A Stochastic Parabolic Obstacle Problem

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Abstract

This thesis is devoted to the study of a white-noise driven semilinear stochastic partial differential equation with reflection, introduced by Nualart and Pardoux. The main result is a completely rigorous interpretation of such equation as an infinite-dimensional Skorokhod problem. The relationship between the Nualart-Pardoux equation and the excursion measure, i.e. the 3-d Bessel Bridge, is explored: in particular, a fundamental tool is an integration by parts formula on the excursion measure, where explicit infinite-dimensional boundary terms appear. Other applications are given, to integration by parts formulae with respect to δ-d Bessel Bridges, $\delta > 3$, to reaction-diffusion SPDEs with singular nonlinearities and to integration by parts formulae w.r.t. δ-Brownian Bridges, $\mathbb{N} \ni \delta \geq 3$, along vector fields which do not belong to the Cameron Martin space.

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Chapter 1

Introduction

In [NP 92], Nualart and Pardoux prove existence and uniqueness of a solution of a stochastic partial differential equation (SPDE) with reflection, namely of a pair (u, η), where u is a continuous function of $(t, \xi) \in \mathcal{O} := [0, +\infty) \times [0, 1]$ and η a positive measure on $\mathcal{O}$, satisfying:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \xi^2} + \dfrac{\partial^2 W}{\partial t \partial \xi} + \eta(t, \xi) \\ \\ u(0, \xi) = x(\xi), \;\; u(t, 0) = u(t, 1) = 0 \\ \\ u \geq 0, \;\; d\eta \geq 0, \;\; \int_{\mathcal{O}} u \, d\eta = 0. \end{cases} \tag{1.1}$$

where $x : [0, 1] \mapsto [0, \infty)$ is continuous with $x(0) = x(1) = 0$, $\{W(t, \xi) : (t, \xi) \in \mathcal{O}\}$ is a Brownian sheet. The solution u is obtained by penalization, i.e. u is the pathwise limit as $\varepsilon \downarrow 0$ of the increasing family u^ε, where for all $\varepsilon > 0$, u^ε is the unique solution of the following approximating problem:

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u^\varepsilon}{\partial \xi^2} + \dfrac{\partial^2 W}{\partial t \partial \xi} + \dfrac{(u^\varepsilon)^-}{\varepsilon} \\ \\ u^\varepsilon(0, \xi) = x(\xi), \;\; u^\varepsilon(t, 0) = u^\varepsilon(t, 1) = 0. \end{cases} \tag{1.2}$$

This thesis starts from the results of Nualart and Pardoux, but takes advantage of a supplementary structure of the approximating processes u^ε. Indeed, it turns out that u^ε is a gradient system in $H := L^2(0, 1)$, i.e. a Markov process associated with a symmetric Dirichlet Form of gradient type and with state space H. Moreover, the unique invariant probability measure of u^ε is absolutely continuous with respect to the Gaussian measure $\mu :=$

$\mathcal{N}(0,Q)$, where $D(Q^{-1}) := H^2 \cap H_0^1(0,1)$, $Q^{-1} := -d^2/d\xi^2$, and the density can be explicitly computed. It is therefore reasonable to ask whether this structure is maintained letting $\varepsilon \to 0$.

First, we prove that u is a symmetric process, with respect to an explicit invariant probability measure: the law ν on $H := L^2(0,1)$ of the 3-d Bessel Bridge, i.e. the law of the modulus of a 3-dimensional Brownian Motion $(B^3(\tau))_{\tau\in[0,1]}$, conditioned to be 0 at time $\tau = 1$. This probability measure is also known as the Excursion Measure, in connection with the Theory of Excursions of the linear Brownian Motion: see [RY 91]. Therefore, our result provides an unexpected connection between the analytic theory of SPDEs and the probabilistic study of Brownian Motion.

We mention that this result has been obtained independently by Funaki and Olla in a recent paper, [FO 01], where SPDEs with reflection are applied to the study of fluctuations of an interface on a hard wall: see the Introduction to Chapter 3.

Since the process $x \mapsto u(t,\cdot)$, $t > 0$, satisfies a Strong Feller property in H, then the law of $u(t,\cdot)$ for $t > 0$ is absolutely continuous w.r.t. ν on $L^2(0,1)$: in particular, for $(t,\xi) \in (0,\infty) \times (0,1)$, the law of $u(t,\xi)$ is absolutely continuous w.r.t. to the measure $y^2 dy$ on $[0,\infty)$. Recall that in [DMP 97], Donati-Martin and Pardoux consider the minimal solutions (u,η) of a class of SPDEs with reflection and multiplicative white-noise and prove that, for $(t,\xi) \in (0,\infty) \times (0,1)$, the law of $u(t,\xi)$ is absolutely continuous w.r.t. to the Lebesgue measure on $(0,\infty)$: see Section 3.4.

The next step is the characterization of η, the measure term in (1.1), as a family of Additive Functionals of u: see Section 3.6. A key tool is an integration by parts formula on the excursion measure ν: for all smooth function $\varphi : H \mapsto \mathbb{R}$ and $h \in C_c^2(0,1)$ we have

$$\int_{K_0} \partial_h \varphi \, d\nu \;=\; -\int_{K_0} \varphi(x) \, \langle x, h'' \rangle \, \nu(dx) \;-\; \int_0^1 dr \;h(r) \int_{K_0} \varphi(x) \, \sigma_0(r, dx), \tag{1.3}$$

where $K_0 := \{x \in H : x \geq 0\}$, $\partial_h \varphi$ is the directional derivative of φ along the direction $h \in H$ and h'' is the second derivative of h, and the measure $\sigma_0(r,\cdot)$ is explicitly defined in terms of two independent 3-d Bessel Bridges, respectively on $[0,r]$ and on $[0,1-r]$, glued at $r \in (0,1)$: see (2.5) below. The proof of (1.3) is based essentially on a result of Biane in [Bi 86], about a connection between the Brownian Bridge and the 3-d Bessel Bridge: see Section 2.1.

Recall that, by the Divergence Theorem in finite dimension, we have:

$$\int_O (\partial_h \varphi)\, \rho\, dx \;=\; -\int_O \varphi\, (\partial_h \log \rho)\, \rho\, dx \;-\; \int_{\partial O} \varphi\, \langle n, h\rangle\, \rho\, d\sigma \tag{1.4}$$

where O is a regular bounded open subset of $\mathbb{R}^d$, $h \in \mathbb{R}^d$, $\varphi, \rho \in C_b^1(O)$, $0 < \lambda \le \rho \le \Lambda < \infty$, n is the inward-pointing normal vector to the boundary ∂O and σ is the surface measure of ∂O.

For all $r \in (0,1)$, $\sigma_0(r,\cdot)$ is concentrated on the set:

$$\partial_r K_0 \;:=\; \Big\{ x : [0,1] \mapsto \mathbb{R}^+ \text{ continuous} : \, x(0) = x(r) = x(1) = 0,$$
$$x(\xi) > 0 \quad \forall \xi \in (0,1) \backslash \{r\} \Big\}$$

and the measure $\int_0^1 dr\, \sigma_0(r,\cdot)$ is concentrated on $\partial K_0 := \bigcup_{r\in(0,1)} \partial_r K_0$. On the other hand $\nu(\partial K_0) = 0$.

Compare (1.3) and (1.4): in the right hand side of (1.3), the first term shows that the logarithmic derivative of ν is $x \mapsto x''$, and the second one can be interpreted as a boundary term. Moreover the factor $\langle n, h\rangle$ in finite dimension corresponds to $h(r) = \langle \delta_r, h\rangle$ in the infinite-dimensional case, where δ_r is the Dirac delta at r.

Therefore, (1.3) provides the following interpretation: the boundary of $K_0 = \{x \in H, x \ge 0\}$ w.r.t. ν, is equal to ∂K_0, and for all $x \in \partial_r K_0$, the inward-pointing normal vector to ∂K_0 at x is equal to the Dirac delta δ_r at r. See the Introduction of Chapter 2.

Getting back to equation (1.1), formula (1.3) allows to prove that u is associated with a gradient-type Dirichlet Form on the space (K_0, ν), and:

1. For all Borel set $I \subseteq (0,1)$, the process $t \mapsto \eta([0,t] \times I)$ is an Additive Functional of u, with Revuz-measure $\frac{1}{2}\int_I dr\, \sigma_0(r,\cdot)$

2. There exists a Borel set $S \subseteq \mathbb{R}^+$ and a map $r : S \mapsto (0,1)$, such that $\eta((\mathbb{R}^+ \backslash S) \times (0,1)) = 0$, and for all $s \in S$, $u(s,\cdot) \in \partial_{r(s)} K_0$, i.e.

$$u(s, r(s)) \;=\; 0, \quad u(s,\xi) \;>\; 0 \quad \forall\, \xi \in (0,1) \backslash \{r(s)\}.$$

3. The measure η admits the decomposition:

$$\eta(ds, d\xi) \;=\; \delta_{r(s)}(d\xi)\, \eta(ds, (0,1)). \tag{1.5}$$

Therefore, we can write equation (1.1) as an infinite-dimensional Skorokhod problem:

$$du \,=\, \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2}\, dt \,+\, dW \,+\, \frac{1}{2}\, n(u)\cdot dL \tag{1.6}$$

where n is the inward-pointing normal vector to the boundary, i.e. $n(x) = \delta_r$ if $x \in \partial_r K_0$, and $L_t := 2\,\eta([0,t]\times(0,1))$ is the Additive Functional associated with the boundary measure $\int dr\, \sigma_0(r,\cdot)$. Recall that if O is a regular bounded open subset of $\mathbb{R}^d$, n is the inward-pointing normal vector to the boundary ∂O, and $(B^d(t))_{t\geq 0}$, a d-dimensional Brownian Motion, then a solution of the Skorokhod problem is a pair (X, L), where $(X(t))_{t\geq 0}$ is a $\overline{O}$-valued continuous process, and L is a continuous non-decreasing process which increases only when $X(t) \in \partial O$, solving:

$$X(t) \,=\, X(0) \,+\, B(t) \,+ \frac{1}{2}\int_0^t n(X(s))\, dL_s, \qquad t \geq 0.$$

The process X is called the Reflecting Brownian Motion in O and L is called the Local Time of X at ∂O: see [Ta 67], [LS 84], [FOT 94], [BH 90] and references therein.

The proof of (1.5) and (1.6) is based on the theory of symmetric Dirichlet Forms, and in particular on the theory of Additive Functionals and smooth measures for non-locally compact state spaces, developed by Ma and Röckner in [MR 92] and by Fukushima in [Fu 99]: see section 3.6.

We mention here that SDEs with reflecting boundary in an Abstract Wiener Space have been recently studied by Fukushima in [Fu 00]: see the Introduction to Chapter 3.

Once (1.3) is obtained, we prove the following integration by parts formula w.r.t. the law π_δ of the δ-d Bessel Bridge, $\delta > 3$:

$$\int_{K_0} \partial_h \varphi \, d\pi_\delta \,=\, -\int_{K_0} \varphi(x)\, \left(\langle x, h''\rangle \,+\, \frac{(\delta-3)(\delta-1)}{4}\langle x^{-3}, h\rangle \right) \pi_\delta(dx). \tag{1.7}$$

Moreover, we prove that for all $\delta > 3$, there exists a unique solution u to the following SPDE:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \xi^2} + \dfrac{(\delta-3)(\delta-1)}{8u^3} + \dfrac{\partial^2 W}{\partial t \partial \xi} \\[2ex] u(0,\cdot) = x \geq 0, \;\; u(t,0) = u(t,1) = 0 \end{cases} \tag{1.8}$$

and the process $x \mapsto u(t,\cdot)$, $t \geq 0$, is symmetric with respect to its unique invariant probability measure π_δ.

We give a final application to the law $\mu^{\otimes\delta}$ of a δ-dimensional Brownian Bridge, $\mathbb{N} \ni \delta \geq 3$: formulae (1.3) and (1.7) allow to write an integration by parts formula with respect to $\mu^{\otimes\delta}$ along a vector field which does not belong to the Cameron Martin space.

The results of this thesis are presented in Chapters 2, 3 and 4. Chapter 2 is devoted to the proof of the integration by parts formulae on the law of δ-d Bessel Bridges, $\delta \geq 3$. In Chapter 3, it is proved that the solution u of equation (1.1) is the Markov process associated with a gradient-type Dirichlet Form on (K_0, ν). This result allows to apply the Theory of Additive Functionals to η, and obtain the decomposition (1.5). In Chapter 4 we give applications of the results of Chapters 2-3, to equation (1.8), and to integration by parts formulae w.r.t. the law of δ-dimensional Brownian Bridges, $\mathbb{N} \ni \delta \geq 3$.

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1.1 Definitions and notations

We introduce the following notations: $(t,\xi) \in \mathcal{O} := [0,+\infty) \times [0,1]$, $H := L^2(0,1)$ with the canonical scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$,

$$\langle h,k\rangle := \int_0^1 h(\xi)k(\xi)\,d\xi, \qquad \|h\|^2 := \langle h,h\rangle,$$

$$C_0 := C_0(0,1) := \{c : [0,1] \mapsto \mathbb{R} \ \text{ continuous}, \ c(0) = c(1) = 0\},$$

$$A : D(A) \subset H \mapsto H, \quad D(A) := H^2 \cap H^1_0(0,1), \quad A := \frac{1}{2}\frac{\partial^2}{\partial\xi^2}.$$

We denote by $C_c^k(0,1)$, $k \in \mathbb{N} \cup \{\infty\}$, the subset of $C_0(0,1)$ of all C^k functions with support being compact in $(0,1)$.

We set $K_\alpha := \{h \in H : \ h \geq -\alpha\}$ with $\alpha \geq 0$, and we denote by $\Pi_{K_\alpha} : H \mapsto K_\alpha$ the projection from H onto the closed convex set $K_\alpha \subset H$. Recall that Π_{K_α} is 1-Lipschitz continuous.

We introduce the following function spaces:

- If $D \subseteq H$, we denote by $C_b(D)$ the space of all $\varphi : D \mapsto \mathbb{R}$ being bounded and uniformly continuous in the norm of H. If $D \subseteq H$ and $\varphi \in C_b(D)$, we denote the modulus of continuity of φ by $\omega_\varphi : [0,\infty) \mapsto [0,1]$:

$$\omega_\varphi(r) := \sup\{|\varphi(x) - \varphi(x')| \wedge 1 : x, x' \in D, \ \|x - x'\| \leq r\}.$$

  We let $\|\varphi\|_\infty := \sup|\varphi|$. Then $(C_b(D), \|\cdot\|_\infty)$ is a Banach space.

- For all $\alpha \geq 0$, we identify $C_b(K_\alpha)$ with a subspace of $C_b(H)$ by means of the injection: $C_b(K_\alpha) \ni \varphi \mapsto \varphi \circ \Pi_{K_\alpha} \in C_b(H)$. If $0 \leq \alpha \leq \beta$, then $C_b(K_\alpha) \subseteq C_b(K_\beta)$.

- We denote by $\mathrm{Exp}_A(H)$ the linear span of $\{1, \cos(\langle\cdot, h\rangle), \sin(\langle\cdot, h\rangle) : h \in D(A)\}$; $\mathrm{Exp}_A(K_\alpha)$ is equal to the restrictions of $\mathrm{Exp}_A(H)$ to K_α.

- If $D \subseteq H$, the space $\mathrm{Lip}(D)$ is the set of all $\varphi \in C_b(D)$ such that:

$$\|\varphi\|_{\mathrm{Lip}} := \|\varphi\|_\infty + \sup_{r>0} \frac{\omega_\varphi(r)}{r} < \infty.$$

- The space $C_b^1(H)$ is defined as the set of all Fréchet-differentiable $\varphi \in C_b(H)$, with continuous gradient $\nabla\varphi : H \mapsto H$; finally, $C_b^1(K_\alpha) \subset C_b(K_\alpha)$ is equal to the set of all φ such that:

  1. For all $x \in K_\alpha$, there exists a vector $\nabla\varphi(x) \in H$ such that for all $h \in K_0$, we have:

$$\lim_{t \downarrow 0} \frac{1}{t}(\varphi(x + t\,h) - \varphi(x)) = \langle \nabla\varphi(x), h\rangle.$$

  2. $K_\alpha \ni x \mapsto \nabla\varphi(x) \in H$ is continuous and bounded.

  For all $\varphi \in C_b^1(K_\alpha)$ we call $\nabla\varphi : H \mapsto H$ the gradient of φ.

If $\{m_n\}_n \cup \{m\}$ is a sequence of probability measures on $(H, \mathcal{B}(H))$, where $\mathcal{B}(H)$ is the Borel σ-field of H, we say that m_n converges weakly to m, if:

$$\lim_{n\to\infty} \int_H \varphi \, dm_n \;=\; \int_H \varphi \, dm, \qquad \forall \varphi \in C_b(H).$$

Given a Markov process $\{Y(t,x) : t \geq 0, x \in D\}$ on $D \subseteq H$, we say that a probability measure m on D is symmetrizing for Y, or that Y is symmetric w.r.t. m, if, setting for all $\varphi \in C_b(D)$: $P_t^Y \varphi(x) := \mathbb{E}[\varphi(Y(t,x))]$, $x \in D$, we have:

$$\int_D \varphi \, P_t^Y \psi \, dm \;=\; \int_D \psi \, P_t^Y \varphi \, dm \qquad \forall \varphi, \psi \in C_b(D).$$

A symmetrizing measure is in particular invariant, i.e.:

$$\int_D P_t^Y \varphi \, dm \;=\; \int_D \varphi \, dm, \qquad \forall \varphi \in C_b(D).$$

We denote by $1_D(\cdot)$ the characteristic function of a set D. We sometimes write: $m(\varphi)$ for $\int_H \varphi \, dm$, $\varphi \in C_b(H)$.

By $W = \{W(t,\xi) : (t,\xi) \in \mathcal{O}\}$ we denote a two-parameter Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. W is a Gaussian process with zero mean and covariance function

$$\mathbb{E}\left[W(t,\xi)W(t',\xi')\right] = (t \wedge t')(\xi \wedge \xi'), \qquad (t,\xi),(t',\xi') \in \mathcal{O}.$$

We denote by $\mathcal{F}_t$ the σ-field generated by the random variables $\{W(s,\xi) : (s,\xi) \in [0,t] \times [0,1]\}$.

A Gaussian measure $\mathcal{N}(0,Q)$ on H with 0 mean and covariance operator $Q : H \mapsto H$, with Q symmetric, positive and of trace-class, is defined as the unique probability measure on H with Fourier transform:

$$\int_H e^{i\langle h,x\rangle} \, \mathcal{N}(0,Q)(dx) \;=\; \exp\left(-\frac{1}{2}\langle Qh, h\rangle\right), \qquad h \in H,$$

see e.g. [DPZ 92].

Let $(B_t)_{t\geq 0}$ a linear Brownian Motion, and $(B_t^\delta)_{t\geq 0}$ a $\mathbb{R}^\delta$-valued BM, $\mathbb{N} \ni \delta \geq 2$. We fix the following notations:

- μ is the law on $L^2(0,1)$ of the Brownian Bridge between 0 and 0 on $[0,1]$, i.e. the law of $(B_\tau)_{\tau\in[0,1]}$ conditioned on $B_1 = 0$.

- ν is the law on $L^2(0,1)$ of the 3-d Bessel Bridge between 0 and 0 on $[0,1]$, i.e. the law of $(|B^3_\tau|)_{\tau\in[0,1]}$, conditioned on $B^3_1 = 0$.

- π_δ, $\delta > 1$, is the law on $L^2(0,1)$ of the δ-d Bessel Bridge between 0 and 0 on $[0,1]$, i.e. the law of the unique process $(x_\delta(\tau))_{\tau\in[0,1]}$ solving the SDE:

$$d\,x_\delta = \frac{\delta-1}{2x_\delta}\,d\tau \;-\; \frac{x_\delta}{1-\tau}\,d\tau \;+\; dB, \quad \tau\in[0,1[, \quad x_\delta(0)=0.$$

  If $\delta\in\mathbb{N}$, $\delta\geq 2$, then π_δ is equal to the law of $(|B^\delta_\tau|)_{\tau\in[0,1]}$, conditioned on $B^\delta_1 = 0$.

- For all $\alpha > 0$, $\nu_\alpha := \mu(\cdot|K_\alpha)$, i.e. $\nu_\alpha(D) := \mu(D\cap K_\alpha)/\mu(K_\alpha)$.

Recall that $\mu(K_\alpha) = 1-\exp(-2\alpha^2)$, so that ν_α is well defined for $\alpha > 0$.

1.2 Preliminary results

We will use the following:

Lemma 1.1 *Let T be a Polish metric space, and let $\{m_n\}_n\cup\{m\}$, respectively $\{\varphi_n\}_n$, a sequence of probability measures, resp. of real-valued continuous functions, on T, satisfying:*

- *m_n converges weakly to m.*

- *The family $\{\varphi_n\}_n$ is uniformly bounded and equicontinuous on T.*

- *$\varphi_n(x)$ has a limit $\varphi(x)$ as $n\to\infty$, for all $x\in S$, with $S\subseteq T$ Borel and $m(S) = 1$.*

Then:

$$\lim_{n\to\infty}\int_T \varphi_n\,dm_n \;=\; \int_S \varphi\,dm.$$

Proof—We can suppose that $0\leq\varphi_n\leq 1$ for all n. By Prokhorov's Theorem, there exists for every $\delta > 0$ a compact set $Q_\delta\subset T$ such that eventually $m_n(Q_\delta)\geq 1-\delta$. Let $\{\varphi_{n_k}\}_k$ any subsequence of $\{\varphi_n\}_n$. On Q_δ we can apply Ascoli-Arzelà Theorem and obtain uniform convergence of a sub-subsequence $\{\varphi_{n_{k(l)}}\}_l$ to a continuous function $f: Q_\delta\mapsto\mathbb{R}$. Then:

$$\int_T \varphi_{n_{k(l)}}\,dm_{n_{k(l)}} - \int_T \varphi_{n_{k(l)}}\,dm \leq m_{n_{k(l)}}(T-Q_\delta) + \int_{Q_\delta}\varphi_{n_{k(l)}}\left[dm_{n_{k(l)}} - dm\right]$$

$$\leq \ \delta + 2\delta + \int_{Q_\delta} f \ \left[dm_{n_{k(l)}} - dm\right]$$

where for $l \geq l_0$, $\sup_{Q_\delta} |\varphi_{n_{k(l)}} - f| \leq \delta$. Since $m(T-S) = 0$ and Q_δ is closed:

$$\lim_{n\to\infty} \int_T \varphi_n \, dm = \int_S \varphi \, dm, \quad \limsup_{n\to\infty} \int_{Q_\delta} f \, dm_n \leq \int_{Q_\delta} f \, dm, \quad \text{and therefore :}$$

$$\limsup_{l\to\infty} \int_T \varphi_{n_{k(l)}} \, dm_{n_{k(l)}} \ \leq \ \int_S \varphi \, dm.$$

Changing φ_n with $1 - \varphi_n$, we obtain the thesis. □

The next Lemma identifies the Gaussian measure on H, $\mathcal{N}(0, (-2A)^{-1})$, with a well-known probability measure on $C_0(0,1)$: the law μ of the Brownian Bridge. Recall that the law of the Brownian Bridge is concentrated on $C_0(0,1)$ and is the unique Gaussian measure on $\mathbb{R}^{[0,1]}$ with 0 mean and covariance function: $\Gamma(\tau, \sigma) = \tau \wedge \sigma - \tau\sigma, \quad \sigma, \tau \in [0,1]$, (see [RY 91], Chap I).

Lemma 1.2 *The Gaussian measure $\mathcal{N}(0, (-2A)^{-1})$ coincides with the law μ of the Brownian Bridge.*

Proof—Notice that the measure μ is concentrated on $C([0,1]) \subset H$. By definition of Gaussian measures, the following holds for all $h, k \in H$:

$$\int_H \langle x, h\rangle \langle x, k\rangle \, \mathcal{N}(0, (-2A)^{-1})(dx) \ = \ \langle (-2A)^{-1} h, k\rangle \tag{1.9}$$

Since the operator $(-2A)^{-1}$ can be expressed as an integral operator with kernel: $\xi \wedge \sigma - \xi\sigma$, $\xi, \sigma \in [0,1]$, then setting in (1.9) $h = \chi_{[0,t]}$, $k = \chi_{[0,s]}$, $s, t \in [0,1]$, and differentiating with respect to t and s, we obtain:

$$\int_{C([0,1])} x(t) \, x(s) \, d\mu(x) \ = \ t \wedge s - ts. \qquad \square$$

We shall use also the following Proposition, see e.g. [DPZ 96], Chap. 8.

Proposition 1.1

- *The positive symmetric bilinear form:*

$$\varphi, \psi \in C_b^1(H) \mapsto \frac{1}{2} \int_H \langle \nabla\varphi, \nabla\psi\rangle d\mu$$

is closable in $L^2(H, \mu)$. We denote by $(\Lambda, W^{1,2}(H, \mu))$ its closure.

- *The semigroup associated with Λ is the Ornstein-Uhlenbeck semigroup $(\Pi_t)_{t\geq 0}$, given by the Mehler Formula:*

$$\Pi_t\varphi(z) := \int_H \varphi(y)\,\mathcal{N}(e^{tA}z, Q_t)(dy), \qquad \forall\varphi\in C_b(H),\ z\in H, \quad (1.10)$$

where $Q_t := \int_0^t e^{2sA}\,ds$. The infinitesimal generator $(\mathcal{M}, D(\mathcal{M}))$ of $(\Pi_t)_{t\geq 0}$ is the closure in $L^2(H,\mu)$ of the Ornstein-Uhlenbeck operator:

$$M\varphi(x) := \frac{1}{2}\mathrm{Tr}\left[D^2\varphi(x)\right] + \langle x, A\nabla\varphi(x)\rangle, \quad \varphi\in \mathrm{Exp}_A(H). \quad (1.11)$$

- *$(\Pi_t)_{t\geq 0}$ is the transition semigroup of the Markov process $\{Z(t,x) : t\geq 0, x\in H\}$ in H, satisfying the linear SPDE:*

$$\begin{cases} \dfrac{\partial Z}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 Z}{\partial\xi^2} + \dfrac{\partial^2 W}{\partial t\partial\xi}, \\ Z(t,x)\in C_0, \quad t>0 \\ Z(0,x) = x \in H \end{cases} \quad (1.12)$$

- *For all $t>0$ and $\varphi\in L^2(H,\mu)$, we have $\Pi_t\varphi\in W^{1,2}(H,\mu)$ and:*

$$W^{1,2}(H,\mu) = \left\{\varphi\in L^2(H,\mu) : \sup_{t>0}\Lambda(\Pi_t\varphi,\Pi_t\varphi) < \infty\right\}. \quad (1.13)$$

- *For all $t>0$ and $\varphi\in L^\infty(H,\mu)$, we have $\Pi_t\varphi\in C_b^1(H)$: in particular, Π is Strong Feller. Moreover, for all $\varphi\in C_b(H)$ and $x\in H$, the map $0\leq t\mapsto\Pi_t\varphi(x)$ is continuous, i.e. Π is weakly continuous: see [Ce 94].*

By (1.13) we have that every Lipschitz function on H is in $W^{1,2}(H,\mu)$, since

$$\|\nabla\Pi_t\varphi(x)\| = \sup_{\|h\|\leq 1}|\langle\nabla\Pi_t\varphi(x), h\rangle|$$

$$= \sup_{\|h\|\leq 1}\lim_{s\downarrow 0}\frac{1}{s}\left|\int\left(\varphi\left(e^{tA}(x+sh)+y\right)-\varphi\left(e^{tA}x+y\right)\right)\mathcal{N}(0,Q_t)(dy)\right|$$

$$\leq \|\varphi\|_{\mathrm{Lip}}$$

Finally, we have the following:

Lemma 1.3 *Let $g^1_\rho, g^2_\rho : [0,1]\times\mathbb{R} \mapsto \mathbb{R}$ be jointly measurable, $\rho > 0$. Suppose that:*

- *For all $\rho > 0$ there exists $c_\rho \geq 0$ such that for all $\xi \in [0,1]$:*
$$|g^1_\rho| \leq c_\rho, \quad |g^1_\rho(\xi,y) - g^1_\rho(\xi,y')| \leq c_\rho\,|y-y'|, \qquad \forall\ y,y' \in \mathbb{R},$$
- *$\mathbb{R} \ni y \mapsto g^2_\rho(\xi,y)$ is non-increasing and there exists $c \geq 0$ such that:*
$$|g^2_\rho(\xi,y)| \leq c(1+|y|), \qquad \forall\ y \in \mathbb{R}.$$

For all $\rho > 0$, set $g_\rho := g^1_\rho + g^2_\rho$ and for all $a \in \mathbb{R}$, let u^a_ρ be the unique solution of the following SPDE:

$$\begin{cases} \dfrac{\partial u^a_\rho}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u^a_\rho}{\partial \xi^2} + g_\rho(\xi, u^a_\rho(t,\xi)) + \dfrac{\partial^2 W}{\partial t \partial \xi} \\ \\ u^a_\rho(t,0) = u^a_\rho(t,1) = a \\ \\ u^a_\rho(0,\cdot) = x(\cdot)\ \in\ L^2(0,1). \end{cases} \tag{1.14}$$

Then, a.s. for all $t \geq 0$, $\xi \in [0,1]$, we have:

- *If $\rho \mapsto g_\rho(\cdot,\cdot)$ is monotone non-decreasing, then $\rho \mapsto u^a_\rho(\cdot,\cdot)$ is monotone non-decreasing for all $a \in \mathbb{R}$.*
- *If $\rho \mapsto g_\rho(\cdot,\cdot)$ is monotone non-increasing, then $\rho \mapsto u^a_\rho(\cdot,\cdot)$ is monotone non-increasing for all $a \in \mathbb{R}$.*
- *$a \mapsto u^a_\rho(\cdot,\cdot)$ is monotone non-decreasing for all $\rho > 0$.*

Proof–We prove the first assertion: the others follow analogously. Let $\rho_1 \geq \rho_2 > 0$ and set $b := (u^a_{\rho_2} - u^a_{\rho_1})^-$. Then:

$$\begin{aligned} \frac{d}{dt}\|b\|^2 &= 2\,\langle b, A(u^a_{\rho_2} - u^a_{\rho_1})\rangle + 2\langle b, g_{\rho_2}(u^a_{\rho_2}) - g_{\rho_1}(u^a_{\rho_1})\rangle \\ &= -\,\|b'\|^2 + 2\langle b, g_{\rho_2}(u^a_{\rho_2}) - g_{\rho_2}(u^a_{\rho_1})\rangle + 2\langle b, g_{\rho_2}(u^a_{\rho_1}) - g_{\rho_1}(u^a_{\rho_1})\rangle \\ &\leq -\pi^2\|b\|^2 + 2\langle b, g^1_{\rho_2}(u^a_{\rho_2}) - g^1_{\rho_2}(u^a_{\rho_1})\rangle \leq -\pi^2\|b\|^2 + 4\,c_{\rho_2}\,\|b\| \\ &\leq -\frac{\pi^2}{2}\|b\|^2 + C \end{aligned}$$

for some $C \geq 0$. Since $\|b(0,\cdot)\| = 0$, by Gronwall's Lemma we have $b \equiv 0$, so that $u^a_{\rho_1} \geq u^a_{\rho_2}$. $\square$

Chapter 2

Integration by parts formulae

In this chapter we prove the following Integration by Parts Formulae (**IbPF**) on the convex sets of paths $K_\alpha := \{x : [0,1] \mapsto \mathbb{R} : x \geq -\alpha\}$, $\alpha \geq 0$:

- with respect to the law μ of the Brownian Bridge on K_α, $\alpha > 0$:

$$\int_{K_\alpha} \partial_h \varphi \, d\mu \;=\; -\int_{K_\alpha} \varphi(x)\, \langle x, h'' \rangle \, d\mu \;-\; \int_0^1 dr \;\; h(r) \int \varphi(x)\, \sigma_\alpha(r, dx) \tag{2.1}$$

- with respect to the law ν of the 3-d Bessel Bridge on K_0:

$$\int_{K_0} \partial_h \varphi \, d\nu \;=\; -\int_{K_0} \varphi(x)\, \langle x, h'' \rangle \, d\nu \;-\; \int_0^1 dr \;\; h(r) \int \varphi(x)\, \sigma_0(r, dx). \tag{2.2}$$

- with respect to the law π_δ of the δ-d Bessel Bridge, $\delta > 3$, on K_0:

$$\int_{K_0} \partial_h \varphi \, d\pi_\delta = -\int_{K_0} \varphi(x) \left(\langle x, h'' \rangle + \frac{(\delta-3)(\delta-1)}{4} \langle x^{-3}, h \rangle \right) \pi_\delta(dx). \tag{2.3}$$

In (2.1), (2.2) and (2.3), and in the whole chapter, $\varphi : H \mapsto \mathbb{R}$ is bounded and Fréchet differentiable, $h \in C_c^2(0,1) \subset H$ and $h'' \in H$ is the second derivative of h. Moreover we set for $\alpha > 0$, $r \in (0,1)$:

$$\int \varphi(x)\, \sigma_\alpha(r, dx) \;:=\; \frac{\sqrt{2}\, \alpha^2 \, e^{-\alpha^2/(2r(1-r))}}{\sqrt{\pi r^3 (1-r)^3}} \, \mathbb{E}\left[\varphi\left(e_{0,\alpha}^r \oplus_r \hat{e}_{0,\alpha}^{1-r} - \alpha \right) \right] \tag{2.4}$$

$$\int \varphi(x)\, \sigma_0(r, dx) \;:=\; \frac{1}{\sqrt{2\pi r^3 (1-r)^3}} \, \mathbb{E}\left[\varphi\left(e_{0,0}^r \oplus_r \hat{e}_{0,0}^{1-r} \right) \right] \tag{2.5}$$

where $e^r_{0,a}$, $\hat{e}^r_{0,a}$ are two independent copies of the 3-d Bessel Bridge on $[0,r]$ between 0 and $a \geq 0$, and for $(y,z) \in L^2(0,r) \times L^2(0,1-r)$:

$$y \oplus_r z \in H, \quad [y \oplus_r z]\,(\tau) := y(r-\tau)\, 1_{[0,r]}(\tau) \, + \, z(\tau - r)\, 1_{(r,1]}(\tau). \tag{2.6}$$

Recall that, by the Divergence Theorem in finite dimension, we have:

$$\int_O (\partial_h \varphi)\, \rho\, dx \;=\; - \int_O \varphi\, (\partial_h \log \rho)\, \rho\, dx \;-\; \int_{\partial O} \varphi\, \langle n, h \rangle\, \rho\, d\sigma \tag{2.7}$$

where O is a regular bounded open subset of $\mathbb{R}^d$, $h \in \mathbb{R}^d$, $\varphi, \rho \in C^1_b(O)$, $0 < \lambda \leq \rho \leq \Lambda < \infty$, n is the inward-pointing normal vector to the boundary ∂O and σ is the surface measure.

Since, by Lemma 1.2, μ is equal to the Gaussian measure $\mathcal{N}(0,(-2A)^{-1})$, the Cameron-Martin Theorem gives:

$$\int_H \partial_h \varphi\, d\mu \;=\; - \int_H \varphi(x)\, \langle x, h'' \rangle\, d\mu.$$

Therefore, the first term in the right-hand side of (2.1) comes from the well-known fact that the measure μ admits as logarithmic derivative the map $x \mapsto x''$.

On the other hand, the second term in the right-hand side of (2.1) is essentially of a different type, and can be interpreted as a boundary term: indeed, it is concentrated on the set $\{x \in C_0(0,1) : \inf x = -\alpha\}$, i.e. the topological boundary of $K_\alpha \cap C_0(0,1)$ in the sup-norm, which has zero μ-measure.

Recall that a.s. the Brownian Bridge β attains its minimum on $[0,1]$ at an unique time ζ, and ζ is uniformly distributed on $[0,1]$: a trajectory $x(\cdot) \in K_\alpha$ of β lies on the boundary of K_α if and only if $x(\zeta(x)) = -\alpha$. We define for all $r \in (0,1)$:

$$\begin{aligned} \partial^*_r K_\alpha \;&:=\; \{x : [0,1] \mapsto [-\alpha, \infty) \text{ continuous} : \\ &\qquad\qquad x(0) = x(1) = 0,\; x(\xi) = -\alpha \Longleftrightarrow \xi = r\,\}, \end{aligned}$$

and $\partial^* K_\alpha := \bigcup_{r \in (0,1)} \partial^*_r K_\alpha$. Then $\partial^*_r K_\alpha$, $r \in (0,1)$, are the faces with lowest co-dimension in $\partial^* K_\alpha$. Moreover, the factor $h(r) = \langle \delta_r, h \rangle$ corresponds in the finite-dimensional case (2.7) to the scalar product $\langle n, h \rangle$, where n is the inward-pointing normal vector to the boundary: this suggests that the inward-pointing normal vector to $\partial^* K_\alpha$ is equal to the Dirac mass δ_r at r, on each face $\partial^*_r K_\alpha$, $r \in (0,1)$. Notice that $\delta_r \notin H$, which is related to the fact that K_α is not a C^1 domain in H.

Following De Giorgi, we say that $\partial^* K_\alpha$ is the *μ-reduced boundary* of K_α. This terminology is justified, since $\partial^* K_\alpha$ is smaller than the boundary of K_α in any reasonable topology.

Consider now (2.2). Analogous considerations hold for the right-hand side: the first term gives the logarithmic derivative of ν and the second one the boundary measure σ_0 on K_0. In particular, we can define the ν-reduced boundary $\partial^* K_0 := \bigcup_{r\in(0,1)} \partial^*_r K_0$ and for all $x \in \partial^*_r K_0$, we can interpret δ_r as the inward-pointing normal vector to $\partial^* K_0$ at x. In this case σ_0 is σ-finite but not finite. Notice that, because of the boundary conditions, $K_0 \cap C_0$ has empty interior in the sup-norm topology, and therefore the definition of a ν-reduced boundary is particularly interesting.

The fact that the logarithmic derivative of ν is $x \mapsto x''$, i.e. the same as in the case of the Gaussian measure μ, suggests that ν can be considered as a natural reference measure for SPDEs with values in the cone of positive functions on $[0,1]$.

We turn to (2.3). Notice that no more boundary term appears if $\delta > 3$: on the other hand, the logarithmic derivative of π_δ becomes $x \mapsto x'' + \kappa(\delta)x^{-3}$, with $\kappa(\delta) := (\delta-3)(\delta-1)/4$. Since π_δ converges weakly to $\pi_3 = \nu$ as $\delta \downarrow 3$, comparing (2.2) and (2.3) we obtain the remarkable formula:

$$\lim_{\delta\downarrow 3} \frac{\delta-3}{2} \int \varphi(x)\, \langle x^{-3}, h\rangle\, \pi_\delta(dx) \;=\; \int_0^1 dr\ h(r) \int \varphi(x)\, \sigma_0(r, dx) \qquad (2.8)$$

There is a simple finite-dimensional analogue to this fact: consider the measures on $\mathbb{R}$

$$m_1(dx) \;:=\; 1_{[0,\infty)}(x)dx, \qquad m_\delta(dx) \;:=\; 1_{[0,\infty)}(x)\, x^{\delta-1}\, dx, \qquad \delta > 1.$$

Then, the following integration by parts formulae hold for all smooth φ with compact support in $\mathbb{R}$:

$$\int \varphi'\, dm_1 \;=\; -\varphi(0), \qquad (2.9)$$

$$\int \varphi'\, dm_\delta \;=\; -\int \varphi(x)\, \frac{\delta-1}{x}\, dm_\delta, \qquad \delta > 1. \qquad (2.10)$$

The space $[0,\infty)$ has $\{0\}$ as boundary with respect to the measure m_1. On the other hand, if $\delta > 1$, the boundary term does not appear and is replaced by an absolutely continuous term, with density $(\delta-1)/x$ diverging as $x \downarrow 0$. However, as $\delta \downarrow 1$, m_δ converges weakly to m_1, and therefore the right-hand side of (2.10) converges to the boundary term of (2.9).

2.1 The Brownian Bridge conditioned on K_0

The main tools in the proof of (2.1) and (2.2) are the following Theorems:

Theorem 2.1 (Durrett, Iglehart, Miller, [DIM 77]) *For all continuous $\varphi : C_0 \mapsto \mathbb{R}$ such that for some $\omega < \pi^2$, $|\varphi(x)| \le e^{\omega \|x\|^2}$ for all $x \in H$, we have: $\int \varphi \, d\nu_\alpha \to \int \varphi \, d\nu_0$ as $\alpha \downarrow 0$. In particular, ν_α converges weakly as $\alpha \downarrow 0$ to $\nu_0 = \nu$.*

Theorem 2.2 (Biane, [Bi 86]) *Let $(e_\tau)_{\tau \in [0,1]}$ be a 3-d Bessel Bridge, and let ζ be a random variable with uniform distribution on $[0,1]$ and independent of e. Then the process:*

$$(\beta_\tau)_{\tau \in [0,1]}, \qquad \beta_\tau := e_{\tau \oplus \zeta} - e_\zeta,$$

where $\oplus$ denotes the sum mod 1*, is a Brownian Bridge.*

Recall that $\nu_\alpha := \mu(\cdot | K_\alpha)$. Notice that

$$\left\{ e_{(\,\cdot\,\oplus r)} - e_r \ge -\alpha \right\} = \{ e_r \le \alpha \}, \qquad r \in [0,1]. \tag{2.11}$$

Indeed, $e_{(\tau \oplus r)} > 0$ for all $\tau \in [0,1] \backslash \{1-r\}$ and $e_{(1-r \oplus r)} = e(1) = 0$. In particular, an infinite-dimensional information, namely that $\beta(\tau) \ge -\alpha$ for all $\tau \in [0,1]$, is reduced to an information on two independent real valued random variables, namely ζ and e_r, $r \in]0,1[$. As we shall see in the next section, formula (2.1) can be seen as a strengthening of this remarkable simplification. Now, we give a proof of Theorem 2.1 based on Theorem 2.2.

Notice first that, by (2.11) and Theorem 2.2, we have for $\alpha > 0$, $\varphi \in C_b(H)$:

$$\nu_\alpha(\varphi) = \frac{1}{1 - \exp\{-2\alpha^2\}} \int_0^1 \nu\left(\varphi\left(e_{(\cdot\,\oplus r)} - e_r \right) 1_{(e_r \le \alpha)} \right) \, dr. \tag{2.12}$$

Lemma 2.1 *There exist regular conditional distributions $\{\nu(\,\cdot\,|\, e_r = y) : y \ge 0\}$ of ν given e_r, $r \in [0,1]$, such that, setting:*

$$\delta_\varphi(r,y) := \nu\left(\varphi\left(e_{(\cdot\,\oplus r)} - e_r \right) \, | \, e_r = y \right), \quad \varphi \in C_b(H), \, r \in \,]0,1[, \; y \ge 0,$$

we have for all $y \ge 0$:

$$\lim_{r \downarrow 0} \delta_\varphi\left(r, \sqrt{r(1-r)}\; y \right) \; = \; \lim_{r \uparrow 1} \delta_\varphi\left(r, \sqrt{r(1-r)}\; y \right) \; = \; \nu(\varphi).$$

Proof—Let $(B_\tau)_{\tau\in[0,\infty)}$ and $(\hat B_\tau)_{\tau\in[0,\infty)}$ be two independent 3-d Brownian Motions and $r \in \,]0,1[$. Set $b := |B|$, $\hat b := |\hat B|$,

$$\beta(z)(\tau) := B_\tau - \tau B_1 + \tau z, \quad \hat\beta(z)(\tau) := \hat B_\tau - \tau \hat B_1 + \tau z, \quad \tau \in [0,1], \quad z \in \mathbb{R}^3,$$

$$\pi_r, \hat\pi_r : L^2(0,\infty) \times L^2(0,\infty) \mapsto L^2(0,1),$$

$$\begin{cases} \pi_r(c,d)(\tau) := 1_{[0,r]}(\tau)c(\tau) + 1_{]r,1]}(\tau)d(1-\tau), \\ \\ \hat\pi_r(c,d)(\tau) := 1_{[0,1-r]}(\tau)d(1-r-\tau) + 1_{]1-r,1]}(\tau)c(\tau+r-1). \end{cases} \tag{2.13}$$

For all $\varphi \in C_b(H)$, we set:

$$\nu\left(\varphi(e)\,|\,e_r = y\right) \;:=\; \mathbb{E}\left[\varphi\left(\pi_r(b,\hat b)\right)\;\Big|\; b_r = y = \hat b_{1-r}\right], \quad y \geq 0, \tag{2.14}$$

$$\mathbb{E}[\varphi(b)|\,b(1) = y] \;:=\; \int_{S^2} \sigma(dn)\;\mathbb{E}\left[\varphi\left(|\beta(yn)|\right)\right], \quad y \geq 0, \tag{2.15}$$

where S^2 is the unitary sphere in $\mathbb{R}^3$ and $\sigma(dn)$ is the normalized uniform distribution on S^2. Then (2.14), respectively (2.15), is a regular conditional distribution of ν given e_r, resp. of $\mathbb{P}(b \in \,\cdot\,)$ given $b(1)$. In particular, the law of $|\beta(0)|$ is equal to ν. By (2.13) and (2.14) we have:

$$\nu\left(\varphi\left(e_{(\cdot\,\oplus r)} - e_r\right)\,|\,e_r = y\right) \;=\; \mathbb{E}\left[\varphi\left(\hat\pi_r(b,\hat b) - y\right)\;\Big|\; b_r = y = \hat b_{1-r}\right]. \tag{2.16}$$

Identifying $h \in L^2(0,1)$ with $h\,1_{[0,1]} \in L^2(0,\infty)$, we set $\varphi_r : H \times H \mapsto \mathbb{R}$,

$$\varphi_r(h,k) := \varphi\left(\hat\pi_r\left(\sqrt{r}\,h\left(\cdot/r\right),\,\sqrt{1-r}\,k\left(\cdot/(1-r)\right)\right) - \sqrt{r(1-r)}\;y\right).$$

Since for $\gamma > 0$, $\sqrt{\gamma}\,B_{(\cdot\,/\gamma)}$ is still a 3-d Brownian Motion, we obtain by (2.16):

$$\begin{aligned} \delta_\varphi\left(r, \sqrt{r(1-r)}\;y\right) \;&=\; \nu\left(\varphi\left(e_{(\cdot\,\oplus r)} - e_r\right)\;\Big|\; e_r = \sqrt{r(1-r)}\;y\right) \\ &= \mathbb{E}\left[\varphi\left(\hat\pi_r(b,\hat b) - \sqrt{r(1-r)}\;y\right)\;\Big|\; b_r = \sqrt{r(1-r)}\;y = \hat b_{1-r}\right] \\ &= \mathbb{E}\left[\varphi_r(b,\hat b)\;\Big|\; b_1 = \sqrt{1-r}\;y,\; \hat b_1 = \sqrt{r}\;y\right]. \end{aligned} \tag{2.17}$$

Since for all $n \in S^2$ and $y \geq 0$:

$$\lim_{r\downarrow 0} \varphi_r\left(\left|\,\beta\left(\sqrt{1-r}\,yn\right)\right|,\,\left|\,\hat\beta\left(\sqrt{r}\,yn\right)\right|\right) \;=\; \varphi\left(\left|\hat\beta(0)(1-\,\cdot\,)\right|\right),$$

$$\lim_{r\uparrow 1} \varphi_r\left(\left|\,\beta\left(\sqrt{1-r}\,yn\right)\right|,\,\left|\,\hat\beta\left(\sqrt{r}\,yn\right)\right|\right) \;=\; \varphi\left(|\beta(0)|\right),$$

and since ν is invariant by the time-change $\tau \mapsto 1-\tau$, the thesis follows by Dominated Convergence Theorem and (2.15)-(2.17). $\square$

Proof of Theorem 2.1—We split the integral on $[0,1]$ in (2.12), into two integrals on $[0,1/2]$ and $[1/2,1]$, respectively. Conditioning with respect to e_r and setting $c_\alpha := (1-\exp\{-2\alpha^2\})$, we obtain:

$$
\begin{aligned}
&\frac{1}{c_\alpha}\int_0^{1/2} dr\ \nu\left(\varphi\left(e_{(\cdot\,\oplus r)}-e_r\right) 1_{(e_r\le\alpha)}\right)\\
&=\frac{1}{c_\alpha}\int_0^{1/2} dr\int_0^{\alpha} dy\ \sqrt{\frac{2}{\pi[r(1-r)]^3}}\ y^2\ \exp\left\{-\frac{y^2}{2r(1-r)}\right\}\ \delta_\varphi(r,y)\\
&=\frac{1}{c_\alpha}\int_0^{1/2} dr\int_0^{\alpha/\sqrt{r(1-r)}} dy\ \sqrt{\frac{2}{\pi}}\ y^2\ \exp\left\{-\frac{y^2}{2}\right\}\ \delta_\varphi\left(r,\sqrt{r(1-r)}\ y\right)\\
&=\ \frac{1}{c_\alpha}\sqrt{\frac{2}{\pi}}\int_0^{2\alpha} dy\ \exp\left\{-\frac{y^2}{2}\right\}\ y^2\int_0^{1/2} dr\ \delta_\varphi\left(r,\sqrt{r(1-r)}\ y\right)\\
&+\ \frac{\alpha^2}{c_\alpha}\sqrt{\frac{2}{\pi}}\int_{2\alpha}^{+\infty} dy\ \exp\left\{-\frac{y^2}{2}\right\}\ \left(\frac{y}{\alpha}\right)^2\int_0^{\rho(\alpha,y)} dr\ \delta_\varphi\left(r,\sqrt{r(1-r)}\ y\right)\\
&=:\ I_1(\alpha)\ +\ I_2(\alpha),\qquad \rho(\alpha,y):=\frac{1}{2}\left(1-\sqrt{1-\left(\frac{2\alpha}{y}\right)^2}\right)\sim\left(\frac{\alpha}{y}\right)^2
\end{aligned}
$$

as $\alpha\downarrow 0$, $y>0$. It is easy to see that $\lim_{\alpha\downarrow 0} I_1(\alpha)=0$, while $I_2(\alpha)$ tends to $(1/2)\nu(\varphi)$ by Lemma 2.1 and Dominated Convergence Theorem. Since analogous computations hold for the integral on $[1/2,1]$, we obtain that $\nu_\alpha(\varphi)$ converges to $\nu(\varphi)$ and Theorem 2.1 is proved. $\square$

2.2 IbPF on the 3-d Bessel Bridge

In this section we prove formulae (2.1) and (2.2). In the previous section we have given a proof of Theorem 2.1 based on Theorem 2.2. The key observation there is that $\beta = e_{\cdot\oplus\zeta}-e_\zeta\in K_\alpha$ if and only if $e_\zeta\le\alpha$. Now formula (2.1) says in particular that $\beta=e_{\cdot\oplus\zeta}-e_\zeta$ is in the boundary of K_α if and only if $e_\zeta=\alpha$: the proof of (2.1) is formalization of this intuitive fact.

Proof of (2.1) and (2.2)—Recall the notations given in (2.4), (2.5) and (2.6). For $x\in H$, we set $x^+\in H$, $x^+(\tau):=\sup\{x(\tau),0\}$, $x^-:=x^+-x$.

Notice that $h_\lambda := \lambda(\lambda - A)^{-1}h$ converges to h in $D(A)$ as $\lambda \to \infty$. Moreover, we have $h_\lambda = (h^+)_\lambda - (h^-)_\lambda$, with $(h^+)_\lambda, (h^-)_\lambda \in D(A)$, $(h^+)_\lambda, (h^-)_\lambda \geq 0$ and:

$$\partial_{h_\lambda}\varphi \;=\; \langle \nabla\varphi, h_\lambda\rangle \;=\; \langle \nabla\varphi(x), (h^+)_\lambda\rangle \;-\; \langle \nabla\varphi(x), (h^-)_\lambda\rangle,$$

Then, we can suppose that $h \geq 0$, so that $K_\alpha \subseteq K_\alpha - th$, $t \geq 0$. Moreover, since φ is bounded and $\nabla(\varphi - \inf\varphi) = \nabla\varphi$, we can suppose $\varphi \geq 0$. Recall that $\partial_h\varphi(x) = \lim_{t\downarrow 0}(\varphi(x) - \varphi(x - th))/t$. By the Cameron Martin Theorem:

$$\begin{aligned} \frac{1}{t}\int_{K_\alpha} (\varphi(x) - \varphi(x - th))\,\mu(dx) \;&=\; -\frac{1}{t}\int_{(K_\alpha - th)\setminus K_\alpha} \varphi(x)\,\mu(dx) \\ &\quad + \;\frac{1}{t}\int_{K_\alpha - th} \varphi(x)\,\left(1 - \exp\left(\frac{1}{2}\|th'\|^2 + t\langle x, h''\rangle\right)\right)\,\mu(dx). \end{aligned} \tag{2.18}$$

Let $n \in \mathbb{N}$, $c_n \geq c_{n-1} \geq \cdots \geq c_1 \geq c_0 := 0$, $\{I_1, \ldots, I_n\}$ a Borel partition of $[0,1]$ and $I_0 := \emptyset$, and set:

$$h_i := \sum_{j=1}^{n} (c_j \wedge c_i)\, 1_{I_j}, \qquad i = 1, \ldots, n.$$

The key point is the following: for $i = 1, \ldots, n$, since $h_i \geq h_{i-1}$, and $h_i = h_{i-1}$ on $\bigcup_{j=1}^{i-1} I_j$, then for all $r \in (0,1)$

$$e_{\cdot\oplus r} - e_r \;\in\; (K_\alpha - th_i)\setminus(K_\alpha - th_{i-1}) \quad \Longleftrightarrow$$

$$e_{\cdot\oplus r} - e_r \;\in\; K_\alpha - th_i,\;\; 1 - r \in \bigcup_{j=i}^{n} I_i \;\;\text{and}\;\; e_r \in\; [\alpha + tc_{i-1}, \alpha + tc_i).$$

Indeed, recall that $e_{\cdot\oplus r} - e_r$ attains its minimum $-e_r$ only at time $1 - r$. Applying Theorem 2.2 we obtain for all $t \geq 0$ and $i = 1, \ldots, n$:

$$\begin{aligned} \int_{(K_\alpha - th_i)\setminus K_\alpha} \varphi(x)\,\mu(dx) \;&=\; \int_0^1 \mathbb{E}\left[\varphi \cdot 1_{(K_\alpha - th_i)\setminus K_\alpha}\;(e_{\cdot\oplus r} - e_r)\right]\,dr \\ =\;& \int_0^1 \mathbb{E}\left[\varphi \cdot \left[1_{(K_\alpha - th_{i-1})\setminus K_\alpha} + 1_{(K_\alpha - th_i)\setminus(K_\alpha - th_{i-1})}\right](e_{\cdot\oplus r} - e_r)\right]\,dr \\ =\;& \int_0^1 \mathbb{E}\left[\varphi \cdot 1_{(K_\alpha - th_{i-1})\setminus K_\alpha}\,(e_{\cdot\oplus r} - e_r)\right]\,dr \\ &+ \int_{1 - \cup_{j=i}^{n} I_j} \mathbb{E}\left[\varphi \cdot 1_{(K_\alpha - th_i)}\,(e_{\cdot\oplus r} - e_r)\,1_{[\alpha + tc_{i-1}, \alpha + tc_i[}(e_r)\right]\,dr, \end{aligned}$$

where $1-I := \{1-\tau : \tau \in I\}$. Proceeding by induction on n we obtain:

$$\begin{aligned}
&\int_{(K_\alpha - th_n)\setminus K_\alpha} \varphi(x)\,\mu(dx) \\
= \;& \sum_{i=1}^n \int_{1-\cup_{j=i}^n I_j} \mathbb{E}\left[\varphi \cdot 1_{(K_\alpha - th_i)}\,(e_{\cdot\oplus r} - e_r)\, 1_{[\alpha+tc_{i-1},\alpha+tc_i[}(e_r)\right]\,dr \\
= \;& \sum_{i=1}^n \sum_{j=i}^n \int_{1-I_j} dr \int_{\alpha+tc_{i-1}}^{\alpha+tc_i} da\ \lambda(r,a)\ \mathbb{E}\left[\varphi \cdot 1_{(K_\alpha - th_i)}\,(e_{\cdot\oplus r} - e_r)\ \Big|\, e_r = a\right]
\end{aligned}$$

where $\lambda(r,a)da$, law of e_r, is defined by

$$\lambda(r,a) := \sqrt{\frac{2}{\pi r^3 (1-r)^3}}\; a^2 \exp\left(-\frac{a^2}{2r(1-r)}\right), \qquad r \in [0,1],\ a \geq 0,$$

and for all bounded Borel $\psi : H \mapsto \mathbb{R}$ and $a \geq 0$:

$$\mathbb{E}\left[\psi\,(e_{\cdot\oplus r} - e_r)\ \Big|\, e_r = a\right] := \mathbb{E}\left[\psi\left(e^r_{0,a} \oplus_r \hat{e}^{1-r}_{0,a} - a\right)\right]. \tag{2.19}$$

The measure defined by (2.19) depends continuously on $a \geq 0$. Then we obtain, since $\lambda(1-r,a) = \lambda(r,a)$:

$$\begin{aligned}
&\lim_{t\downarrow 0} \frac{1}{t} \int_{(K_\alpha - th_n)\setminus K_\alpha} \varphi(x)\,\mu(dx) \\
= \;& \sum_{i=1}^n \sum_{j=i}^n (c_i - c_{i-1}) \int_{I_j} \lambda(r,\alpha)\,\mathbb{E}\left[\varphi\left(e_{\cdot\oplus(1-r)} - \alpha\right)\ \Big|\, e_{1-r} = \alpha\right]\,dr \\
= \;& \sum_{j=1}^n \int_{I_j} c_j\,\lambda(r,\alpha)\,\mathbb{E}\left[\varphi\left(e_{\cdot\oplus(1-r)} - \alpha\right)\ \Big|\, e_{1-r} = \alpha\right]\,dr \\
= \;& \int_0^1 dr\, h_n(r)\,\lambda(r,\alpha)\,\mathbb{E}\left[\varphi\left(e^r_{0,\alpha} \oplus_r \hat{e}^{1-r}_{0,\alpha} - \alpha\right)\right] \\
= \;& \int_0^1 dr\, h_n(r) \int \varphi(z)\,\sigma_\alpha(r,dz).
\end{aligned}$$

Set now $I_i := h^{-1}([(i-1)/n, i/n))$, $i \in \mathbb{N}$,

$$f_n := \sum_{i=1}^\infty \frac{i-1}{n}\, 1_{I_i}, \quad g_n := \sum_{i=1}^\infty \frac{i}{n}\, 1_{I_i},$$

where both sums are finite, since h is bounded. Then $f_n \le h \le g_n$, f_n and g_n converge uniformly on $[0,1]$ to h as $n \to \infty$ and: $K_\alpha - tf_n \subseteq K_\alpha - th \subseteq K_\alpha - tg_n$, $t \ge 0$. Therefore we have, since $\varphi \ge 0$,

$$\int_0^1 dr\, f_n(r) \int \varphi(z)\, \sigma_\alpha(r,dz) \,\le\, \liminf_{t\downarrow 0} \frac{1}{t} \int_{(K_\alpha - th)\setminus K_\alpha} \varphi(x)\, \mu(dx)$$

$$\le \limsup_{t\downarrow 0} \frac{1}{t} \int_{(K_\alpha - th)\setminus K_\alpha} \varphi(x)\, \mu(dx) \,\le\, \int_0^1 dr\, g_n(r) \int \varphi(z),\, \sigma_\alpha(r,dz)$$

and by (2.18) :
$$\int_{K_\alpha} \partial_h \varphi\, d\mu \,=\, \lim_{t\downarrow 0} \frac{1}{t} \int_{K_\alpha} (\varphi(x) - \varphi(x-th))\mu(dx)$$

$$= -\int_{K_\alpha} \varphi(x)\langle x, h''\rangle d\mu \,-\, \int_0^1 dr\, h(r) \int \varphi(z)\, \sigma_\alpha(r,dz)$$

so that (2.1) is proved. In order to prove (2.2), we recall that $\mu(K_\alpha) = 1 - \exp(-2\alpha^2)$. We divide (2.1) by $\mu(K_\alpha)$ and let $\alpha \downarrow 0$: in the second term of the right-hand side, we have for all $r \in]0,1[$:

$$\lim_{\alpha\downarrow 0} \frac{1}{2\alpha^2} \lambda(r,\alpha) = \lambda(r,0) := \frac{1}{\sqrt{2\pi r^3(1-r)^3}}, \quad \frac{|h(r)|}{2\alpha^2}\lambda(r,\alpha) \le |h(r)|\lambda(r,0),$$

which is integrable, since $h \in H^2 \cap H_0^1(0,1)$ implies $|h(r)| \le C\, r(1-r)$, $r \in [0,1]$, for some $C \ge 0$. Moreover, the laws of $e^r_{0,\alpha}$ are continuous in $\alpha \ge 0$. Then we apply Theorem 2.1 to the first and second term in (2.1) and the proof of (2.2) is complete. $\Box$

Remark 2.1 We have in fact proved that formula (2.1) holds for all $\varphi \in C_b^1(K_\alpha)$ and formula (2.2) holds for all $\varphi \in \cup_{\alpha>0} C_b^1(K_\alpha)$.

Corollary 2.1 *For all $\varphi \in C_b(H)$, $\alpha > 0$, $h \in H^2 \cap H_0^1(0,1)$:*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_H \varphi(x)\, \langle (x+\alpha)^-, h\rangle\, \exp\left(-\frac{\|(x+\alpha)^-\|^2}{\varepsilon}\right)\, \mu(dx)$$

$$= \frac{1}{2} \int_0^1 dr\, h(r) \int \varphi(z)\, \sigma_\alpha(r,dz).$$

Proof—We can suppose $h \ge 0$. If $\varphi \in C_b^1(H)$, then by (2.1):

$$\frac{1}{\varepsilon} \int_H \varphi(x)\langle h, (x+\alpha)^-\rangle \exp\left(-\frac{\|(x+\alpha)^-\|^2}{\varepsilon}\right)\, \mu(dx) \tag{2.20}$$

$$= -\frac{1}{2}\int_H \left(\langle \nabla\varphi(x), h\rangle + \langle x, h''\rangle\, \varphi(x)\right) \exp\left(-\frac{\|(x+\alpha)^-\|^2}{\varepsilon}\right)\, \mu(dx)$$

$$\to -\frac{1}{2}\int_{K_\alpha} \left(\langle \nabla\varphi(x), h\rangle + \langle x, h''\rangle\, \varphi(x)\right)\, \mu(dx) \quad \text{as } \varepsilon \downarrow 0. \tag{2.21}$$

Setting $\varphi \equiv 1$, we see that the family of finite measures on H defined by (2.20) have equibounded mass by (2.21). Since $C_b^1(H)$ is dense in $C_b(H)$ in the uniform norm, the thesis follows for all $\varphi \in C_b(H)$. $\square$

Recall the definition of the Ornstein-Uhlenbeck operator:

$$M\varphi(x) := \frac{1}{2}\mathrm{Tr}\left[D^2\varphi(x)\right] + \langle x, A\nabla\varphi(x)\rangle, \quad \varphi \in \mathrm{Exp}_A(H). \tag{2.22}$$

Corollary 2.2 *For all $\psi \in C_b^1(H)$, $\varphi(x) \in \mathrm{Exp}_A(H)$,*

$$\frac{1}{2}\int_{K_\alpha} \langle \nabla\varphi, \nabla\psi\rangle\, d\mu = -\int_{K_\alpha} \psi\, M\varphi\, d\mu - \frac{1}{2}\int_0^1 dr \int \langle \nabla\varphi, \delta_r\rangle\, \psi\, d\sigma_\alpha(r,\cdot) \tag{2.23}$$

$$\frac{1}{2}\int_{K_0} \langle \nabla\varphi, \nabla\psi\rangle\, d\nu = -\int_{K_0} \psi\, M\varphi\, d\nu - \frac{1}{2}\int_0^1 dr \int \langle \nabla\varphi, \delta_r\rangle\, \psi\, d\sigma_0(r,\cdot), \tag{2.24}$$

where, denoting by i the imaginary unit, for $\varphi = \exp(i\langle h,\cdot\rangle)$, $h \in D(A)$:

$$\langle \nabla\varphi(x), \delta_r\rangle := i\, h(r)\, \exp(i\langle h, x\rangle), \qquad x \in H. \tag{2.25}$$

2.3 IbPF on the δ-d Bessel Bridge

The aim of this section is to prove (2.3). We define:

$$c(\delta) := \frac{\delta-3}{2}, \quad \kappa(\delta) := \frac{(\delta-3)(\delta-1)}{4} \qquad \delta > 3. \tag{2.26}$$

We fix $\delta > 3$ throughout the section. The proof of (2.3) will be divided into several steps. We can assume without loss of generality that:

$$\varphi \geq 0, \qquad h \geq 0. \tag{2.27}$$

Step 1. We consider a fixed linear Brownian Motion $(B_\tau)_{\tau\in[0,1]}$. Recall that the $\delta-$dimensional Bessel Bridge between 0 and 0 on $[0,r]$, for $\delta > 1$ and

$r \in (0,1]$, is defined as the unique solution of the SDE:

$$\begin{cases} d\,x_\delta^r = \dfrac{\delta-1}{2x_\delta^r}\,d\tau \;-\; \dfrac{x_\delta^r}{r-\tau}\,d\tau \;+\; dB, \quad \tau \in]0,r[\\ \\ x_\delta^r(0) = 0 \end{cases} \tag{2.28}$$

Then we define for all $r \in (0,1]$, $\delta > 1$:

$$\pi_\delta^r \;:=\; \text{law of} \quad x_\delta^r \quad \text{on } L^2(0,r), \tag{2.29}$$

and we write $x_\delta := x_\delta^1$, $\pi_\delta = \pi_\delta^1$. If $Q_\delta := (x_\delta)^2$, then Q_δ satisfies:

$$\begin{cases} d\,Q_\delta = \delta\,d\tau \;-\; \dfrac{Q_\delta}{1-\tau}\,d\tau \;+\; 2\sqrt{Q_\delta}\,dB, \;\; \tau \in]0,1[\\ \\ Q_\delta(0) = 0 \end{cases} \tag{2.30}$$

We set the following SDE for $\epsilon > 0$:

$$\begin{cases} d\,q_\epsilon = \left(3 \,+\, 2c(\delta)\dfrac{\sqrt{|q_\epsilon|}}{\epsilon + \sqrt{|q_\epsilon|}} \;-\; \dfrac{q_\epsilon}{1-\tau}\right) d\tau \;+\; 2\sqrt{|q_\epsilon|}\,dB_\tau, \;\; \tau \in]0,1[\\ \\ q_\epsilon(0) = 0 \end{cases} \tag{2.31}$$

(2.31) is a one-dimensional SDE with $\frac{1}{2}$-Hölder continuous diffusion coefficient and bounded drift on every interval $[0,a[$. Then by Theorem IX.3.5 of [RY 91] pathwise existence and uniqueness holds for (2.31). Moreover, by comparison with (2.30) we obtain by Theorem IX.3.7 of [RY 91] that $Q_\delta \geq q_\epsilon \geq Q_3$. In particular $q_\epsilon(\tau) > 0$ for all $\tau \in]0,1[$ and we can apply Itô's formula to $z_\epsilon := \sqrt{q_\epsilon}$, obtaining:

$$\begin{cases} d\,z_\epsilon = \dfrac{1}{z_\epsilon}\,d\tau \;+\; \dfrac{c(\delta)}{\epsilon + z_\epsilon}\,d\tau \;-\; \dfrac{z_\epsilon}{1-\tau}\,d\tau \;+\; dB, \quad \tau \in]0,1[\\ \\ z_\epsilon(0) = 0 \end{cases} \tag{2.32}$$

Step 2. We list a few properties of equation (2.32):

A. Pathwise uniqueness holds for (2.32).

B. Uniqueness in law holds for (2.32).

C. $]0,\infty[\ni \epsilon \mapsto z_\epsilon$ is monotone non-increasing.

D. $z_\epsilon \uparrow x_\delta$ uniformly on $[0,1]$ as $\epsilon \downarrow 0$.

E. $]0,\infty[\ni \epsilon \mapsto \epsilon + z_\epsilon$ is monotone non-decreasing.

F. $\epsilon + z_\epsilon \downarrow x_\delta$ uniformly on $[0,1]$ as $\epsilon \downarrow 0$.

Proof

A. Let z'_ϵ be a solution of (2.32). Then, setting $b := (z_\epsilon - z'_\epsilon)^+$ we have:

$$\frac{1}{2}\frac{d}{d\tau}(b)^2 = (z_\epsilon - z'_\epsilon)^+ \left[\left(\frac{1}{z_\epsilon} - \frac{1}{z'_\epsilon}\right) + \left(\frac{c(\delta)}{\epsilon + z_\epsilon} - \frac{c(\delta)}{\epsilon + z'_\epsilon}\right)\right] - \frac{b^2}{1-\tau} \le 0$$

which implies $z_\epsilon \ge z'_\epsilon$ and, by symmetry, $z_\epsilon = z'_\epsilon$.

B. This follows by (A) and Yamada-Watanabe's Theorem.

C. Consider $\epsilon_1 \ge \epsilon_2 > 0$, and set $b := (z_{\epsilon_1} - z_{\epsilon_2})^+$: then

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}(b)^2 &= (z_{\epsilon_1} - z_{\epsilon_2})^+ \left[\left(\frac{1}{z_{\epsilon_1}} - \frac{1}{z_{\epsilon_2}}\right) + c(\delta)\left(\frac{1}{\epsilon_1 + z_{\epsilon_1}} - \frac{1}{\epsilon_1 + z_{\epsilon_2}}\right)\right] \\ &\quad + b\,c(\delta)\left(\frac{1}{\epsilon_1 + z_{\epsilon_2}} - \frac{1}{\epsilon_2 + z_{\epsilon_2}}\right) - \frac{b^2}{1-\tau} \le 0\end{aligned}$$

which implies $z_{\epsilon_1} \le z_{\epsilon_2}$.

D. By (C), $]0,\infty[\ni \epsilon \mapsto q_\epsilon = (z_\epsilon)^2$ is monotone non-increasing. By Dominated Convergence Theorem, we find that $q := \lim_{\epsilon\downarrow 0} q_\epsilon$ satisfies (2.30). Since pathwise uniqueness holds for (2.30), we find $q = Q_\delta$ and therefore $x_\delta = \lim_{\epsilon\downarrow 0} z_\epsilon$. Since x_δ is continuous, by Dini's Theorem the convergence is uniform on $[0,1]$.

E. Consider $\epsilon_1 \ge \epsilon_2 > 0$, and set $z'_\epsilon := \epsilon + z_\epsilon$, $b := (z'_{\epsilon_2} - z'_{\epsilon_1})^+$. Then by (C) and $c(\delta) \ge 0$:

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}(b)^2 &= (z'_{\epsilon_2} - z'_{\epsilon_1})^+ \left[\left(\frac{1}{z_{\epsilon_2}} - \frac{1}{z_{\epsilon_1}}\right) + c(\delta)\left(\frac{1}{z'_{\epsilon_2}} - \frac{1}{z'_{\epsilon_1}}\right)\right] \\ &\quad - b\,\frac{z_{\epsilon_2} - z_{\epsilon_1}}{1-\tau} \le 0\end{aligned}$$

which implies $z'_{\epsilon_1} \ge z'_{\epsilon_2}$.

F. This follows from (D) and (E).

Step 3. We prove now that for all $\varphi \in C_b(H)$:

$$\mathbb{E}[\varphi(z_\epsilon)] \;=\; \mathbb{E}[\varphi(x_3)\,\Gamma_\epsilon], \qquad \text{where :} \tag{2.33}$$

$$\Gamma_\epsilon := \exp\left(\int_0^1 \frac{c(\delta)}{\epsilon + x_3}\,dB \;-\; \frac{1}{2}\int_0^1 \left[\frac{c(\delta)}{\epsilon + x_3}\right]^2 d\tau\right). \tag{2.34}$$

First notice that $0 \le (\epsilon + x_3)^{-1} \le \epsilon^{-1}$. Then $\mathbb{E}[\Gamma_\epsilon] = 1$, and the right hand side of (2.33) defines a probability measure on H. If we set:

$$\hat{B}(\tau) := B(\tau) - \int_0^\tau \frac{c(\delta)}{\epsilon + x_3}\,ds, \quad \tau \in [0,1]$$

then by the Girsanov Theorem $\hat{B}$ is a Brownian Motion under $\Gamma_\epsilon d\mathbb{P}$. Therefore x_3 is, under $\Gamma_\epsilon d\mathbb{P}$, a weak solution of (2.32), and by (B) above (2.33) is proved.

Step 4. Notice that we can write Γ_ϵ as a function of x_3 only. Indeed, by (2.28) with $\delta = 3$:

$$dB \;=\; dx_3 - \frac{1}{x_3}\,d\tau \;+\; \frac{x_3}{1-\tau}\,d\tau,$$

$$d\,\log(\epsilon + x_3) \;=\; \frac{dx_3}{\epsilon + x_3} \;-\; \frac{1}{2}\frac{d\tau}{(\epsilon + x_3)^2}, \quad \log(\epsilon + x_3(0)) = \log(\epsilon + x_3(1)),$$

$$\int_0^1 \frac{dB}{\epsilon + x_3} \;=\; \int_0^1 \left(\frac{1}{2(\epsilon + x_3)^2} - \frac{1}{x_3(\epsilon + x_3)} + \frac{1}{1-\tau}\frac{x_3}{\epsilon + x_3}\right) d\tau.$$

We obtain that

$$\Gamma_\epsilon \;=\; \gamma_\epsilon(x_3) \tag{2.35}$$

$$:= \;\exp\left(c(\delta)\int_0^1 \left(\frac{1 - c(\delta)}{2(\epsilon + x_3)^2} - \frac{1}{x_3(\epsilon + x_3)} + \frac{1}{1-\tau}\frac{x_3}{\epsilon + x_3}\right) d\tau\right),$$

where $\gamma_\epsilon : K_0 \mapsto \mathbb{R}$ is in $L^1(\nu)$ and (2.33) becomes:

$$\mathbb{E}[\varphi(z_\epsilon)] \;=\; \mathbb{E}[\varphi(x_3)\gamma_\epsilon(x_3)] \;=\; \int \varphi(x)\gamma_\epsilon(x)\,\nu(dx). \tag{2.36}$$

Notice that γ_ϵ is not in $C_b^1(K_\alpha)$ for any $\alpha > 0$. If we set for all $\rho > 0$:

$$K_0 \ni x \mapsto \gamma_\epsilon^\rho(x) := \tag{2.37}$$

$$\exp\left(c(\delta)\int_0^1 \left(\frac{1-c(\delta)}{2(\epsilon+x)^2} - \frac{1}{(\rho+x)(\epsilon+x)} + \frac{1}{1+\rho-\tau}\frac{x}{\epsilon+x}\right) d\tau\right).$$

then for all $\alpha \in [0, \rho\wedge\epsilon)$, γ_ϵ^ρ is in $C_b^1(K_\alpha)$ and

$$\begin{aligned} \nabla \log \gamma_\epsilon^\rho(x) &= c(\delta)\left[\frac{c(\delta)-1}{(\epsilon+x)^3} + \frac{1}{(\rho+x)^2(\epsilon+x)} + \frac{1}{(\rho+x)(\epsilon+x)^2}\right. \\ &\qquad \left. + \frac{\epsilon}{(1+\rho-\tau)(\epsilon+x)^2}\right], \qquad x \in K_\alpha. \end{aligned} \tag{2.38}$$

By (2.2) and Remark 2.1, we obtain for all $h \in C_c^2(0,1)$:

$$\begin{aligned} \int_{K_0} \langle \nabla\varphi, h\rangle \, \gamma_\epsilon^\rho \, d\nu &= -\int_{K_0} \varphi(x)\left[\langle x, h''\rangle + \langle \nabla \log \gamma_\epsilon^\rho, h\rangle\right]\gamma_\epsilon^\rho \, \nu(dx) \\ &\quad - \int_0^1 dr \; h(r) \int \varphi(x)\, \gamma_\epsilon^\rho(x)\, \sigma_0(r, dx). \end{aligned} \tag{2.39}$$

Step 5. We want to let $\rho \downarrow 0$ in (2.39). Notice that:

$$\gamma_\epsilon^\rho(x) \le \gamma_\epsilon(x) \exp\left(\frac{1}{\epsilon}\int_0^1 \frac{d\tau}{x}\right) \quad \text{for } \nu-\text{a.e. } x, \tag{2.40}$$

$$\begin{aligned} &\int \langle h, x^{-2}\rangle \, \gamma_\epsilon(x) \exp\left(\frac{1}{\epsilon}\int_0^1 \frac{d\tau}{x}\right) \nu(dx) \\ &= \mathbb{E}\left[\langle h, (z_\epsilon)^{-2}\rangle \exp\left(\frac{1}{\epsilon}\int_0^1 \frac{d\tau}{z_\epsilon}\right)\right] \le \mathbb{E}\left[\langle h, (x_3)^{-2}\rangle \exp\left(\frac{1}{\epsilon}\int_0^1 \frac{d\tau}{x_3}\right)\right] \\ &\le \mathbb{E}\left[\langle h, (x_3)^{-2p}\rangle\right]^{1/p} \mathbb{E}\left[\exp\left(\frac{q}{\epsilon}\int_0^1 \frac{d\tau}{x_3}\right)\right]^{1/q} \end{aligned}$$

for every $p, q > 1$ with $(1/p) + (1/q) = 1$. Recall that by (2.28):

$$\begin{aligned} \mathbb{E}\left[\exp\left(\frac{q}{\epsilon}\int_0^1 \frac{d\tau}{x_3}\right)\right] &\le \mathbb{E}\left[\exp\left(\frac{2q}{\epsilon}\int_0^{1/2} \frac{d\tau}{x_3}\right)\right] \\ &\le \mathbb{E}\left[\exp\left(\frac{2q}{\epsilon}\left(x_3(1/2) + \int_0^{1/2} \frac{x_3}{1-\tau} d\tau - B(1/2)\right)\right)\right] < \infty \end{aligned} \tag{2.41}$$

$$\mathbb{E}\left[\langle h,(x_3)^{-2p}\rangle\right] \;=\; \int_0^1 d\tau\, h(\tau)\int_0^\infty dy\,\frac{C}{\sqrt{\tau^3(1-\tau)^3}}\,y^{2(1-p)}e^{-y^2/2\tau(1-\tau)}<\infty$$

for $p<2$, since h has compact support in $(0,1)$. Then we can by Dominated Convergence Theorem let $\rho\downarrow 0$ in (2.39) and obtain:

$$\begin{aligned}\int_{K_0}\langle\nabla\varphi,h\rangle\,\gamma_\epsilon\,d\nu \;&=\; -\int_{K_0}\varphi(x)\left[\langle x,h''\rangle\;+\;\langle\nabla\log\gamma_\epsilon,h\rangle\right]\gamma_\epsilon(x)\,\nu(dx)\\ &\quad -\int_0^1 dr\;h(r)\int\varphi(x)\,\gamma_\epsilon(x)\,\sigma_0(r,dx),\end{aligned}\tag{2.42}$$

where we set for ν-a.e. x:

$$\begin{aligned}\langle h,\nabla\;\log\gamma_\epsilon(x)\rangle\;&:=\;\int_0^1 h(\tau)\,c(\delta)\left[\frac{c(\delta)-1}{(\epsilon+x)^3}\;+\;\frac{1}{x^2(\epsilon+x)}\right.\\ &\qquad\left.+\frac{1}{x(\epsilon+x)^2}\;+\;\frac{\epsilon}{(1-\tau)(\epsilon+x)^2}\right](\tau)\,d\tau.\end{aligned}$$

Step 6. Recall that $\gamma_\epsilon\,d\nu$ is the law of z_ϵ, so that:

$$\int_{K_0}\varphi(x)\,\langle\nabla\log\gamma_\epsilon(x),h\rangle\gamma_\epsilon(x)\,\nu(dx)\;=\;\mathbb{E}[\varphi(z_\epsilon)\,\langle\nabla\log\gamma_\epsilon(z_\epsilon),h\rangle].$$

We prove now that:

$$\lim_{\epsilon\downarrow 0}\mathbb{E}[\varphi(z_\epsilon)\,\langle\nabla\log\gamma_\epsilon(z_\epsilon),h\rangle]\;=\;\kappa(\delta)\;\mathbb{E}\left[\varphi(x_\delta)\,\langle h,\frac{1}{(x_\delta)^3}\rangle\right].\tag{2.43}$$

Set:

$$\zeta_\epsilon\;:=\;\frac{1}{(z_\epsilon)^2(\epsilon+z_\epsilon)}\;+\;\frac{1}{z_\epsilon(\epsilon+z_\epsilon)^2}\;\geq\;\frac{2}{(\epsilon+z_\epsilon)^3}.\tag{2.44}$$

Then (2.43) is implied by (2.45)-(2.46)-(2.47):

$$\lim_{\epsilon\downarrow 0}\mathbb{E}\left[\varphi(z_\epsilon)\;\langle h,\zeta_\epsilon\rangle\right]\;=\;\mathbb{E}\left[\varphi(x_\delta)\langle h,\frac{2}{(x_\delta)^3}\rangle\right]\tag{2.45}$$

$$\lim_{\epsilon\downarrow 0}\mathbb{E}\left[\varphi(z_\epsilon)\;\langle h,\frac{c(\delta)-1}{(\epsilon+z_\epsilon)^3}\rangle\right]\;=\;\mathbb{E}\left[\varphi(x_\delta)\langle h,\frac{c(\delta)-1}{(x_\delta)^3}\rangle\right]\tag{2.46}$$

$$\lim_{\epsilon\downarrow 0}\mathbb{E}\left[\varphi(z_\epsilon)\;\langle h,\frac{\epsilon}{(1-\tau)(\epsilon+z_\epsilon)^2}\rangle\right]\;=\;0.\tag{2.47}$$

First, (2.46) and (2.47) follow by (E), (F) and Dominated Convergence Theorem: indeed $\epsilon + z_\epsilon \geq x_\delta$, and since $\delta > 3$:

$$\int \langle h, x^{-3} \rangle \, \pi_\delta(dx) \;=\; \int_0^1 d\tau \, h(\tau) \int x^{-3}(\tau) \, \pi_\delta(dx)$$

$$= \int_0^1 d\tau \, h(\tau) \int_0^\infty dy \, \frac{1}{y^3} \, \frac{C_\delta \, y^{\delta-1}}{[\tau(1-\tau)]^{\delta/2}} \exp\left\{ -\frac{y^2}{2\tau(1-\tau)} \right\} < \infty,$$

since h has compact support in $(0,1)$. On the other hand, (2.45) is natural but not immediate: indeed, the map

$$]0, \infty[\, \ni \epsilon \mapsto \frac{1}{(z_\epsilon)^2(\epsilon + z_\epsilon)} + \frac{1}{z_\epsilon(\epsilon + z_\epsilon)^2}$$

is not monotone a priori; moreover the easy estimate:

$$\frac{1}{(z_\epsilon)^2(\epsilon + z_\epsilon)} + \frac{1}{z_\epsilon(\epsilon + z_\epsilon)^2} \leq \frac{2}{(x_3)^3},$$

is not useful for a Dominated Convergence argument, since $\mathbb{E}[\langle h, (x_3)^{-3} \rangle] = +\infty$ if $h \geq 0$ and h is not equal to 0 a.e. In order to prove (2.45), we have to proceed in a different way. By (2.46):

$$\lim_{\epsilon \downarrow 0} \mathbb{E}\left[\varphi(z_\epsilon) \; \langle h, \frac{2}{(\epsilon + z_\epsilon)^3} \rangle \right] \;=\; \mathbb{E}\left[\varphi(x_\delta) \langle h, \frac{2}{(x_\delta)^3} \rangle \right]. \tag{2.48}$$

Now by (2.27) and (2.44) we have:

$$\begin{aligned}
&\left| \mathbb{E}\left[\varphi(z_\epsilon) \langle h, \zeta_\epsilon \rangle \right] \; - \; \mathbb{E}\left[\varphi(z_\epsilon) \; \langle h, \frac{2}{(\epsilon + z_\epsilon)^3} \rangle \right] \right| \\
&\leq \|\varphi\|_\infty \; \mathbb{E}\left[\left| \langle h, \zeta_\epsilon - \frac{2}{(\epsilon + z_\epsilon)^3} \rangle \right| \right] \\
&\leq \|\varphi\|_\infty \; \left(\mathbb{E}\left[\langle h, \zeta_\epsilon \rangle \right] - \mathbb{E}\left[\langle h, \frac{2}{(\epsilon + z_\epsilon)^3} \rangle \right] \right)
\end{aligned} \tag{2.49}$$

By Itô's formula we find:

$$0 \;=\; \frac{h}{\epsilon + z_\epsilon} \Bigg|_{\tau=0}^{\tau=1} \;=\; \int_0^1 d\left(\frac{h}{\epsilon + z_\epsilon} \right) \;=\; \int_0^1 \frac{h'}{\epsilon + z_\epsilon} \, d\tau$$

$$+ \int_0^1 \left[- \frac{h}{(\epsilon + z_\epsilon)^2} \left[\left(\frac{1}{z_\epsilon} - \frac{z_\epsilon}{1-\tau} + \frac{c(\delta)}{\epsilon + z_\epsilon} \right) d\tau + dB \right] + \frac{h}{(\epsilon + z_\epsilon)^3} \, d\tau \right]$$

so that we can compute:

$$\begin{aligned}\mathbb{E}\left[\int_0^1 \frac{h}{z_\epsilon(\epsilon+z_\epsilon)^2}\right] &= \mathbb{E}\left[\int_0^1 \frac{h'}{\epsilon+z_\epsilon} + \frac{h\,z_\epsilon}{(1-\tau)(\epsilon+z_\epsilon)^2} + \frac{(1-c(\delta))h}{(\epsilon+z_\epsilon)^3}\right] \\ &\to \mathbb{E}\left[\int_0^1 \frac{h'}{x_\delta} + \frac{h}{(1-\tau)x_\delta} + (1-c(\delta))\,\frac{h}{(x_\delta)^3}\right] \qquad (2.50)\end{aligned}$$

as $\epsilon \downarrow 0$, where the convergence of each term in the right hand side of the first line of (2.50) can be justified by either Dominated or Monotone convergence. Analogously, we obtain for all $\alpha > 0$:

$$\begin{aligned}\left.\frac{h}{\alpha+z_\epsilon}\right|_{\tau=0}^{\tau=1} = 0 &= \int_0^1 \left(\frac{h'}{\alpha+z_\epsilon} + \frac{h}{(\alpha+z_\epsilon)^3}\right) d\tau \\ &\quad - \int_0^1 \frac{h}{(\alpha+z_\epsilon)^2}\left[\left(\frac{1}{z_\epsilon} - \frac{z_\epsilon}{1-\tau} + \frac{c(\delta)}{\epsilon+z_\epsilon}\right) d\tau + dB\right],\end{aligned}$$

$$\begin{aligned}&\mathbb{E}\left[\int_0^1 \frac{h}{(\alpha+z_\epsilon)^2(\epsilon+z_\epsilon)}\right] \\ &= \frac{1}{c(\delta)}\,\mathbb{E}\left[\int_0^1 \frac{h'}{\alpha+z_\epsilon} + \frac{h}{(1-\tau)z_\epsilon} + \frac{h}{(\alpha+z_\epsilon)^3} - \frac{h}{z_\epsilon(\alpha+z_\epsilon)^2}\right] \\ &\le \frac{1}{c(\delta)}\,\mathbb{E}\left[\int_0^1 \frac{h'}{\alpha+z_\epsilon} + \frac{h}{(1-\tau)z_\epsilon}\right],\end{aligned}$$

$$\begin{aligned}\left.\frac{h}{\alpha+x_\delta}\right|_{\tau=0}^{\tau=1} = 0 &= \int_0^1 \left(\frac{h'}{\alpha+x_\delta} + \frac{h}{(\alpha+x_\delta)^3}\right) d\tau \\ &\quad - \int_0^1 \left[\frac{h}{(\alpha+x_\delta)^2}\left(\frac{c(\delta)+1}{x_\delta} - \frac{x_\delta}{(1-\tau)}\right) d\tau \;+\; dB\right].\end{aligned}$$

$$\mathbb{E}\left[\int_0^1 \left(\frac{(c(\delta)+1)h}{x_\delta(\alpha+x_\delta)^2} - \frac{h}{(\alpha+x_\delta)^3}\right)\right] = \mathbb{E}\left[\int_0^1 \frac{h'}{\alpha+x_\delta} + \frac{h\,x_\delta}{(1-\tau)(\alpha+x_\delta)^2}\right].$$

Letting first $\alpha \downarrow 0$ and then $\epsilon \downarrow 0$ we obtain:

$$\limsup_{\epsilon\downarrow 0} \mathbb{E}\left[\int_0^1 \frac{h}{z_\epsilon^2(\epsilon+z_\epsilon)}\right] \le \frac{1}{c(\delta)}\mathbb{E}\left[\int_0^1 \frac{h'}{x_\delta} + \frac{h}{(1-\tau)x_\delta}\right] \qquad (2.51)$$

$$\mathbb{E}\left[\int_0^1 \frac{h}{(x_\delta)^3}\right] = \frac{1}{c(\delta)}\,\mathbb{E}\left[\int_0^1 \frac{h'}{x_\delta} + \frac{h}{(1-\tau)x_\delta}\right], \qquad (2.52)$$

so that by (2.50), (2.51) and (2.52):

$$0 \,\le\, \limsup_{\epsilon\downarrow 0} \mathbb{E}\left[\langle h, \zeta_\epsilon\rangle \,-\, \langle h, \frac{2}{(\epsilon+z_\epsilon)^3}\rangle\right] \,\le\, 0$$

and by (2.48) and (2.49), (2.45) is proved.

Step 7. We turn to the last term in (2.42). Notice that

$$\int \varphi(x)\,\gamma_\epsilon(x)\sigma_0(r,dx) \,=$$

$$= \frac{1}{\sqrt{2\pi r^3(1-r)^3}} \int \varphi\,(w \oplus_r z)\;\gamma_\epsilon^r(w)\,\gamma_\epsilon^{1-r}(z)\mathcal{Z}_\epsilon^r(w)\,\pi_3^r(dw)\otimes\pi_3^{1-r}(dz),$$

where $\gamma_\epsilon^r, \mathcal{Z}_\epsilon^r \in L^1(\pi_3^r)$ are defined by:

$$\gamma_\epsilon^r(w) \,:=\, \exp\left(c(\delta)\int_0^r \left(\frac{1-c(\delta)}{2(\epsilon+w)^2} - \frac{1}{w(\epsilon+w)} + \frac{1}{r-\tau}\frac{w}{\epsilon+w}\right)\,d\tau\right),$$

$$\mathcal{Z}_\epsilon^r(w) \,:=\, \exp\left(c(\delta)\int_0^r \left(\frac{1}{1-\tau} - \frac{1}{r-\tau}\right)\frac{w}{\epsilon+w}\,d\tau\right) \,\le\, \exp\left(\frac{c(\delta)}{1-r}\right).$$

Arguing as in steps 3-4, by (2.28) $\gamma_\epsilon^r\,d\pi_3^r$ is the law of w_ϵ^r, where:

$$\begin{cases} d\,w_\epsilon^r = \dfrac{1}{w_\epsilon^r}\,d\tau \,+\, \dfrac{\delta-3}{2(\epsilon+w_\epsilon^r)}\,d\tau \,-\, \dfrac{w_\epsilon^r}{r-\tau}\,d\tau \,+\, dB, \quad \tau\in\,]0,r[\\ \\ w_\epsilon^r(0) = 0 \end{cases}$$

and $w_\epsilon^r \uparrow x_\delta^r$ as $\epsilon\downarrow 0$. Moreover, since a.s. $w_1^r(\tau) > 0$ for all $\tau\in(0,1)$, we have $\mathcal{Z}_\epsilon^r(w_\epsilon^r) \to 0$ a.s. as $\epsilon\downarrow 0$ and by Dominated Convergence Theorem we obtain

$$\lim_{\epsilon\downarrow 0} \int_0^1 dr\; h(r) \int \varphi(x)\,\gamma_\epsilon(x)\,\sigma_0(r,dx)$$

$$= \lim_{\epsilon\downarrow 0} \int_0^1 dr\, \frac{h(r)}{\sqrt{2\pi r^3(1-r)^3}}\, \mathbb{E}\left[\varphi(w_\epsilon^r \oplus_r w_\epsilon^{1-r})\;\mathcal{Z}_\epsilon^r(w_\epsilon^r)\right] \,=\, 0.$$

Chapter 3

SPDEs with reflection

Consider the solution (u, η), where u is a continuous function of $(t, \xi) \in \mathcal{O} := [0, +\infty) \times [0, 1]$ and η a measure on $\mathcal{O}$, of the Nualart-Pardoux equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \xi^2} - f(\xi, u(t,\xi)) + \dfrac{\partial^2 W}{\partial t \partial \xi} + \eta(t, \xi) \\ \\ u(0, \xi) = x(\xi), \ u(t, 0) = u(t, 1) = 0 \\ \\ u \geq 0, \ d\eta \geq 0 \ \int_{\mathcal{O}} u \, d\eta = 0. \end{cases} \tag{3.1}$$

where $x : [0, 1] \mapsto [0, \infty)$ is continuous with $x(0) = x(1) = 0$, $\{W(t, \xi) : (t, \xi) \in \mathcal{O}\}$ is a Brownian sheet and $f : [0, 1] \times \mathbb{R} \mapsto \mathbb{R}$. The aim of this chapter is to prove the following decomposition theorem for η:

$$\eta(ds, d\xi) \ = \ \delta_{r(s)}(d\xi) \, \eta(ds, (0, 1)), \tag{3.2}$$

where $r : S \mapsto (0, 1)$, $\eta((\mathbb{R}^+ \backslash S) \times (0, 1)) = 0$ and $r(s)$ is the unique $\xi \in (0, 1)$ such that $u(s, \xi) = 0$. Moreover, for all Borel subset of $(0, 1)$, we characterize $t \mapsto \eta([0, t] \times I)$ as an Additive Functional of u with a Revuz-measure explicitly written in terms of the boundary measure σ_0 of (2.2). Recalling the introduction of Chapter 2, we can write (3.1) as a Skorokhod problem:

$$du \ = \ \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2} \, dt \ - \ f(u) \, dt \ + \ dW \ + \ \frac{1}{2} \, n(u) \cdot dL, \tag{3.3}$$

where $n(x) = \delta_r$ for $x \in \partial_r^* K_0$ is the inward-pointing normal vector to the boundary, and $L_t := 2 \, \eta([0, t] \times (0, 1))$ is the Additive Functional associated with the boundary measure $\int_0^1 dr \, \sigma_0(r, \cdot)$.

The results of this chapter establish a precise connection between the Nualart-Pardoux equation (3.1) and the theory of stochastic differential inclusions: recall that, in [DP 00], Da Prato considers a convex, lower semi-continuous $U : H \mapsto \mathbb{R} \cup \{+\infty\}$, satisfying suitable integrability conditions with respect to μ and:

$$\mu(x : \ U(x) < +\infty \ \text{ and } \ \partial U(x) \neq \emptyset) = 1$$

where $\partial U(x)$, the subdifferential of U at x, is defined as the subset of H:

$$\partial U(x) := \{y \in H : \ U(x+h) \geq U(x) + \langle h, y \rangle, \ \ \forall h \in H\}.$$

For all $x \in H$, $\partial U(x)$ is a closed convex subset of H: therefore, if $\partial U(x) \neq \emptyset$, then there exists an element of minimal norm $\partial_0 U(x) \in \partial U(x)$. The main result of [DP 00] is the existence of a symmetric semigroup $(P_t)_{t \geq 0}$ on $L^2\left(H \, , \, (\int_H e^{-2U} d\mu)^{-1} e^{-2U} d\mu \right)$, associated with the stochastic differential inclusion:

$$dX \ \in \ \left(\frac{1}{2} \frac{\partial^2 X}{\partial \xi^2} - \partial U(X) \right) \, dt \, + \, \frac{\partial^2 W}{\partial t \partial \xi}, \qquad X(0) = x \in H. \tag{3.4}$$

This result is obtained, proving that the infinite-dimensional elliptic operator on $\text{Exp}_A(H)$:

$$\frac{1}{2} \text{Tr}[D^2 \varphi(x)] + \langle x, A \nabla \varphi(x) \rangle - \langle \partial_0 U(x), \nabla \varphi(x) \rangle, \quad x \in H,$$

is essentially self-adjoint in $L^2\left(H, (\int_H e^{-2U} d\mu)^{-1} e^{-2U} d\mu\right)$. Using the Theory of Dirichlet Forms, one obtains the existence of a weak solution of the equation:

$$dX \ = \ \left(\frac{1}{2} \frac{\partial^2 X}{\partial \xi^2} - \partial_0 U(X) \right) \, dt \, + \, \frac{\partial^2 W}{\partial t \partial \xi}, \qquad X(0) = x \in H, \tag{3.5}$$

and therefore of the differential inclusion (3.4), since $\partial_0 U(X) \in \partial U(X)$. The technique is based on the study of the approximating operator, $\varepsilon > 0$:

$$\frac{1}{2} \text{Tr}[D^2 \varphi(x)] + \langle x, A \nabla \varphi(x) \rangle - \langle \nabla U_\varepsilon(x), \nabla \varphi(x) \rangle, \quad x \in H,$$

defined on $\text{Exp}_A(H)$, associated with the SPDE:

$$dX^\varepsilon \ = \ \left(\frac{1}{2} \frac{\partial^2 X^\varepsilon}{\partial \xi^2} \ - \ \nabla U_\varepsilon(X^\varepsilon) \right) \, dt \, + \, \frac{\partial^2 W}{\partial t \partial \xi}, \qquad X^\varepsilon(0) = x \in H, \tag{3.6}$$

where U_ε are the Yosida-approximations of U:

$$U_\varepsilon(x) := \inf_{y\in H}\left(U(y) + \frac{|x-y|^2}{2\varepsilon}\right), \quad x \in H,\ \varepsilon > 0,$$

and the following properties of U_ε are used (see [DP 76]):

$$\text{as } \varepsilon \downarrow 0: \quad U_\varepsilon \to U, \quad \nabla U_\varepsilon \to \partial_0 U, \quad |\nabla U_\varepsilon| \le |\partial_0 U|.$$

Recall now that the solution u of (1.1) is constructed in [NP 92] by means of a penalization technique, i.e. the following approximating problem is introduced for all $\varepsilon > 0$:

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u^\varepsilon}{\partial \xi^2} - f(\xi, u^\varepsilon(t,\xi)) + \dfrac{\partial^2 W}{\partial t \partial \xi} + \dfrac{(u^\varepsilon)^-}{\varepsilon} \\ \\ u^\varepsilon(0,\xi) = x(\xi),\ u^\varepsilon(t,0) = u^\varepsilon(t,1) = 0. \end{cases} \tag{3.7}$$

Then for all $t > 0$, $\xi \in [0,1]$, $\varepsilon \mapsto u^\varepsilon(t,\xi)$ is monotone non-decreasing, and $u := \lim_{\varepsilon\downarrow 0} u^\varepsilon$ is continuous. Moreover, the measure $\eta^\varepsilon(dt, d\xi) := (u^\varepsilon)^- \, dt\, d\xi$ converges to a positive measure η on $\mathbb{R}^+ \times (0,1)$ as $\varepsilon \downarrow 0$, and (u,η) solves (1.1). Equation (3.7) can be written in the following form:

$$\begin{cases} du^\varepsilon = \left(\dfrac{1}{2}\dfrac{\partial^2 u^\varepsilon}{\partial \xi^2} - \nabla F(u^\varepsilon) - \nabla U_\varepsilon(u^\varepsilon)\right) dt \ + \ \dfrac{\partial^2 W}{\partial t \partial \xi} \\ \\ u^\varepsilon(0,x) \ = \ x \in L^2(0,1) \end{cases} \tag{3.8}$$

$$\text{where} \quad F, U_\varepsilon : H \mapsto \mathbb{R}, \quad F(x) := \int_0^1 d\xi \int_0^{x(\xi)} f(\xi, s)\, ds,$$

$$U_\varepsilon(x) := \frac{1}{2\varepsilon}\int_0^1 \left[(x(\xi))^-\right]^2 d\xi = \frac{1}{2\varepsilon}\left[d(x, K_0)\right]^2,$$

and U_ε are the Yosida-approximations of:

$$U(x) \ := \begin{cases} 0 & \text{if} \quad x \ge 0 \\ \\ +\infty & \text{otherwise} \end{cases} \tag{3.9}$$

Therefore, the approximating problem (3.7) of the Nualart-Pardoux equation, is an example of (3.6), apart from the trivial drift term ∇F. However, since $\mu(K_0) = 0$, in the case of (3.7) $\mu(U = +\infty) = 1$ and the measure "$(\int_H e^{-2F-2U} d\mu)^{-1} \exp(-2F - 2U)\, d\mu$" is not well defined.

Our first result is that a natural Gibbs-type measure for (3.1) is provided by:

$$\frac{1}{Z} \exp\left\{-2 \int_0^1 d\xi \int_0^{x(\xi)} f(\xi, s)\, ds\right\} \nu(dx), \qquad x \in C_0(0,1), \tag{3.10}$$

where Z is a normalization constant. Notice that ν is not Gaussian and is even singular with respect to the reference Gaussian measure μ. Moreover, the support of ν is a closed convex set, having empty interior both in the topologies of $L^2(0,1)$ and $C([0,1])$.

This result, together with a Strong-Feller property satisfied by u in H, implies that the law of $u(t,\xi)$, for all $t > 0$ and $\xi \in (0,1)$, is absolutely continuous w.r.t. $y^2\, dy$ on $[0,\infty)$: see Section 3.4.

Notice that, in general, it is a difficult task, to prove that the solution X^ε of (3.6) converges as $\varepsilon \downarrow 0$, and in fact this is not the content of [DP 00]. The monotonicity of $\varepsilon \mapsto u^\varepsilon(t,\xi)$, where u^ε is the solution of (3.7), is one of the key tools in [NP 92], and strongly depends on the specific form of the nonlinearity $\frac{1}{\varepsilon}x^-$. However, as already mentioned, in the case of (3.7), $\mu(U < \infty) = 0$, and this makes most of the techniques of [DP 00] non-effective in this situation.

The key tool, which allows in this thesis to combine the probabilistic approach of [NP 92] and the analytic approach of [DP 00], is the integration by parts formula on ν proved in the previous chapter:

$$\int_{K_0} \partial_h \varphi \, d\nu = -\int_{K_0} \varphi(x)\, \langle x, h'' \rangle \, d\nu - \int_0^1 dr\ h(r) \int \varphi(x)\, \sigma_0(r, dx). \tag{3.11}$$

Indeed, we can prove that: $x \mapsto u(t,\cdot)$, $t \geq 0$ is the Markov process associated with the Dirichlet Form:

$$\mathcal{E}^0(\varphi, \psi) := \frac{1}{2\nu(e^{-2F})} \int_{K_0} \langle \nabla\varphi, \nabla\psi \rangle\, e^{-2F}\, d\nu,$$

and for all interval I in $(0,1)$, the process $t \mapsto \eta([0,t] \times I)$ is an Additive Functional of u, with Revuz measure:

$$\frac{1}{2\nu(e^{-2F})} \int_I dr\, e^{-2F(x)}\, \sigma_0(r, dx). \tag{3.12}$$

Moreover, the specific form of (3.12) and the Theory of Additive Functionals allow to prove the decomposition of η, (3.2). On one hand, the use of the explicit knowledge of the boundary measure σ_0 seems to be crucial. On the other hand, our proof of (3.2) is also essentially based on the Theory of Additive Functionals and in particular on the one-to-one correspondence between Additive Functionals and $\mathcal{E}^0$-smooth measures: see Section 3.6. Since the state space K_0 of $x \mapsto u$ is not locally compact, the theory of quasi-regular symmetric Dirichlet Forms is needed, in order to apply the theory of Additive Functionals. In particular, quasi-regularity of $\mathcal{E}^0$ has to be proved. Recall that in Theorems IV.3.5 and IV.5.1 of [MR 92], Ma and Röckner prove that quasi-regularity is a necessary and sufficient condition for a Dirichlet Form to be properly associated with a nice Markov process. In the present situation, the process is given by the Nualart-Pardoux solution of (3.1): since we prove that u is properly associated with $\mathcal{E}^0$, then quasi-regularity of $\mathcal{E}^0$ follows by the necessity-part of Ma-Röckner's Theorem.

By (3.3), equation (3.1) can be interpreted as an example of (3.4), provided we define for all $x \in K$, $\tilde{\partial} U(x)$ as the subset of the dual space M of $C_0(0,1)$, $M := \{\text{signed measures on } (0,1)\}$,

$$\begin{aligned} \tilde{\partial} U(x) &:= \{m \in M : U(x+z) \geq U(x) + \langle z, m\rangle, \, \forall z \in C_0(0,1)\} \\ &= -\left\{ m \in M : m \geq 0, \, \int_{[0,1]} x(\xi)\, m(d\xi) \, = \, 0 \right\}. \end{aligned}$$

Then (3.1) can be written formally as a differential inclusion of measures on $(0,1)$:

$$\left(d_t u \, - \, \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2}\, dt \, + \, f(\xi, u(t,\xi))\, dt \right) d\xi \, - \, W(dt, d\xi) \, \in \, - \, \frac{1}{2}\, \tilde{\partial} U(u(t, \, \cdot\,)) \cdot dL, \tag{3.13}$$

with $L_t = 2\,\eta([0,t] \times (0,1))$. The appearance of dL instead of the classical dt in (3.13) is not surprising, since also in finite-dimensions the local time at the boundary of a reflecting BM is not absolutely continuous w.r.t. the Lebesgue measure. On the other hand, the fact that we have $\tilde{\partial} U$ instead of ∂U is peculiar of this infinite-dimensional model, and is in particular related to the fact, mentioned above, that the normal vector to the boundary $\partial^* K_0$ of K_0 is a measure on $(0,1)$, but not an element of $H = L^2(0,1)$.

To our knowledge, it is still unknown, whether $\eta([0,t] \times (0,1))$ is finite or infinite for $t > 0$. Using the theory of Additive Functionals, we can say that a.s. $\eta([0,t]\times(0,1))<\infty$ for all $t\geq 0$, if and only if the measure $\int_0^1 dr\,\sigma_0(r,\cdot)$ is $\mathcal{E}^0$-smooth: see Definition VI.2.3 in [MR 92] and also [Fu 99]. However, we have not been able yet to prove the $\mathcal{E}^0$-smoothness of $\int_0^1 dr\,\sigma_0(r,\cdot)$. On the other hand, in [NP 92] Nualart and Pardoux prove the estimate:

$$\int_0^T\int_0^1 \xi(1-\xi)\,\eta(dt,d\xi)\;<\;\infty,\qquad \forall T\geq 0.$$

Recall the definition (2.5) of σ_0, and in particular the factor $(r(1-r))^{-3/2}$, $r\in(0,1)$: this allows to improve the Nualart-Pardoux estimate, and obtain for all $\rho:(0,1)\mapsto\mathbb{R}^+$:

$$\int_0^1 \frac{\rho(\xi)}{[\xi(1-\xi)]^{3/2}}\,d\xi\;<\;\infty \Longrightarrow \int_0^T\int_0^1 \rho(\xi)\,\eta(dt,d\xi)\;<\;\infty,\qquad \forall T\geq 0.$$

Recently, Funaki and Olla in [FO 01] and Otobe in [Ot 01], have applied the Nualart-Pardoux equations to the study of fluctuations of an interface on a hard wall. We recall the setting of Funaki and Olla: if $\Gamma_N := \{1/N,\, 2/N,\,\dots\,,(N-1)/N\}$, and $w_t^N := (w_t^N(\gamma))_{\gamma\in\Gamma_N}$, $t\geq 0$, is a $\mathbb{R}^{N-1}$-valued Brownian Motion, then $\phi_t^N:\Gamma_N\mapsto\mathbb{R}^+$ $t\geq 0$, is defined as the unique stationary solution of the following system of SDEs with reflection:

$$\begin{aligned} d\phi_t^N(\gamma) \;&=\; -\left(V'\left(\phi_t^N(\gamma)-\phi_t^N(\gamma-1/N)\right)+V'\left(\phi_t^N(\gamma)-\phi_t^N(\gamma+1/N)\right)\right)dt \\ &\qquad +\sqrt{2}\,dw_t^N(\gamma)\;+\;dl_t^N(\gamma),\quad \gamma\in\Gamma_N \end{aligned} \tag{3.14}$$

with $\phi_t^N(0)=\phi_t^N(1):=0$, subject to the conditions:

(i) $\phi_t^N(\gamma)\geq 0$,

(ii) $t\mapsto l_t(\gamma)$ is non-decreasing,

(iii) $\displaystyle\int_0^\infty \phi_t^N(\gamma)\,dl_t^N(\gamma)\;=\;0,$

for every $\gamma\in\Gamma_N$. Moreover, in equation (3.14) the potential $V\in C^2(\mathbb{R})$ is pair and strictly convex: $0<c_-\leq V''\leq c_+<\infty$. Then, setting:

$$\Phi^N(t,\xi)\;:=\;\frac{1}{\sqrt{N}}\sum_{\gamma\in\Gamma_N}\phi_{N^2t}^N(\gamma)\,1_{[\gamma-\frac{1}{2N},\gamma+\frac{1}{2N})}(\xi),\quad \xi\in[0,1],$$

Funaki and Olla prove that Φ^N converges in law, as $N \to \infty$, to the unique stationary solution Φ of the Nualart-Pardoux equation:

$$\begin{cases} \dfrac{\partial \Phi}{\partial t} = q \dfrac{\partial^2 \Phi}{\partial \xi^2} + \sqrt{2}\, \dfrac{\partial^2 W}{\partial t \partial \xi} + \theta(t, \xi) \\ \\ \Phi(t,0) = \Phi(t,1) = 0 \\ \\ \Phi \geq 0, \ d\theta \geq 0 \ \int_{\mathcal{O}} \Phi \, d\theta = 0, \end{cases} \tag{3.15}$$

$$\text{where :} \quad q := \left(\int_{\mathbb{R}} e^{-V(y)} \, dy \right) \cdot \left(\int_{\mathbb{R}} y^2 \, e^{-V(y)} \, dy \right)^{-1}.$$

Our results, and in particular the decomposition (3.2), hold also for the measure θ of (3.15): therefore, we hope that the results of this thesis find applications also in the study of such problems.

In [Fu 00], M. Fukushima has recently given a theory of stochastic equations in domains with reflecting boundary on an Abstract Wiener Space. The results presented here are in the same spirit. However, the Abstract Wiener Space setting allows to study Hilbert space-valued SDEs, but not SPDEs. Moreover, the main results of this chapter are based on the explicit knowledge of the invariant measure ν and of the boundary measure σ_0, rather than on general arguments.

3.1 Setting of the problem

In this section we recall the definition given by Nualart and Pardoux in [NP 92] of a solution of the following SPDE with reflection at $-\alpha \leq 0$:

$$\begin{cases} \dfrac{\partial u_\alpha}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 u_\alpha}{\partial \xi^2} - f(\xi, u_\alpha(t,\xi)) + \dfrac{\partial^2 W}{\partial t \partial \xi} + \eta_\alpha(t, \xi) \\ \\ u_\alpha(0,\xi) = x(\xi), \ u_\alpha(t,0) = u_\alpha(t,1) = 0 \\ \\ u_\alpha + \alpha \geq 0, \ d\eta_\alpha \geq 0 \ \int_{\mathcal{O}} (u_\alpha + \alpha) \, d\eta_\alpha = 0. \end{cases} \tag{3.16}$$

where $\alpha \geq 0$, $x : [0,1] \mapsto [-\alpha, +\infty)$ is continuous and $x(0) = x(1) = 0$.

We assume in the following that:

(H1) $f : [0,1] \times \mathbb{R} \mapsto \mathbb{R}$ is jointly measurable.

(H2) $f(\xi, \cdot)$ is continuously differentiable for all $\xi \in [0,1]$, and there exists $c > 0$ such that

$$|f| \le c, \quad |\partial_y f(\xi, y)| \le c, \ \forall \xi \in [0,1], \ y \in \mathbb{R}.$$

(H3) There exists $C \ge 0$ such that for all $\xi \in [0,1]$:

$$\left| \int_0^t f(\xi, u)\, du \right| \le C, \qquad \forall t \ge 0.$$

Hypothesis (H1)-(H3) do not aim at the greatest generality. We have in mind mainly the following example:

$$f(\xi, y) = -a(\xi) \frac{1}{(\epsilon + y)^b}, \ y \ge 0$$

with a bounded and non-negative, $\epsilon > 0$ and $b \ge 0$, and $f(\cdot, \cdot)$ defined on $(0,1) \times (-\infty, 0)$ in any way which fits (H1)-(H3). This example will be of interest in section 4.

Following [NP 92], we set:

Definition 3.1 *A pair (u_α, η_α) is said to be a solution of equation (3.16) with reflection in $-\alpha \le 0$ and initial value $x \in K_\alpha \cap C_0(0,1)$, if:*

(i) $u_\alpha : \mathcal{O} \mapsto [-\alpha, \infty)$, $\{u_\alpha(t,\xi) : (t,\xi) \in \mathcal{O}\}$ *is a continuous and adapted process, i.e.* $u_\alpha(t,\xi)$ *is* $\mathcal{F}_t$*-measurable for all* $(t,\xi) \in \mathcal{O}$*, and a.s.* $u_\alpha(\cdot,\cdot)$ *is continuous on* $\mathcal{O}$*,* $u_\alpha(t,\cdot) \in K_\alpha \cap C_0(0,1)$ *for all* $t \ge 0$*, and* $u_\alpha(0,\cdot) = x$*.*

(ii) η_α *is a random positive measure on* $\mathcal{O}$ *such that* $\eta_\alpha([0,T] \times [\delta, 1-\delta]) < +\infty$ *for all* $T, \delta > 0$*, and* η_α *is adapted, i.e.* $\eta_\alpha(B)$ *is* $\mathcal{F}_t$*-measurable for every Borel set* $B \subset [0,t] \times [0,1]$*.*

(iii) For all $t \ge 0$ *and* $\varphi \in C_c^\infty(0,1)$*:*

$$\langle u_\alpha(t,\cdot), \varphi\rangle - \int_0^t \langle u_\alpha(s,\cdot), A\varphi\rangle ds + \int_0^t \langle f(\cdot, u_\alpha(s,\cdot)), \varphi\rangle ds =$$

$$= \langle x, \varphi\rangle + \int_0^t \int_0^1 \varphi(\xi)\, dW_{s,\xi} + \int_0^t \int_0^1 \varphi(\xi)\, \eta_\alpha(dt, d\xi).$$

(iv) $\int_{\mathcal{O}} (u_\alpha + \alpha) d\eta_\alpha = 0.$

3.2 The approximating problem

In the proof of existence of solutions of equation (3.1), the following approximating problem is introduced:

$$\begin{cases} \dfrac{\partial u_\alpha^\varepsilon}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u_\alpha^\varepsilon}{\partial \xi^2} - f(\cdot, u_\alpha^\varepsilon(t,\cdot)) + \dfrac{\partial^2 W}{\partial t \partial \xi} + \dfrac{(\alpha + u_\alpha^\varepsilon)^-}{\varepsilon} \\ \\ u_\alpha^\varepsilon(0,\cdot) = x \in H, \quad u_\alpha^\varepsilon(t,0) = u_\alpha^\varepsilon(t,1) = 0, \ \forall t \geq 0. \end{cases} \tag{3.17}$$

with $\varepsilon > 0$, $(r)^- := \sup\{-r, 0\}$ and $\alpha \geq 0$. This is a SPDE in $L^2(0,1)$ with additive noise and monotone or Lipschitz-continuous drift terms, for which existence and uniqueness of a solution are well known: see e.g. [DPZ 92]. We write:

$$X_\alpha^\varepsilon(t,x) \ := \ u_\alpha^\varepsilon(t,\cdot) \ \in \ H \quad t \geq 0, \ x \in H.$$

Let $\alpha, \varepsilon > 0$, and set:

$$F : H \mapsto \mathbb{R}, \quad F(x) \ := \int_0^1 d\xi \int_0^{x(\xi)} f(\xi, s)\, ds. \tag{3.18}$$

By (H1)-(H3), $F \in C_b^1(H)$ and $\nabla F(x) = f(\cdot, x(\cdot))$ for all $x \in H$. Set also:

$$\mu_\alpha^\varepsilon(dx) \ := \ \frac{1}{Z_\alpha^\varepsilon} \exp\left(-\, 2F(x) \ - \ \frac{\|(x+\alpha)^-\|^2}{\varepsilon} \right) \ \mu(dx), \quad x \in H,$$

$$E^{\alpha,\varepsilon}(\varphi, \psi) \ := \ \frac{1}{2} \int_H \langle \nabla\varphi, \nabla\psi \rangle \, d\mu_\alpha^\varepsilon, \qquad \forall \varphi, \psi \in C_b^1(H),$$

$$L\varphi(x) \ := \ M\varphi(x) \ - \ \langle \nabla F(x), \nabla\varphi(x) \rangle, \quad \varphi \in \mathrm{Exp}_A(H),$$

$$L_\alpha^\varepsilon \varphi(x) \ := \ L\varphi(x) \ + \ \frac{1}{\varepsilon}\langle (x+\alpha)^-, \nabla\varphi(x) \rangle, \quad \varphi \in \mathrm{Exp}_A(H),$$

where Z_α^ε is a normalization constant such that $\mu_\alpha^\varepsilon(H) = 1$ and M is the Ornstein-Uhlenbeck operator defined in (2.22). Finally, set for all $\varphi \in C_b(H)$:

$$R_\alpha^\varepsilon(\lambda)\varphi(x) := \int_0^\infty e^{-\lambda t}\, \mathbb{E}\left[\varphi(X_\alpha^\varepsilon(t,x))\right] dt, \qquad x \in H, \ \lambda > 0.$$

Recall Proposition 1.1. Then we have the following:

Theorem 3.1

1. *$(E^{\alpha,\varepsilon}, \mathrm{Exp}_A(H))$ is closable in $L^2(\mu_\alpha^\varepsilon)$: we denote by $(\mathcal{E}^{\alpha,\varepsilon}, D(\mathcal{E}^{\alpha,\varepsilon}))$ the closure. We have $W^{1,2}(H,\mu) \subseteq D(\mathcal{E}^{\alpha,\varepsilon})$ with continuous immersion.*

2. $(L^\varepsilon_\alpha, \mathrm{Exp}_A(H))$ *is essentially self-adjoint in* $L^2(\mu^\varepsilon_\alpha)$*: we denote the closure by* $(\mathcal{L}^\varepsilon_\alpha, D(\mathcal{L}^\varepsilon_\alpha))$*. We have that* $D(\mathcal{E}^{\alpha,\varepsilon}) = D((-\mathcal{L}^\varepsilon_\alpha)^{1/2})$ *and:*

$$\int_H \varphi\, \mathcal{L}^\varepsilon_\alpha \psi \, d\mu^\varepsilon_\alpha \;=\; -\,\mathcal{E}^{\alpha,\varepsilon}(\varphi,\psi), \quad \forall \varphi \in D(\mathcal{E}^{\alpha,\varepsilon}),\ \psi \in D(\mathcal{L}^\varepsilon_\alpha). \tag{3.19}$$

3. *The process* $(X^\varepsilon_\alpha(t,x))_{t\geq 0, x\in H}$ *is the diffusion generated by* $\mathcal{E}^{\alpha,\varepsilon}$*, i.e. for all* $\lambda > 0$ *and* $\varphi \in C_b(H)$*,* $R^\varepsilon_\alpha(\lambda)\varphi \in D(\mathcal{E}^{\alpha,\varepsilon})$ *and:*

$$\lambda \int_H R^\varepsilon_\alpha(\lambda)\varphi\, \psi \, d\mu^\varepsilon_\alpha + \mathcal{E}^{\alpha,\varepsilon}(R^\varepsilon_\alpha(\lambda)\varphi, \psi) \;=\; \int_H \varphi\, \psi\, d\mu^\varepsilon_\alpha, \quad \forall \psi \in D(\mathcal{L}^\varepsilon_\alpha).$$

4. μ^ε_α *is the unique invariant probability measure of* X^ε_α*. Moreover,* X^ε_α *is symmetric with respect to* μ^ε_α*.*

Proof–This result is well known. For the reader's convenience, we sketch the proof.

Let $\{\varphi_n\} \subset \mathrm{Exp}_A(H)$ such that $\|\varphi_n\|_{L^2(\mu^\varepsilon_\alpha)} \to 0$, as $n \to \infty$. Then, integrating by parts:

$$E^{\alpha,\varepsilon}(\varphi_n, \psi) \;=\; -\int_H \varphi_n\, L^\varepsilon_\alpha \psi\, d\mu^\varepsilon_\alpha \;\to 0 \quad \text{as } n \to \infty, \quad \forall \psi \in \mathrm{Exp}_A(H),$$

and this implies closability of $(E^{\alpha,\varepsilon}, \mathrm{Exp}_A(H))$ by Lemma I.3.4 in [MR 92].

Essential self-adjointness can be proven as in [DP 98] or [DP 00].

Applying Itô's formula to $\varphi(X^\varepsilon_\alpha(t,x))$ for $\varphi \in \mathrm{Exp}_A(H)$, we obtain by point (2.) that R^ε_α is the strongly continuous resolvent associated with $\mathcal{E}^{\alpha,\varepsilon}$.

From the previous points, we obtain that μ^ε_α is invariant for X^ε_α, and that X^ε_α is symmetric. Let m^1 and m^2 be two invariant probability measures for X_α, and let q^1 and q^2 be K_α-valued random variables, such that the law of q^i is m^i and $\{q^1, q^2, W\}$ is an independent family. Setting $b := \|X^\varepsilon_\alpha(t,q^1) - X^\varepsilon_\alpha(t,q^2)\|$ we have:

$$\frac{d}{dt} b^2 \;\leq\; -\pi^2 b^2 \;+\; cb \;\leq\; -\frac{\pi^2}{2} b^2 + C^2,$$

$$\|X^\varepsilon_\alpha(t,q^1) - X^\varepsilon_\alpha(t,q^2)\| \;\leq\; C\, e^{-\pi^2 t/4} \|q^1 - q^2\|, \quad \forall t \geq 0. \tag{3.20}$$

Since the law of $X^\varepsilon_\alpha(t,q^i)$ is equal to m^i for all $t \geq 0$, (3.20) implies $m^1 = m^2$.
□

3.3 The process X_α, $\alpha \geq 0$

In [NP 92], the following theorem is proved:

Theorem 3.2 *Assume that f satisfies (H1),(H2),(H3) and let $x \in K_\alpha \cap C_0(0,1)$. Then there exists a unique solution (u_α, η_α) of equation (3.16) with reflection in $-\alpha$ and initial value x.*

The existence statement is proved in the following way: let u_α^ε be the solution of (3.17). Then:

(a) $u_\alpha^\varepsilon(t,\cdot) \in C_0(0,1)$ for all $t \geq 0$, and u_α^ε is continuous on $\mathcal{O}$.

(b) By Lemma 1.3, the map $0 < \varepsilon \mapsto u_\alpha^\varepsilon(t,\xi)$ is non-increasing for all $(t,\xi) \in \mathcal{O}$. The limit $\lim_{\varepsilon\downarrow 0} u_\alpha^\varepsilon(t,\xi) = \sup_{\varepsilon>0} u_\alpha^\varepsilon(t,\xi) =: u_\alpha(t,\xi)$ is finite for all $(t,\xi) \in \mathcal{O}$, $u_\alpha(t,\cdot) \in K_\alpha \cap C_0(0,1)$ for all $t \geq 0$, and u_α is continuous on $\mathcal{O}$.

(c) The measure on $\mathcal{O}$, $\eta_\alpha^\varepsilon(dt,d\xi) := (1/\varepsilon)(\alpha + u_\alpha^\varepsilon)^- dt d\xi$, converges distributionally as $\varepsilon \downarrow 0$ to a Radon measure $\eta_\alpha(dt,d\xi)$ on $\mathcal{O}$.

(d) The pair (u_α, η_α) is the solution of (3.16) with reflection in $-\alpha$ and initial value $x \in K_\alpha \cap C_0(0,1)$.

We set for all $t \geq 0$, $\alpha \geq 0$, $\varepsilon > 0$:

- $X_\alpha(t,x) \in C_0(0,1)$, $X_\alpha(t,x)(\xi) := u_\alpha(t,\xi)$, $x \in K_\alpha \cap C_0(0,1)$,
- $X_\alpha^\varepsilon(t,x) \in H$, $X_\alpha^\varepsilon(t,x)(\xi) := u_\alpha^\varepsilon(t,\xi)$, $x \in H$.

Lemma 3.1 *For all $\alpha \geq 0$, $\varepsilon > 0$, $t \geq 0$, we have: $\forall x, x' \in C_0(0,1)$,*

$$\|X_\alpha^\varepsilon(t,x) - X_\alpha^\varepsilon(t,x')\| \leq e^{ct}\|x - x'\| \tag{3.21}$$

where $\|\cdot\|$ denotes the norm in H and $c > 0$ is the constant of (H2).

Proof—By the monotonicity properties of $(\cdot)^-$ and $f_2(\xi,\cdot)$ we have:

$$\frac{1}{2}\frac{d}{dt}\|X_\alpha^\varepsilon(t,x) - X_\alpha^\varepsilon(t,x')\|^2 \leq c\|X_\alpha^\varepsilon(t,x) - X_\alpha^\varepsilon(t,x')\|^2$$

and the thesis follows from Gronwall's Lemma. $\square$

Therefore, the same estimate holds for X_α, $\alpha \geq 0$: $\forall x, x' \in K_\alpha \cap C_0(0,1)$,

$$\|X_\alpha(t,x) - X_\alpha(t,x')\| \leq e^{ct}\|x - x'\| \tag{3.22}$$

and we can uniquely extend $X^\varepsilon_\alpha(t,\cdot)$, respectively $X_\alpha(t,\cdot)$, to maps from H to H, resp. from K_α to K_α, that we denote by the same symbols, satisfying (3.21) for all $x, x' \in H$, resp. (3.22) for all $x, x' \in K_\alpha$. We set for all $\alpha \geq 0$, $\varepsilon > 0$, $\varphi \in C_b(H)$, $t \geq 0$:

$$P^\varepsilon_\alpha(t)\varphi : H \mapsto \mathbb{R}, \quad P^\varepsilon_\alpha(t)\varphi(x) := \mathbb{E}[\varphi(X^\varepsilon_\alpha(t,x))], \quad x \in H, \tag{3.23}$$

$$P_\alpha(t)\varphi : K_\alpha \mapsto \mathbb{R}, \quad P_\alpha(t)\varphi(x) := \mathbb{E}[\varphi(X_\alpha(t,x))], \quad x \in K_\alpha. \tag{3.24}$$

Lemma 3.2 *For all $\alpha \geq 0$, $\varepsilon > 0$, $\varphi \in C_b(H)$, $t \geq 0$, we have:*

$$P^\varepsilon_\alpha(t)\varphi \in C_b(H), \qquad \omega_{P^\varepsilon_\alpha(t)\varphi}(r) \leq \omega_\varphi(e^{ct}r) \qquad \forall r \geq 0, \tag{3.25}$$

$$P_\alpha(t)\varphi \in C_b(K_\alpha), \qquad \omega_{P_\alpha(t)\varphi}(r) \leq \omega_\varphi(e^{ct}r) \qquad \forall r \geq 0, \tag{3.26}$$

$$\lim_{\varepsilon\downarrow 0} P^\varepsilon_\alpha(t)\varphi(x) = P_\alpha(t)\varphi(x) \qquad \forall x \in K_\alpha, \tag{3.27}$$

$$P_\alpha(s)P_\alpha(t)\varphi(x) = P_\alpha(t+s)\varphi(x), \qquad \forall x \in K_\alpha. \tag{3.28}$$

In particular, $(P_\alpha(t))_{t\geq 0}$ is a Markov semigroup acting on $C_b(K_\alpha)$.

Proof—For (3.26), notice that, by (3.22), for all $x, x' \in K_\alpha$:

$$|P_\alpha(t)\varphi(x) - P_\alpha(t)\varphi(x')| \;\leq\; \mathbb{E}[|\varphi(X_\alpha(t,x)) - \varphi(X_\alpha(t,x'))|] \;\leq$$

$$\leq\; \mathbb{E}[\omega_\varphi(\|X_\alpha(t,x) - X_\alpha(t,x')\|)] \;\leq\; \omega_\varphi(e^{ct}\|x - x'\|),$$

and (3.25) follows analogously. (3.27) is a consequence of (b) in the proof of Theorem 3.2 and (3.25). It is well known that $(P^\varepsilon_\alpha(t))_{t\geq 0}$ is a semigroup acting on $C_b(H)$: since the family of probability measures $\{m^\varepsilon\}_{\varepsilon>0}$, where m^ε is the law of $X^\varepsilon_\alpha(s,x)$, and the family of functions $\{P^\varepsilon_\alpha(t)\varphi\}_{\varepsilon>0}$ satisfy the Hypothesis of Lemma 1.1, (3.28) follows. □

Lemma 3.3 *For all $\varphi \in C_b(H)$, $\lim_{\alpha\downarrow 0} P_\alpha(t)\varphi(x) = P_0(t)\varphi(x)$, $t \geq 0$, $x \in K_0$.*

Proof—If $x \in K_0 \cap C_0(0,1)$, then the map $0 < \alpha \mapsto X^\varepsilon_\alpha(t,x)(\xi)$ is non-decreasing for all $(t,\xi) \in \mathcal{O}$, $\varepsilon > 0$. Therefore:

$$\lim_{\alpha\downarrow 0} X_\alpha(t,x)(\xi) \;=\; \sup_{\alpha>0} X_\alpha(t,x)(\xi) \;=\; \sup_{\alpha>0}\sup_{\varepsilon>0} X^\varepsilon_\alpha(t,x)(\xi)$$

$$=\; \sup_{\varepsilon>0}\sup_{\alpha>0} X^\varepsilon_\alpha(t,x)(\xi) \;=\; \sup_{\varepsilon>0} X^\varepsilon_0(t,x)(\xi) \;=\; X_0(t,x)(\xi),$$

since $\sup_{\alpha>0} u^\varepsilon_\alpha(t,\xi) = u^\varepsilon_0(t,\xi)$ by the uniqueness of solutions of (3.17). The general case follows by (3.26) and a density argument. □

Proposition 3.1 *For all $\alpha \geq 0$ and $\varphi : H \mapsto \mathbb{R}$ bounded and Borel we have:*

$$|P_\alpha(t)\varphi(x) - P_\alpha(t)\varphi(y)| \,\leq\, C\,\|\varphi\|_\infty (1\wedge t)^{-\frac{1}{2}}\|x-y\|, \qquad x,y\in K_\alpha,\ t>0.$$

In particular, the process X_α is Strong Feller.

Proof—Fix $\varepsilon > 0$, $\alpha \geq 0$ and set for $\gamma > 0$:

$$s_\gamma : \mathbb{R} \mapsto \mathbb{R}, \qquad s_\gamma(r) := \begin{cases} [(r)^-]^{1+\gamma}, & r \leq (1+\gamma)^{-1/\gamma} \\ \\ r - \gamma(1+\gamma)^{-1-1/\gamma}, & r \geq (1+\gamma)^{-1/\gamma} \end{cases}$$

Then s_γ is $C^1(\mathbb{R})$, monotone non-decreasing, and for all $r \in \mathbb{R}$, $s_\gamma(r) \uparrow (r)^-$ as $\gamma \downarrow 0$. Consider the following equation:

$$\begin{cases} \dfrac{\partial \tilde u_\gamma}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 \tilde u_\gamma}{\partial \xi^2} + \dfrac{\partial^2 W}{\partial t \partial \xi} - f(\tilde u_\gamma) + \dfrac{s_\gamma(\alpha + \tilde u_\gamma)}{\varepsilon} \\ \\ \tilde u_\gamma(0,\cdot) = x \in H, \quad \tilde u_\gamma(t,0) = \tilde u_\gamma(t,1) = 0,\ \forall t \geq 0. \end{cases} \tag{3.29}$$

We set $\tilde X_\gamma(t,x) := \tilde u_\gamma(t,\cdot)$. Equation (3.29) is a white-noise driven SPDE with differentiable non-linearity of Nemytskii type, satisfying the hypothesis of Proposition 8.3.3 of [Ce 01]. Then, we have for $\varphi \in C_b(H)$, $x, y \in H$:

$$|\mathbb{E}[\varphi(\tilde X_\gamma(t,x))] - \mathbb{E}[\varphi(\tilde X_\gamma(t,y))]| \,\leq\, C\,\|\varphi\|_\infty (1\wedge t)^{-\frac{1}{2}}|x-y|. \tag{3.30}$$

By the monotonicity properties of s_γ, Lemma 1.3 and the uniqueness of solutions of (3.17), we have that $\tilde u_\gamma \uparrow u^\varepsilon_\alpha$ as $\gamma \downarrow 0$. Then letting $\gamma \downarrow 0$ in (3.30), we obtain:

$$|P^\varepsilon_\alpha \varphi(x) - P^\varepsilon_\alpha \varphi(y)| \,\leq\, C\,\|\varphi\|_\infty (1\wedge t)^{-\frac{1}{2}}|x-y|.$$

The thesis follows letting $\varepsilon \downarrow 0$ and using the Monotone Class Theorem. □

3.4 The invariant measure of X_α

We have the following:

Theorem 3.3 *For all $\alpha \geq 0$, the measure*

$$d\nu^F_\alpha \,:=\, \frac{1}{\nu_\alpha(e^{-2F})}\, \exp(-2F(x))\,\nu_\alpha(dx) \tag{3.31}$$

is the unique invariant probability measure of X_α. In particular, X_α is ν^F_α-ergodic. Moreover, X_α is symmetric with respect to ν^F_α.

Proof—If m^1 and m^2 are two invariant probability measures for X_α, then for all $t \geq 0$, arguing as for (3.20):

$$\|X_\alpha(t,p^1) - X_\alpha(t,p^2)\| = \lim_{\varepsilon\downarrow 0} \|X^\varepsilon_\alpha(t,p^1) - X^\varepsilon_\alpha(t,p^2)\| \leq C\, e^{-\pi^2/4t}\|p^1 - p^2\|, \tag{3.32}$$

where p^1 and p^2 are K_α-valued random variables, such that the law of p^i is m^i and $\{p^1, p^2, W\}$ is an independent family. Since the law of $X_\alpha(t,p^i)$ is equal to m^i for all $t \geq 0$, (3.32) implies $m^1 = m^2$.

Let $\alpha > 0$. By (H1)-(H3) above, we have that e^{-2F} is bounded and continuous on H. Therefore, by Dominated Convergence Theorem μ^ε_α converges weakly to ν^F_α as $\varepsilon \downarrow 0$. Moreover, the families $\{\psi\, P^\varepsilon_\alpha(t)\varphi\}_{\varepsilon>0} \cup \{\psi\, P_\alpha(t)\varphi\}$ and $\{\mu^\varepsilon_\alpha\}_{\varepsilon>0} \cup \{\nu^F_\alpha\}$, $\psi, \varphi \in C_b(H)$, satisfy the Hypothesis of Lemma 1.1 by (3.25), (2.1) and $\nu^F_\alpha(K_\alpha) = 1$. Therefore:

$$\nu^F_\alpha\left(\psi\, P_\alpha(t)\varphi\right) = \lim_{\varepsilon\downarrow 0} \mu^\varepsilon_\alpha\left(\psi\, P^\varepsilon_\alpha(t)\varphi\right) = \lim_{\varepsilon\downarrow 0} \mu^\varepsilon_\alpha\left(\varphi\, P^\varepsilon_\alpha(t)\psi\right) = \nu^F_\alpha\left(\varphi\, P_\alpha(t)\psi\right).$$

Therefore, ν^F_α is symmetrizing measure for $\{X_\alpha(t,x) : t \geq 0, x \in K_\alpha\}$. By Theorem 2.1, ν^F_α converges to ν^F_0 as $\alpha \downarrow 0$, and by Lemma 1.1 and Proposition 3.1, the thesis follows analogously. □

Remark 3.1 Since we assumed (H3), we have: $e^{-2C} \leq \exp(-2F) \leq e^{2C}$ on H. This is in fact not really necessary, but it simplifies several technical points: see e.g. the proof of Theorem 3.5 below. Moreover, we stress again that (H3) is enough to handle the SPDEs which appear in the proof of Theorem 4.1 below.

By Theorem 3.3 and Proposition 3.1 we obtain:

Corollary 3.1 *For all $t > 0$, $x \in H$, the law of $X_\alpha(t,x)$ is absolutely continuous with respect to ν_α. In particular, for all $t > 0$ and $\xi \in]0,1[$, the law of $u_0(t,\xi)$ is equal to*

$$\rho_{t,\xi}(y)\; y^2\; \exp\left\{-\frac{y^2}{2\xi(1-\xi)}\right\}\, dy, \qquad y \geq 0$$

with $\rho_{t,\xi} \geq 0$.

Proof—This is well known, since if S is a Borel set of H such that $\nu_\alpha(S) = 0$, then:

$$0 = \int 1_S\, e^{-2F}\, d\nu_\alpha = \int [P_\alpha(t)1_S]\; e^{-2F}\, d\nu_\alpha.$$

Since $e^{-2F} \geq e^{-2C}$ and $P_\alpha(t)1_S$ is continuous, we have for all $x \in H$:

$$P_\alpha(t)1_S(x) = \mathbb{P}(X_\alpha(t,x) \in S) = 0. \quad \square$$

Recall that Donati-Martin and Pardoux prove in [DMP 93] the existence of a minimal solution (v, θ) of the following semilinear SPDE with reflection at 0:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 v}{\partial \xi^2} - f(v(t,\xi)) + \sigma(v(t,\xi))\,\dfrac{\partial^2 W}{\partial t \partial \xi} + \theta(t,\xi) \\ \\ v(0,\xi) = x(\xi), \;\; v(t,0) = v(t,1) = 0 \\ \\ v \geq 0, \;\; d\theta \geq 0 \;\; \int_{\mathcal{O}} v \, d\eta = 0. \end{cases} \tag{3.33}$$

and in [DMP 97], under the assumptions that f, σ are differentiable on $\mathbb{R}$ with bounded derivative, that for all $t > 0$, $\xi \in (0,1)$, the law of $v(t,\xi)$ is absolutely continuous w.r.t. the Lebesgue measure dy on $(0,\infty)$. Corollary 3.1 shows that, if $\sigma \equiv 1$, then for all $t > 0$, $\xi \in (0,1)$, the law of $v(t,\xi)$ is absolutely continuous w.r.t. the measure $y^2\,dy$ on the whole of $[0,\infty)$.

3.5 The Dirichlet Form $\mathcal{E}^\alpha$, $\alpha \geq 0$

The aim of this section is to apply (2.1) and (2.2) to the symmetric bilinear forms

$$C_b^1(H) \ni \varphi, \psi \mapsto E^\alpha(\varphi,\psi) \,:=\, \frac{1}{2} \int \langle \nabla\varphi, \nabla\psi \rangle \, d\nu_\alpha^F, \quad \alpha \geq 0.$$

The main result is that E^α is closable in $L^2(\nu_\alpha)$ for all $\alpha \geq 0$, and X_α is the associated diffusion. We refer to [FOT 94] and [MR 92] for all basic definitions.

We set for all $f \in C_b(H)$:

$$R_\alpha(\lambda) f(x) := \int_0^\infty e^{-t}\mathbb{E}[f(X_\alpha(t,x))]\,dt, \qquad x \in K_\alpha, \;\; \lambda > 0.$$

Notice that by Theorem 3.3, $R_\alpha(\lambda)$ extends to a bounded linear operator in $L^2(\nu_\alpha^F)$ for all $\lambda > 0$: we denote also such extension by $R_\alpha(\lambda)$. We also set:

$$\sigma_\alpha^F(r,dx) \,:=\, \frac{1}{\nu_\alpha(e^{-2F})}\,\exp(-2F(x))\,\sigma_\alpha(r,dx), \qquad \alpha \geq 0, \;\; r \in (0,1). \tag{3.34}$$

Theorem 3.4 *Let $\alpha > 0$. Set for all $\varphi, \psi \in W^{1,2}(H,\mu)$:*

$$E^\alpha(\varphi,\psi) := \frac{1}{2}\int_{K_\alpha} \langle \nabla\varphi, \nabla\psi\rangle \, d\nu_\alpha^F.$$

Then the positive symmetric bilinear form

$$C_b^1(K_\alpha) \ni \varphi, \psi \mapsto E^\alpha(\varphi,\psi)$$

is closable in $L^2(\nu_\alpha^F)$. We denote by $(\mathcal{E}^\alpha, W^{1,2}(\nu_\alpha^F))$ the closure. The family $(R_\alpha(\lambda))_{\lambda>0}$ on $L^2(\nu_\alpha^F)$ is the strongly continuous resolvent associated with $\mathcal{E}^\alpha$, $\mathrm{Lip}(K_\alpha) \subseteq W^{1,2}(\nu_\alpha^F)$ *and* $\mathrm{Exp}_A(K_\alpha)$ *is a core for $\mathcal{E}^\alpha$.*

Proof Let $\varphi \in \mathrm{Exp}_A(H)$, $f \in C_b(H)$. Set $V_\alpha^\varepsilon := R_\alpha^\varepsilon(1)f$, $V_\alpha := R_\alpha(1)f$. Then, by Lemma 1.1, Lemma 3.2 and Corollary 2.1:

$$\begin{aligned}
\mathcal{E}^{\alpha,\varepsilon}(V_\alpha^\varepsilon,\varphi) &= -\int_H V_\alpha^\varepsilon \, \mathcal{L}_\alpha^\varepsilon \varphi \, d\mu_\alpha^\varepsilon \\
&= -\int_H V_\alpha^\varepsilon \, L\varphi \, d\mu_\alpha^\varepsilon - \int_H V_\alpha^\varepsilon(x) \, \frac{1}{\varepsilon}\langle (x+\alpha)^-, \nabla\varphi(x)\rangle \, \mu_\alpha^\varepsilon(dx) \\
&\to -\int_{K_\alpha} V_\alpha \, L\varphi \, d\nu_\alpha^F - \frac{1}{2}\int_0^1 dr \int V_\alpha(z) \, \langle \nabla\varphi(z), \delta_r\rangle \, \sigma_\alpha^F(r,dz),
\end{aligned}$$

as $\varepsilon \downarrow 0$. On the other hand, we have

$$\int V_\alpha^\varepsilon \, \varphi \, d\mu_\alpha^\varepsilon + \mathcal{E}^{\alpha,\varepsilon}(V_\alpha^\varepsilon,\varphi) = \int f \, \varphi \, d\mu_\alpha^\varepsilon, \quad \text{so that :}$$

$$\int_{K_\alpha} (V_\alpha - f)\,\varphi \, d\nu_\alpha^F = \int_{K_\alpha} V_\alpha \, L\varphi \, d\nu_\alpha^F + \frac{1}{2}\int_0^1 dr \int V_\alpha \, \langle \nabla\varphi, \delta_r\rangle \, \sigma_\alpha^F(r,dz). \tag{3.35}$$

Notice that $V_\alpha \circ \Pi_{K_\alpha}$ is Lipschitz on H: therefore it is in $W^{1,2}(H,\mu)$. Set $\{\gamma_n\}_n := \Pi_{1/n}(V_\alpha \circ \Pi_{K_\alpha})$, where $(\Pi_t)_{t\geq 0}$ is the Ornstein-Uhlenbeck semigroup defined in (1.10). Then $\{\gamma_n\} \subseteq C_b^1(H)$, $\sup_n \|\gamma_n\|_\infty < \infty$ and γ_n converges to $V_\alpha \circ \Pi_{K_\alpha}$ in $W^{1,2}(H,\mu)$ and pointwise. By (2.23) and (3.35):

$$\begin{aligned}
-E^\alpha(V_\alpha,\varphi) &= -\lim_n E^\alpha(\gamma_n,\varphi) \\
&= \lim_n \left(\int_{K_\alpha} \gamma_n \, L\varphi \, d\nu_\alpha^F + \frac{1}{2}\int_0^1 dr \int \gamma_n(z) \, \langle \nabla\varphi(z), \delta_r\rangle \, \sigma_\alpha^F(r,dz) \right) \\
&= \int_{K_\alpha} V_\alpha \, L\varphi \, d\nu_\alpha^F + \frac{1}{2}\int_0^1 dr \int V_\alpha(z) \, \langle \nabla\varphi(z), \delta_r\rangle \, \sigma_\alpha^F(r,dz) \\
&= \int_{K_\alpha} (V_\alpha - f)\,\varphi \, d\nu_\alpha^F.
\end{aligned}$$

Let now $\psi \in \mathrm{Lip}(K_\alpha)$: then $\psi \circ \Pi_{K_\alpha} \in \mathrm{Lip}(H)$, and by Proposition 1.1 we can find a sequence $\varphi_n \in \mathrm{Exp}_A(H)$ converging to $\psi \circ \Pi_{K_\alpha}$ in $W^{1,2}(H,\mu)$. Then we obtain

$$E^\alpha(V_\alpha, \psi) \;=\; \int_{K_\alpha} (f \;-\; V_\alpha)\, \psi \; d\nu_\alpha^F \qquad \forall \psi \in \mathrm{Lip}(K_\alpha),$$

and analogously, $R_\alpha(\lambda) f \circ \Pi_{K_\alpha} \in W^{1,2}(H,\mu)$ and for all $\lambda > 0$:

$$E^\alpha(R_\alpha(\lambda) f, \psi) \;=\; \int_{K_\alpha} (f \;-\; \lambda R_\alpha(\lambda) f)\, \psi \; d\nu_\alpha^F \quad \forall \psi \in \mathrm{Lip}(K_\alpha). \tag{3.36}$$

Since $(R_\alpha(\lambda))_{\lambda>0}$ is a strongly-continuous resolvent in $L^2(\nu_\alpha^F)$, then there exists a Dirichlet Form $(\tilde{E}^\alpha, D(\tilde{E}^\alpha))$ with $D(\tilde{E}^\alpha)$ dense in $L^2(\nu_\alpha^F)$, associated with $(R_\alpha(\lambda))_{\lambda>0}$. Consider $f \in \mathrm{Lip}(K_\alpha)$: by the general theory of Dirichlet Forms,

$$f \in D(\tilde{E}^\alpha) \iff \sup_{\lambda>0} \int_{K_\alpha} \lambda\, (f - \lambda R_\alpha(\lambda) f)\; f \; d\nu_\alpha^F \;<\; \infty \tag{3.37}$$

By (3.26) and (3.36), we have:

$$\int_{K_\alpha} \lambda\, (f - \lambda R_\alpha(\lambda) f)\; f \; d\nu_\alpha^F \;=\; E^\alpha(\lambda R_\alpha(\lambda) f, f) \;\le\; C\, \|f\|^2_{\mathrm{Lip}(K_\alpha)},$$

for some $C \ge 0$, so that $\mathrm{Lip}(K_\alpha) \subseteq D(\tilde{E}^\alpha)$. Then, by (3.36), E^α is closable on $R_\alpha(1)(\mathrm{Lip}(K_\alpha))$, the closure $(\mathcal{E}^\alpha, W^{1,2}(\nu_\alpha^F))$ coincides with $(\tilde{E}^\alpha, D(\tilde{E}^\alpha))$ and $(R_\alpha(\lambda))_{\lambda>0}$ is the resolvent associated with $(\mathcal{E}^\alpha, W^{1,2}(\nu_\alpha^F))$.

Finally, since for all $\psi \in \mathrm{Lip}(K_\alpha)$ there exists a sequence $\varphi_n \in \mathrm{Exp}_A(H)$ converging to $\psi \circ \Pi_{K_\alpha}$ in $W^{1,2}(H,\mu)$, then we have that $\mathrm{Exp}_A(H)$ is a core for $\mathcal{E}^\alpha$, and the Theorem is proved. $\square$

We turn now to the case $\alpha = 0$. We set $H^3 = \oplus_{i=1}^3 H = L^2(0,1;\mathbb{R}^3)$,

$$H^3 \ni y \mapsto |y| \in K_0, \quad |y|(\tau) := |y(\tau)|_{\mathbb{R}^3}, \quad \tau \in (0,1).$$

$$C_b(K_0) \ni \varphi \mapsto \overline{\varphi} \in C_b(H^3), \quad \overline{\varphi}(y) \;=\; \varphi(|y|).$$

Recall that the image measure of $\mu^{\otimes 3}$ under $|\cdot|$ is ν, i.e.

$$\int_{H^3} \varphi(|y|)\, \mu^{\otimes 3}(dy) \;=\; \int_{K_0} \varphi(x)\, \nu(dx), \qquad \forall \varphi \in C_b(H). \tag{3.38}$$

In particular, there exists a measurable set $\Omega_0 \subseteq H^3$ with $\mu^{\otimes 3}(\Omega_0) = 1$, such that for all $y \in \Omega_0$, $|y| > 0$ on $(0,1)$. Then, we can define for all $h \in K_0$:

$$\Omega_0 \ni y \mapsto h\,\frac{y}{|y|} \in H^3, \qquad \left[h\,\frac{y}{|y|}\right](\tau) := h(\tau)\,\frac{y(\tau)}{|y(\tau)|_{\mathbb{R}^3}}.$$

Notice that an analogue of Proposition 1.1 also holds for the Gaussian space $(H^3, \mu^{\otimes 3})$: in particular, we denote by $W^{1,2}(H^3, \mu^{\otimes 3})$ the domain of the closure Λ^3 of the form:

$$C_b^1(H^3) \ni \Phi, \Psi \mapsto \frac{1}{2}\int_{H^3} \langle \overline{\nabla}\,\Phi, \overline{\nabla}\,\Psi\rangle_{H^3} d\mu^{\otimes 3}, \tag{3.39}$$

where $\overline{\nabla}\,\Phi \in C_b(H^3; H^3)$ is the usual gradient of Φ. If $\Phi \in W^{1,2}(H^3, \mu^{\otimes 3})$, then we denote the generalized gradient of Φ by $\overline{\nabla}\Phi \in L^2(H^3, \mu^{\otimes 3}; H^3)$. Moreover, if $\varphi \in \mathrm{Lip}(K_0)$, then $\overline{\varphi} \in \mathrm{Lip}(H^3) \subseteq W^{1,2}(H^3, \mu^{\otimes 3})$.

Lemma 3.4 *For all $\varphi \in \mathrm{Lip}(K_0)$ and $h \in K_0$ there exists the limit in $L^2(\nu)$:*

$$\lim_{t \downarrow 0} \frac{1}{t}(\varphi(x + t\,h) - \varphi(x)) =: \langle \nabla\varphi(x), h\rangle \qquad x \in K_0. \tag{3.40}$$

We call $\nabla\varphi \in L^\infty(K_0, \nu; H)$ the generalized gradient of φ. For $\mu^{\otimes 3}$-a.e. $y \in H^3$ we have:

$$\langle \nabla\varphi(|y|), h\rangle = \langle \overline{\nabla}\,\overline{\varphi}(y), h\,\frac{y}{|y|}\rangle_{H^3}, \qquad \forall h \in K_0. \tag{3.41}$$

Finally, for all $\varphi, \psi \in \mathrm{Lip}(K_0)$ and $\gamma \in L^2(\nu)$:

$$\int_{K_0} \langle \nabla\varphi, \nabla\psi\rangle\,\gamma\,d\nu = \int_{H^3} \langle \overline{\nabla}\,\overline{\varphi}, \overline{\nabla}\,\overline{\psi}\rangle_{H^3}\,\gamma(|\cdot|)\,d\mu^{\otimes 3} \tag{3.42}$$

Proof—By Proposition 1.1, for all $\Phi \in \mathrm{Lip}(H^3)$, there exists a sequence $\Phi_n \in C_b^1(H^3)$, such that

$$\|\Phi_n\|_{\mathrm{Lip}(H^3)} \le \|\Phi\|_{\mathrm{Lip}(H^3)}, \quad \Phi_n \to \Phi \quad \text{in } W^{1,2}(H^3, \mu^{\otimes 3}).$$

Then, by a density argument, for all $\Phi \in \mathrm{Lip}(H^3)$:

$$\lim_{t \downarrow 0} \frac{1}{t}\left[\Phi\left(y + t\,h\,\frac{y}{|y|}\right) - \Phi(y)\right] = \langle \overline{\nabla}\,\Phi(y), h\,\frac{y}{|y|}\rangle_{H^3} \qquad \text{in } L^2(\mu^{\otimes 3}).$$

Then, for $\Phi := \overline{\varphi}$:

$$\langle \nabla \varphi(|y|), h\rangle := \lim_{t\downarrow 0} \frac{1}{t}(\varphi(|y| + t\,h) - \varphi(|y|))$$

$$= \lim_{t\downarrow 0} \frac{1}{t}\left[\overline{\varphi}\left(y + t\,h\,\frac{y}{|y|}\right) - \overline{\varphi}(y)\right] = \langle \overline{\nabla}\,\overline{\varphi}(y), h\,\frac{y}{|y|}\rangle_{H^3} \qquad \text{in } L^2(\mu^{\otimes 3}).$$

By (3.38), we obtain (3.40) and (3.41). Finally, (3.42) follows by (3.41): indeed, denoting the scalar product in $\mathbb{R}^3$ by $(\,\cdot\,,\,\cdot\,)_{\mathbb{R}^3}$, for $\mu^{\otimes 3}$-a.e. $y \in H^3$ we have:

$$\begin{aligned} \langle h\,\frac{y}{|y|}, k\,\frac{y}{|y|}\rangle_{H^3} &= \int_0^1 h(\tau)k(\tau)\left(\frac{y(\tau)}{|y|(\tau)}\,,\,\frac{y(\tau)}{|y|(\tau)}\right)_{\mathbb{R}^3} d\tau \\ &= \langle h, k\rangle, \qquad\qquad \forall h,k \in K_0.\ \Box \end{aligned}$$

Theorem 3.5 *Set for all $\varphi, \psi \in \mathrm{Lip}(K_0)$:*

$$E^0(\varphi,\psi) := \frac{1}{2}\int_{K_0} \langle \nabla\varphi, \nabla\psi\rangle\, d\nu_0^F,$$

where $\nabla\varphi$ and $\nabla\psi$ are defined by (3.40). Then the positive symmetric bilinear form $(E^0, \mathrm{Lip}(K_0))$ is closable in $L^2(\nu_0^F)$. We denote the closure of $(E^0, \mathrm{Lip}(K_0))$ by $(\mathcal{E}^0, W^{1,2}(\nu_0^F))$. The family $(R_0(\lambda))_{\lambda>0}$ on $L^2(\nu_0^F)$ is the strongly continuous resolvent associated with $\mathcal{E}^0$ and $\mathrm{Exp}_A(K_0)$ is a core for $\mathcal{E}^0$. Finally, $\mathcal{E}^0$ is the image of the Dirichlet form

$$W^{1,2}(H^3,\mu^{\otimes 3}) \ni \Phi,\Psi \mapsto \frac{1}{2\int e^{-2F(|\cdot|)}d\mu^{\otimes 3}} \int_{H^3} \langle \overline{\nabla}\Phi, \overline{\nabla}\Psi\rangle_{H^3}\, e^{-2F(|\cdot|)}\, d\mu^{\otimes 3}$$

under the map $|\cdot|$, i.e. $W^{1,2}(\nu_0^F) = \{\varphi \in L^2(\nu_0^F) : \overline{\varphi} = \varphi(|\cdot|) \in W^{1,2}(H^3,\mu^{\otimes 3})\}$ and for all $\varphi,\psi \in W^{1,2}(\nu_0^F)$:

$$\mathcal{E}^0(\varphi,\psi) = \frac{1}{2\int e^{-2F(|\cdot|)}d\mu^{\otimes 3}} \int_{H^3} \langle \overline{\nabla}\,\overline{\varphi}, \overline{\nabla}\,\overline{\psi}\rangle_{H^3}\, e^{-2F(|\cdot|)}\, d\mu^{\otimes 3}. \tag{3.43}$$

Proof—By (3.42), the form

$$\mathrm{Lip}(K_0) \ni \varphi,\psi \mapsto \frac{1}{2}\int_{K_0} \langle \nabla\varphi, \nabla\psi\rangle\, d\nu \tag{3.44}$$

is closable in $L^2(\nu)$, since by (3.42) it is the image of the Dirichlet Form

$$W^{1,2}(H^3,\mu^{\otimes 3}) \ni \Phi,\Psi \mapsto \Lambda^3(\Phi,\Psi) := \frac{1}{2}\int_{H^3} \langle \overline{\nabla}\,\Phi, \overline{\nabla}\,\Psi\rangle_{H^3} d\mu^{\otimes 3} \tag{3.45}$$

under the map $|\cdot|$. By (H1)-(H3), we have that $0 < e^{-2C} \le \exp(-2F) \le e^{2C} < \infty$, so that $L^2(\nu) = L^2(\nu_0^F)$ with equivalence of norms, and the form in (3.44) is equivalent to E^0 on $\mathrm{Lip}(K_0)$. This implies closability of $(E^0, \mathrm{Lip}(K_0))$ in $L^2(\nu_0^F)$: we denote the closure by $(\mathcal{E}^0, W^{1,2}(\nu_0^F))$.

As in the proof of Theorem 3.4, let $\varphi(x) \in \mathrm{Exp}_A(H)$ and $f \in C_b(H)$. Set $V^\varepsilon := R_0^\varepsilon(1)f$, $V := R_\alpha(1)f$. Then, by Lemma 1.1 and Lemma 3.2, letting $\varepsilon \downarrow 0$ in (3.35) we obtain:

$$\int_{K_0} (V - f)\, \varphi \, d\nu_0^F \;=\; \int_{K_0} V\, L\varphi \, d\nu_0^F \;+\; \frac{1}{2}\int_0^1 dr \int V\, \langle \nabla\varphi, \delta_r\rangle\, \sigma_0^F(r, dz). \tag{3.46}$$

By Corollary 2.2, for all $\psi \in C_b^1(H)$ and $h \in D(A)$, denoting the imaginary unit by i and setting $\varphi_h := \exp(i\langle h, x\rangle)$:

$$\begin{aligned} T_t\psi \;:=\; & \frac{i}{2}\int_{K_0} \frac{1}{t}(\psi(x + t\,h) - \psi(x))\, \varphi_h(x)\, \nu_0^F(dx) \\ =\; & \frac{1}{2}\int_0^1 ds \int_{K_0} \langle \nabla\psi(x + s\,t\,h), i\,h\rangle\, \varphi_h\, \nu_0^F(dx) \\ =\; & \frac{1}{2}\int_0^1 ds \int_{K_0} \langle \nabla\psi(x + s\,t\,h), \nabla\varphi_h\rangle\, \nu_0^F(dx) \\ =\; & -\int_0^1 ds \left[\int_{K_0} \psi(\,\cdot\, + s\,t\,h)\, M\varphi_h \, d\nu_0^F \right. \\ & \left. + \frac{1}{2}\int_0^1 dr \int \psi(\,\cdot\, + st\,h)\, \langle \nabla\varphi_h, \delta_r\rangle\, \sigma_0^F(r, dz)\right], \\ & |T_t\psi| \;\le\; C\,\|\psi\|_\infty, \end{aligned}$$

with $C > 0$ independent of $\psi \in C_b^1(H)$. By the density of $C_b^1(H)$ in $C_b(H)$ in the sup-norm, by (3.46) and (3.40), we obtain:

$$E^0(V, \varphi_h) \;=\; \lim_{t\downarrow 0} \frac{i}{2}\int_{K_0} \frac{1}{t}(V(\cdot + t\,h) - V(x))\, \varphi_h \, d\nu_0^F \;=\; \int_{K_0} (f - V)\, \varphi_h \, d\nu_0^F,$$

and for all $\varphi \in \mathrm{Exp}_A(H)$:

$$E^0(V, \varphi) \;=\; \int_{K_0} (f - V)\, \varphi \, d\nu_0^F,$$

and analogously for all $\lambda > 0$, $\varphi \in \mathrm{Exp}_A(H)$:

$$E^0(R_0(\lambda)f, \varphi) \;=\; \int_{K_0} (f \;-\; \lambda R_0(\lambda) f)\, \varphi \, d\nu_0^F. \tag{3.47}$$

The thesis follows once we have proved that (3.47) holds for all $\varphi \in \mathrm{Lip}(K_0)$. Let $\{e_k\}_{k\in\mathbb{N}}$ be any complete orthonormal system in H, and set

$$H \ni x \mapsto x_n := \sum_{i=1}^{n} \langle x, e_i\rangle\, e_i,$$

and for $\psi \in C_b(H)$, $\psi_n \in C_b(H)$, $\psi_n(x) := \psi(x_n)$. Then:

$$\lim_{n\to\infty} \psi_n(x) = \psi(x), \quad \sup_{n,x} |\psi_n(x)| < \infty.$$

Let $\psi \in \mathrm{Lip}(H)$: then $\psi_n \in \mathrm{Lip}(H)$, and $\|\psi_n\|_{W^{1,2}(\nu_0^F)} \le \|\psi\|_{W^{1,2}(\nu_0^F)}$: therefore, there exists a subnet $\{n_i\}_{i\in I}$ such that $(\psi_{n_i})_{i\in I}$ converges to ψ weakly in $W^{1,2}(\nu_0^F)$. Fix $i \in I$: then ψ_{n_i} can be identified with a function in $\mathrm{Lip}(\mathbb{R}^{n_i})$. By convolution between ψ_{n_i} and a smooth mollifier $\rho_{n_i,\varepsilon} \in C_b^1(\mathbb{R}^{n_i})$, we obtain a family $(\psi_{n_i,\varepsilon})_{\varepsilon>0} \subset C_b^1(H)$, such that

$$\lim_{\varepsilon\downarrow 0} \psi_{n_i,\varepsilon}(x) = \psi_{n_i}(x), \quad \forall x \in H, \quad \sup_{\varepsilon>0,x} (|\psi_{n_i,\varepsilon}(x)| + |\nabla\psi_{n_i,\varepsilon}(x)|) < \infty.$$

Again, there exists a subnet $(\psi_{n_i,\varepsilon_j})_{j\in J}$ converging to ψ_{n_i} weakly in $W^{1,2}(\nu_0^F)$. Since ψ_{n_i,ε_j} depends only on finitely many coordinates, it is well known that for all $i \in I$, $j \in J$, there exists a net $(\varphi_k^{i,j})_{k\in K} \subseteq \mathrm{Exp}_A(H)$, such that:

$$\lim_{k\in K} \varphi_k^{i,j}(x) = \psi_{n_i,\varepsilon_j}(x), \quad \lim_{k\in K} \nabla\varphi_k^{i,j}(x) = \nabla\psi_{n_i,\varepsilon_j}(x), \quad \forall x \in H,$$

$$\sup_{k,x} (|\varphi_k^{i,j}(x)| + |\nabla\varphi_k^{i,j}(x)|) < \infty.$$

Let $\varphi := \varphi_k^{i,j}$ in (3.47): then by convergence in the weak topology of $W^{1,2}(\nu_0^F)$:

$$\lim_i \lim_j \lim_k E^0(R_0(\lambda)f, \varphi_k^{i,j}) = E^0(R_0(\lambda)f, \psi),$$

and by Dominated Convergence Theorem:

$$\lim_i \lim_j \lim_k \int_{K_0} (f - \lambda R_0(\lambda)f)\, \varphi_k^{i,j}\, d\nu_0^F = \int_{K_0} (f - \lambda R_0(\lambda)f)\, \psi\, d\nu_0^F,$$

and (3.47) is proved for all $\varphi \in \mathrm{Lip}(H)$. Notice that we have also proved that $\mathrm{Exp}_A(H)$ is dense in $\mathrm{Lip}(H)$ with respect to the weak topology of $W^{1,2}(\nu_0^F)$: by Hahn-Banach Theorem, this implies that $\mathrm{Exp}_A(H)$ is a core in $W^{1,2}(\nu_0^F)$.
$\square$

Corollary 3.2 *Formulae (2.1) and (2.2) hold for all $\varphi \in \mathrm{Lip}(H)$, where for fixed $\alpha \geq 0$, $\partial_h \varphi := \langle \nabla \varphi, h \rangle \in L^2(\nu_\alpha)$.*

Proof–For $\alpha > 0$, the proof of (2.1) yields also this result. For $\alpha = 0$, the thesis follows from (2.2) and the existence of a net $\{\varphi_n\} \subseteq \mathrm{Exp}_A(H)$ such that $\varphi_n \to \varphi$ weakly in $W^{1,2}(\nu_0)$ and:

$$\lim_n \varphi_n(x) = \varphi(x) \qquad \forall x \in H, \quad \sup_{n,x} |\varphi_n(x)| \, < \, \infty. \, \square$$

For all $\alpha \geq 0$, the Dirichlet Form $\mathcal{E}^\alpha$ enjoys the following properties:

(i) $\mathrm{Lip}(K_\alpha)$ is dense in $W^{1,2}(\nu_\alpha^F)$.

(ii) $\mathrm{Exp}_A(K_\alpha)$ separates the points of K_α and is contained in $W^{1,2}(\nu_\alpha^F)$.

By Definition IV.3.1 in [MR 92], $\mathcal{E}^\alpha$ is **quasi-regular** if moreover:

(iii) There exists a sequence of compact sets F_k in K_α, such that the set:

$$\bigcup_k \{\varphi \in W^{1,2}(\nu_\alpha^F) : \, \varphi = 0 \; \nu_\alpha^F - \text{a.e. on } K_\alpha \backslash F_k\}$$

is dense in $W^{1,2}(\nu_\alpha^F)$.

On the other hand, by Nualart-Pardoux's Theorem 3.2, the process X_α is continuous, with infinite life-time and Strong Markov. Therefore X_α is a Hunt process on K_α, properly associated with $\mathcal{E}^\alpha$, see Chapter IV in [MR 92]: indeed, for all Borel bounded $\varphi : K_\alpha \mapsto \mathbb{R}$ and $t > 0$, $P_\alpha(t)\varphi \in C_b(K_\alpha)$, and by Theorems 3.4-3.5, P_α is the semigroup associated with $\mathcal{E}^\alpha$. Then we have:

Theorem 3.6 *Let $\alpha \geq 0$. The process $\{X_\alpha(\,\cdot\,, x)\}_x$ is a continuous Hunt process on K_α with infinite life-time, properly associated with the Dirichlet form $\mathcal{E}^\alpha$. In particular, $\mathcal{E}^\alpha$ is quasi-regular.*

The last assertion in Theorem 3.6 is a consequence of Theorem IV.5.1 in [MR 92], which states the necessity of quasi regularity for a Dirichlet Form to be properly associated with a nice Markov process. Theorem 3.6 plays a crucial role in the next section. Finally, we have:

Corollary 3.3 *The Log-Sobolev and the Poincaré inequalities hold for the Nualart-Pardoux equation (3.16) for all $\alpha \geq 0$, i.e. there exists $C > 0$ such that for all $\varphi \in W^{1,2}(\nu_\alpha^F)$:*

$$\int_{K_\alpha} \left|\varphi - \nu_\alpha^F(\varphi)\right|^2 d\nu_\alpha^F \, \leq \, C \int_{K_\alpha} \|\nabla \varphi\|^2 \, d\nu_\alpha^F,$$

$$\int_{K_\alpha} \varphi^2 \log(\varphi^2)\, d\nu_\alpha^F \;\le\; C \int_{K_\alpha} \|\nabla\varphi\|^2\, d\nu_\alpha^F \;+\; \|\varphi\|^2_{L^2(\nu_\alpha^F)} \log(\|\varphi\|^2_{L^2(\nu_\alpha^F)}).$$

For the proof, see e.g. [St 93] and [DP 01].

3.6 The Revuz-measure of η

The aim of this section is to characterize η_α as a family of Positive Continuous Additive Functionals of X_α and to prove the decomposition formula (3.2). Notice that Theorem 3.6 has the following important consequences: by the transfer method of Chapter VI in [MR 92], several statements of the theory of Dirichlet Form can rephrased from the classical locally-compact case into our setting. In particular, we can apply the results of Chapter 5 in [FOT 94]. We refer to [MR 92] and [FOT 94] for all basic definitions.

Let now $E := C([0,\infty); H)$ and define $X_t : E \mapsto H$, $t \ge 0$, $X_t(e) := e(t)$,

$$\mathcal{N}^0_\infty \;:=\; \sigma\{X_s,\, s \in [0,\infty)\}, \quad \mathcal{N}^0_t \;:=\; \sigma\{X_s,\, s \in [0,t]\}.$$

Fix $\alpha \ge 0$. For all $x \in K_\alpha$, we denote by $\mathbb{P}_x$ the law of $X_\alpha(\cdot, x)$ on $(E, \mathcal{N}^0_\infty)$, and for all probability measure λ on K_α, we define the probability measure $\mathbb{P}_\lambda$ on $(E, \mathcal{N}^0_\infty)$:

$$\mathcal{N}^0_\infty \ni \Lambda \,\mapsto\, \mathbb{P}_\lambda := \int_{K_\alpha} \mathbb{P}_x(\Lambda)\, \lambda(dx).$$

Then we denote by $\mathcal{N}^\lambda_\infty$ (resp. $\mathcal{N}^\lambda_t$) the completion of $\mathcal{N}^0_\infty$ (resp. completion of $\mathcal{N}^0_t$ in $\mathcal{N}^\lambda_\infty$) with respect to $\mathbb{P}_\lambda$. We also set $\mathcal{N}_\infty := \cap_{\lambda \in \mathcal{P}(K_\alpha)} \mathcal{N}^\lambda_\infty$, $\mathcal{N}_t := \cap_{\lambda \in \mathcal{P}(K_\alpha)} \mathcal{N}^\lambda_t$, where $\mathcal{P}(K_\alpha)$ denotes the set of probability measures on K_α. By an Additive Functional (**AF**) of X_α, we mean a family of functions $A(t) : E \mapsto \mathbb{R}^+$, $t \ge 0$, such that:

(A.1) $A(t)$ is $(\mathcal{N}_t)$-adapted

(A.2) There exist a set $\Lambda \in \mathcal{N}_\infty$ and a $\mathcal{E}^\alpha$-exceptional set $V \subset K_\alpha$, such that $\mathbb{P}_x(\Lambda) = 1$ for all $x \in K_\alpha \backslash V$, $\theta_t(\Lambda) \subseteq \Lambda$ for all $t \ge 0$, and for all $\omega \in \Lambda$: $A(\cdot)(\omega)$ is continuous, $A(0)(\omega) = 0$ and for all $t, s \ge 0$:

$$A(t)(\omega) \;<\; \infty, \quad A(t+s)(\omega) = A(s)(\omega) + A(t)(\theta_t\omega). \tag{3.48}$$

where $(\theta_s)_{s \ge 0}$ is the time-translation semigroup on $C([0,\infty); H)$. We say that an AF A is a Positive Continuous Additive Functional (**PCAF**) is A satisfies moreover:

(A.3) For all $\omega \in \Lambda$: $A(\cdot)(\omega)$ is non-decreasing.

Two AFs A_1 and A_2 are said to be equivalent if for each $t > 0$, $\mathbb{P}_x(A_1(t) = A_2(t)) = 1$, for $\mathcal{E}^\alpha$-q.e. x. Moreover, we say that A is a **PCAF in the strict sense** if one can choose $V = \emptyset$ in (A.1). Recall that X_α is Strong Feller and Corollary 3.1 implies the "absolute continuity condition" of [FOT 94]. This condition often allows to avoid the restriction: $x \in K_\alpha \backslash V$ of (A.2) above: see e.g. Theorems 5.1.6 and 5.1.7.

In the sequel, when it is necessary to stress the dependence of η_α on the initial datum x and the Brownian sheet W, we write η_α^x or $\eta_\alpha^{x,W}$. By the uniqueness statement of Theorem 3.2, we have a.s. for all $t \geq 0$:

$$\eta_\alpha^{x,W}([0,t+s],l) \;=\; \eta_\alpha^{x,W}([0,t],l) \;+\; \eta_\alpha^{X_\alpha(t,x),W^t}([0,s],l) \tag{3.49}$$

where $W^t := W(\,\cdot\, + t,\,\cdot\,) - W(t,\,\cdot\,)$ is a Brownian sheet, independent of $\mathcal{F}_t$. Notice that Formula (3.49) is reminiscent of (3.48). However, it is not clear whether η_α is a PCAF of X_α: in fact, η_α is adapted to the filtration of the noise W, but a priori not to the natural filtration of X_α. We can clarify this point by means of the following example: for all $y : [0,\infty) \mapsto \mathbb{R}$ continuous, $y(0) \geq 0$, by Skorokhod Lemma, see Lemma VI.2.1 in [RY 91], we have that setting

$$[0,\infty) \ni t \mapsto a(t) := \sup_{s \leq t}(-y(s)) \vee 0, \qquad [0,\infty) \ni t \mapsto z(t) := y(t) - a(t),$$

then (z,a) is the unique solution of:

1. $z \geq 0$ continuous
2. a continuous non-decreasing, $a(0) = 0$
3. $\int_0^\infty z(t)\, da(t) \;= 0$.

Therefore, if $y(t) = x + B_t$, where $x \geq 0$ and $(B_t)_{t \geq 0}$ is a linear Brownian Motion, then $X := z$ is the reflecting Brownian Motion and $L := 2a$ its local time at 0. L is a PCAF of X, in particular adapted to the filtration of X: indeed, it is well known that:

$$L_t \;=\; \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t 1_{[0,\varepsilon]}(X_s)\, ds \qquad \forall t \geq 0,\ \text{a.s.}$$

However, this is true only almost surely: indeed, we have:

$$y_1 \equiv 0 \implies z_1 \equiv 0,\ a_1 \equiv 0,$$

$$y_2(t) = -t \quad \forall t \geq 0 \implies z_2 \equiv 0, \; a_2(t) = t \quad \forall t \geq 0.$$

In this case, $z_1 \equiv z_2$, but $a_1 \neq a_2$, so that a is not a function of z. The same can happen to η: two different paths of W can give rise to different paths of X, but same η. The Theory of Additive Functionals provides the tools which allow to prove that that a.s. this does not happen.

Recall that, since X_α is conservative with unique invariant measure ν_α^F, the Revuz-measure of a AF $(A(t))_{t\geq 0}$ is defined as:

$$C_b(K_\alpha) \ni \varphi \mapsto \int_{K_\alpha} \mathbb{E}\left[\int_0^1 \varphi(X_\alpha(t,x))\, dA^x(t)\right] \nu_\alpha^F(dx),$$
$$\text{where}: \quad A^x(t) := A(t)\left(X_\alpha(\cdot,x)\right), \qquad t \geq 0.$$

In the proof of Theorem 3.8 below, we use several results of the Theory Additive Functionals. In the next theorem we collect the results we need, stating them in our setting in order to avoid notational confusion. For the general statements and the proofs, we refer to Theorem 2.4 in [MR 92] and Theorem 4.2 in [Fu 99].

Theorem 3.7 *Let $\alpha \geq 0$. For $\varphi \in W^{1,2}(\nu_\alpha^F)$, the next three conditions are equivalent:*

(i) For some constant $C > 0$ we have:

$$|\mathcal{E}^\alpha(\varphi,\psi)| \;\leq\; C\|\psi\|_\infty, \qquad \forall \psi \in W^{1,2}(\nu_\alpha^F) \cap L^\infty(\nu_\alpha^F). \tag{3.50}$$

(ii) There exists a finite signed measure m charging no $\mathcal{E}^\alpha$-exceptional set such that:

$$\mathcal{E}_1^\alpha(\varphi,\psi) \;=\; \int \psi\, dm, \qquad \forall \psi \in W^{1,2}(\nu_\alpha^F) \cap C_b(K_\alpha), \tag{3.51}$$

where $\mathcal{E}_1^\alpha := (\cdot,\cdot)_{L^2(\nu_\alpha^F)} + \mathcal{E}^\alpha$. We say that m is a $\mathcal{E}^\alpha$-smooth measure with 1-potential φ.

(iii) There exists a AF $(A(t))_t$ of X_α, unique up to equivalence, such that $\varphi(x) = \mathbb{E}_x[\int_0^\infty e^{-t} dA(t)]$ for $\mathcal{E}^\alpha$-q.e. x.

If (i)-(iii) hold, then m in (ii) is the Revuz-measure of $(A(t))_t$ in (iii).

Moreover, we shall use that for all AF A with Revuz-measure m and for all φ bounded and Borel, we have that $(f \cdot A)(t) := \int_0^t f(X_t)\, dA(t)$, $t \geq 0$, is a AF with Revuz-measure $f \cdot dm$: see e.g. Lemma 5.1.3 in [FOT 94].

Recall that $u_\alpha(t,\cdot) = X_\alpha(t,x)(\cdot)$ is the solution of (3.16). The main result of this section is the following:

Theorem 3.8

1. *Let $\alpha > 0$, $x \in K_\alpha \cap C_0$. Almost surely, there exist a measurable random set $S_\alpha \subseteq \mathbb{R}^+$ with $\eta^x_\alpha(\mathbb{R}^+ \backslash S_\alpha, (0,1)) = 0$, and a measurable map $r_\alpha : S_\alpha \mapsto (0,1)$, such that:*

$$\forall t \in S_\alpha,\ u_\alpha(t, r_\alpha(t)) = -\alpha, \ \text{and} \quad u_\alpha(t,\xi) > -\alpha \quad \forall \xi \in (0,1)\backslash\{r_\alpha(t)\}.$$

Almost surely, for all continuous l with compact support in $[0,\infty) \times (0,1)$, we have:

$$\int_{\mathcal{O}} l \, d\eta^x_\alpha \ = \ \int_0^\infty l(t, r_\alpha(t)) \, \eta^x_\alpha(dt, (0,1)), \tag{3.52}$$

$$i.e. \qquad \eta^x_\alpha(dt, d\xi) = \delta_{r_\alpha(t)}(d\xi) \, \eta^x_\alpha(dt,(0,1)) \quad \text{on } \mathcal{O}.$$

Finally, $t \mapsto \eta_\alpha([0,t] \times (0,1))$ is a PCAF in the strict sense of X_α with Revuz measure given by $\frac{1}{2}\sigma^F_\alpha((0,1),\cdot)$: i.e. there exists a PCAF in the strict sense of X_α, $(A_{(0,1)}(t))_{t\geq 0}$, such that

$$\eta^x_\alpha([0,t] \times (0,1)) \ = \ A_{(0,1)}(t)(X_\alpha(\cdot, x)) \qquad \forall\ t \geq 0,\ x \in K_\alpha,$$

$$\int_{K_\alpha} \mathbb{E}\left[\int_0^1 \varphi(X_\alpha(t,x)) \, dA^x_{(0,1)}(t)\right] \nu^F_\alpha(dx) = \frac{1}{2}\int_0^1 dr \int \varphi(z) \, \sigma^F_\alpha(r, dz).$$

2. *Let $\alpha = 0$, $x \in K_0 \cap C_0$. Almost surely, there exist a measurable random set $S_0 \subseteq \mathbb{R}^+$ with $\eta^x_0(\mathbb{R}^+ \backslash S_0, (0,1)) = 0$, and a measurable map $r_0 : S_0 \mapsto (0,1)$, such that:*

$$\forall t \in S_0, \quad u_0(t, r_0(t)) = 0, \quad \text{and} \quad u_0(t,\xi) > 0 \quad \forall \xi \in (0,1)\backslash\{r_0(t)\}.$$

Almost surely, for any $\delta \in (0,1/2)$ and for all continuous l with compact support in $\mathbb{R}^+ \times [\delta, 1-\delta]$, we have:

$$\begin{aligned} \int_{\mathcal{O}} l \, d\eta^x_0 \ &= \ \int_0^\infty l(t, r_0(t)) \, \eta^x_0(dt, (0,1)) \qquad (3.53) \\ &:= \ \int_0^\infty l(t, r_0(t)) \, \eta^x_0(dt, [\delta, 1-\delta]), \end{aligned}$$

$$i.e. \qquad \eta^x_0(dt, d\xi) = \delta_{r_0(t)}(d\xi) \, \eta^x_0(dt,(0,1)) \quad \text{on } \mathcal{O}.$$

Finally, for all $\delta \in (0,1/2)$, there exists a PCAF in the strict sense $(A_{[\delta,1-\delta]}(t))_{t\geq 0}$ of X_0 with Revuz measure given by $\frac{1}{2}\sigma^F_0([\delta,1-\delta],\cdot)$ such that:

$$\eta^x_0([0,t] \times [\delta, 1-\delta]) \ = \ A_{[\delta,1-\delta]}(t)(X_0(\cdot, x)) \qquad \forall\ t \geq 0,\ x \in K_0,$$

$$\int \mathbb{E}\left[\int_0^1 \varphi(X_0(t,x))\, dA^x_{[\delta,1-\delta]}(t)\right] \nu^F_0(dx) \;=\; \frac{1}{2}\int_\delta^{1-\delta} dr \int \varphi\, \sigma^F_0(r,dz),$$

The family $(A_{[\delta,1-\delta]})_{\delta\in(0,1/2)}$ satisfies the consistency condition:

$$A_{[\delta',1-\delta']}(t) \;=\; \int_0^t 1_{[\delta',1-\delta']}(r_0(s))\, dA_{[\delta,1-\delta]}(s), \qquad \forall\; 0<\delta\le\delta'<\frac{1}{2}.$$

Proof–We divide the proof into two steps.

Step 1. Let $\alpha \ge 0$ and $h \in C^2_c(0,1)$, $h \ge 0$. We claim that there exists a PCAF $(A_h(t))_{t\ge0}$ of X_α with Revuz measure:

$$\int_{K_\alpha} \mathbb{E}\left[\int_0^1 \varphi(X_\alpha(t,x))\, A^x_h(dt)\right] \nu^F_\alpha(dx) \;=\; \frac{1}{2}\int_0^1 dr\, h(r) \int \varphi\, \sigma^F_\alpha(r,dz)$$

and such that for $\mathbb{P}$-a.e. ω:

$$\int_0^1 h(\xi)\, \eta^x_\alpha([0,t],d\xi)(\omega) \;=:\; \eta^x_\alpha([0,t],h)(\omega) \;=\; A_h(t)\,(X_\alpha(\cdot,x)(\omega))\,.$$

In particular, $t \mapsto \eta_\alpha([0,t],h)$ is adapted to the filtration of X_α.

We can restrict to a dense countable family $\{h_n\} \subset D(A)$. We set for all $x \in H$:

$$U^{\varepsilon,\alpha}(x) := \mathbb{E}\left[\int_0^\infty e^{-t}\frac{1}{\varepsilon}\langle h, (X^\varepsilon_\alpha(t,x)+\alpha)^-\rangle dt\right],$$

$$U^\alpha(x) := \mathbb{E}\left[\int_0^\infty e^{-t}\eta^x_\alpha(dt,h)\right].$$

Then we have for all $\varphi \in \mathrm{Exp}_A(H)$:

$$\frac{1}{\varepsilon}\int_H \varphi(x)\langle h,(x+\alpha)^-\rangle\, \mu^\varepsilon_\alpha(dx)$$

$$= \int U^{\varepsilon,\alpha}\,\varphi\, d\mu^\varepsilon_\alpha \,+\, E^{\varepsilon,\alpha}(U^{\varepsilon,\alpha},\varphi) \;=\; \int U^{\varepsilon,\alpha}(\varphi \,-\, L^\varepsilon_\alpha\varphi)\, d\mu^\varepsilon_\alpha$$

For $\alpha > 0$, letting $\varepsilon \downarrow 0$, we find by Corollary 2.1:

$$\int U^\alpha(\varphi \,-\, L\varphi)\, d\nu^F_\alpha \;-\; \frac{1}{2}\int_0^1 dr \int \langle\nabla\varphi(z),\delta_r\rangle\, U^\alpha(z)\, \sigma^F_\alpha(r,dz)$$

$$= \frac{1}{2}\int_0^1 dr\, h(r)\int \varphi(z)\, \sigma^F_\alpha(r,dz), \tag{3.54}$$

Notice that we have for all $\alpha \geq 0$:

$$\int_0^\infty e^{-t}\eta_\alpha^x(dt,h) \;=\; -\langle x,h\rangle + \int_0^\infty e^{-t}\langle X_\alpha(t,x), h - Ah\rangle\, dt$$
$$+ \int_0^\infty e^{-t}\langle f(X_\alpha(t,x)), h\rangle\, dt - \int_0^\infty e^{-t} h(\xi)\, W(dt,d\xi),$$

$$U^\alpha(x) \;=\; -\langle x,h\rangle + \int_0^\infty e^{-t}\,\mathbb{E}\left[\langle X_\alpha(t,x), h - Ah\rangle + \langle f(X_\alpha(t,x)), h\rangle\right] dt.$$

Then $U^\alpha(x) \to U^0(x)$ as $\alpha \downarrow 0$ for all $x \in K_\alpha$, and by Proposition 3.1 $\{U^\alpha\}_{\alpha\geq 0}$ is an equi-Lipschitz family. Moreover, $0 \leq U^\alpha \circ \Pi_{K_\alpha} \leq C(1+|\cdot|)$. Then Lemma 1.1 and Theorem 2.1 yield (3.54) for every $\alpha \geq 0$. Moreover, $U^\alpha \in \mathrm{Lip}(K_\alpha) \subseteq D(\mathcal{E}^\alpha)$. By (3.54), Corollary 3.2 and the density of $\mathrm{Exp}_A(K_\alpha)$ in $D(\mathcal{E}^\alpha)$, we obtain for all $\varphi \in D(\mathcal{E}^\alpha) \cap C_b(K_\alpha)$, $\alpha \geq 0$:

$$\mathcal{E}_1^\alpha(U^\alpha,\varphi) \;=\; \frac{1}{2}\int_0^1 h(r)\int \varphi(z)\,\sigma_\alpha^F(r,dz) \;=:\; \frac{1}{2}\int \varphi(z)\,\sigma_\alpha^F(h,dz),$$

$$|\mathcal{E}_1^\alpha(U^\alpha,\varphi)| \;\leq\; \sigma_\alpha^F(h,K_\alpha)\,\|\varphi\|_\infty, \quad \sigma_\alpha^F(h,K_\alpha) \;<\; \infty, \tag{3.55}$$

where $\mathcal{E}_1^\alpha := (\cdot,\cdot)_{L^2(\nu_\alpha^F)} + \mathcal{E}^\alpha$. If $\varphi \in D(\mathcal{E}^\alpha) \cap L^\infty(\nu_\alpha^F)$, we set $\varphi_n := P_\alpha(1/n)\varphi$. Since P_α is Strong Feller, letting $n \to \infty$, we obtain that (3.55) holds for all $\varphi \in D(\mathcal{E}^\alpha) \cap L^\infty(\nu_\alpha^F)$. Now we can apply Theorem 3.7: U^α satisfies (i), $\sigma_\alpha^F(h,\cdot)$ is equal to the measure given by (ii), so that by (iii) there exists a PCAF $(A_h(t))_{t\geq 0}$ with Revuz measure $\sigma_\alpha^F(h,\cdot)$ and α-potential equal to U^α: in particular, we have $U^\alpha(x) = \mathbb{E}\left[\int_0^\infty e^{-t}\, dA_h^x(t)\right]$ for all $x \in K_\alpha \backslash V_\alpha$, for some $\mathcal{E}^\alpha$-properly exceptional set V_α.

Since U^α is continuous and therefore locally bounded on K_α, we can repeat the proof of Theorem 5.1.6 in [FOT 94], and extend $(A_h(t))_{t\geq 0}$ to a PCAF in the strict sense, which we still denote by $(A_h(t))_{t\geq 0}$. In particular, $U^\alpha(x) = \mathbb{E}\left[\int_0^\infty e^{-t}\, dA_h^x(t)\right]$ for all $x \in K_\alpha$. Now we can mimic the proof of Theorem 5.1.2 in [FOT 94]: by (3.49), we have for all $x \in K_\alpha$:

$$\mathbb{E}\left[\left(\int_0^\infty e^{-t}\, dA_h^x(t)\right)^2\right] \;=\; 2\,\mathbb{E}\left[\int_0^\infty e^{-2t} U^\alpha(X_\alpha(t,x))\, dA_h^x(t)\right]$$
$$=\; 2\,\mathbb{E}\left[\int_0^\infty e^{-t}\left(\int_t^\infty e^{-u}\eta_\alpha^x(du,h)\right) dA_h^x(t)\right],$$

$$\mathbb{E}\left[\left(\int_0^\infty e^{-t}\eta_\alpha^x(dt,h)\right)^2\right] \;=\; 2\,\mathbb{E}\left[\int_0^\infty e^{-2t} U^\alpha(X_\alpha(t,x))\,\eta_\alpha^x(dt,h)\right]$$

$$= 2\,\mathbb{E}\left[\int_0^\infty e^{-t}\left(\int_t^\infty e^{-u} dA_h^x(u)\right)\,\eta_\alpha^x(dt,h)\right]$$

$$= 2\,\mathbb{E}\left[\int_0^\infty e^{-t}\left(\int_0^t e^{-u}\eta_\alpha^x(du,h)\right)\,dA_h^x(t)\right],$$

$$\mathbb{E}\left[\int_0^\infty e^{-t}\eta_\alpha^x(dt,h)\int_0^\infty e^{-t}\,dA_h^x(t)\right]$$

$$= \mathbb{E}\left[\int_0^\infty e^{-t}\left(\left[\int_0^t + \int_t^\infty\right] e^{-u}\eta_\alpha^x(du,h)\right)\,dA_h^x(t)\right],$$

so that : $\mathbb{E}\left[\left(\int_0^\infty e^{-t}\eta_\alpha^x(dt,h) - \int_0^\infty e^{-t}\,dA_h^x(t)\right)^2\right] = 0$, and analogously :

$$\mathbb{E}\left[\left(\int_0^\infty e^{-\lambda t}\eta_\alpha^x(dt,h) - \int_0^\infty e^{-\lambda t}\,dA_h^x(t)\right)^2\right] = 0, \quad \forall \lambda > 0,$$

which implies $\eta_\alpha^x([0,t],h) = A_h^x(t)$ for all x, t, a.s.

Step 2. Let $\alpha > 0$, and $I \subseteq (0,1)$ be an interval. Denote by ψ_I the indicator function of the Borel set $\{x \in K_\alpha \cap C_0 : x \cdot 1_{(0,1)\setminus I} > -\alpha\}$. The key point is that the following holds:

$$\int_0^1 dr \int \varphi\,\psi_I(z)\,\sigma_\alpha^F(r,dz) = \int_I dr \int \varphi(z)\,\sigma_\alpha^F(r,dz), \qquad \forall \varphi \in C_b(K_\alpha). \tag{3.56}$$

Set now:

$$A_I^x(t) := \int_0^t \psi_I(X_\alpha(s,x))\,\eta_\alpha^x(ds,(0,1)), \quad t \geq 0,\ x \in H.$$

By Step 1, we have that A_I is a PCAF of X_α with Revuz measure equal to $\psi_I(z) \cdot \sigma_\alpha^F((0,1),dz)$. In particular, by (3.56):

$$\int_{K_\alpha} \mathbb{E}\left[\int_0^1 \varphi\psi_I(X_\alpha(s,x))\,\eta_\alpha^x(ds,(0,1))\right]\,\nu_\alpha(dx)$$

$$= \frac{1}{2}\int_0^1 dr \int \varphi\,\psi_I(z)\,\sigma_\alpha^F(r,dz) = \frac{1}{2}\int_I dr \int \varphi(z)\,\sigma_\alpha^F(r,dz)$$

which is the Revuz measure of $t \mapsto \eta_\alpha^x([0,t],I)$. By (iii) in Theorem 3.7, A_I and $\eta_\alpha(\,\cdot\,,I)$ are equivalent, i.e. there exists a $\mathcal{E}^\alpha$-properly exceptional set V_α, such that for all $x \in K_\alpha \backslash V_\alpha$, and for every interval $I \subseteq (0,1)$ with rational extremes, we have

$$\eta_\alpha^x([0,T],I) \,=\, \int_0^T \psi_I(X_\alpha(s,x))\,\eta_\alpha^x(ds,(0,1)) \quad \forall T \geq 0, \text{ a.s.}. \tag{3.57}$$

We claim that (3.57) holds for all $x \in K_\alpha \cap C_0$. First, $\eta_\alpha^z(\{0\},(0,1)) = 0$ for all $z \in K_\alpha$. Moreover, by Corollary 3.1, if $t > 0$ then $\mathbb{P}(X_\alpha(t,x) \in V_\alpha) = 0$. Then, for all $x \in K_\alpha \cap C_0$, $t > 0$, a.s. $X_\alpha(t,x) \in K_\alpha \backslash V_\alpha$ and by (3.57):

$$\begin{aligned}
\eta_\alpha^x([t,T],I) &= \eta_\alpha^{X_\alpha(t,x),W^t}([0,T-t],I) \\
&= \int_{]0,T-t]} \psi_I(X_\alpha(s+t,x))\,\eta_\alpha^{X_\alpha(t,x),W^t}(ds,(0,1)) \\
&= \int_t^T \psi_I(X_\alpha(s,x))\,\eta_\alpha^x(ds,(0,1)), \quad \forall T \geq t > 0, \text{ a.s.}
\end{aligned}$$

and the claim is proved. Now, fix $x \in K_\alpha \cap C_0$ and consider a regular conditional distribution of η on $[0,\infty)\times(0,1)$, w.r.t. the Borel map $(t,\xi) \mapsto t$: i.e., a measurable kernel $(t,J) \mapsto \gamma(t,J)$, where $t \geq 0$, $J \subseteq (0,1)$ Borel, such that

$$\eta_\alpha^x([t,T],J) \,=\, \int_t^T \gamma(s,J)\,\eta_\alpha^x(ds,(0,1)), \tag{3.58}$$

for all $0 \leq t \leq T$, $J \subseteq (0,1)$ Borel. By (3.57) and (3.58) we obtain that a.s. and for $\eta_\alpha^x(ds,(0,1))$-a.e. s:

$$\gamma(s,[a_n,b_n]) \,=\, \psi_{[a_n,b_n]}(X_\alpha(s,x)), \quad \forall a_n, b_n \in \mathbb{Q} \cap (0,1). \tag{3.59}$$

Notice that, since ψ_I is an indicator function, the right hand side of (3.59) assumes only the values 0 and 1. Therefore the measure $I \mapsto \gamma(s,I)$ takes only the values 0 and 1 on all intervals I with rational extremes in $(0,1)$, and the value 1 is assumed, since $\psi_{(0,1)} = 1_{K_\alpha \cap C_0}$. Then $\gamma(s,\,\cdot\,)$ is a Dirac mass at some point $r_\alpha(s) \in (0,1)$.

Consider $s \in S$, $q_n, p_n \in \mathbb{Q}$, $q_n \uparrow r_\alpha(s)$, $p_n \downarrow r(s)$, and set $I_n := [q_n, p_n]$: then $1 = \gamma(s,I_n) = \psi_{I_n}(X_\alpha(s,x))$, which means $u_\alpha(s,\xi) > -\alpha$ for all $\xi \in (0,1)\backslash I_n$. Therefore, $r_\alpha(s)$ is the unique $\xi \in (0,1)$ such that $X_\alpha(s,x)(\xi) = u_\alpha(s,\xi) = -\alpha$.

Let now $\alpha = 0$, and for all interval $I \subset (0,1)$ define ψ_I as the indicator function of the Borel set $\{x \in K_0 \cap C_0 : x(\xi) > 0, \forall \xi \in (0,1)\backslash I\}$. Notice that in this case it is not known whether $\eta_0^x([0,T],(0,1))$ is finite or not.

However, $\eta_0^x([0,T],(\delta,1-\delta)) < \infty$ for all $\delta > 0$. Therefore, the proof can proceed as in the case of $\alpha > 0$, provided one replaces $\eta_0^x(dt,(0,1))$ with $\eta_0^x(dt,(1/n,1-1/n))$ and then let $n \to \infty$. $\square$

Corollary 3.4 *For all Borel $\rho : (0,1) \mapsto \mathbb{R}^+$:*

$$\int_0^1 \frac{\rho(\xi)}{[\xi(1-\xi)]^{3/2}}\,d\xi \;<\; \infty \Longrightarrow \int_0^T \int_0^1 \rho(\xi)\,\eta_0(dt,d\xi) \;<\; \infty, \qquad \forall T \geq 0.$$

Chapter 4

Applications

This chapter is devoted to some applications of the results obtained in Chapters 2 and 3.

In section 4.1, we consider the following white-noise driven semilinear heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \xi^2} + \dfrac{(\delta-1)(\delta-3)}{8u^3} + \dfrac{\partial^2 W}{\partial t \partial \xi} \\ \\ u(0,\xi) = x(\xi), \ u(t,0) = u(t,1) = 0 \end{cases} \tag{4.1}$$

where $u \geq 0$, $x : [0,1] \mapsto [0,\infty)$ is continuous and satisfies $x(0) = x(1) = 0$, W is a Brownian sheet and $\delta > 3$. Notice that $(3,\infty) \ni \delta \mapsto (\delta-1)(\delta-3) \in (0,\infty)$ is increasing and bijective.

We prove that, for all $\delta > 3$, there exists a unique solution u of (4.1) and the process $x \mapsto u(t,\cdot)$, $t \geq 0$, is symmetric w.r.t. its unique invariant measure: the δ-dimensional Bessel Bridge on $[0,1]$. Notice that (4.1) is a SPDE with reaction-diffusion dissipative non-linearities and additive white-noise, whose solutions are non-negative. The non-linearity in (4.1) is singular enough to make the standard techniques non-effective.

We recall that Mueller in [Mu 98] and Mueller and Pardoux in [MP 99] study the following SPDE on $\mathbb{S}^1 := \mathbb{R}/\mathbb{Z}$ with periodic boundary condition:

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 \hat{u}}{\partial \xi^2} + \hat{u}^{-\alpha} + g(\hat{u})\,\dfrac{\partial^2 W}{\partial t \partial \xi} \\ \\ \hat{u}(0,\cdot) = \hat{x} \end{cases} \tag{4.2}$$

where $\alpha > 0$, $\hat{x} : \mathbb{S}^1 \mapsto \mathbb{R}$ is continuous, $\inf \hat{x} > 0$ and g satisfies suitable growth conditions. The results of [Mu 98] and [MP 99] are that $\alpha = 3$ is the critical exponent for $\hat{u}$ to hit zero in finite time. More precisely, the following is proved:

1. If $\alpha > 3$, then a.s. $\hat{u}(t,\xi) > 0$ for all $t \geq 0$, $\xi \in \mathbb{S}^1$.

2. If $\alpha < 3$, then with positive probability, there exist $t > 0$, $\xi \in \mathbb{S}^1$, such that $\hat{u}(t,\xi) = 0$.

However, the critical case $\alpha = 3$ is not covered by [Mu 98] and [MP 99]. The study of (4.1) allows to prove, that, in the critical case $\alpha = 3$ with $g \equiv 1$, $\hat{u}$ has a well-defined solution for all time $t \geq 0$ and $\hat{u}^{-3} \in L^1_{loc}(\mathcal{O})$. Indeed, if u solves (4.1), with $\delta = 5$, so that $(\delta-1)(\delta-3) = 8$, and $x \equiv 0$, then $\hat{u} \geq u$, and since $u^{-3} \in L^1_{loc}(\mathcal{O})$, then $\hat{u}^{-3} \in L^1_{loc}(\mathcal{O})$.

In section 4.2 we prove that, for $\mathbb{N} \ni \delta \geq 3$, the law $\mu^{\otimes\delta}$ of the δ-dimensional Brownian Bridge admits an integration by parts formula along the vector field $H^\delta \ni y \mapsto h\, y/|y|$, where $h \in C^2_c(0,1)$ and $|y|(\tau) := |y(\tau)|_{\mathbb{R}^\delta}$, see (4.15) below. However, the resulting measure $\Sigma^\delta(h,\cdot)$ is known explicitly only on radial functions $\Phi = \varphi(|\cdot|)$, $\varphi \in C_b(K_0)$. A complete explicit description of $\Sigma^\delta(h,\cdot)$ would lead to an Itô's formula for the modulus of the solution $Z_\delta : [0,\infty) \times [0,1] \times L^2(0,1;\mathbb{R}^\delta) \mapsto \mathbb{R}^\delta$ of the system of linear SPDEs:

$$\begin{cases} \dfrac{\partial Z_\delta}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 Z_\delta}{\partial \xi^2} + \dfrac{\partial^2 W^\delta}{\partial t \partial \xi}, \\[2ex] Z_\delta(t,\cdot,x) \in (C_0)^\delta, \quad t > 0 \\[2ex] Z_\delta(0,\cdot,x) = x \in H^\delta \end{cases} \tag{4.3}$$

$W^\delta := (W_1,\ldots,W_\delta)$ and $\{W_i\}_i$ are independent Brownian Sheets.

4.1 SPDE generated by the δ-Bessel Bridge, $\delta > 3$

Recall that in section 2.3 we defined for all $\delta > 3$:

$$c(\delta) := \frac{\delta-3}{2}, \qquad \kappa(\delta) := \frac{(\delta-3)(\delta-1)}{4}.$$

This section is devoted to the proof of the following

Theorem 4.1 *Let $\delta > 3$. For all $x \in K_0 \cap C_0$, there exists a unique random continuous $u : [0,\infty) \times [0,1] \mapsto \mathbb{R}^+$, such that $u^{-3} \in L^1_{loc}$ and:*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \xi^2} + \dfrac{\kappa(\delta)}{2u^3} + \dfrac{\partial^2 W}{\partial t \partial \xi} \\ \\ u(0,\xi) = x(\xi), \;\; u(t,0) = u(t,1) = 0 \end{cases} \tag{4.4}$$

Moreover, u is adapted and the process $x \mapsto u$ admits as unique invariant measure the law π_δ of the δ-dimensional Bessel Bridge on $[0,1]$. The symmetric bilinear form

$$C^1_b(K_0) \ni \varphi, \psi \;\mapsto\; \frac{1}{2}\int_{K_0} \langle \nabla\varphi, \nabla\psi\rangle \, d\pi_\delta$$

is closable, and the process $x \mapsto u$ is properly associated with the closure and Strong Feller.

Remark 4.1 Let u^δ be the unique solution of (4.4), for all $\delta > 3$.

- If $\delta_1 > \delta_2 > 3$, then the law of u^{δ_1} and u^{δ_2} are singular: indeed, u^{δ_i} is Strong Feller, and therefore the law of $u^{\delta_i}(t,\cdot)$ is absolutely continuous w.r.t. π_{δ_i}, and π_{δ_1}, π_{δ_2} are singular. Analogously, the law of u^δ and the law of u, solution of the Nualart-Pardoux equation (3.1), are singular.

- For all $(t,\xi) \in \mathcal{O}$, $(3,\infty) \ni \delta \mapsto u^\delta(t,\xi)$ is non-decreasing, and by the uniqueness stated in Theorem 3.2, as $\delta \downarrow 3$: $u^\delta \downarrow u$ uniformly on $[0,T] \times [0,1]$, $T \geq 0$, and

$$\frac{\delta - 3}{4\,(u^\delta)^3}\, dt\, d\xi \;\to\; \eta(dt, d\xi) \qquad \text{distributionally on } \mathcal{O},$$

  where (u,η) is the solution of equation (3.1), with $f \equiv 0$.

Proof of Theorem 4.1–Uniqueness is clear by the dissipativity of the coefficients in (4.4). We divide the proof of existence into several steps. The idea is to approximate the solution u from below by means of functions u_ϵ, $\epsilon > 0$, solving equations of Nualart-Pardoux type with suitable reaction-diffusion coefficients f_ϵ. Indeed, we can choose f_ϵ so that the invariant measure of u_ϵ is the law of z_ϵ, see (2.32), which converges to π_δ. The results of section 2.3 are used extensively.

Step 1. We define for $0 < \rho \le \epsilon$: $f_\epsilon^\rho,\ f_\epsilon\ :\ [0,1]\times[0,\infty)\mapsto[0,\infty)$,

$$\begin{aligned} f_\epsilon^\rho(\tau,a) \quad &:= \quad \frac{c(\delta)}{2}\left[\frac{c(\delta)-1}{(\epsilon+a)^3}+\frac{1}{(\rho+a)^2(\epsilon+a)}+\frac{1}{(\rho+a)(\epsilon+a)^2}\right. \\ &\qquad\qquad \left. +\frac{\epsilon}{(1+\rho-\tau)(\epsilon+a)^2}\right], \end{aligned} \tag{4.5}$$

$$f_\epsilon := \sup_{\rho>0} f_\epsilon^\rho \ = \ \lim_{\rho\downarrow 0} f_\epsilon^\rho.$$

Recall (2.37) and (2.38).

Notice that $-f_\epsilon^\rho$ and $\mathbb{R}^+ \ni y \mapsto -\kappa(\delta)(\epsilon+y)^{-3}/2$ satisfy (H1)-(H3) of Section 3.1. We set the following SPDEs with reflection at 0 for $x \in K_0 \cap C_0$:

$$\begin{cases} \dfrac{\partial u_\epsilon^\rho}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u_\epsilon^\rho}{\partial \xi^2} + f_\epsilon^\rho(\xi, u_\epsilon^\rho) + \dfrac{\partial^2 W}{\partial t \partial \xi} + \eta_{\epsilon,\rho}(t,\xi) \\ u_\epsilon^\rho(0,\xi) = x(\xi),\ u_\epsilon^\rho(t,0) = u_\epsilon^\rho(t,1) = 0 \\ u_\epsilon^\rho \ge 0,\ d\eta_{\epsilon,\rho} \ge 0,\ \int u_\epsilon^\rho \, d\eta_{\epsilon,\rho} = 0 \end{cases} \tag{4.6}$$

$$\begin{cases} \dfrac{\partial v_\epsilon}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 v_\epsilon}{\partial \xi^2} + \dfrac{\kappa(\delta)}{2(\epsilon+v_\epsilon)^3} + \dfrac{\partial^2 W}{\partial t \partial \xi} + \theta_\epsilon(t,\xi) \\ v_\epsilon(0,\xi) = x(\xi),\ v_\epsilon(t,0) = v_\epsilon(t,1) = 0 \\ v_\epsilon \ge 0,\ d\theta_\epsilon \ge 0,\ \int v_\epsilon \, d\theta_\epsilon = 0 \end{cases} \tag{4.7}$$

and we set $X_\epsilon^\rho(t,x) := u_\epsilon^\rho(t,\cdot)$. By Theorem 3.3, the unique invariant measures of u_ϵ^ρ is $\gamma_\epsilon^\rho\, d\nu$, defined in (2.37). By Theorem 3.8, the Revuz-measures of $\eta_{\epsilon,\rho}$ is given by $\frac{1}{2\nu(\gamma_\epsilon^\rho)}\,\gamma_\epsilon^\rho\,\sigma_0(r,\cdot)$.

We shall prove that u_ϵ^ρ converges to a process u, which is symmetric with respect to π_δ and satisfies (4.4). The proof of this is based only on monotonicity arguments and on the computations already performed in section 2.3. Then, we shall prove that v_ϵ converges to a continuous process v, and $u \equiv v$. First, notice that:

(a1) $a \mapsto f_\epsilon^\rho(\tau,a)$ is non-increasing

(a2) $\rho \mapsto f_\epsilon^\rho(\tau,a)$ is non-increasing

(a3) $(0,a) \ni \epsilon \mapsto f_\epsilon^\rho(\tau,a-\epsilon)$ is non-decreasing (a4) $f_\epsilon^\rho(\tau, a) \geq \kappa(\delta)(\epsilon + a)^{-3}/2$.

By Lemma 1.3 and (a1)-a(4), we obtain:

(b1) $\rho \mapsto u_\epsilon^\rho$ is non-increasing

(b2) $\epsilon \mapsto \epsilon + u_\epsilon^\rho$ is non-decreasing

(b3) $\epsilon \mapsto v_\epsilon$ is non-increasing, $\epsilon \mapsto \epsilon + v_\epsilon$ is non-decreasing.

(b4) $u_\epsilon^\rho \geq v_\epsilon$.

Step 2. By (b1) we obtain that, almost surely, there exists the limit $u_\epsilon := \lim_{\rho\downarrow 0} u_\epsilon^\rho = \sup_{\rho>0} u_\epsilon^\rho$. We set $X_\epsilon(t,x) := \lim_{\rho\downarrow 0} X_\epsilon^\rho(t,x)$. We claim that a.s. for every $t \geq 0$, $u_\epsilon(t,\cdot) \in L^2(0,1)$. Indeed, we have:

$$\|X_\epsilon^\rho(t,0)\| \leq \|X_\epsilon^\rho(t,x)\| \leq \|X_\epsilon^\rho(t,0)\| + \|x\|, \tag{4.8}$$

The first inequality follows from a comparison argument, and the second one by the Lipschitz continuity of $X_\epsilon^\rho(t,\cdot)$. Therefore, we can reduce to the case $x \equiv 0$. Integrating the first inequality in (4.8) with respect to $(\gamma_\epsilon^\rho \, d\nu) \otimes \mathbb{P}$, where the invariant measure $\gamma_\epsilon^\rho \, d\nu$ of X_ϵ^ρ, we obtain:

$$\mathbb{E}[\|X_\epsilon^\rho(t,0)\|] \leq \int \mathbb{E}[\|X_\epsilon^\rho(t,x)\|] \, \gamma_\epsilon^\rho(x) \, \nu(dx) = \int \|x\| \gamma_\epsilon^\rho(x) \, \nu(dx)$$

By Beppo-Levi and (2.40)-(2.41), we obtain:

$$\mathbb{E}[\|X_\epsilon(t,0)\|] \leq \int \|x\| \gamma_\epsilon(x) \, \nu(dx) = \mathbb{E}[\|z_\epsilon\|] < \infty,$$

and the claim is proved. By Dominated Convergence, we obtain:

$$\int \mathbb{E}[\varphi(X_\epsilon(t,x))] \, \gamma_\epsilon(x) \, \nu(dx) = \int \varphi(x) \gamma_\epsilon(x) \, \nu(dx), \quad \forall \, \varphi \in C_b(H),$$

i.e. $\gamma_\epsilon \, d\nu$ is an invariant probability measure for X_ϵ. Moreover, $X_\epsilon(t,\cdot)$ is 1-Lipschitz. Analogously, by (b2) $\epsilon \mapsto \epsilon + u_\epsilon$ is non-decreasing. Therefore, there exist the limits $u := \lim_{\epsilon\downarrow 0} u_\epsilon$, $H \ni X(t,x) := \lim_{\epsilon\downarrow 0} X_\epsilon(t,x)$ for all $t \geq 0$. Be the equicontinuity of $X_\epsilon(t,\cdot)$, we have for $\varphi \in C_b(H)$:

$$\mathbb{E}[\varphi(X(t,x_\delta))] = \lim_{\epsilon\downarrow 0} \mathbb{E}[\varphi(X_\epsilon(t,z_\epsilon))] = \lim_{\epsilon\downarrow 0} \mathbb{E}[\varphi(z_\epsilon)] = \mathbb{E}[\varphi(x_\delta)],$$

i.e. π_δ is an invariant probability measure for X.

Step 3. By Dominated Convergence, we obtain for $h \in C_c^2(0,1)$, $h \geq 0$:

$$\langle X_\epsilon(t,x), h\rangle - \langle x, h\rangle - \frac{1}{2}\int_0^t \langle h'', X_\epsilon(s,x)\rangle\, ds - \langle h, W(t)\rangle \geq \int_0^t \int_0^1 \frac{h\,\kappa(\delta)}{2(\epsilon + X_\epsilon)^3} \tag{4.9}$$

Since $\epsilon \mapsto \epsilon + u_\epsilon$ is non-decreasing we can let $\epsilon \downarrow 0$, and obtain by Beppo-Levi Theorem:

$$\langle X(t,x), h\rangle - \langle x, h\rangle - \frac{1}{2}\int_0^t \langle h'', X(s,x)\rangle\, ds - \langle h, W(t)\rangle \geq \frac{\kappa(\delta)}{2}\int_0^t \int_0^1 \frac{h}{(X)^3} \tag{4.10}$$

Recall that $\gamma_\epsilon^\rho\, d\nu$ is invariant for X_ϵ^ρ. Then:

$$\int \mathbb{E}\Big[\langle X_\epsilon^\rho(t,x), h\rangle - \langle x, h\rangle - \frac{1}{2}\int_0^t \langle h'', X_\epsilon^\rho(s,x)\rangle\, ds - \langle h, W(t)\rangle\Big]\, \gamma_\epsilon^\rho(x)\, \nu(dx)$$

$$= \int \mathbb{E}\Big[\int_0^t \langle h, f_\epsilon^\rho(X_\epsilon^\rho(s,x))\rangle\, ds + \int_{[0,t]\times[0,1]} h\, d\eta_{\epsilon,\rho}^x\Big]\gamma_\epsilon^\rho(x)\, \nu(dx)$$

$$= t\int \langle h, \mathbb{E}[f_\epsilon^\rho(x)]\rangle \gamma_\epsilon^\rho(x)\, \nu(dx) + \frac{t}{2}\int_0^1 dr\, h(r) \int \gamma_\epsilon^\rho(x)\, \sigma_0(r,dx)$$

First, we have by Step 7 of the proof in section 2.3:

$$\lim_{\epsilon\downarrow 0}\lim_{\rho\downarrow 0} \int_0^1 dr\, h(r)\int \gamma_\epsilon^\rho(x)\,\sigma_0(r,dx) = \lim_{\epsilon\downarrow 0}\int_0^1 dr\, h(r)\int \gamma_\epsilon(x)\,\sigma_0(r,dx)$$

$$= \lim_{\epsilon\downarrow 0}\int_0^1 dr\, h(r)\int \mathcal{Z}_\epsilon^r(w)\,\nu^r(dw) = 0.$$

Furthermore, by (2.50), (2.51) and (2.52):

$$\lim_{\epsilon\downarrow 0}\lim_{\rho\downarrow 0}\int \langle h, \mathbb{E}[f_\epsilon^\rho(x)]\rangle \gamma_\epsilon^\rho(x)\,\nu(dx) = \lim_{\epsilon\downarrow 0}\mathbb{E}[f_\epsilon(z_\epsilon)] = \frac{\kappa(\delta)}{2}\,\mathbb{E}\left[\langle h\frac{1}{(x_\delta)^3}\rangle\right]$$

By Dominated Convergence, we have:

$$\lim_{\rho\downarrow 0}\int \mathbb{E}\Big[\langle X_\epsilon^\rho(t,\cdot), h\rangle - \langle \cdot, h\rangle - \frac{1}{2}\int_0^t \langle h'', X_\epsilon^\rho(s,\cdot)\rangle\, ds - \langle h, W(t)\rangle\Big]\,\gamma_\epsilon^\rho\, d\nu$$

$$= \mathbb{E}\Big[\langle X_\epsilon(t,z_\epsilon), h\rangle - \langle z_\epsilon, h\rangle - \frac{1}{2}\int_0^t \langle h'', X_\epsilon(s,z_\epsilon)\rangle\, ds - \langle h, W(t)\rangle\Big].$$

By the 1-Lipschitz continuity of $X_\epsilon(t,\cdot)$, we can let $\epsilon \downarrow 0$ and obtain:

$$\mathbb{E}\Big[\langle X(t,x_\delta),h\rangle - \langle x_\delta,h\rangle - \frac{1}{2}\int_0^t \langle h'', X(s,x_\delta)\rangle\, ds - \langle h, W(t)\rangle\Big]$$

$$= \frac{t}{2}\,\kappa(\delta)\,\mathbb{E}\left[\langle h\frac{1}{(x_\delta)^3}\rangle\right] \tag{4.11}$$

On the other hand, since $\epsilon \mapsto X_\epsilon(t,z_\epsilon)$ is non-increasing, we have by (4.10):

$$0 \le \frac{1}{2}\int_0^t \langle h, \frac{\kappa(\delta)}{(\epsilon + X_\epsilon(s,z_\epsilon))^3}\rangle\, ds$$

$$\le \langle X_\epsilon(t,z_\epsilon),h\rangle - \langle z_\epsilon, h\rangle - \frac{1}{2}\int_0^t\int_0^1 h'' X_\epsilon(s,z_\epsilon)\, ds\, d\xi - \langle h, W(t)\rangle$$

$$\le \langle X(t,x_\delta),h\rangle \,+\, \|h''\| \int_0^t \|X(s,x_\delta)\|\, ds \,-\, \langle h, W(t)\rangle,$$

$$\mathbb{E}\left[\langle X(t,x_\delta),h\rangle \,+\, \|h''\| \int_0^t \|X(s,x_\delta)\|\, ds \,-\, \langle h, W(t)\rangle\right]$$

$$= \mathbb{E}\left[\langle x_\delta, h\rangle \,+\, t\,\|h''\|\,\|x_\delta\|\right] \,<\, \infty,$$

and by Dominated Convergence Theorem we obtain:

$$\mathbb{E}\Big[\int_0^t \langle h, \frac{1}{(X(s,x_\delta))^3}\rangle\, ds\Big] \;=\; \lim_{\epsilon\downarrow 0}\mathbb{E}\Big[\int_0^t \langle h, \frac{1}{(\epsilon + X_\epsilon(s,z_\epsilon))^3}\rangle\, ds\Big]$$

$$= \lim_{\epsilon\downarrow 0} t\,\mathbb{E}\Big[\langle h, \frac{1}{(\epsilon + z_\epsilon)^3}\rangle\Big] \;=\; t\,\mathbb{E}\left[\langle h\frac{1}{(x_\delta)^3}\rangle\right]. \tag{4.12}$$

By (4.10), (4.11) and (4.12), we obtain that there exists $\Lambda \subseteq K_0 \times \Omega$ with $\pi_\delta \otimes \mathbb{P}(\Lambda) = 1$, such that for every $(x,\omega) \in \Lambda$, we have for all $t \ge 0$:

$$\langle X(t,x),h\rangle = \langle x,h\rangle + \frac{1}{2}\int_0^t \left(\langle h'', X(s,x)\rangle + \langle \frac{\kappa(\delta)}{(X(s,x))^3}, h\rangle\right) ds + \langle h, W(t)\rangle \tag{4.13}$$

Step 4. By (b4) $u_\epsilon \ge v_\epsilon$, and by (b3) there exists the limit in the uniform norm of v_ϵ as $\epsilon \downarrow 0$. Setting $v := \lim_{\epsilon\downarrow 0} v_\epsilon$, we have that v is a.s. continuous, and $u \geq v$. Setting $Y(t,x) := v(t,\cdot)$, we have that $Y(t,\cdot)$ is 1-Lipschitz. Moreover, by Beppo-Levi, for $h \geq 0$:

$$\langle v(t,\cdot), h\rangle \geq \langle x, h\rangle + \frac{1}{2}\int_0^t \left(\langle v(s,\cdot), h''\rangle + \langle \frac{\kappa(\delta)}{(v(s,\cdot))^3}, h\rangle\right) ds + \langle W(t), h\rangle.$$

If now $h(\xi) := \xi(1-\xi)$, $\xi \in [0,1]$, then $h \in H^2 \cap H_0^1(0,1)$, $h > 0$ on $(0,1)$ and $h'' \leq 0$. By (4.13) for all $(x,\omega) \in \Lambda$, $t \geq 0$:

$$\begin{aligned}
\langle u(t,\cdot), h\rangle &\geq \langle v(t,\cdot), h\rangle \\
&\geq \langle x, h\rangle + \frac{1}{2}\int_0^t \left(\langle v(s,\cdot), h''\rangle + \langle \frac{\kappa(\delta)}{(v(s,\cdot))^3}, h\rangle\right) ds + \langle W(t), h\rangle \\
&\geq \langle x, h\rangle + \frac{1}{2}\int_0^t \left(\langle u(s,\cdot), h''\rangle + \langle \frac{\kappa(\delta)}{(u(s,\cdot))^3}, h\rangle\right) ds + \langle W(t), h\rangle \\
&= \langle u(t,\cdot), h\rangle
\end{aligned}$$

so that for all $(x,\omega) \in \Lambda$, for all $t \geq 0$ $u(t,\cdot) = v(t,\cdot)$ in H. By Fubini-Tonelli Theorem, for $\mathbb{P}$-a.e. ω, we have $\pi_\delta(\Lambda \cap (K_0 \times \{\omega\})) = 1$. By the continuity of $X(t,\cdot)$ and $Y(t,\cdot)$, we obtain that $u(t,\cdot) = v(t,\cdot)$ in H for all $t \geq 0$, $x \in K_0$, a.s. Therefore, v is a.s. continuous, solves (4.4) and is symmetric with respect to its unique invariant measure π_δ.

The last assertions of Theorem 4.1 can be proved arguing as in the proof of Theorems 3.4 and 3.5. □

4.2 IbPF on δ-d Brownian Bridge, $\delta \geq 3$

Let $\mathbb{N} \ni \delta \geq 3$ and denote the δ-dimensional Brownian Bridge $(\beta^\delta_\tau)_{\tau\in[0,1]}$. We set $H^\delta = \oplus_{i=1}^\delta H = L^2(0,1;\mathbb{R}^3)$,

$$H^\delta \ni y \mapsto |y| \in K_0, \quad |y|(\tau) := |y(\tau)|_{\mathbb{R}^\delta}, \quad \tau \in (0,1).$$

$$C_b(K_0) \ni \varphi \mapsto \overline{\varphi} \in C_b(H^\delta), \quad \overline{\varphi}(y) = \varphi(|y|).$$

Recall that the image measure of $\mu^{\otimes\delta}$ under $|\cdot|$ is π_δ, i.e.

$$\int_{H^\delta} \varphi(|y|)\, \mu^{\otimes\delta}(dy) = \int_{K_0} \varphi(x)\, \pi_\delta(dx), \qquad \forall \varphi \in C_b(H). \tag{4.14}$$

We denote the gradient of $\Phi \in W^{1,2}(H^\delta, \mu^{\otimes\delta})$ by $\overline{\nabla}\Phi$. Then:

Theorem 4.2 *For all $h \in H^2 \cap H^1_0(0,1)$, there exists a finite signed measure $\Sigma^\delta(h,\cdot)$ on H^δ such that for all $\Phi \in W^{1,2}(H^\delta, \mu^{\otimes\delta}) \cap C_b(H^\delta)$:*

$$\int_{H^\delta} \langle \overline{\nabla}\Phi(y), h\,\frac{y}{|y|}\rangle_{H^\delta}\, \mu^{\otimes\delta}(dy) \;=\; -\int_{H^\delta} \Phi(y)\, \Sigma^\delta(h,dy). \tag{4.15}$$

The measure $\Sigma^\delta(h,\cdot)$ is equal, on the radial functions $\Phi = \varphi(|\cdot|)$, $\varphi \in C_b(K_0)$, to:

$$\int_{H^\delta} \Phi(y)\, \Sigma^\delta(h,dy) \;=\; \int_{H^\delta} \varphi(|y|)\, \Sigma^\delta(h,dy)$$

$$= \int_{K_0} \varphi(x)\, \langle x, h''\rangle\, \pi_\delta(dx) \;+\; \int_0^1 dr\;\, h(r) \int \varphi(x)\, \sigma_0(r,dx), \quad \text{if } \delta = 3,$$

$$\int_{H^\delta} \Phi(y)\, \Sigma^\delta(h,dy) \;=\; \int_{H^\delta} \varphi(|y|)\, \Sigma^\delta(h,dy)$$

$$= \int_{K_0} \varphi(x)\, \big(\langle x, h''\rangle \,+\, \kappa(\delta)\, \langle x^{-3}, h\rangle\big)\, \pi_\delta(dx) \quad \text{if } \delta \,\geq\, 4.$$

Proof–Let:

$$W^{1,2}(H^\delta, \mu^{\otimes\delta}) \ni \Phi, \Psi \mapsto \Lambda^\delta(\Phi,\Psi) \;:=\; \frac{1}{2}\int_{H^\delta} \langle \overline{\nabla}\Phi, \overline{\nabla}\Psi\rangle\, d\mu^{\otimes\delta},$$

Let $\mathcal{D}$ be the closure in $W^{1,2}(H^\delta, \mu^{\otimes\delta})$ of $\{\overline{\varphi} := \varphi(|\cdot|) : \varphi \in \mathrm{Lip}(K_0)\}$ and $\Pi : W^{1,2}(H^\delta, \mu^{\otimes\delta}) \mapsto \mathcal{D}$ the unique symmetric orthogonal projector w.r.t. $\Lambda^\delta_1 := (\cdot,\cdot)_{L^2(\mu^{\otimes\delta})} + \Lambda^\delta$. Arguing as in Lemma 3.4, we find:

$$\frac{1}{2}\int_{K_0} \langle \nabla\varphi, \nabla\psi\rangle\, d\pi_\delta \;=\; \Lambda^\delta(\overline{\varphi}, \overline{\psi}), \quad \forall \varphi, \psi \in W^{1,2}(\pi_\delta).$$

Since $(\mathcal{D}, \Lambda^\delta_1)$ is a Dirichlet form, Π satisfies a maximum principle:

$$\|\Pi\Phi\|_{L^\infty(\mu^{\otimes\delta})} \;\leq\; \|\Phi\|_{L^\infty(\mu^{\otimes\delta})}, \quad \forall \Phi \in W^{1,2}(H^\delta, \mu^{\otimes\delta}) \cap L^\infty(\mu^{\otimes\delta}).$$

Let $\psi_h(x) := \langle x, h\rangle$, $x \in K_0$, then:

$$\overline{\nabla}\,\overline{\psi}_h(y) \;=\; h\,\frac{y}{|y|}, \quad \text{for } \mu^{\otimes\delta} - \text{a.e. } y.$$

Let $\Phi \in W^{1,2}(H^\delta, \mu^{\otimes\delta}) \cap L^\infty(\mu^{\otimes\delta})$. Then $\Pi\Phi = \overline{\varphi}$ for some $\varphi \in W^{1,2}(K_0, \pi_\delta) \cap L^\infty(\pi_\delta)$, and

$$\int_{H^\delta} \langle \overline{\nabla}\Phi(y), h\,\frac{y}{|y|}\rangle_{H^\delta}\, \mu^{\otimes\delta}(dy) \;=\; \int_{H^\delta} \langle \overline{\nabla}\Phi, \overline{\nabla}\,\overline{\psi}_h\rangle_{H^\delta}\, d\mu^{\otimes\delta},$$

$$\Lambda_1^\delta(\Phi,\overline{\psi}_h) \;=\; \Lambda_1^\delta(\Pi\Phi,\overline{\psi}_h) \;=\; \int_{K_0} \varphi\,\psi_h\,d\pi_\delta \;+\; \frac{1}{2}\int_{K_0} \langle\nabla\varphi, h\rangle\,d\pi_\delta,$$

where $\Lambda_1^\delta := (\cdot,\cdot)_{L^2(\mu^{\otimes\delta})} + \Lambda^\delta$. By Theorem 4.1, there exists a sequence $\varphi_n \in \mathrm{Lip}(K_0) \cap W^{1,2}(K_0,\pi_\delta)$, such that: φ converges to φ pointwise and weakly in $W^{1,2}(K_0,\pi_\delta)$, and $\|\varphi_n\|_\infty$ is uniformly bounded. By (2.2) for $\delta = 3$ and (2.3) for $\delta \geq 4$ we obtain:

$$\left|\int_{H^\delta} \langle\overline{\nabla}\Phi(y), h\,\frac{y}{|y|}\rangle_{H^\delta}\,\mu^{\otimes\delta}(dy)\right| \;\leq\; C_h\|\Pi\Phi\|_{L^\infty(\mu^{\otimes\delta})} \;\leq\; C_h\|\Phi\|_{L^\infty(\mu^{\otimes\delta})},$$

and by the analogue of (ii) in Theorem 3.7 we have the thesis. □

The interest of Theorem 4.2 is that the field $H^\delta \ni y \mapsto hy/|y|$ does not take values in the Cameron-Martin space of the Gaussian measure $\mu^{\otimes\delta}$, i.e. $(H_0^1(0,1))^\delta$. Let now $Z_\delta : [0,\infty)\times[0,1]\times L^2(0,1;\mathbb{R}^\delta) \mapsto \mathbb{R}^\delta$ be the solution of the system of linear SPDEs:

$$\begin{cases} \dfrac{\partial Z_\delta}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 Z_\delta}{\partial\xi^2} + \dfrac{\partial^2 W^\delta}{\partial t\partial\xi}, \\[2ex] Z_\delta(t,\cdot,z) \in (C_0)^\delta, \quad t>0 \\[2ex] Z_\delta(0,\cdot,z) \;=\; z \;\in\; H^\delta \end{cases} \tag{4.16}$$

$W^\delta := (W_1,\ldots,W_\delta)$, and $\{W_i\}_i$ are independent Brownian Sheets. By Proposition 1.1, Z_δ is associated with the Dirichlet Form Λ^δ. Then, by Theorem 4.2 we can apply the celebrated Fukushima decomposition and, by Theorem 6.2 in [Fu 99], obtain:

Corollary 4.1 *For all $h \in H^2 \cap H_0^1(0,1)$, we have the following decomposition:*

$$\langle |Z_\delta(t,x)|, h\rangle_H \;=\; \langle |x|, h\rangle_H \;+\; N^h(t) \;+\; \langle W(t), h\rangle_H$$

where $(W(t))_{t\geq 0}$ is a cylindrical white-noise in $L^2(0,1)$ and N^h is a bounded-variation Additive Functional of Z_δ with associated Revuz-measure equal to $\Sigma^\delta(h,\cdot)$.

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