Approximation of Hunt Processes Associated to Generalized Dirichlet Forms

Diplomarbeit

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Introduction

In this master's thesis we would like to explore a deeper connection of generalized Dirichlet forms and associated Markov processes. We are motivated by the results in [MRZ98], where processes associated to sectorial Dirichlet forms are approximated in a canonical way using the Yosida approximation. We wish to extend these methods and results for generalized Dirichlet forms, as developed in [Sta99b]. In the following we have two aims.

The first one is to approximate a given Hunt process associated with a generalized Dirichlet form in a canonical way. We have to assume our generalized Dirichlet form to be strictly quasi-regular, since this property characterizes all forms which are associated with a Hunt process (cf. [Tru05]). We give here a short approximation scheme, which is taken from [MRZ98] and [EK86, IV.2]: Consider a Markov chain Y^β with some transition kernel, and a Poisson process $(\Pi_t^\beta)_{t\geq 0}$ with parameter β. Then the compound process $X_t^\beta := Y^\beta(\Pi_t^\beta)$ is a strong Markov process, and the corresponding transition semigroup can be calculated explicitely. If the transistion kernel of Y^β is given by βR_β, then $(X_t^\beta)_{t\geq 0}$ is associated to the approximate forms $\mathcal{E}^\beta$ for $\mathcal{E}$. Furthermore, we will prove that the laws of (X_t^β) are relatively compact and hence, there exists a weak limit along a subsequence which can be shown to be unique. This means, that the processes converge in distribution. Finally, we will prove that the so constructed limit process is a Hunt process.

The second aim is to give a new existence proof of a Hunt process associated to a strictly quasi-regular generalized Dirichlet form. Since we start our construction with a strictly quasi-regular generalized Dirichlet form, by the approximation above we obtain the existence of a Hunt process.

In chapter 1 we introduce all the necessary notions used in this text, in particular the notion of strict quasi-regularity. Also we give some examples for strictly quasi-regular generalized Dirichlet forms.
In chapter 2 we introduce the approximation scheme and the process (X^β).

Furthermore, we make use of a compactification method, which makes things easier to handle. In particular, we prove that the laws of (X^β) are relatively compact. The proof of Theorem 2.13 has been modificated.
In chapter 3 we will prove our key theorem which will then be used in chapter 4. This will be done with the help of nice excessive functions. The proof of Lemma 3.4 is new and the one of Lemma 3.8 has been modificated.
In chapter 4 we prove our main result, namely that there exists a Hunt process associated to a generalized Dirichlet form. Furthermore, if we are already given a Hunt process, we can approximate this process in distribution in a canonical way.

Finally, I would like to thank Prof. Dr. M. Röckner, who led me to the interesting field of Dirichlet forms. I also want to thank him for his encouragement during my whole mathematical education. Furthermore, I would like to thank Dr. G. Trutnau for several discussions and helpful comments during the preperation of this master's thesis.

Chapter 1

Dirichlet Forms and Potential Theory

In this chapter we introduce the basic notions from Dirichlet form theory. We also present some results concerning the potential theory of generalized Dirichlet forms. Furthermore, we introduce a new capacity. As we will see, the strict quasi-regularity plays an important role in our considerations. We also present some examples. For details we refer to [Sta99b] and [Tru05].

1.1 Generalized Dirichlet Forms

Let $\mathcal{H}$ be a real separable Hilbert space and let $(\mathcal{A}, \mathcal{V})$ be a coercive closed form on $\mathcal{H}$, i.e. $\mathcal{V}$ is a dense linear subspace of $\mathcal{H}$ and $\mathcal{A} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is a positive definite bilinear form such that $\mathcal{V}$ is complete w.r.t. the norm given by $(\mathcal{A}(\cdot,\cdot) + (\cdot,\cdot)_{\mathcal{H}})^{\frac{1}{2}}$ and $(\mathcal{A}, \mathcal{V})$ satisfies the weak sector condition, i.e. there exists a constant $K > 0$ such that

$$|\mathcal{A}_1(u, v)| \leq K \cdot \mathcal{A}^{\frac{1}{2}}(u, u)\mathcal{A}^{\frac{1}{2}}(v, v) \quad \forall\, u, v \in \mathcal{V}.$$

A coercive closed form on $L^2(E; m)$ is called a Dirichlet form if for all $v \in \mathcal{V}$ one has $v^+ \wedge 1 \in \mathcal{V}$ and

$$\begin{aligned} \mathcal{A}(v + v^+ \wedge 1, v - v^+ \wedge 1) &\geq 0 \text{ and} \\ \mathcal{A}(v - v^+ \wedge 1, v + v^+ \wedge 1) &\geq 0. \end{aligned}$$

We stress that $\mathcal{A}$ needs not to be symmetric. Since in applications the weak sector condition turns out to be very restrictive, it is natural to consider forms without this property.

Since $\mathcal{V} \subset \mathcal{H}$ continuously and densely and by identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we obtain an embedding $\mathcal{V} \hookrightarrow \mathcal{H} = \mathcal{H}' \hookrightarrow \mathcal{V}'$ which again is continuously and densely. For a linear operator Λ defined on a linear subspace D of one of the Hilbert spaces $\mathcal{V},\ \mathcal{H},\ \mathcal{V}'$ we will use the notation (Λ, D).
Let Λ be a linear operator with domain $D(\Lambda, \mathcal{H})$ on $\mathcal{H}$ with the following properties:

D1 (*i*) $(\Lambda, D(\Lambda, \mathcal{H}))$ generates a $\mathcal{C}_0-$ semigroup of contractions $(U_t)_{t\geq 0}$.

(*ii*) $\mathcal{V}$ is $\Lambda-$ admissible, i.e. $(U_t)_{t\geq 0}$ can be restricted to a $\mathcal{C}_0-$ semigroup on $\mathcal{V}$.

We obtain that $D(\Lambda, \mathcal{H}) \cap \mathcal{V}$ is a dense subset of $\mathcal{V}$. And by [Sta99b, I.2.3] $\Lambda : D(\Lambda, \mathcal{H}) \cap \mathcal{V} \to \mathcal{V}'$ is closable. So we denote its closure by $(\Lambda, \mathcal{F})$. Then $\mathcal{F}$ itself is a real Hilbert space with corresponding norm

$$\|u\|_{\mathcal{F}}^2 := \|u\|_{\mathcal{V}}^2 + \|\Lambda u\|_{\mathcal{V}'}^2,$$

and we have $\mathcal{F} \subset \mathcal{V}$. Furthermore, by [Sta99b, I.2.4] we have that the adjoint semigroup $(\hat{U}_t)_{t\geq 0}$ of $(U_t)_{t\geq 0}$ can be extended to a $\mathcal{C}_0$-semigroup on $\mathcal{V}'$. The corresponding generator $(\hat{\Lambda}, D(\hat{\Lambda}, \mathcal{V}'))$ is the dual operator of $(\Lambda, D(\Lambda, \mathcal{V}))$. Then $\hat{\mathcal{F}} := D(\hat{\Lambda}, \mathcal{V}') \cap \mathcal{V}$ is again a real Hilbert space with corresponding norm

$$\|u\|_{\hat{\mathcal{F}}}^2 := \|u\|_{\mathcal{V}}^2 + \|\hat{\Lambda} u\|_{\mathcal{V}'}^2.$$

Definition 1.1. *Let*

$$\mathcal{E}(u,v) := \begin{cases} \mathcal{A}(u,v) - {}_{\mathcal{V}'}\langle \Lambda u, v\rangle_{\mathcal{V}} & if\ u \in \mathcal{F},\ u \in \mathcal{V} \\ \mathcal{A}(u,v) - {}_{\mathcal{V}'}\langle \hat{\Lambda} v, u\rangle_{\mathcal{V}} & if\ u \in \mathcal{V},\ v \in \hat{\mathcal{F}} \end{cases}$$

and $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha(u,v)_{\mathcal{H}}$ *for* $\alpha > 0$, *where* ${}_{\mathcal{V}'}\langle\cdot,\cdot\rangle_{\mathcal{V}}$ *denotes the dualization between* $\mathcal{V}$ *and* $\mathcal{V}'$. *We call* $\mathcal{E}$ *the bilinear form associated with* $(\mathcal{A}, \mathcal{V})$ *and* $(\Lambda, D(\Lambda, \mathcal{H}))$.

Note that, in general, $\mathcal{E}$ does not satisfy the weak sector condition.

Proposition 1.2. *There exists a unique* $\mathcal{C}_0$*-resolvent* $(G_\alpha)_{\alpha>0}$ *and a unique* $\mathcal{C}_0$*-coresolvent* $(\hat{G}_\alpha)_{\alpha>0}$ *on* $\mathcal{H}$ *such that for all* $\alpha > 0,\ f \in \mathcal{H}$ *and* $u \in \mathcal{V}$

$$G_\alpha(\mathcal{H}) \subset \mathcal{F}, \quad \hat{G}_\alpha(\mathcal{H}) \subset \hat{\mathcal{F}},$$

$$\mathcal{E}_\alpha(G_\alpha f, u) = \mathcal{E}_\alpha(u, \hat{G}_\alpha f) = (f,u)_{\mathcal{H}}.$$

$\hat{G}_\alpha$ is the adjoint of G_α and αG_α, $\alpha\hat{G}_\alpha$ are contraction operators on $\mathcal{H}$. Also, we have for $u \in \mathcal{V}$ that

$$\lim_{\alpha\to\infty} \alpha G_\alpha u = u$$

strongly in $\mathcal{V}$.

Proof. See [Sta99b, section I.3] □

The $\mathcal{C}_0$-semigroup of contractions $(T_t)_{t\geq 0}$ corresponding to $(G_\alpha)_{\alpha>0}$ is called the semigroup associated with $\mathcal{E}$. The corresponding generator $(L, D(L))$ is called the generator associated with $\mathcal{E}$.

Definition 1.3.

(i) A bounded linear operator $G : \mathcal{H} \to \mathcal{H}$ is called positivity preserving (resp. sub-Markovian) if $Gf \geq 0$ (resp. $0 \leq Gf \leq 1$) for all $f \in \mathcal{H}$ with $f \geq 0$ (resp. $0 \leq f \leq 1$).

(ii) A $\mathcal{C}_0$-resolvent $(G_\alpha)_{\alpha>0}$ is called positivity preserving (resp. sub-Markovian) if αG_α is positivity preserving (resp. sub-Markovian) for all $\alpha > 0$.

(iii) A $\mathcal{C}_0$-semigroup $(T_t)_{t\geq 0}$ is called positivity preserving (resp. sub-Markovian) if T_t is positivity preserving (resp. sub-Markovian) for all $t \geq 0$.

(iv) A linear operator $(L, D(L))$ is called a Dirichlet operator if $(Lu, (u-1)^+)_{\mathcal{H}} \leq 0$ for all $u \in D(L)$.

Proposition 1.4. *The resolvent $(G_\alpha)_{\alpha>0}$ associated with $\mathcal{E}$ is sub-Markovian if and only if*

$$\textbf{D2} \quad u \in \mathcal{F} \Rightarrow u^+ \wedge 1 \in \mathcal{V} \text{ and } \mathcal{E}(u, u - u^+ \wedge 1) \geq 0.$$

Proof. [Sta99b, I.4.5] □

A criterion for D2 to be satisfied gives us the next proposition, which is an analogue of [MR92, I.4.4].

Proposition 1.5. *If $(\mathcal{A}, \mathcal{V})$ is a Dirichlet form and $(\Lambda, D(\Lambda, \mathcal{H}))$ is a Dirichlet operator then D2 is satisfied.*

Proof. [Sta99b, I.4.7] □

Definition 1.6. *The bilinear form $\mathcal{E}$ associated with $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda, \mathcal{H}))$ is called a generalized Dirichlet form if D2 is satisfied.*

We now give some examples of generalized Dirichlet forms, which include the case of non-symmetric sectorial Dirichlet forms.

Example 1.7.

(i) Let $(\mathcal{A}, \mathcal{V})$ be a Dirichlet form and $\Lambda = 0$. Then with $\mathcal{F} = \mathcal{V} = \hat{\mathcal{F}}$ and $\mathcal{E} = \mathcal{A}$, it follows that $\mathcal{E}$ is a generalized Dirichlet form.

(ii) Let $\mathcal{A} = 0$ on $\mathcal{V} = \mathcal{H}$ and $(\Lambda, D(\Lambda, \mathcal{H}))$ be a Dirichlet operator generating a $\mathcal{C}_0$-semigroup of contractions on $\mathcal{H}$. In this case, $\mathcal{F} = D(\Lambda)$, $\hat{\mathcal{F}} = D(\hat{\Lambda})$ and the corresponding bilinear form $\mathcal{E}(u, v) = -(\Lambda u, v)_{\mathcal{H}}$ if $u \in D(\Lambda)$, $v \in \mathcal{H}$, and $\mathcal{E}(u, v) = -(u, \hat{\Lambda} v)_{\mathcal{H}}$ if $u \in \mathcal{H}$, $v \in D(\hat{\Lambda})$, is a generalized Dirichlet form.

1.2 Analytic Potential Theory

Let E be a Hausdorff topological space such that the Borel σ-algebra $\mathcal{B}(E)$ is generated by the set $\mathcal{C}(E)$ of all continuous functions on E. We stress that no other assumptions will be made on E. Let m be a σ-finite measure on $(E, \mathcal{B}(E))$ and let $(\mathcal{E}, \mathcal{F})$ be a generalized Dirichlet form on $\mathcal{H} := L^2(E; m)$ with inner product $(\cdot, \cdot)_{\mathcal{H}}$ and coercive part $(\mathcal{A}, \mathcal{V})$.

We recall some facts from analytic potential theory of generalized Dirichlet forms as developed in [Sta99b, ch. III] and [Tru05]. For $\alpha > 0$ we call an element $u \in \mathcal{H}$ α-excessive if $\beta G_{\beta+\alpha} u \leq u$ for all $\beta > 0$. This definition is equivalent with $e^{-\alpha t} T_t u \leq u$ for all $t \geq 0$, and if $u \in \mathcal{V}$ this is equivalent with $\mathcal{E}_\alpha(u, v) \geq 0$ for $v \in \hat{\mathcal{F}}$, $v \geq 0$. Denote by $\mathcal{P}_\alpha$ the set of all α-excessive elements in $\mathcal{V}$. For an element $h \in \mathcal{H}$ let $\mathcal{L}_h = \{v \mid v \in \mathcal{H},\ v \geq h\}$. Suppose that $\mathcal{L}_h \cap \mathcal{F} \neq \emptyset$ then we denote by e_h the smallest 1-excessive function greater or equal than h. We have $e_h \in \mathcal{L}_h \cap \mathcal{P}_1$ and e_h is called the 1-reduced function of h.

For $U \subset E$ open and $f \in \mathcal{H}$ with $\mathcal{L}_{f \cdot 1_U} \cap \mathcal{F} \neq \emptyset$ let $f_U := e_{(f \cdot 1_U)}$ be the 1-reduced function of $f \cdot 1_U$. Now an increasing sequence of closed subsets $(F_k)_{k \geq 1}$ is called an $\mathcal{E}$-nest if for every element $u \in \mathcal{F} \cap \mathcal{P}_1$ it follows that $\lim_{k \to \infty} u_{F_k^c} = 0$ in $\mathcal{H}$. A subset $N \subset E$ is called $\mathcal{E}$-exceptional if there exists an $\mathcal{E}$-nest $(F_k)_{k \geq 1}$ such that $N \subset \bigcap_{k \geq 1} F_k^c$. Finally, we say that a property of points in E holds $\mathcal{E}$-quasi everwhere ($\mathcal{E}$-q.e.) if the property holds for all points outside some $\mathcal{E}$-exceptional set. For an $\mathcal{E}$-nest $(F_k)_{k\geq 1}$ we define

$$\mathcal{C}(\{F_k\}) := \left\{ f : A \to \mathbb{R} \mid \bigcup_{k\geq 1} F_k \subset A \subset E, f_{|F_k} \text{ is cont. } \forall\, k \right\}.$$

An $\mathcal{E}$-q.e. defined function f is called $\mathcal{E}$-quasi-continuous ($\mathcal{E}$-q.c.) if $f \in \mathcal{C}(\{F_k\})$ for some $\mathcal{E}$-nest $(F_k)_{k\geq 1}$.

Definition 1.8. *The generalized Dirichlet form $\mathcal{E}$ is called quasi-regular if:*

(i) There exists an $\mathcal{E}$-nest $(E_k)_{k\geq 1}$ such that $E_k,\ k \geq 1$, is compact in E.

(ii) There exists a dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$-q.c. m-versions.

(iii) There exist $u_n \in \mathcal{F},\ n \in \mathbb{N}$, having $\mathcal{E}$-q.c. m-versions $\tilde{u}_n,\ n \in \mathbb{N}$, and an $\mathcal{E}$-exceptional set $N \subset E$ such that $\{\tilde{u}_n \mid n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

The quasi-regularity is the analytic characterization of those generalized Dirichlet forms which are associated to an m-tight special standard process, if in addition $\mathcal{E}$ satisfies the following (cf. [Sta99b, chap. IV]):

D3 There exists a linear subspace $\mathcal{Y} \subset \mathcal{H} \cap L^\infty(E;m)$ such that $\mathcal{Y} \cap \mathcal{F}$ is dense in $\mathcal{F}$, $\lim_{\alpha\to\infty} e_{\alpha G_\alpha u - u} = 0$ in $\mathcal{H}$ for all $u \in \mathcal{Y}$ and for the closure $\bar{\mathcal{Y}}$ of $\mathcal{Y}$ in $L^\infty(E;m)$ it follows that $u \wedge \alpha \in \bar{\mathcal{Y}}$ for $u \in \bar{\mathcal{Y}}$ and $\alpha \geq 0$.

We give an example of a quasi-regular generalized Dirichlet form, which is taken from [Sta99b, II.1].

Example 1.9 (Time dependent potentials)**.** *Let $d \geq 1$, and $V : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $V \in L^1_{loc}(\mathbb{R} \times \mathbb{R}^d, dt \otimes dx)$, $V \geq 0$, be a time dependent potential. Let $(\mathcal{E}, \mathcal{C}_0^\infty(\mathbb{R}^{d+1}))$ be the bilinear form*

$$\mathcal{E}(u,v) := \int_{\mathbb{R}} \int_{\mathbb{R}^d} \langle \nabla u, \nabla v \rangle\, dx\ dt + \int_{\mathbb{R}} \int_{\mathbb{R}^d} uvV\ dx\ dt - \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{\partial u}{\partial t} v\ dx\ dt;\ u, v \in \mathcal{C}_0^\infty(\mathbb{R}^{d+1}), \quad (1.1)$$

where ∇u means gradient w.r.t. x. Our aim is to construct a generalized Dirichlet form on $\mathcal{H} := L^2(\mathbb{R} \times \mathbb{R}^d, dx \otimes dt)$ extending the bilinear form $\mathcal{E}$. To this end define

$$\mathcal{A}^0(u,v) := \int_{\mathbb{R}} \int_{\mathbb{R}^d} \langle \nabla u, \nabla v \rangle \, dx \, dt; \; u, v \in \mathcal{C}_0^\infty(\mathbb{R}^{d+1})$$

and

$$\mathcal{A}(u,v) := \int_{\mathbb{R}} \int_{\mathbb{R}^d} \langle \nabla u, \nabla v \rangle \, dx \, dt + \int_{\mathbb{R}} \int_{\mathbb{R}^d} uvV \, dx \, dt; \; u, v \in \mathcal{C}_0^\infty(\mathbb{R}^{d+1}).$$

It is easy to see that $(\mathcal{A}^0, \mathcal{C}_0^\infty(\mathbb{R}^{d+1}))$ is closable in $\mathcal{H}$, but by [Sta99b, II.1.1] we obtain that $(\mathcal{A}, \mathcal{C}_0^\infty(\mathbb{R}^{d+1}))$ is also closable in $\mathcal{H}$ and the closure $(\mathcal{A}, \mathcal{V})$ is a symmetric Dirichlet form. Now, we assume the following on V:

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{\partial u}{\partial t} uV \, dx \, dt \leq c \cdot \mathcal{A}_1(u,u) \text{ for all } u \in \mathcal{C}_0^\infty(\mathbb{R}^{d+1}) \tag{1.2}$$

Then by [Sta99b, II.1.2] $\mathcal{V}$ is $\frac{\partial}{\partial t}$-admissible, if V satisfies (1.2). In this case the bilinear form associated with $(\mathcal{A}, \mathcal{V})$ and $(\frac{\partial}{\partial t}, D(\frac{\partial}{\partial t}, \mathcal{H}))$ extends the bilinear form $\mathcal{E}$, and $\mathcal{E}$ is a generalized Dirichlet form. Furthermore, this form is quasi-regular. Indeed, let $(U_t)_{t\geq 0}$ denote the semigroup corresponding to $\big(\frac{\partial}{\partial t}, D(\frac{\partial}{\partial t}, \mathcal{H})\big)$. Since $U_t(\mathcal{C}_0^\infty(\mathbb{R}^{d+1})) \subset \mathcal{C}_0^\infty(\mathbb{R}^{d+1})$, $t \geq 0$, and $\mathcal{C}_0^\infty(\mathbb{R}^{d+1}) \subset D(\frac{\partial}{\partial t}, \mathcal{V})$ it follows from [RS80, Theorem X.49] that $\mathcal{C}_0^\infty(\mathbb{R}^{d+1}) \subset D(\frac{\partial}{\partial t}, \mathcal{V})$ dense w.r.t. the graph norm. In particular, $\mathcal{C}_0^\infty(\mathbb{R}^{d+1}) \subset \mathcal{F}$ dense. Hence if $(K_n)_{n\geq 1}$ is an increasing sequence of compact subsets with $\mathbb{R}^{d+1} = \bigcup_{n\geq 1} K_n$ and $K_n \subset \mathring{K}_{n+1}$, $n \geq 1$, it follows that $\mathcal{C}_0^\infty(\mathbb{R}^{d+1}) \subset \bigcup_{n\geq 1} \mathcal{F}_{K_n}$ and consequently, $(K_n)_{n\geq 1}$ is an $\mathcal{E}$-nest. Hence $\mathcal{E}$ is a quasi-regular generalized Dirichlet form. Moreover, D3 is satisfied for $\mathcal{Y} = \mathcal{C}_0^\infty(\mathbb{R}^{d+1})$.

In the following we adjoin an extra point Δ to our state space E and we denote by E_Δ the set $E \cup \{\Delta\}$. The point Δ serves as a cemetery for our Markov process. If the space E is locally compact then there are two ways of defining a topology on E_Δ. Either we consider Δ as an isolated point of E_Δ, or we consider E_Δ with the one point compactification. We fix one of the two topologies. Of course, if E is not locally compact then we consider Δ as an isolated point of E_Δ. So our framework can be easily extended to E_Δ : We extend any function on E to E_Δ by setting $f(\Delta) = 0$. In the same way our measure m is extended to $(E_\Delta, \mathcal{B}(E_\Delta))$ by putting $m(\Delta) = 0$. For an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ we define

$$\mathcal{C}_\infty(\{F_k\}) := \left\{ f : A \to \mathbb{R} \mid \bigcup_{k\geq 1} \subset A \subset E, \; f_{|F_k \cup \{\Delta\}} \text{ is continuous for every } k \in \mathbb{N} \right\}.$$

Of course, if we consider Δ as an isolated point then $\mathcal{C}_\infty(\{F_k\})$ coincides with $\mathcal{C}(\{F_k\})$.

We now introduce strict notions, which already have been considered in [MR92, ch. V] and in [Tru05], and therefore we need a new capacity. From now on fix $\varphi \in L^1(E,m) \cap \mathcal{B}(E)$ with $0 < \varphi(z) \leq 1$ for every $z \in E$ and set $g := \hat{G}_1\varphi$.

Definition 1.10. *For $U \subset E$, U open, set*

$$Cap_{1,g}(U) := \int_E e_U \varphi \, dm,$$

where $e_U := \lim_{k\to\infty}(kG_1\varphi \wedge 1)_U$ exists in $L^\infty(E;m)$. If $A \subset E$ arbitrary then $Cap_{1,g}(A) := \inf\{Cap_{1,g}(U) \mid U \supset A,\ U\,open\}$.

Note that e_U is not necessarily in $\mathcal{H}$.

By [Tru05, Thm.1] $Cap_{1,g}$ is a finite Choquet capacity, i.e. $Cap_{1,g}$ has the following properties:

(i) If $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence of subsets of E then

$$Cap_{1,g}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sup_{n\geq 1} Cap_{1,g}(A_n).$$

(ii) If $(K_n)_{n\in\mathbb{N}}$ is a decreasing sequence of compact subsets of E then

$$Cap_{1,g}\left(\bigcap_{n\in\mathbb{N}} K_n\right) = \inf_{n\geq 1} Cap_{1,g}(K_n).$$

A subset $N \subset E$ is called strictly $\mathcal{E}$-exceptional if $Cap_{1,g}(N) = 0$. An increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E is called a strict $\mathcal{E}$-nest if $Cap_{1,g}(F_k^c) \downarrow 0$ as $k \to \infty$. A property of points in E holds strictly $\mathcal{E}$-quasi-everywhere (s. $\mathcal{E}$-q.e.) if the property holds outside some strictly $\mathcal{E}$-exceptional set. A function f defined up to some strictly $\mathcal{E}$-exceptional set $N \subset E$ is called strictly $\mathcal{E}$-quasi-continuous (s. $\mathcal{E}$-q.c.) if there exists a strict $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that $f \in \mathcal{C}_\infty(\{F_k\})$. For a subset $\mathcal{D} \subset \mathcal{H}$ denote by $\tilde{\mathcal{D}}^{str}$ all s. $\mathcal{E}$-q.c. m-versions of elements in $\mathcal{D}$. After all these preparations we are able to give the essential definition for our purposes.

Definition 1.11. *The generalized Dirichlet form $\mathcal{E}$ is called strictly quasi-regular if:*

(i) There exists a strict $\mathcal{E}$-nest $(E_k)_{k\geq 1}$ such that $E_k \cup \{\Delta\}$, $k \geq 1$, is compact in E_Δ.

(ii) There exists a dense subset of $\mathcal{F}$ whose elements have s. $\mathcal{E}$-q.c. m-versions.

(iii) There exist $u_n \in \mathcal{F}$, $n \in \mathbb{N}$, having s. $\mathcal{E}$-q.c. m-versions $\tilde{u}_n$, $n \in \mathbb{N}$, and a strictly $\mathcal{E}$-exceptional set $N \subset E$ such that $\{\tilde{u}_n \mid n \in \mathbb{N}\}$ separates the points of $E_\Delta \setminus N$.

Although the next two propositions will not be used in the sequel, we include them for completeness.

Proposition 1.12. *Let E be a locally compact separable metric space. Let the generalized Dirichlet form $(\mathcal{E}, \mathcal{F})$ be regular, i.e. $\mathcal{C}_0(E) \cap \mathcal{F}$ is dense in $\mathcal{F}$ w.r.t. $\|\cdot\|_{\mathcal{F}}$ as well as in $\mathcal{C}_0(E)$ w.r.t the uniform norm. Then it is strictly quasi-regular.*

Proof. See [Tru05, Prop. 1] □

Proposition 1.13. *Assume that $(\mathcal{E}, \mathcal{F})$ is a quasi-regular generalized Dirichlet form on $\mathcal{H}$ such that $1 \in \mathcal{F}$ and Δ is adjoined to E as an isolated point of E_Δ. Then $(\mathcal{E}, \mathcal{F})$ is strictly quasi-regular.*

Proof. See [Tru05, Prop. 3] □

We now give a condition, which is sufficient for our purposes:

SD3 There exists an algebra of functions $\mathcal{G} \subset \mathcal{H}_b$ such that $\mathcal{G} \cap \mathcal{F}$ is dense in $\mathcal{F}$ and $\lim_{\alpha\to\infty} e_{u-\alpha G_\alpha u} + e_{\alpha G_\alpha u - u} = 0$ for every $u \in \mathcal{G}$.

If SD3 is satisfied then the strict quasi-regularity of a generalized Dirichlet form characterizes all m-tight Hunt processes which are associated with $\mathcal{E}$ (cf. [Tru05]).

Having introduced all necessary notions and results, we now turn to the preparation for the next chapter. Also the next three lemmas will be important for the rest of this work.

Define

$$Y_1 := \bigcup_{k\geq 1} E_k,$$

where $(E_k)_{k\geq 1}$ is such as in Definition 1.11. By the arguments in [MR92, IV.3, Remark 3.2(iii)] Y_1 can always be assumed as a Lusin topological space.

From now on the generalized Dirichlet form $(\mathcal{E}, \mathcal{F})$ will be assumed to be strictly quasi-regular.

Proposition 1.14. *Let $\mathcal{E}$ be a strictly quasi-regular. Then every $u \in \mathcal{F}$ admits a s.$\mathcal{E}$-q.c. m-version $\tilde{u}$. In particular we have for any $\varepsilon > 0$*

$$Cap_{1,g}(\{|\tilde{u}| > \varepsilon\}) \leq \varepsilon^{-1}\|e_u + e_{-u}\|_{\mathcal{H}}\|\varphi\|_{\mathcal{H}}.$$

Let $\mathcal{E}$ satisfy SD3 in addition. Then every element in $\mathcal{G}$ admits a s. $\mathcal{E}$-q.c. m-version.

Proof. See [Tru05, Prop. 2] □

Lemma 1.15. *[Tru05, Lemma 2]Let $\mathcal{E}$ be strictly quasi-regular. Let $\alpha > 0$. There exists a kernel $\tilde{R}_\alpha$ from $(E, \mathcal{B}(E))$ to $(Y_1, \mathcal{B}(Y_1))$ such that*

(i) $\tilde{R}_\alpha f$ is a s. $\mathcal{E}$-q.c. m-version of $G_\alpha f$ for all $f \in \mathcal{H}$,

(ii) $\alpha\tilde{R}_\alpha(z, Y_1) \leq 1$ for all $z \in E$.

The kernel $\tilde{R}_\alpha$ is unique in the sense that, if K is another kernel from $(E, \mathcal{B}(E))$ to $(Y_1, \mathcal{B}(Y_1))$ satisfying (i) and (ii), it follows that $K(z, \cdot) = \tilde{R}_\alpha(z, \cdot)$ s. $\mathcal{E} -$ q.e.

Proof. Fix $\alpha > 0$. Let $sq\mathcal{C}(E)$ denote the set of s. $\mathcal{E}$-q.c. functions defined s. $\mathcal{E}$-q.e. on E. Let $T : \mathcal{H} \to sq\mathcal{C}(E),\ f \mapsto \tilde{G_\alpha} f$, where $\tilde{G_\alpha} f$ is a s. $\mathcal{E}$-q.c. m-version of $G_\alpha f$. By Proposition 1.14 $\tilde{G_\alpha} f$ exists and we will show that T is quasi-linear. Indeed, let $c_1, c_2 \in \mathbb{R},\ f_1, f_2 \in \mathcal{H}$. By Proposition 1.14 we have for any $\varepsilon > 0$

$$Cap_{1,g}\Big(\{|\tilde{G_\alpha}(c_1 f_1 + c_2 f_2) - c_1\tilde{G_\alpha} f_1 - c_2\tilde{G_\alpha} f_2| > \varepsilon\}\Big) \leq \varepsilon^{-1}\|e_0 + e_{-0}\|_{\mathcal{H}}\|\varphi\|_{\mathcal{H}} = 0.$$

Hence $\tilde{G_\alpha}(c_1 f_1 + c_2 f_2) = c_1\tilde{G_\alpha} f_1 + c_2\tilde{G_\alpha} f_2$ s. $\mathcal{E}$-q.e. If $f_n \downarrow 0,\ f_n \in \mathcal{H}$ then $f_n \to 0$ in $\mathcal{H}$. Hence $e_{G_\alpha f_n} + e_{-G_\alpha f_n} \to 0$ in $\mathcal{H}$ and therefore again by Proposition 1.14

$$Cap_{1,g}(\{|\tilde{G_\alpha} f_n| > \varepsilon\}) \longrightarrow 0$$

for all $\varepsilon > 0$ as $n \to \infty$. Finally, if $f \geq 0$ m-a.e., $f \in \mathcal{H}$ then $\tilde{G_\alpha} f \geq 0$ s. $\mathcal{E}$-q.e. Indeed, since $G_\alpha f \geq 0$ m-a.e. and $\tilde{G_\alpha} f = G_\alpha f$ m-a.e. we have by Lemma A.4 that $\tilde{G_\alpha} f \geq 0$ s. $\mathcal{E}$-q.e. This completes the proof that T is quasi-linear. By [AM91, Thm. 4.2] there exists a unique kernel K from $(E, \mathcal{B}(E))$ to $(Y_1, \mathcal{B}(Y_1))$ such that $Kf = Tf$ s. $\mathcal{E}$-q.e. for all $f \in \mathcal{H}$. Since m is σ-finite it follows in particular

$$\alpha K 1_{Y_1} = \alpha T 1_{Y_1} = \alpha \tilde{G_\alpha} 1_{Y_1} \leq 1 \quad s.\ \mathcal{E} - q.e.,$$

and hence there exists a s. $\mathcal{E}$-exceptional set $N \in \mathcal{B}(E)$ such that $\alpha K(z, \cdot) \leq 1$ for all $z \in E \setminus N$. Now let

$$\tilde{R}_\alpha(z, \cdot) := 1_{E \setminus N} K(z, \cdot).$$

□

Let $\mathcal{E}$ satisfy SD3. The condition SD3 implies the condition D3. Hence, we can adapt the arguments in [Sta99b, IV.2] to the strictly quasi-regular case to get the following results.

Lemma 1.16. *There exists a countable family J_0 of bounded strictly $\mathcal{E}$-quasi-continuous 1-excessive functions and a Borel set $Y \subset Y_1$ satisfying:*

(i) *If $u, v \in J_0$, $\alpha, c_1, c_2 \in \mathbb{Q}_+^*$, then $\tilde{R}_\alpha u$, $u \wedge v$, $u \wedge 1$, $(u+1) \wedge v$, $c_1 u + c_2 v$ are all in J_0.*

(ii) *$N := E \setminus Y$ is strictly $\mathcal{E}$-exceptional and $\tilde{R}_\alpha(x, N) = 0$, for all $x \in Y$, $\alpha \in \mathbb{Q}_+^*$.*

(iii) *J_0 separates the points of Y_Δ.*

(iv) *If $u \in J_0$, $x \in Y$, then $\beta \tilde{R}_{1+\beta} u(x) \leq u(x)$ for all $\beta \in \mathbb{Q}_+^*$, $\tilde{R}_\alpha u(x) - \tilde{R}_\beta u(x) = (\beta - \alpha) \tilde{R}_\alpha \tilde{R}_\beta u(x)$ for all $\alpha, \beta \in \mathbb{Q}_+^*$, $\lim_{\alpha \to \infty} \alpha \tilde{R}_\alpha u(x) = u(x)$.*

We now want to extend our kernel $\tilde{R}_\alpha$ to the point Δ. So define for $\alpha \in \mathbb{Q}_+^*$, $A \in \mathcal{B}(Y_\Delta) := \mathcal{B}(E_\Delta) \cap Y_\Delta$

$$R_\alpha(x, A) := \begin{cases} \tilde{R}_\alpha(x, A \cap Y) + \left(\frac{1}{\alpha} - \tilde{R}_\alpha(x, Y) \right) 1_A(\Delta), & if\ x \in Y \\ \frac{1}{\alpha} 1_A(\Delta), & if x = \Delta \end{cases}$$

and set

$$J := \{u + c1_{Y_\Delta} \mid u \in J_0, c \in \mathbb{Q}_+\}.$$

Since J_0 separates the points of Y_Δ, so does J.

Lemma 1.17. *Let $(R_\alpha)_{\alpha\in\mathbb{Q}_+^*}$ and J be as above. Then the statements of Lemma 1.16 remain true with J_0, Y and $\tilde{R}_\alpha$ replaced by J, Y_Δ and R_α respectively.*

We close this chapter with two important examples.

1.3 Examples of s. Quasi-Regular Dirichlet Forms

These examples are taken from [Tru05]. The first one shows that the generalized Dirichlet form defined in [Sta99a] is actually strictly quasi-regular.

Example 1.18 (Non-symmetric perturbations through divergence free vector fields). *Let $U \subset \mathbb{R}^d$ be open and let $H_0^{1,2}(U)$ be the closure of $\mathcal{C}_0^\infty(U)$ in $L^2(U;dx)$ w.r.t. the norm given by $\int_U |\nabla u|^2 + u^2\, dx$. Let $H_{loc}^{1,2}(U)$ be the space of all elements u such that $u_\chi \in H_0^{1,2}(U)$ for all $\chi \in \mathcal{C}_0^\infty(U)$. Let $\rho \in H_{loc}^{1,2}(U)$ such that the measure $m := \rho^2 dx$ on U has full support. Similarly to the above $H_{loc}^{1,2}(U,m)$ and $H_0^{1,2}(U,m)$ are defined.*
Let $A = (a_{ij})_{1\le i,j\le d}$ with $a_{ij} \in H_{loc}^{1,2}(U,m)$ be a symmetric matrix and locally uniformly elliptic, i.e. for any V relatively compact in U there exists $\nu_V > 0$ such that

$$\nu_V^{-1}|h|^2 \le \langle A(x)h, h\rangle \le \nu_V |h|^2 \text{ for all } h \in \mathbb{R}^d,\ x \in V. \tag{1.3}$$

Consider the closure of

$$\mathcal{E}^0(u,v) = \frac{1}{2}\int \langle A\nabla u, \nabla v\rangle\, dm, \quad u, v \in \mathcal{C}_0^\infty(U)$$

on $L^2(U,m)$ which we denote by $(\mathcal{E}^0, D(\mathcal{E}^0))$. Let $(L^0, D(L^0))$ be the associated generator. By construction we have that $\mathcal{C}_0^\infty(U) \subset D(L^0)$ and

$$L^0 = \frac{1}{2}\sum_{i,j=1}^d a_{ij}\partial_i\partial_j u + \frac{1}{2}\sum_{i=1}^d \beta_{A,i}\partial_i u,\ u \in \mathcal{C}_0^\infty(U),$$

where

$$\beta_{A,i} = \sum_{j=1}^d (\partial_j a_{ij} + \frac{2a_{ij}\partial_j\rho}{\rho}).$$

Let $B = (b_1, \dots b_d) \in L^2_{loc}(U, \mathbb{R}^d, m)$*, i.e.* $\int_V \langle B, B\rangle\, dm < \infty$ *for all* V *relatively compact in* U*, and such that*

$$\int \langle B, \nabla u\rangle\, dm = 0 \text{ for all } u \in \mathcal{C}_0^\infty(U). \tag{1.4}$$

For a subset $W \in L^2(U;m)$ *let* W_0 *denote the space of all* $u \in W$ *such that* $\operatorname{supp}|u|m$ *is compact in* U *and let* $W_{0,b} = W_0 \cap W_b$*. Define*

$$Lu := L^0u + \sum_{i,j=1}^d b_i\partial_i u, \qquad u \in D(L^0)_{0,b}.$$

By [Sta99a, Thm. 1.5] there exists a closed extension $(\bar{L}, D(\bar{L}))$ *of* $(L, D(L^0)_{0,b})$ *on* $L^1(U, m)$ *generating a strongly continuous resolvent* $(\bar{G}_\alpha)_{\alpha>0}$ *on* $L^1(U, m)$ *which is sub-Markovian. Furthermore, we have* $D(\bar{L})_b \subset D(\mathcal{E}^0)$ *and*

$$\mathcal{E}^0(u, v) - \int \langle B, \nabla u\rangle v\, dm = -\int \bar{L}uv\, dm, \qquad u \in D(\bar{L})_b,\ v \in D(\mathcal{E}^0)_{0,b}. \tag{1.5}$$

Now, let $(L, D(L))$ *be the part of* $(\bar{L}, D(\bar{L}))$ *in* $L^2(U, m)$*, i.e.*

$$D(L) = \{f \in D(\bar{L}) \cap L^2(E;m) \mid \bar{L}f \in L^2(E;m)\},$$

$$Lf := \bar{L}f,\ f \in D(L).$$

Let $(L', D(L'))$ *be the adjoint operator of* $(L, D(L))$ *in* $L^2(U;m)$*. Let* $(G_\alpha)_{\alpha>0}$ *(resp.* $(G'_\alpha)_{\alpha>0}$*) be the associated resolvent to* $(L, D(L))$ *(resp.* $(L', D(L'))$*).* $(\bar{G}_\alpha)_{\alpha>0}$ *and* $(G_\alpha)_{\alpha>0}$ *coincide on* $L^1(U;m)\cap L^2(U;m)$*. According to our basic example* $(L, D(L))$ *is associated to a generalized Dirichlet form on* $D(L) \times L^2(U;m) \cup L^2(U;m) \times D(L')$ *given by*

$$\mathcal{E}(u, v) := \begin{cases} (-Lu, v)_\mathcal{H} & \text{for } u \in D(L),\ v \in L^2(U;m) \\ (u, -L'v)_\mathcal{H} & \text{if } u \in L^2(U;m),\ v \in D(L'). \end{cases}$$

In this case $\mathcal{F} = D(L)$*. Then by [Tru05, Thm. 6] it follows that* $\mathcal{E}$ *is strichtly quasi-regular.*

Example 1.19 (Time inhomogeneous diffusions on infinite dimensional space)**.** *Let* E *be a separable real Banach space with norm* $\|\cdot\|_E$ *and let* E' *equipped with the operator norm* $\|\cdot\|_{E'}$ *be its dual. In particular, we have* $\mathcal{B}(E) = \sigma(E')$*. Let* $(\mathcal{H}, (\cdot,\cdot)_\mathcal{H})$ *be a separable real Hilbert space such that* $\mathcal{H} \subset E$ *densely and continuously. Identifying* $\mathcal{H}$ *with its dual* $\mathcal{H}'$ *we obtain* $E' \subset \mathcal{H} \subset E$ *densely and continuously. The corresponding dualization* ${}_{E'}\langle\cdot,\cdot\rangle_E$ *:*

$E' \times E \to \mathbb{R}$ restricted to $E' \times \mathcal{H}$ coincides with $(\cdot,\cdot)_{\mathcal{H}}$.
Let $\mathcal{C}^1_{0,b}(\mathbb{R}\times\mathbb{R}^m)$ denote the one times continuously differentiable functions on $\mathbb{R}\times\mathbb{R}^m$ with all partial derivatives in space bounded and compact support in time. Let $\mathcal{C}^1_{0,b}([0,\infty)\times\mathbb{R}^m)$ denote the restrictions to $[0,\infty)\times\mathbb{R}^m$ of functions in $\mathcal{C}^1_{0,b}(\mathbb{R}\times\mathbb{R}^m)$. Let us now define the finitely based time-dependent functions as

$$\mathcal{FTC}^1_{0,b} := \left\{ f(t, l_1, \cdots, l_m) \mid m \in \mathbb{N},\ f \in \mathcal{C}^1_{0,b}([0,\infty)\times\mathbb{R}^m),\ l_1, \cdots l_m \in E' \right\}.$$

For $u \in \mathcal{FTC}^1_{0,b},\ k \in E$, let

$$\frac{\partial u}{\partial k}(t,z) := \frac{d}{ds} u(t, z+sk)_{|s=0} \in [0,\infty)\times E$$

denote the Gâteaux-derivative of u in the direction k.
If $u(t,z) = f(t, l_1(z), \ldots, l_m(z))$, then

$$\frac{\partial u}{\partial k}(t,z) = \sum_{i=1}^m \frac{\partial f}{\partial x_i}(t, l_1(z), \ldots, l_m(z))\ {}_{E'}\langle l_i, k\rangle_E.$$

Hence, if $k \in \mathcal{H}$, then there exists by the Riesz representation theorem a unique element $\nabla_{\mathcal{H}} u(t,z) \in \mathcal{H}$ such that

$$(\nabla_{\mathcal{H}} u(t,z), k)_{\mathcal{H}} = \frac{\partial u}{\partial k}(t,z).$$

Let ds denote the Lebesgue measure on $[0,\infty)$. Let μ be a finite positive measure with full support on $(E, \mathcal{B}(E))$. Let $\rho : [0,\infty)\times E \to \mathbb{R}$ be $\mathcal{B}([0,\infty))\otimes \mathcal{B}(E)$-measurable, $\rho > 0$ $d\mu ds$-a.e., and $\int_K \int_E \rho(s,z)\mu(dz)ds < \infty$ for any compact set $K \subset [0,\infty)$. Let $\mathcal{C}^1_0([0,\infty))$ consist of restrictions to $[0,\infty)$ of the one times continuously differentiable functions with compact support on $\mathbb{R}$. Note that $\mathcal{FTC}^1_{0,b}$ contains functions of the form $fg,\ f \in \mathcal{C}^1_0([0,\infty)),\ g \in \mathcal{FC}^\infty_b := \{f(l_1, \cdots, l_m) \mid m \in \mathbb{N},\ f \in \mathcal{C}^\infty_b(\mathbb{R}^m),\ l_1, \cdots l_m \in E'\}$, the finitely based smooth functions. Since $\mathcal{FC}^\infty_b$ separates the points of E by the Hahn-Banach theorem, it is clear that $\mathcal{FTC}^1_{0,b}$ separates the points of $[0,\infty)\times E$. Furthermore $\mathcal{FTC}^1_{0,b}$ is an algebra of functions. Thus, by monotone class arguments

$$\mathcal{FTC}^1_{0,b} \subset \mathcal{H} := L^2([0,\infty)\times E, \rho d\mu ds) \text{ densely.}$$

Let $c \in L^\infty([0,\infty)\times E, \rho d\mu ds),\ c \geq 0$, and assume that the following densely defined positive-definite symmetric bilinear form

$$\begin{aligned} \mathcal{A}(u,v) := &\frac{1}{2}\int_0^\infty \int_E (\nabla_{\mathcal{H}} u(s,z), \nabla_{\mathcal{H}} v(s,z))_{\mathcal{H}} \rho(s,z)\mu(dz)ds \\ &+ \int_0^\infty \int_E c(s,z)u(s,z)v(s,z)\rho(s,z)\mu(dz)ds; \quad u,v \in \mathcal{FTC}^1_{0,b} \end{aligned} \tag{1.6}$$

is closable on $\mathcal{H}$. The closure $(\mathcal{A}, \mathcal{V})$ is then a symmetric Dirichlet form. Let us now define the semigroup corresponding to the perturbation of $\mathcal{A}$. Let $d \geq 0$ be a constant. For $u(s,z) = f(s, l_1(z), \ldots, l_m(z))$ let first either

$$U_t u(s,z) := f(se^{dt}, l_1(z), \ldots, l_m(z)); \quad t \geq 0$$

or

$$U_t u(s,z) := u(s+t, z); \quad t \geq 0.$$

It is clear that $(U_t)_{t\geq 0}$ has the semigroup property on $\mathcal{FTC}^1_{0,b}$ and this space is invariant under U_t, $t \geq 0$. Furthermore U_t is sub-Markovian for all $t \geq 0$. Let us assume the following:

$$\rho(s,\cdot) \leq \rho(t,\cdot) \ \mu - a.e. \ \forall \ s \leq t$$

and

$$c(s,\cdot)\rho(s,\cdot) \leq c(t,\cdot)\rho(t,\cdot) \ \mu - a.e. \ \forall \ s \leq t.$$

In case of the first semigroup we have the following contraction property for $u \in \mathcal{FTC}^1_{0,b}$

$$\begin{aligned}
& \int_0^\infty \int_E U_t u(s,z)^2 \rho(s,z)\mu(dz)ds \\
= & \int_0^\infty \int_E f(se^{dt}, l_1(z), \ldots, l_m(z))^2 \rho(s,z)\mu(dz)ds \\
= & \int_0^\infty \int_E u(s,z)^2 e^{-dt} \rho(se^{-dt}, z)\mu(dz)ds \\
\leq & \int_0^\infty \int_E u(s,z)^2 \rho(s,z)\mu(dz)ds,
\end{aligned}$$

by our assumptions on ρ. In the second case the contraction property is even easier to see. Hence we proved $\|U_t u\|^2_{\mathcal{H}} \leq \|u\|^2_{\mathcal{H}}$. Since $\mathcal{FTC}^1_{0,b} \subset \mathcal{H}$ densely, $(U_t)_{t\geq 0}$ above induces a sub-Markovian semigroup of contractions on $\mathcal{H}$ which we also denote by $(U_t)_{t\geq 0}$. This is a $\mathcal{C}_0$-semigroup on $\mathcal{H}$. Indeed, in the first case we have

$$|U_t u(s,z) - u(s,z)| = |f(se^{dt}, l_1(z), \ldots, l_m(z)) - f(s, l_1(z), \ldots, l_m(z))|$$

and $|U_t u(s,z) - u(s,z)|$ converges pointwise to zero as $t \to 0$. Let the support of f be contained in $K \times E$ with $K \subset [0,\infty)$, K compact. For any $t \geq 0$, the support of $f(se^{dt}, l_1(z), \ldots, l_m(z))$ is also contained in $K \times E$ since d is a positive constant. Therefore $|U_t u - u|$ is bounded by the integrable function $2\|f\|_\infty 1_{K\times E}$. Hence, $(U_t)_{t\geq 0}$ is strongly continuous on $\mathcal{H}$. In the second case *this is again even easier to see. The corresponding generator* $(\Lambda, D(\Lambda, \mathcal{H}))$ *on* $\mathcal{H}$ *is an extension of*

$$\Lambda u(t,z) = td\partial_t u(t,z)$$

on $\mathcal{FTC}^1_{0,b}$ *in the first case and of*

$$\Lambda u(t,z) = \partial_t u(t,z)$$

in the second case. Furthermore, $(U_t)_{t\geq 0}$ *can be restricted to a* $\mathcal{C}_0$*-semigroup on* $\mathcal{V}$*. The last clearly follows from*

$$\|U_t u\|^2_{\mathcal{V}} \leq \|u\|^2_{\mathcal{V}},$$

which again follows from our assumptions on ρ*. Now we have the following*

Theorem 1.20. *Let* $\Lambda = \partial_t$ *and* $1 \in \mathcal{F}$*. Then* $(\mathcal{E}, \mathcal{F})$ *is strictly quasi-regular.*

Proof. See [Tru05, Prop. 5] □

Chapter 2

Processes Associated with $\mathcal{E}^{(\beta)}$

Let $(\mathcal{E}, \mathcal{F})$ be a strictly quasi-regular generalized Dirichlet form satisfying SD3. Let J, Y_Δ and $(R_\alpha)_{\alpha \in \mathbb{Q}_+^*}$ be as in Lemma 1.17. The aim of this chapter is to construct a family of processes (X^β), $\beta > 0$, which is a compostion of a Poisson process and a Markov chain and which is associated to the approximate forms $\mathcal{E}^{(\beta)}$ for $\mathcal{E}$. The process (X^β), $\beta > 0$ will serve to approximate a Hunt process that is associated to $\mathcal{E}$ and therefore leads to a new proof of existence (cf. chapter 4).

2.1 Basic Definitions

In the following we will deal with a special class of Markov processses, namely Hunt processes, so we give a precise definition:

Definition 2.1. $\mathbb{M} = (\Omega, \mathcal{M}, (X_t)_{t \geq 0}, (P_x)_{x \in E_\Delta})$ *is called a Hunt process with state space E, lifetime ζ and corresponding filtration $(\mathcal{M}_t)_{t \geq 0}$, if*

(M.1) $X_t : \Omega \to E_\Delta$ *is* $\mathcal{M}_t/\mathcal{B}(E_\Delta)$*-measurable for all* $t \geq 0$ *and* $X_t(\omega) = \Delta \Leftrightarrow t \geq \zeta(\omega)$ *for all* $\omega \in \Omega$, *where* $\zeta : \Omega \to [0, \infty]$.

(M.2) For all $t \geq 0$ *there exists a map* $\theta_t : \Omega \to \Omega$ *such that* $X_s \circ \theta_t = X_{s+t}$ *for all* $s \geq 0$.

(M.3) $(P_x)_{x \in E_\Delta}$ *is a family of probability measures on* $(\Omega, \mathcal{M})$, *such that* $x \mapsto P_x[B]$ *is* $\mathcal{B}(E_\Delta)^*$*-measurable for all* $B \in \mathcal{M}$ *and* $\mathcal{B}(E_\Delta)$*-measurable for all* $B \in \sigma(X_t \mid t \geq 0)$ *and* $P_\Delta[X_0 = \Delta] = 1$.

(M.4) *For all* $A \in \mathcal{B}(E_\Delta)$, $s, t \geq 0$ *and* $x \in E_\Delta$

$$P_x[X_{t+s} \in A \mid \mathcal{M}_t] = P_{X_t}[X_s \in A] \quad P_x\text{-}a.s$$

(M.5) $P_x[X_0 = x] = 1$ *for all* $x \in E_\Delta$.

(M.6) *For each* $\omega \in \Omega$, $t \mapsto X_t(\omega)$ *is right continuous on* $[0, \infty)$.

(M.7) $(\mathcal{M}_t)_{t \geq 0}$ *is a right continuous filtration and for every* $(\mathcal{M}_t)$*-stopping time* τ *and every* $\mu \in \mathcal{P}(E_\Delta)$

$$P_\mu[X_{\tau+t} \in A \mid \mathcal{M}_\tau] = P_{X_\tau}[X_t \in A] \quad P_\mu - a.s.$$

for all $A \in \mathcal{B}(E_\Delta)$, $t \geq 0$.

(M.8) $X_{t-} := \lim_{s \uparrow t} X_s$ *exists in* E_Δ *for all* $t > 0$ P_μ*-a.s. for all* $\mu \in \mathcal{P}(E_\Delta)$.

(M.9) $\lim_{n \to \infty} X_{\tau_n} = X_\tau$ P_μ*-a.s. on* $\{\tau < \infty\}$ *and* X_τ *is* $\bigvee_{n \geq 1} \mathcal{F}_{\tau_n}^{P_\mu}$*-measurable for every increasing sequence* $(\tau_n)_{n \geq 1}$ *of* $(\mathcal{F}_t^{P_\mu})$*-stopping times with limit* τ *and for all* $\mu \in \mathcal{P}(E_\Delta)$.

$\mathbb{M}$ *is called a right process if it satisfies (M.1)-(M.7) above. The process* $\mathbb{M}$ *is called strictly m-tight if there exists an increasing sequence* $(K_n)_{n \in \mathbb{N}}$ *of compact metrizable sets in* E *such that*

$$P_{\varphi \cdot m}\left[\lim_{n \to \infty} \sigma_{E \setminus K_n} < \infty\right] = 0.$$

For a right process $\mathbb{M}$,

$$p_t f(x) := E_x[f(X_t)] \quad x \in E,\ t \geq 0,\ f \in \mathcal{B}(E)^+$$

defines a sub-Markovian semigroup of kernels on $(E, \mathcal{B}(E))$. Furthermore, we define

$$U_\alpha f(x) := \int_0^\infty e^{-\alpha t} p_t f(x)\, dt,$$

called the resolvent of $\mathbb{M}$.

Definition 2.2. *A Hunt process $\mathbb{M}$ with resolvent U_α is called associated (in the resolvent sense) with $\mathcal{E}$ if $U_\alpha f$ is an m-version of $G_\alpha f$ for $\alpha > 0$ and $f \in \mathcal{B}_b(E) \cap \mathcal{H}$. $\mathbb{M}$ is called properly associated (in the resolvent sense) with $\mathcal{E}$ if in addition $U_\alpha f$ is $\mathcal{E}$-q.c. for $\alpha > 0$ and $f \in \mathcal{B}_b(E) \cap \mathcal{H}$. The process is called strictly properly associated if $U_\alpha f$ is strictly $\mathcal{E}$-q.c.*

One can prove that $\mathbb{M}$ is associated to $\mathcal{E}$ if and only if $p_t f$ is an m-version of $T_t f$ for all $t \geq 0$ and $f \in \mathcal{B}_b(E) \cap \mathcal{H}$.

2.2 The Construction

First, we want to define a metric on Y_Δ. For this let $J = \{u_n \mid n \in \mathbb{N}\}$ be the countable family of functions from Lemma 1.17. For the rest of this chapter set $g_n := R_1 u_n,\ n \in \mathbb{N}$. Furthermore, define for all $x, y \in Y_\Delta$

$$\rho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |g_n(x) - g_n(y)| \wedge 1.$$

By Lemma 1.16(ii) and Lemma1.17 $\{g_n \mid n \in \mathbb{N}\}$ separates the points of Y_Δ and hence ρ defines a metric on Y_Δ. Note that by definition we have

$$\rho - \mathcal{B}(Y_\Delta) = \sigma(g_n \mid n \in \mathbb{N}).$$

Since Y_Δ is a topological Lusin space, it follows by [Sch73, Lemma 18, p.108] that $\mathcal{B}(Y_\Delta) = \sigma(g_n \mid n \in \mathbb{N})$. Hence, the ρ-topology and the original topology generate the same Borel σ-algebra on Y_Δ. By the same arguments we obtain that $\sigma(J) = \mathcal{B}(Y_\Delta)$.

Let $(\Sigma, \mathcal{M}, P)$ be a probability space. For a fixed $\beta \in \mathbb{Q}_+^*$, let $\{Y^\beta(k),\ k = 0, 1, \ldots\}$ be a Markov chain on $(\Sigma, \mathcal{M}, P)$ with values in Y_Δ with some initial distribution ν and transition function βR_β. Furthermore, let $(\Pi_t^\beta)_{t \geq 0}$ be a Poisson process with parameter β, i.e.

$$P[\Pi_t^\beta = k] = e^{-\beta t} \frac{(\beta t)^k}{k!}.$$

Assume that $(\Pi_t^\beta)_{t \geq 0}$ is independent of $\{Y^\beta(k),\ k = 0, 1, \ldots\}$, i.e. the σ-algebras generated by them are independent, and define

$$X_t^\beta = Y^\beta(\Pi_t^\beta), \quad t \geq 0.$$

Let $\mathcal{F}_t^{\Pi^\beta} = \sigma(\Pi_s^\beta \mid s \leq t)$, and $\mathcal{F}_t^{X^\beta} = \sigma(X_s^\beta \mid s \leq t)$, and finally

$$\mathcal{M}_t^\beta = \mathcal{F}_t^{\Pi^\beta} \bigvee \mathcal{F}_t^{X^\beta}.$$

Proposition 2.3. *$(X_t^\beta)_{t\geq 0}$ is a Markov process w.r.t. $\mathcal{M}_t^\beta$ in Y_Δ.*

Proof. (cf. [EK86, IV.2]) Denote by $\mathcal{B}_b(Y_\Delta)$ the set of all bounded Borel functions on Y_Δ. By Markov property of Y^β we have for all $f \in \mathcal{B}_b(Y_\Delta)$

$$E[f(Y^\beta(k+l)) \mid Y^\beta(0), \ldots, Y^\beta(l))] = (\beta R_\beta)^k f(Y^\beta(l))$$

for $k, l = 0, 1, 2, \ldots$, and we claim that

$$E[f(Y^\beta(k + \Pi^\beta(t)) \mid \mathcal{M}_t^\beta] = (\beta R_\beta)^k f(X_t^\beta) \quad P - a.s.$$

for $k = 0, 1, 2, \ldots,\ t \geq 0$ and for all $f \in \mathcal{B}_b(Y_\Delta)$.
To see this, let $A \in \mathcal{F}_t^{\Pi^\beta}$ and $B \in \mathcal{F}_l^{Y^\beta} := \sigma(Y^\beta(k) \mid k \leq l)$. Then

$$\begin{aligned}
\int_{A\cap B\cap\{\Pi_t^\beta = l\}} f(Y^\beta(k+\Pi_t^\beta))\, dP &= \int_{A\cap B\cap\{\Pi_t^\beta = l\}} f(Y^\beta(k+l))\, dP \\
&= P(A \cap \{\Pi_t^\beta = l\}) \int_B f(Y^\beta(k+l))\, dP \\
&= P(A \cap \{\Pi_t^\beta = l\}) \int_B (\beta R_\beta)^k f(Y^\beta(l))\, dP \\
&= \int_{A\cap B\cap\{\Pi_t^\beta = l\}} (\beta R_\beta)^k f(X_t^\beta)\, dP.
\end{aligned}$$

Since $\{A \cap B \cap \{\Pi_t^\beta = l\} \mid A \in \mathcal{F}_t^{\Pi^\beta},\ B \in \mathcal{F}_l^{Y^\beta},\ l = 0, 1, \ldots\}$ is closed under finite intersections and generates $\mathcal{M}_t^\beta$, by the Dynkin class theorem we have

$$\int_A f(Y^\beta(k+\Pi_t^\beta))\, dP = \int_A (\beta R_\beta)^k f(X_t^\beta)\, dP$$

for all $A \in \mathcal{M}_t^\beta$. So we have that $(X_t^\beta)_{t\geq 0}$ is a Markov process. □

For all $f \in \mathcal{B}_b(Y_\Delta)$ define

$$P_t^\beta f := e^{-\beta t} \sum_{k=0}^{\infty} \frac{(\beta t)^k}{k!} (\beta R_\beta)^k f \quad \forall\, t \geq 0. \tag{2.1}$$

Proposition 2.4. *$(P_t^\beta)_{t\geq 0}$ is the transition semigroup of $(X_t^\beta)_{t\geq 0}$, i.e. for all $f \in \mathcal{B}_b(Y_\Delta)$, $t, s \geq 0$ we have*

$$E[f(X_{t+s}^\beta) \mid \mathcal{M}_t^\beta] = (P_s^\beta f)(X_t^\beta).$$

Proof. (cf. [EK86, IV.2])We have

$$\begin{aligned}
E[f(X_{t+s}^\beta) \mid \mathcal{M}_t^\beta] &= E[f(Y^\beta(\Pi_{t+s}^\beta - \Pi_t^\beta + \Pi_t^\beta)) \mid \mathcal{M}_t^\beta] \\
&\underset{(\star)}{=} \sum_{k=0}^{\infty} e^{-\beta s} \frac{(\beta s)^k}{k!} E[f(Y^\beta(k + \Pi_t^\beta)) \mid \mathcal{M}_t^\beta] \\
&= \sum_{k=0}^{\infty} e^{-\beta s} \frac{(\beta s)^k}{k!} (\beta R_\beta)^k f(X_t^\beta) \\
&= P_s^\beta f(X_t^\beta).
\end{aligned}$$

The equality in $(\star)$ follows, because for $A \in \mathcal{F}_t^{\Pi^\beta},\ B \in \mathcal{F}_t^{Y^\beta}$ we have

$$\begin{aligned}
&\textstyle\int_{A\cap B} \quad f(Y^\beta(\Pi_{t+s}^\beta - \Pi_t^\beta + \Pi_t^\beta))\, dP \\
&= \sum_k \int_{A\cap B\cap\{\Pi_{t+s}^\beta - \Pi_t^\beta = k\}} f(Y^\beta(k + \Pi_t^\beta))\, dP \\
&= \sum_k \underbrace{P(A \cap \{\Pi_{t+s}^\beta - \Pi_t^\beta = k\})}_{=P(A)\cdot e^{-\beta s}\frac{(\beta s)^k}{k!}} \int_B f(Y^\beta(k + \Pi_t^\beta))\, dP \\
&= \sum_k e^{-\beta s} \frac{(\beta s)^k}{k!} \int_{A\cap B} f(Y^\beta(k + \Pi_t^\beta))\, dP.
\end{aligned}$$

□

Remark 2.5. $(P_t^\beta)_{t\geq 0}$ *is a strongly continuous contraction semigroup on the Banach space* $(\mathcal{B}_b(Y_\Delta), \|\cdot\|_\infty)$ *and the corresponding generator is given by*

$$L^\beta u(x) = \beta(\beta R_\beta u(x) - u(x)) \quad \forall\ u \in \mathcal{B}_b(Y_\Delta).$$

Indeed, for all $x \in Y_\Delta$ *we have*

$$\begin{aligned}
\frac{d}{dt} P_t^\beta f(x)\Big|_{t=0} &= -\beta e^{-\beta t} f(x) + e^{-\beta t} \sum_{k=0}^{\infty} \frac{\beta k (\beta t)^{k-1}}{k!} (\beta R_\beta)^k f(x)\Big|_{t=0} \\
&= \beta(\beta R_\beta f - f)(x).
\end{aligned}$$

Now we have the following

Theorem 2.6. (X_t^β) *is a strong Markov process w.r.t.* $\mathcal{M}_{t+}^\beta := \bigcap_{\varepsilon>0} \mathcal{M}_{t+\varepsilon}^\beta$.

Proof. Since the ρ-topology and the original topology generate the same Borel σ-algebra, we consider the ρ-topology. In this case $R_\alpha f$ is uniformly continuous on Y_Δ for each $\alpha \in \mathbb{Q}_+^*$ and $f \in J$. Set

$$W := \Big\{ f \in \mathcal{B}_b(Y_\Delta) \mid P[t \mapsto R_\alpha f(X_t^\beta) \\ \text{is right continuous on } [0,\infty)] = 1 \ \forall\ \alpha \in \mathbb{Q}_+^* \Big\}. \quad (2.2)$$

Then $J \subset W$ and W is a linear vector space. To prove that W is a monotone vector space, consider $Z_t^f := e^{-\alpha t} R_\alpha f(X_t^\beta) + \int_0^t e^{-\alpha s} f(X_s^\beta)\, ds$ for $f \in W$. Then it can be easily seen that $(Z_t^f)_{t\geq 0}$ is a right continuous martingale for all $f \in W$. If $f_n \in W,\ n \in \mathbb{N}$ such that $0 \leq f_n \uparrow f$ bounded then it follows by monotone convergence theorem that $Z_t^{f_n} \uparrow Z_t^f$ for all $t \geq 0$. By [DM82, Thm. VI.18] we have that Z_t^f is indistinguishable from a right continuous process, hence $f \in W$, i.e. W is a monotone vector space. By the monotone class theorem we obtain that $\mathcal{B}_b(Y_\Delta)$ is contained in W. Now the strong Markov property follows from [Sha88, Thm. 7.4]. □

For each $\beta \in \mathbb{Q}_+^*$ have constructed a strong Markov process with a special transition semigroup. We would like to find forms to which this process is associated. Define the forms $\mathcal{E}^{(\beta)},\ \beta > 0$, by

$$\mathcal{E}^{(\beta)}(u,v) := \beta(u - \beta G_\beta u, v)_{\mathcal{H}}, \quad u, v \in \mathcal{H},$$

where $(G_\beta)_{\beta>0}$ is the resolvent of $\mathcal{E}$. It is known, that the semigroup associated to $\mathcal{E}^{(\beta)}$ is given by

$$T_t^\beta f = e^{-\beta t} \sum_{j=0}^{\infty} \frac{(\beta t)^j}{j!} (\beta G_\beta)^j f \quad \forall\ f \in L^2(E;m). \quad (2.3)$$

From Proposition 2.4, (2.1) and (2.3) we see that (X_t^β) is associated with $\mathcal{E}^{(\beta)}$.

2.3 Hunt Processes

For an arbitrary subset $M \subset E_\Delta$ let $\Omega_M := D_M[0,\infty)$ be the space of all càdlàg functions from $[0,\infty)$ to M. Let $(X_t)_{t\geq 0}$ be the coordinate process on Ω_{E_Δ}, i.e. $X_t(\omega) = \omega(t)$ for $\omega \in \Omega_{E_\Delta}$. Let P_x^β be the law of X^β on Ω_{E_Δ} with initial distribution δ_x for $x \in Y_\Delta$, i.e.

$$P_x^\beta[\cdot] := P[\ \cdot \mid X_0^\beta = x];$$

and for $x \in E_\Delta \setminus Y_\Delta$ let P_x^β be the Dirac measure on Ω_{E_Δ} such that $P_x^\beta[X_t = x \text{ for all } t \geq 0] = 1$. Finally, let $(\mathcal{F}_t)_{t\geq 0}$ be the completion w.r.t. $(P_x^\beta)_{x\in E_\Delta}$ of the natural filtration of $(X_t)_{t\geq 0}$,i.e.

$$\mathcal{F}_t = \bigcap_{x\in E_\Delta} \sigma(X_s \mid s \leq t)^{P_x^\beta}.$$

Proposition 2.7. *$\mathbb{M}^\beta := (\Omega_{E_\Delta},\ (X_t)_{t\geq 0},\ (\mathcal{F}_t)_{t\geq 0},\ (P_x^\beta)_{x\in E_\Delta})$ is a Hunt process associated with $\mathcal{E}^{(\beta)}$, i.e. for all $t \geq 0$ and any m-version of $u \in L^2(E;m)$, $x \mapsto \int u(X_t)\, dP_x^\beta$ is an m-version of $T_t^\beta u$.*

Proof. (cf. [MR92, IV.3.21]) By construction it is clear that $\mathbb{M}^\beta$ is a right process. And, of course, the left limits of X_t exist in E_Δ. So we only have to prove the quasi-left continuity up to ∞.
Let $(\tau_n)_{n\geq 1}$ be an increasing sequence of $(\mathcal{F}_t)$-stopping times such that $\tau_n \uparrow \tau$. Define

$$V(\omega) := \begin{cases} \lim_{n\to\infty} X_{\tau_n}(\omega) & if\ \tau(\omega) < \infty \\ \Delta & if\ \tau(\omega) = \infty. \end{cases}$$

So, we have to prove that $V = X_\tau$ P_μ-a.s. for all $\mu \in \mathcal{P}(E_\Delta)$.
Step1 : Assume first that τ is bounded. For all $f, g \in \mathcal{C}_b(E_\Delta)$, $x \in E_\Delta$ we have

$$\begin{aligned}
E_x^\beta[g(V)R_\alpha f(X_\tau)] &= E_x^\beta\left[g(V)E_{X_\tau}^\beta\left[\int_0^\infty e^{-\alpha t} f(X_t)\ dt\right]\right] \\
&= E_x^\beta\left[g(V)\int_0^\infty e^{-\alpha t}\ \underbrace{E_{X_\tau}^\beta[f(X_t)]}_{=E_x^\beta[f(X_{\tau+t})|\mathcal{F}_\tau]\ P_x^\beta-a.s.}\ dt\right] \\
&= E_x^\beta\left[g(V)\int_\tau^\infty e^{-\alpha(t-\tau)}E_x^\beta\left[f(X_t)\mid \mathcal{F}_\tau\right]\ dt\right] \\
&= E_x^\beta\left[g(V)e^{\alpha\tau}\int_\tau^\infty e^{-\alpha t} f(X_t)\ dt\right] \\
&= \lim_{n\to\infty} E_x^\beta\left[g(X_{\tau_n})e^{\alpha\tau_n}\int_{\tau_n}^\infty e^{-\alpha t} f(X_t)\ dt\right] \\
&= \lim_{n\to\infty} E_x^\beta\left[g(X_{\tau_n})R_\alpha f(X_{\tau_n})\right] \\
&= E_x^\beta\left[g(V)R_\alpha f(V)\right],
\end{aligned}$$

where in the last step we used Lebesgue's dominated convergence theorem. By a monotone class argument we have

$$E_x^\beta[g(V)\alpha R_\alpha f(X_\tau)] = E_x^\beta[g(V)\alpha R_\alpha f(V)]$$

for all Borel measurable bounded functions g. Since Y is Borel measurable, we may replace g by $1_Y \cdot g$ and if we let α tend to infinity, we obtain by monotone convergence theorem that for all $f \in \mathcal{C}_b(E_\Delta),\ g \in \mathcal{B}_b(E_\Delta)$

$$E_x^\beta[g(V)f(X_\tau)1_{\{V\in Y\}}] = E_x^\beta[g(V)f(V)1_{\{V\in Y\}}].$$

Again by a monotone class argument we obtain that

$$E_x^\beta[h(V,X_\tau)1_{\{V\in Y\}}] = E_x^\beta[h(V,V)1_{\{V\in Y\}}]$$

for all $\mathcal{B}_b(E_\Delta \times E_\Delta)$-measurable bounded functions h. Let now h to be the indicator function of the diagonal in $E_\Delta \times E_\Delta$. So, the assertion follows.
Step2 : For arbitrary τ we have

$$\begin{aligned}
& P_x^\beta\left[V \neq X_\tau, V \in Y\right] \\
= \ & P_x^\beta\left[V \neq X_\tau,\ V \in Y,\ \tau < \infty\right] \\
= \ & \sum_{n=1}^\infty P_x^\beta\left[V \neq X_\tau,\ V \in Y,\ n-1 \leq \tau < n\right] \\
= \ & \sum_{n=1}^\infty P_x^\beta\left[\lim_{k\to\infty} X_{\tau_k \wedge n} \neq X_{\tau\wedge n},\ \lim_{k\to\infty} X_{\tau_k\wedge n} \in Y,\ n-1 \leq \tau < n\right] \\
= \ & 0,
\end{aligned}$$

where all summands are zero by step 1. Hence $(X_t)_{t\geq 0}$ is quasi-left continuous up to ∞ and so $\mathbb{M}^\beta$ is a Hunt process. □

Our next aim is to prove the relative compactness of the family $\{P_x^\beta \mid \beta \in \mathbb{Q}_+^*\}$. For technical reasons we will make use of a compactification method. So we construct a compact superset of Y_Δ by completion w.r.t. ρ. Set $\bar{E} := \overline{Y_\Delta}^\rho$.

Proposition 2.8. *$(\bar{E},\rho)$ is a compact metric space.*

Proof. Since J separates the points of Y_Δ,

$$\Phi : x \to \left(\frac{g_n(x)}{\|g_n\|_\infty}\right)_{n\in\mathbb{N}}$$

defines an isometry from (Y_Δ,ρ) to $[0,1]^\mathbb{N}$ with the product metric. By Tychonoff's theorem $[0,1]^\mathbb{N}$ is compact, hence so is $(\bar{E},\rho)$. □

We extend the kernel $(R_\alpha)_{\alpha\in\mathbb{Q}_+^*}$ to the space $\bar{E}$ by setting for $\alpha \in \mathbb{Q}_+^*,\ A \in \mathcal{B}(\bar{E})$,

$$R_\alpha(x,A) := \begin{cases} R_\alpha(x, A\cap Y_\Delta), & x \in Y_\Delta \\ \frac{1}{\alpha}1_A(x), & x \in \bar{E}\setminus Y_\Delta. \end{cases}$$

$(X_t^\beta)_{t\geq 0}$ can be regarded as a càdlàg process with state space $\bar{E}$. We use the same notation as before: P_x^β denotes the law of $(X_t^\beta)_{t\geq 0}$ in $\Omega_{\bar{E}}$ with initial distribution δ_x. Each g_n is uniformly continuous w.r.t. ρ and hence extends uniquely to a continuous function on $\bar{E}$ which we denote again by g_n.

2.4 The Skorohod Topology

Let (M, d) be an arbitrary metric space and let $D_M[0,\infty)$ be the space of all càdlàg functions from $[0,\infty)$ to M. Since we are concerned with càdlàg functions, we will introduce a topology on this space, the so-called Skorohod topology. The first observation is that càdlàg functions do not behave as bad as we might think of them.

Lemma 2.9. *[EK86, chap. III, Lemma 5.1] Every $x \in D_M[0,\infty)$ has at most a countable number of discontinuities.*

Proof. Set for $n \geq 1$ $A_n := \{t > 0 \mid d(x_t, x_{t-}) > \frac{1}{n}\}$. Then A_n is countable, since $\lim_{s\nearrow t} x(s)$ and $\lim_{s\searrow t} x(s)$ exist for all $t \geq 0$. The set of all discontinuities of x is $\bigcup_{n\geq 1} A_n$, and hence countable. □

Let $\mathcal{L}$ be the collection of all real-valued increasing functions λ on $[0,\infty)$ such that $\lambda(0) = 0$. Such a function λ is called a time change. Define for $\lambda \in \mathcal{L}$

$$\|\lambda\| := \sup_{s\neq t}\left| log\left(\frac{\lambda(s)-\lambda(t)}{s-t}\right)\right| + \sup_{t\geq 0}|\lambda(t) - t|.$$

Now, define for $a, b \in D_E[0,\infty)$ the Skorohod metric

$$s(a,b) = \inf_{\lambda\in\mathcal{L}}\left\{\|\lambda\| + \sup_{t\geq 0} e^{-t} d(a_t, b_{\lambda(t)})\right\}.$$

Theorem 2.10. *If M is separable, then $D_M[0,\infty)$ is separable. If (M,d) is complete then $(D_M[0,\infty), s)$ is complete.*

Proof. See [EK86, chap. III, Thm. 5.6]. □

2.5 Relative Compactness of $\{P_x^\beta \mid \beta \in \mathbb{Q}_+^*\}$

To prove the relative compactness we need some results which are taken from [EK86].

Theorem 2.11. *Let (M,d) be a complete and separable metric space and let (X^α) be a family of processes with sample paths in Ω_M. Suppose that the compact containment condition holds. That is, for every $\eta > 0$ and $T > 0$ there exists a compact set $\Gamma_{\eta,T} \subset E$ for which*

$$\inf_\alpha P\left[X_t^\alpha \in \Gamma_{\eta,T} \text{ for } 0 \leq t \leq T\right] \geq 1 - \eta.$$

Let H be a dense subset of $\mathcal{C}_b(M)$ in the topology of uniform convergence on compact sets. Then (X^α) is relatively compact if and only if $(f \circ X^\alpha)$ is relatively compact for each $f \in H$.

Proof. See [EK86, chap. III, Thm. 9.1]. □

If the metric space is compact then the compact containment condition holds. Although the next theorem holds for arbitrary metric spaces, we will formulate it in the case of our compact metric space $(\bar{E}, \rho)$.

For each α, let X^α be a process with sample paths in $\Omega_{\bar{E}}$ defined on a probability space $(\Sigma^\alpha, \mathcal{F}^\alpha, Q^\alpha)$ and adapted to a filtration $(\mathcal{F}_t^\alpha)$. Let $\mathbb{L}^\alpha$ be the Banach space of real-valued $(\mathcal{F}_t^\alpha)$-progressive processes with norm $\|Y\| := \sup_{t\geq 0} E[|Y_t|] < \infty$. Let

$$\hat{\mathcal{M}}_\alpha = \left\{(Y,Z) \in \mathbb{L}^\alpha \times \mathbb{L}^\alpha \middle| Y_t - \int_0^t Z_s\, ds \text{ is an } \mathbb{L}^\alpha - \text{martingale}\right\}.$$

Theorem 2.12. *Let $(\bar{E}, \rho)$ be the compact metric space from Proposition 2.8, and let (X^α) be a family of processes as above. Let C_a be a subalgebra of $\mathcal{C}(\bar{E})$ and let D be the collection of $f \in \mathcal{C}(\bar{E})$ such that for every $\varepsilon > 0$ and $T > 0$ there exist $(Y^\alpha, Z^\alpha) \in \hat{\mathcal{M}}_\alpha$ with*

$$\sup_\alpha E\left[\sup_{t\in[0,T]\cap\mathbb{Q}} |Y_t^\alpha - f(X_t^\alpha)|\right] < \varepsilon$$

and

$$\sup_\alpha E[\|Z^\alpha\|_{p,T}] < \infty \quad \text{for some } p \in (1,\infty],$$

where $\|h\|_{p,T} = \left(\int_0^T |h(t)|^p\right)^{\frac{1}{p}}$ if $p < \infty$ and $\|h\|_{\infty,T} = ess\sup_{0\leq t\leq T}|h(t)|$. If C_a is contained in $\overline{D}^{\|\cdot\|_\infty}$, then $\{f \circ X^\alpha\}$ is relatively compact for each $f \in C_a$.

Proof. See [EK86, chap. III, Thm. 9.4]. □

Now we have the following important theorem:

Theorem 2.13. *$\{P_x^\beta \mid \beta \in \mathbb{Q}_+^*\}$ is relatively compact for any $x \in \bar{E}$.*

Proof. Recall that g_n was defined by $g_n = R_1 u_n$, $n \in \mathbb{N}$, where $J = \{u_n \mid n \in \mathbb{N}\}$. Since $g_i \in D(L^\beta)$ for all $i \in \mathbb{N}$, we can define for all $i \in \mathbb{N}$

$$M_t^{\beta,i} := g_i(X_t^\beta) - \int_0^t L^\beta g_i(X_s^\beta)\, ds, \quad t \geq 0.$$

It follows that $(M_t^{\beta,i})_{t\geq 0}$ is an $(\mathcal{M}_t^\beta)$-martingale. Indeed, by Proposition 2.4 we have

$$\begin{aligned}
& E[M_{t+s}^{\beta,i} \mid \mathcal{M}_t^\beta] \\
= \;& E[g_i(X_{t+s}^\beta) \mid \mathcal{M}_t^\beta] - E\left[\int_0^{t+s} L^\beta g_i(X_r^\beta)\, dr \mid \mathcal{M}_t^\beta\right] \\
= \;& E[g_i(X_{t+s}^\beta) \mid \mathcal{M}_t^\beta] - \int_0^t L^\beta g_i(X_r^\beta)\, dr - \int_t^{t+s} \underbrace{E[L^\beta g_i(X_r^\beta) \mid \mathcal{M}_t^\beta]}_{=\frac{d}{dr}P_r^\beta(g_i(X_t^\beta))}\, dr \\
= \;& E[g_i(X_{t+s}^\beta) \mid \mathcal{M}_t^\beta] - \int_0^s \frac{d}{dr} P_r^\beta(g_i(X_t^\beta))\, dr - \int_0^t L^\beta g_i(X_r^\beta)\, dr \\
= \;& E[g_i(X_{t+s}^\beta) \mid \mathcal{M}_t^\beta] - P_s^\beta g_i(X_t^\beta) + P_0^\beta g_i(X_t^\beta) - \int_0^t L^\beta g_i(X_r^\beta)\, dr \\
= \;& g_i(X_t^\beta) - \int_0^t L^\beta g_i(X_r^\beta)\, dr = M_t^{\beta,i} \quad P - a.s.
\end{aligned}$$

Moreover, we have

$$L^\beta g_i(x) = 1_{Y_\Delta} \beta R_\beta(g_i - u_i)(x).$$

Therefore, we conclude for all $i \in \mathbb{N}$

$$\begin{aligned}
\sup_{\beta\in\mathbb{Q}_+^*} \|L^\beta(g_i)\|_\infty &= \sup_{\beta\in\mathbb{Q}_+^*} \|1_{Y_\Delta}\beta R_\beta(g_i - u_i)\|_\infty \\
&\leq \|1_{Y_\Delta}(g_i - u_i)\|_\infty < +\infty.
\end{aligned}$$

So, we proved that $\{g_n \mid n \in \mathbb{N}\}$ is contained in D, where D is defined as in Theorem 2.12. Every $u \in J$ is ρ-uniformly continuous on Y_Δ and hence has a unique ρ-continuous extension $\bar{u}$ to $\bar{E}$. Set $\overline{J} := \{\bar{u} \in \mathcal{C}_b(\bar{E}) \mid \bar{u}_{|Y_\Delta} \in J\}$. Consider for $u \in J$

$$R_1\alpha(u - \alpha R_{\alpha+1}u)(x) = \alpha R_{\alpha+1}u(x) \uparrow u(x) \quad \forall\, x \in Y_\Delta.$$

The extension of $\alpha R_{\alpha+1}u$ to $\bar{E}$ is pointwise increasing in α on $\bar{E}$. By the Dini theorem $\alpha R_{\alpha+1}u \uparrow v$ uniformly on $\bar{E}$ for some $v \in \mathcal{C}_b(\bar{E})$. By uniqueness of the extension we obtain $v = \bar{u}$. It follows that $\overline{J}$ is contained in $\overline{D}^{\|\cdot\|_\infty}$. Since $\overline{D}^{\|\cdot\|_\infty}$ is a linear space, we have $\overline{J} - \overline{J} \subset \overline{D}^{\|\cdot\|_\infty}$. Furthermore, the set $\overline{J} - \overline{J}$ contains the constant functions, is inf-stable and separates the points of $\bar{E}$. Hence, by the Stone-Weierstraß theorem we obtain that $\overline{J} - \overline{J}$ is dense in $\mathcal{C}_b(\bar{E})$. Since

$$\mathcal{C}_b(\bar{E}) = \overline{\overline{J} - \overline{J}}^{\|\cdot\|_\infty} \subset \overline{D}^{\|\cdot\|_\infty} \subset \mathcal{C}_b(\bar{E}),$$

it follows that $\mathcal{C}_b(\bar{E}) = \overline{D}^{\|\cdot\|_\infty}$. By Theorem 2.12 we have that $\{f \circ X^\beta \mid \beta \in \mathbb{Q}_+^*\}$ is relatively compact for all $f \in \mathcal{C}_b(\bar{E})$. And by Theorem 2.11 this is equivalent with the relative compactness of $\{X^\beta \mid \beta \in \mathbb{Q}_+^*\}$. In particular, $\{P_x^\beta \mid \beta \in \mathbb{Q}_+^*\}$ is relatively compact for all $x \in \bar{E}$. $\square$

Chapter 3

The Key Theorem

This chapter is devoted to the proof of Theorem 3.1 (the key theorem). In particular, we will make use of the potential theory developed in chapter 1. Let R_α, Y be as in Lemma 1.17 and $\mathbb{M}^\beta := (\Omega_{E_\Delta},\ (X_t)_{t\geq 0},\ (\mathcal{F}_t)_{t\geq 0},\ (P_x^\beta)_{x\in E_\Delta})$ be the Hunt process from Proposition 2.7. Furthermore, let $(\mathcal{E}, \mathcal{F})$ be a strictly quasi-regular generalized Dirichlet form satisfying SD3. In particular, we follow [MRZ98, Section 3]. All the lemmas there remain true in the case of generalized Dirichlet forms.

For a Borel subset $S \subset Y$, we shall write S_Δ for $S \cup \{\Delta\}$. The topology on S_Δ is, except otherwise stated, the one induced by the metric ρ.

Theorem 3.1. *There exists a Borel subset $Z \subset Y$ and a Borel subset $\Omega \subset \Omega_{\bar{E}}$ with the following properties:*

(i) $E \setminus Z$ is strictly $\mathcal{E}$-exceptional.

(ii) $R_\alpha(x, \bar{E} \setminus Z_\Delta) = 0,\ \forall\ x \in Z_\Delta,\ \alpha \in \mathbb{Q}_+^*$.

(iii) If $\omega \in \Omega$, then $\omega_t,\ \omega_{t-} \in Z_\Delta$ for all $t \geq 0$. Moreover, each $\omega \in \Omega$ is cadlag in the original topology of Y_Δ and $\omega^0_{t-} = \omega_{t-}$ for all $t > 0$, where ω^0_{t-} denotes the left limit in the original topology.

(iv) If $x \in Z_\Delta$ and P_x is a weak limit of some sequence $(P_x^{\beta_j})_{j\in\mathbb{N}}$ with $\beta_j \in \mathbb{Q}_+^*,\ \beta_j \uparrow \infty$, then $P_x[\Omega] = 1$.

We explain why this theorem is crucial. This theorem provides two Borel sets Z and Ω with the property that all paths from Ω take their values and left-limits in Z_Δ. Z and Ω are big enough, in the sense that $E \setminus Z$ is strictly $\mathcal{E}$-exceptional and $P_x[\Omega] = 1$. Restricting our process to Ω we get a Hunt process, as we will see in chapter 4. The proof of the key theorem will be accomplished through several lemmas, which are contained in the next section.

3.1 Construction of Excessive Functions

In this section we construct a Borel set S and a family of 2-excessive functions, such that these functions have nice properties on S. The details are contained in the next lemmas.

For our purposes we will use a description of strictly $\mathcal{E}$-exceptional sets by μ-zero sets, where μ is taken from a special class of measures. For details we refer to [Tru05].

For an arbitrary subset $D \in \mathcal{H}$ let $\hat{\mathcal{P}}_{1,D}$ denote the set of all 1-coexcessive functions in $\mathcal{V}$, which are dominated by some function in D. Furthermore, let $\tilde{\mathcal{P}}^{str}_{1,\mathcal{F}}$ denote the set of all s.$\mathcal{E}$-q.c. m-versions of 1-excessive elements in $\mathcal{V}$ which are dominated by elements in $\mathcal{F}$.

Theorem 3.2. *Let $\hat{u} \in \hat{\mathcal{P}}_{1,\hat{\mathcal{F}}}$. Then there exists a unique σ-finite and positive measure $\mu^{str}_{\hat{u}}$ on $(E, \mathcal{B}(E))$ charging no strictly $\mathcal{E}$-exceptional sets, such that*

$$\int_E \tilde{f}\, d\mu^{str}_{\hat{u}} = \lim_{\alpha\to\infty} \mathcal{E}_1(f, \alpha \hat{G}_{\alpha+1}\hat{u}) \quad \forall\ \tilde{f} \in \tilde{\mathcal{P}}^{str}_{1,\mathcal{F}} - \tilde{\mathcal{P}}^{str}_{1,\mathcal{F}}.$$

Proof. See [Tru05, Thm. 4]. □

According to the notation in Theorem 3.2 we introduce the following class of measures

$$\hat{S}^{str}_{00} := \{\mu^{str}_{\hat{u}} \mid \hat{u} \in \hat{\mathcal{P}}_{1,\hat{G}_1\mathcal{H}^+_b} \text{ and } \mu^{str}_{\hat{u}}(E) < \infty\},$$

where $\hat{G}_1\mathcal{H}^+_b := \{\hat{G}_1 h \mid h \in \mathcal{H}^+_b\}$ and $\mathcal{H}^+_b$ denotes the set of all positive and bounded elements in $\mathcal{H}$.

Theorem 3.3. *For $B \in \mathcal{B}(E)$ the following statements are equivalent:*

1. *B is strictly $\mathcal{E}$-exceptional.*

2. *$\mu(B) = 0$ for all $\mu \in \hat{S}^{str}_{00}$.*

Proof. See [Tru05, Thm. 5]. □

Recall that

$$Cap_{1,g}(U) = \int e_U \varphi \, dm,$$

where $e_U \in L^\infty(E; m)$. If u is in $\mathcal{P}_1$ then $\bar{u}$ will denote the fixed s.$\mathcal{E}$-q.l.s.c m-version (regularization) defined by

$$\bar{u} := \sup_{n \in \mathbb{N}} n\tilde{R}_{n+1} u.$$

Lemma 3.4. *Let $U_n \subset E$, $n \geq 1$ be a decreasing sequence of open sets. If $Cap_{1,g}(U_n) \to 0$, as $n \to \infty$, then we can find m-versions e_n of e_{U_n} such that*

(i) $e_n \geq 1$ strictly $\mathcal{E}$-q.e. on U_n, $n \geq 1$.

(ii) $\alpha\tilde{R}_{\alpha+1}(e_n) \leq e_n$ strictly $\mathcal{E}$-q.e. for $\alpha \in \mathbb{Q}_+^*$, $n \geq 1$.

(iii) $e_n \downarrow 0$ strictly $\mathcal{E}$-q.e. as $n \to \infty$.

Proof. Let $(U_n)_{n\geq 1}$ be the decreasing sequence of open sets in E such that $Cap_{1,g}(U_n) \to 0$. Define

$$e_n := \lim_{k\to\infty} \overline{(kG_1\varphi \wedge 1)_{U_n}}.$$

We have $\overline{(kG_1\varphi \wedge 1)_{U_n}} \geq \overline{kG_1\varphi \wedge 1} = k\tilde{R}_1\varphi \wedge 1$ $m-$a.e. on U_n. By Lemma A.4 we obtain

$$\overline{(kG_1\varphi \wedge 1)_{U_n}} \geq k\tilde{R}_1\varphi \wedge 1 \quad s.\mathcal{E} - q.e. \text{ on } U_n.$$

Note that $\tilde{R}_1\varphi > 0$ s.$\mathcal{E}$-q.e. Hence, by letting $k \to \infty$ we obtain (i).

We have, since $\tilde{R}_{\alpha+1}$ is kernel,

$$\begin{aligned} \alpha\tilde{R}_{\alpha+1}(e_n) &= \alpha\tilde{R}_{\alpha+1}\left(\lim_k \overline{(kG_1\varphi \wedge 1)_{U_n}}\right) \\ &= \lim_k \alpha\tilde{R}_{\alpha+1}\left(\overline{(kG_1\varphi \wedge 1)_{U_n}}\right) \\ &\leq \lim_k \overline{(kG_1\varphi \wedge 1)_{U_n}} \\ &= e_n \quad s.\mathcal{E} - q.e. \end{aligned}$$

This proves (ii). By assumption we have $Cap_{1,g}(U_n) \downarrow 0$ as $n \to \infty$. So it follows that $e_{n_k}\varphi \to 0$ m-a.e. for a subsequence $(n_k)_{k\geq 1}$. But by monotonicity we get $e_n\varphi \to 0$ m-a.e. and since $\varphi > 0$ we obtain $e_n \to 0$ m-a.e. Next, applying Theorem 3.2 we have for $\mu_{\hat{u}} \in \hat{S}_{00}^{str}$ and $h \in \mathcal{H}_b^+$ such that $\hat{u} \le \hat{G}_1 h$

$$\begin{aligned}
\int e_n \wedge \tilde{R}_1\varphi \, d\mu_{\hat{u}} &= \lim_{\alpha\to\infty} \int \alpha \tilde{R}_{\alpha+1}(e_n \wedge \tilde{R}_1\varphi)\, d\mu_{\hat{u}} \\
&= \lim_{\alpha\to\infty} \lim_{\beta\to\infty} \mathcal{E}_1(\alpha G_{\alpha+1}(e_n \wedge \tilde{R}_1\varphi), \beta \hat{G}_{\beta+1}\hat{u}) \\
&\le \lim_{\alpha\to\infty} \mathcal{E}_1(\alpha G_{\alpha+1}(e_n \wedge \tilde{R}_1\varphi), \hat{G}_1 h) \\
&= \lim_{\alpha\to\infty} (\alpha G_{\alpha+1}(e_n \wedge \tilde{R}_1\varphi), h)_{\mathcal{H}} \\
&= \int (e_n \wedge \tilde{R}_1\varphi) h \, dm \to_{n\to\infty} 0,
\end{aligned}$$

since $e_n \to 0$ m-a.e. So it follows that $e_{n_k} \wedge \tilde{R}_1\varphi \to 0$ $\mu_{\hat{u}}$-a.e. and again by monotonicity we get $e_n \wedge \tilde{R}_1\varphi \to 0$ $\mu_{\hat{u}}$-a.e. Since $\tilde{R}_1\varphi > 0$ we obtain that $e_n \to 0$ $\mu_{\hat{u}}$-a.e. Define now $N := \{x \mid e_n(x) \nrightarrow 0\}$. Then we have $\mu_{\hat{u}}(N) = 0$ for all $\mu_{\hat{u}} \in \hat{S}_{00}^{str}$ and by Theorem 3.3 this is equivalent with $e_n \to 0$ s.$\mathcal{E}$-q.e. □

Lemma 3.5. *In the situation of Lemma 3.4 there exists $S \in \mathcal{B}(E)$, $S \subset Y$ such that $E \setminus S$ is strictly $\mathcal{E}$-exceptional and the following holds:*

(i) $\tilde{R}_\alpha(x, Y \setminus S) = 0\ \forall x \in S,\ \alpha \in \mathbb{Q}_+^*$.

(ii) $e_n(x) \ge 1$ *for* $x \in S \cap U_n,\ n \ge 1$.

(iii) $\alpha \tilde{R}_{\alpha+1}(e_n)(x) \le e_n(x),\ \forall x \in S,\ \alpha \in \mathbb{Q}_+^*,\ n \ge 1$.

(iv) $e_n \downarrow 0,\ \forall x \in S$.

Proof. By Lemma 3.4, there exists a Borel set $S_1 \subset Y$ such that assertions (ii)-(iv) hold pointwise on S_1 and $Y \setminus S_1$ is strictly $\mathcal{E}$-exceptional. Thus by Proposition A.4 we can find a Borel set $S_2 \subset S_1$ such that

$$\tilde{R}_\alpha(x, Y \setminus S_1) = 0\ \forall x \in S_2,\ \alpha \in \mathbb{Q}_+^*$$

and $E \setminus S_2$ is strictly $\mathcal{E}$-exceptional. Repeating this argument, we get a decreasing sequence $(S_n)_{n\ge1}$ of Borel sets such that $E \setminus S_n$ is strictly $\mathcal{E}$-exceptional and $\tilde{R}_\alpha(x, Y \setminus S_n) = 0\ \forall x \in S_{n+1},\ \alpha \in \mathbb{Q}_+^*$. Clearly, $S := \bigcap_{n\ge1} S_n$ is strictly $\mathcal{E}$-exceptional and

$$\tilde{R}_\alpha(x, Y \setminus S) = 0 \quad \forall\, x \in S,\ \alpha \in \mathbb{Q}_+^*.$$

□

Lemma 3.6. *Let $S \in \mathcal{B}(E)$ such that Lemma 3.5 (i) holds. Then*

$$P_x^\beta[X_t \in S_\Delta,\ X_{t-} \in S_\Delta\ \forall\ t \geq 0] = 1\ \forall\ x \in S_\Delta.$$

Proof. Lemma 3.5 (i) implies that

$$(\beta R_\beta)^n(x, \bar{E} \setminus S_\Delta) = 0,\ \forall x \in S_\Delta,\ \beta \in \mathbb{Q}_+^*,\ n \geq 1.$$

Therefore, if $Y^\beta(k),\ k = 1, 2, \ldots$ is a Markov chain starting from some $x \in S_\Delta$ with transition function βR_β, then

$$P[Y^\beta(k) \in \bar{E} \setminus S_\Delta \text{ for some } k] = 0.$$

Clearly, this implies

$$P[Y^\beta(\Pi_t^\beta) \in \bar{E} \setminus S_\Delta \text{ for some } t \geq 0] = 0$$

and

$$P[Y^\beta(\Pi_{t-}^\beta) \in \bar{E} \setminus S_\Delta \text{ for some } t \geq 0] = 0,$$

since (Π_t^β) is càdlàg. Hence, the assertion follows. □

Lemma 3.7. *Let $\beta \in \mathbb{Q}_+^*,\ \beta \geq 2,\ n \geq 1$. Then e_n is a (P_t^β)-2-excessive function on S_Δ, i.e.*

$$e^{-2t} P_t^\beta e_n(x) \leq e_n(x) \quad \textit{and}$$
$$\lim_{t \to 0} e^{-2t} P_t^\beta e_n(x) = e_n(x)\ \forall\ x \in S_\Delta.$$

Proof. We have by Lemma 3.5 (i)and (iii) $((\beta - 1)R_\beta)^k(e_n)(x) \leq e_n(x)\ \forall\ x \in S_\Delta$. Hence $\forall\ x \in S_\Delta$

$$\begin{aligned}
P_t^\beta(e_n)(x) &= e^{-\beta t} \sum_{k=0}^{\infty} \frac{(\beta t)^k}{k!} (\beta R_\beta)^k e_n(x) \\
&= e^{-\beta t} \sum_{k=0}^{\infty} \frac{(\beta t)^k}{k!} \left(\frac{\beta}{\beta - 1}\right)^k ((\beta - 1)R_\beta)^k e_n(x) \\
&\leq e^{-\beta t} \sum_{k=0}^{\infty} \frac{\left(\frac{\beta^2}{\beta - 1} t\right)^k}{k!} e_n(x) \\
&= e^{\left(1 + (\frac{1}{\beta - 1})t\right)} e_n(x) \leq e^{2t} e_n(x),
\end{aligned}$$

since $\beta \geq 2$. This gives

$$e^{-2t} P_t^\beta e_n(x) \leq e_n(x),\ \forall\ x \in S_\Delta.$$

But

$$\lim_{t\to 0} e^{-2t} P_t^\beta f(x) = f(x)$$

holds $\forall\ x \in \bar{E}$ and $f \in \mathcal{B}_b(\bar{E})$. Hence, e_n is a (P_t^β)-2-excessive function. $\square$

Define for $n \in \mathbb{N}$ the stopping time

$$\tau_n := \inf\{t \geq 0 \mid X_t \in U_n\},$$

called the first entry time of (X_t) in U_n.

Lemma 3.8. *Let $\beta \in \mathbb{Q}_+^*,\ \beta \geq 2$ and $\mathbb{M}^\beta := (\Omega_{\bar{E}},\ (X_t)_{t\geq 0},\ (P_x^\beta)_{x\in\bar{E}})$ be the canonical realization of the Markov process (X_t^β). Then*

$$E_x^\beta[e^{-2\tau_n}] \leq e_n(x),\ \forall\ x \in S_\Delta.$$

Proof. Since by Lemma 3.6 S_Δ is invariant set of $\mathbb{M}^\beta$, the resriction $\mathbb{M}^\beta_{S_\Delta}$ of $\mathbb{M}^\beta$ to S_Δ is still a Hunt process. We first prove that $(e^{-2t}e_n(X_t))_{t\geq 0}$ is an $(\mathcal{F}_t)$-supermartingale. By the Markov property we have for $s \leq t$

$$\begin{aligned} E_x^\beta[e^{-2t}e_n(X_t) \mid \mathcal{F}_s] &= E_{X_s}^\beta[e^{-2t}e_n(X_{t-s})] \\ &= e^{-2t}P_{t-s}^\beta e_n(X_s) = e^{-2s}e^{-2(t-s)}P_{t-s}^\beta e_n(X_s) \\ &\leq e^{-2s}e_n(X_s), \end{aligned}$$

where the last inequality follows from Lemma 3.7. So by the optional sampling theorem we have

$$E_x^\beta[e^{-2\tau_n}e_n(X_{\tau_n})] \leq E_x^\beta[e^{-2\cdot 0}e_n(X_0)] = e_n(x),\ x \in S_\Delta.$$

By Lemma 3.5 we have that $e_n(x) \geq 1$ for all $x \in U_n$. In view of Lemma 3.7 we may apply [FOT94, Thm. A.2.5] and obtain

$$P_x^\beta[t \mapsto e_n(X_t) \text{ is right continuous}] = 1 \quad \forall\ x \in S_\Delta.$$

Hence, for all $x \in S_\Delta$ we have $e_n(X_{\tau_n}) \geq 1$ P_x^β-a.s. It follows that for all $x \in S_\Delta$

$$e^{-2\tau_n} \leq e^{-2\tau_n}e_n(X_{\tau_n}) \quad P_x^\beta - a.s$$

Hence,

$$E_x^\beta[e^{-2\tau_n}] \leq E_x^\beta[e^{-2\tau_n}e_n(X_{\tau_n})] \leq e_n(x) \quad \forall\ x \in S_\Delta.$$

$\square$

Now we are well prepared for the proof of Theorem 3.1.

3.2 Proof of the Key Theorem

Proof. Take a strict $\mathcal{E}$-nest $(F_k^{(1)})_{k\in\mathbb{N}}$ such that $J_0 \in \mathcal{C}_\infty(\{F_k^{(1)}\})$, $F_k^{(1)} \cup \{\Delta\}$ is compact and

$$\bigcup_{k\geq 1} F_k^{(1)} \subset Y.$$

Let $U_k := E \setminus F_k^{(1)}$ and $\tau_k := \inf\{t \geq 0 \mid X_t \in U_k\}$. $(U_k)_{k\in\mathbb{N}}$ is a decreasing sequence of open sets and

$$Cap_{1,g}(U_k) \longrightarrow 0 \; as \; k \to \infty.$$

Hence the assumptions in Lemma 3.5 are satisfied and we can find a subset $S^{(1)} \in \mathcal{B}(E)$ satisfying Lemma 3.5 (i)-(iv). Without loss of generality we can assume that $S^{(1)} \subset \bigcup_{k\geq 1} F_k^{(1)}$. Fix any $T > 0$, $\beta \in \mathbb{Q}_+^*$, $\beta \geq 2$, $k \in \mathbb{N}$ and $x \in S_\Delta^{(1)}$. By Lemma 3.8 we have

$$P_x^\beta[\tau_k < T] = E_x^\beta[1_{\{\tau_k < T\}}] = E_x^\beta[e^{-2\tau_k}]e^{2T} \leq e^{2T} e_k(x).$$

Now, consider the canonical projection $\Pi : \Omega_{\bar{E}} \times [0, T) \to \bar{E}$. Clearly, Π is continuous and therefore

$$B_k^T := \{\omega \in \Omega_{\bar{E}} \mid \omega_t \in F_k^{(1)} \cup \{\Delta\}, \forall\, t < T\}$$

is a closed subspace of $\Omega_{\bar{E}}$, since $B_k^T = \Pi^{-1}(F_k^{(1)} \cup \{\Delta\})$ and since the trace topology of $\bar{E}$ on $F_k^{(1)} \cup \{\Delta\}$ is the same as the original one. Thus, if P_x is a weak limit of some sequence $(P_x^{\beta_j})_{j\in\mathbb{N}}$, $\beta_j \uparrow \infty$, $\beta_j \in \mathbb{Q}_+^*$, and if $(f_n)_{n\in\mathbb{N}} \in \mathcal{C}_b(\bar{E})$ is a positive monotone sequence such that $f_n \downarrow 1_{B_k^T}$, then

$$\begin{aligned}
P_x[B_k^T] &= E_x[1_{B_k^T}] = E_x\left[\lim_{n\to\infty} f_n\right] \\
&= \lim_n E_x[f_n] = \lim_n \lim_{j\to\infty} \underbrace{E_x^{\beta_j}[f_n]}_{\geq E_x^{\beta_j}[1_{B_k^T}]} \\
&\geq \limsup_{j\to\infty} E_x^{\beta_j}[1_{B_k^T}] \geq \limsup_{j\to\infty} P_x^{\beta_j}[\tau_k \geq T] \\
&= \limsup_{j\to\infty} (1 - P_x^{\beta_j}[\tau_k < T]) \geq 1 - e^{2T} e_k(x)
\end{aligned}$$

But by Lemma 3.5 (iv) it follows that

$$P_x\Big[\bigcup_{k\geq 1} B_k^T\Big] = \lim_{k\to\infty} P_x[B_k^T] \geq \limsup_{k\to\infty}(1 - e^{2T}e_k(x)) = 1.$$

Let $\Omega_1 := \bigcap_{N\geq 1}\bigcup_{k\geq 1} B_k^N$. Then $P_x[\Omega_1] = 1$ for $x \in S^{(1)}$ and Ω_1 satisfies Theorem 3.1 (iii) with Z_Δ replaced by $\bigcup_{k\geq 1} F_k^{(1)}$.

Now take another strict $\mathcal{E}$-nest $(F_k^{(2)})_{k\geq 1}$ such that $F_k^{(2)} \subset F_k^{(1)}\ \forall\ k$, and

$$\bigcup_{k\geq 1} F_k^{(2)} \subset S^{(1)}.$$

Repeating the above argument we get $S^{(2)} \subset \bigcup_{k\geq 1} F_k^{(2)}$ and $\Omega_2 \subset \Omega_1$, satisfying the same property as above.

Repeating the procedure we obtain the following: strict $\mathcal{E}$-nests $(F_k^{(n)})_{k\geq 1}$, Borel sets $S^{(n)} \subset E$ such that Theorem key (ii) holds with Z_Δ replaced by $S_\Delta^{(n)}$ and

$$Y \supset \bigcup_{k\geq 1} F_k^{(1)} \supset S^{(1)} \supset \ldots \supset \bigcup_{k\geq 1} F_k^{(n)} \supset S^{(n)} \supset \ldots,$$

and finally Borel sets $\Omega_n \subset D_{\bar{E}}[0,\infty)$ such that

$$D_{\bar{E}}[0,\infty) \supset \Omega_1 \supset \Omega_2 \supset \ldots \Omega_n \supset \ldots.$$

Ω_n satisfies (iii) with Z_Δ replaced by $\bigcup_{k\geq 1} F_k^{(n)} \cup \{\Delta\}$, satisfies (iv) with Z_Δ replaced by $S_\Delta^{(n)}$. We now define $\Omega := \bigcap_{n\geq 1}\Omega_n,\ Z := \bigcap_{n\geq 1} S^{(n)}$. Then Z and Ω satisfy (i)-(iv). $\square$

Chapter 4

Hunt Processes Associated with $(\mathcal{E}, \mathcal{F})$

Again, let $(\mathcal{E}, \mathcal{F})$ be a strictly quasi-regular generalized Dirichlet form satisfying SD3. The notation is the same as in the previous chapters. This chapter contains the main result. Let Ω, Z_Δ and $(P_x)_{x \in Z_\Delta}$ be as in Theorem 3.1. Consider a process $\mathbb{M} := (\Omega, (X_t)_{t \geq 0}, (P_x)_{x \in Z_\Delta})$. We will prove that this process is associated to $(\mathcal{E}, \mathcal{F})$ and unique. Furthermore, we will prove that $\mathbb{M}$ is a Hunt process. We follow, as before, [MRZ98, Section 4].

Lemma 4.1. *Define for* $\alpha, \beta \ \in \mathbb{Q}_+^*$

$$R_\alpha^\beta f(x) := E_x^\beta \left[\int_0^\infty e^{-\alpha t} f(X_t) \, dt \right], \quad f \in \mathcal{B}_b(\bar{E}), \ x \in \bar{E}.$$

Then

$$R_\alpha^\beta f = \left(\frac{\beta}{\alpha + \beta} \right)^2 R_{\frac{\alpha\beta}{\alpha+\beta}} f + \frac{1}{\alpha + \beta} f. \tag{4.1}$$

Proof. Denote by $A_\alpha^\beta f$ the r.h.s of (4.1). Then we just start to calculate and use $L^\beta f = \beta(\beta R_\beta f - f)$ and the resolvent equation.

$$\begin{aligned}
(A_\alpha^\beta(\alpha - L^\beta))f &= \left(\frac{\beta}{\alpha+\beta}\right)^2 \alpha R_{\frac{\alpha\beta}{\alpha+\beta}} f - \left(\frac{\beta}{\alpha+\beta}\right)^2 \beta^2 R_{\frac{\alpha\beta}{\alpha+\beta}} R_\beta f \\
&\quad + \left(\frac{\beta}{\alpha+\beta}\right)^2 \beta R_{\frac{\alpha\beta}{\alpha+\beta}} f + \frac{\alpha}{\alpha+\beta} f \\
&\quad - \frac{\beta^2}{\alpha+\beta} R_\beta f + \frac{\beta}{\alpha+\beta} f \\
&= \frac{\beta^2}{\alpha+\beta} \underbrace{\left[\frac{\alpha}{\alpha+\beta} R_{\frac{\alpha\beta}{\alpha+\beta}} f + \frac{\beta}{\alpha+\beta} R_{\frac{\alpha\beta}{\alpha+\beta}} f\right]}_{=R_{\frac{\alpha\beta}{\alpha+\beta}} f} - \frac{\beta^2}{\alpha+\beta} R_{\frac{\alpha\beta}{\alpha+\beta}} f \\
&\quad - \frac{\beta^2}{\alpha+\beta} R_\beta f + \frac{\beta^2}{\alpha+\beta} R_\beta f + f \\
&= f.
\end{aligned}$$

In the same way we obtain

$$((\alpha - L^\beta)A_\alpha^\beta)f = f.$$

Hence, we obtain that $R_\alpha^\beta f = (\alpha - L^\beta)^{-1} f = A_\alpha^\beta f$ for $f \in \mathcal{B}_b(\bar{E})$. □

Lemma 4.2. *Let $x \in \bar{E}$ and let P_x be a weak limit of a subsequence $(P_x^{\beta_j})_{j\geq 1}$ with $\beta_j \uparrow \infty$, $\beta_j \in \mathbb{Q}_+^*$. Define the kernel*

$$P_t f(x) := E_x[f(X_t)] \quad \forall\ f \in \mathcal{B}_b(\bar{E}).$$

Then

$$\int_0^\infty e^{-\alpha t} P_t f(x)\, dt = R_\alpha f(x), \quad \forall\ f \in \mathcal{B}_b(\bar{E}),\ \alpha \in \mathbb{Q}_+^*. \tag{4.2}$$

In particular, the kernels P_t, $t \geq 0$, are independent of the subsequence $(P_x^{\beta_j})_{j\geq 1}$.

Proof. Since $P_x^{\beta_j} \to P_x$ weakly in $\Omega_{\bar{E}}$, we have by [EK86, III.7.8] and Lebesgue's dominated convergence theorem

$$\begin{aligned}
E_x^{\beta_j}\left[\int_0^\infty e^{-\alpha t} f(X_t)\, dt\right] &= \int_0^\infty e^{-\alpha t} E_x^{\beta_j}[f(X_t)]\, dt \\
&= \int_0^\infty e^{-\alpha t} \underbrace{E_x^{\beta_j}[f(X_t)]}_{\to E_x[f(X_t)]}\, dt \\
&\underset{j\to\infty}{\longrightarrow} \int_0^\infty e^{-\alpha t} P_t f(x)\, dt
\end{aligned}$$

for all $f \in \mathcal{C}_b(\bar{E})$ and $\alpha > 0$. But by Lemma 4.1 we have

$$
\begin{aligned}
E_x^{\beta_j}\left[\int_0^\infty e^{-\alpha t} f(X_t)\, dt\right] &= R_\alpha^{\beta_j} f(x) \\
&= \left(\frac{\beta_j}{\alpha+\beta_j}\right)^2 R_{\frac{\alpha\beta_j}{\alpha+\beta_j}} f(x) + \frac{1}{\alpha+\beta_j} f(x) \\
&\underset{j\to\infty}{\longrightarrow} R_\alpha f(x) \qquad (4.3)
\end{aligned}
$$

for all $f \in \mathcal{B}_b(\bar{E})$, $x \in \bar{E}$. The convergence follows from the resolvent equation:

$$\left(\frac{\beta_j}{\alpha+\beta_j}\right)^2 R_{\frac{\alpha\beta_j}{\alpha+\beta_j}} f = \underbrace{\frac{\alpha^2\beta_j^2}{(\alpha+\beta_j)^3} R_{\frac{\alpha\beta_j}{\alpha+\beta_j}}}_{=:A^{\beta_j}} R_\alpha f + R_\alpha f \underset{j\to\infty}{\longrightarrow} R_\alpha f,$$

since

$$A^{\beta_j} f = \underbrace{\frac{\alpha\beta_j}{(\alpha+\beta_j)^2}}_{\to 0} \underbrace{\frac{\alpha\beta_j}{\alpha+\beta_j} R_{\frac{\alpha\beta_j}{\alpha+\beta_j}}}_{contraction} f \longrightarrow 0 \ as\ \beta_j \to \infty.$$

Hence the assertion holds for all $f \in \mathcal{C}_b(\bar{E})$. Define

$$V := \left\{ f \in \mathcal{B}_b(\bar{E}) \middle| \int_0^\infty e^{-\alpha t} P_t f(x)\, dt = R_\alpha f(x)\ \forall\ x \in \bar{E} \right\}.$$

Then V is a monotone vector space and we proved that V contains $\mathcal{C}_b(\bar{E})$. Hence, by monotone class theorem, it follows that V contains all bounded, $\sigma(\mathcal{C}_b(\bar{E}))$-measurable functions. But $\sigma(\mathcal{C}_b(\bar{E})) = \mathcal{B}_b(\bar{E})$. So the assertion holds for all $f \in \mathcal{B}_b(\bar{E})$. The last statement follows by the right continuity of $P_t f(x)$ in t for $f \in \mathcal{C}_b(\bar{E})$ and the uniqueness of the Laplace transform. □

Theorem 4.3. *For every $x \in Z_\Delta$ the relatively compact set $\{P_x^\beta \mid \beta \in \mathbb{Q}_+^*\}$ has a unique limit P_x for $\beta \uparrow \infty$. The process $(\Omega_{\bar{E}}, (X_t)_{t\geq 0}, (P_x)_{x\in Z_\Delta})$ is a Markov process with the transition semigroup $(P_t)_{t\geq 0}$ determined by (4.2). Moreover,*

$$P_x[X_t \in Z_\Delta, X_{t-} \in Z_\Delta \ for\ all\ t \geq 0] = 1$$

for all $x \in Z_\Delta$.

Proof. The last assertion follows from Theorem 3.1. By the previous Lemma P_x is unique, since it is independent of the chosen subsequence. So we only prove that $(\Omega_{\bar{E}}, (X_t)_{t\geq 0}, (P_x)_{x\in Z_\Delta})$ is a Markov process. For that we have to prove

$$E_x[f_1(X_{t_1}) \ldots f_n(X_{t_1+\ldots t_n})] = P_{t_1}[f_1 P_{t_2}[f_2 \ldots P_{t_n}[f_n] \ldots](x) \qquad (4.4)$$

for any $n \geq 1,\ t_1, \dots t_n \geq 0$ and $f_1, \dots f_n \in \mathcal{B}_b(Z_\Delta)$, which is equivalent for $(X_t)_{t\geq 0}$ to be a Markov process. The equation (4.4) follows by induction from

$$E_x[f_1(X_{t_1}) \dots f_n(X_{t_1+\dots t_n})] \\ = E_x[f_1(X_{t_1}) \dots f_{n-1}(X_{t_1+\dots t_{n-1}}) P_{t_n}(f_n)(X_{t_1+\dots t_{n-1}})]. \quad (4.5)$$

Assume first that $f_1, \dots f_{n-1} \in \mathcal{C}_b(Z_\Delta)$ and $f_n = R_\alpha R_1 u$ for some $\alpha \in \mathbb{Q}_+^*,\ \alpha > 1,\ u \in J$. In this case

$$P_t^\beta f_n = e^{-\beta t} \sum_{k=0}^{\infty} \frac{(\beta t)^k}{k!} (\beta R_\beta)^k R_\alpha R_1 u \quad \in \mathcal{C}_b(Z_\Delta \times [0, T])$$

for $\beta \in \mathbb{Q}_+^*$ and any $T > 0$. For $\beta_1, \beta_2 \in \mathbb{Q}_+^*$ we have

$$\begin{aligned} P_t^{\beta_1} f_n - P_t^{\beta_2} f_n &= \int_0^t \frac{d}{ds}(P_s^{\beta_1} P_{t-s}^{\beta_2} f_n)\, ds \\ &= \int_0^t P_s^{\beta_1}(P_{t-s}^{\beta_2})(L^{\beta_1} - L^{\beta_2}) f_n\, ds. \end{aligned} \quad (4.6)$$

By a calculation we get

$$(L^{\beta_1} - L^{\beta_2}) f_n = R_{\beta_1} w - R_{\beta_2} w, \quad (4.7)$$

where $w := R_1(\alpha R_\alpha u - u) - (\alpha R_\alpha u - u)$. By (4.6) and (4.7) it follows that

$$\begin{aligned} \sup_{t\leq T} \sup_x |P_t^{\beta_1} f_n(x) - P_t^{\beta_2} f_n(x)| &\leq \sup_{t\leq T} \sup_x \int_0^t \underbrace{|P_s^{\beta_1} P_{t-s}^{\beta_2}(L^{\beta_1} - L^{\beta_2}) f_n(x)|}_{\leq \sup_x |L^{\beta_1} f_n(x) - L^{\beta_2} f_n(x)|}\, ds \\ &\leq T \sup_x |L^{\beta_1} f_n(x) - L^{\beta_2} f_n(x)| \\ &= T \sup_x |R_{\beta_1} w(x) - R_{\beta_2} w(x)| \\ &\leq T \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \sup_x |w(x)| \end{aligned}$$

From this and (4.3) we have

$$\sup_{t\leq T} \sup_x |P_t^\beta f_n(x) - P_t f_n(x)| \underset{\beta\to\infty}{\longrightarrow} 0. \quad (4.8)$$

In particular $P_t f_n(x)$ is jointly continuous in (t, x).
Set $\psi(t_1, \dots, t_n) := e^{-\alpha_1 t_1 - \dots - \alpha_n t_n} f_1(X_{t_1}) \dots f_{n-1}(X_{t_1+\dots+t_{n-1}})$. Since $P_x^{\beta_j} \to$

P_x weakly by [EK86, III.7.8] and Lebesgue's dominated convergence theorem we have

$$\begin{aligned}
&\lim_{j\to\infty} E_x^{\beta_j}\left[\int_0^\infty \ldots \int_0^\infty \psi(t_1,\ldots,t_n)P_{t_n}(f_n)(X_{t_1+\ldots+t_{n-1}})\,dt_1\ldots dt_n\right] \\
= &\lim_j E_x^{\beta_j}\left[\int_0^\infty \ldots \int_0^\infty \psi(t_1,\ldots,t_n)P_{t_n}(f_n)(X_{t_1+\ldots+t_{n-1}})\,dt_1\ldots dt_n\right] \\
= &\lim_j E_x^{\beta_j}\left[\int_0^\infty \ldots \int_0^\infty \psi(t_1,\ldots,t_n)P_{t_n}(f_n)(X_{t_1+\ldots+t_{n-1}})\,dt_1\ldots dt_n\right] \\
= &\lim_j E_x^{\beta_j}\left[\int_0^\infty \ldots \int_0^\infty \psi(t_1,\ldots,t_n)P_{t_n}(f_n)(X_{t_1+\ldots+t_{n-1}})\,dt_1\ldots dt_n\right] \\
= &E_x\left[\int_0^\infty \ldots \int_0^\infty \psi(t_1,\ldots,t_n)P_{t_n}(f_n)(X_{t_1+\ldots+t_{n-1}})\,dt_1\ldots dt_n\right] \\
= &E_x\left[\int_0^\infty \ldots \int_0^\infty \psi(t_1,\ldots,t_n)P_{t_n}(f_n)(X_{t_1+\ldots+t_{n-1}})\,dt_1\ldots dt_n\right].
\end{aligned}$$

From here and (4.8) it follows that

$$\begin{aligned}
&\lim_j E_x^{\beta_j}\left[\int_0^\infty \ldots \int_0^\infty \psi(t_1,\ldots,t_n)P_{t_n}(f_n)(X_{t_1+\ldots+t_{n-1}})\,dt_1\ldots dt_n\right] \\
= &\lim_j E_x^{\beta_j}\left[\int_0^\infty \ldots \int_0^\infty \psi(t_1,\ldots,t_n)P_{t_n}^{\beta_j}(f_n)(X_{t_1+\ldots+t_{n-1}})\,dt_1\ldots dt_n\right] \\
= &\lim_j E_x^{\beta_j}\left[\int_0^\infty \ldots \int_0^\infty \psi(t_1,\ldots,t_n)f_n(X_{t_1+\ldots+t_n})\,dt_1\ldots dt_n\right] \\
= &E_x\left[\int_0^\infty \ldots \int_0^\infty \psi(t_1,\ldots,t_n)f_n(X_{t_1+\ldots+t_n})\,dt_1\ldots dt_n\right],
\end{aligned}$$

where we used Proposition 2.4. From here we conclude by Fubini's theorem

$$\begin{aligned}
&\int_0^\infty \ldots \int_0^\infty e^{-\alpha_1 t_1-\ldots-\alpha_n t_n} E_x[f_1(X_{t_1})\ldots f_n(X_{t_1+\ldots+t_n})]\,dt_1\ldots dt_n \\
&\qquad = \int_0^\infty \ldots \int_0^\infty e^{-\alpha_1 t_1-\ldots-\alpha_n t_n} E_x[f_1(X_{t_1})\ldots f_{n-1}(X_{t_1+\ldots+t_{n-1}}) \\
&\qquad\qquad\qquad\qquad P_{t_n}(f_n)(X_{t_1+\ldots+t_{n-1}})]\,dt_1\ldots dt_n
\end{aligned}$$

Since the above integrands are right-continuous, by the uniqueness of the

Laplace-transform we get successively

$$\begin{aligned}&E_x[f_1(X_{t_1})\dots f_n(X_{t_1+\dots+t_n})]\\ &\qquad = E_x[f_1(X_{t_1})\dots f_{n-1}(X_{t_1+\dots+t_{n-1}})P_{t_n}(f_n)(X_{t_1+\dots+t_{n-1}})],\end{aligned} \tag{4.9}$$

for $f_1, \dots f_{n-1} \in \mathcal{C}_b(Z_\Delta)$, $f_n = R_\alpha R_1 u$, $u \in J$. Multiplying this equation by α and letting $\alpha \to \infty$, we obtain by Lemma 1.17

$$\begin{aligned}&E_x[f_1(X_{t_1})\dots R_1 u(X_{t_1+\dots+t_n})]\\ &\qquad = E_x[f_1(X_{t_1})\dots f_{n-1}(X_{t_1+\dots+t_{n-1}})P_{t_n}(R_1 u)(X_{t_1+\dots+t_{n-1}})].\end{aligned} \tag{4.10}$$

Now we have for $u \in J$ that $u - \alpha R_{\alpha+1} u \in J$ and $R_1 \alpha(u - \alpha R_{\alpha+1} u) = \alpha R_{\alpha+1} u \uparrow u$, hence by monotone convergence theorem it follows that

$$\begin{aligned}&E_x[f_1(X_{t_1})\dots u(X_{t_1+\dots+t_n})]\\ &\qquad = E_x[f_1(X_{t_1})\dots f_{n-1}(X_{t_1+\dots+t_{n-1}})P_{t_n}(u)(X_{t_1+\dots+t_{n-1}})]\end{aligned} \tag{4.11}$$

holds for all $f_1, \dots f_{n-1} \in \mathcal{C}_b(Z_\Delta)$, for all $u \in J$. Define now

$$V := \{u \mid (4.11) \text{ holds for } u\}.$$

Then V is a monotone vector space and we have $J \subset V$. It follows that $J - J \subset V$. The set $J - J$ is a positive conves cone, inf-stable and contains 1. Hence, we obtain by monotone class theorem that V contains all bounded $\sigma(J - J)$-measurable functions. But by [Sch73, Lemma 18] we have that $\mathcal{B}(Z_\Delta) \subset \sigma(J) \subset \sigma(J - J)$. We conclude that

$$\begin{aligned}&E_x[f_1(X_{t_1})\dots f_n(X_{t_1+\dots+t_n})]\\ &\qquad = E_x[f_1(X_{t_1})\dots f_{n-1}(X_{t_1+\dots+t_{n-1}})P_{t_n}(f_n)(X_{t_1+\dots+t_{n-1}})]\end{aligned} \tag{4.12}$$

for all $f_1, \dots f_{n-1} \in \mathcal{C}_b(Z_\Delta)$ and $f_n \in \mathcal{B}_b(Z_\Delta)$. And now it is easy to obtain, again by monotone class arguments, the assertion for $f_1, \dots f_n \in \mathcal{B}_b(Z_\Delta)$. $\square$

In what follows let $(P_x)_{x \in Z_\Delta}$ be as in Theorem 4.3. Let Ω and Z_Δ be specified by Theorem 3.1. Since $P_x[\Omega] = 1$ for all $x \in Z_\Delta$, we may restrict P_x and the coordinate process $(X_t)_{t\geq 0}$ to Ω. Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of $(X_t)_{t\geq 0}$. Finally, we are well prepared to state and to prove our main result. Here we use the same arguments as in chapter 2. For completeness we give all the details.

Theorem 4.4. $\mathbb{M} := (\Omega, (X_t)_{t\geq 0}, (\mathcal{F}_t)_{t\geq 0}, (P_x)_{x\in Z_\Delta})$ *is a Hunt process with respect to both the ρ-topology and the original topology.*

Proof. The ρ-topology and the original topology generate the same Borel sets. Hence, we only discuss the ρ-topology case. In this case $R_\alpha f$ is uniformly continuous on Y_Δ for each $\alpha \in \mathbb{Q}_+^*$ and $f \in J$. Set

$$W := \Big\{ f \in \mathcal{B}_b(Y_\Delta) \mid P_x[t \mapsto R_\alpha f(X_t) \\ \text{is right continuous on } [0,\infty)] = 1 \ \forall\, \alpha \in \mathbb{Q}_+^*,\ \forall\, x \in \bar{E} \Big\}. \tag{4.13}$$

Then $J \subset W$ and W is a linear vector space. To prove that W is a monotone vector space, consider $Z_t^f := e^{-\alpha t} R_\alpha f(X_t) + \int_0^t e^{-\alpha s} f(X_s)\ ds$ for $f \in W$. Then it can be easily seen that $(Z_t^f)_{t\geq 0}$ is a right continuous martingale for all $f \in W$. If $f_n \in W,\ n \in \mathbb{N}$ such that $0 \leq f_n \uparrow f$ bounded then it follows by monotone convergence theorem that $Z_t^{f_n} \uparrow Z_t^f$ for all $t \geq 0$. By [DM82, Thm. VI.18] we have that Z_t^f is indistinguishable from a right continuous process, hence $f \in W$, i.e. W is a monotone vector space. By the monotone class theorem we obtain that $\mathcal{B}_b(Y_\Delta)$ is contained in W. Now the strong Markov property follows from [Sha88, Thm. 7.4]. So, it remains to prove the quasi-left-continuity of $(X_t)_{t\geq 0}$. Assume first that τ is bounded. Then let $(\tau_n)_{n\geq 1}$ be an increasing sequence of stopping times such that $\tau_n \uparrow \tau$. Define

$$V(\omega) := \begin{cases} \lim_{n\to\infty} X_{\tau_n}(\omega) & if\ \tau(\omega) < \infty \\ \Delta & if\ \tau(\omega) = \infty. \end{cases}$$

So, we have to prove that $V = X_\tau\ P_\mu$-a.s. For all $f, g \in \mathcal{C}_b(E_\Delta),\ x \in Z_\Delta$ we have

$$\begin{aligned} E_x[g(V) R_\alpha f(X_\tau)] &= E_x\left[g(V) E_{X_\tau}\left[\int_0^\infty e^{-\alpha t} f(X_t)\ dt\right]\right] \\ &= E_x\left[g(V) \int_0^\infty e^{-\alpha t} \underbrace{E_{X_\tau}[f(X_t)]}_{=E_x[f(X_{\tau+t})|\mathcal{F}_\tau]\ P_x^\beta - a.s.}\ dt\right] \\ &= E_x\left[g(V) \int_\tau^\infty e^{-\alpha(t-\tau)} E_x[f(X_t) \mid \mathcal{F}_\tau]\ dt\right] \\ &= E_x\left[g(V) e^{\alpha\tau} \int_\tau^\infty e^{-\alpha t} f(X_t)\ dt\right] \\ &= \lim_{n\to\infty} E_x\left[g(X_{\tau_n}) e^{\alpha\tau_n} \int_{\tau_n}^\infty e^{-\alpha t} f(X_t)\ dt\right] \\ &= \lim_{n\to\infty} E_x\left[g(X_{\tau_n}) R_\alpha f(X_{\tau_n})\right] \\ &= E_x\left[g(V) R_\alpha f(V)\right], \end{aligned}$$

where in the last step we used Lebesgue's dominated convergence theorem. By a monotone class argument we have

$$E_x[g(V)\alpha R_\alpha f(X_\tau)] = E_x[g(V)\alpha R_\alpha f(V)]$$

for all Borel measurable bounded functions g. Since Y is Borel measurable, we may replace g by $1_Y \cdot g$ and if we let α tend to infinity, we obtain by monotone convergence theorem that for all $f \in \mathcal{C}_b(E_\Delta),\ g \in \mathcal{B}_b(E_\Delta)$

$$E_x[g(V)f(X_\tau)1_{\{V\in Y\}}] = E_x[g(V)f(V)1_{\{V\in Y\}}].$$

Again by a monotone class argument we obtain that

$$E_x[h(V, X_\tau)1_{\{V\in Y\}}] = E_x[h(V, V)1_{\{V\in Y\}}]$$

for all $\mathcal{B}_b(E_\Delta \times E_\Delta)$-measurable bounded functions h. Let now h to be the indicator function of the diagonal in $E_\Delta \times E_\Delta$. So, the assertion follows.
Step2 : For arbitrary τ we have

$$\begin{aligned}
& P_x\left[V \neq X_\tau, V \in Y\right] \\
= \ & P_x\left[V \neq X_\tau,\ V \in Y,\ \tau < \infty\right] \\
= \ & \sum_{n=1}^{\infty} P_x\left[V \neq X_\tau,\ V \in Y,\ n-1 \leq \tau < n\right] \\
= \ & \sum_{n=1}^{\infty} P_x\left[\lim_{k\to\infty} X_{\tau_k \wedge n} \neq X_{\tau\wedge n},\ \lim_{k\to\infty} X_{\tau_k\wedge n} \in Y,\ n-1 \leq \tau < n\right] \\
= \ & 0,
\end{aligned}$$

where all summands are 0 by step 1. Hence $(X_t)_{t\geq 0}$ is a Hunt process. $\square$

Note that by Lemmas 1.15 and 4.2 $\mathbb{M}$ is even strictly properly associated in the resolvent sense to $\mathcal{E}$.

Chapter A

Appendix

The next results are strict versions of some results from [Sta99b].

Lemma A.1. *[Sta99b, III.3.5] Let S be a countable family of s.$\mathcal{E}$-q.c. functions. Then there exists a s.$\mathcal{E}$-nest $(F_k)_{k\geq 1}$ such that $S \subset \mathcal{C}(\{F_k\})$.*

Proof. Let $S = \{f_l \mid l \in \mathbb{N}\}$. For every $l \in \mathbb{N}$ there exists a strict $\mathcal{E}$-nest $(F_{lk})_{k\geq 1}$ such that $f_l \in \mathcal{C}(\{F_{lk}\})$ and $Cap_{1,g}(F_{lk}^c) < \frac{1}{2^l k}$. Let $F_k := \bigcap_{l\geq 1} F_{lk},\ k \in \mathbb{N}$. Then each F_k is closed and

$$Cap_{1,g}(F_k^c) \leq \sum_{l\geq 1} Cap_{1,g}(F_k^c) \leq \frac{1}{k}.$$

Hence, $(F_k)_{k\geq 1}$ is a s.$\mathcal{E}$-nest and obviously $S \subset \mathcal{C}(\{F_k\})$. □

Definition A.2. *(i) A(n) (strict) $\mathcal{E}$-nest $(F_k)_{k\geq 1}$ is called regular if for all $k \in \mathbb{N},\ U \subset E,\ U$ open, $m(U \cap F_k) = 0$ implies that $U \subset F_k^c$.*

(ii) A subspace $A \subset E$ is called the topological support of a measure μ, if for every open set $\emptyset \neq U \subset A$ we have $\mu(U) > 0$. Notation: $\operatorname{supp}\mu := A$.

(iii) A topological space is called strongly Lindelöf if every open covering of an open set U has a countable subcover.

Lemma A.3. *[Sta99b, III.2.4] Let $(F_k)_{k\geq 1}$ be a s.$\mathcal{E}$-nest such that the relative topology on F_k is strongly Lindelöf for all k, and define $\tilde{F}_k := \operatorname{supp}[1_{F_k} \cdot m]$. Then $(\tilde{F}_k)_{k\geq 1}$ is a regular strict $\mathcal{E}$-nest with $\tilde{F}_k \subset F_k$ for all k.*

Proof. Fix $k \in \mathbb{N}$. Note that $\tilde{F}_k$ is smallest closed set F such that $m(F^c \cap F_k) = 0$. In particular, $\tilde{F}_k \subset F_k$ and $m(F_k \setminus \tilde{F}_k) = m(\tilde{F}_k^c \cap F_k) = 0$. Hence $Cap_{1,g}(F_k^c) = Cap_{1,g}(\tilde{F}_k)$. If $U \subset E$ open with $m(U \cap \tilde{F}_k) = 0$ then $m(U \cap F_k) = 0$, since $m(F_k \setminus \tilde{F}_k) = 0$. Consequently, $U^c \supset \tilde{F}_k$ and this is equivalent with $U \subset \tilde{F}_k^c$. Hence, $(\tilde{F}_k)_{k\geq 1}$ is a strict regular $\mathcal{E}$-nest. □

Lemma A.4. *[Sta99b, III.3.3] If f is s.$\mathcal{E}$-q.s.l.c. and $f \leq 0$ m-a.e. on an open set $U \subset E$, then $f \leq 0$ s.$\mathcal{E}$-q.e. on U.*

Proof. Let $(F_k)_{k\geq 1}$ be a strict $\mathcal{E}$-nest such that $f \in \mathcal{C}(\{F_k\})$ and let $E_k,\ k \in \mathbb{N}$, be the sets from Definition 1.11 which can be assumed to be metrizable. Then set $F_k' := F_k \cap E_k,\ k \in \mathbb{N}$. Then F_k' is strongly Lindelöf as a second-countable space. Hence, by Lemma A.3 we have $\tilde{F}_k := \text{supp}[1_{F_k'} \cdot m],\ k \in \mathbb{N}$ forms a strict regular $\mathcal{E}$-nest. For $k \in \mathbb{N}$ $\{f > 0\} \cup F_k^c \cap U$ is open and $m(\{f > 0\} \cup F_k^c \cap U \cap \tilde{F}_k) = 0$, since $f \leq 0$ m-a.e. Hence $\{f > 0\} \cup F_k^c \cap U \subset \tilde{F}_k^c$, since $\tilde{F}_k$ is a strict regular $\mathcal{E}$-nest. So, we obtain $\{f > 0\} \cap U \subset \tilde{F}_k$, which is equivalent with $\tilde{F}_k \cap U \subset \{f \leq 0\}$. Consequently,

$$\bigcup_{k\geq 1} \tilde{F}_k \cap U \subset \{f \leq 0\}.$$

□

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