We construct the time evolution of Kawasaki dynamics for a spatial infinite particle system in terms of generating functionals. This is carried out by an Ovsjannikov-type result in a scale of Banach spaces, which leads to a local (in time) solution. An application of this approach to Vlasov-type scaling in terms of generating functionals is considered as well.

1. Introduction

Originally, Bogoliubov generating functionals (GF for short) were introduced by N. N. Bogoliubov in [2] to define correlation functions for statistical mechanics systems. Apart from this specific application, and many others, GF are, by themselves, a subject of interest in infinite dimensional analysis. This is partially due to the fact that to a probability measure $\mu$ defined on the space $\Gamma$ of locally finite configurations $\gamma \subset \mathbb{R}^d$ one may associate a GF

$$B_\mu(\theta) := \int_\Gamma d\mu(\gamma) \prod_{x \in \gamma} (1 + \theta(x)),$$

yielding an alternative method to study the stochastic dynamics of an infinite particle system in the continuum by exploiting the close relation between measures and GF [4, 9].

Existence and uniqueness results for the Kawasaki dynamics through GF arise naturally from Picard-type approximations and a method suggested in [6, Appendix 2, A2.1] in a scale of Banach spaces (see e.g. [5, Theorem 2.5]). This method, originally presented for equations with coefficients time independent, has been extended to an abstract and general framework by T. Yamanaka in [12] and L. V. Ovsjannikov in [10] in the linear case, and many applications were exposed by F. Treves in [11]. As an aside, within an analytical framework outside of our setting, all these statements are very closely related to variants of the abstract Cauchy-Kovalevskaya theorem. However, all these abstract forms only yield a local solution, that is, a solution which is defined on a finite time interval. Moreover, starting with an initial condition from a certain Banach space, in general the solution evolves on larger Banach spaces.

As a particular application, this work concludes with the study of the Vlasov-type scaling proposed in [3] for general continuous particle systems and accomplished in [1] for the Kawasaki dynamics. The general scheme proposed in [3] for correlation functions yields a limiting hierarchy which possesses a chaos preservation property, namely, starting with a Poissonian (non-homogeneous) initial state this structural property is preserved during the time evolution. In Section 4 the same problem is formulated in terms of GF.
and its analysis is carried out by the general Ovsjannikov-type result in a scale of Banach spaces presented in [5, Theorem 4.3].

2. General Framework

In this section we briefly recall the concepts and results of combinatorial harmonic analysis on configuration spaces and Bogoliubov generating functionals needed throughout this work (for a detailed explanation see [7, 9]).

2.1. Harmonic analysis on configuration spaces. Let $\Gamma := \Gamma_{\mathbb{R}^d}$ be the configuration space over $\mathbb{R}^d$, $d \in \mathbb{N}$,

$$
\Gamma := \{ \gamma \subset \mathbb{R}^d : |\gamma \cap \Lambda| < \infty \text{ for every compact } \Lambda \subset \mathbb{R}^d \},
$$

where $|\cdot|$ denotes the cardinality of a set. We identify each $\gamma \in \Gamma$ with the non-negative Radon measure $\sum x \in \gamma \delta_x$ on the Borel $\sigma$-algebra $B(\mathbb{R}^d)$, where $\delta_x$ is the Dirac measure with mass at $x$, which allows to endow $\Gamma$ with the vague topology and the corresponding Borel $\sigma$-algebra $B(\Gamma)$.

For any $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ let

$$
\Gamma^{(n)} := \{ \gamma \in \Gamma : |\gamma| = n \}, \quad n \in \mathbb{N}, \quad \Gamma^{(0)} := \{ \emptyset \}.
$$

Clearly, each $\Gamma^{(n)}$, $n \in \mathbb{N}$, can be identify with the symmetrization of the set $\{(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n : x_i \neq x_j \text{ if } i \neq j\}$, which induces a natural (metrizable) topology on $\Gamma^{(n)}$ and the corresponding Borel $\sigma$-algebra $B(\Gamma^{(n)})$. In particular, for the Lebesgue product measure $(dx)^\otimes n$ fixed on $(\mathbb{R}^d)^n$, this identification yields a measure $m^{(n)}$ on $(\Gamma^{(n)}, B(\Gamma^{(n)}))$.

For $n = 0$ we set $m^{(0)}(\{\emptyset\}) := 1$. This leads to the definition of the space of finite configurations

$$
\Gamma_0 := \bigcup_{n=0}^{\infty} \Gamma^{(n)}
$$

endowed with the topology of disjoint union of topological spaces and the corresponding Borel $\sigma$-algebra $B(\Gamma_0)$, and to the so-called Lebesgue-Poisson measure on $(\Gamma_0, B(\Gamma_0))$,

$$
\lambda := \lambda_{dx} := \sum_{n=0}^{\infty} \frac{1}{m^{(n)}} m^{(n)}.
$$

Let $B_c(\mathbb{R}^d)$ be the set of all bounded Borel sets in $\mathbb{R}^d$ and, for each $\Lambda \in B_c(\mathbb{R}^d)$, let $\Gamma_\Lambda := \{ \eta \in \Gamma : \eta \subset \Lambda \}$. Evidently $\Gamma_\Lambda = \bigcup_{n=0}^{\infty} \Gamma^{(n)}_\Lambda$, where $\Gamma^{(n)}_\Lambda := \Gamma_\Lambda \cap \Gamma^{(n)}$, $n \in \mathbb{N}_0$. Given a complex-valued $B(\Gamma_0)$-measurable function $G$ such that $G|_{\Gamma_\Lambda} \equiv 0$ for some $\Lambda \in B_c(\mathbb{R}^d)$, the $K$-transform of $G$ is a mapping $KG : \Gamma \to \mathbb{C}$ defined at each $\gamma \in \Gamma$ by

$$
(KG)(\gamma) := \sum_{|\eta| < \infty} G(\eta).
$$

It has been shown in [7] that the $K$-transform is a linear and invertible mapping.

Let $M^b_{\text{fin}}(\Gamma)$ be the set of all probability measures $\mu$ on $(\Gamma, B(\Gamma))$ with finite local moments of all orders, i.e.,

$$
\int_{\Gamma} d\mu(\gamma) |\gamma \cap \Lambda|^n < \infty \quad \text{for all } n \in \mathbb{N} \text{ and all } \Lambda \in B_c(\mathbb{R}^d),
$$

and let $B_{bs}(\Gamma_0)$ be the set of all complex-valued bounded $B(\Gamma_0)$-measurable functions with bounded support, i.e., $G^{1}_{\Gamma_0} \in B_{bs}(\Gamma_0)^N \equiv 0$ for some $N \in \mathbb{N}_0, \Lambda \in B_c(\mathbb{R}^d)$. Given
Additionally, if $f$ has compact support we have

$$
(Ke_\lambda(f)) (\gamma) := \prod_{x \in \gamma} (1 + f(x))
$$

for all $\gamma \in \Gamma$, while for functions $f$ such that $e_\lambda(f) \in L^1(\Gamma_0, \rho_\mu)$ equality (2.5) holds, but only for $\mu$-a.e. $\gamma \in \Gamma$. Concerning the Lebesgue-Poisson measure (2.1), we observe that $e_\lambda(f) \in L^p(\Gamma_0, \lambda)$ whenever $f \in L^p := L^p(\mathbb{R}^d, dx)$ for some $p \geq 1$. In this case, $\| e_\lambda(f) \|_{L^p} = \exp(\|f\|_{L^p})$. In particular, for $p = 1$, in addition we have

$$
\int_{\Gamma_0} d\lambda(\eta) e_\lambda(f, \eta) = \exp \left( \int_{\mathbb{R}^d} dx f(x) \right),
$$

for all $f \in L^1$. For more details see [8].

2.2. **Bogoliubov generating functionals.** Given a probability measure $\mu$ on $(\Gamma, B(\Gamma))$ the so-called Bogoliubov generating functional (GF for short) $B_\mu$ corresponding to $\mu$ is the functional defined at each $B(\mathbb{R}^d)$-measurable function $\theta$ by

$$
B_\mu(\theta) := \int_{\Gamma} d\mu(\gamma) \prod_{x \in \gamma} (1 + \theta(x)),
$$

provided the right-hand side exists. It is clear from (2.6) that the domain of a GF $B_\mu$ depends on the underlying measure $\mu$ and, conversely, the domain of $B_\mu$ reflects special properties over the measure $\mu$. Throughout this work we will consider GF defined on the whole complex $L^1$ space. This implies, in particular, that the underlying measure $\mu$ has finite local exponential moments, i.e.,

$$
\int_{\Gamma} d\mu(\gamma) e^{\alpha |x|^\Lambda} < \infty \quad \text{for all } \alpha > 0 \text{ and all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d),
$$

and thus $\mu \in \mathcal{M}_{1_{\text{fin}}}^b(\Gamma)$. According to the previous subsection, this implies that to such a measure $\mu$ one may associate the correlation measure $\rho_\mu$, which leads to a description of the functional $B_\mu$ in terms of either the measure $\rho_\mu$:

$$
B_\mu(\theta) = \int_{\Gamma} d\mu(\gamma) (Ke_\lambda(\theta)) (\gamma) = \int_{\Gamma_0} d\rho_\mu(\eta) e_\lambda(\theta, \eta),
$$

or the so-called correlation function $k_\mu := \frac{d\rho_\mu}{dx}$ corresponding to the measure $\mu$, if $\rho_\mu$ is absolutely continuous with respect to the Lebesgue–Poisson measure $\lambda$:

$$
B_\mu(\theta) = \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) k_\mu(\eta).
$$

1Throughout this work all $L^p$-spaces, $p \geq 1$, consist of complex-valued functions.
Throughout this work we will assume, in addition, that GF are entire on the $L^1$ space \([9]\), which is a natural environment, namely, to recover the notion of correlation function. For a generic entire functional $B$ on $L^1$, this assumption implies that $B$ has a representation in terms of its Taylor expansion,

$$B(\theta_0 + z\theta) = \sum_{n=0}^{\infty} \frac{z^n}{n!} d^n B(\theta_0; \theta, ..., \theta), \quad z \in \mathbb{C}, \theta \in L^1,$$

being each differential $d^n B(\theta_0; \cdot), n \in \mathbb{N}, \theta_0 \in L^1$ defined by a symmetric kernel

$$\delta^n B(\theta_0; \cdot) \in L^\infty(\mathbb{R}^{dn}) := L^\infty((\mathbb{R}^d)^n, (dx)^\otimes n),$$
called the variational derivative of $n$-th order of $B$ at the point $\theta_0$. That is,

$$(2.8) \quad d^n B(\theta_0; \theta_1, ..., \theta_n) := \left. \frac{\partial^n}{\partial z_1 \cdots \partial z_n} B \left( \theta_0 + \sum_{i=1}^n z_i \theta_i \right) \right|_{z_1 = \cdots = z_n = 0} =: \int_{(\mathbb{R}^d)^n} dx_1 \cdots dx_n \delta^n B(\theta_0; x_1, \ldots, x_n) \prod_{i=1}^n \theta_i(x_i)$$

for all $\theta_1, ..., \theta_n \in L^1$. Moreover, the operator norm of the bounded $n$-linear functional $d^n B(\theta_0; \cdot)$ is equal to $\|\delta^n B(\theta_0; \cdot)\|_{L^\infty(\mathbb{R}^n)}$ and for all $r > 0$ one has

$$(2.9) \quad \|\delta^n B(\theta_0; \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{r} \sup_{\|\theta\|_{L^1} \leq r} |B(\theta_0 + \theta')|$$

and, for $n \geq 2$,

$$(2.10) \quad \|\delta^n B(\theta_0; \cdot)\|_{L^\infty(\mathbb{R}^n)} \leq n! \left(\frac{e}{r}\right)^n \sup_{\|\theta\|_{L^1} \leq r} |B(\theta_0 + \theta')|.$$
hops to a site \( y \) according to a rate given by \( a(x-y) \exp(-E(y,\gamma)) \), where \( E(y,\gamma) \) is a relative energy of interaction between the site \( y \) and the configuration \( \gamma \) defined by

\[
E(y,\gamma) := \sum_{x \in \gamma} \phi(x-y) \in [0, +\infty].
\]

Informally, the behavior of such an infinite particle system is described by

\[
(LF)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy \, a(x-y) e^{-E(y,\gamma)} \left( F(\gamma \setminus \{x\} \cup \{y\}) - F(\gamma) \right).
\]

Given an infinite particle system, as the Kawasaki dynamics, its time evolution in terms of states is informally given by the so-called Fokker-Planck equation,

\[
\frac{d\mu_t}{dt} = L^* \mu_t, \quad \mu_t|_{t=0} = \mu_0,
\]

where \( L^* \) is the dual operator of \( L \). Technically, the use of definition (2.3) allows an alternative approach to the study of (3.2) through the corresponding correlation functions \( k_t := k_{\mu_t}, \quad t \geq 0, \) provided they exist. This leads to the Cauchy problem

\[
\frac{\partial}{\partial t} k_t = \tilde{L}^* k_t, \quad k_{t|_{t=0}} = k_0,
\]

where \( k_0 \) is the correlation function corresponding to the initial distribution \( \mu_0 \) and \( \tilde{L}^* \) is the dual operator of \( \tilde{L} := K^{-1} L K \) in the sense

\[
\int_{\Gamma_0} d\lambda(\eta) (\tilde{L}G) (\eta)(k(\eta)) = \int_{\Gamma_0} d\lambda(\eta) G(\eta) (\tilde{L}^* k)(\eta).
\]

Through the representation (2.7), this gives us a way to express the dynamics also in terms of the GF \( B_t \) corresponding to \( \mu_t \), i.e., informally,

\[
\frac{\partial}{\partial t} B_t(\theta) = \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta,\eta) \left( \frac{\partial}{\partial t} k_t(\eta) \right) = \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta,\eta) (\tilde{L}^* k_t)(\eta)
\]

\[
= \int_{\Gamma_0} d\lambda(\eta) \left( \tilde{L}e_\lambda(\theta)(\eta) \right) k_t(\eta) =: (\tilde{L}B_t)(\theta).
\]

This leads to the time evolution equation

\[
\frac{\partial B_t}{\partial t} = \tilde{L}B_t,
\]

where, in the case of the Kawasaki dynamics, \( \tilde{L} \) is given cf. [4] by

\[
(\tilde{L}B)(\theta)
\]

\[
= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \, a(x-y) e^{-\phi(x-y)}(\theta(y) - \theta(x)) \delta B(\theta e^{-\phi(y-\cdot)} + e^{-\phi(y-\cdot)} - 1; x).
\]

**Theorem 3.1.** Given an \( \alpha_0 > 0 \), let \( B_0 \in \mathcal{E}_{\alpha_0} \). For each \( \alpha \in (0, \alpha_0) \) there is a \( T > 0 \) (which depends on \( \alpha, \alpha_0 \)) such that there is a unique solution \( B_t, \quad t \in [0, T), \) to the initial value problem (3.4), (3.5), \( B_{t|_{t=0}} = B_0 \) in the space \( \mathcal{E}_{\alpha} \).

This theorem follows as a particular application of an abstract Ovsjannikov-type result in a scale of Banach spaces which can be found e.g. in [5, Theorem 2.5], and the following estimate of norms.

**Proposition 3.2.** Let \( 0 < \alpha < \alpha_0 \) be given. If \( B \in \mathcal{E}_{\alpha''} \) for some \( \alpha'' \in (\alpha, \alpha_0] \), then \( \tilde{L}B \in \mathcal{E}_{\alpha'} \) for all \( \alpha \leq \alpha' < \alpha'' \), and we have

\[
\| \tilde{L}B \|_{\alpha'} \leq 2 e^{|0|_{\alpha'}^{\alpha''}_{\alpha_0}} \|a\|_{\alpha''} \frac{\alpha_0}{\alpha'' - \alpha'} \|B\|_{\alpha''}.
\]
To prove this result as well as other forthcoming ones the next lemma shows to be useful.

**Lemma 3.3.** Let \( \varphi, \psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) be such that, for a.a. \( y \in \mathbb{R}^d \), \( \varphi(y, \cdot) \in L^\infty : = L^\infty(\mathbb{R}^d) \), \( \psi(y, \cdot) \in L^1 \) and \( \| \varphi(y, \cdot) \|_{L^\infty} \leq c_0 \), \( \| \psi(y, \cdot) \|_{L^1} \leq c_1 \) for some constants \( c_0, c_1 > 0 \) independent of \( y \). For each \( \alpha > 0 \) and all \( B \in \mathcal{E}_\alpha \) let

\[
(L_0 B)(\theta) := \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \, a(x - y) e^{-k \rho(x-y)} (\theta(y) - \theta(x)) \delta B(\varphi(y, \cdot) \theta + \psi(y, \cdot); x),
\]

\( \theta \in L^1 \). Here \( a \) and \( \rho \) are defined as before and \( k \geq 0 \) is a constant. Then, for all \( \alpha' > 0 \) such that \( c_0 \alpha' < \alpha \), we have \( L_0 B \in \mathcal{E}_{\alpha'} \) and

\[
\| L_0 B \|_{\alpha'} \leq 2 e^{\frac{c_1}{\alpha}} \| a \|_L \frac{\alpha'}{\alpha - c_0 \alpha'} \| B \|_{\alpha}.
\]

**Proof.** First we observe that from the considerations done in Subsection 2.2 it follows that \( L_0 B \) is an entire functional on \( L^1 \) and, in addition, that for all \( r > 0 \), \( \theta \in L^1 \), and a.a. \( x, y \in \mathbb{R}^d \),

\[
| \delta B(\varphi(y, \cdot) \theta + \psi(y, \cdot); x) | \leq \| \delta B(\varphi(y, \cdot) \theta + \psi(y, \cdot); \cdot) \|_{L^\infty} \leq \frac{1}{r} \sup_{\| \theta \|_{L^1} \leq r} | B(\varphi(y, \cdot) \theta + \psi(y, \cdot) + \theta_0) |,
\]

where, for all \( \theta_0 \in L^1 \) such that \( \| \theta_0 \|_{L^1} \leq r \),

\[
| B(\varphi(y, \cdot) \theta + \psi(y, \cdot) + \theta_0) | \leq \| B \|_{\alpha} e^{\frac{\| \varphi(y, \cdot) \theta + \psi(y, \cdot) \|_{L^1}}{\alpha} \frac{c_0 \alpha + c_1 + 1 + r}{1 + \frac{c_0 \alpha}{\alpha}} \leq \| B \|_{\alpha} e^{c_0 \| \varphi \|_{L^1} + c_1 + \frac{c_0 \alpha}{\alpha}}.
\]

As a result, due to the positiveness of \( \rho \) and to the fact that \( \alpha \) is an even function, for all \( \theta \in L^1 \) one has

\[
| (L_0 B)(\theta) | \leq 1 e^{\frac{c_1}{\alpha}} \| a \|_L \frac{c_0 \alpha + c_1 + 1 + r}{1 + \frac{c_0 \alpha}{\alpha}} \| B \|_{\alpha} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dy \, a(x - y) e^{-k \rho(x-y)} | \theta(y) - \theta(x) |.
\]

Thus,

\[
\| L_0 B \|_{\alpha'} = \sup_{\theta \in L^1} \left( e^{-\frac{1}{\alpha'} \| \theta \|_{L^1}} | (L_0 B)(\theta) | \right)
\]

\[
\leq 2 e^{\frac{c_1}{\alpha}} \| a \|_L \| B \|_{\alpha} \sup_{\theta \in L^1} \left( e^{-\frac{c_0 \alpha}{\alpha}} \| \theta \|_{L^1} \right)
\]

where the supremum is finite provided \( \frac{1}{\alpha'} - \frac{c_0 \alpha}{\alpha} > 0 \). In such a situation, the use of the inequality \( xe^{-mx} \leq \frac{1}{cm}, \ x \geq 0, \ m > 0 \) leads for each \( r > 0 \) to

\[
\| L_0 B \|_{\alpha'} \leq 2 e^{\frac{c_1 + r}{\alpha}} \| a \|_L \| B \|_{\alpha} \frac{c_0 \alpha + c_1}{e^{\alpha} - c_0 \alpha} \| B \|_{\alpha}.
\]

The required estimate of norms follows by minimizing the expression \( \frac{1}{2} e^{\frac{c_1 + r}{\alpha}} \) in the parameter \( r \), that is, \( r = \alpha \).

**Proof of Proposition 3.2.** In Lemma 3.3 replace \( \varphi \) by \( e^{-\phi} \) and \( \psi \) by \( e^{-\phi} - 1 \), and consider \( k = 1 \). Due to the positiveness and integrability properties of \( \phi \) one has \( e^{-\phi} \leq 1 \) and \( |e^{-\phi} - 1| = 1 - e^{-\phi} \leq \phi \in L^1 \), ensuring the conditions to apply Lemma 3.3. \( \square \)

**Remark 3.4.** Concerning the initial conditions considered in Theorem 3.1, observe that, in particular, \( B_0 \) can be an entire GF \( B_{\beta_0} \) on \( L^1 \) such that, for some constants \( \alpha_0, C > 0 \), \( |B_{\beta_0}(\theta) | \leq C \exp \left( \frac{\| \theta \|_{L^1}}{\alpha_0} \right) \) for all \( \theta \in L^1 \). In such a situation an additional analysis is need in order to guarantee that for each \( t \) the local solution \( B_t \) given by Theorem 3.1 is a GF (corresponding to some measure). For more details see e.g. [5, 9] and references therein.
4. **Vlasov scaling**

We proceed to investigate the Vlasov-type scaling proposed in [3] for generic continuous particle systems and accomplished in [1] for the Kawasaki dynamics. As explained in both references, we start with a rescaling of an initial correlation function \( k_0 \), denoted by \( k_0^{(\varepsilon)} \), \( \varepsilon > 0 \), which has a singularity with respect to \( \varepsilon \) of the type \( k_0^{(\varepsilon)}(\eta) \sim \varepsilon^{-|\eta|}r_0(\eta), \eta \in \Gamma_0 \), being \( r_0 \) a function independent of \( \varepsilon \). The aim is to construct a scaling of the operator \( L \) defined in (3.1), \( \varepsilon > 0 \), in such a way that the following two conditions are fulfilled. The first one is that under the scaling \( L \mapsto L_\varepsilon \) the solution \( k_t^{(\varepsilon)}(\eta) \), \( t \geq 0 \), to

\[
\frac{\partial}{\partial t} k_t^{(\varepsilon)} = \hat{L}_\varepsilon k_t^{(\varepsilon)}, \quad k_t^{(\varepsilon)}|_{t=0} = k_0^{(\varepsilon)}
\]

preserves the order of the singularity with respect to \( \varepsilon \), that is, \( k_t^{(\varepsilon)}(\eta) \sim \varepsilon^{-|\eta|}r_t(\eta), \eta \in \Gamma_0 \). The second condition is that the dynamics \( r_0 \mapsto r_t \) preserves the Lebesgue-Poisson exponents, that is, if \( r_0 \) is of the form \( r_0 = e_\lambda(\rho_0) \), then each \( r_t, t > 0 \), is of the same type, i.e., \( r_t = e_\lambda(\rho_t) \), where \( \rho_t \) is a solution to a non-linear equation (called a Vlasov-type equation).

The previous scheme was accomplished in [1] through the scale transformation \( \phi \mapsto \varepsilon \phi \) of the operator \( L \), that is,

\[
(L_\varepsilon F)(\gamma) := \sum_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} dy \ a(x-y)e^{-\varepsilon E(y,\gamma)} (F(\gamma \setminus \{x\} \cup \{y\}) - F(\gamma)).
\]

As shown in [3, Example 12], [1], the corresponding Vlasov-type equation is given by

\[
\frac{\partial}{\partial t} \rho_t(x) = (\rho_t * a)(x)e^{-(\rho_t * \phi)(x)} - \rho_t(x)(a * e^{-(\rho_t * \phi)})(x), \quad x \in \mathbb{R}^d,
\]

where \( * \) denotes the usual convolution of functions. Existence of classical solutions \( 0 \leq \rho_t \leq L^\infty \) to (4.1) has been discussed in [1]. Therefore, it is natural to consider the same scaling, but in GF.

To proceed towards GF, we consider \( k_t^{(\varepsilon)} \) defined as before and \( k_{t,\text{ren}}^{(\varepsilon)}(\eta) := \varepsilon |\eta| k_t^{(\varepsilon)}(\eta) \). In terms of GF, these yield

\[
B_t^{(\varepsilon)}(\theta) := \int_{\Gamma_0} \ d\lambda(\eta) e_\lambda(\theta, \eta) k_t^{(\varepsilon)}(\eta),
\]

and

\[
B_{t,\text{ren}}^{(\varepsilon)}(\theta) := \int_{\Gamma_0} \ d\lambda(\eta) e_\lambda(\theta, \eta) k_{t,\text{ren}}^{(\varepsilon)}(\eta) = \int_{\Gamma_0} \ d\lambda(\eta) e_\lambda(\varepsilon \theta, \eta) k_t^{(\varepsilon)}(\eta) = B_t^{(\varepsilon)}(\varepsilon \theta),
\]

leading, as in (3.3), to the initial value problem

\[
\frac{\partial}{\partial t} B_{t,\text{ren}}^{(\varepsilon)} = \hat{L}_{\varepsilon,\text{ren}} B_{t,\text{ren}}^{(\varepsilon)}, \quad B_{t,\text{ren}}^{(\varepsilon)}|_{t=0} = B_{0,\text{ren}}^{(\varepsilon)}.
\]

**Proposition 4.1.** For all \( \varepsilon > 0 \) and all \( \theta \in L^1 \), we have

\[
(\hat{L}_{\varepsilon,\text{ren}} B)(\theta) = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \ a(x-y)e^{-\varepsilon \phi(x-y)}(\theta(y) - \theta(x))
\times \delta B \left( \theta e^{-\varepsilon \phi(y^-)} + \frac{e^{-\varepsilon \phi(y^-)} - 1}{\varepsilon} ; x \right).
\]

**Proof.** Since

\[
(\hat{L}_{\varepsilon,\text{ren}} B)(\theta) = \int_{\Gamma_0} \ d\lambda(\eta) (\hat{L}_{\varepsilon,\text{ren}} e_\lambda(\theta))(\eta) k(\eta),
\]
first we have to calculate \((\hat{L}_{c,\text{ren}} e_\lambda(\theta))(\eta) := \varepsilon^{-|\eta|} \hat{L}_c(e_\lambda(\varepsilon \theta, \eta)), \hat{L}_c = K^{-1} L_c\) cf. [3].

Similar calculations done in [4, Subsection 4.2.1] show

\[
(\hat{L}_{c,\text{ren}} e_\lambda(\theta))(\eta) = \sum_{x \in \eta} \int_{\mathbb{R}^d} dy \ a(x - y) e^{-c \phi(x - y)} (\theta(y) - \theta(x)) \\
\times e_\lambda \left( \theta e^{-c \phi(y^-)} + \frac{e^{-c \phi(y^-)}}{\varepsilon} - 1, \eta \setminus \{x\} \right),
\]

and thus, using the relation between variational derivatives derived in [9, Proposition 11], one finds

\[
(\hat{L}_{c,\text{ren}} B)(\theta) = \int_{\Gamma_0} d\lambda(\eta) \ k(\eta) \sum_{x \in \eta} \int_{\mathbb{R}^d} dy \ a(x - y) e^{-c \phi(x - y)} (\theta(y) - \theta(x)) \\
\times e_\lambda \left( \theta e^{-c \phi(y^-)} + \frac{e^{-c \phi(y^-)}}{\varepsilon} - 1, \eta \setminus \{x\} \right)
\]

\[
= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \ a(x - y) e^{-c \phi(x - y)} (\theta(y) - \theta(x)) \\
\times \delta B \left( \theta e^{-c \phi(y^-)} + \frac{e^{-c \phi(y^-)}}{\varepsilon} - 1; x \right).
\]

\[\square\]

**Proposition 4.2.** (i) If \(B \in \mathcal{E}_\alpha\) for some \(\alpha > 0\), then, for all \(\theta \in L^1\), \((\hat{L}_{c,\text{ren}} B)(\theta)\) converges as \(\varepsilon\) tends to zero to

\[
(\hat{L}_V B)(\theta) := \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \ a(x - y)(\theta(y) - \theta(x)) \delta B(\theta - \phi(y - \cdot); x).
\]

(ii) Let \(\alpha_0 > \alpha > 0\) be given. If \(B \in \mathcal{E}_{\alpha''}\) for some \(\alpha'' \in (\alpha, \alpha_0]\), then \((\hat{L}_{c,\text{ren}} B, \hat{L}_V B) \subset \mathcal{E}_{\alpha'}\) for all \(\alpha \leq \alpha' < \alpha''\), and we have

\[
\|\hat{L}_{\#} B\|_{\alpha'} \leq 2\|a\|_{L^1} \frac{\alpha_0}{(\alpha'' - \alpha')} e^{|\eta|+1} \|B\|_{\alpha''}
\]

where \(\hat{L}_{\#} = \hat{L}_{c,\text{ren}}\) or \(\hat{L}_{\#} = \hat{L}_V\).

**Proof.** (i) To prove this result we first analyze the pointwise convergence of the variational derivative (4.3) appearing in \(\hat{L}_{c,\text{ren}}\). For this purpose we will use the relation between variational derivatives derived in [9, Proposition 11], i.e.,

\[
\delta B(\theta_1 + \theta_2; x) = \int_{\Gamma_0} d\lambda(\eta) \delta^{(|\eta|+1)} B(\theta_1; \eta \cup \{x\}) e_\lambda(\theta_2, \eta), \quad \text{a.a. } x \in \mathbb{R}^d, \theta_1, \theta_2 \in L^1,
\]

which allows to rewrite (4.3) as

\[
\delta B \left( \theta e^{-c \phi(y^-)} + \frac{e^{-c \phi(y^-)}}{\varepsilon} - 1; x \right)
\]

\[
(4.4) \quad = \int_{\Gamma_0} d\lambda(\eta) \delta^{(|\eta|+1)} B(\theta - \phi(y - \cdot); \eta \cup \{x\}) \\
\times e_\lambda \left( \theta \left( e^{-c \phi(y^-)} - 1 \right) + \frac{e^{-c \phi(y^-)}}{\varepsilon} + \phi(y - \cdot, \eta) \right),
\]
for a.a. \( x, y \in \mathbb{R}^d \). Concerning the function
\[
f_\varepsilon := f_\varepsilon(\phi, y) := \varepsilon \left( e^{-\varepsilon \phi(x-y)} - 1 \right) + e^{-\varepsilon \phi(y-x)} - 1 + \phi(y - \cdot)
\]
which appears in (4.4), for a.a. \( y \in \mathbb{R}^d \), one clearly has \( \lim_{\varepsilon \to 0} f_\varepsilon = 0 \) a.e. in \( \mathbb{R}^d \). By definition (2.4), the latter implies that \( e_\lambda(f_\varepsilon) \) converges \( \lambda \)-a.e. to \( e_\lambda(0) \). Moreover, for the whole integrand function in (4.4), estimates (2.9), (2.10) yield for any \( r > 0 \) and \( \lambda \)-a.a. \( \eta \in \Gamma_0 \),
\[
\|\delta^{[n+1]} B(\theta - \phi(y - \cdot) ; \eta \cup \{ x \}) e_\lambda(f_\varepsilon, \eta) \|_{L^\infty(\mathbb{R}^d; \varepsilon\eta)} \leq \sum_{n=0}^{\infty} (n+1) e^{-\varepsilon \phi(y - \cdot)} \theta \| \lambda \|_{L^1} \leq r \| B(\theta - \phi(y - \cdot) + \theta_0) \|_{L^1} \leq \sum_{n=0}^{\infty} (n+1) e^{-\varepsilon \phi(y - \cdot)} \theta \| \lambda \|_{L^1} \leq r \| B\|_{L^1} \]
with
\[
\int_{\Gamma_0} d\lambda(\eta) (|\eta|+1)! \left( e^{-\varepsilon \phi(y - \cdot)} \theta \| \lambda \|_{L^1} \right) = \sum_{n=0}^{\infty} (n+1) e^{-\varepsilon \phi(y - \cdot)} \theta \| \lambda \|_{L^1} \leq r \| B\|_{L^1} \]
being finite for any \( r > \varepsilon (\| \lambda \|_{L^1} + 2 \| \phi \|_{L^1}) \).

As a result, by an application of the Lebesgue dominated convergence theorem we have proved that, for a.a. \( x, y \in \mathbb{R}^d \), (4.4) converges as \( \varepsilon \) tends to zero to
\[
\int_{\Gamma_0} d\lambda(\eta) \delta^{[n+1]} B(\theta - \phi(y - \cdot) ; \eta \cup \{ x \}) e_\lambda(0, \eta) = \delta B(\theta - \phi(y - \cdot) ; x).
\]
In addition, for the integrand function which appears in \( (\tilde{L}_{\varepsilon, \text{ren}} B)(\theta) \) we have
\[
\left| a(x - y) e^{-\varepsilon \phi(x-y)} (\theta(y) - \theta(x)) \delta B \left( \theta e^{-\varepsilon \phi(y-x)} + e^{-\varepsilon \phi(y-x)} - 1 ; x \right) \right| \leq \frac{\varepsilon}{\alpha} a(x - y) \| \theta(y) - \theta(x) \| \| B \|_{\alpha} \exp \left( \frac{1}{\alpha} \| \theta \|_{L^1} + \frac{1}{\alpha} \| \phi \|_{L^1} \right),
\]
for all \( \varepsilon > 0 \) and a.a. \( x, y \in \mathbb{R}^d \), leading through a second application of the Lebesgue dominated convergence theorem to the required limit.

(ii) In Lemma 3.3 replace \( \varphi \) by \( e^{-\varepsilon \phi} \), \( \psi \) by \( \frac{s - \varepsilon \phi}{1 - \varepsilon} \), and \( k \) by \( \varepsilon \). Arguments similar to prove Proposition 3.2 complete the proof for \( \tilde{L}_{\varepsilon, \text{ren}} \). A similar proof holds for \( \tilde{L}_V \). \( \square \)

Proposition 4.2 (ii) provides similar estimates of norms for \( \tilde{L}_{\varepsilon, \text{ren}} \), \( \varepsilon > 0 \), and the limiting mapping \( \tilde{L}_V \). According to the Ovsjannikov-type result used to prove Theorem 3.1, this means that given any \( B_{0,V}, B_{\varepsilon, \text{ren}} \in \mathcal{E}_\alpha, \varepsilon > 0 \), for each \( \alpha \in (0, \alpha_0) \) there is a \( T > 0 \) such that there is a unique solution \( B_{\varepsilon, \text{ren}}: [0, T) \to \mathcal{E}_\alpha, \varepsilon > 0 \), to each initial value problem (4.2) and a unique solution \( B_{\varepsilon, V}: [0, T) \to \mathcal{E}_\alpha \) to the initial value problem
\[
\frac{\partial}{\partial t} B_{\varepsilon, V} = \tilde{L}_V B_{\varepsilon, V}, \quad B_{\varepsilon, V}|_{t=0} = B_{0,V}.
\]
In other words, independent of the initial value problem under consideration, the solutions obtained are defined on the same time-interval and with values in the same Banach space. For more details see e.g. Theorem 2.5 and its proof in [5]. Therefore, it is natural to analyze under which conditions the solutions to (4.2) converge to the solution to (4.5). This follows from a general result presented in [5] (Theorem 4.3). However, to proceed to an application of this general result one needs the following estmate of norms.
Proposition 4.3. Assume that $0 \leq \phi \in L^1 \cap L^\infty$ and let $\alpha_0 > \alpha > 0$ be given. Then, for all $B \in \mathcal{L}_\alpha$, $\alpha'' \in (\alpha, \alpha_0)$, the following estimate holds

$$
\| \bar{L}_{\text{e,rn}} B - \bar{L} V B \|_{\alpha'} 
\leq 2\varepsilon |a|_{L^1} \| \phi \|_{L^\infty} \frac{\varepsilon \alpha_0}{\alpha} |B|_{\alpha''} e^{-\frac{\varepsilon \alpha_0}{\alpha}} \left( (2\varepsilon |\phi|_{L^1} + \frac{\alpha_0}{\varepsilon}) \frac{1}{\alpha'' - \alpha'} + \frac{8\alpha_0^2}{(\alpha'' - \alpha')^2} \right)
$$

for all $\alpha'$ such that $\alpha \leq \alpha' < \alpha''$ and all $\varepsilon > 0$.

Proof. First we observe that

$$
| (\bar{L}_{\text{e,rn}} B)(\theta) - (\bar{L} V B)(\theta) | 
\leq \int \int dy \int dx \partial_t C \left( \theta e^{-\varepsilon \phi(y - \cdot)} + \frac{e^{-\varepsilon \phi(y - \cdot)} - 1}{\varepsilon} x \right) - \delta B (\theta - \phi(y - \cdot); x)
$$

where

$$
e^{-\varepsilon \phi(x - y)} \delta B \left( \theta e^{-\varepsilon \phi(y - \cdot)} + \frac{e^{-\varepsilon \phi(y - \cdot)} - 1}{\varepsilon} x \right) - \delta B (\theta - \phi(y - \cdot); x)
$$

(4.6)

In order to estimate (4.6), given any $\theta_0, \theta_1, \theta_2 \in L^1$, let us consider the function $C_{\theta_1, \theta_2}(t) = dB (\theta_1 + (1 - t) \theta_2; \theta_0), t \in [0, 1]$, where $dB$ is the first order differential of $B$, defined in (2.8). One has

$$
\frac{\partial}{\partial t} C_{\theta_1, \theta_2}(t) = \frac{\partial}{\partial s} C_{\theta_0, \theta_1, \theta_2}(t + s) \bigg|_{s=0}
$$

$$
= \frac{\partial}{\partial s} dB (\theta_2 + t(\theta_1 - \theta_2) + s(\theta_1 - \theta_2); \theta_0) \bigg|_{s=0}
$$

$$
= \frac{\partial^2}{\partial s_1 \partial s_2} B (\theta_2 + t(\theta_1 - \theta_2) + s_1(\theta_1 - \theta_2) + s_2 \theta_0) \bigg|_{s_1=s_2=0}
$$

$$
= \int \int dx \int \int dy (\theta_1(x) - \theta_2(x)) \theta_0(y) \delta^2 B (\theta_2 + t(\theta_1 - \theta_2); x, y),
$$

leading to

$$
|dB(\theta_1; \theta_0) - dB(\theta_2; \theta_0)|
$$

$$
= |C_{\theta_1, \theta_2}(1) - C_{\theta_0, \theta_1, \theta_2}(0)|
$$

$$
\leq \max_{t \in [0, 1]} \int \int dx \int \int dy |\theta_1(x) - \theta_2(x)| |\theta_0(y)| |\delta^2 B (\theta_2 + t(\theta_1 - \theta_2); x, y)|
$$

$$
\leq \|\theta_1 - \theta_2\|_{L^1} \|\theta_0\|_{L^1} \max_{t \in [0, 1]} \|\delta^2 B (\theta_2 + t(\theta_1 - \theta_2); \cdot)\|_{L^\infty(\mathbb{R}^d)}
$$

where, through estimate (2.10) with $r = \alpha''$, we have

$$
\|\delta^2 B (\theta_2 + t(\theta_1 - \theta_2); \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq 2 \frac{\varepsilon^3}{\alpha \beta^2} \|B\|_{\alpha''} \exp \left( \frac{\varepsilon |\phi|_{L^1} + (1 - t) |\theta_2|_{L^1}}{\alpha''} \right).
$$

As a result,

$$
|dB(\theta_1; \theta_0) - dB(\theta_2; \theta_0)|
$$

$$
\leq 2 \frac{\varepsilon^3}{\alpha \beta^2} \|\theta_1 - \theta_2\|_{L^1} \|\theta_0\|_{L^1} \|B\|_{\alpha''} \max_{t \in [0, 1]} \left( \frac{t |\theta_1|_{L^1} + (1 - t) |\theta_2|_{L^1}}{\alpha''} \right).
$$

Continuing, we have

$$
\leq 2 \varepsilon |\phi|_{L^\infty} \frac{\varepsilon \alpha_0}{\alpha} |B|_{\alpha''} e^{-\frac{\varepsilon \alpha_0}{\alpha}} \left( (2\varepsilon |\phi|_{L^1} + \frac{\alpha_0}{\varepsilon}) \frac{1}{\alpha'' - \alpha'} + \frac{8\alpha_0^2}{(\alpha'' - \alpha')^2} \right).
$$

This completes the proof.
for all \( \theta_0, \theta_1, \theta_2 \in L^1 \). In particular, this shows that for all \( \theta_0 \in L^1 \),

\[
\left| d\theta \left( \frac{e^{-e\phi(y-\cdot)} + e^{-e\phi(y-\cdot)}}{\varepsilon} - 1 \right); \theta_0 \right| \\
\leq 2\varepsilon^3 \sup_{L^1} \| \phi \| L^\infty \| B \| _{\alpha\alpha} \left( \| \theta \| _{L^1} + \| \phi \| _{L^1} \right) \| \theta_0 \| _{L^1} \\
\times \max_\varepsilon \exp \left( \frac{1}{\alpha^{\nu}} \left( t \left( \| \theta \| _{L^1} + \| \phi \| _{L^1} \right) + (1 - t) \left( \| \theta \| _{L^1} + \| \phi \| _{L^1} \right) \right) \right) \\
= 2\varepsilon^3 \sup_{L^1} \| \phi \| L^\infty \| B \| _{\alpha\alpha} \left( \| \theta \| _{L^1} + \| \phi \| _{L^1} \right) \exp \left( \frac{1}{\alpha^{\nu}} \left( \| \theta \| _{L^1} + \| \phi \| _{L^1} \right) \right) \| \theta_0 \| _{L^1}.
\]

where we have used the inequalities

\[
\left\| \theta e^{-e\phi(y-\cdot)} - \theta \right\| _{L^1} \leq \varepsilon \left\| \phi \right\| _{L^\infty} \left\| \theta \right\| _{L^1}, \\
\left\| \frac{e^{-e\phi(y-\cdot)}}{\varepsilon} + \phi(y-\cdot) \right\| _{L^1} \leq \varepsilon \left\| \phi \right\| _{L^\infty} \left\| \phi \right\| _{L^1}, \\
\left\| \theta e^{-e\phi(y-\cdot)} + \frac{e^{-e\phi(y-\cdot)}}{\varepsilon} - 1 \right\| _{L^1} \leq \left\| \theta \right\| _{L^1} + \left\| \phi \right\| _{L^1}.
\]

In other words, we have shown that the norm of the bounded linear functional on \( L^1 \)

\[
L^1 \ni \theta_0 \mapsto d\theta \left( \frac{e^{-e\phi(y-\cdot)} + e^{-e\phi(y-\cdot)}}{\varepsilon} - 1 \right); \theta_0 - d\theta \left( \theta - \phi(y-\cdot); \theta_0 \right)
\]

is bounded by

\[
Q := 2\varepsilon^3 \sup_{L^1} \| \phi \| L^\infty \| B \| _{\alpha\alpha} \left( \| \theta \| _{L^1} + \| \phi \| _{L^1} \right) \exp \left( \frac{1}{\alpha^{\nu}} \left( \| \theta \| _{L^1} + \| \phi \| _{L^1} \right) \right) \\
\]
and thus
\[
\|\tilde{L}_{\varepsilon, \text{ren}} B - L_V B\|_{\alpha'} \\
\leq 2\varepsilon\|\phi\|_{L^\infty} \|\alpha\|_{L^1} \frac{e}{\alpha'} e^{\frac{\varepsilon\|\phi\|_{L^1}}{\alpha'}} \left( 2 \frac{e^2}{\alpha'} \sup_{\theta \in L^1} \left( \|\theta\|_{L^2}^2, \exp \left( \|\theta\|_{L^1} \left( \frac{1}{\alpha'} - \frac{1}{\alpha''} \right) \right) \right) \right) \\
+ \left( \frac{2e^2}{\alpha'} \|\phi\|_{L^1} + 1 \right) \sup_{\theta \in L^1} \left( \|\theta\|_{L^1} \exp \left( \|\theta\|_{L^1} \left( \frac{1}{\alpha'} - \frac{1}{\alpha''} \right) \right) \right) \|B\|_{\alpha''},
\]
and the proof follows using the inequalities \(xe^{-mx} \leq \frac{1}{m}e\) and \(x^2e^{-mx} \leq \frac{4}{m^2}e\) for \(x \geq 0, m > 0\). \(\square\)

We are now in conditions to state the following result.

**Theorem 4.4.** Given an \(0 < \alpha < \alpha_0\), let \(B_{1, \varepsilon}^{(c)}(t, \varepsilon), t \in [0, T), \) be the local solutions in \(E_{\alpha}\) to the initial value problems (4.2), (4.5) with \(B_{0, \varepsilon}^{(c)}(0, \varepsilon), B_{0, \varepsilon}^1 \in E_{\alpha_0}\). If \(0 \leq \phi \in L^1 \cap L^\infty\) and \(\lim_{\varepsilon \to 0} \|B_{0, \varepsilon}^{(c)} - B_{0, \varepsilon}\|_{\alpha_0} = 0\), then, for each \(t \in [0, T)\),
\[
\lim_{\varepsilon \to 0} \|B_{1, \varepsilon}^{(c)}(t, \varepsilon) - B_{1, \varepsilon}(t)\|_{\alpha} = 0.
\]
Moreover, if \(B_{0, \varepsilon}(\theta) = \exp \left( \int_\mathbb{R}^d dx \rho_\varepsilon(x) \theta(x) \right), \theta \in L^1,\) for some function \(0 \leq \rho_\varepsilon \in L^\infty\) such that \(\|\rho_\varepsilon\|_{L^\infty} \leq \frac{1}{\alpha_0}\), then for each \(t \in [0, T)\),
\[
B_{1, \varepsilon}(\theta) = \exp \left( \int_\mathbb{R}^d dx \rho_t(x) \theta(x) \right), \theta \in L^1,
\]
where \(0 \leq \rho_t \in L^\infty\) is a classical solution to the equation (4.1).

**Proof.** The first part follows directly from Proposition 4.3 and [5, Theorem 4.3], taking in [5, Theorem 4.3] \(p = 2\) and
\[
N_\varepsilon = 2\varepsilon\|\alpha\|_{L^1} \|\phi\|_{L^\infty} \frac{\alpha_0}{\alpha} e^{\frac{\varepsilon\|\phi\|_{L^1}}{\alpha}} \max \left\{ 2e\|\phi\|_{L^1} + \frac{\alpha_0}{\alpha}, 8\alpha_0^2 \right\}.
\]

Concerning the last part, we begin by observing that it has been shown in [1, Subsection 4.2] that given a \(0 \leq \rho_\varepsilon \in L^\infty\) such that \(\|\rho_\varepsilon\|_{L^\infty} \leq \frac{1}{\alpha_0}\), there is a solution \(0 \leq \rho_t \in L^\infty\) to (4.1) such that \(\|\rho_t\|_{L^\infty} \leq \frac{1}{\alpha_0}\). This implies that \(B_{1, \varepsilon}\), given by (4.7), does not leave the initial Banach space \(E_{\alpha_0} \subset E_{\alpha}\). Then, by an argument of uniqueness, to prove the last assertion we need to show that \(B_{1, \varepsilon}\) solves equation (4.5). For this purpose we note that for any \(\theta, \theta_1 \in L^1\) we have
\[
\frac{\partial}{\partial z_1} B_{1, \varepsilon}(\theta + z_1 \theta_1) \bigg|_{z_1 = 0} = B_{1, \varepsilon}(\theta) \int_\mathbb{R}^d dx \rho_t(x) \theta_1(x),
\]
and thus \(\delta B_{1, \varepsilon}(\theta; x) = B_{1, \varepsilon}(\theta) \rho_t(x)\). Hence, for all \(\theta \in L^1,\)
\[
(\tilde{L}_V B_{1, \varepsilon})(\theta) = B_{1, \varepsilon}(\theta) \left( \int_\mathbb{R}^d dx \int_\mathbb{R}^d dy a(x-y) (\theta(y) - \theta(x)) \rho_t(x) e^{-(\rho_t \ast \phi)(y)} \right)
\]
\[
= B_{1, \varepsilon}(\theta) \left( \int_\mathbb{R}^d dy \theta(y) (a \ast \rho_t)(y) e^{-(\rho_t \ast \phi)(y)} \\
- \int_\mathbb{R}^d dx \theta(x) (a \ast e^{-(\rho_t \ast \phi)})(x) \rho_t(x) \right).
\]

Since \(\rho_t\) is a classical solution to (4.1), \(\rho_t\) solves a weak form of equation (4.1), that is, the right-hand side of the latter equality is equal to
\[
B_{1, \varepsilon}(\theta) \frac{d}{dt} \int_\mathbb{R}^d dx \rho_t(x) \theta(x) = \frac{d}{dt} B_{1, \varepsilon}(\theta). \square
\]
References


Institute of Mathematics, Ukrainian National Academy of Sciences, 01601 Kiev, Ukraine
E-mail address: fdl@imath.kiev.ua

Fakultät für Mathematik, Universität Bielefeld, D 33615 Bielefeld, Germany; Forschungszentrum BiBoS, Universität Bielefeld, D 33615 Bielefeld, Germany
E-mail address: kondrat@mathematik.uni-bielefeld.de

Universidade Aberta, P 1269-001 Lisbon, Portugal; CMAF, University of Lisbon, P 1649-003 Lisbon, Portugal
E-mail address: oliveira@ci.uc.pt