exceptional set $\mathbb{N} \subset E$ such that $\{\tilde{u}_n | n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

For the existence of an associated process we need one further condition. Then we have the following theorem.

A.15. THEOREM. *Let $\mathcal{E}$ be a quasi-regular Dirichlet form satisfying the following condition:*

There exists a linear subspace $\mathcal{Y} \subset \mathcal{H} \cap L^\infty(E, m)$ such that $\mathcal{Y} \cap \mathcal{F}$ is dense in $\mathcal{F}$, $\lim_{\alpha\to\infty}(\alpha G_\alpha u - u)_E = 0$ in $\mathcal{H}$ for all $u \in \mathcal{Y}$ and for the closure $\bar{\mathcal{Y}}$ of $\mathcal{Y}$ in $L^\infty(E, m)$ it follows that $u \wedge \alpha \in \bar{\mathcal{Y}}$ for $u \in \mathcal{Y}$ and $\alpha \geq 0$.

Then there exists an m-tight special standard process $\mathbf{M}$ which is properly associated in the resolvent sense with $\mathcal{E}$, i.e. for the resolvent $(R_\alpha)_{\alpha>0}$ we have that $R_\alpha f$ is $\mathcal{E}$-q.c. and an m-version of $G_\alpha f$ for all $\alpha > 0$ and $f \in \mathcal{B}_b(E) \cap \mathcal{H}$.

APPENDIX B

Some Complements

1. A Feller process with right-continuous paths has the strong Markov property

The contents of the following proposition are well-known. However for the version presented here we could not find a reference, but only one with slightly stronger assumptions (cf. [**ReYo91**, III (3.1) Theorem]). Therefore we will present a complete proof which is essentially the proof of the abovementioned reference.

B.1. PROPOSITION. *Let* $(\Omega, \mathcal{F}_\infty, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ *be a Markov process with the natural filtration* $(\mathcal{F}_t)_{t\geq 0}$ *(i.e.* $\mathcal{F}_t^0 = \sigma\{X_s | 0 \leq s \leq t\}$ *and* $\mathcal{F}_t = \bigcap_{\mu\in\mathcal{P}(E)} (\mathcal{F}_t^0)^{P_\mu}$*, where* $\mathcal{P}(E)$ *denotes the set of all probability measures on* $(E, \mathcal{B}(E))$*, and* $\mathcal{A}^P$ *denotes the completion of the* σ*-algebra* $\mathcal{A}$ *with respect to* P*). Suppose its transition semigroup* $(p_t)_{t\geq 0}$ *has the Feller property (i.e.* $p_t C(E) \subset C(E)$*), then it has the strong Markov property with respect to* $(\mathcal{F}_{t+})$

PROOF. We will show that for any $\mathcal{F}_\infty$-measurable positive f and for any $(\mathcal{F}_{t+})$-stopping time we have that

$$E_{P_x}[f \circ \theta_T | \mathcal{F}_{T+}] = E_{P_{X_T}}[f].$$

Step 1. $T(\Omega) = D$*, where* D *is countable.* In this case we have by the Markov property

$$\begin{aligned}
E_{P_\nu}[f \circ \theta_T | \mathcal{F}_T] &= E_{P_\nu}\left[\sum_{d\in D} 1_{\{T=d\}} f \circ \theta_T \middle| \mathcal{F}_T\right] \\
&= \sum_{d\in D} E_{P_\nu}[1_{\{T=d\}} f \circ \theta_t | \mathcal{F}_T] \\
&= \sum_{d\in D} E_{P_\nu}[1_{\{T=d\}} f \circ \theta_t | \mathcal{F}_t] \\
&= \sum_{d\in D} 1_{\{T=d\}} E_{P_\nu}[f \circ \theta_t | \mathcal{F}_t] \\
&= \sum_{d\in D} 1_{\{T=d\}} E_{P_{X_d}}[f] \\
&= E_{P_{X_T}}[f].
\end{aligned}$$

Step 2. $f \in \mathcal{F}^0_\infty$. Define $T_n := \frac{[2^n T]+1}{2^n}$ where $[x] := \max\{z \in \mathbb{Z} | z \leq x\}$ then we have $T_n \downarrow T$ as $n \to \infty$. And since $\{T_n \leq t\} = \{T < \frac{[2^n t]}{2^n}\}$ we have that T_n is a $(\mathcal{F}_t)$-stopping time such that $T_n(\Omega)$ is countable.

Take now $f_i \in C_b(E)$, $i = 1, \dots, k$ and $t_1 < \cdots < t_k$. And define $g(x) := \int p_{t_1}(x, dx_1) f_1(x_1) \int p_{t_2-t_1}(x_1, dx_2) \cdots \int p_{t_k - t_{k-1}}(x_{k-1}, dx_k) f_k(x_k)$.

Then we have by Step 1 for every $n \in \mathbb{N}$

$$E_{P_\nu}\Big[\prod_{i=1}^k f_i(X_{t_i}) \circ \theta_{T_n} \Big| \mathcal{F}_{T_n}\Big] = E_{P_{X_{T_n}}}\Big[\prod_{i=1}^k f_i(X_{t_i})\Big] = g(X_{T_n}).$$

By taking the limit $n \to \infty$ we get by the continuity of g, f_i, $i = 1, \dots, k$ and by the right-continuity of X. that

$$E_{P_\nu}\Big[\prod_{i=1}^k f_i(X_{t_i}) \circ \theta_T \Big| \mathcal{F}_{T+}\Big] = g(X_T) = E_{P_{X_T}}\Big[\prod_{i=1}^k f_i(X_{t_i})\Big].$$

By a monotone class argument we then get for every $f \in \mathcal{F}^0_\infty$

$$E_{P_\nu}[f \circ \theta_T | \mathcal{F}_{T+}] = E_{P_{X_T}}[f].$$

Step 3. $f \in \mathcal{F}_\infty$. Take now $f \in \mathcal{F}_\infty$ and let $\mu \in \mathcal{P}(E)$ be the distribution of X_T under P_ν. Then there exist $f', f'' \in \mathcal{F}^0_\infty$ such that $f' \leq f \leq f''$ and $P_\mu[f'' - f' > 0] = 0$.

Then we have by Step 2

$$\begin{aligned} P_\nu[f'' \circ \theta_T - f' \circ \theta_T > 0] &= E_{P_\nu}[P_\nu[f'' \circ \theta_t - f' \circ \theta_T > 0 | \mathcal{F}_T]] \\ &= E_{P_\nu}[P_{X_T}[f'' - f' > 0]] \\ &= P_\mu[f'' - f' > 0] \\ &= 0. \end{aligned}$$

So we have that $f \circ \theta_T$ is $\mathcal{F}_\infty$-measurable and $E_{P_\nu}[f \circ \theta_T | \mathcal{F}_T]$ can be calculated. We get

$$E_{P_\nu}[f' \circ \theta_T | \mathcal{F}_T] \leq E_{P_\nu}[f \circ \theta_T | \mathcal{F}_T] \leq E_{P_\nu}[f'' \circ \theta_T | \mathcal{F}_T]$$

and thus

$$E_{P_{X_T}}[f'] \leq E_{P_\nu}[f \circ \theta_T | \mathcal{F}_T] \leq E_{P_{X_T}}[f'']$$

and by the above equation

$$E_{P_\nu}[f \circ \theta_T | \mathcal{F}_T] = E_{P_{X_T}}[f] P_\nu\text{-a.s.}$$

□

2. Exercise 2.14 of [BlGe68]

B.2. PROPOSITION. *Let $(\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x \in E})$ be a Markov process with filtration $(\mathcal{F}_t)_{t \geq 0}$. Suppose $\mathcal{M} \subset \mathcal{F}$ is a σ-algebra that is for every t independent of $\mathcal{F}_t$. Then $(\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ is a Markov process with respect to the filtration $(\sigma(\mathcal{M}, \mathcal{F}_t))_{t \geq 0}$.*

PROOF. We have to show that for every $f \in \mathcal{B}_b(E)$ we have $E_{P_x}[f \circ X_{s+t} | \sigma(\mathcal{M} \cup \mathcal{F}_t)] = E_{P_{X_t}}[f(X_s)]$. Since the right hand is $\mathcal{F}_t$-measurable by assumption, it is also $\sigma(\mathcal{M} \cup \mathcal{F}_t)$-measurable. Hence we have only to show that for every $\sigma(\mathcal{M} \cup \mathcal{F}_t)$-measurable g we have that $E_{P_x}[g \cdot E_{P_{X_t}}[f(X_s)]] = E_{P_x}[g \cdot f(X_s)]$. By a monotone class argument we can assume that $g = \hat{g} \cdot \tilde{g}$ where $\hat{g}$ is $\mathcal{M}$-measurable and $\tilde{g}$ is $\mathcal{F}_t$-measurable. Then we get

$$
\begin{aligned}
&E_{P_x}[g \cdot E_{P_{X_t}}[f(X_s)]] \\
&\qquad = E_{P_x}[\hat{g} \cdot \tilde{g} \cdot E_{P_{X_t}}[f(X_s)]] \\
&\qquad = E_{P_x}[\hat{g}] \cdot E_{P_x}[\tilde{g} \cdot E_{P_{X_t}}[f(X_s)]] \quad && (\mathcal{M} \text{ and } \mathcal{F}_t \text{ are independent}) \\
&\qquad = E_{P_x}[\hat{g}] \cdot E_{P_x}[\tilde{g} \cdot f(X_{s+t})] && (\tilde{g} \text{ is } \mathcal{F}_t\text{-measurable,} \\
& && \quad \text{Markov property}) \\
&\qquad = E_{P_x}[\hat{g} \cdot \tilde{g} \cdot f(X_{s+t})] && (\mathcal{M} \text{ and } \mathcal{F}_t \text{ are independent}) \\
&\qquad = E_{P_x}[g \cdot f(X_{s+t})].
\end{aligned}
$$

□

Bibliography

[Ba74] Bauer, H., Wahrscheinlichkeitstheorie und Grundzüge der Maßtheorie, 2.Auflage, Berlin-New York: de Gruyter 1974.

[Ba91] Bauer, H., Wahrscheinlichkeitstheorie, 4. Auflage, Berlin-New York: de Gruyter 1991.

[BlGe68] Blumenthal, R.M., Getoor, R.K., Markov Processes and Potential Theory, New York and London: Academic press 1968

[BoKrRö] Bogachev, V.I., Krylov, N.V., Röckner, M., On Regularity of Transition Probabilities and Invariant Measures of Singular Diffusions under Minimal Conditions, SFB-343-Preprint 99-141, to appear in Comm. PDE

[DaRö01] DaPrato, G., Röckner, M., Singular dissipative stochastic equations in Hilbert spaces, in preparation.

[Eb99] Eberle, A. Uniqueness and non-uniqueness of semigroups generated by singular diffusion operators.Berlin: Springer 1999

[MaRö92] Ma, Z.-M., Röckner, M., Introduction to the Theory of (Non-Symmetric) Dirichlet Forms. Berlin: Springer 1992.

[ReYo91] Revuz, D., Yor, M., Continuous Martingales and Brownian Motion. Berlin: Springer 1991

[St99a] Stannat, W. The theory of generalized Dirichlet forms and its applications in analysis and stochastics. Providence, RI : American Mathematical Society 1999.

[St99b] Stannat, W. (Nonsymmetric) Dirichlet Operators on L^1: Existence, Uniqueness and Associated Markov Processes. Ann. Scuola Norm. Sup. Pisa Cl. Sci., (4) 28 (1999), no. 1, 99-140

[We81] Wentzell, A.D. A Course in the Theory of Stochastic Processes. New York: McGraw-Hill, 1981.

[Wi79] Williams, D. Diffusions, Markov Processes, and Martingales. Chichester, New York, Brisbane, Toronto: John Wiley & Sons, 1979.