Smoluchowski-Kramers Approximation for Stochastic Differential Equations with non-Lipschitzian coefficients

Diplomarbeit

vorgelegt von

Norbert Breimhorst

Fakultät für Mathematik
Universität Bielefeld

Januar 2009

Contents

Introduction

In this Diploma Thesis the existence and uniqueness of solutions of stochastic differential equations are studied. Differing from most of the existing literature, (see [IW81, Str84, Pro90, KS91]), we concentrate on the case of non-Lipschitz coefficients.

More precisely, the aim of this thesis is to extend the corresponding results of [FZ05] to the case of time dependent coefficients. Those extended results we will apply to the Newton system, which describes the evolution of a small particle moving in a random field. A further step is to use our results for stochastic differential equations to prove the Smoluchowski-Kramers approximation, which states the convergence of the solutions of this Newton system.

The first three chapters of this thesis are based on the paper [FZ05], which was written by Fang and Zhang and published in 2005. The results of this paper are adapted to our framework and are extended to the time dependent case. As an application of the obtained results, we study in chapter 4 the Newton system, which was first considered by [Nel67]. Finally we prove the convergence of Smoluchowski-Kramers approximation in our more general case.

We give a short summary of each chapter. In chapter 1 we outline our notations and repeat a few basic concepts of Stochastic Analysis. In section 1.2 we establish the definition of weak and strong solutions of stochastic differential equations. We do this closely to the presentation in the book of Karatzas and Shreve [KS91]. Furthermore, we recall some well known theorems from Stochastic Calculus and martingale theory.

In chapter 2 we study conditions to gain a unique solution of a given stochastic differential equation. We examine the following stochastic differential equation

$$dX_t = \sigma(t, X_t)dW_t + b(t, X_t)dt, \quad X_0 = x_0 \in \mathbb{R}^d. \tag{0.1}$$

This is a generalization of the paper of Fang and Zhang, [FZ05], where the coefficients σ and b are not time dependent:

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt, \quad X_0 = x_0 \in \mathbb{R}^d. \tag{0.2}$$

We prove that even in the general case, there exists a unique strong solution of (0.1). This will be proved by two theorems. The key idea is to show that the solution does not explode, where we assume conditions similar to those stated in the paper (cf. Theorem A), and pathwise uniqueness (cf. Theorem B). We point out, that the assumption on the function b, (H1) and (H2), are noticeable weaker than that of Fang and Zhang.

In this more general framework, we prove that two solutions to the same stochastic differential equation with different starting points will not coincide P-a.s. The details are expounded in sections 2.4 and 2.5. As a preparation we extend the result of Ikeda Watanabe, Theorem 1.8, to our time dependent case. For a clearer presentation and easier understanding we put some technical lemmas in section 2.3. Furthermore we prove Theorem 2.11 about continuous dependence on initial data.

In Chapter 3, we will prove that the solution of the stochastic differential equation is continuously dependent on both, the initial value and time (cf. Theorem D). To this end we construct an Euler approximation of the solution of the stochastic differential equation (0.2) in section 3.1. Here, we assume σ and b to be bounded. As in [FZ05] we obtain the uniform convergence in $t \in [0,T]$ (see Theorem 3.1). The next step (cf. section 3.2) is to establish some technical lemmas again for a better readability of the following proof. We work out some steps of the proof in more detail than Fang and Zhang did.

Finally, we consider in chapter 4 the Newton system

$$\begin{aligned} dX_t^\mu &= Y_t^\mu \, dt, \\ \mu \, dY_t^\mu &= b(t, X_t^\mu) \, dt + \sigma(t, X_t^\mu) \, dW_t - dX_t^\mu, \\ X_0^\mu &= \zeta_1, \quad Y_0^\mu = \zeta_2 \end{aligned} \tag{0.3}$$

and prove the existence of a unique strong solution. The Newton system describes the behavior of a small particle with mass μ, $0 < \mu \ll 1$, placed in a force field. The force field consists of a deterministic part $b(t, X_t^\mu)$, which only depends on the position of the particle and the time, and a random part, where $\sigma(t, X_t^\mu) \, dW_t$ represents the stochastic differential. The term dX_t^μ describes the friction of the particle. In section 4.2 we consider along with the Newton system above the stochastic differential equation (0.1). We prove that the solution of (0.3) converges in probability to that of (0.1), if the mass μ tends to zero. This property, (cf. Theorem 4.5), is called Smoluchowski-Kramers approximation. It allows us to solve a one-dimensional stochastic differential equation like (0.1) instead of solving the 2-dimensional Newton system. Clearly, this reduces the complexity of our problem.

The Smoluchowski-Kramer approximation has first been discussed rigorously by Nelson [Nel67, chapter 10] in 1967. He assumed that the function b satisfies a global Lipschitz condition and $\sigma \equiv 1$ and proved then (4.10). Thereafter Z. Schuss [Sch80, chapter 6] proved a Smoluchowski-Kramers approximation for the Langevin equation, an equation well known by physicists. There, σ is still a constant, but the coefficient b describing the force does not need to be necessarily Lipschitz. The main contribution in the last years is due to M. Freidlin, S. Cerrai and Z. Chen, [Fre04, CF05, CF06a, CF06b]. These papers concentrate mostly on properties around the Smoluchowski-Kramers approximation, but did not deal with best possible solvability. M. Freidlin [Fre04] assumes a non trivial σ, but still a Lipschitz condition. So does R. Westermann [Wes06]. We consider the Smoluchowski-Kramer approximation for a similar system and impose considerably weaker assumptions. Furthermore, we use a new technique to prove the approximation.

1 Mathematical Basis and Tools

1.1 Basic Definitions and Notations

First we want to introduce some frequently used notations. By $|x|$ we mean the Euclidean norm for a vector $x \in \mathbb{R}^d$. Let $M(d \times m, \mathbb{R})$ denote the set of all real $d \times m$-matrices.

Definition 1.1 (norm). *For a matrix $\sigma \in M(d \times m, \mathbb{R})$ we denote by $\|\sigma\|$ its Hilbert-Schmidt norm:*

$$\|\sigma\|^2 := \sum_{ij} \sigma_{ij}^2,$$

(which is equivalent to the usual operator norm $\mathbb{R}^m \to \mathbb{R}^d$).

In the following, we consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t < \infty}, P)$ that satisfies the *usual hypotheses* as follows:

(i) $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$;

(ii) $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$, for all $0 \le t < \infty$; that is, the filtration $(\mathcal{F}_t)_{0 \le t < \infty}$ is right continuous.

A *stochastic process* X on $(\Omega, \mathcal{F}, P)$ is a collection of random variables $(X_t)_{0 \le t < \infty}$. The process X is said to be *adapted* if $X_t \in \mathcal{F}_t$. This means that X_t is $\mathcal{F}_t$-measurable for each t.

Definition 1.2 (Brownian motion). *An adapted process $B = (B_t)_{0 \le t < \infty}$ taking values in $\mathbb{R}^m$ is called an m-*dimensional Brownian motion *if:*

(i) for $0 \le s < t < \infty$, $B_t - B_s$ is independent of $\mathcal{F}_s$

(ii) for $0 \le s < t$, the increment $B_t - B_s$ is normally distributed with mean zero and covariance matrix equal to $(t-s)Id$, where Id is the $(m \times m)$ identity matrix.

The Brownian motion starts at *x if $P(B_0 = x) = 1$.*

Every Brownian motion has a continuous modification cf. [Pro90, Thrm. 26, p.17].

We will now give a general version of Itô's formula, later we will use a special case of it.

Theorem 1.3 (Itô's formula). *We assume that:*
$\sigma : [0,T] \times \Omega \to M(d \times m, \mathbb{R})$ is predictable with $P(\int_0^t \|\sigma_s\|^2\, ds < \infty) = 1$, $t \in [0,T]$,
$b : [0,T] \times \Omega \to \mathbb{R}^d$ is a predictable and P-a.s. Bochner integrable process on $[0,T]$,
$F : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is twice Fréchet differentiable with derivatives $F_t := \frac{\partial}{\partial t}$, $F_{xx} := \frac{\partial^2}{\partial x^2}$ which are continuous on bounded subsets. Under these assumptions the process

$$X_t = X_0 + \int_0^t \sigma_s\, dW_s + \int_0^t b_s\, ds, \quad t \in [0,T],$$

is well defined and there exists a P-null set $N \in \mathcal{F}$, such that the following formula is fulfilled on N^c for all $t \in [0, T]$:

$$F(t, X_t) = F(0, X_0) + \int_0^t \langle F_x(s, X_s), \sigma_s dW_s \rangle$$
$$+ \int_0^t F_t(s, X_s) + \langle F_x(s, X_s), b_s \rangle + \frac{1}{2} tr[F_{xx}(s, X_s)\sigma_s \sigma_s^*] \, ds,$$

where tr denotes the trace of the operator.

Proof. see [Röc06, 2.4.5] or the standard reference [DPZ92, Thrm. 4.17, p.105]. □

1.2 Solutions of SDEs

We follow the exposition in [KS91, p. 285, 300]. Consider an m-dimensional Brownian motion $W = \{W_t, \mathcal{F}_t^W; 0 \le t < \infty\}$ on a probability space $(\Omega, \mathcal{F}, P)$. For a random vector $\xi \in \mathbb{R}^d$, we consider the left-continuous filtration

$$\mathcal{G}_t := \sigma(\xi) \vee \mathcal{F}_t^W = \sigma(\xi, W_s; 0 \le s \le t); \quad 0 \le t < \infty.$$

By adding a collection of null sets

$$\mathcal{N} := \{N \subseteq \Omega | \, \exists G \in \mathcal{G}_\infty \text{ with } N \subseteq G \text{ and } P(G) = 0\},$$

we create the *augmented filtration*

$$\mathcal{F}_t := \sigma(\mathcal{G}_t \cup \mathcal{N}), \quad 0 \le t < \infty; \quad \mathcal{F}_\infty := \sigma\left(\bigcup_{t \ge 0} \mathcal{F}_t\right).$$

This filtration $\{\mathcal{F}_t\}$ is a normal filtration, that means it fulfills the usual hypotheses. In the following we define two different types of solution: weak and strong solutions.

Definition 1.4 (strong solution)**.** *A* strong solution *of the stochastic differential equation* (0.1) *on the given probability space $(\Omega, \mathcal{F}, P)$ with the fixed Brownian motion W and initial condition ξ, is a process $X = \{X_t; 0 \le t < \infty\}$ with continuous sample paths and with the following properties:*

(i) X is adapted to the normal filtration $\{\mathcal{F}_t\}$ defined above;

(ii) $P(X_0 = \xi) = 1$;

(iii)

$$\int_0^t |b(s, X)| \, ds < \infty \quad P\text{-a.e. for all } t \in [0, \infty)$$
$$\int_0^t \|\sigma(s, X)\|^2 \, ds < \infty \quad P\text{-a.e. for all } t \in [0, \infty);$$

(iv) *the integral version*

$$X_t = X_0 + \int_0^t b(s,X)\,ds + \int_0^t \sigma(s,X)\,dW_s \quad 0 \le t < \infty,$$

holds almost surely.

One can interpret the strong solution of the stochastic differential equation (0.1) as an output of a machine, which consists of the functions b and σ. This machine will be fed by the initial condition ξ and a Brownian motion W and produces then with every input a solution X.

Definition 1.5 (weak solution)**.** *A* weak solution *of equation* (0.1) *is a triple* (X,W), $(\Omega,\mathcal{F},P)$, $\{\mathcal{F}_t\}$, *where*

(i) $(\Omega,\mathcal{F},P)$ *is a probability space and* $\{\mathcal{F}_t\}$ *is a filtration which satisfies the usual hypotheses;*

(ii) $X = \{X_t,\mathcal{F}_t\,;0 \le t < \infty\}$ *is a continuous, adapted* $\mathbb{R}^d$*-valued process,* $W = \{W_t,\mathcal{F}_t\,;0 \le t < \infty\}$ *is an* m*-dimensional Brownian motion;*

(iii)

$$\int_0^t |b(s,X)|\,ds < \infty \quad P\text{-a.e. for all } t \in [0,\infty)$$
$$\int_0^t \|\sigma(s,X)\|^2\,ds < \infty \quad P\text{-a.e. for all } t \in [0,\infty);$$

(iv) *the integral version*

$$X_t = X_0 + \int_0^t b(s,X)\,ds + \int_0^t \sigma(s,X)\,dW_s \quad 0 \le t < \infty$$

holds almost surely.

In contrast to the strong solution we do not have the automatism, that we can choose any Brownian motion W and initial condition ξ and get a solution X in the case of weak solutions. Here it is possible that there exists no weak solution to a given Brownian motion on a probability space. This is due to the fact that the filtration $\{\mathcal{F}_t\}$ does not need to be the augmentation of the filtration $\mathcal{G}_t = \sigma(\xi) \vee \mathcal{F}_t^W$. Therefore the weak solution $X_t(\omega)$ does not need to be a measurable functional of the Brownian motion and the initial condition.

Definition 1.6 (pathwise uniqueness)**.** *We say that* pathwise uniqueness *holds for* (0.1)*, if whenever* X *and* X' *are two (weak) solutions on the same stochastic basis* $(\Omega,\mathcal{F},P,(\mathcal{F}_t))$ *and with the same* $(\mathcal{F}_t)$*-Wiener process* $W(t)$*,* $t \in [0,\infty)$ *on* $(\Omega,\mathcal{F},P)$ *such that* $X(0) = X'(0)$ P*-a.e., then* P*-a.e.*

$$X(t) = X'(t),\ t \in [0,\infty).$$

Sometimes one does not have a solution for all times, but just locally. Then it blows up in finite time. Therefore it is more convenient to modify the notion of a solution to include solutions admitting explosions similarly to [IW81, p.158].

Definition 1.7 (explosion time). *Let* $\hat{\mathbb{R}}^d := \mathbb{R}^d \cup \{\triangle\}$ *be the one-point compactification of* $\mathbb{R}^d$ *and*

$$\hat{W}^d := \{w|\,[0,\infty) \ni t \mapsto w(t) \in \mathbb{R}^d \text{ is continuous and such that if } \\ w(t) = \triangle, \text{ then } w(t') = \triangle \text{ for all } t' \geq t\}.$$

Let $\mathcal{B}(\hat{W}^d)$ *be the* σ*-field generated by all Borel cylinder sets. For* $w \in \hat{W}^d$*, we set*

$$\zeta(w) := \inf\{t|\, w(t) = \triangle\}$$

and call $\zeta(w)$ *the* explosion time *of the trajectory* w*.*

Now we recall an important theorem that ensures the existence of a weak solution only with continuous coefficients.

Theorem 1.8. *Given continuous* $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^m$ *and* $b : \mathbb{R}^d \to \mathbb{R}^d$*, we consider the equation* (0.2)*. Then for any probability* μ *on* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ *with compact support, there exists a weak solution* X_t *of 0.2 such that the law of* X_0 *coincides with* μ *i.e.,* $P(X_0 \in A) = \mu(A)$ *for any* $A \in \mathcal{B}(\mathbb{R}^d)$*.*

Proof. see [IW81, Theorem 2.3 on p. 159]. □

1.3 Tools from Stochastic Calculus and martingale theory

The connection between the two kinds of solution is given by the theorem of Yamada and Watanabe:

Let $W^d := \mathcal{C}([0,\infty) \to \mathbb{R}^d)$. Let $\mathcal{B}_t(W^d)$ denote the σ-algebra generated by all maps π_s, $0 \leq s \leq t$, where $\pi_s(w) := w(s)$, $w \in W^d$. Let $\mathcal{A}^{d,m}$ denote the set of all $\mathcal{B}([0,\infty)) \otimes \mathcal{B}(W^d)/\mathcal{B}(M(d\times m, \mathbb{R}))$-measurable maps $\alpha : [0,\infty) \times W^d \to M(d \times m, \mathbb{R})$ such that for each $t \in [0,\infty)$ the map

$$W^d \ni w \mapsto \alpha(t,w) \in M(d \times m, \mathbb{R})$$

is $\mathcal{B}_t(W^d)/\mathcal{B}(M(d\times m,\mathbb{R}))$-measurable.

Theorem 1.9 (Yamada-Watanabe). *Let* $\sigma \in \mathcal{A}^{d,m}$ *and* $b \in \mathcal{A}^{d,1}$*. Then the equation* (0.1) *has a unique strong solution if and only if the following two properties hold:*

(i) For every probability measure μ *on* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ *there exists a (weak) solution* (X, W) *of* (0.1) *such that* μ *is the distribution of* $X(0)$*.*

(ii) Pathwise uniqueness holds for (0.1)*.*

Proof. See [PR07, Appendix E] or [Röc06, Appendix E]. □

In fact, it is enough to consider initial distributions only of the form $\mu = \delta_{x_0}$ for all $x_0 \in \mathbb{R}^d$. There is the following refinement of the Yamada-Watanabe theorem:

Theorem 1.10 (Kallenberg). *Let weak existence and pathwise uniqueness hold for the equation* (0.2) *for every initial data* μ *on* $\mathbb{R}^d$. *Then strictly strong existence and uniqueness in law hold for any initial probability distribution* μ *on* $\mathbb{R}^d$.

Proof. See [Kal96]. □

In particular, we have unique strong solvability for any initial data $x_0 \in \mathbb{R}^d$

The next Theorem claims the existence of a continuous modification of a stochastic process under certain boundary conditions. Recall that a process X_t where every sample path is right-continuous on $[0, \infty)$ with finite left-hand limits on $(0, \infty)$ is called a *càdlàg* process.

Theorem 1.11 (Kolmogorov's modification). *Let* $(X_t^a)_{t \geq 0, a \in \mathbb{R}^d}$ *be a parameterized family of stochastic processes such that* $t \to X_t^a$ *is càdlàg almost surely for each* $a \in \mathbb{R}^d$. *Suppose that*

$$E[\sup_{s \leq t} |X_s^a - X_t^b|^\alpha] \leq C(t)|a - b|^{d+\beta}$$

for some α, $\beta > 0$, $C(t) > 0$. *Then there exists a version* $\hat{X}_t^a$ *of* X_t^a *which is* $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}$ *measurable and which is càdlàg in t and uniformly continuous in a on compacts and is such that for all* $a \in \mathbb{R}^d$, $t \geq 0$,

$$\hat{X}_t^a = X_t^a \quad \textit{almost surely.}$$

Proof. See [Pro90, p.173]. □

We define

$$Z_t(X) := \exp\left[\int_0^t X_s \, dW_s - \frac{1}{2}\int_0^t |X|^2 \, ds\right], \tag{1.1}$$

which is a local martingale cf.[KS91, p.198]. A sufficient condition for $Z_t(X)$ to be a martingale is known as the Novikov criterium. We give its version for Brownian motion, for the general version see [KS91, Prop.5.12, p.198].

Theorem 1.12 (Novikov criterium). *Let* $W = (W_t, \mathcal{F}_t\,; 0 \leq t < \infty)$ *be an m-dimensional Brownian motion, and let* $X = (X_t, \mathcal{F}_t\,; 0 \leq t < \infty)$ *be a measurable, adapted* $\mathbb{R}^d$*-valued process satisfying* $P(\int_0^t |X_s|^2 \, ds < \infty) = 1$ *for all* $t \in [0, \infty)$. *If*

$$E\left[\exp\left(\frac{1}{2}\int_0^t |X_s|^2 \, ds\right)\right] < \infty, \quad 0 \leq t < \infty,$$

then $Z_t(X)$ *defined by* (1.1) *is a martingale.*

Proof. see [KS91, Thrm. 5.13, p.199]. □

Here we recall a useful fact about functions of bounded variation (which easily can be checked):

Lemma 1.13. *Let* $F = F(t)$, $t \in [a, b]$ *be such that its derivative* $F'(t)$ *exists for all* $t \in [a, b]$.

Moreover, F' is integrable on $[a,b]$. Then F is of bounded variation and

$$var_{[a,b]}(F) = \int_a^b |F'(t)| \, dt.$$

We also need some facts about quadratic variation: Let τ_n be a sequence of partitions of $[0,t]$, whose mesh tends to zero as n tends to infinity. Then the quadratic variation (along (τ_n)) of a real valued, continuous process $t \mapsto X_t$, $t \in [0,\infty)$ is defined as

$$\langle X \rangle_t := \lim_{n\to\infty} \sum_{\substack{t_i \in \tau_n \\ t_i \le t}} (X_{t_{i+1}} - X_{t_i})^2,$$

where the limit (provided it exists) is taken in probability.

Lemma 1.14. *Let X_t be a real valued continuous process such that the quadratic variation $\langle X \rangle_t$ exists for all $t \geq 0$ and is continuous on $[0,\infty)$.*

(i) Let $F \in C^1(\mathbb{R})$. Then $t \mapsto F(X_t)$ has (finite) quadratic variation

$$\langle F(X) \rangle_t = \int_0^t (F'(X_s))^2 \, d\langle X \rangle_s.$$

(ii) If $M_t := X_t + A_t, t \geq 0$, for some $t \mapsto A_t$ continuous and $\langle A \rangle \equiv 0$, then

$$\langle M \rangle_t = \langle X \rangle_t$$

Proof. see [Röc07, Lemma 1.2.9]. □

2 Existence and Uniqueness of strong solutions

2.1 Weak solutions for time dependent equations

Let $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^m$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ be continuous functions. Since we have by [IW81, Theorem 2.3 on p. 159] only a weak solution for the time-independent equation (0.2), we have to use a trick to get it for the time-dependent one. So, our aim is to construct a weak solution for

$$X_t = x_0 + \int_0^t \sigma(s, X_s)\, dW_s + \int_0^t b(s, X_s)\, ds, \tag{2.1}$$

where $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m$ and $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ are continuous and $x_0 \in \mathbb{R}^d$ is an initial value.

Theorem 2.1. *There exists a weak solution up to an explosion time ζ of the stochastic differential equation* (2.1)

Proof. Instead of (2.1) we consider the following, $\mathbb{R}^{d+1}$-valued equation:

$$Z_t = Z_0 + \int_0^t \bar{\sigma}(Z_s)\, dW_s + \int_0^t \bar{b}(Z_s)\, ds \tag{2.2}$$

with $Z_t = (t, X_t) \in \mathbb{R}^{d+1}$, $t \in \mathbb{R}$, $X_t \in \mathbb{R}^d$. The coefficients are given by the functions $\bar{b} : \mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}^d$,

$$\bar{b}(z) := \bar{b}(t,x) := \begin{cases} (1, b(t,x)), & t \in [0, +\infty) \\ 0, & t < 0 \end{cases}$$

and $\bar{\sigma} : \mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{d+1} \otimes \mathbb{R}^m$:

$$\bar{\sigma}(z) := \bar{\sigma}(t,x) := \begin{cases} \left(\begin{array}{ccc} 0 & \cdots & 0 \\ & \left(\begin{array}{c} \\ \sigma(t,x) \\ \\ \end{array} \right) & \end{array} \right), & t \in [0, +\infty), \\ 0, & t < 0. \end{cases}$$

Obviously the functions $\bar{\sigma}$ and $\bar{b}$ are continuous. So we can apply Theorem 1.8, which tells us there exists an $\mathbb{R}^{d+1}$-valued weak solution Y_t of (2.2) up to the explosion time ζ which satisfies

$$Y_t = Y_0 + \int_0^t \bar{\sigma}(Y_s)\, dW_s + \int_0^t \bar{b}(Y_s)\, ds,$$

where $Y_0 = (0, x_0)$. Now we define projections:

$$\begin{aligned}
&P_d : \mathbb{R}^{d+1} \to \mathbb{R}^d, \quad P_d(t, x_1, \dots, x_d) = (x_1, \dots, x_d),\\
&\bar{X}_t := P_d Y_t\\
&\xi_t := P_0 Y_t := (\mathbb{1} - P_d) Y_t,
\end{aligned}$$

so that $Y_t = (\xi_t, \bar{X}_t)$. Now we show that $\bar{X}_t = X_t$ and $\xi_t = t$:

$$\begin{aligned}
\bar{X}_t = P_d Y_t &= P_d Y_0 + P_d \int_0^t \bar{\sigma}(Y_s)\, dW_s + P_d \int_0^t \bar{b}(Y_s)\, ds\\
&= P_d Y_0 + P_d \begin{pmatrix} \sum_{j=1}^m \int_0^t \bar{\sigma}_{1j}(Y_s)\, dW_s^{(j)} \\ \vdots \\ \sum_{j=1}^m \int_0^t \bar{\sigma}_{d+1j}(Y_s)\, dW_s^{(j)} \end{pmatrix} + P_d \begin{pmatrix} \int_0^t \bar{b}_1(Y_s)\, ds \\ \vdots \\ \int_0^t \bar{b}_{d+1}(Y_s)\, ds \end{pmatrix}\\
&= x_0 + \begin{pmatrix} \sum_{j=1}^m \int_0^t \sigma_{1j}(Y_s)\, dW_s^{(j)} \\ \vdots \\ \sum_{j=1}^m \int_0^t \sigma_{dj}(Y_s)\, dW_s^{(j)} \end{pmatrix} + \begin{pmatrix} \int_0^t b_1(Y_s)\, ds \\ \vdots \\ \int_0^t b_d(Y_s)\, ds \end{pmatrix}\\
&= x_0 + \int_0^t \sigma(Y_s)\, dW_s + \int_0^t b(Y_s)\, ds\\
&= X_t,
\end{aligned}$$

$$\begin{aligned}
\xi_t = P_0 Y_t &= P_0 Y_0 + P_0 \int_0^t \bar{\sigma}(Y_s)\, dW_s + P_0 \int_0^t \bar{b}(Y_s)\, ds\\
&= P_0 Y_0 + P_0 \begin{pmatrix} \sum_{j=1}^m \int_0^t \bar{\sigma}_{1j}(Y_s)\, dW_s^{(j)} \\ \vdots \\ \sum_{j=1}^m \int_0^t \bar{\sigma}_{d+1j}(Y_s)\, dW_s^{(j)} \end{pmatrix} + P_0 \begin{pmatrix} \int_0^t \bar{b}_1(Y_s)\, ds \\ \vdots \\ \int_0^t \bar{b}_{d+1}(Y_s)\, ds \end{pmatrix}\\
&= 0 + \int_0^t 1\, ds\\
&= t.
\end{aligned}$$

Therefore, we have a weak solution of (2.1) up to the explosion time ζ. $\square$

2.2 Main results about strong solvability

In the global Lipschitz case, it is clear, that a strong solution exists. Fang and Zhang [FZ05] showed in their paper, that we have for the time-independent case also a strong solution under certain boundary conditions on the functions σ and b. We will generalize this condition to time dependent σ and b. The first result tells us that our weak solution does not explode in finite time P-a.s. even in the time-dependent case.

Theorem A. *Let ρ be a strictly positive, C^1-function defined on $(0, +\infty)$, satisfying*

(i)
$$\lim_{s \to +\infty} \rho(s) = +\infty, \tag{2.3}$$

(ii)
$$\lim_{s\to+\infty}\frac{s\rho'(s)}{\rho(s)}=0 \qquad \text{and} \tag{2.4}$$

(iii)
$$\int_0^{+\infty}\frac{ds}{s\rho(s)+1}=+\infty. \tag{2.5}$$

Assume that there exist $C, K > 0$, such that for all $x \in \mathbb{R}^d$ and $t \in [0,\infty)$

$$\begin{aligned}\|\sigma(t,x)\|^2 &\le C(|x|^2\rho(|x|^2)+1),\\ \langle x, b(t,x)\rangle &\le C(|x|^2\rho(|x|^2)+1).\end{aligned} \tag{H1}$$

Then the weak solution of the stochastic differential equation (2.1) *with the initial distribution δ_{x_0}, $x_0 \in \mathbb{R}^d$, has no explosion, that means $P(\zeta = +\infty) = 1$ where $\zeta = \sup_{R>0}\tau_R$ and $\tau_R := \inf\{t > 0 | \xi_t > R\}$.*

Remark 2.2. *(i) For example the function $\rho(s) := \log(1+s)$ satisfies the conditions (i)-(iii). This fact will be checked later in section 2.3 (see Lemma 2.4).*

(ii) In Theorem A we may always assume that $\rho \ge 1$.

The second result is about pathwise uniqueness of the weak solution. Here we claim that semi-monotonicity for b and certain bounds, which hold uniformly in t, are enough to have pathwise uniqueness.

Theorem B. *Let r be a strictly positive, C^1-function defined on an interval $(0, c_0]$ with $c_0 \ge 0$, satisfying*

(i)
$$\lim_{s\to 0} r(s)=+\infty, \tag{2.6}$$

(ii)
$$\lim_{s\to 0}\frac{sr'(s)}{r(s)}=0 \qquad \text{and} \tag{2.7}$$

(iii)
$$\int_0^{a}\frac{ds}{sr(s)}=+\infty \quad \forall a>0. \tag{2.8}$$

Assume that there exists $C > 0$, such that for $|x-y| \le c_0$ and all $t \in [0,+\infty)$

$$\begin{aligned}\|\sigma(t,x)-\sigma(t,y)\|^2 &\le C|x-y|^2 r(|x-y|^2),\\ \langle x-y, b(t,x)-b(t,y)\rangle &\le C|x-y|^2 r(|x-y|^2).\end{aligned} \tag{H2}$$

Then pathwise uniqueness holds for the weak solution of the stochastic differential equation (2.1).

Remark 2.3. *The second condition in* (H1) *is called* coercivity. *The second condition in* (H2) *is called* semi-monotonicity. *A typical example of a function r satisfying the conditions (i)-(iii) is given by $r(s) := \log 1/s$. This we prove in the next section, see Lemma 2.5.*

Combining the two Theorems A and B, it follows by the Yamada-Watanabe and Kallenberg theorems, (cf. Theorems 1.9 and 1.10), that there exists a unique strong solution of (2.1).
Our next result is the same what Fang and Zhang [FZ05] claimed in their paper but under our weaker conditions in Theorem B and with time-dependent coefficients. It says, that two solutions with different starting points will P-a.s. never meet each other.

Theorem C. *Let the hypothesis of Theorem B hold with*

$$|\langle x-y, b(t,x)-b(t,y)\rangle| \le C|x-y|^2 r(|x-y|^2), \tag{2.9}$$

which is stronger than the semi-monotonicity assumption in (H2). *Suppose that the solution does not explode at a finite time. Then for $x_0 \neq y_0$, almost surely $X_t(x_0) \neq X_t(y_0)$ for all $t > 0$.*

The last result is about continuous dependence of the solution with respect to the initial data.

Theorem D. *Assume that there exist $C, c_0 > 0$ such that for all x, y with $|x-y| \le c_0$ and all $t > 0$*

$$\begin{aligned} \|\sigma(t,x)-\sigma(t,y)\|^2 &\le C|x-y|^2 r(|x-y|^2), \\ |b(t,x)-b(t,y)| &\le C|x-y| r(|x-y|^2) \end{aligned} \tag{H3}$$

with $r(s) = \log 1/s$. Suppose that the stochastic differential equation has no-explosion. Then there exists a version $\tilde{X}_t(x_0)$ of $X_t(x_0)$ such that $(t, x_0) \to \tilde{X}_t(x_0)$ is continuous over $[0,+\infty) \times \mathbb{R}^d$ almost surely.

2.3 Preparing Lemmas

Before we prove the theorems we need to prove some technical lemmas. First we prove that our examples really fulfill the conditions.

Lemma 2.4. *The function $\rho(s) = \log(1+s)$ satisfies the conditions* (2.3)-(2.5) *in Theorem A.*

Proof. (i): It is clear,
(ii): Noting that $\rho'(s) = \frac{1}{1+s}$, we have

$$\frac{s\rho'(s)}{\rho(s)} = \frac{s \cdot \frac{1}{1+s}}{\log(1+s)} \le \frac{1}{\log(1+s)} \underset{s\to\infty}{\longrightarrow} 0.$$

(iii): We need to show: $\int_0^{+\infty} \frac{ds}{s\log(s+1)+1} = +\infty$.
We have that

$$\int_0^\infty \frac{ds}{s\log(1+s)+1} \ge \int_0^\infty \frac{ds}{(1+s)\log(1+s)+1} = \int_1^\infty \frac{ds}{s\log s+1}.$$

We observe that

$$\begin{aligned} \frac{1}{s\log s+1} &\geq \frac{1}{2}\frac{1}{s\log s} \quad \forall s \geq 2 \\ \Leftrightarrow s\log s+1 &\leq 2s\log s \\ \Leftrightarrow \qquad 1 &\leq s\log s. \end{aligned}$$

Therefore

$$\begin{aligned} \int_1^{+\infty} \frac{ds}{s\log s+1} &\geq \int_2^{+\infty} \frac{1}{2}\frac{1}{s\log s}\,ds + \int_1^2 \frac{1}{s\log s+1}\,ds \\ &\geq \frac{1}{2}\int_2^{+\infty} \frac{1}{s\log s}\,ds + \frac{1}{2\log 2+1} \\ &\geq \frac{1}{2}\int_2^{+\infty} (\log s)'\frac{1}{\log s}\,ds \\ &= \frac{1}{2}\int_{\log 2}^{+\infty} \frac{1}{y}\,dy \\ &= \frac{1}{2}\log y|_{\log 2}^{\infty} \\ &= \frac{1}{2}(\log\infty - \log\log 2) = \infty. \end{aligned}$$

So all three claims are fulfilled. $\square$

Lemma 2.5. *The function* $r(s) = \log 1/s$ *satisfies the conditions* (2.6)-(2.8) *in Theorem B.*

Proof. (i):

$$\lim_{s\to 0}\log\frac{1}{s} = \log\lim_{s\to 0}\frac{1}{s} = \infty.$$

(ii):

$$\lim_{s\to 0}\frac{sr'(s)}{r(s)} = \lim_{s\to 0}\frac{s(-\frac{1}{s})}{\log 1/s} = -\lim_{s\to 0}\frac{1}{\log 1/s} = 0.$$

(iii):

$$\begin{aligned} \int_0^a \frac{1}{s\log 1/s}\,ds &= -\int_0^a \frac{1}{s\log s}\,ds \\ &= -\lim_{\varepsilon\to 0}\int_\varepsilon^a \frac{1}{s\log s}\,ds \\ &= -\lim_{\varepsilon\to 0}\int_{log\varepsilon}^{\log a} \frac{1}{y}\,dy \\ &= -\lim_{\varepsilon\to 0}(\log|\log a| - \log|\log\varepsilon|) \\ &= -(\log|\log a| - \log|(\lim_{\varepsilon\to 0}\log\varepsilon)|) \\ &= \infty. \end{aligned}$$

$\square$

Let $(\Omega, \mathcal{F}, P)$ be a probability space, endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$. Let $(W_t)_{t\geq 0}$ be a $\mathcal{F}_t$-Brownian motion taking values in $\mathbb{R}^m$. Consider the following Itô process in $\mathbb{R}^d$:

$$\eta_t = \eta_0 + \int_0^t e_s \, dW_s + \int_0^t f_s \, ds, \quad \eta_0 \in \mathbb{R}^d, \tag{2.10}$$

where $(e_t(\omega))_{t\geq 0}$ is an $M(d\times m, \mathbb{R})$-valued adapted process such that $\int_0^T \|e_s\|^2 \, ds < +\infty$ for any $T > 0$ and $(f_t(\omega))_{t\geq 0}$ is an $\mathbb{R}^d$-valued adapted process such that $\int_0^T |f_s| \, ds < +\infty$ for any $T > 0$.

The following lemma is a particular case of Itô's formula for the square of norm. For a general formulation of Itô's formula see Theorem 1.3. For completeness of exposition we give a proof.

Lemma 2.6 (Itô's formula). *Let $\xi_t := |\eta_t|^2$, $t \in [0, \infty)$. Then*

$$d\xi_t = 2\langle e_t^* \eta_t, dW_t\rangle + 2\langle \eta_t, f_t\rangle \, dt + \|e_t\|^2 \, dt \tag{2.11}$$

where e_t^* denotes the transpose matrix of e_t. The stochastic contraction (i.e. quadratic variation) $\langle d\xi_t\rangle$ (see the definition before Lemma 1.14) is given by

$$\langle d\xi_t\rangle = 4|e_t^*\eta_t|^2 dt. \tag{2.12}$$

Proof. For the function $F : [0,T]\times\mathbb{R}^d \to \mathbb{R}$, $(t,x) \mapsto |x|^2 = \sum_{i=1}^d x_i^2$ we have the derivatives: $F_x = 2x$, $F_t = 0$ and $F_{xx} = 2 \cdot Id$. Thus with the Itô-formula in Theorem 1.3:

$$\begin{aligned}
|\eta_t|^2 &= |\eta_0|^2 + \int_0^t \langle 2\eta_s, e_s dW_s\rangle + \langle 2\eta_s, f_s\rangle + \frac{1}{2} tr[2e_s e_s^*] \, ds \\
&= \xi_0 + \int_0^t 2\langle \eta_s, e_s dW_s\rangle + \int_0^t 2\langle \eta_s, f_s\rangle + tr[e_s e_s^*] \, ds \\
&= \xi_0 + 2\int_0^t \langle e_s^*\eta_s, dW_s\rangle + 2\int_0^t \langle \eta_s, f_s\rangle \, ds + \int_0^t \|e_s\|^2 \, ds,
\end{aligned}$$

where we used the fact, that $Q = Id$ and the calculation below:

$$\begin{aligned}
tr[(e_s Q^{1/2})(e_s Q^{1/2})^*] &= \sum_{i=1}^d \langle e_s Q^{1/2} (e_s Q^{1/2})^* l_i, l_i\rangle \\
&= \sum_{i=1}^d \langle Q^{1/2} (e_s Q^{1/2})^* l_i, e_s^* l_i\rangle \\
&= \sum_{i=1}^d \langle Q^{1/2} \underbrace{Q^{(1/2)*}}_{=Q^{1/2}} e_s^* l_i, e_s^* l_i\rangle \\
&= \sum_{i=1}^d \underbrace{\langle e_s^* l_i, e_s^* l_i\rangle}_{=|e_s^* l_i|^2} = \sum_{i=1}^d \langle e_s e_s^* l_i, l_i\rangle = tr[e_s e_s^*] \\
&= \|e_s^*\|^2 = \|e_s\|^2.
\end{aligned}$$

Here $\{l_i\}_{i=1}^d$ is an orthonormal basic of $\mathbb{R}^d$. To check the second claim we show first, that $\int_0^t \langle \eta_s, f_s \rangle ds$ and $\int_0^t \|e_s\|^2 ds$ are of bounded variation. Since e_t is continuous by assumption, we see that $\int_0^t \|e_s\|^2\, ds$ is continuous, too, even continuously differentiable with derivative $\|e_t\|^2$. Therefore, by Lemma 1.13 we have for $t \in [0,T]$

$$var_{[0,t]}(\int_0^t \|e_s\|^2\, ds) = \int_0^t \|e_s\|^2\, ds < \infty,$$

hence the quadratic variation $\langle \int_0^t \|e_s\|^2\, ds \rangle = 0$. The same result follows from the additivity property of the integral.

The continuity of $\int_0^t 2\langle \eta_s, f_s \rangle\, ds$ is obvious, since e_s, f_s and W_s are, too. It is even continuously differentiable with derivative $2\langle \eta_t, f_t \rangle$. Again by Lemma 1.13

$$\begin{aligned} var_{[0,t]}(\int_0^t 2\langle \eta_s, f_s \rangle\, ds) &= \int_0^t |2\langle \eta_s, f_s \rangle|\, ds \\ &\leq 2\int_0^t |\eta_s||f_s|\, ds < \infty, \end{aligned}$$

hence the quadratic variation $\langle \int_0^t 2\langle \eta_s, f_s \rangle\, ds \rangle = 0$. Now we apply Lemma 1.14 and get

$$\langle \xi \rangle_t = \langle \int_0^t 2\langle e_s^* \eta_s, dW_s \rangle \rangle = 4|e_t^* \eta_t|^2 t,$$

which completes the proof. □

The next two lemmas give us useful upper bounds for $E[\Phi(\xi(t))]$ under different assumptions on the function Φ and its derivatives. These are due to [FZ05].

Lemma 2.7. *Let ρ be a continuous function on $[0,+\infty)$ such that $\rho \geq 1$. Let Φ be a strictly positive, C^2-function on $[0,+\infty)$ satisfying the following conditions*

$$|\Phi'(\xi)| \leq \frac{C_1 \Phi(\xi)}{\xi\rho(\xi)+1}, \qquad \Phi''(\xi) \leq \frac{C_2\Phi(\xi)\rho(\xi)}{(\xi\rho(\xi)+1)^2}, \quad \xi \in [0,\infty), \tag{2.13}$$

where C_1, C_2 are two positive constants. Keeping the notations in Lemma 2.6, assume that for all $t \geq 0$.

$$\|e_t\|^2 \leq C_3(\xi_t\rho(\xi_t)+1), \tag{2.14}$$

$$|\langle \eta_t, f_t \rangle| \leq C_4(\xi_t\rho(\xi_t)+1), \tag{2.15}$$

where C_3, C_4 are two positive constants. Set

$$K := (C_1 + 2C_2)C_3 + 4C_1C_4, \tag{2.16}$$

then the following bound holds uniformly for all $t > 0$, $R > 0$

$$E(\Phi(\xi_{t\wedge\tau_R})) \leq \Phi(|\eta_0|^2)e^{Kt}, \tag{2.17}$$

where $\tau_R := \inf\{t > 0 | \xi_t \geq R\}$ is a stopping time.
If additionally $\Phi'(\xi) > 0$, $\xi \in [0,+\infty)$, then the result still holds under the weaker (than (2.15)*) assumption*

$$\langle \eta_t, f_t \rangle \leq C_4(\xi_t\rho(\xi_t)+1) \tag{2.18}$$

Proof. We use Itô's formula (Theorem 1.3) with $F = \Phi$ and (2.11), (2.12):

$$\begin{aligned}
\Phi(\xi_{t\wedge\tau_R}) - \Phi(\xi_0) &= \int_0^{t\wedge\tau_R} \Phi'(\xi_s)\, d\xi_s + \frac{1}{2}\int_0^{t\wedge\tau_R} \Phi''(\xi_s)\, d\langle \xi_s\rangle \\
&= 2\int_0^{t\wedge\tau_R} \Phi'(\xi_s)\langle e_s^*\eta_s, dW_s\rangle + 2\int_0^{t\wedge\tau_R} \Phi'(\xi_s)\langle \eta_s, f_s\rangle\, ds \\
&\quad + \int_0^{t\wedge\tau_R} \Phi'(\xi_s)\|e_s\|^2\, ds + \frac{1}{2}4\int_0^{t\wedge\tau_R} \Phi''(\xi_s)|e_s^*\eta_s|^2\, ds \\
&= I_1(t) + I_2(t) + I_3(t) + I_4(t)
\end{aligned}$$

In the next steps we estimate the single terms.
(i) **Claim**: $I_1(t)$ is a martingale
Proof of claim (i): Since

$$\begin{aligned}
I_1(t) &= 2\int_0^{t\wedge\tau_R} \Phi'(\xi_s)\langle e_s^*\eta_s, dW_s\rangle \\
&= 2\int_0^{t\wedge\tau_R} \langle e_s^*\eta_s, \Phi'(\xi_s)dW_s\rangle,
\end{aligned}$$

it is clearly a local martingale. To prove that $I_1(t)$ is indeed a martingale, we need to check that $E[\langle I_1\rangle_t] < \infty$ for all $t \geq 0$. Let us recall the following fact from [Pro90]. If X is a local martingale, then $X^2 - \langle X\rangle$ is a local martingale, too. If in addition $E[\langle X\rangle_t] < \infty\ \forall t \geq 0$ holds, then X is a quadratic integrable martingale. It remains to check the integrability of $\langle I_1\rangle_t$: By [Röc06, Lemma 2.4.4]

$$\langle I_1\rangle_t = \int_0^{t\wedge\tau_R} |\Phi'(\xi_s)|^2 \cdot \|e_s^*\eta_s\|^2\, ds$$

and hence by (2.13), (2.14) and (2.15)

$$\begin{aligned}
E[\langle I_1\rangle_t] &\leq E\left[\int_0^{t\wedge\tau_R} (\|e_s^*\| \cdot |\eta_s| \cdot |\Phi'(\xi_s)|)^2\, ds\right] \\
&\leq E\left[\int_0^{t\wedge\tau_R} C_3(\xi_s\rho(\xi_s)+1)\xi_s \left(\frac{C_1\Phi(\xi_s)}{\xi_s\rho(\xi_s)+1}\right)^2 ds\right] \\
&= E\left[\int_0^{t\wedge\tau_R} C_3C_1^2\xi_s\Phi(\xi_s)^2 \frac{1}{\xi_s\rho(\xi_s)+1}\, ds\right] \\
&\leq C_1^2C_3E\left[\int_0^{t\wedge\tau_R} \xi_s\Phi(\xi_s)^2\frac{1}{\xi_s+1}\, ds\right] \\
&\leq C_1^2C_3E\left[\int_0^t \Phi(\xi_{s\wedge\tau_R})^2\, ds\right] < \infty.
\end{aligned}$$

The expectation in the last line is finite because $\Phi \in C^2$ is uniformly bounded on the interval $[0, R]$. △

(ii) Estimate of $I_2(t)$. First we observe that

$$\begin{aligned}|\Phi'(\xi_s)\langle\eta_s, f_s\rangle| &\le |\Phi'(\xi_s)| \cdot |\langle\eta_s, f_s\rangle| \\ &\le \frac{C_1\Phi(\xi_s)}{\xi_s\rho(\xi_s)+1} C_4(\xi_s\rho(\xi_s)+1) \\ &\le C_1C_4\Phi(\xi_s),\end{aligned}$$

where we used (2.13) and (2.14). If $\Phi' > 0$, then we can use here the weaker assumption (2.18).
Hence, we get the following estimate:

$$\begin{aligned}E(I_2(t)) &= E\left[2\int_0^{t\wedge\tau_R} \Phi'(\xi_s)\langle\eta_s, f_s\rangle\, ds\right] \\ &\le 4C_1C_4 \int_0^{t\wedge\tau_R} E[\Phi(\xi_s)]\, ds \\ &\le 4C_1C_4 \int_0^t E[\Phi(\xi_{s\wedge\tau_R})]\, ds. \qquad \triangle\end{aligned}$$

(iii) Estimate $I_3(t)$ by direct calculation:

$$\begin{aligned}E[I_3(t)] &\le E\left[\int_0^{t\wedge\tau_R} |\Phi'(\xi_s)| \cdot \|e_s\|^2\, ds\right] \\ &\le E\left[\int_0^{t\wedge\tau_R} \frac{C_1\Phi(\xi_s)}{\xi_s\rho(\xi_s)+1} C_3(\xi_s\rho(\xi_s)+1)\, ds\right] \\ &= E\left[\int_0^{t\wedge\tau_R} C_1C_3\Phi(\xi_s)\, ds\right] \\ &\le C_1C_3 \int_0^t E[\Phi(\xi_{s\wedge\tau_R})]\, ds. \qquad \triangle\end{aligned}$$

(iv) Estimate of $I_4(t)$. We observe that by (2.13)

$$\Phi''(\xi_s) \le \frac{C_2\Phi(\xi_s)\rho(\xi_s)}{(\xi_s\rho(\xi_s)+1)^2} \le \frac{C_2\Phi(\xi_s)}{\xi_s(\xi_s\rho(\xi_s)+1)},$$

which holds because of $\frac{\rho(\xi)}{\xi\rho(\xi)+1} \le \frac{1}{\xi}$ and $\rho \ge 1$. Thus with (2.14)

$$\begin{aligned}\Phi''(\xi_s)|e_s^*\eta_s|^2 &\le \frac{C_2\Phi(\xi_s)}{\xi_s(\xi_s\rho(\xi_s)+1)}|e_s^*\eta_s|^2 \\ &\le \frac{C_2\Phi(\xi_s)}{\xi_s(\xi_s\rho(\xi_s)+1)}\|e_s^*\|^2|\eta_s|^2 \\ &\le \frac{C_2\Phi(\xi_s)}{\xi_s(\xi_s\rho(\xi_s)+1)}\xi_s C_3(\xi_s\rho(\xi_s)+1) \\ &\le C_2C_3\Phi(\xi_s).\end{aligned}$$

This implies the final estimate

$$\begin{aligned} E[I_4] &\le E\left[2\int_0^{t\wedge\tau_R} C_2C_3\Phi(\xi_s)\,ds\right] \\ &\le 2C_2C_3\int_0^t E[\Phi(\xi_{s\wedge\tau_R})]\,ds. \qquad \triangle \end{aligned}$$

Putting all four parts (i)-(iv) together, we have

$$\begin{aligned} E[\Phi(\xi_{t\wedge\tau_R})] &= E[\Phi(\xi_0)+I_1+I_2+I_3+I_4] \\ &\le \Phi(\xi_0)+0+(\underbrace{4C_1C_4+C_1C_3+2C_2C_3}_{=K})\int_0^t E(\Phi(\xi_{s\wedge\tau_R}))\,ds \end{aligned}$$

By Gronwall's inequality we get that for all $t \ge 0$ and $R > 0$

$$E[\Phi(\xi_{t\wedge\tau_R})] \le \Phi(\xi_0)e^{Kt},$$

which completes the proof of the lemma. □

Lemma 2.8. *Let r be a continuous function defined on a neighborhood of zero, say $(0, c_0]$, such that $r \ge 1$. Let Φ be a strictly positive, C^2-function defined on $[0, c_0]$. Suppose that there exists $\delta > 0$ such that for $\xi \in [0, c_0]$*

$$|\Phi'(\xi)| \le \frac{C_1\Phi(\xi)}{\xi r(\xi)+\delta}, \qquad \Phi''(\xi) \le \frac{C_2\Phi(\xi)r(\xi)}{(\xi r(\xi)+\delta)^2}. \tag{2.19}$$

Keeping the notations in Lemma 2.6, suppose that $|\eta_0|^2 < c_0$. Define the stopping time

$$\tau = \inf\{t > 0 | \xi_t \ge c_0\}.$$

Assume that for $t < \tau$,

$$\|e_t\|^2 \le C_3(\xi_t r(\xi_t)+\delta), \tag{2.20}$$
$$|\langle \eta_t, f_t\rangle| \le C_4(\xi_t r(\xi_t)+\delta). \tag{2.21}$$

Let

$$K = (C_1+2C_2)C_3 + 4C_1C_4, \tag{2.22}$$

then

$$E[\Phi(\xi_{t\wedge\tau})] \le \Phi(|\eta_0|^2)e^{Kt}, \quad \text{for any } t \ge 0.$$

If additionally $\Phi'(\xi) > 0$ for all $\xi \in [0, c_0]$, then in (2.21) it suffices to assume

$$\langle \eta_t, f_t\rangle \le C_4(\xi_t r(\xi_t)+\delta) \tag{2.23}$$

Proof. Using Itô's formula with $F = \Phi$ and according to (2.11) and (2.12):

$$\begin{aligned}
\Phi(\xi_{t\wedge\tau}) &= \Phi(\xi_0) + \int_0^{t\wedge\tau} \Phi'(\xi_s)\, d\xi_s + \frac{1}{2}\int_0^{t\wedge\tau} \Phi''(\xi_s)\, d\langle\xi_s\rangle \\
&= \Phi(\xi_0) + 2\int_0^{t\wedge\tau} \Phi'(\xi_s)\langle e_s^*\eta_s, dW_s\rangle + 2\int_0^{t\wedge\tau} \Phi'(\xi_s)\langle\eta_s, f_s\rangle\, ds \\
&\quad + \int_0^{t\wedge\tau} \Phi'(\xi_s)\|e_s\|^2\, ds + 2\int_0^{t\wedge\tau} \Phi''(\xi_s)|e_s^*\eta_s|^2\, ds \\
&= \Phi(\xi_0) + I_1(t) + I_2(t) + I_3(t) + I_4(t).
\end{aligned}$$

By assumption (2.20), for any $s < \tau$,

$$|e_s^*\eta_s|^2 \le \|e_s^*\|^2|\eta_s|^2 \le C_3(\xi_s r(\xi_s) + \delta)\xi_s.$$

According to (2.19), for any $s < \tau$ and $0 \le \xi_s < c_0$, we have

$$\begin{aligned}
|\Phi'(\xi_s)e_s^*\eta_s|^2 &\le |\Phi'(\xi_s)|^2\|e_s^*\|^2|\eta_s|^2 \\
&\le \left|\frac{C_1\Phi(\xi_s)}{\xi_s r(\xi_s) + \delta}\right|^2 C_3(\xi_s r(\xi_s) + \delta)\xi_s \\
&= C_1^2 C_3 \Phi(\xi_s)^2 \frac{\xi_s}{\xi_s r(\xi_s) + \delta} \\
&\le C_1^2 C_3 \Phi(\xi_s)^2 < \infty,
\end{aligned}$$

where we used that $\Phi \in C^2[0, c_0]$. As in the previous lemma, we thus get that I_1 is a martingale and $E[I_1] = 0$. With the assumptions (2.19), (2.20) and (2.21) we obtain

$$\begin{aligned}
|\Phi'(\xi_s)\langle\eta_s, f_s\rangle| &\le |\Phi'(\xi_s)| \cdot |\langle\eta_s, f_s\rangle| \le \frac{C_1\Phi(\xi_s)}{\xi_s r(\xi_s) + \delta} C_4(\xi_s r(\xi_s) + \delta) = C_1 C_4 \Phi(\xi_s), \\
|\Phi'(\xi_s)| \cdot \|e_s\|^2 &\le \frac{C_1\Phi(\xi_s)}{\xi_s r(\xi_s) + \delta} C_3(\xi_s r(\xi_s) + \delta) = C_1 C_3 \Phi(\xi_s)
\end{aligned}$$

and

$$\begin{aligned}
\Phi''(\xi_s)|e_s^*\eta_s|^2 &\le \Phi''(\xi_s)\|e_s\|^2|\eta_s|^2 \\
&\le \frac{C_2\Phi(\xi_s)r(\xi_s)}{(\xi_s r(\xi_s) + \delta)^2} C_3(\xi_s r(\xi_s) + \delta)\xi_s \\
&= C_2 C_3 \Phi(\xi_s)\, \underbrace{\frac{r(\xi_s)\xi_s}{\xi_s r(\xi_s) + \delta}}_{<1} \le C_2 C_3 \Phi(\xi_s).
\end{aligned}$$

If additionally $\Phi' > 0$ then by (2.23) we have

$$\Phi'(\xi_s)\langle\eta_s, f_s\rangle \le C_1 C_4 \Phi(\xi_s).$$

Let K be the constant defined in (2.22). Then we get as in Lemma 2.7

$$E[\Phi(\xi_{t\wedge\tau})] \le \Phi(|\eta_0|^2) + K\int_0^t E[\Phi(\xi_{s\wedge t})]\, ds.$$

Finally, by Gronwall's inequality, it follows that $E[\Phi(\xi_{t\wedge\tau})] \le \Phi(|\eta_0|^2)e^{Kt}$ for all $t > 0$. $\square$

The next Lemma will be used in the proof of Theorem 3.1.

Lemma 2.9. *Keeping the same notation, assume that the coefficients e and f are bounded, namely*

$$\|e_t(\omega)\| \leq A, \qquad |f_t(\omega)| \leq B \quad \textit{uniformly for all } t \in [0,\infty) \textit{ and } \omega \in \Omega.$$

Assume $\eta_0 = 0$. Then for any $T > 0$ and $R > \sqrt{d}BT$, we have

$$P\left(\sup_{0\leq s\leq T} |\xi_s| \geq R^2\right) \leq 2de^{-(R-\sqrt{d}BT)^2/2dA^2T}. \tag{2.24}$$

This is a classical result, its proof can be found in [Str84, p. 81]. For the reader's convenience, we present a detailed proof adapted to our concrete setup.

Proof. We have

$$\begin{aligned} P(\sup_{0\leq s\leq T} |\xi_s| \geq R^2) &= P(\sup_{0\leq s\leq T} |\eta_s| \geq R) \\ &= P\left(\sup_{0\leq t\leq T}\left|\int_0^t e_s\,dW_s + \int_0^t f_s\,ds\right| \geq R\right) \end{aligned}$$

since $\eta_0 = 0$ and

$$\eta_t = \int_0^t e_s\,dW_s + \int_0^t f_s\,ds.$$

Let

$$\bar{\zeta}_t := \eta_t - \int_0^t f_s\,ds = \int_0^t e_s\,dW_s.$$

(i) **Claim**: The process

$$\exp\left(\langle\theta, \bar{\zeta}_t\rangle - \frac{1}{2}\int_0^t |e_s^*\theta|^2\,ds\right) = \exp\left(\int_0^t \langle\theta, e_s\,dW_s\rangle - \frac{1}{2}\int_0^t |e_s^*\theta|^2\,ds\right) \tag{2.25}$$

is a martingale for all $\theta \in \mathbb{R}^d$.
We will show, that (2.25) is an exponential martingale. Define $Y_t := \int_0^t\langle\theta, e_s\,dW_s\rangle = \int_0^t\langle e_s^*\theta, dW_s\rangle$, which is obviously a local martingale. By [Röc06, Lemma 2.4.4]

$$\langle Y_t\rangle = \int_0^t |e_s^*\theta|^2\,ds.$$

The Novikov criterium (Theorem 1.12) is fulfilled:

$$E\left[\exp(\frac{1}{2}\int_0^t \underbrace{|e_s^*\theta|^2}_{\leq\|e_s^*\|^2|\theta|^2}\,ds)\right] \leq E\left[\exp(\frac{1}{2}\int_0^t \|e_s^*\|^2\,ds)\right] < e^{\frac{1}{2}tA^2}.$$

Therefore $\exp(Y_t - \frac{1}{2}\langle Y_t\rangle) = \exp(\theta\int_0^t e_s\,dW_s - \frac{1}{2}\int_0^t |e_s^*\theta|^2\,ds)$ is a martingale. $\triangle$

(ii) Let S^{d-1} be the $d-1$-dimensional sphere in $\mathbb{R}^d$. For fixed $\theta \in S^{d-1}$ and for all $\lambda > 0$ we obtain

$$\begin{aligned}
&P(\sup_{0\leq t\leq T} \langle \theta, \eta_t \rangle \geq R) \\
&= P(\sup_{0\leq t\leq T} \langle \theta, \eta_t \rangle - BT \geq R - BT) \\
&\leq P(\sup_{0\leq t\leq T} \langle \theta, \eta_t \rangle - \langle \theta, \int_0^t f_s \, ds \rangle \geq R - BT) \\
&= P(\sup_{0\leq t\leq T} \langle \theta, \bar{\zeta}_t \rangle \geq R - BT) \\
&= P(\sup_{0\leq t\leq T} \lambda \langle \theta, \bar{\zeta}_t \rangle \geq \lambda(R - BT)) \\
&= P\left(\sup_{0\leq t\leq T} \left(\lambda \langle \theta, \bar{\zeta}_t \rangle - \frac{\lambda^2}{2} \int_0^t |e_s^* \theta|^2 \, ds\right) \geq \lambda(R - BT) - \frac{\lambda^2}{2} \underbrace{\int_0^t |e_s^* \theta|^2 \, ds}_{\leq \int_0^T A^2 \, ds = A^2 T}\right) \\
&\leq P\left(\sup_{0\leq t\leq T} \left(\lambda \langle \theta, \bar{\zeta}_t \rangle - \frac{\lambda^2}{2} \int_0^t |e_s^* \theta|^2 \, ds\right) \geq \lambda(R - BT) - \frac{\lambda^2 A^2 T}{2}\right) \\
&= P\left(\sup_{0\leq t\leq T} \exp\left(\lambda \langle \theta, \bar{\zeta}_t \rangle - \frac{\lambda^2}{2} \int_0^t |e_s^* \theta|^2 \, ds\right) \geq \exp\left(\lambda(R - BT) - \frac{\lambda^2 A^2 T}{2}\right)\right).
\end{aligned}$$

Since

$$Z_{\lambda,t} := \exp(\lambda Y_t - \frac{\lambda^2}{2} \langle Y_t \rangle) = \exp\left(\lambda \langle \theta, \bar{\zeta}_t \rangle - \frac{\lambda^2}{2} \int_0^t |e_s^* \theta|^2 \, ds\right)$$

is a martingale, we can use Doob's maximal inequality and get,

$$P\Big(\sup_{0\leq s\leq T} \theta \eta_t \geq R\Big) \leq \exp\left(-\lambda(R - BT) + \frac{\lambda^2 A^2 T}{2}\right), \tag{2.26}$$

keeping in mind that $E[Z_{\lambda,t}] = 1$. Taking $\lambda = \frac{R-BT}{A^2 T}$, we arrive at

$$\begin{aligned}
\exp\left(-\lambda(R - BT) + \frac{\lambda^2 A^2 T}{2}\right) &= \exp\left(-\frac{R - BT}{A^2 T}(R - BT) + \frac{(\frac{R-BT}{A^2 T})^2 A^2 T}{2}\right) \\
&= \exp\left(-\frac{(R - BT)^2}{A^2 T} + \frac{(R - BT)^2}{2A^2 T}\right) \\
&= \exp\left(-\frac{(R - BT)^2}{2A^2 T}\right). \qquad (2.27)
\end{aligned}$$

Let $\{e_i\}_{i=1}^d$ be an ONB of $\mathbb{R}^d$. Then we have

$$\begin{aligned}
\{\sup_{0\leq s\leq T} |\eta_s| \geq R\} &= \left\{\sup_{0\leq s\leq T} \sum_{i=1}^d \langle \eta_s, e_i \rangle^2 \geq R^2\right\} \\
&\subset \left\{\sum_{i=1}^d \sup_{0\leq s\leq T} \langle \eta_s, e_i \rangle^2 \geq R^2\right\}
\end{aligned}$$

$$\subset \bigcup_{i=1}^{d} \left[\left\{ \sup_{0\le s\le T} \langle \eta_s, e_i\rangle \ge \frac{R}{\sqrt{d}} \right\} \cup \left\{ \sup_{0\le s\le T} \langle \eta_s, e_i\rangle \le -\frac{R}{\sqrt{d}} \right\} \right]. \tag{2.28}$$

Hence

$$\begin{aligned}
&P\Big(\sup_{0\le s\le T} |\eta_s| \ge R\Big) \\
&\le \sum_{i=1}^{d} \left[P\left(\sup_{0\le s\le T} \langle \eta_s, e_i\rangle \ge \frac{R}{\sqrt{d}} \right) + P\left(\sup_{0\le s\le T} \langle \eta_s, e_i\rangle \le -\frac{R}{\sqrt{d}} \right) \right] \\
&\le \sum_{i=1}^{d} \left[\sup_{1\le i\le d} P\left(\sup_{0\le s\le T} \langle \eta_s, e_i\rangle \ge \frac{R}{\sqrt{d}} \right) + \sup_{1\le i\le d} P\left(\sup_{0\le s\le T} \langle \eta_s, e_i\rangle \le -\frac{R}{\sqrt{d}} \right) \right] \\
&\le d \left[\sup_{\theta\in S^{d-1}} P\left(\sup_{0\le s\le T} \langle \eta_s, \theta\rangle \ge \frac{R}{\sqrt{d}} \right) + \sup_{\theta\in S^{d-1}} P\left(\sup_{0\le s\le T} \langle \eta_s, \theta\rangle \le -\frac{R}{\sqrt{d}} \right) \right] \\
&\le 2d \sup_{\theta\in S^{d-1}} P\left(\sup_{0\le s\le T} \langle \theta, \eta_s\rangle \ge \frac{R}{\sqrt{d}} \right).
\end{aligned}$$

Putting (2.26), (2.27) and (2.28) together we prove the claim:

$$\begin{aligned}
P(\sup_{0\le s\le T} |\eta_s| \ge R) &\le 2d \sup_{\theta\in S^{d-1}} P\left(\sup_{0\le s\le T} \langle \theta, \eta_s\rangle \ge \frac{R}{\sqrt{d}} \right) \\
&\le 2d \sup_{\theta\in S^{d-1}} \exp\left(-\frac{\left(\frac{R}{\sqrt{d}} - BT\right)^2}{2A^2T} \right) \\
&= 2d \exp\left(-\frac{(R-\sqrt{d}BT)^2}{2dA^2T} \right).
\end{aligned}$$

□

2.4 Proof of non-explosion

The aim of this section is to prove Theorem A. Therefore we recall our setup. Let $\sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^m$ and $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ be continuous functions. Let (X_t, W_t) be a weak solution of the Itô stochastic differential equation

$$dX_t = \sigma(t, X_t)\, dW_t + b(t, X_t)\, dt, \quad X_0 = x_0 \in \mathbb{R}^d, \tag{2.29}$$

up to the explosion time ζ. Such solution exists by Theorem 2.1. We need to show that $\zeta = +\infty$.

Proof of Theorem A. Without loss of generality we may always assume that $\rho \ge 1$, see Remark 2.2 (ii). Let us define the functions

$$\Psi(\xi) := \int_0^{\xi} \frac{ds}{s\rho(s)+1} \quad \text{and} \quad \Phi(\xi) := e^{\Psi(\xi)}, \quad \xi \ge 0.$$

We calculate their derivatives:

$$\Phi'(\xi) = e^{\Psi(\xi)}\Psi'(\xi) = \Phi(\xi)\frac{d}{d\xi}\int_0^\xi \frac{ds}{s\rho(s)+1} = \Phi(\xi)\frac{1}{\xi\rho(\xi)+1}$$

and

$$\Phi''(\xi) = \frac{\Phi'(\xi)(\xi\rho(\xi)+1) - \Phi(\xi)(\xi\rho'(\xi)+\rho(\xi))}{(\xi\rho(\xi)+1)^2} = \frac{\Phi(\xi)(1-\xi\rho'(\xi)-\rho(\xi))}{(\xi\rho(\xi)+1)^2}.$$

This shows, that Φ is in C^2 and it's clear, that Φ and Φ' are strictly positive. Since $\rho \geq 1$ and ρ obeys conditions (2.3), (2.4), the following estimate holds:

$$|1-\rho(\xi)-\xi\rho'(\xi)| \leq \underbrace{|1-\rho(\xi)|}_{\leq\rho(\xi)} + \underbrace{|\xi\rho'(\xi)|}_{\leq\tilde{C}\rho(\xi)} \leq (1+\tilde{C})\rho(\xi) = C_1\rho(\xi)$$

So we have

$$\Phi''(\xi) \leq C_1\frac{\Phi(\xi)\rho(\xi)}{(\xi\rho(\xi)+1)^2} \qquad \text{for all } \xi \geq 0.$$

This means that the conditions in (2.13) are satisfied. Let now $\eta_t := X_t$, $\xi_t := |\eta_t|^2$ according to the notation in Lemma 2.7. Then, by comparison of (2.29) and (2.10) we have $e_t = \sigma(t, X_t)$ and $f_t = b(t, X_t)$. By hypothesis (H1),

$$\|e_t\|^2 \leq C(\xi_t\rho(\xi_t)+1), \quad \langle f_t, \eta_t\rangle \leq C(\xi_t\rho(\xi_t)+1), \quad t \in [0,\infty).$$

So the conditions in (2.14) and (2.18) are fulfilled, too. Now we define the stopping time

$$\tau_R := \inf\{t > 0 | \xi_t \geq R\}, \quad R > 0.$$

It is clear, that τ_R tends to the explosion time ζ as $R \to +\infty$. Now we can use Lemma 2.7 which gives us the existence of a constant $C_2 > 0$ such that

$$E[\Phi(\xi_{t\wedge\tau_R})] \leq \Phi(\xi_0)e^{C_2 t}.$$

Thus, employing the continuity of ξ_t, we have for every $t > 0, R > 0$,

$$E[\mathbb{1}_{(\zeta\leq t)}\Phi(R)] \leq E[\mathbb{1}_{(\zeta\leq t)}\Phi(\xi_{t\wedge\tau_R})] \leq \Phi(\xi_0)e^{C_2 t}$$

Letting $R \to \infty$ we have by the condition (2.5) that $\Phi(+\infty) = +\infty$. Hence, $P(\zeta \leq t) = 0$ for any $t > 0$ which implies $P(\zeta = +\infty) = 1$. □

Let $X_t(x_0)$ be a solution of SDE (2.29) with initial value x_0.

Theorem 2.10. *Under the hypothesis of Theorem A but with*

$$|\langle x, b(t,x)\rangle| \leq C(|x|^2\rho(|x|^2)+1)$$

instead of the second condition in (H1) *we have*

$$\lim_{|x_0|\to+\infty} |X_t(x_0)| = +\infty \textit{ in probability,} \tag{2.30}$$

that means that $\lim_{|x_0|\to\infty} P(|X_t(x_0)| \geq R) = 1$ *for all* $R > 0$.

Proof. Let Ψ be the same as in the proof of Theorem A, that is $\Psi(\xi) := \int_0^\xi \frac{ds}{s\rho(s)+1}$. The function Φ now is defined by

$$\Phi(\xi) := e^{-\Psi(\xi)}.$$

We see that Φ is a decreasing function (because Ψ is increasing) with the derivatives

$$\begin{aligned}\Phi'(\xi) &= -\frac{\Phi(\xi)}{\xi\rho(\xi)+1} < 0, \qquad |\Phi'(\xi)| = \frac{\Phi(\xi)}{\xi\rho(\xi)+1},\\ \Phi''(\xi) &= -\frac{\Phi'(\xi)(\xi\rho(\xi)+1) - \Phi(\xi)(\rho(\xi)+\xi\rho'(\xi))}{(\xi\rho(\xi)+1)^2} = \frac{\Phi(\xi)(1+\rho(\xi)+\xi\rho'(\xi))}{(\xi\rho(\xi)+1)^2}.\end{aligned}$$

Because of $\rho \geq 1$ and condition (2.4) we have

$$\underbrace{1}_{\leq\rho(\xi)} + \rho(\xi) + \underbrace{\xi\rho'(\xi)}_{\leq C\rho(\xi)} \leq \rho(\xi)(C+2).$$

Thus,

$$\Phi''(\xi) \leq C_1 \frac{\Phi(\xi)\rho(\xi)}{(\xi\rho(\xi)+1)^2},$$

with $C_1 = C+2 > 0$. Hence, the conditions in (2.13) are satisfied. Let R, M be two positive constants such that $R < |x_0| < M$. Define

$$\hat{\tau}_R := \inf\{t > 0 | |X_t(x_0)| \leq R\} \qquad \text{and} \qquad \tau_M := \inf\{t > 0 | |X_t(x_0)| \geq M\}.$$

By Theorem A we know that $\tau_M \uparrow +\infty$ as $M \uparrow +\infty$. Let $\eta_t = X_{t\wedge\hat{\tau}_R}$, which is an Itô process. According to notations in Lemma 2.6, we have

$$e_s(\omega) = \mathbb{1}_{\{\hat{\tau}_R \geq s\}}\sigma(s, X_s), \qquad f_s(\omega) = \mathbb{1}_{\{\hat{\tau}_R \geq s\}}b(s, X_s).$$

By hypothesis (H1) it follows

$$\|e_s\|^2 \leq C(\xi_s\rho(\xi_s)+1), \qquad \langle \eta_s, f_s\rangle \leq C(\xi_s\rho(\xi_s)+1).$$

Using Lemma 2.7 we have

$$E[\Phi(\xi_{t\wedge\tau_M})] \leq \Phi(|x_0|^2)e^{Ct}.$$

Letting $M \to \infty$ and repeating the arguments from the proof of Theorem A, we get

$$E[\Phi(|X_{t\wedge\hat{\tau}_R}(x_0)|^2)] \leq \Phi(|x_0|^2)e^{Ct}. \tag{2.31}$$

Because of $X_{t\wedge\hat{\tau}_R} = X_{\hat{\tau}_R}$ on the set $\{t \geq \hat{\tau}_R\}$ and $|X_{\hat{\tau}_R}| \leq R$, we have $|X_{t\wedge\hat{\tau}_R}|^2 \leq R^2$. Since Φ is decreasing, it follows $\Phi(|X_{t\wedge\hat{\tau}_R}|^2) \geq \Phi(R^2)$. Combining this with (2.31) we get

$$\begin{aligned}P(\hat{\tau}_R \leq t)\Phi(R^2) &= \int \mathbb{1}_{\{\hat{\tau}_R \leq t\}}\Phi(R^2)\,dt \;\leq\; \int \mathbb{1}_{\{\hat{\tau}_R \leq t\}}\Phi(|X_{\hat{\tau}_R}|^2)\,dt\\ &\leq \int \Phi(|X_{t\wedge\hat{\tau}_R}|^2)\,dt \;=\; E[\Phi(|X_{t\wedge\hat{\tau}_R}|^2)] \;\leq\; \Phi(|x_0|^2)e^{Ct}.\end{aligned}$$

Therefore for every fixed R,

$$\begin{aligned} P(\inf_{0\le s\le t}|X_s(x_0)|\le R) \le P(\hat{\tau}_R\le t) \;&\le\; \frac{\Phi(|x_0|^2)}{\Phi(R^2)}e^{Ct} \\ &= e^{Ct}e^{-(\Psi(|x_0|^2)-\Psi(R^2))} \\ &\le e^{Ct}\exp\Big\{-\underbrace{\int_{R^2}^{|x_0|^2}\frac{ds}{s\rho(s)+1}}_{\to+\infty}\Big\} \\ &\to 0 \end{aligned}$$

when $|x_0|$ tends to $+\infty$. Then convergence in probability follows. □

2.5 Proof of pathwise uniqueness and non contact property

In this section we will prove Theorem B and C. The proofs mainly follow the idea of the paper of Fang and Zhang [FZ05]. Afterwards, we will prove Theorem 2.11 whose arguments of proof will be used for the Smoluchowski-Kramer approximation (see Section 4.2).

Proof of Theorem B. Without loss of generality we can assume that the explosion time ζ of SDE (2.29) is infinite, otherwise we have the pathwise uniqueness up to the explosion time. Let X_t and Y_t be two solutions of (2.29) having the same initial data. Consider the deviation process $\eta_t = X_t - Y_t$ and $\xi_t = |\eta_t|^2$, $t\in[0,\infty)$. According to the notations in Lemma 2.6,

$$e_t = \sigma(t,X_t)-\sigma(t,Y_t), \qquad\qquad f_t = b(t,X_t)-b(t,Y_t).$$

Let $\tau=\inf\{t>0|\xi_t\ge c_0^2\}$. By hypothesis (H2), for $\delta>0$ and $t\le\tau$

$$\begin{aligned} \|e_t\|^2 = \|\sigma(t,X_t)-\sigma(t,Y_t)\|^2 \;&\le\; C|X_t-Y_t|^2 r(|X_t-Y_t|^2) \\ = C\xi_t r(\xi_t) \;&\le\; C(\xi_t r(\xi_t)+\delta), \end{aligned}$$

and

$$\begin{aligned} \langle\eta_t,f_t\rangle &= \langle X_t-Y_t, b(t,X_t)-b(t,Y_t)\rangle \\ &\le C|X_t-Y_t|^2 r(|X_t-Y_t|^2) \;=\; C\xi_t r(\xi_t) \;\le\; C(\xi_t r(\xi_t)+\delta), \end{aligned}$$

where r is defined in Theorem B. According to condition (2.6) on the function r, we assume that $r(\xi)\ge 1$ for all $\xi\in(0,c_0]$. Otherwise we choose a smaller c_0. We define for $\delta\ge 0$

$$\Psi_\delta(\xi)=\int_0^\xi \frac{ds}{sr(s)+\delta} \qquad\text{and}\qquad \Phi_\delta(\xi)=e^{\Psi_\delta(\xi)}.$$

Condition (2.8) on r leads to

$$\Phi_0(\xi)=e^{\Psi_0(\xi)}=\exp\Big(\int_0^\xi\frac{ds}{sr(s)+0}\Big)=e^{+\infty}=+\infty \quad \forall\xi>0. \tag{2.32}$$

Calculation as in the proof of Theorem A implies

$$\Phi_\delta'(\xi) = \frac{\Phi_\delta(\xi)}{\xi r(\xi)+\delta} > 0, \qquad \Phi_\delta''(\xi) = \Phi_\delta(\xi)\frac{1-r(\xi)-\xi r'(\xi)}{(\xi r(\xi)+\delta)^2}.$$

Conditions (2.6) and (2.7) ensures that there exists a large constant $C_1 > 0$ such that

$$|1-r(\xi)-\xi r'(\xi)| \le |1-r(\xi)| + |\xi r'(\xi)| \le C_1 r(\xi), \qquad \xi \in (0, c_0].$$

Therefrom it follows that for $\xi \in (0, c_0]$

$$\Phi_\delta''(\xi) \le C_1 \frac{\Phi_\delta(\xi) r(\xi)}{(\xi r(\xi)+\delta)^2}.$$

The conditions in (2.19) are fulfilled. Now the Lemma 2.8 tells us that there exists a constant $C_2 > 0$ such that for any $t > 0$

$$E[\Phi_\delta(\xi_{t\wedge\tau})] \le \underbrace{\Phi_\delta(|\eta_0|^2)}_{=\Phi_\delta(0)=e^0} e^{C_2 t} = e^{C_2 t}.$$

Letting $\delta \downarrow 0$ and applying Fatou's lemma we have

$$\begin{aligned}
\liminf_{\delta\downarrow 0} E[\Phi_\delta(\xi_{t\wedge\tau})] &\ge E[\liminf_{\delta\downarrow 0} \Phi_\delta(\xi_{t\wedge\tau})] \\
&= E\left[\exp\left(\lim_{\delta\downarrow 0}\int_0^{\xi_{t\wedge\tau}} \frac{1}{sr(s)+\delta}\, ds\right)\right] \\
&= E\left[\exp\left(\int_0^{\xi_{t\wedge\tau}} \frac{1}{sr(s)}\, ds\right)\right]. \qquad (2.33)
\end{aligned}$$

In regard of (2.32), this implies that for any given t,

$$\xi_{t\wedge\tau} = 0 \quad \text{almost surely.} \qquad (2.34)$$

Recall that η_t and hence ξ_t is continuous in t. Hence, by continuity $\xi_\tau = 0$ on $\{\tau < \infty\}$ P-a.s. But again by continuity $\xi_\tau \ge c_0^2$. Hence $P(\tau < \infty) = 0$. □

Proof of Theorem C. Without loss of generality we may assume $|x_0 - y_0| < c_0/2$. Let $0 < \varepsilon < |x_0 - y_0|$ and define the random times

$$\hat\tau_\varepsilon := \inf\{t > 0 \,|\, |X_t(x_0) - X_t(y_0)| \le \varepsilon\}, \quad \hat\tau := \inf\{t > 0 \,|\, X_t(x_0) = X_t(y_0)\}. \qquad (2.35)$$

It's clear that $\hat\tau_\varepsilon \uparrow \hat\tau$ as $\varepsilon \downarrow 0$. Let

$$\tau := \inf\left\{t > 0 \,\middle|\, |X_t(x_0) - X_t(y_0)| \ge \frac{3}{4}c_0\right\}.$$

Consider

$$\eta_t = X_{t\wedge\hat\tau_\varepsilon}(x_0) - X_{t\wedge\hat\tau_\varepsilon}(y_0) \qquad \text{and} \qquad \xi_t = |\eta_t|^2. \qquad (2.36)$$

Again we use the notation from Lemma 2.6:

$$e_t = \mathbb{1}_{\hat{\tau}_\varepsilon \geq t}(\sigma(t, X_t(x_0)) - \sigma(t, X_t(y_0))), \quad f_t = \mathbb{1}_{\hat{\tau}_\varepsilon \geq t}(b(t, X_t(x_0)) - b(t, X_t(y_0))).$$

By hypothesis (H2), it follows for $t < \tau$ and $\delta > 0$,

$$\|e_t\|^2 \leq C\xi_t r(\xi_t) \ \leq C(\xi_t r(\xi_t) + \delta)$$

and

$$\begin{aligned} |\langle \eta_t, f_t \rangle| &= |\langle \eta_t, \mathbb{1}_{\hat{\tau}_\varepsilon \geq t}(b(t, X_t(x_0)) - b(t, X_t(y_0))) \rangle| \\ &\leq |\langle X_{t\wedge\hat{\tau}_\varepsilon}(x_0) - X_{t\wedge\hat{\tau}_\varepsilon}(y_0), b(t\wedge\hat{\tau}_\varepsilon, X_{t\wedge\hat{\tau}_\varepsilon}(x_0)) - b(t\wedge\hat{\tau}_\varepsilon, X_{t\wedge\hat{\tau}_\varepsilon}(y_0)) \rangle| \\ &\leq C\xi_t r(\xi_t) \\ &\leq C(\xi_t r(\xi_t) + \delta). \end{aligned}$$

We define the functions

$$\Psi_\delta(\xi) := \int_\xi^{c_0} \frac{ds}{sr(s) + \delta} \qquad \text{and} \qquad \Phi_\delta := e^{\Psi_\delta(\xi)}$$

for $\xi \leq c_0$. Like as in Theorem B we have

$$\Phi_\delta'(\xi) < 0, \qquad |\Phi_\delta'(\xi)| = |\Phi_\delta(\xi)\Psi_\delta'(\xi)| = \Phi_\delta(\xi)\frac{1}{\xi r(\xi) + \delta}$$

and for a constant C_1 large enough

$$\Phi_\delta''(\xi) = \Phi_\delta(\xi)\frac{1 + r(\xi) + \xi r'(\xi)}{(\xi r(\xi) + \delta)^2} \ \leq \ C_1 \frac{\Phi_\delta(\xi) r(\xi)}{(\xi r(\xi) + \delta)^2}, \qquad \xi \in [0, c_0].$$

So the conditions in (2.19) and (2.20) are fulfilled and we can apply Lemma 2.8 to get

$$E[\Phi_\delta(\xi_{t\wedge\tau})] \leq \Phi_\delta(\xi_0)e^{C_2 t},$$

for some $C_2 > 0$ and for all $t > 0$. Letting $\delta \downarrow 0$, we get by Fatou's lemma

$$\begin{aligned} E[\Phi_0(\xi_{t\wedge\tau})] &= E[\liminf_{\delta\downarrow 0} \Phi_\delta(\xi_{t\wedge\tau})] \\ &\leq \liminf_{\delta\downarrow 0} E[\Phi_\delta(\xi_{t\wedge\tau})] \ \leq \ \lim_{\delta\downarrow 0} E[\Phi_\delta(\xi_{t\wedge\tau})] \\ &\leq \lim_{\delta\downarrow 0} \Phi_\delta(\xi_0)e^{C_2 t} = \Phi_0(\xi_0)e^{C_2 t}. \end{aligned}$$

Writing out ξ_t we have

$$E[\Phi_0(|X_{t\wedge\hat{\tau}_\varepsilon\wedge\tau}(x_0) - X_{t\wedge\hat{\tau}_\varepsilon\wedge\tau}(y_0)|^2)] \leq \Phi_0(\xi_0)e^{C_2 t}.$$

On the subset $\{\hat{\tau}_\varepsilon < t \wedge \tau\}$ we have by the definition (2.35) of $\hat{\tau}_\varepsilon$

$$|X_{t\wedge\hat{\tau}_\varepsilon\wedge\tau}(x_0) - X_{t\wedge\hat{\tau}_\varepsilon\wedge\tau}(y_0)| = |X_{\hat{\tau}_\varepsilon}(x_0) - X_{\hat{\tau}_\varepsilon}(y_0)| = \varepsilon.$$

Therefore

$$\begin{aligned}P(\hat{\tau}_\varepsilon < t \wedge \tau)\Phi_0(\varepsilon^2) &= P(\hat{\tau}_\varepsilon < t \wedge \tau)\Phi_0\left(|X_{\hat{\tau}_\varepsilon}(x_0) - X_{\hat{\tau}_\varepsilon}(y_0)|^2\right) \\ &\le E[\Phi_0(|X_{t\wedge\hat{\tau}_\varepsilon}(x_0) - X_{t\wedge\hat{\tau}_\varepsilon}(y_0)|^2)] \\ &\le \Phi_0(\xi_0)e^{C_2 t}.\end{aligned}$$

Since $\xi_0 > \varepsilon^2$, the latter implies that

$$\begin{aligned}P(\hat{\tau}_\varepsilon < t \wedge \tau) &\le \frac{\Phi_0(\xi_0)}{\Phi_0(\varepsilon^2)}e^{C_2 t} \\ &= \exp\left(-\int_{\varepsilon^2}^{\xi_0} \frac{1}{sr(s)}\,ds\right)e^{C_2 t} \\ &\longrightarrow 0, \qquad \text{as } \varepsilon \downarrow 0,\end{aligned}$$

because of the condition (2.8) on r. Since $\hat{\tau}_\varepsilon \uparrow \hat{\tau}$ we have that $P(\hat{\tau} < t \wedge \tau) = 0$ for all t. Letting $t \to \infty$ we get $P(\hat{\tau} < \tau) = 0$. So, we see that ξ_t is positive almost surely on the interval $[0, \tau]$. This means that the deviation ξ_t first becomes bigger than $\frac{3c_0}{4}$ instead of becoming zero almost surely. Now define $T_0 := 0$ and

$$T_1 := \tau, \qquad T_2 = \inf\left\{t > 0 \,\middle|\, |X_t(x_0) - X_t(y_0)| \le \frac{c_0}{2}\right\}$$

and generally

$$\begin{aligned}T_{2n} &:= \inf\left\{t > T_{2n-1} \,\middle|\, |X_t(x_0) - X_t(y_0)| \le \frac{c_0}{2}\right\}, \\ T_{2n+1} &:= \inf\left\{t > T_{2n} \,\middle|\, |X_t(x_0) - X_t(y_0)| \ge \frac{3c_0}{4}\right\}.\end{aligned}$$

By definition $T_n \to \infty$ as $n \to \infty$. By pathwise uniqueness of solutions by Theorem B, X enjoys the strong Markovian property, cf. [KS91, Theorem. 4.20, p.322]. By definition ξ_t is positive on the interval $[T_{2n-1}, T_{2n}]$. With help of the strong Markovian property, we start again from T_{2n} and apply the same arguments as in the first part of the proof. This shows that ξ_t is positive almost surely also on the interval $[T_{2n}, T_{2n+1}]$. So the proof is completed. □

Theorem 2.11. *Under the same hypothesis as in Theorem B, for any $\varepsilon > 0$, we have*

$$\lim_{y_0 \to x_0} P(\sup_{0 \le s \le t} |X_s(x_0) - X_s(y_0)| > \varepsilon) = 0. \tag{2.37}$$

Proof. Let x_0, y_0 be such that $|x_0 - y_0| < \varepsilon < c_0$, where c_0 is the parameter in definition of function r in Theorem B. Without loss of generality let $c_0 < 1$ and define

$$\xi_t := |X_t(y_0) - X_t(x_0)|^2 \qquad \text{and} \qquad \tau(x_0, y_0) := \inf\{t > 0 | \xi_t > \varepsilon^2\}.$$

Let Φ_δ and Ψ_δ be defined like as in the proof of Theorem B:

$$\Phi_\delta(\xi) := \exp\left(\int_0^\xi \frac{1}{sr(s) + \delta}\,ds\right), \qquad \Psi_\delta(\xi) := \int_0^\xi \frac{ds}{sr(s) + \delta}, \qquad \xi \in (0, +\infty).$$

It is important to note that $\Phi'_\delta(\xi) > 0$. Similarly to the proof of Theorem B we can use Lemma 2.8, which gives us a constant $C > 0$ such that

$$E[\Phi_\delta(\xi_{t\wedge\tau(x_0,y_0)})] \leq \Phi_\delta(\xi_0)e^{Ct} \quad \text{for all } t > 0 \text{ and } \delta > 0.$$

Let $\delta = |x_0 - y_0|$, then from the above inequality

$$\begin{aligned} E[\Phi_\delta(\xi_{t\wedge\tau(x_0,y_0)})] &\leq e^{\Psi_\delta(\delta^2)}e^{Ct} \\ &\leq e^\delta e^{Ct}. \end{aligned} \tag{2.38}$$

In the last line we used the estimate

$$\int_0^{\delta^2} \frac{1}{sr(s)+\delta}\,ds \leq \delta^2 \sup_{s\in[0,\delta^2]} \frac{1}{sr(s)+\delta} \leq \delta.$$

Taking into account that $\xi_{t\wedge\tau} = \xi_\tau > \varepsilon^2$ on the set $\{\tau < t\}$ and that Φ_δ is increasing, we obtain from (2.38)

$$\begin{aligned} P(\tau(x_0,y_0) < t)\Phi_\delta(\varepsilon^2) &\leq P(\tau(x_0,y_0) < t)\Phi_\delta(\xi_{t\wedge\tau(x_0,y_0)}) \\ &\leq E[\Phi_\delta(\xi_{t\wedge\tau(x_0,y_0)})] \\ &\leq e^\delta e^{Ct}. \end{aligned}$$

Therefore

$$\begin{aligned} P(\sup_{0\leq s\leq t}|X_s(x_0) - X_s(y_0)| > \varepsilon) &= P(\sup_{0\leq s\leq t}\sqrt{\xi_s} > \varepsilon) = P(\sup_{0\leq s\leq t}\xi_s > \varepsilon^2) \\ &= P(\tau(x_0,y_0) < t) \\ &\leq e^\delta e^{Ct}\frac{1}{\Phi_\delta(\varepsilon^2)} = e^\delta e^{Ct}\exp\left(-\int_0^{\varepsilon^2}\frac{1}{sr(s)+\delta}\,ds\right) \\ &\xrightarrow[\delta\to 0]{} 0, \end{aligned}$$

which completes the proof. $\square$

3 Continuous dependence of initial data

In this chapter we are interested in continuous modification of a solution $X_t(x_0)$ of the stochastic differential equation (2.29). Therefore, in the first section, we construct the strong solution via Euler approximation. In the second section we will prove some technical lemmas which thereafter will be used to prove Theorem D in Section 3.3.

3.1 Euler approximation

We assume that for all $t \geq 0$ and $x, y \in \mathbb{R}^d$, $0 \leq |x-y| \leq c_0$, where $c_0 > 0$ is a small constant,

$$\begin{aligned} \|\sigma(t,x) - \sigma(t,y)\|^2 &\leq C|x-y|^2 \log \tfrac{1}{|x-y|}, \\ |b(t,x) - b(t,y)| &\leq C|x-y| \log \tfrac{1}{|x-y|}. \end{aligned} \tag{3.1}$$

Then (H2) holds and by Theorem B the stochastic differential equation (2.29) obeys a pathwise unique solution up to the explosion time ζ. We now construct the strong solution directly via uniform Euler approximation under the additional assumption that the coefficients σ and b are bounded.

Theorem 3.1. *Let σ and b satisfy the condition (3.1) and be bounded:*

$$\|\sigma(t,x)\| \leq A, \qquad |b(t,x)| \leq B \quad \text{for all } x \in \mathbb{R}^d \text{ and } t > 0.$$

Fix an initial data $x_0 \in \mathbb{R}^d$. For $n \geq 1$ define $(X_n(t))_{n\geq 1}$ by $X_n(0) = x_0$ and

$$X_n(t) := X_n(k2^{-n}) + \sigma(k2^{-n}, X_n(k2^{-n}))(W_t - W_{k2^{-n}}) + b(k2^{-n}, X_n(k2^{-n}))(t - k2^{-n})$$

for $k2^{-n} \leq t \leq (k+1)2^{-n}$. Then for any $T > 0$, almost surely, $X_n(t)$ converges uniformly in $t \in [0,T]$ to the solution X_t of stochastic differential equation (2.29).

Proof. Define

$$\phi_n(t) := k2^{-n} \quad \text{for } t \in [k2^{-n}, (k+1)2^{-n}),\ k \geq 0.$$

Then by definition of the stochastic integral we have

$$\int_{\phi_n(t)}^t \underbrace{\sigma(k2^{-n}, X_n(\phi_n(s)))}_{=const\ \forall s \in [\phi_n(t), t]} \, dW_s = \sigma(k2^{-n}, X_n(\phi_n(t)))(W_t - W_{\phi_n(t)}).$$

By iteration over k, $X_n(t)$ can be expressed by

$$
\begin{aligned}
X_n(t) &= X_n\left(k2^{-n}\right) + \sigma\left(k2^{-n}, X_n(k2^{-n})\right)(W_t - W_{k2^{-n}}) + b\left(k2^{-n}, X_n(k2^{-n})\right)(t - k2^{-n}) \\
&= X_n\left(k2^{-n}\right) + \int_{\phi_n(t)}^t \sigma(k2^{-n}, X_n(\phi_n(s)))\, dW_s + \int_{\phi_n(t)}^t b(k2^{-n}, X_n(\phi_n(s)))\, ds \\
&= X_n\left((k-1)2^{-n}\right) + \int_{(k-1)2^{-n}}^{k2^{-n}} \sigma((k-1)2^{-n}, X_n(\phi_n(s)))\, dW_s \\
&\quad + \int_{(k-1)2^{-n}}^{k2^{-n}} b((k-1)2^{-n}, X_n(\phi_n(s)))\, ds \\
&\quad + \int_{k2^{-n}}^t \sigma(k2^{-n}, X_n(\phi_n(s)))\, dW_s + \int_{k2^{-n}}^t b(k2^{-n}, X_n(\phi_n(s)))\, ds \\
&= \ldots = X_n(0) + \int_0^t \bar{\sigma}(s, X_n(\phi_n(s)))\, dW_s + \int_0^t \bar{b}(s, X_n(\phi_n(s)))\, ds, \qquad (3.2)
\end{aligned}
$$

where we set $\bar{\sigma}(s,x) := \sigma(k2^{-n}, x)$ and $\bar{b}(s,x) := b(k2^{-n}, x)$ for $k2^{-n} \leq s \leq (k+1)2^{-n}$. Now let $1 < a < \sqrt{2}$. Introduce the stopping time

$$\tau_n = \inf\{t > 0 \,\big|\, |X_n(t) - X_n(\phi_n(t))| \geq a^{-n}\}.$$

For $t \in [k2^{-n}, (k+1)2^{-n})$, by expression (3.2), we have

$$
\begin{aligned}
&X_n(t) - X_n(\phi_n(t)) \\
&= \int_0^t \bar{\sigma}(s, X_n(\phi_n(s)))\, dW_s + \int_0^t \bar{b}(s, X_n(\phi_n(s)))\, ds \\
&\quad - \int_0^{\phi_n(t)} \bar{\sigma}(s, X_n(\phi_n(s)))\, dW_s - \int_0^{\phi_n(t)} \bar{b}(s, X_n(\phi_n(s)))\, ds \\
&= \int_0^t \bar{\sigma}(s, X_n(\phi_n(s)))\, dW_s - \int_0^{\phi_n(t)} \bar{\sigma}(s, X_n(\phi_n(s)))\, dW_s + \int_{\frac{k}{2^n}}^t \bar{b}(s, X_n(\phi_n(s)))\, ds \\
&= \int_{\frac{k}{2^n}}^t \bar{\sigma}(X_n(s, \phi_n(s)))\, dW_s + \int_{\frac{k}{2^n}}^t \bar{b}(s, X_n(\phi_n(s)))\, ds \\
&= \int_0^{t-\frac{k}{2^n}} \bar{\sigma}\Big(\frac{k}{2^n} + s, X_n\left(\phi_n(\frac{k}{2^n} + s)\right)\Big)\, d(\underbrace{W_{\frac{k}{2^n}+s} - W_{\frac{k}{2^n}}}_{=:\tilde{W}_s}) \\
&\quad + \int_0^{t-\frac{k}{2^n}} \bar{b}\Big(\frac{k}{2^n} + s, X_n\left(\phi_n(\frac{k}{2^n} + s)\right)\Big)\, ds.
\end{aligned}
$$

Then we apply Lemma 2.9 with $\|e_t(\omega)\| = \|\bar{\sigma}(t,x)\| \leq A$ and $|f_t(\omega)| = |\bar{b}(t,x)| \leq B$. Choosing $T = \frac{1}{2^n}$ and $R = \frac{1}{a^n}$ ($\frac{2}{a} > 1$, so $R > \sqrt{d}BT$ holds for n large enough), we get

$$
\begin{aligned}
&P\left(\sup_{\frac{k}{2^n} \leq t < \frac{(k+1)}{2^n}} |X_n(t) - X_n(\phi_n(t))| \geq R\right) \\
&\leq 2de^{-(R-\sqrt{d}BT)^2/2dA^2T}
\end{aligned}
$$

$$
\begin{aligned}
&= 2d \exp\left(-\frac{(a^{-n} - \sqrt{d}B2^{-n})^2}{2dA^2 2^{-n}}\right) \\
&= 2d \exp\left(-\frac{(a^{-n} - \sqrt{d}B2^{-n})^2 2^n}{2dA^2}\right) \\
&= 2d \exp\left(-(\frac{2}{a^2})^n \Big(1 - \sqrt{d}B(\frac{2}{a})^{-n}\Big)^2 / 2dA^2\right).
\end{aligned}
$$

The last equality holds because of:

$$
\begin{aligned}
(\frac{2}{a^2})^n \left[1 - \sqrt{d}B(\frac{2}{a})^{-n}\right]^2 &= \frac{2^n}{a^{2n}}\Big(1 - \sqrt{d}B(\frac{2}{a})^{-n} + dB^2(\frac{2}{a})^{-2n}\Big) \\
&= \frac{2^n}{a^{2n}} - \sqrt{d}B\frac{a^n}{a^{2n}} + dB^2\frac{2^n}{2^{2n}} \\
&= \frac{2^n}{a^{2n}} - \sqrt{d}B\frac{1}{a^n} + dB^2 2^{-n} \\
&= 2^n\left(\frac{1}{a^{2n}} - \sqrt{d}B\frac{1}{a^n 2^n} + dB^2 2^{-2n}\right) \\
&= 2^n\Big(a^{-n} - \sqrt{d}B2^{-n}\Big)^2.
\end{aligned}
$$

Let $c = 2/a^2$, which is strictly bigger than 1 (cf. the above definition of a). Therefore we have for large n,

$$
\begin{aligned}
&P\left(\sup_{\frac{k}{2^n} \le t < \frac{(k+1)}{2^n}} |X_n(t) - X_n(\phi_n(t))| \ge a^{-n}\right) \\
&\le 2d \exp\left(-c^n(1 - \sqrt{d}B(\frac{2}{a})^{-n})^2 / 2dA^2\right) \\
&\le 2d \exp\left(-c^n / 4dA^2\right),
\end{aligned}
$$

where the last inequality holds because $\frac{2}{a} > 1$ and hence $(\frac{2}{a})^{-n} \to 0$ as $n \to \infty$. Therefore, the term $(1 - \sqrt{d}B(\frac{2}{a})^{-n})^2 \to 1$ as $n \to \infty$, so it is in especially bigger than $\frac{1}{2}$. On the other hand, we have for integer $T > 0$

$$
\begin{aligned}
P(\tau_n \le T) &= P\left(\inf_{t>0}\{|X_n(t) - X_n(\phi_n(t))| \ge a^{-n}\} \le T\right) \\
&= P\left(\sup_{0<t\le T} |X_n(t) - X_n(\phi_n(t))| \ge a^{-n}\right) \\
&= P\left(\sup_{\substack{k\in\mathbb{N} \\ 0<k\le T2^n}} \sup_{\frac{k-1}{2^n}<t\le\frac{k}{2^n}} |X_n(t) - X_n(\phi_n(t))| \ge a^{-n}\right) \\
&= P\left(\bigcup_{\substack{k\in\mathbb{N} \\ 0<k\le T2^n}} \left\{\sup_{\frac{k-1}{2^n}<t\le\frac{k}{2^n}} |X_n(t) - X_n(\phi_n(t))| \ge a^{-n}\right\}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{2^n T} P\left(\sup_{\frac{k-1}{2^n} < t \leq \frac{k}{2^n}} |X_n(t) - X_n(\phi_n(t))| \geq a^{-n} \right) \\
&\leq T 2^n 2d \exp\left(-c^n / 4dA^2\right).
\end{aligned}
$$

We know that $\frac{c^n}{(n+1)\log 2 + \log Td} \to \infty$ when $n \to \infty$. Therefore,

$$
\begin{aligned}
& & \frac{c^n}{(n+1)\log 2 + \log Td} &\geq 8dA^2 \\
&\Leftrightarrow & \frac{c^n}{8dA^2} &\geq \log(2^{n+1} Td) \\
&\Leftrightarrow & \exp\left(\frac{c^n}{8dA^2}\right) &\geq 2^{n+1} Td \\
&\Leftrightarrow & \exp\left(-\frac{c^n}{8dA^2}\right) \exp\left(\frac{c^n}{4dA^2}\right) &\geq 2^{n+1} Td \\
&\Leftrightarrow & \exp\left(-\frac{c^n}{8dA^2}\right) &\geq 2^n T 2d \exp\left(-\frac{c^n}{4dA^2}\right).
\end{aligned}
$$

Using this with the inequality above we arrive at

$$
P(\tau_n \leq T) \leq \exp\left(-c^n / 8dA^2\right). \tag{3.3}
$$

Defining now

$$
\eta_n(t) := X_{n+1}(t) - X_n(t), \qquad \xi_n(t) := |\eta_n(t)|^2
$$

and

$$
\begin{aligned}
e_t &:= \bar{\sigma}(t, X_{n+1}(\phi_{n+1}(t))) - \bar{\sigma}(t, X_n(\phi_n(t))) \\
f_t &:= \bar{b}(t, X_{n+1}(\phi_{n+1}(t))) - \bar{b}(t, X_n(\phi_n(t)))
\end{aligned}
$$

By Lemma 2.6 we have that

$$
d\xi_n(t) = 2\langle e_t^* \eta_n(t), dW_t \rangle + 2\langle \eta_n(t), f_t \rangle \, dt + \|e_t\|^2 \, dt
$$

has the stochastic contraction

$$
\langle d\xi_t(t) \rangle = 4 |e_t^* \eta_n(t)|^2 \, dt.
$$

Define the stopping time

$$
\zeta_n := \inf\left\{ t > 0 \,\middle|\, \xi_n(t) \geq \frac{1}{n^{2\beta}} \right\}
$$

with the parameter $\beta > 1$. Then for $s \leq \tau_{n+1}$ and n large enough (such that $a^{-(n+1)} < 1/e$), we can use (3.1) to obtain

$$
\begin{aligned}
&\|\bar{\sigma}(s, X_{n+1}(\phi_{n+1}(s))) - \bar{\sigma}(s, X_{n+1}(s))\|^2 \\
&\leq C |X_{n+1}(\phi_{n+1}(s)) - X_{n+1}(s)|^2 \log(1 / |X_{n+1}(\phi_{n+1}(s)) - X_{n+1}(s)|) \\
&\leq C a^{-2(n+1)} \log(1 / a^{-(n+1)}) \\
&\leq C a^{-2n} \log a^n,
\end{aligned}
$$

where we used the fact that $s \to s \log 1/s$ is an increasing function for $s \in [0, 1/e]$. The same arguments leads to

$$\|\bar{\sigma}(s, X_n(\Phi_n(t))) - \bar{\sigma}(s, X_n(s))\|^2 \leq C a^{-2n} \log a^n, \tag{3.4}$$

and hence with the parallelogram law for $s \leq \tau_n \wedge \tau_{n+1} \wedge \zeta_n$,

$$\begin{aligned}
\|e_s\|^2 &= \|\bar{\sigma}(s, X_{n+1}(\phi_{n+1}(s))) - \bar{\sigma}(s, X_{n+1}(s)) + \bar{\sigma}(s, X_{n+1}(s)) - \bar{\sigma}(s, X_n(s)) \\
&\quad + \bar{\sigma}(s, X_n(s)) - \bar{\sigma}(X_n(s, \phi_n(s)))\|^2 \\
&\leq 2[\|\bar{\sigma}(s, X_{n+1}(\phi_{n+1}(s))) - \bar{\sigma}(s, X_{n+1}(s))\|^2 \\
&\qquad + \|\bar{\sigma}(s, X_{n+1}(s)) - \bar{\sigma}(s, X_n(s)) + \bar{\sigma}(s, X_n(s)) - \bar{\sigma}(s, X_n(\phi_n(s)))\|^2] \\
&\quad - \|\bar{\sigma}(s, X_{n+1}(\phi_{n+1}(s))) - \bar{\sigma}(s, X_{n+1}(s)) \\
&\qquad - (\bar{\sigma}(s, X_{n+1}(s)) - \bar{\sigma}(s, X_n(s)) + \bar{\sigma}(s, X_n(s)) - \bar{\sigma}(s, X_n(\phi_n(s))))\|^2 \\
&\leq 2[\|\bar{\sigma}(s, X_{n+1}(\phi_{n+1}(s))) - \bar{\sigma}(s, X_{n+1}(s))\|^2 \\
&\qquad + 2(\|\bar{\sigma}(s, X_{n+1}(s)) - \bar{\sigma}(s, X_n(s))\|^2 + \|\bar{\sigma}(s, X_n(s)) - \bar{\sigma}(s, X_n(\phi_n(s)))\|^2) \\
&\qquad - \underbrace{\|\bar{\sigma}(s, X_{n+1}(s)) - \bar{\sigma}(s, X_n(s)) - (\bar{\sigma}(s, X_n(s)) - \bar{\sigma}(s, X_n(\phi_n(s))))\|^2]}_{>0} \\
&\leq 4[\|\bar{\sigma}(s, X_{n+1}(\phi_{n+1}(s))) - \bar{\sigma}(s, X_{n+1}(s))\|^2 + \|\bar{\sigma}(s, X_{n+1}(s)) - \bar{\sigma}(s, X_n(s))\|^2 \\
&\qquad + \|\bar{\sigma}(s, X_n(s)) - \bar{\sigma}(s, X_n(\phi_n(s)))\|^2] \\
&\leq 4\left[2Ca^{-2n} \log a^n + C|X_{n+1}(s) - X_n(s)|^2 \log(1/|X_{n+1}(s) - X_n(s)|)\right] \\
&\leq 4C\left(2a^{-2n} \log a^n + \xi_n(s) \log(|\eta_n(s)|^{-1})\right) \\
&\leq 4C\left(2a^{-2n} \log a^n + \xi_n(s) \log(1/\xi_n(s))\right).
\end{aligned}$$

Here we used (3.4) in the 5th line and the last inequality results from the relation $s \leq \zeta_n$ and then $\xi_n \leq \frac{1}{n^{2\beta}} \leq 1$. Therefore,

$$|\eta_n(t)|^2 = \xi_n(t) \leq \sqrt{\xi_n(t)} \leq 1.$$

On the other hand we have for $t \leq \tau_n \wedge \tau_{n+1} \wedge \zeta_n$ and n large enough so that $a^{-(n+1)} < 1/e$,

$$\begin{aligned}
|\langle \eta_n(t), f_t \rangle| &\leq |\eta_n(t)||f_t| = |\eta_n(t)||\bar{b}(t, X_{n+1}(\phi_{n+1}(t))) - \bar{b}(t, X_n(\phi_n(t)))| \\
&\leq |\eta_n(t)|[\underbrace{|\bar{b}(t, X_{n+1}(\phi_{n+1}(t))) - \bar{b}(t, X_{n+1}(t))|}_{\underset{(3.1)}{\leq} C|X_{n+1}(\phi_{n+1}(t)) - X_{n+1}(t)| \log(1/|X_{n+1}(\phi_{n+1}(t)) - X_{n+1}(t)|)} \\
&\qquad + \underbrace{|\bar{b}(t, X_{n+1}(t)) - \bar{b}(t, X_n(t))|}_{\underset{(3.1)}{\leq} \dots} + \underbrace{|\bar{b}(t, X_n(t)) - \bar{b}(t, X_n(\phi_n(t)))|}_{\underset{(3.1)}{\leq} \dots}] \\
&\leq |\eta_n(t)| C \Big[\underbrace{|X_{n+1}(\phi_{n+1}(t)) - X_{n+1}(t)| \log(1/|X_{n+1}(\phi_{n+1}(t)) - X_{n+1}(t)|)}_{\leq a^{-(n+1)} \log(1/a^{-(n+1)}) \leq a^{-n} \log(1/a^{-n})} \\
&\qquad + |\eta_n(t)| \log(1/|\eta_n(t)|) \\
&\qquad + \underbrace{|X_n(\phi_n(t)) - X_n(t)| \log(1/|X_n(\phi(t)) - X_n(t)|)}_{\leq a^{-n} \log(1/a^{-n})} \Big]
\end{aligned}$$

$$
\begin{aligned}
&\leq |\eta_n(t)| C\left[2a^{-n}\log(a^n) + |\eta_n(t)|\log(1/|\eta_n(t)|)\right] \\
&\leq C\Big[2\underbrace{|\eta_n(t)|}_{\leq\frac{1}{n^\beta}} a^{-n}\log(a^n) + \underbrace{|\eta_n(t)|^2}_{=\xi_n(t)}\underbrace{\log(1/|\eta_n(t)|)}_{\leq\log(1/\xi_n(t))}\Big] \\
&\leq C\left[\frac{2}{n^\beta}a^{-n}\log a^n + \xi_n(t)\log(1/\xi_n(t))\right],
\end{aligned}
$$

where we used again that $s \to s\log(1/s)$ is increasing. Define the parameter ρ_n by:

$$\rho_n := \frac{2}{n^\beta}a^{-n}\log a^n.$$

We have to show that the conditions (2.20) of Lemma 2.8 are satisfied with $C_3 = 4C$, $C_4 = C$ and $\delta = \rho_n$. Indeed,

$$
\begin{aligned}
\rho_n \geq 2a^{-2n}\log a^n \quad &\Leftrightarrow \quad 2a^{-2n}\log a^n \leq \frac{2}{n^\beta}a^{-n}\log a^n \\
&\Leftrightarrow \quad a^{-2n} \leq \frac{a^{-n}}{n^\beta} \\
&\Leftrightarrow \quad \frac{a^{-2n}}{a^{-n}} \leq \frac{1}{n^\beta} \\
&\Leftrightarrow \quad a^{-n} \leq \frac{1}{n^\beta},
\end{aligned}
$$

which is true for n large enough.

Consider now the functions $\Psi_n(\xi) := \int_0^\xi \frac{ds}{s\log(1/s)+\rho_n}$ and $\Phi_n(\xi) := e^{4\Psi_n(\xi)}$. We have

$$\Phi_n'(\xi) = \frac{4\Phi_n(\xi)}{\xi\log(1/\xi) + \rho_n}$$

and

$$
\begin{aligned}
\Phi_n''(\xi) &= \frac{4\Phi_n'(\xi)(\xi\log(1/\xi)+\rho_n) - 4\Phi_n(\xi)(\log(1/\xi)-1)}{(\xi\log(1/\xi)+\rho_n)^2} \\
&= \frac{4\cdot 4\Phi_n(\xi) - 4\Phi_n(\xi)(\log(1/\xi)-1)}{(\xi\log(1/\xi)+\rho_n)^2} \\
&= \frac{4\Phi_n(\xi)(4-\log(1/\xi)+1)}{(\xi\log(1/\xi)+\rho_n)^2} \\
&= \frac{4\Phi_n(\xi)(5+\log(\xi))}{(\xi\log(1/\xi)+\rho_n)^2}.
\end{aligned}
$$

If $\xi \leq e^{-5} = c_0$, then $\Phi_n''(\xi) \leq 0$ and the conditions in (2.19) are satisfied with $C_1 = 4$, $C_2 = 0$. Consider the stopping time

$$\tilde{\tau}_n = \tau_n \wedge \tau_{n+1} \wedge \zeta_n.$$

For n large enough, we have $\xi_n(t\wedge\tilde{\tau}_n) \leq c_0$. Let $K = 32C$. Then by Lemma 2.8 we get

$$E[\Phi_n(\xi_n(t\wedge\tilde{\tau}_n))] \leq e^{Kt} \quad \text{for all } t,$$

from which we conclude

$$E[\mathbb{1}_{\{\tau_n\wedge\tau_{n+1}\geq T,\zeta_n\leq T\}}\Phi_n(\xi_n(T\wedge\tilde{\tau}_n))]\leq e^{KT}.$$

On the set $\{T\geq\zeta_n\}$, by definition of $\tilde{\tau}_n$ it holds $T\wedge\tilde{\tau}_n=\tilde{\tau}_n=\zeta_n$ because $T\leq\tau_n\wedge\tau_{n+1}$. So

$$\begin{aligned}
&& E[\mathbb{1}_{\{\tau_n\wedge\tau_{n+1}\geq T,\zeta_n\leq T\}}\Phi_n(\xi_n(\zeta_n))]\leq e^{KT}\\
\Rightarrow && E[\mathbb{1}_{\{\tau_n\wedge\tau_{n+1}\geq T,\zeta_n\leq T\}}\Phi_n(\frac{1}{n^{2\beta}})]\leq e^{KT}\\
\Leftrightarrow && \Phi_n(\frac{1}{n^{2\beta}})P(\tau_n\wedge\tau_{n+1}\geq T,\zeta_n\leq T)\leq e^{KT}\\
\Leftrightarrow && e^{4\Psi_n(1/n^{2\beta})}P(\tau_n\wedge\tau_{n+1}\geq T,\zeta_n\leq T)\leq e^{KT}
\end{aligned}$$

Thus we have

$$P(\tau_n\wedge\tau_{n+1}\geq T,\zeta_n\leq T)\leq e^{Kt}\exp\left(-4\int_0^{n^{-2\beta}}\frac{1}{s\log(1/s)+\rho_n}\,ds\right). \tag{3.5}$$

Note that $n^{-\beta}>a^{-n}$ for large n. Using again that $s\log 1/s$ is increasing over $[0,1/e]$, we have therefore $n^{-\beta}2\log n^{\beta}>2a^{-n}\log a^n$. So we see that $\rho_n<n^{-2\beta}\log n^{\beta}<n^{-2\beta}\log n^{2\beta}$ holds for large n. Hence there exists a $c_n\in(0,n^{-2\beta})$ such that

$$c_n\log\frac{1}{c_n}=\rho_n=\frac{2}{n^{\beta}}a^{-n}\log a^n<a^{-n}\log a^n.$$

Since $s\log 1/s$ is still increasing over $[0,1/e]$, we see that $0<c_n<a^{-n}$. Now

$$\begin{aligned}
&& c_n<s\\
\Leftrightarrow && c_n\log 1/c_n<s\log 1/s\\
\Leftrightarrow && s\log 1/s+c_n\log 1/c_n<2s\log 1/s\\
\Leftrightarrow && \frac{1}{s\log 1/s+c_n\log 1/c_n}>\frac{1}{2}\frac{1}{s\log 1/s}.
\end{aligned}$$

With the above we get

$$\begin{aligned}
\int_0^{n^{-2\beta}}\frac{1}{s\log 1/s+\rho_n}\,ds &= \int_0^{n^{-2\beta}}\frac{1}{s\log 1/s+c_n\log 1/c_n}\,ds\\
&\geq \int_{c_n}^{n^{-2\beta}}\frac{1}{2}\frac{1}{s\log 1/s}\,ds\\
&= -\frac{1}{2}\int_{c_n}^{n^{-2\beta}}\frac{1}{s\log s}\,ds\\
&= -\frac{1}{2}\int_{c_n}^{n^{-2\beta}}(\log s)'\frac{1}{\log s}\,ds\\
&= -\frac{1}{2}\int_{\log c_n}^{\log n^{-2\beta}}\frac{1}{y}dy \;=\; -\;\frac{1}{2}\Big|\log y\Big|_{\log c_n}^{\log n^{-2\beta}}\\
&= -\frac{1}{2}\log\left(\frac{\log n^{-2\beta}}{\log c_n}\right).
\end{aligned}$$

This and (3.5) imply

$$
\begin{aligned}
P(\tau_n \wedge \tau_{n+1} \geq T, \zeta_n \leq T) &\leq e^{KT} \exp\Big\{ -4 \int_0^{n^{-2\beta}} \frac{1}{s \log 1/s + \rho_n} \, ds \Big\} \\
&\leq e^{KT} \exp\Big\{ -4(-\frac{1}{2} \log \left(\frac{\log n^{-2\beta}}{\log c_n} \right)) \Big\} \\
&= e^{KT} \exp\left\{ \log \left(\frac{\log n^{-2\beta}}{\log c_n} \right)^2 \right\} \\
&= e^{KT} \left(\frac{\log n^{-2\beta}}{\log c_n} \right)^2 \\
&= e^{KT} \left(-\frac{2\beta \log n}{\log c_n} \right)^2 \\
&\leq e^{KT} \left(\frac{2\beta \log n}{n \log a} \right)^2,
\end{aligned}
$$

where we used that $-\log c_n > -\log a^{-n} = \log a^n > 0$. Defining $C_5 := e^{KT} \frac{4\beta^2}{(\log a)^2}$ and taking n large enough, we conclude

$$
P(\tau_n \wedge \tau_{n+1} \geq T, \zeta_n \leq T) \leq C_5 \left(\frac{\log n}{n} \right)^2. \tag{3.6}
$$

After putting (3.3) and (3.6) together, we see that for n large enough

$$
\begin{aligned}
P(\zeta_n \leq T) &= P(\{\tau_n \wedge \tau_{n+1} \leq T, \zeta_n \leq T\} \cup \{\tau_n \wedge \tau_{n+1} \geq T, \zeta_n \leq T\}) \\
&\leq P(\tau_n \wedge \tau_{n+1} \leq T, \zeta_n \leq T) + P(\tau_n \wedge \tau_{n+1} \geq T, \zeta_n \leq T) \\
&\leq P(\tau_n \leq T) + P(\tau_{n+1} \leq T) + P(\tau_n \wedge \tau_{n+1} \geq T, \zeta_n \leq T) \\
&\leq \exp\left(-c^n/8dA^2\right) + \exp\left(-c^{n+1}/8dA^2\right) + C_5 \left(\frac{\log n}{n} \right)^2 \\
&\leq 2e^{-c^n/8dA^2} + C_5 \left(\frac{\log n}{n} \right)^2 \\
&\leq \frac{1}{n^\gamma}, \quad \text{for some } \gamma > 1
\end{aligned}
$$

where we used that $c > 1$ and, for any $\varepsilon > 0$, $\frac{(\log n)^2}{n^{1+\varepsilon}} \to 0$, as $n \to \infty$. This means that

$$
P\Big(\sup_{0 \leq t \leq T} |X_{n+1}(t) - X_n(t)|^2 \geq \frac{1}{n^{2\beta}} \Big) \leq \frac{1}{n^\gamma} \underset{n \to \infty}{\longrightarrow} 0.
$$

It is well known that $\sum_{n=1}^\infty \frac{1}{n^\gamma} < \infty$ for $\gamma > 1$. Therefore, by Borel-Cantelli Lemma, almost surely

$$
\sup_{0 \leq t \leq T} |X_{n+1}(t) - X_n(t)| \leq \frac{1}{n^\beta} \quad \text{for large } n.
$$

It follows that the series

$$X_t := \sum_{n\geq 1}(X_{n+1}(t) - X_n(t)) + X_1(t)$$

converges uniformly in $t \in [0,T]$. The only thing remains to check is that X_t is really a solution of stochastic differential equation (2.29). Since the coefficients $\sigma(t,x)$ and $b(t,x)$ are continuous and bounded, we obtain together with (3.2) that

$$\begin{aligned}
X_t &= \sum_{n\geq 1}(X_{n+1}(t) - X_n(t)) + X_1(t) = \lim_{n\to\infty} X_n(t) \\
&= \lim_{n\to\infty}\left(X_n(\frac{k}{2^n}) + \sigma\Big(\frac{k}{2^n}, X_n(\frac{k}{2^n})\Big)(W_t - W_{\frac{k}{2^n}}) + b\Big(\frac{k}{2^n}, X_n(\frac{k}{2^n})\Big)(t - \frac{k}{2^n})\right) \\
&= \lim_{n\to\infty}\left(X_n(0) + \int_0^t \bar\sigma(s, X_n(\phi_n(s)))dW_s + \int_0^t \bar b(s, X_n(\phi_n(s)))\,ds\right) \\
&= x + \lim_{n\to\infty}\int_0^t \bar\sigma(s, X_n(\phi_n(s)))dW_s + \lim_{n\to\infty}\int_0^t \bar b(s, X_n(\phi_n(s)))\,ds \\
&= x + \int_0^t \lim_{n\to\infty}\bar\sigma(s, X_n(\phi_n(s)))\,dW_s + \int_0^t \lim_{n\to\infty}\bar b(s, X_n(\phi_n(s)))\,ds \\
&= x + \int_0^t \sigma(s, X_s)\,dW_s + \int_0^t b(s, X_s)\,ds.
\end{aligned}$$

□

3.2 Preparing Lemmas

In this section we will prove several technical lemmas needed in the proof of Theorem D.

Lemma 3.2. *Let r be a strictly positive continuous function defined on $(0,c_0]$, where $0 < c_0 < 1$. Assume that the coefficients σ and b are compactly supported, say*

$$\sigma(t,x) = 0 \quad \textit{and} \quad b(t,x) = 0 \quad \textit{for all} \quad |x| \geq R, 0 \leq t \leq T \tag{3.7}$$

with some $0 < R, T < \infty$, and satisfy the hypothesis (H3) in theorem D. Let $p \geq 1$. If $s \to r(s)$ is decreasing on $(0,c_0]$, then there exists a constant $C_p > 0$ such that for all $|x| \leq R+1$, $|y| \leq R+1$, $0 \leq t \leq T$

$$\begin{cases} \|\sigma(t,x) - \sigma(t,y)\|^2 & \leq C_p|x-y|^2 r\left(\frac{|x-y|^{2p}}{M^p}\right) \\ |b(t,x) - b(t,y)| & \leq C_p|x-y| r\left(\frac{|x-y|^{2p}}{M^p}\right) \end{cases} \tag{3.8}$$

where $M = \frac{4(R+1)^2}{c_0}$.

Proof. The similarity of these two inequalities show, that it is enough to prove only the one for b. So if $|x-y| \leq c_0$, by (H3)

$$\begin{aligned}
|b(t,x) - b(t,y)| &\leq C|x-y| r(|x-y|^2) \\
&\leq C|x-y| r\left(\left(\frac{|x-y|^2}{M}\right)^p\right),
\end{aligned} \tag{3.9}$$

because

$$\begin{aligned} M &= \frac{4(R+1)^2}{c_0} > 1 \\ \Rightarrow 0 &< \frac{|x-y|^2}{M} < |x-y|^2 \le c_0^2 < 1 \\ \Rightarrow \left(\frac{|x-y|^2}{M}\right)^p &< |x-y|^2 < |x-y| \end{aligned}$$

and r is decreasing. We observe that

$$\begin{aligned} \inf_{c_0 \le \xi \le 2(R+1)} \xi \cdot r\left(\left[\frac{\xi^2}{M}\right]^p\right) &= \inf_{c_0 \le \xi \le 2(R+1)} \xi \cdot r\left(\left[\frac{\xi^2 c_0}{4(R+1)^2}\right]^p\right) \\ &\ge \inf_{c_0 \le \xi \le 2(R+1)} \xi \cdot r\left(c_0^p\right) \end{aligned}$$

Hence, using that r is decreasing,

$$\begin{aligned} \inf_{c_0 \le \xi \le 2(R+1)} \xi \cdot r\left(\left[\frac{\xi^2}{M}\right]^p\right) &\ge c_0 r\left(\left[\frac{(2(R+1))^2}{M}\right]^p\right) \\ &= c_0 r(c_0^p) \;\; > 0. \end{aligned}$$

On the other hand,

$$\sup_{x,y} |b(t,x) - b(t,y)| \le \sup_{x,y} ||b(x)| + |b(y)|| \;\; \le \;\; 2\|b\|_\infty,$$

where $\|b\|_\infty$ denotes the uniform norm of b over $[0,T] \times \mathbb{R}^d$. Now let $|x-y| \ge c_0$. Choosing a large C_p which fulfills $C_p c_0 r(c_0^p) \ge 2\|b\|_\infty$, we then have

$$|b(x) - b(y)| \le C_p |x-y| \cdot r\left(\left[\frac{|x-y|^2}{M}\right]^p\right). \tag{3.10}$$

Both inequalities (3.9) and (3.10) together gives us the result. $\square$

Lemma 3.3. *Let σ and b be continuous functions satisfying the support condition (3.7). If the stochastic differential equation (2.29) has the pathwise uniqueness, then for any $|x_0| \le R+1$, it holds $|X_t(x_0)| \le R+1$ almost surely for all $t \in [0,T]$.*

Proof. We define the stopping time

$$\tau := \inf\{t > 0 | |X_t(x_0)| \ge R+1\}. \tag{3.11}$$

Set $Y_t := X_{t\wedge\tau}(x_0)$, $0 \le t \le T$. Then $X_s = Y_s$ if $s \le t \wedge \tau$, so that

$$\begin{aligned} X_{t\wedge\tau}(x) &= x_0 + \int_0^{t\wedge\tau} \sigma(s, X_s)\, dW_s + \int_0^{t\wedge\tau} b(s, X_s)\, ds \\ \Rightarrow Y_t(x) &= x_0 + \int_0^{t\wedge\tau} \sigma(s, Y_s)\, dW_s + \int_0^{t\wedge\tau} b(s, Y_s)\, ds. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}E[\int_0^t \|\sigma(s,Y_s)\|^2(\mathbb{1}_{(s<\tau)}-1)^2\,ds] &= E[\int_0^t \|\sigma(s,Y_s)\|^2\mathbb{1}_{(s\geq\tau)}\,ds]\\ &= E[\int_\tau^t \|\sigma(s,Y_s)\|^2\,ds]\\ &= E[\int_\tau^t \|\sigma(s,X_\tau)\|^2\,ds].\end{aligned}$$

Since X_τ takes values on the sphere of Radius $R+1$, so $\sigma(X_\tau)=0$ because σ is compactly supported. Therefore the last term in the above equality is zero. Herefrom it follows that

$$\begin{aligned}\int_0^{t\wedge\tau}\sigma(s,Y_s)\,dW_s &= \int_0^{t\wedge\tau}\sigma(s,X_{s\wedge\tau})\,dW_s\\ &= \begin{cases}\int_0^t \sigma(s,X_{s\wedge\tau})\,dW_s, & \text{if } t<\tau,\\ \int_0^\tau \sigma(s,X_{s\wedge\tau})\,dW_s = \int_0^t \sigma(s,X_{s\wedge\tau})\,dW_s, & \text{if } \tau<t,\end{cases}\\ &= \int_0^t \sigma(s,Y_s)\,dW_s\end{aligned}$$

and

$$\int_0^{t\wedge\tau} b(s,Y_s)\,ds = \int_0^{t\wedge\tau} b(s,X_{s\wedge\tau})\,ds = \int_0^t b(s,X_{s\wedge\tau})\,ds = \int_0^t b(s,Y_s)\,ds$$

almost surely. Finally

$$Y_t = x_0 + \int_0^t \sigma(s,Y_s)\,dW_s + \int_0^t b(s,Y_s)\,ds.$$

This means that $(Y_t, 0\leq t\leq T)$ satisfies the same stochastic differential equation as $(X_t, 0\leq t\leq T)$. By pathwise uniqueness, we conclude that $Y_t = X_t$ almost surely for all $0\leq t\leq T$. By the definition (3.11) of τ, this means that if $|x_0|\leq R+1$, $|X_t(x_0)| = |Y_t(x_0)| = |X_{t\wedge\tau}(x_0)|\leq R+1$ almost surely for all $t\in[0,T]$. □

Lemma 3.4. *Assume the same hypothesis as in Lemma 3.2 and furthermore let r satisfy the conditions (2.6)-(2.8) in Theorem B and let $\xi\to\xi r(\xi)$ be concave over $(0,c_0]$. Let $p\geq 2$ be an integer. For $|x_0|\leq R+1$ and $|y_0|\leq R+1$, set*

$$\eta_t := X_t(x_0)-X_t(y_0), \qquad \xi_t := |\eta_t|^2 \quad \textit{and} \quad z_t := \left(\frac{\xi_t}{M}\right)^p, \qquad 0\leq t\leq T, \tag{3.12}$$

where M is the constant defined in Lemma 3.2. Put $\phi(t)=Ez_t$. Then for some constant C_p

$$\phi'(t)\leq C_p\phi(t)r(\phi(t)), \qquad 0\leq t\leq T.$$

Proof. We can apply Lemmas 3.2 and 3.3 and get that for all $0\leq t\leq T$

$$z_t = \frac{\xi_t^p}{M^p} = \left(\frac{|X_t(x_0)-X_t(y_0)|^2}{M}\right)^p \leq \left(\frac{|\overbrace{|X_t(x_0)|}^{\leq R+1}+\overbrace{|X_t(y_0)|}^{\leq R+1}|^2}{M}\right)^p \leq \left(\frac{4(R+1)^2}{M}\right)^p = c_0^p,$$

so z_t is a bounded process. Let $e_t = \sigma(X_t(x_0)) - \sigma(X_t(y_0))$ and $f_t = b(X_t(x_0)) - b(X_t(y_0))$. Then with the help of Lemma 2.6 and Itô's formula, (cf. Theorem 1.3, where we used that $p \geq 2$) we have for all $0 \leq t \leq T$

$$\begin{aligned} dz_t = d\Big(\frac{\xi_t^p}{M^p}\Big) &= \frac{1}{M^p} d\xi_t^p \\ &= \frac{1}{M^p}[p\xi_t^{p-1} d\xi_t + \frac{1}{2}p(p-1)\xi_s^{p-2} d\langle\xi_t\rangle] \\ &= \frac{1}{M^p}[2p\langle e_t^*\eta_t, dW_t\rangle \xi_t^{p-1} + 2\langle\eta_t, f_t\rangle p\xi_t^{p-1} dt \\ &\qquad + p\|e_t\|^2\xi_t^{p-1}dt + 2p(p-1)\xi_t^{p-2}|e_t^*\eta_t|^2 dt]. \end{aligned}$$

Now we will estimate the single terms: By Lemma 3.3 $X_t(x_0)$ and $X_t(y_0)$ are bounded by $R+1$. So we can apply the result (3.8) of Lemma 3.2,

$$\begin{aligned} \frac{\xi_s^{p-1}}{M^p}|\langle\eta_s, f_s\rangle| &\leq \frac{\xi_s^{p-1}}{M^p}|\eta_s||f_s| \\ &\leq \frac{\xi_s^{p-1}}{M^p}|\eta_s| C_p \underbrace{|X_s(x_0) - X_s(y_0)|}_{=|\eta_s|} r\left(\frac{|X_s(x_0) - X_s(y_0)|^{2p}}{M^p}\right) \\ &\leq \frac{\xi_s^{p-1}}{M^p} C_p \underbrace{|\eta_s|^2}_{=\xi_s} r\left(\frac{|\eta_s|^{2p}}{M^p}\right) \\ &= \frac{\xi_s^p}{M^p} C_p r\left(\frac{\xi_s^p}{M^p}\right) \;=\; C_p z_s r(z_s) \end{aligned}$$

and similarly

$$\begin{aligned} \frac{\xi_s^{p-1}}{M^p}\|e_s\|^2 &= \frac{\xi_s^{p-1}}{M^p}\|\sigma(X_s(x_0)) - \sigma(X_s(y_0))\|^2 \\ &\leq \frac{\xi_s^{p-1}}{M^p} C_p |X_s(x_0) - X_s(y_0)|^2 r\left(\frac{|X_s(x_0) - X_s(y_0)|^{2p}}{M^p}\right) \\ &= C_p \frac{\xi_s^p}{M^p} r\left(\frac{\xi_s^p}{M^p}\right) \\ &= C_p z_s r(z_s). \end{aligned}$$

Finally we use that $\|T\| = \|T^*\|$ for any bounded linear operator T, which yields

$$\begin{aligned} \frac{\xi_s^{p-2}}{M^p}|e_s^*\eta_s|^2 &= \frac{\xi_s^{p-2}}{M^p}|\left(\sigma(X_s(x_0)) - \sigma(X_s(y_0))\right)^* \left(X_s(x_0) - X_s(y_0)\right)|^2 \\ &\leq \frac{\xi_s^{p-2}}{M^p}\|\left(\sigma(X_s(x_0)) - \sigma(X_s(y_0))\right)^*\|^2 |X_s(x_0) - X_s(y_0)|^2 \\ &\leq \frac{\xi_s^{p-2}}{M^p} C_p |X_s(x_0) - X_s(y_0)|^2 r\left(\frac{|X_s(x_0) - X_s(y_0)|^{2p}}{M^p}\right) \underbrace{|X_s(x_0) - X_s(y_0)|^2}_{=\xi_s} \\ &= C_p \frac{\xi_s^p}{M^p} r(\frac{\xi_s^p}{M^p}) \;=\; C_p z_s r(z_s). \end{aligned}$$

Note that $\xi r(\xi)$ is in $C^1((0,c_0])$ by the same property of $r(\xi)$. By assumption, $\xi r(\xi)$ is concave, hence every tangent of $\xi r(\xi)$ is bigger or equal to its value at every point of the domain. More precisely $\xi r(\xi) \le g(\xi)$ where $g(\xi)$ is a linear function defined by

$$g(\xi) = r(c_0) + (r(c_0) + c_0 r'(c_0))(\xi - c_0).$$

Therefore

$$\sup_{0<\xi\le c_0} \xi r(\xi) \le \sup_{0<\xi\le c_0} g(\xi) < \infty.$$

Thus the first term in the expression of dz_t is a martingale and then $\phi(t) = Ez_t$ is a differentiable function with respect to t and its derivative can be estimated by

$$\begin{aligned}
\phi'(t) &= \frac{d}{dt}E[z_t] \\
&= \frac{1}{M^p}(\underbrace{E[2p\xi_t^{p-1}\langle e_t^*\eta_t, dW_t\rangle]}_{=0 \text{ (martingale)}} + E[2p\xi_t^{p-1}\langle \eta_t, f_t\rangle] \\
&\qquad + E[p\xi_t^{p-1}\|e_t\|^2] + E[2p(p-1)\xi_t^{p-2}|e_t^*\eta_t|^2]) \\
&= \frac{1}{M^p}(E[2p\xi_t^{p-1}\langle \eta_t, f_t\rangle] + E[p\xi_t^{p-1}\|e_t\|^2] \\
&\qquad + E[2p(p-1)\xi_t^{p-2}|e_t^*\eta_t|^2]) \\
&\le pC_pE[z_t r(z_t)](2 + 1 + 2(p-1)) \\
&= (p + 2p^2)C_pE[z_t r(z_t)] \\
&\le (p + 2p^2)C_pE[z_t]r(E[z_t]), \qquad 0 \le t \le T.
\end{aligned}$$

Note that in the last line we used Jensen's inequality. Then the Lemma is proved by taking $\tilde{C}_p = (p + 2p^2)C_p$ as a new C_p. □

3.3 Proof of Theorem D

Proof of Theorem D. Recall that under the hypothesis (H3) the equation (2.1) has a unique strong solution. So, it is enough to prove the continuity of $(t, x_0) \mapsto X_t(x_0)$ on the domain $[0,T] \times \mathbb{R}^d$ for each $T > 0$. We make two steps.

Step 1. Assume that σ and b are compactly supported, say

$$\sigma(t,x) = 0 \quad \text{and} \quad b(t,x) = 0 \quad \text{for } |x| \ge R.$$

Let ϕ be defined as $\phi(t) = Ez_t$, $t \in [0,T]$, where z_t is defined in (3.12). Let us check the conditions of the Lemma 3.4. Obviously, $r(s) = \log(1/s)$ satisfies (2.6)-(2.8), see Lemma 2.5. Furthermore $r(s) > 0$ for all $s \in (0, c_0]$, $c_0 < 1$. The function $f : s \mapsto sr(s)$ is concave, which follows by direct calculation: $f'' = -\frac{1}{s} \le 0$. So we can apply the Lemma 3.4 and get

$$\phi'(t) \le C_p\phi(t)r(\phi(t)) = C_p\phi(t)\log(\frac{1}{\phi(t)}), \qquad 0 \le t \le T.$$

Solving the differential inequality:

$$\begin{aligned} \phi'(t) &\le C_p \phi(t) \log \frac{1}{\phi(t)} \\ \Leftrightarrow (\log \phi(t))' &\le -C_p \log \phi(t), \end{aligned}$$

we get that

$$\phi(t) \le \phi(0)^{e^{-C_p t}}$$

or explicitly

$$\begin{aligned} & E\left[\left(\frac{\xi_t}{M}\right)^p\right] \le E\left[\left(\frac{\xi_0}{M}\right)^p\right]^{e^{-C_p t}} \\ \Leftrightarrow \qquad & E[|X_t(x_0) - X_t(y_0)|^{2p}] \le \Big(E[\underbrace{|X_0(x_0) - X_0(y_0)|^{2p}}_{=|x_0-y_0|^{2p}}]\Big)^{e^{-C_p t}} \cdot M^{p(1-e^{-C_p t})} \\ \Leftrightarrow \qquad & E[|X_t(x_0) - X_t(y_0)|^{2p}] \le \left(|x_0 - y_0|^{2p}\right)^{e^{-C_p t}} \cdot M^{p(1-e^{-C_p t})} \\ & \qquad \le M^p |x_0 - y_0|^{2pe^{-C_p t}} \\ & \qquad \le C'_p |x_0 - y_0|^{2pe^{-C_p t}}, \quad \text{for } |x_0 - y_0| \le 1, \end{aligned} \tag{3.13}$$

where C'_p is the maximum of C_p and M^p. Now we show some upper bounds on the integral terms: for all $t, s \in [0, T], |t - s| \le 1$,

$$\Big| \int_s^t b(r, X_r)\, dr \Big|^{2p} \le |t-s|^{2p} \|b\|_\infty^{2p} \le |t-s|^p \|b\|_\infty^{2p},$$

$$\begin{aligned} E\left[\Big| \int_s^t \sigma(r, X_r)\, dW_r \Big|^{2p}\right] &\le C(p) E\left[\Big(\int_s^t \|\sigma(r, X_r)\|^2\, dr \Big)^p\right] \\ &\le C(p) |t-s|^p \|\sigma\|_\infty^{2p}. \end{aligned} \tag{3.14}$$

In deriving (3.14) we used the Burkholder-Davis-Gundy inequality (see Theorem 3.28, p. 166, [KS91]). Recall that b, σ are compactly supported and continuous, therefore bounded on $[0, T] \times \mathbb{R}^d$, i.e. there exists $K > 0$ such that $\|b\|_\infty \le K, \|\sigma\|_\infty \le K$. Then with help of (3.13) and (3.14) we see that

$$E[|X_t(x_0) - X_s(y_0)|^{2p}] \le C''_p \left(|t-s|^p + |x_0 - y_0|^{2pe^{-C_p t}} \right),$$

with certain constant $C''_p > 0$, uniformly for all $x_0, y_0 \in \mathbb{R}^d$, $|x_0 - y_0| \le 1$, and all $s, t \in [0, T]$, $|s - t| \le 1$.

Fix $p > d + 1$. Choose a constant $T_0 \in (0, T]$ small enough ($T_0 < 1/C_p \log 2$), such that $2pe^{-C_p T_0} > d + 1$. Applying Kolmogorov's modification Theorem 1.11 with $\beta = 2p - 1$, $\alpha = 2p$ and $c = C_p$, we conclude that there exists $\hat{X}_t(x_0)$, a version of $X_t(x_0)$, which is continuous in (t, x_0), $t \in [0, T_0]$, $|x_0| \le R + 1$ almost surely. But from pathwise uniqueness it is obvious that

$$X_t(x_0, \omega) \equiv x_0 \quad \text{if} \quad |x_0| > R.$$

We conclude that $(t, x_0) \to \tilde{X}_t(x_0, \omega)$ can be extended continuously to $[0, T_0] \times \mathbb{R}^d$. Let $(\theta_{T_0}\omega)(t) = \omega(t + T_0) - \omega(T_0)$. Define for $0 < t \leq T_0$,

$$\tilde{X}_{T_0+t}(x_0, \omega) = \tilde{X}_t(\tilde{X}_{T_0}(x_0, \omega), \theta_{T_0}\omega).$$

Then $\tilde{X}_{T_0+\cdot}(x_0, \omega)$ satisfies the stochastic differential equation (2.29) driven by the Brownian motion $\theta_{T_0}\omega$ with the initial condition $\tilde{X}_{T_0}(x_0, \omega)$. Since by Theorem B we have pathwise uniqueness, it holds that $\tilde{X}_{T_0+t}(x_0, \omega) = X_{T_0+t}(x_0, \omega)$ almost surely for all $t \in [0, T_0]$. This means that $\tilde{X}_t(x_0, \omega)$ is a continuous version of $X_t(x_0, \omega)$ over $[0, 2T_0] \times \mathbb{R}^d$. Proceeding in this way, we get a continuous version on the whole space $[0, T] \times \mathbb{R}^d$.

Step 2: General case. For $R > 0$, let $f_R(x)$ denote a smooth function with compact support satisfying

$$f_R(x) = 1 \quad \text{for} \quad |x| \leq R \quad \text{and} \quad f_R(x) = 0 \quad \text{for} \quad |x| > R + 1.$$

Define

$$\sigma_R(x) = \sigma(x) f_R(x) \quad \text{and} \quad b_R(x) = b(x) f_R(x).$$

Let $X_t^R(x, \omega)$ be the unique solution of the stochastic differential equation (2.29) with σ and b replaced by σ_R and b_R. Let $\tilde{X}_t^R(x, \omega)$ denote a continuous version of $X_t^R(x, \omega)$. Such a version exists according to step 1. For $K > 0$, set

$$\tau_K^R(x) := \inf\left\{t \in (0, T] \,\middle|\, |\tilde{X}_t^R(x, \omega)| \geq K\right\}, \quad \tau_K(x) := \inf\left\{t \in (0, T] \,||X_t(x, \omega)| \geq K\right\}.$$

If $|x| \leq R$, by pathwise uniqueness we have P-a.s.

$$X_t(x, \omega) = \tilde{X}_t^N(x, \omega) \quad \text{for all} \quad N > R + 1 \quad \text{and} \quad t < \tau_{R+1}^N.$$

Since $N > R + 1$, we conclude

$$\begin{aligned} \tau_{R+1}(x) &= \inf\{t \in (0, T] | |X_t(x, \omega)| \geq R + 1\} \\ &= \inf\{t \in (0, T] | |\tilde{X}_t^N(x, \omega)| \geq R + 1\} \\ &= \tau_{R+1}^N(x). \end{aligned}$$

For $|x| \leq R$, we define

$$\tilde{X}_t(x, \omega) := \tilde{X}_t^{R+2}(x, \omega) \quad \text{for} \quad t \in [0, \tau_{R+1}^{R+2}(x))$$

Then on this set, $\tilde{X}_\cdot(x, \omega)$ is a version of $X_\cdot(x, \omega)$. It remains to show that $\tilde{X}_t(x, \omega)$ is continuous in $(t, x) \in [0, T] \times \mathbb{R}^d$ for almost all ω. Fix x_0 with $|x_0| \leq R$. Since the explosion time of the solution is infinite, there exists $R > 0$ such that $\tau_{R+1}^{R+2}(x_0) > t + \varepsilon$ for a small $\varepsilon > 0$. The later implies that

$$\sup_{0 \leq s \leq t+\varepsilon} \left|\tilde{X}_s^{R+2}(x_0, \omega)\right| < R + 1.$$

By the continuity we can find a neighborhood $B_\delta(x_0)$ of x_0 such that

$$\sup_{0 \leq s \leq t+\varepsilon} \left|\tilde{X}_s^{R+2}(x, \omega)\right| < R + 1 \quad \text{or} \quad \tau_{R+1}^{R+2}(x) > t + \varepsilon$$

for all $x \in B_\delta(x_0)$. Hence, $\tilde{X}_s(x_0, \omega) = \tilde{X}_s^{R+2}(x, \omega)$ for all $x \in B_\delta(x_0)$ and $s \leq t + \varepsilon$, which implies that $\tilde{X}_s(x_0, \omega)$ is continuous at the point (t, x_0). $\square$

4 Smoluchowski-Kramer Approximation

In Section 4.1 we introduce the Newton system describing a motion of a small particle in a force field and a notion of its solution. For the mathematical background and physical motivation see the paper of M. Freidlin [Fre04]. Besides, we shall use the related results of the diploma thesis by R. Westermann [Wes06], completed in Bielefeld in 2006. As compared with the [Wes06], we impose considerably weaker assumptions on the interaction coefficients.

4.1 The Newton system

We consider the motion of a particle of small mass μ, $0 < \mu \ll 1$ in a force field. We assume that the differential of the force is given by $b(s, X_s)\, ds + \sigma(s, X_s)\, dW_s$, where $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and W_s is an one-dimensional Wiener process.
The motion of the particle is described by its position and velocity (X_t^μ, Y_t^μ). By Newton's law, X_t^μ is governed by the following system of stochastic differential equations:

$$\begin{aligned} dX_t^\mu &= Y_t^\mu\, dt, \\ \mu\, dY_t^\mu &= b(t, X_t^\mu)\, dt + \sigma(t, X_t^\mu)\, dW_t - dX_t^\mu, \\ X_0^\mu &= \zeta_1, \quad Y_0^\mu = \zeta_2, \end{aligned} \tag{4.1}$$

where $\zeta_1, \zeta_2 \in \mathbb{R}$ are the initial condition.
One can interpret the Newton system as follows: The momentum of a particle is defined by mass times velocity. The Newton system says now that the increment of the momentum of a particle is given by the differential of a force field from which we subtract the differential of the friction. The force field consists of a deterministic part $b(t, X_t^\mu)$, which only depends on the path of the particle and the time, and a random part, where $\sigma(t, X_t^\mu)\, dW_t$ represents the stochastic differential. The friction is represented by the term dX_t^μ.

Definition 4.1. *A* weak solution of the Newton system (4.1) *is an $\mathcal{F}_t$-adapted, real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and a pair of real-valued stochastic processes $(X, Y) = \{(X_t, Y_t) | t \geq 0\}$, each of them has continuous sample paths and is adapted to the filtration $\mathcal{F}_t$. It satisfies the initial conditions $X_0^\mu = \zeta_1$ and $Y_0^\mu = \zeta_2$. Moreover the process Y_t^μ defined by $Y_t^\mu\, dt = dX_t^\mu$ for all $t \geq 0$ P-a.s. should be a weak solution (in sense of Definition 1.5) of*

$$dY_t^\mu = \frac{1}{\mu}\sigma(t, X_t^\mu)\, dW_t + \left(\frac{1}{\mu} b(t, X_t^\mu) - \frac{1}{\mu} Y_t^\mu \right) dt.$$

The later means that for all $t \geq 0$ P-a.s.

$$Y_t^\mu = \zeta_2 + \frac{1}{\mu} \int_0^t \sigma(s, X_s^\mu)\, dW_s + \frac{1}{\mu} \int_0^t \left(b(s, X_s^\mu) - Y_s^\mu \right) ds.$$

Definition 4.2. *A* strong solution of the Newton system (4.1) *on the given probability space* $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$ *with respect to the fixed Brownian motion* W *is a pair of real-valued stochastic processes* $(X,Y) = \{(X_t, Y_t) | t \geq 0\}$, *each of them with continuous sample paths and adapted to the filtration* $\mathcal{F}_t$. *Moreover, it should hold* $X_0^\mu = \zeta_1$ *as well as* $Y_0^\mu = \zeta_2$ *and the process* Y_t^μ *defined by* $Y_t^\mu \, dt = dX_t^\mu$ *for all* $t \geq 0$ P*-a.s. should be a strong solution (in sense of Definition 1.4) of*

$$dY_t^\mu = \frac{1}{\mu}\sigma(t, X_t^\mu)\, dW_t + \left(\frac{1}{\mu} b(t, X_t^\mu) - \frac{1}{\mu} Y_t^\mu\right) dt.$$

This is to say that for all $t \geq 0$ P*-a.s.*

$$Y_t^\mu = \zeta_2 + \frac{1}{\mu}\int_0^t \sigma(s, X_s^\mu)\, dW_s + \frac{1}{\mu}\int_0^t \left(b(s, X_s^\mu) - Y_s^\mu\right) ds.$$

In her diploma thesis [Wes06], Ramona Westermann claimed that there exists a unique strong solution (X_t^μ, Y_t^μ) on $[0,T]$ under the following conditions on the functions b and σ:

(i) b, σ Lipschitz continuous on $[0,T]\times\mathbb{R}^d$, i.e. there exists a $D > 0$, such that for all $x_0, x_1 \in \mathbb{R}$, $t_0, t_1 \in [0,T]$

$$|b(t_0,x_0) - b(t_1,x_1)| + |\sigma(t_0,x_0) - \sigma(t_1,x_1)| \;\leq\; D(|x_0 - x_1| + |t_0 - t_1|). \tag{4.2}$$

(ii) there exists a $K > 0$, such that for all $x \in \mathbb{R}, t \in [0,T]$ the linear growth condition holds:

$$|b(t,x)|^2 + |\sigma(t,x)|^2 \;\leq\; K^2(1 + |x|^2), \tag{4.3}$$

Our aim is to give weaker conditions, under which the unique solvability still holds. We can equivalently write (4.1) as the two-dimensional system

$$d\begin{bmatrix} X_t^\mu \\ Y_t^\mu \end{bmatrix} = \begin{pmatrix} Y_t^\mu \\ \frac{1}{\mu}b(t,X_t^\mu) - \frac{1}{\mu}Y_t^\mu \end{pmatrix} dt + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\mu}\sigma(t,X_t^\mu) \end{pmatrix} dW_t.$$

We define $z \in \mathbb{R}^2$, $f : [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}^2$ and $g : [0,T]\times\mathbb{R}\times\mathbb{R} \to M(2\times 2, \mathbb{R})$, by

$$z := \begin{pmatrix} x \\ y \end{pmatrix}, \quad f(t,x,y) := \begin{pmatrix} y \\ \frac{1}{\mu}b(t,x) - \frac{1}{\mu}y \end{pmatrix}, \quad g(t,x,y) := \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\mu}\sigma(t,x) \end{pmatrix}.$$

Hence our second order system of stochastic differential equations can be written as

$$dZ_t = f(t, X_t^\mu, Y_t^\mu)\, dt + g(t, X_t^\mu, Y_t^\mu)\, dW_t. \tag{4.4}$$

We impose the following conditions, which are weaker than (4.2),(4.3). Let f, g be continuous, and satisfy the following:

- For a constant $C > 0$ and all $z \in \mathbb{R}^2$, $t \in [0,T]$,

$$\begin{aligned} \|g(t,z)\|^2 &\leq C(|z|^2\rho(|z|^2) + 1), \\ \langle z, f(t,z)\rangle &\leq C(|z|^2\rho(|z|^2) + 1); \end{aligned} \tag{4.5}$$

- There exists $c_0 > 0$ such that for all $t \in [0,T]$ and all $v, w \in \mathbb{R}^2$, $|v-w| \leq c_0$,

$$\begin{array}{rcl} \|g(t,v)-g(t,w)\|^2 & \leq & C|v-w|^2 r(|v-w|^2), \\ \langle v-w, f(t,v)-f(t,w)\rangle & \leq & C|v-w|^2 r(|v-w|^2). \end{array} \tag{4.6}$$

Here $\rho : [0,\infty) \to [1,\infty)$ is a continuously differentiable function satisfying the following three conditions

(i) $\lim_{s\to\infty} \rho(s) = \infty$,

(ii) $\lim_{s\to\infty} \frac{s\rho'(s)}{\rho(s)} = 0$,

(iii) $\int_0^{+\infty} \frac{1}{s\rho(s)+1}\, ds = \infty$

and $r : (0, c_0] \to [1,\infty)$ is a continuously differentiable function satisfying also three conditions:

(i) $\lim_{s\to 0} r(s) = \infty$,

(ii) $\lim_{s\to 0} \frac{sr'(s)}{r(s)} = 0$,

(iii) $\int_0^a \frac{1}{sr(s)}\, ds = \infty\ \forall a > 0$.

Since f and g are continuous, the equation (4.4) is a special case of (2.1). Hence, by Theorem 2.1 we have a weak solution (X_t^μ, Y_t^μ) up to an explosion time ζ.

Theorem A'. *Assume the conditions above are fulfilled, then the weak solution of* (4.4) *has no explosion, that means,* $P(\zeta = \infty) = 1$.

Proof. Since (4.4) is a special case of (2.1), we only have to check whether the conditions of Theorem A are satisfied. The function ρ fulfills all the required conditions (2.3)-(2.5). The assumptions (4.5) are chosen such that they fulfill (H1). All conditions are satisfied, so Theorem A applies. $\square$

Theorem B'. *Assume that the conditions above are fulfilled. Then the pathwise uniqueness holds for the weak solution of* (4.4).

Proof. As seen before, (4.4) fits in (2.1). Furthermore, the function r fulfills (2.6)-(2.8). Comparing (4.6) with (H2), we see that indeed they are the same. We checked that all conditions of Theorem B are satisfied and thus the pathwise uniqueness holds. $\square$

The existence of a unique strong solution then follows by the theorems of Yamada-Watanabe and Kallenberg, (cf. Theorems 1.9 and 1.10).

Proposition 4.3. *Instead of considering functions f and g, it is possible to assume directly that for all $x, t \in \mathbb{R}$*

$$\begin{array}{rcl} \|\sigma(t,x)\|^2 & \leq & C(|x|^2\rho(|x|^2)+1), \\ |b(t,x)| & \leq & C(|x|\rho(|x|^2)+1), \end{array} \tag{4.7}$$

and for all $x, y \in \mathbb{R}$, $|x-y| \le c_0$,

$$\begin{aligned} \|\sigma(t,x)-\sigma(t,y)\|^2 &\le C|x-y|^2 r(|x-y|^2), \\ |b(t,x)-b(t,y)| &\le C|x-y| r(|x-y|^2). \end{aligned} \tag{4.8}$$

Then we have the same estimates for the functions f and g, but we have to assume additionally that the functions $\rho(s)$, $sr(s)$ and $sr(s^2)$ are increasing. For example, the function $r(s) := \log \frac{1}{s}$ fulfills the required property for $s \in]0, \frac{1}{e}]$ and the function $\rho(s) := 1 + \log(1+s)$ respectively for $s \in (0, +\infty)$. In especially we still have a unique strong solution.

Proof. First we show, that the function $g(t,z)$ satisfies the conditions in (4.7) and (4.8): For $z = (x,y) \in \mathbb{R}^2$ we have

$$\begin{aligned} \|g(t,z)\|^2 = \sum_{ij} g(t,z)_{ij}^2 \;&=\; \frac{1}{\mu^2}\sigma(t,x)^2 \\ &\le \frac{1}{\mu^2} C(x^2\rho(x^2)+1) \\ &\le \frac{C}{\mu^2}((x^2+y^2)\rho(x^2+y^2)+1) \\ &= \frac{C}{\mu^2}(|z|^2\rho(|z|^2)+1). \end{aligned}$$

Let $z_i = (x_i, y_i) \in \mathbb{R}^2$ for $i = 1, 2$, then

$$\begin{aligned} \|g(t,z_1)-g(t,z_2)\|^2 &= \left\| \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\mu}\sigma(t,x_1) \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\mu}\sigma(t,x_2) \end{pmatrix} \right\|^2 \\ &= \frac{1}{\mu^2}(\sigma(t,x_1)-\sigma(t,x_2))^2 \\ &\le \frac{1}{\mu^2} C|x_1-x_2|^2 r(|x_1-x_2|^2) \\ &\le \frac{1}{\mu^2} C(|x_1-x_2|^2+|y_1-y_2|^2) r(|x_1-x_2|^2+|y_1-y_2|^2) \\ &\le \frac{1}{\mu^2} C|z_1-z_2|^2 r(|z_1-z_2|^2). \end{aligned}$$

Note that the last inequality works only if $sr(s)$ is increasing. So g fulfills the required inequalities.
Next, we prove that $f(t,z)$ fulfills (4.7) and (4.8). Let $z = (x,y) \in \mathbb{R}^2$, we have

$$\begin{aligned} |f(t,z)| &= \left(y^2 + \frac{1}{\mu^2}(b(t,x)-y)^2 \right)^{1/2} \\ &\le \left(y^2 + \frac{1}{\mu^2}(C(|x|\rho(x^2)+1)-y)^2 \right)^{1/2} \\ &\le \left(y^2 + \frac{2}{\mu^2}(C^2(|x|\rho(x^2)+1)^2+y^2) \right)^{1/2} \\ &= \left(y^2 + \frac{2}{\mu^2}y^2 + \frac{2C^2}{\mu^2}(|x|\rho(x^2)+1)^2 \right)^{1/2} \end{aligned}$$

$$
\begin{aligned}
&= \left(\left(1+\frac{2}{\mu^2}\right) y^2 + \frac{2C^2}{\mu^2}(|x|\rho(x^2)+1)^2\right)^{1/2} \\
&\leq \left(\left(1+\frac{2}{\mu^2}\right) \left(y^2 + C^2(|x|\rho(x^2)+1)^2\right)\right)^{1/2} \\
&= \sqrt{1+\frac{2}{\mu^2}}\sqrt{y^2 + C^2(|x|\rho(x^2)+1)^2} \\
&\leq C\sqrt{1+\frac{2}{\mu^2}}\Big(\underbrace{y^2 + |x|^2\rho(x^2)^2}_{\leq(|x|^2+y^2)\rho(x^2)^2} + 2\underbrace{|x|}_{\leq\sqrt{x^2+y^2}}\rho(x^2)+1\Big)^{1/2} \\
&\leq C\sqrt{1+\frac{2}{\mu^2}}\Big(\Big(\sqrt{x^2+y^2}\underbrace{\rho(x^2)}_{\underset{\rho\nearrow}{\leq}\rho(x^2+y^2)} + 1\Big)^2\Big)^{1/2} \\
&\leq C\sqrt{1+\frac{2}{\mu^2}}\left(\sqrt{x^2+y^2}\rho(x^2+y^2)+1\right) \\
&= \tilde{C}(|z|\rho(|z|^2)+1).
\end{aligned}
$$

Now we will show (4.8): In the following, we use the abbreviation $\Delta x := (x_1 - x_2)^2$ and $\Delta y := (y_1 - y_2)^2$. Thus

$$
\begin{aligned}
|f(t,z_1) - f(t,z_2)| &= \left((y_1-y_2)^2 + \frac{1}{\mu^2}(b(t,x_1) - b(t,x_2) - (y_1-y_2))^2\right)^{1/2} \\
&\leq \left(\Delta y + \frac{1}{\mu^2}\left(C(x_1-x_2)r(\Delta x) - (y_1-y_2)\right)^2\right)^{1/2} \\
&\leq \left(\Delta y + \frac{2}{\mu^2}\left(C^2\Delta x\cdot r(\Delta x)^2 + \Delta y\right)\right)^{1/2} \\
&= \left((1+\frac{2}{\mu^2})\Delta y + \frac{2C^2}{\mu^2}\Delta x\cdot r(\Delta x)^2\right)^{1/2} \\
&\leq \left((1+\frac{2}{\mu^2})\left(\Delta y + C^2\Delta x\cdot r(\Delta x)^2\right)\right)^{1/2} \\
&\leq C\sqrt{1+\frac{2}{\mu^2}}\left(\Delta y + \Delta x\cdot r(\Delta x)^2\right)^{1/2} \\
&= C\sqrt{1+\frac{2}{\mu^2}}\left(\Delta y + [\sqrt{\Delta x}\cdot r(\Delta x)]^2\right)^{1/2} \\
&\underset{*}{\leq} C\sqrt{1+\frac{2}{\mu^2}}\left(\Delta y + [\sqrt{\Delta x+\Delta y}\cdot r(\Delta x+\Delta y)]^2\right)^{1/2} \\
&\leq C\sqrt{1+\frac{2}{\mu^2}}\left(2(\Delta y+\Delta x)r(\Delta x+\Delta y)^2\right)^{1/2} \\
&= C\sqrt{2(1+\frac{2}{\mu^2})}\sqrt{\Delta y+\Delta x}\cdot r(\Delta x+\Delta y) \\
&= C|z_1-z_2|r(|z_1-z_2|^2),
\end{aligned}
$$

where we used that $sr(s^2)$ is increasing in the inequality marked with *. We checked, that if we have the Lipschitz-type estimates for the coefficients of the single SDE, then we have the same estimates for the Newton system, too. □

Remark 4.4. *The properties of coercivity and semi-monotonicity for $b(t,x)$ however do not apply the same property for $f(t,z)$ in* (4.5)*,* (4.6)*.*

4.2 The Smoluchowski-Kramer Approximation

Along with the Newton system (4.1), we will consider in this section also the following one-dimensional differential equation

$$dX_t = \sigma(t, X_t)\, dW_t + b(t, X_t)\, dt, \tag{4.9}$$

with initial condition $X_0 = \zeta_1$. We assume the functions $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ to be continuous and bounded, i.e.

$$\|b\|_\infty := \sup_{t\in[0,T],\ x\in\mathbb{R}} b(t,x) < \infty, \qquad \|\sigma\|_\infty := \sup_{t\in[0,T],\ x\in\mathbb{R}} \sigma(t,x) < \infty.$$

Furthermore, we assume that σ is even Lipschitz-continuous, i.e. there exists $C_1 > 0$, such that for all $t \geq 0$ and $x, y \in \mathbb{R}$

$$\|\sigma(t,x) - \sigma(t,y)\| \leq C_1|x-y|.$$

Concerning b we assume that it satisfies the continuity condition (4.8) with $r(s) = \log(1/s)$ and that there exists $C_2 > 0$, such that for all $t \geq 0$ and $x, y \in \mathbb{R}$

$$(b(t,x) - b(t,y))(x-y) \leq C_2|x-y|^2.$$

Then (4.9) is a special case of (4.4), and by the section above we have a unique strong solution X_t.
The following theorem will compare these two equations and point out that the Newton system converges in probability to the first order equation when μ tends to zero. Therefore this theorem is the justification for using the first order equation (4.9) to describe the motion of a small particle disturbed by a Wiener process instead of using the two-dimensional Newton system (4.1). Clearly, it is much easier to analyze the first order equation.

Theorem 4.5 (Smoluchowski-Kramers approximation)**.** *The first component of the solution of* (4.1)*, X_t^μ, converges in probability uniformly on $[0,T]$ to the solution X_t of* (4.9)*. This means that we have for all $T > 0$, $\varepsilon > 0$*

$$\lim_{\mu\to 0} P\left(\sup_{0\leq s\leq T} |X_s^\mu - X_s| > \varepsilon\right) = 0. \tag{4.10}$$

Proof. We use the idea of the proof of Theorem 2.11. Let $\eta_t := X_t^\mu - X_t$. We define the stopping time $\tau := \inf\{t > 0 | \eta_t^2 > \varepsilon^2\}$. By [Fre04] (see also [Wes06, Prop. 3.2, p.30]) we have that

$$\begin{aligned} X_t^\mu = \zeta_1 + \mu\zeta_2\left(1 - e^{-\frac{t}{\mu}}\right) + \int_0^t b(s, X_s^\mu)\, ds - e^{-\frac{t}{\mu}} \int_0^t e^{\frac{s}{\mu}} b(s, X_s^\mu)\, ds \\ + \int_0^t \sigma(s, X_s^\mu)\, dW_s - e^{-\frac{t}{\mu}} \int_0^t e^{\frac{s}{\mu}} \sigma(s, X_s^\mu)\, dW_s. \end{aligned}$$

Since X_t is a strong solution, by Definition 1.4 we have P-a.s.

$$X_t = \zeta_1 + \int_0^t \sigma(s, X_s)\, dW_s + \int_0^t b(s, X_s)\, ds.$$

Therefore, we have the representation $\eta_t = \beta_t + \gamma_t$, where

$$\begin{aligned}
\beta_t &:= \int_0^t [b(s, X_s^\mu) - b(s, X_s)]\, ds + \int_0^t [\sigma(s, X_s^\mu) - \sigma(s, X_s)]\, dW_s, \\
\gamma_t &:= \zeta_2 \int_0^t e^{-\frac{s}{\mu}}\, ds + e^{-\frac{t}{\mu}} \int_0^t e^{\frac{s}{\mu}} b(t, X_t^\mu)\, ds + e^{-\frac{t}{\mu}} \int_0^t e^{\frac{s}{\mu}} \sigma(t, X_t^\mu)\, dW_s.
\end{aligned}$$

Similarly to the proof of Lemma 2.6 and 2.8 we use Itô's formula with $F = Id$ and get

$$\begin{aligned}
\beta_{t\wedge\tau}^2 &= \beta_0^2 + \int_0^{t\wedge\tau} 1\, d\beta_s^2 + \frac{1}{2}\int_0^{t\wedge\tau} 0\, d\langle \beta_s^2 \rangle \\
&= \beta_0^2 + 2\int_0^{t\wedge\tau} \beta_s [b(s, X_s^\mu) - b(t, X_s)]\, ds \\
&\quad + 2\int_0^{t\wedge\tau} \beta_s [\sigma(s, X_s^\mu) - \sigma(t, X_s)]\, dW_s + \int_0^{t\wedge\tau} [\sigma(s, X_s^\mu) - \sigma(t, X_s)]^2\, ds.
\end{aligned}$$

As in Claim (i) in the proof of Lemma 2.8, we have that the process $\int_0^{t\wedge\tau} \beta_s[\sigma(s, X_s^\mu) - \sigma(t, X_s)]\, dW_s$ is a martingale. Now, keeping in mind that $\beta_t = \eta_t - \gamma_t$, we take expectations:

$$\begin{aligned}
E[\beta_{t\wedge\tau}^2] &= 2E\left[\int_0^{t\wedge\tau} (\eta_s - \gamma_s)(b(s, X_s^\mu) - b(t, X_s))\, ds\right] \\
&\quad + E\left[\int_0^{t\wedge\tau} (\sigma(s, X_s^\mu) - \sigma(t, X_s))^2\, ds\right] \\
&\le 2C_2 E\left[\int_0^{t\wedge\tau} \eta_s^2\, ds\right] + 4E\left[\|b\|_\infty \int_0^{t\wedge\tau} |\gamma_s|\, ds\right] + C_1 E\left[\int_0^{t\wedge\tau} \eta_s^2\, ds\right] \\
&\le (C_1 + 2C_2)\int_0^t E[\eta_{s\wedge\tau}^2]\, ds + 4\|b\|_\infty t \sup_{0\le s\le t} E[\gamma_s].
\end{aligned}$$

But $\eta_s^2 \le 2(\beta_s^2 + \gamma_s^2)$ and hence

$$E[\eta_{t\wedge\tau}^2] \le 2(C_1 + 2C_2)\int_0^t E[\eta_{s\wedge\tau}^2]\, ds + 8\|b\|_\infty t \sup_{0\le s\le t} E[\gamma_s] + 2\sup_{0\le s\le t} E[\gamma_s^2], \qquad t \in \mathbb{R}.$$

Set

$$K_t := 8\|b\|_\infty t \sup_{0\le s\le t} E[\gamma_s] + 2\sup_{0\le s\le t} E[\gamma_s^2].$$

We claim that for each $t > 0$ we have $K_t \to 0$ as $\mu \to 0$. Indeed, if $\mu \to 0$

$$\begin{aligned}
&\left|\zeta_2 \int_0^t e^{-\frac{s}{\mu}}\, ds\right| = |\zeta_2|\mu(1 - e^{-\frac{t}{\mu}}) \le |\zeta_2|\mu \to 0, \\
&\left|e^{-\frac{t}{\mu}} \int_0^t e^{\frac{s}{\mu}} b(s, X_s^\mu)\, ds\right| \le \left|e^{-\frac{t}{\mu}} \|b\|_\infty \int_0^t e^{\frac{s}{\mu}}\, ds\right| = \left|e^{-\frac{t}{\mu}} \|b\|_\infty (e^{\frac{t}{\mu}} - 1)\mu\right| \le \|b\|_\infty \mu \to 0, \\
&E\left[\left|e^{-\frac{t}{\mu}} \int_0^t e^{\frac{s}{\mu}} \sigma(t, X_t^\mu)\, dW_s\right|^2\right] = E\left[\int_0^t e^{-\frac{2(t-s)}{\mu}} \sigma^2(t, X_t^\mu)\, ds\right] \le \frac{\mu}{2}\|\sigma\|^2 \to 0,
\end{aligned}$$

where we used Itô isometry. Thus, by Gronwall's inequality,

$$E[\eta_{t\wedge\tau}^2] \leq K_t e^{2(C_1+2C_2)t}, \qquad t \geq 0.$$

Therefore, by definition of the stopping time τ

$$P(\tau < t)\varepsilon^2 \leq E[\mathbb{1}_{\tau<t}\eta_\tau^2] \leq E[\eta_{t\wedge\tau}^2] \leq K_t e^{2(C_1+2C_2)t}$$

Finally:

$$\begin{aligned} P(\sup_{0\leq s\leq t} |X_s^\mu - X_s| > \varepsilon) &= P(\sup_{0\leq s\leq t} \eta_s^2 > \varepsilon^2) = P(\tau < t) \\ &\leq K_t e^{2(C_1+2C_2)t}\frac{1}{\varepsilon^2} \underset{\mu\to 0}{\longrightarrow} 0, \end{aligned}$$

because $K_t \to 0$ as $\mu \to 0$. $\square$

Bibliography

[Bau02] H. Bauer. *Wahrscheinlichkeitstheorie*, volume 5. de Gruyter, Berlin, New York, 2002.

[CF05] Z. Chen and M. Freidlin. Smoluchowski-Kramers approximation and exit problems. *Stochastics and Dynamics*, 5(4): 569–585, 2005.

[CF06a] S. Cerrai and M. Freidlin. On the Smoluchowski-Kramers approximation for a system with an infinite number of degrees of freedom. *Probability Theory and Related Fields*, 135(3): 363–394, 2006.

[CF06b] S. Cerrai and M. Freidlin. Smoluchowski-Kramers approximation for a general class of SPDEs. *Journal of Evolution Equations*, 6(4): 657–689, 2006.

[DPZ92] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions*. Cambridge University Press, Cambridge, 1992.

[Fre04] M. Freidlin. Some Remarks on the Smoluchowski-Kramers Approximation. *Journal of Statistical Physics*, 117(3/4): 617–634, November 2004.

[FZ05] S. Fang and T. Zhang. A study of a class of stochastic differential equations with non-Lipschitzian coefficients. *Probability Theory Related Fields*, 132: 356–390, 2005.

[IW81] I. Ikeda and S. Watanabe. *Stochastic differential equations and Diffusion processes*. North-Holland, Amsterdam, 1981.

[Kal96] O. Kallenberg. On the existence of universal functional solutions to classical sde's. *Ann. Probab.*, 24(1): 196–205, 1996.

[KS91] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*. Springer Verlag, New York, Berlin, second edition, 1991.

[Nel67] E. Nelson. *Dynamical Theories of Brownian Motion*. Princeton University Press, Princeton, New Jersey, 1967.

[PR07] C. Prévôt and M. Röckner. *A Concise Course on Stochastic Partial Differential Equations*, volume 1905 of *Lecture Notes in Mathmatics*. Springer Verlag, Berlin, Heidelberg, 2007.

[Pro90] P. Protter. *Stochastic Integration and Differential Equations A new approach*. Springer Verlag, Berlin, Heidelberg, New York, 1990.

[Röc05] M. Röckner. *Wahrscheinlichkeitstheorie I und II*, 2005. Lecture Notes.

[Röc06] M. Röckner. *Introduction to Stochastic Partial Differential Equations*, 2006. Lecture Notes.

[Röc07] M. Röckner. *Introduction to Stochastic Analysis*, 2007. Lecture Notes.

[Sch80] Z. Schuss. *Theory and applications of stochastic differential equations.* Wiley series in probability and mathmatical statistics. Wiley, New York, 1980.

[Str84] D. W. Stroock. *An Introduction to the Theory of Large Deviations.* Springer Verlag, 1984.

[Wes06] R. Westermann. *Smoluchowski-Kramers Approximation for Stochastic Differential Equations and Applications in Behavioral Finance.* Diploma thesis, Universität Bielefeld, June 2006.