Convergence of non-symmetric forms with changing reference measures

Diploma thesis
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Contents

1. Introduction

In this thesis the author examines convergence problems of non-symmetric forms defined on different Hilbert spaces. The aim is to provide necessary and sufficient conditions for convergence of the associated resolvents and semigroups. The convergence of processes shall also be considered. We deal with various concepts of convergence "along" a "sequence of Hilbert spaces", including the concept of *generalized convergence* of non-symmetric forms or, more precisely, so-called *generalized Dirichlet forms*. Our notion of convergence is a generalization of the famous concept of *Mosco convergence*, a variational convergence of symmetric quadratic forms introduced by U. Mosco (cf. [Mos94]). Not only that the considered forms are fitting the general framework of generalized Dirichlet forms introduced by W. Stannat (cf. [Sta98], [Sta99]), they are also assumed to be defined on different Hilbert spaces. This idea is due to K. Kuwae and T. Shioya who developed this framework in [KS03] as a consequence of research on convergence of metric measure spaces. To understand this conceptual difference to former research on convergence of forms (except a few papers published recently), we would like to explain what we mean with "convergence of Hilbert spaces" in applications. An abstract functional analytic introduction to this new framework and consequences can be found in Chapter 2.1, where many proofs are taken either from [KS03] or from [Kol05a] and can be found in Appendix A in order to make this thesis as self-contained as possible.

Let E be an infinite dimensional locally convex (real) topological vector space. Let μ_n, $n \in \mathbb{N}$, μ be fully supported Borel probability measures such that $\mu_n \to \mu$ weakly. Define $H_n := L^2(E;\mu_n)$, $n \in \mathbb{N}$, $H := L^2(E;\mu)$ and let $C := \mathscr{F}C_b^\infty(E)$ be the space of so-called cylindrical test functions. Assume that E is Souslinean, so that C is dense in H. Now $\{H_n\}$ converges to H in the following sense:
There exists a sequence $\{\Phi_n : C \subset H \to H_n\}$ of injective linear operators with dense linear domain C such that

$$\lim_n \|\Phi_n(u)\|_{H_n} = \|u\|_H, \quad \forall u \in C.$$

Clearly, if we take each Φ_n as the identity operator on C, H_n converges to H in this sense by weak convergence of measures.
If we let $E := \mathbb{R}^d$ and μ_n, $n \in \mathbb{N}$, μ be fully supported Borel measures such that $\mu_n \to \mu$ vaguely, we clearly obtain by setting $C := C_0^\infty(\mathbb{R}^d)$, $H_n := L^2(\mathbb{R}^d;\mu_n)$, $n \in \mathbb{N}$, $H := L^2(\mathbb{R}^d;\mu)$ that H_n converges to H in this sense. Let us denote the disjoint union of Hilbert spaces by $\mathscr{H} := \dot{\bigcup}_n H_n \dot{\cup} H$.

Now one can define the concept of strong and weak convergence of vectors $\{u_n\}$ to u with $u_n \in H_n$, $n \in \mathbb{N}$, $u \in H$. This is done via approximation in C, since in a way we deal with "isometry classes of Hilbert spaces" which itselves are not normed but somehow equipped with a Gromov-Hausdorff-type topology.

Definition 1.1 (Strong and weak convergence). A sequence of vectors $\{u_n\}$, $u_n \in H_n$, $n \in \mathbb{N}$ is said to *strongly converge* to $u \in H$ if there exists a approximating sequence $\{\varphi_m\} \subset C$ with $\varphi_m \to u$ in H and

$$\lim_m \overline{\lim_n} \|\Phi_n(\varphi_m) - u_n\|_{H_n} = 0.$$

A sequence of vectors $\{u_n\}$, $u_n \in H_n$, $n \in \mathbb{N}$ is said to *weakly converge* to $u \in H$ if

$$\lim_n (u_n, v_n)_{H_n} = (u, v)_H$$

for every $v_n \to v$ strongly convergent.

Carrying on in this framework we also define weak and strong convergence of bounded linear operators from H_n to H_n, $n \in \mathbb{N}$ and the analog to Mosco convergence (resp. Γ-convergence) in this framework. This has mainly be done by K. Kuwae and T. Shioya in [KS03]. The analysis of Dirichlet forms by convergence problems goes back to [DGS73], [DGF75] where the so-called Γ-convergence introduced by E. De Georgi was used to obtain asymptotic properties of Dirichlet forms (see [DM93] for a complete disquisition on Γ-convergence on arbitrary topological spaces). In [Mos94] U. Mosco examined the Mosco convergence formerly known as strong Γ-convergence. His main result was to identify Mosco convergence as a necessary and sufficient condition on a sequence of symmetric closed forms $\{\mathscr{E}^n\}$ such that the associated C_0-contraction resolvents $\{G_\alpha^n\}$ $\alpha > 0$ converge in the strong operator topology. (For an introduction to symmetric Dirichlet forms we refer to [FOT94] or [MR92]).

Definition 1.2 (Mosco convergence - standard version). We say that a sequence of symmetric closed forms $\{\mathscr{E}^n\}$ on some common Hilbert space H_0 *Mosco converges* to a symmetric closed form $\mathscr{E}$ on the same Hilbert space H_0 if the following two conditions are fulfilled:

(M1) For every weakly convergent sequence of vectors $u_n \to u$ we have

$$\mathscr{E}(u) \leq \underline{\lim}_n \mathscr{E}^n(u_n).$$

(M2) For every $v \in \mathscr{D}(\mathscr{E})$ there exists a strongly convergent sequence $v_n \to v$ such that

$$\mathscr{E}(v) = \lim_n \mathscr{E}^n(v_n),$$

where $\mathscr{D}(\mathscr{E})$ denotes the domain of $\mathscr{E}$. We extend every $\mathscr{E}^n$, $n \in \mathbb{N}$, $\mathscr{E}$ to H_0 by setting $\mathscr{E}(u) := +\infty$ if $u \in H_0 \setminus \mathscr{D}(\mathscr{E})$ (similarly for each $\mathscr{E}^n$).

The convergence of the associated C_0-contraction semigroups $\{T_t^n\}$, $t \geq 0$ in the strong operator topology is either obtained by analysis of the associated spectral measures or the more general Theorem of T. Kato (cf. [Kat66, Theorem IX.2.16]).

Combining both the concepts of [Mos94] and [KS03] almost analog results for Mosco convergence can be obtained (see Chapter 2.2 of this thesis). There has been some research on this new approach, since (as can be seen in the examples above) a change of reference measures resp. speed measures can be considered here. We refer to the papers of A.V. Kolesnikov [Kol05a], [Kol06] and [Kol05b], where also other useful completions of the Kuwae-Shioya framework have been proved. They can be found in Chapter 3, since Mosco convergence of symmetric parts of non-symmetric forms helps to establish generalized convergence (see Definition 1.3 below). The proofs are omitted with some important exceptions.

Convergence problems for Dirichlet forms, resp. associated operators have been examined in [AHKS80], [AKS86], [CES02], [Can75], [Hin98], [Kas05], [Kol05a], [Kol06], [Kol05b], [KS03], [KU97], [KU96], [LZ93], [LZ94], [LZ96], [Mat99], [Mer94], [Mos94], [OTT02], [Pos96], [PZ04], [RZ97], [Str88], [Sun99] and [Uem95].

We would like to point out that S. Mataloni in [Mat99] and P. Mertens in [Mer94] first considered an abstract convergence of sectorial forms (though non-symmetric cases can be found also in [RZ96] and other papers) and that M. Hino in [Hin98] first stated abstract conditions on generalized forms (as in [Sta99]). In all of these three papers the strong convergence of the associated resolvents was proved. (For an introduction to sectorial forms we refer to [MR92]; the theory of generalized forms can be found in [Sta99]).

Our notion of generalized convergence as found below is a generalization of M. Hino's conditions for the Kuwae-Shioya framework. The author would like to express his gratitude to this advance in research. Although we shall use the case of generalized forms later, we would like to formulate this notion for sectorial forms here:

Definition 1.3 (Generalized convergence). A sequence of coercive closed forms $\{(\mathscr{E}^n, \mathscr{D}(\mathscr{E}^n))\}$ defined on H_n resp. *converges to* a coercive closed form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ on H *in the generalized sense* if the sector constants K_n of the $\mathscr{E}^n$'s are uniformly bounded and $\mathscr{D}(\mathscr{E}) \subset H$ is dense w.r.t. $\tilde{\mathscr{E}}_1^{1/2}$. And, moreover, the following two conditions hold:

(F1) For every weakly convergent sequence $u_n \to u$ with $\underline{\lim}_n \tilde{\mathscr{E}}_1(u_n) < \infty$ we get that $u \in \mathscr{D}(\mathscr{E})$.

(F2) For every $w \in \mathscr{D}(\mathscr{E})$ and for every weakly convergent sequence $u_n \to u$ with $u_n \in \mathscr{D}(\mathscr{E}^n)$, $u \in \mathscr{D}(\mathscr{E})$ there exists a strongly convergent sequence $w_n \to w$ such that

$$\lim_n \mathscr{E}(w_n, u_n) = \mathscr{E}(w, u)$$

We prove in the Kuwae-Shioya framework, that this is equivalent to the strong convergence of the associated resolvents and the weak convergence of the associated coresolvents. This has not been done before, although the proof relies on the results of M. Hino. The condition that the sector constants are uniformly bounded can be relaxed into a weaker version of (F1).

Let us now give a brief overview of the structure of this thesis and the main results.

In Chapter 2.1 we develop the Kuwae-Shioya framework of convergence of spectral structures along a sequence of Hilbert spaces. Chapter 2.1.1 introduces us to the original rather abstact setting, whereas in view of our later applications we will use the setting of Chapter 2.1.2 exclusively (this is just the setting as described above, from an abstact point of view). Here we obtain as a main result (cf. Theorem 2.10) the (complete and separable) metric structure of our space $\mathscr{H}$ and a sequence of isometric isomorphisms $\{\Psi_n : H \to H_n\}$ with the property that $u_n \to u$ strongly if and only if $\lim_n \|\Psi_n(u) - u_n\|_{H_n} = 0$. The proof is essentially due to A.V. Kolesnikov (cf. [Kol05a, Proposition 7.2]), but has been completed, simplified and entirely rewritten. All other Lemmas and Propositions from this Chapter are taken from K. Kuwae and T. Shioya in [KS03, Chapter 2]. All proofs (partially completed and rewritten) can be found in Appendix A (as well as some of the proofs of A.V. Kolesnikov in [Kol05a] and [Kol06] and one proof of H. Attouch in [Att84]). We propose to the interested reader to carry through Chapters 2.1–2.2 and Appendix A parallelly.

In Chapter 2.1.4 we prove an entirely new result, namely the generalization of T. Kato's Theorem (as mentioned above) for the Kuwae-Shioya case. It gives the equivalence of strong convergence of resolvents and semigroups.

Chapter 2.2 is taken from [KS03] and [Kol05a].

In Chapter 2.3 we arrive at the main result for generalized (non-symmetric) forms. The result (Theorem 2.41) is already described above. We would like to point out that our version differs from the above, and, being more general, gives necessary and sufficient conditons on the convergence of resolvents. After that we examine the relation between Mosco convergence and generalized convergence. This is also entirely new.

Chapter 2.4 examines the connection of generalized convergence and strong graph convergence of the infinitesimal generators of our forms. Equivalence is proved as well. Additionally, we give a new characterization of strong graph convergence for closed operators associated with generalized forms. All this is done in the Kuwae-Shioya framework, and particularly, since dealing with the non-symmetric case, is entirely new.

Chapter 2.5 deals with the contraction and Dirichlet properties of our forms, and shows that generalized convergence is sufficiently strong to obtain the Dirichlet property of the limiting form.

Chapter 3 reviews the results of [Kol05a] and [Kol06] by reasons mentioned already.

Chapter 4 containes the main application results, which are also totally new. We consider finite and infinite dimensional elliptic (and sectorial) $a_{i,j}$-forms and give sufficient conditions on the generalized convergence of these forms.

Chapter 5 gives an short application to stochastics, namely, that the associated processes converge weakly (in the sense of the generating path measures) provided the associated forms converge in the generalized sense and the collection of path measures is tight. This is also new in this detail in the Kuwae-Shioya framework, and we would like to remark that here the concept of changing reference measures unfolds its strength in a very distinct manner.

Appendix A is a collection of proofs (which may have been completed and rewritten) from some reference papers.

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2. General functional analytic theory

2.1. Convergence of spectral structures

We shall first start with some notation. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ resp. denote the natural, integer, rational, real, complex numbers resp. We write $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$ if we do not want to specify whether we use the real or complex numbers. Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, i.e., the corresponding compactified numbers. For two sets A, B we write $A \dot{\cup} B := A \cup B$, if A and B are disjoint, i.e., $A \cap B = \emptyset$. This notation also extends to arbitrary unions of pairwise disjoint sets. For a Banach space E we denote by E' its (topological) dual. For Banach spaces E, F denote the set of all bounded linear operators from E to F by $\mathscr{L}(E, F)$ with operator norm $\| \; \|_{\mathscr{L}(E,F)}$. For convenience we set $\mathscr{L}(E) := \mathscr{L}(E, E)$. We abbreviate $\alpha := \alpha \operatorname{Id}$ for any $\alpha \in \mathbb{K}$.

We follow a framework developed recently by K. Kuwae and T. Shioya in [KS03]. It provides a beautiful functional analytic theory introducing convergence along a sequence of Hilbert spaces, and applications to convergence of spectral structures and forms. The most important proofs are repeated in Appendix A.

2.1.1. Convergence of Hilbert spaces - the general case

The general framework shall be described as follows remarking that we intend to restrict it to a countable index set thereafter. Let $\mathscr{N}$ be an arbitrary index set and $\{H_\nu \mid \nu \in \mathscr{N}\}$ a family of separable Hilbert spaces over $\mathbb{K}$. Assume there exists a family

$$\{\Phi_{\nu,\mu} : \mathscr{C}_\nu \to H_\mu \mid \nu, \mu \in \mathscr{N}\}$$

of linear maps with dense linear domains $\mathscr{C}_\nu \subset H_\nu$ such that $\Phi_{\nu,\nu}$ for each $\nu \in \mathscr{N}$ is the identity operator on $\mathscr{C}_\nu$. Assume that $\{H_\nu \mid \nu \in \mathscr{N}\}$ (i.e., $\mathscr{N}$) has a (not necessarily Hausdorff) topology such that a sequence $\{H_{\nu_n}\}_{n \in \mathbb{N}}$ converges to some H_ν, $\nu \in \mathscr{N}$ if and only if for any $u \in \mathscr{C}_\nu$

$$\lim_n \|\Phi_{\nu,\nu_n} u\|_{H_{\nu_n}} = \|\Phi_{\nu,\nu} u\|_{H_\nu} \quad (= \|u\|_{H_\nu}).$$

Now we can define a topology on the disjoint union $\mathscr{H} := \dot{\bigcup}_{\nu \in \mathscr{N}} H_\nu$ by:

Definition 2.1. Assume that a sequence $\{H_{\nu_n}\}$ converges to an H_ν. We say a that a sequence $\{u_n\}_{n \in \mathbb{N}}$ with $u_n \in H_{\nu_n}$ *(strongly) converges* to a vector $u \in H_\nu$ if for one (and

hence all) sequence(s) $\{\tilde{u}_m\}_{m\in\mathbb{N}} \subset \mathscr{C}_\nu$ tending to u in H_ν such that

$$\lim_m \overline{\lim_n} \|\Phi_{\nu,\nu_n}\tilde{u}_m - u_n\|_{H_{\nu_n}} = 0.$$

This topology is called the *strong topology* on $\mathscr{H}$. Note that this notion of convergence depends explicitly on the sequence $\{\nu_n\}$ and can only be defined in a reasonable way if $H_{\nu_n} \to H_\nu$. "Reasonable" means that "**hence all**" in the above Definition holds, which is a consequence of the easily provable fact that $H_{\nu_n} \to H_\nu$ if and only if for every $\{\tilde{u}_m\} \subset C_\nu$ such that $\tilde{u}_m \to 0 \in H_\nu$ we have

$$\lim_m \overline{\lim_n} \|\Phi_{\nu,\nu_n}\tilde{u}_m\|_{H_{\nu_m}} = 0.$$

Nevertheless, the notion of a subsequence is defined, since, provided $H_{\nu_n} \to H_\nu$, clearly $H_{\nu_{n_k}} \to H_\nu$ for any sequence of natural numbers $n_k \uparrow \infty$, $n_{k+1} > n_k$. It can easily be proved that $u_n \to u$, $u_n \in H_{\nu_n}$, $u \in H_\nu$ strongly if and only if every subsequence $u_{n_k} \to u$, $u_{n_k} \in H_{\nu_{n_k}}$ strongly. On the other hand, it does not make sense to ask for convergence of a sequence $\{u_n\} \subset \mathscr{H}$ with $u_n \in H_{\nu_n}$ for some ν_n such that the sequence $\{H_{\nu_n}\}_{n\in\mathbb{N}}$ does not have a limit.

Lemma 2.2. *The the strong topology on $\mathscr{H}$ is Hausdorff if and only if $\{H_\nu \mid \nu \in \mathscr{N}\}$ is Hausdorff.*

Proof. This is a more detailed proof than the original one found in [KS03, Corollary 2.2]. To prove the "if"-part assume that $\{H_\nu \mid \nu \in \mathscr{N}\}$ is Hausdorff, i.e., a convergent sequence $\{H_{\nu_n}\}$ has at most one limit point H_{ν_0}. Now let $u_n \in H_{\nu_n}$, $u, v \in H_{\nu_0}$ such that $u_n \to u$ and $u_n \to v$ strongly. If we can prove that $u = v$, we are done. Let $\{\tilde{u}_m\}, \{\tilde{v}_m\} \subset \mathscr{C}_{\nu_0}$ as in Definition 2.1 such that $\lim_m \|\tilde{u}_m - u\|_{H_{\nu_0}} = 0$, $\lim_m \|\tilde{v}_m - v\|_{H_{\nu_0}} = 0$,

$$\lim_m \overline{\lim_n} \|\Phi_{\nu_0,\nu_n}\tilde{u}_m - u_n\|_{H_{\nu_n}} = 0 \quad \text{and} \quad \lim_m \overline{\lim_n} \|\Phi_{\nu_0,\nu_n}\tilde{v}_m - u_n\|_{H_{\nu_n}} = 0.$$

Now,

$$\begin{aligned}
\|u - v\|_{H_{\nu_0}} &\le \overline{\lim_m} \left[\|u - \tilde{u}_m\|_{H_{\nu_0}} + \|\tilde{u}_m - \tilde{v}_m\|_{H_{\nu_0}} + \|\tilde{v}_m - v\|_{H_{\nu_0}}\right] \\
&= \overline{\lim_m} \lim_n \|\Phi_{\nu_0,\nu_n}(\tilde{u}_m - \tilde{v}_m)\|_{H_{\nu_n}} = \overline{\lim_m} \lim_n \|\Phi_{\nu_0,\nu_n}\tilde{u}_m - \Phi_{\nu_0,\nu_n}\tilde{v}_m\|_{H_{\nu_n}} \\
&\le \overline{\lim_m} \overline{\lim_n} \|\Phi_{\nu_0,\nu_n}\tilde{u}_m - u_n\|_{H_{\nu_n}} + \overline{\lim_m} \overline{\lim_n} \|u_n - \Phi_{\nu_0,\nu_n}\tilde{v}_m\|_{H_{\nu_n}} \\
&= 0.
\end{aligned}$$

Hence $u = v$.

To prove the "only if"-part, assume that $\{H_\nu \mid \nu \in \mathscr{N}\}$ is not Hausdorff, i.e., there exists a sequence of Hilbert spaces $\{H_{\nu_n}\}$ with $H_{\nu_n} \to H_{\nu_1}$ and $H_{\nu_n} \to H_{\nu_2}$ where $H_{\nu_1} \neq H_{\nu_2}$. Set $u_n := 0 \in H_{\nu_n}$, $n \in \mathbb{N}$. Then clearly $u_n \to 0 \in H_{\nu_1}$ strongly and $u_n \to 0 \in H_{\nu_2}$ strongly. Hence $\mathscr{H}$ is not Hausdorff which proves the "only if"-part. $\square$

We would like to remark that the fact that for every $u \in H_{\nu_0}$ there exists a strongly convergent sequence $\{u_n\}$ along H_{ν_n} is verified in the fundamental case below (see Remark 2.6), but it is not clear whether it holds in the general case. (Theorem 2.10 even tells us that $\mathscr{H}$ is *polish* in this case).

2.1.2. Convergence of Hilbert spaces - the fundamental case

From now on, we shall consider the case that $\mathscr{N} = \mathbb{N} \cup \{\infty\}$. Therefore, let H_n, $n \in \mathbb{N}$, H_∞ resp. be real separable Hilbert spaces with inner product $(\ ,\ )_{H_n}$, $(\ ,\ )_{H_\infty}$ resp. and norm $\|\ \|_{H_n} = (\ ,\ )_{H_n}^{1/2}$, $\|\ \|_{H_\infty} = (\ ,\ )_{H_\infty}^{1/2}$ resp. As a special case of the above, we set $\mathscr{C}_n := H_n$, $n \neq \infty$, and fix some dense linear subspace $\mathscr{C}_\infty \subset H_\infty$. Furthermore, we fix some injective linear operators $\Phi_{\infty,n} : \mathscr{C}_\infty \to H_n$, and for $m, n \in \mathbb{N}$ set $\Phi_{m,n} = \mathrm{Id}$, if $m = n$ and $\Phi_{m,n} \equiv 0$ if $m \neq n$. Let $\mathscr{H} := \dot{\bigcup}_{n\in\mathbb{N}} H_n \dot{\cup} H_\infty$ be the disjoint union of Hilbert spaces. As above we define:

Definition 2.3 (Convergence of Hilbert spaces). Let n_k be (not necessarily increasing) sequence of natural numbers. A sequence of Hilbert spaces

$$\{H_{n_k}\} \subset \{H_n \mid n \in \mathbb{N} \cup \{\infty\}\}$$

is said to *converge* to a Hilbert space H_{n_0}, $n_0 \in \mathbb{N} \cup \{\infty\}$ if

$$\lim_{n\to\infty} \|\Phi_{n_0,n_k} u\|_{H_{n_k}} = \|u\|_{H_{n_0}} \tag{2.1}$$

for every $u \in \mathscr{C}_{n_0}$.

From now on assume:

Assumption 1. *$\{H_n\}_{n\in\mathbb{N}}$ converges to H_∞.*

For the motivation of this assumption and the relation to a sequence of L^2-spaces and forms defined on them we either refer to the introduction or the latter chapters.

Definition 2.4 (Strong convergence). Let $\{n_k\}$ be a (not necessarily increasing) sequence of natural numbers. Let $n_0 \in \mathbb{N} \cup \{\infty\}$. Assume that $H_{n_k} \to H_{n_0}$ in the above sense. $\{u_k\}$, $u_k \in H_{n_k}$ is said to *strongly converge* to some $u \in H_{n_0}$ if there exist $\{\tilde{u}_m\} \subset \mathscr{C}_{n_0}$ with:

$$\lim_m \|\tilde{u}_m - u\|_{H_{n_0}} = 0, \tag{2.2}$$

$$\lim_m \overline{\lim_k} \|\Phi_{n_0,k}\tilde{u}_m - u_k\|_{H_{n_k}} = 0. \tag{2.3}$$

Then (2.3) even holds for every sequence $\{\tilde{v}_m\} \subset \mathscr{C}_{n_0}$ with $\tilde{v}_m \to u$ in H_{ν_0}.

Definition 2.5 (Weak convergence). Let $\{n_k\}$ be a (not necessarily increasing) sequence of natural numbers. Let $n_0 \in \mathbb{N} \cup \{\infty\}$. Assume that $H_{n_k} \to H_{n_0}$ in the above sense. $\{u_k\}$, $u_k \in H_{n_k}$ is said to *weakly converge* to $u \in H_{n_0}$ if

$$(u_k, v_k)_{H_{n_k}} \to (u, v)_{H_{n_0}} \tag{2.4}$$

for every sequence $\{v_k\}$, $v_k \in H_{n_k}$ strongly convergent to $v \in H_{n_0}$.

Strong (weak) convergence creates a topology on $\mathscr{H} = \dot{\bigcup}_{n\in\mathbb{N}} H_n \dot{\cup} H_\infty$, called the *strong (weak) topology.*

Remark 2.6. *We point out that we are in the very special case which in a way restricts strong and weak convergence of "interesting" sequences to those converging along the sequence of Hilbert spaces $H_n \to H_\infty$. To see what this specificly means consider the following statements, which can easily be proved taking into account that for $n, m \in \mathbb{N}$, $\mathscr{C}_n = H_n$, $\Phi_{m,n} = \mathrm{Id}$ if $m = n$ and $\Phi_{m,n} \equiv 0$, if $m \neq n$,*

(1) It can easily be seen that the topology of $\{H_n \mid n \in \mathbb{N} \cup \{\infty\}\}$ is Hausdorff (which also follows from Lemma 2.2 and Theorem 2.10). (It has also been proved that it is second countable, see [KS03, Lemma 2.13]*).*

(2) Let $n_0 \in \mathbb{N}$, ($n_0 \neq \infty$!) $H_{n_k} \to H_{n_0}$ if and only if $\exists k_0 \geq 1$ with $H_{n_k} = H_{n_0}$ $\forall k \geq k_0$.

(3) Let $n_0 \in \mathbb{N}$, ($n_0 \neq \infty$!) and assume $H_{n_k} \to H_{n_0}$. Then $u_k \to u$, $u_k \in H_{n_k}$, $u \in H_{n_0}$ strongly if and only if $\exists k_0 \geq 1$ such that $u_k \in H_{n_0}$ $\forall k \geq k_0$ and

$$\lim_{k\to\infty, k\geq k_0} \|u_k - u\|_{H_{n_0}} = 0.$$

A similar characterization holds for the weak convergence.

As a consequence, in our particular case, depending on the "limiting" Hilbert space, one should understand strong and weak convergence either in the "usual" way, ending up in one "final" Hilbert space, or in the new sense "along H_n" ending in H_∞, introducing a new notion of strong convergence, where "u_n converges strongly to u" means: The "distance" between some approximating sequence $\{\tilde{u}_m\} \subset \mathscr{C}_\infty$ and u is small, and the "distance" between its embeddings in the H_n's via the $\Phi_{\infty,n}$'s and the u_n's is small.

From now on we shall always understand strong and weak convergence, unless stated differently, along the sequence $H_n \to H_\infty$, since by the above remark the other cases are more or less trivial. This convention extends also to the Appendix. **From now on** we also shall set $H := H_\infty$, $C := \mathscr{C}_\infty$, $\Phi_n := \Phi_{\infty,n}$, unless this leads to confusion. Subsequences are also understood in this way.

Now let us start with some useful facts.

Lemma 2.7. *(1) Let $u_n \in H_n$, $n \in \mathbb{N}$. Then $u_n \to 0 \in H$ strongly in $\mathscr{H}$ if and only if $\|u_n\|_{H_n} \to 0$.*

(2) Let $u_n, v_n \in H_n$, $n \in \mathbb{N}$, $u, v \in H$ such that $u_n \to u$ strongly and $v_n \to v$ strongly. Then $\alpha u_n + \beta v_n \to \alpha u + \beta v$ strongly in $\mathscr{H}$ for any $\alpha, \beta \in \mathbb{R}$.

(3) Let $\{u_n\}$ be a sequence with $u_n \in H_n$, $n \in \mathbb{N}$ and $u_n \to u \in H$ strongly. Then $\|u_n\|_{H_n} \to \|u\|_H$. In particular, the sequence of norms $\{\|u_n\|_{H_n}\}$ of a strongly convergent sequence is bounded.

(4) If $u_n, v_n \in H_n$, $n \in \mathbb{N}$, $u, v \in H$ such that $u_n \to u$ and $v_n \to v$ $\mathscr{H}$-strongly, then $(u_n, v_n)_{H_n} \to (u, v)_H$. In particular, every strongly convergent sequence converges weakly.

Proof. See Appendix A. □

Lemma 2.8. *Let $\{u_n\}$, $\{v_n\}$ be two sequences of vectors in $\mathscr{H}$ with $u_n, v_n \in H_n$, $n \in \mathbb{N}$, and let $u \in H$. Suppose that $u_n \to u$ strongly in $\mathscr{H}$. Then $v_n \to u$ strongly in $\mathscr{H}$ if and only if $\|u_n - v_n\|_{H_n} \to 0$.*

Proof. The proof is trivial by Lemma 2.7 (1) and (2). □

Remark 2.9. *(1) One immediately obtains by definition that $\Phi_n(\varphi) \to \varphi$ $\mathscr{H}$-strongly for every $\varphi \in C$.*

*(2) Note that if $v_n \in H_n$, $n \in \mathbb{N}$, with $v_n \to \varphi \in C$ $\mathscr{H}$-strongly, it follows from Lemma 2.8 (or even from "**hence all**" in Definition 2.1), that*

$$\lim_n \|v_n - \Phi_n(\varphi)\|_{H_n} = 0.$$

We will use this later.

The next Theorem is due to A.V. Kolesnikov, taken from [Kol05a, Propositon 7.2]. We would like to point out, that the proof has been rewritten and completed for this paper.

Theorem 2.10. *Assume that all H_n, $n \in \mathbb{N}$ and H are infinite dimensional and separable. Then there exists a complete separable metric $d_{\mathscr{H}}$ on $\mathscr{H}$ such that the convergence in $d_{\mathscr{H}}$ coincides with the strong convergence and there exists a bijective isometry of metric spaces $\Psi : (\mathscr{H}, d_{\mathscr{H}}) \to I \times \ell^2$, where $I := \{0\} \cup \bigcup_{n \in \mathbb{N}} \{\frac{1}{n}\} \subset \mathbb{R}$.*

There also exists a sequence of isometric isomorphisms of Hilbert spaces

$$\Psi_n : H_\infty \to H_n$$

such that $\Psi_\infty = \mathrm{Id}_{H_\infty}$ and $u_n \to u$, $u_n \in H_n$, $u \in H_\infty$ strongly in $\mathscr{H}$ if and only if

$$\lim_n \|\Psi_n u - u_n\|_{H_n} = 0. \tag{2.5}$$

Futhermore, if we fix $n_0 \in \mathbb{N}$, the metric has the property that for $f_m \in H_{n_0}$ for all $m \in \mathbb{N}$, $f \in H_{n_0}$ one has $\|f_m - f\|_{H_{n_0}} \to 0$ as $m \to \infty$ if and only if $d_{\mathscr{H}}(f_m, f) \to 0$ as $m \to \infty$.

Proof. As in the beginning of the paper we write $H_\infty = H$. Let us first construct an orthonormal basis $\{e_i \mid i \in \mathbb{N}\}$ in H_∞ consisting of vectors from C using a standard orthogonalization procedure.

Step 1:
Let us assume for a while that the Ψ_n's are constructed already. Then for fixed $n_0 \in \mathbb{N} \cup \{\infty\}$, $\{\Psi_{n_0}(e_i) \mid i \in \mathbb{N}\}$ clearly is an orthonormal basis of H_{n_0}. Let $u \in H_n$, $v \in H_m$, $n, m \in \mathbb{N} \cup \{\infty\}$. We claim that the metric

$$d_{\mathscr{H}}(u,v) = \sqrt{|\delta_n - \delta_m|^2 + \sum_{i=1}^{\infty}(u_i - v_i)^2},$$

where $u = \sum_{i=1}^{\infty} u_i \Psi_n(e_i) \in H_n$, $v = \sum_{i=1}^{\infty} v_i \Psi_m(e_i) \in H_m$, $\delta_n := \frac{1}{n}$, $\delta_\infty := 0$, is the desired one.

Let us prove that $d_{\mathscr{H}}$ is a metric generating strong convergence on $\mathscr{H}$. Obviously, $d_{\mathscr{H}}$ is symmetric. We would like to prove that $d_{\mathscr{H}}(u,v) = 0$ if and only if $u = v$:
The "if"-part is trivial. To see the "only if"-part, assume that both $|\delta_n - \delta_m|^2 = 0$ and $\sum_{i=1}^{\infty}(u_i - v_i)^2 = 0$. The former gives us that $u, v \in H_{n_0}$ for some fixed $n_0 \in \mathbb{N} \cup \{\infty\}$. and the latter shows that $u = v$.
The triangular inequality follows obviously from the triangular inequality for ℓ^2 (as a metric space). Thus $d_{\mathscr{H}}$ defines a metric on $\mathscr{H}$. Now we would like to prove that convergence in this metric coincides with strong convergence in $\mathscr{H}$.

Case 1: "$H_n \to H_\infty$". Let $u_n \in H_n$, $u \in H_\infty$, $u_n \to u$ strongly. Then by (2.5)

$$\lim_n \|\Psi_n(u) - u_n\|_{H_n} = 0.$$

Let $u_i^{(n)} := (u_n, \Psi_n(e_i))_{H_n}$, $u_i := (u, e_i)_{H_\infty} = (\Psi_n(u), \Psi_n(e_i))_{H_n}$. Thus by Parseval's identity (cf. [RS72, Theorem II.6])

$$\sum_{i=1}^{\infty}(u_i^{(n)} - u_i)^2 = \sum_{i=1}^{\infty} \left|(u_n - \Psi_n(u), \Psi_n(e_i))_{H_n}\right|^2 = \|u_n - \Psi_n(u)\|_{H_n}^2 \to 0 \qquad (2.6)$$

as $n \to \infty$. Clearly, $\lim_n |\delta_n - \delta_\infty|^2 = 0$. Thus $\lim_n d_{\mathscr{H}}(u_n, u) = 0$.
Now assume $\lim_n d_{\mathscr{H}}(u_n, u) = 0$. Let $u_i^{(n)}$, u_i as above. Now we can prove that $\lim_n \|\Psi_n(u) - u_n\|_{H_n} = 0$ by reading (2.6) backwards.

Case 2: "$H_{n_k} \to H_{n_0}$". Let $\{n_k\}$ be a (not necessarily increasing) sequence of natural numbers and $n_0 \in \mathbb{N}$. Assume that H_{n_k} converges to H_{n_0}. Then there exists a $k_0 \in \mathbb{N}$ such that

$$H_{n_k} = H_{n_0} \quad \forall k \geq k_0. \qquad (2.7)$$

Let $u_k \in H_{n_k}$, $u \in H_{n_0}$. We want to prove that $u_k \to u$ strongly if and only if $\lim_k d_{\mathscr{H}}(u_k, u) = 0$.

Assume that $u_k \to u$ strongly in $\mathscr{H}$. By (2.7) $\delta_k = \delta_{n_0}$ $\forall k \geq k_0$. Thus $\lim_k |\delta_k - \delta_{n_0}|^2 = 0$. Let $u_i^{(k)} := (u_k, \Psi_{n_k}(e_i))_{H_{n_k}}$, $u_i := (u, \Psi_{n_0}(e_i))_{H_{n_0}}$. Since $u_k \in H_{n_0}$ $\forall k \geq k_0$, we have by strong convergence

$$
\begin{aligned}
\lim_{k\to\infty, k\geq k_0} d_{\mathscr{H}}(u_k, u) &= \lim_{k\to\infty, k\geq k_0} \sum_{i=1}^{\infty} (u_i^{(k)} - u_i)^2 \\
&= \lim_{k\to\infty, k\geq k_0} \sum_{i=1}^{\infty} \left|(u_k - u, \Psi_{n_0}(e_i))_{H_{n_0}}\right|^2 \\
&= \lim_{k\to\infty, k\geq k_0} \|u_k - u\|^2_{H_{n_0}} = 0.
\end{aligned}
\tag{2.8}
$$

Thus $\lim_k d_{\mathscr{H}}(u_k, u) = 0$.
Suppose now $\lim_k d_{\mathscr{H}}(u_k, u) = 0$. By reading (2.8) backwards and noting the observations above we get $\lim_k \|u_k - u\|^2_{H_{n_0}} = 0$ $\forall k \geq k_0$, which gives the desired result.

By Remark 2.6 all types of strongly convergent sequences are contained in these two cases. The other way round, if $\lim_k d(u_k, u) = 0$ for some $u_k \in \mathscr{H}$, $u \in \mathscr{H}$, we clearly are in one of the two cases above.

Step 2:
It remains to prove the existence of the isometric isomorphisms $\Psi_n : H_\infty \to H_n$ with the property (2.5). To this end, for $k \in \mathbb{N}$ let $\mathscr{L}^k := \mathrm{lin}\{e_1, \dots, e_k\}$, $\mathscr{L}_n^k := \Phi_n(\mathscr{L}^k)$, which is k-dimensional as well since the Φ_n's are one-to-one. Now by (2.1) the sequence of symmetric matrices $\{M_n^k\} \subset \mathbb{R}^k \otimes \mathbb{R}^k$ given by

$$
(M_n^k)_{1\leq i,j\leq k} := ((\Phi_n(e_i), \Phi_n(e_j))_{H_n})_{1\leq i,j\leq k}
$$

tends to $\mathbb{1}_k := \mathrm{diag}\{1, \dots, 1\} \in \mathbb{R}^k \otimes \mathbb{R}^k$ in the usual matrix norm

$$
\|(a_{i,j})_{1\leq i,j\leq k}\|_{\mathbb{R}^k\otimes\mathbb{R}^k} := \max_{1\leq i,j\leq k} |a_{i,j}|,
$$

and thus in the standard operator norm $\|\ \|_{\mathscr{L}_n^k \to \mathscr{L}_n^k}$, i.e.,

$$
\|M_n^k - \mathbb{1}_k\|_{\mathscr{L}_n^k\to\mathscr{L}_n^k} \leq \text{const.} \|M_n^k - \mathbb{1}_k\|_{\mathbb{R}^k\otimes\mathbb{R}^k} < \frac{1}{k} \quad \forall n \geq n(k),
$$

for some increasing sequence of natural numbers $n(k)$. (Here we have used Remark 2.9 (1) and Lemma 2.7 (4)). Let $n \geq n(k)$. For each such n we want to define

$$
\Psi_n : \mathscr{L}^k \subset C \subset H_\infty \to \mathscr{L}_n^k \subset H_n
$$

such that for the operator

$$
(\Psi_n - \Phi_n) \restriction_{\mathscr{L}^k} : \mathscr{L}^k \to \mathscr{L}_n^k \subset H_n
$$

we have
$$\|\Psi_n - \Phi_n\|_{\mathscr{L}^k \to \mathscr{L}^k_n} < \frac{1}{k},$$
provided $n \geq n(k)$. This is needed to establish (2.5).
There exists an orthonormal basis $\{e_1^n, \dots, e_k^n\}$ of $\mathscr{L}^k_n$ and a bijective linear operator $V_n^k : \mathscr{L}^k_n \to \mathscr{L}^k_n$ with
$$\Phi_n(e_i) = \sum_{j=1}^{k} (V_n^k e_i^n, e_j^n)_{H_n} e_j^n,$$
such that $(V_n^k)^* V_n^k = M_n^k$. More precisely, $V_n^k e_i^n := \Phi_n(e_i)$. Note that $\{e_1^n, \dots, e_k^n\}$ depends on the inner product generated by M_n^k.

By the Polar-Decomposition Theorem (cf. [LT85, Theorem 5.7.1]) we can represent $V_n^k = B_n^k U_n^k$ as a composition of a positive definite, self-adjoint operator $B_n^k = \sqrt{V_n^k (V_n^k)^*}$ on $\mathscr{L}^k_n$ and an isometric operator U_n^k on $\mathscr{L}^k_n$. B_n^k is even strictly positive definite, since V_n^k is invertible, hence all eigenvalues $\lambda_{n,i}^k$, $1 \leq i \leq k$, of B_n^k are strictly positive.

For a bounded self-adjoint linear operator A and a polynomial P we have
$$\|P(A)\| = \sup_{\lambda \in \sigma(A)} |P(\lambda)|,$$
where $\sigma(A)$ denotes the spectrum of A (cf. [RS72, Section VII.1 Lemma 2]).

Furthermore, we have $\mathbb{1}_k = U_n^k (U_n^k)^*$ and
$$(B_n^k)^2 = (B_n^k)^* B_n^k = (V_n^k (U_n^k)^*)^* V_n^k (U_n^k)^* = U_n^k (V_n^k)^* V_n^k (U_n^k)^* = U_n^k M_n^k (U_n^k)^*.$$

Hence for each $n \geq n(k)$ we have (using also the fact that U_n^k is isometric)
$$\max_{1 \leq i \leq k} \{|(\lambda_{n,i}^k)^2 - 1|\} = \|(B_n^k)^2 - \mathbb{1}_k\|_{\mathscr{L}^k_n \to \mathscr{L}^k_n} \leq \|M_n^k - \mathbb{1}_k\|_{\mathscr{L}^k_n \to \mathscr{L}^k_n} < \frac{1}{k}.$$

Hence $\sqrt{1 - \frac{1}{k}} < \lambda_{n,i}^k < \sqrt{1 + \frac{1}{k}}$ for all $1 \leq i \leq k$. Define $\Psi_n^k : \mathscr{L}^k \to \mathscr{L}^k_n$ by
$$\Psi_n^k(e_i) := U_n^k e_i^n$$
and linear extension. Then $(\Psi_n^k(e_i), \Psi_n^k(e_j))_{H_n} = \delta_{i,j}$, where $\delta_{i,j}$ denotes the Kronecker-delta. Moreover, for $1 \leq i \leq k$
$$(\Phi_n - \Psi_n^k)(e_i) = (V_n^k - U_n^k)(e_i^n)$$
and consequently for $n \geq n(k)$ again by the above remark about spectra and the fact that U_n^k is isometric
$$\|\Phi_n - \Psi_n^k\|_{\mathscr{L}^k \to \mathscr{L}^k_n} = \|V_n^k - U_n^k\|_{\mathscr{L}^k_n \to \mathscr{L}^k_n} \leq \|B_n^k - \mathbb{1}_k\|_{\mathscr{L}^k_n \to \mathscr{L}^k_n} = \max_{1 \leq i \leq k} \{|\lambda_{n,i}^k - 1|\} \leq \frac{1}{k}.$$

To extend Ψ_n^k to H_∞, pick for every $n \in \mathbb{N}$ some arbitrary isometric isomorphisms $\tilde{\Psi}_n^k : (\mathscr{L}^k)^\perp \to (\mathscr{L}_n^k)^\perp$ and extend Ψ_n^k by setting $\Psi_n^k(v) := \tilde{\Psi}_n^k(v)$ for every $v \in (\mathscr{L}^k)^\perp$. Set $\Psi_\infty := \mathrm{Id}_{H_\infty} : H_\infty \to H_\infty$ and pick some arbitrary isometric isomorphisms $\Psi_n : H_\infty \to H_n$ for $1 \leq n < n(1)$. Now by a standard diagonal argument we can select a sequence of isometric isomorphisms (using the construction above)

$$\Psi_n : H_\infty \to H_n$$

such that

$$\|\Phi_n - \Psi_n\|_{\mathscr{L}^k \to \mathscr{L}_n^k} < \frac{1}{k}$$

for all $n \geq n(k)$. Evidently, for every fixed $k_0 \in \mathbb{N}$ and every $v \in \mathscr{L}_{k_0}$ we have

$$\lim_n \|(\Phi_n - \Psi_n)v\|_{H_n} = 0.$$

Finally, if we can prove (2.5), we are done. So let $u = \sum_{i=1}^\infty \alpha_i e_i \in H_\infty$. Let us first prove that $\Psi_n u \to u$ strongly in $\mathscr{H}$. Therefore, set $\tilde{u}_m := \sum_{i=1}^m \alpha_i e_i \in C$. Then $\tilde{u}_m \to u$ in H_∞ as $m \to \infty$ and

$$\begin{aligned}
&\lim_m \overline{\lim_n} \|\Phi_n \tilde{u}_m - \Psi_n u\|_{H_n} \\
=&\lim_m \overline{\lim_n} \|(\Phi_n - \Psi_n) \sum_{i=1}^m \alpha_i e_i - \Psi_n \sum_{i=m+1}^\infty \alpha_i e_i\|_{H_n} \\
\leq&\lim_m \overline{\lim_n} \|(\Phi_n - \Psi_n) \sum_{i=1}^m \alpha_i e_i\|_{H_n} + \lim_m \overline{\lim_n} \|\Psi_n \sum_{i=m+1}^\infty \alpha_i e_i\|_{H_n} \\
=&\lim_m \|\sum_{i=m+1}^\infty \alpha_i e_i\|_{H_\infty} \\
=&0.
\end{aligned}$$

Now by Lemma 2.8 (since $\Psi_n u \to u$ strongly for any $u \in H_\infty$) for $u_n \in H_n$, $n \in \mathbb{N}$, $u \in H_\infty$, we have $u_n \to u$ $\mathscr{H}$-strongly if and only if

$$\lim_n \|\Psi_n u - u_n\|_{H_n} = 0.$$

In addition, it is clear that the following mapping $\Psi : \mathscr{H} \to I \times \ell^2$, $I := \{0\} \cup \bigcup_{n \in \mathbb{N}} \{\frac{1}{n}\}$ is the desired isometry of metric spaces:

$$\Psi(v) := (\delta_n, (v_i)_{i \in \mathbb{N}}) \in I \times \ell^2,$$

where $v \in H_n$ and $v_i := (v, \Psi_n(e_i))_{H_n}$. It is clearly bijective. Note $I \times \ell^2$ is complete as a product of complete metric spaces (I is complete since it is compact in $\mathbb{R}$) and hence is $(\mathscr{H}, d_{\mathscr{H}})$. Furthermore, $\bigcup_{n \in \mathbb{N}} \mathrm{lin}_{\mathbb{Q}} \bigcup_{i \in \mathbb{N}} \{\Psi_n e_i\}$ is a countable set which is dense in $\mathscr{H}$ w.r.t. $d_{\mathscr{H}}$ (here $\mathrm{lin}_{\mathbb{Q}}$ denotes the *linear span* w.r.t. $\mathbb{Q}$).

The proof is complete. $\square$

Corollary 2.11. *(i) For any $u \in H$ there exists a sequence $\{u_n\}$, $u_n \in H_n$, $n \in \mathbb{N}$, such that $u_n \to u$ strongly.*

(ii) If $n_k \uparrow \infty$ as $k \to \infty$ and $v_k \in H_{n_k}$, $k \in \mathbb{N}$, such that $v_k \to u \in H$ in $d_{\mathscr{H}}$, then there exist $u_n \in H_n$, $n \in \mathbb{N}$, such that $u_n \to u$ in $d_{\mathscr{H}}$ and $u_{n_k} = v_k$ for every $k \in \mathbb{N}$.

Proof. (i): Obvious.

(ii): For $n \notin \{n_k \mid k \in \mathbb{N}\}$ define $u_n := \Psi_n(u)$ and $u_{n_k} := v_k$ for $k \in \mathbb{N}$. Then $\{u_n\}$ is as desired.

□

We would like to remark that the above Theorem enlightens the geometric structure of $\mathscr{H}$ as follows: We start with a rather weak limiting structure of $\mathscr{H}$ based on a "uniform approximation" of H by the H_n's via our embeddings Φ_n (which are not unitary nor even bounded!) on a set of vectors C (having e.g. "nice" or "controllable" properties). The definition of strong convergence (of weak, too) seems natural in this setting, but is yet hard to handle and does not even provide existence of strongly convergent sequences (along the H_n's) for every possible limit $u \in H$, unless much stronger assumptions on the Φ_n's like uniform bound of operator norms are stated. Finally, by the above Theorem, using only basic properties of our concepts and a little linear algebra, we construct a natural metric on $\mathscr{H}$, and moreover, a sequence of isometric isomorphisms $\{\Psi_n\}$ which in a way carry over the geometry of H to each H_n and contain the "asymptotics" of strong convergence (namely, $u_n \to u$ strongly if and only if $\|u_n - \Psi_n(u)\|_{H_n} \to 0$). From now on, strong convergence should always be thought as given by this characterization: along a "limit of orthonormal bases" resp. "geometric structures" via the Ψ_n's. **From now on** we shall always refer to the Ψ_n's of Theorem 2.10 if we use this notation.

As a surprising fact, we would like to mention that in applications the Φ_n's are actually the "easier guys", namely being identity operators. Keeping this in mind, the next two Lemmas and Lemma 2.20 as well as the later convergence Theorems for forms turn out to be very useful for they provide conditions for various properties having to be checked only along $\{\Phi_n(\varphi)\}$, $\varphi \in C$.

Lemma 2.12. *A sequence $\{u_n\}$, $u_n \in H_n$, $n \in \mathbb{N}$ converges to $u \in H$ $\mathscr{H}$-strongly if and only if $\|u_n\|_{H_n} \to \|u\|_H$ and $(u_n, \Phi_n(\varphi))_{H_n} \to (u, \varphi)_H$ for every $\varphi \in C$.*

Proof. See Appendix A. □

Lemma 2.13. *Let $\{u_n\}$, $u_n \in H_n$, $n \in \mathbb{N}$ be a sequence in $\mathscr{H}$ and let $u \in H$. Then $u_n \to u$ weakly if and only if $\sup_n \|u_n\|_{H_n} < \infty$ and $(u_n, \Phi_n(\varphi))_{H_n} \to (u, \varphi)_H$ for every $\varphi \in C$.*

Proof. See Appendix A. □

Lemma 2.14. *(1) Let $\{u_n\}$ be a sequence with $u_n \in H_n$, $n \in \mathbb{N}$. If the sequence of norms $\{\|u_n\|_{H_n}\}$ is bounded, there exists a weakly convergent subsequence of $\{u_n\}$.*

(2) Let $\{u_n\}$, $u_n \in H_n$, $n \in \mathbb{N}$ be a sequence which weakly converges to $u \in H$. Then

$$\sup_n \|u_n\|_{H_n} < \infty, \quad \|u\|_H \leq \varliminf_n \|u_n\|_{H_n}.$$

Moreover, $u_n \to u$ strongly if and only if

$$\|u\|_H = \lim_n \|u_n\|_{H_n}.$$

(3) A sequence $\{u_n\}$, $u_n \in H_n$, $n \in \mathbb{N}$ tends to $u \in H$ $\mathscr{H}$-strongly if and only if

$$(u_n, v_n)_{H_n} \to (u, v)_H$$

for every $\{v_n\}$, $v_n \in H_n$, $n \in \mathbb{N}$ $\mathscr{H}$-weakly tending to $v \in H$.

Proof. See Appendix A. □

2.1.3. Convergence of bounded operators

Definition 2.15 (Convergence of bounded operators). $\{B_n\}$, $B_n \in \mathscr{L}(H_n)$ are said to *strongly (weakly) converge* to $B \in \mathscr{L}(H)$ if for every sequence $\{u_n\}$, $u_n \in H_n$ strongly (weakly) converging to $u \in H$, $\{B_n u_n\}$ strongly (weakly) converges to Bu.

$\{B_n\}$, $B_n \in \mathscr{L}(H_n)$ is said to *compactly converge* to $B \in \mathscr{L}(H)$ if for every sequence $\{u_n\}$, $u_n \in H_n$ weakly converging to $u \in H$, $\{B_n u_n\}$ strongly converges to Bu.

Clearly, compact convergence of a sequence of bounded operators implies both weak and strong convergence of this sequence.

Lemma 2.16. *Let $\{B_n\}$ be a sequence of bounded operators, $B_n \in \mathscr{L}(H_n)$, $B \in \mathscr{L}(H)$. Then we have:*

(1) $B_n \to B$ strongly if and only if

$$\lim_n (B_n u_n, v_n)_{H_n} = (Bu, v)_H \tag{2.9}$$

for any $\{u_n\}$, $\{v_n\}$, u, v such that $u_n \to u$ strongly and $v_n \to v$ weakly.

(2) $B_n \to B$ weakly if and only if (2.9) *holds for any $\{u_n\}$, $\{v_n\}$, u, v such that $u_n \to u$ weakly and $v_n \to v$ strongly.*

(3) $B_n \to B$ compactly if and only if (2.9) *holds for any $\{u_n\}$, $\{v_n\}$, u, v such that $u_n \to u$ weakly and $v_n \to v$ weakly.*

Proof. The lemma follows from the definitions of convergences and Lemma 2.14 (3). □

Lemma 2.17. *(1) If $B_n \to B$ strongly, then*

$$\varliminf_n \|B_n\|_{\mathscr{L}(H_n)} \geq \|B\|_{\mathscr{L}(H)}.$$

(2) If $B_n \to B$ compactly, then

$$\lim_n \|B_n\|_{\mathscr{L}(H_n)} = \|B\|_{\mathscr{L}(H)}.$$

Proof. See Appendix A. □

Denote by $\hat{A}$ the adjoint of an operator A. The following is a direct consequence of Lemma 2.16.

Corollary 2.18. *(1) $B_n \to B$ strongly if and only if $\hat{B}_n \to \hat{B}$ weakly. In particular, strong convergence is equivalent to weak convergence for symmetric operators.*

(2) $B_n \to B$ compactly if and only if $\hat{B}_n \to \hat{B}$ compactly.

It is very important to realize, that in this point the Kuwae-Shioya framework differs from the case of one fixed Hilbert space, where strong operator convergence implies weak operator convergence. Also uniform operator convergence does not make any sense in this framework.

Lemma 2.19. *If $B_n \to B$ compactly, then B and $\hat{B}$ are both compact operators.*

Proof. See Appendix A. □

The next Lemma shows that C contains enough information to verify strong convergence of a sequence of operators, if strong convergence along the sequence $\{\Phi_n(\varphi)\}$, $\varphi \in C$ is assumed. This has not been proved in this setting before.

Lemma 2.20. *Let $B_n \in \mathscr{L}(H_n)$, $n \in \mathbb{N}$, $B \in \mathscr{L}(H)$, such that* $\sup_n \|B_n\|_{\mathscr{L}(H_n)} < \infty$.

(1) If $B_n \Phi_n \varphi \to B\varphi$ $\mathscr{H}$-strongly for every $\varphi \in C$,

(2) or if there exists a dense linear subspace $\tilde{C} \subset H$ (which might be taken equal to H) such that $B_n \Psi_n \psi \to B\psi$ $\mathscr{H}$-strongly for every $\psi \in \tilde{C}$,

then $B_n \to B$ strongly in the sense of Definition 2.15.

Proof. We prove only (2), since the proof of (1) is similar except $B_n\Psi_n\psi_m$ in (2.10) is replaced by $B_n\Phi_n\psi_m$. Let $u \in H$ and $u_n \in H_n$, $n \in \mathbb{N}$ with $u_n \to u$ $\mathscr{H}$-strongly. Pick $\psi_m \to u$ strongly in H with $\psi_m \in \tilde{C}$ for every m. Clearly,

$$\begin{aligned}
&\|B_n u_n - \Psi_n B u\|_{H_n} \\
\leq &\|B_n u_n - B_n \Psi_n \psi_m\|_{H_n} + \|B_n \Psi_n \psi_m - \Psi_n B \psi_m\|_{H_n} + \|\Psi_n B \psi_m - \Psi_n B u\|_{H_n} \\
\leq &\sup_n \|B_n\|_{\mathscr{L}(H_n)} \|u_n - \Psi_n \psi_m\|_{H_n} + \|B_n \Psi_n \psi_m - \Psi_n B \psi_m\|_{H_n} + \|B\|_{\mathscr{L}(H)} \|\psi_m - u\|_H.
\end{aligned} \tag{2.10}$$

The first term tends to 0 as $m, n \to \infty$ by the uniform boundedness of $\{\|B_n\|_{\mathscr{L}(H_n)}\}$ and

$$\|u_n - \Psi_n \psi_m\|_{H_n} \leq \|u_n - \Psi_n u\|_{H_n} + \|\Psi_n u - \Psi_n \psi_m\|_{H_n} = \|u_n - \Psi_n u\|_{H_n} + \|u - \psi_m\|_H.$$

The second term tends to 0 as $m, n \to \infty$ by assumption as clearly the last term, too. This proves the assertion. □

2.1.4. Convergence of semigroups

In this section we prove a generalization of Kato's Theorem for strong convergence of semigroups (cf. [Kat66, Theorem IX.2.16]), i.e., that strong convergence of (not necessarily symmetric) contraction semigroups is equivalent to strong convergence of the associated contraction resolvents. We point out that this result is entirely new for the Kuwae-Shioya framework and that the proof uses techinques turning out to be useful only in this particular framework, for the standard proof cannot be assigned one-to-one.

To this end, let $(A^n, D(A^n))$, $n \in \mathbb{N}$, $(A, D(A))$ resp. be the infinitesimal generators of (not necessarily symmetric) C_0-contraction-semigroups $(T_t^n)_{t\geq 0}$, $n \in \mathbb{N}$, $(T_t)_{t\geq 0}$ resp. defined on subspaces $D(A^n) \subset H_n$, $n \in \mathbb{N}$, $D(A) \subset H$. Let $(G_\alpha^n)_{\alpha>0}$, $n \in \mathbb{N}$, $(G_\alpha)_{\alpha>0}$ resp. be the associated C_0-contraction-resolvents.

Theorem 2.21. *Let $(T_t^n)_{t\geq 0}$, $n \in \mathbb{N}$, $(T_t)_{t\geq 0}$, $(G_\alpha^n)_{\alpha>0}$, $n \in \mathbb{N}$, $(G_\alpha)_{\alpha>0}$ be as above. Then $G_\alpha^n \to G_\alpha$ strongly for any $\alpha > 0$ if and only if $T_t^n \to T_t$ strongly for any $t \geq 0$.*

Proof. Let us first prove the "if"-part: Let $\alpha > 0$, $u_n \in H_n$, $n \in \mathbb{N}$, $u \in H$, $u_n \to u$ $\mathscr{H}$-strongly. Then $T_t^n u_n \to T_t u$ strongly for every $t \geq 0$. We express $G_\alpha^n u_n$ in terms of the following H_n-valued Bochner integral

$$G_\alpha^n u_n = \int_0^\infty e^{-\alpha s} T_s^n u_n ds,$$

(cf. [MR92, Section I.1] and, particularly for Bochner integrals: [Yos78, Section V.5]). Let $v_n \in H_n$, $n \in \mathbb{N}$, $v \in H$ such that $v_n \to v$ weakly in $\mathscr{H}$. It suffices to prove that

$$(G_\alpha^n u_n, v_n)_{H_n} \to (G_\alpha u, v)_H.$$

We note that

$$\sup_n |e^{-\alpha s}(T_s^n u_n, v_n)_{H_n}| \le e^{-\alpha s} \sup_n \|u_n\|_{H_n} \|v_n\|_{H_n} \le C e^{-\alpha s}$$

for some constant $C > 0$ by the contraction property of the T_s^n's, Cauchy's inequality and strong and weak convergence of $\{u_n\}$, $\{v_n\}$ resp. applied to Lemma 2.14 (2). The right-hand side is integrable, so we can use Lebesgue's dominated convergence theorem and the well-known fact, that Bochner integrals interchange with continuous linear functionals (cf. [Yos78, Corollary V.5.2]) to obtain

$$\begin{aligned}
\lim_{n\to\infty} (G_\alpha^n u_n, v_n)_{H_n} &= \lim_{n\to\infty} (\int_0^\infty e^{-\alpha s} T_s^n u_n \, ds, v_n)_{H_n} \\
&= \lim_{n\to\infty} \int_0^\infty e^{-\alpha s} (T_s^n u_n, v_n)_{H_n} ds = \int_0^\infty e^{-\alpha s} (T_s u, v)_H ds \\
&= (\int_0^\infty e^{-\alpha s} T_s u \, ds, v)_H = (G_\alpha u, v)_H.
\end{aligned}$$

The "if"-part is proved.

To prove the "only if"-part, let $G_\alpha^n \to G_\alpha$ strongly for $\alpha > 0$. Recall that for each $\alpha > 0$, $n \in \mathbb{N}$, we have $G_\alpha^n(H_n) = D(A^n)$, $G_\alpha(H) = D(A)$ (cf. [MR92, Section I.1.a)]). We would like to prove for each $\psi \in H$, $t \ge 0$, $\alpha > 0$ that

$$T_t^n \Psi_n G_\alpha \psi \to T_t G_\alpha \psi \tag{2.11}$$

strongly in $\mathscr{H}$. Since $D(A) \subset H$ densely (e.g. by [MR92, Proposition I.1.10]), we can apply Lemma 2.20 (2) and we are done (note that $\sup_n \|T_t^n\|_{\mathscr{L}(H_n)} \le 1 < \infty$).

Throughout the following fix $\alpha > 0$. Now let $u \in H$ and $s \ge 0$. We have

$$\frac{d}{ds} T_s G_\alpha u = T_s A G_\alpha u = -T_s(1 - \alpha G_\alpha) u$$

and the same with T_s^n, A^n and G_α^n replacing T_s, A and G_α resp. Now for $t \ge s \ge 0$

$$\begin{aligned}
&\frac{d}{ds} T_{t-s}^n G_\alpha^n \Psi_n T_s G_\alpha u \\
&= T_{t-s}^n G_\alpha^n \Psi_n \left(\frac{d}{ds} T_s G_\alpha \right) + \frac{d}{ds'} T_{t-s'}^n G_\alpha^n \Psi_n T_s G_\alpha u \bigg|_{s'=s} \\
&= T_{t-s}^n G_\alpha^n \Psi_n (-T_s(1 - \alpha G_\alpha) u) + T_{t-s}^n (1 - \alpha G_\alpha^n) \Psi_n T_s G_\alpha u \\
&= -T_{t-s}^n (G_\alpha^n \Psi_n T_s - \Psi_n T_s G_\alpha) u,
\end{aligned}$$

where the "product rule" is proved the same way as in finite dimensions. Note that we have used that Ψ_n is continuous and the fact that $T_s G_\alpha u = G_\alpha T_s u$ for $u \in H$ (cf.

[Kat66, Chapter IX.]). Applying the fundamental theorem of calculus we obtain

$$
\begin{aligned}
-\int_0^t T^n_{t-s}(G^n_\alpha \Psi_n T_s - \Psi_n G_\alpha T_s)u\,ds &= \int_0^t \frac{d}{ds} T^n_{t-s} G^n_\alpha \Psi_n G_\alpha T_s u\,ds \\
&= T^n_{t-s} G^n_\alpha \Psi_n G_\alpha T_s u \Big|_{s=0}^{s=t} = G^n_\alpha \Psi_n G_\alpha T_t u - T^n_t G^n_\alpha \Psi_n G_\alpha u.
\end{aligned}
$$

Hence by Bochner's inequality (cf. [Yos78, Corollary V.5.1])

$$
\|G^n_\alpha \Psi_n G_\alpha T_t u - T^n_t G^n_\alpha \Psi_n G_\alpha u\|_{H_n} \le \int_0^t \|(G^n_\alpha \Psi_n T_s - \Psi_n G_\alpha T_s)u\|_{H_n} ds. \tag{2.12}
$$

First note that $\|G^n_\alpha \Psi_n T_s u - \Psi_n G_\alpha T_s u\|_{H_n} \to 0$ as $n \to \infty$ by Lemma 2.8 and the fact that both $G^n_\alpha \Psi_n T_s u \to G_\alpha T_s u$ and $\Psi_n G_\alpha T_s u \to G_\alpha T_s u$ $\mathscr{H}$-strongly, which follows from the strong convergence of resolvents. It is easy to see that $\|(G^n_\alpha \Psi_n T_s - \Psi_n T_s G_\alpha)u\|_{H_n} \le \frac{2}{\alpha}\|u\|_H$ for every n. We conclude that by Lebesgue's dominated convergence theorem the right-hand side of (2.12) tends to zero as $n \to \infty$. Altogether, for $u \in H$,

$$
\lim_n \|G^n_\alpha \Psi_n T_t G_\alpha u - T^n_t G^n_\alpha \Psi_n G_\alpha u\|_{H_n} = 0
$$

and since $G_\alpha(H) = D(A)$, and $D(A) \subset H$ densely, we get by a 3-ε-argument, taking into account that $\|G^n_\alpha\|_{H_n} \le \alpha^{-1}$ for every n, that for every $\psi \in H$

$$
\lim_n \|G^n_\alpha \Psi_n T_t \psi - T^n_t G^n_\alpha \Psi_n \psi\|_{H_n} = 0. \tag{2.13}
$$

As another result of strong convergence of resolvents and Lemma 2.8 we get (using the contraction property of the T^n_t's)

$$
\lim_n \|T^n_t \Psi_n G_\alpha \psi - T^n_t G^n_\alpha \Psi_n \psi\|_{H_n} \le \lim_n \|\Psi_n G_\alpha \psi - G^n_\alpha \Psi_n \psi\|_{H_n} = 0, \tag{2.14}
$$

and

$$
\lim_n \|G^n_\alpha \Psi_n T_t \psi - \Psi_n G_\alpha T_t \psi\|_{H_n} = 0. \tag{2.15}
$$

To prove (2.11), let $\psi \in H$, $t \ge 0$, $\alpha > 0$. One easily observes

$$
\begin{aligned}
&\|T^n_t \Psi_n G_\alpha \psi - \Psi_n T_t G_\alpha \psi\|_{H_n} \\
\le &\|T^n_t \Psi_n G_\alpha \psi - T^n_t G^n_\alpha \Psi_n \psi\|_{H_n} + \|T^n_t G^n_\alpha \Psi_n \psi - G^n_\alpha \Psi_n T_t \psi\|_{H_n} + \|G^n_\alpha \Psi_n T_t \psi - \Psi_n G_\alpha T_t \psi\|_{H_n}
\end{aligned}
$$

Gathering (2.14), (2.13) and (2.15) and the last equation above, we get the desired result as $n \to \infty$. The proof is complete. □

2.2. Convergence of symmetric forms

Now we consider three different types of convergence of a sequence of symmetric forms $\{\mathscr{E}_n\}$ along $\mathscr{H}$, where each form is defined on H_n. Recall that a quadratic form is a bilinear mapping $\mathscr{E} : \mathscr{D}(\mathscr{E}) \times \mathscr{D}(\mathscr{E}) \to \mathbb{R}$ defined on some subspace $\mathscr{D}(\mathscr{E}) \subset H$. In this section we consider only *non-negative* and *symmetric* quadratic forms, that is, $\mathscr{E}(u,u) \geq 0$ for every $u \in \mathscr{D}(\mathscr{E})$ and $\mathscr{E}(u,v) = \mathscr{E}(v,u)$ for every $u, v \in \mathscr{D}(\mathscr{E})$. Define for $\alpha > 0$ and a form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ the inner product

$$\mathscr{E}_\alpha(u,v) := \mathscr{E}(u,v) + \alpha(u,v)_H, \quad u, v \in \mathscr{D}(\mathscr{E}),$$

which makes $\mathscr{D}(\mathscr{E})$ a pre-Hilbert space. Recall that a form $\mathscr{E}$ is *closed* if $\mathscr{D}(\mathscr{E})$ equipped with the norm $\mathscr{E}_1^{1/2}$ is complete. We identify a quadratic form $\mathscr{E}$ with the functional

$$\mathscr{E}(u) : u \mapsto \begin{cases} \mathscr{E}(u,u) & , \ u \in \mathscr{D}(\mathscr{E}) \\ \infty & , \ u \notin \mathscr{D}(\mathscr{E}). \end{cases} \tag{2.16}$$

It is well known that $\mathscr{E}$ is closed if and only if $\mathscr{E} : H \to \overline{\mathbb{R}}$ is lower-semicontinuous (see for instance [Mos94, p. 372]). We shall use the notions quadratic form and bilinear form interchangeably if the form is symmetric and non-negative, which is justified by the polarization identity:

$$\mathscr{E}(u,v) = \frac{1}{4}\left[\mathscr{E}(u+v, u+v) - \mathscr{E}(u-v, u-v)\right], \quad u, v \in \mathscr{D}(\mathscr{E}).$$

Also note that $\mathscr{E}(u) := \mathscr{E}(u,u)$, $u \in \mathscr{D}(\mathscr{E})$ is used for the diagonal even for non-symmetric forms.

Definition 2.22. A sequence $\{\mathscr{E}^n : H_n \to \overline{\mathbb{R}}\}$ of symmetric, non-negative, closed forms is said to *Mosco converge* to a quadratic form $\mathscr{E}$ on H if the following two conditions hold:

(M1) If $\{u_n\}$, $u_n \in H_n$, $n \in \mathbb{N}$ weakly converges to $u \in H$ then

$$\mathscr{E}(u) \leq \underline{\lim}_n \mathscr{E}^n(u_n).$$

(M2) For every $u \in H$ there exists a strongly convergent sequence $u_n \to u$, $u_n \in H_n$, $n \in \mathbb{N}$ such that

$$\mathscr{E}(u) = \lim_n \mathscr{E}^n(u_n).$$

Definition 2.23. A sequence $\{\mathscr{E}^n : H_n \to \overline{\mathbb{R}}\}$ of symmetric, non-negative, closed forms is said to Γ-*converge* to a quadratic form $\mathscr{E}$ on H if the following conditions are fulfilled:

(G1) If $\{u_n\}$, $u_n \in H_n$, $n \in \mathbb{N}$ strongly converges to $u \in H$ then

$$\mathscr{E}(u) \leq \underline{\lim}_n \mathscr{E}^n(u_n).$$

(G2) For every $u \in H$ there exists a strongly convergent sequence $u_n \to u$ with $u_n \in H_n$, $n \in \mathbb{N}$ such that
$$\mathscr{E}(u) = \lim_n \mathscr{E}^n(u_n).$$

The above conditions, especially (M2) resp. (G2) make sense by Corollary 2.11 and (2.16). It is clear that Mosco convergence implies Γ-convergence. To determine when they are equivalent, consider the following:

Definition 2.24. A sequence $\{\mathscr{E}^n\}$ is called *asymptotically compact* if for every $\{u_n\}$, $u_n \in H_n$, $n \in \mathbb{N}$ such that
$$\overline{\lim_n}(\mathscr{E}^n(u_n) + \|u_n\|^2_{H_n}) < \infty,$$
there exists a strongly convergent subsequence of $\{u_n\}$.

Lemma 2.25. *Assume that $\{\mathscr{E}^n\}$ is asymptotically compact. Then $\{\mathscr{E}^n\}$ Γ-converges to $\mathscr{E}$ if and only if $\{\mathscr{E}^n\}$ Mosco converges to $\mathscr{E}$.*

Proof. The proof of [Mos94, Lemma 2.3.2] can be extended to our framework easily. □

Note that Γ-convergence can be defined for arbitrary functionals on a topological space with values in $\overline{\mathbb{R}}$ (see for instance [DM93]) and every Γ-limit is lower-semicontinuous. In particular, it means that if the form $\mathscr{E}$ is a Γ-limit, then it is closed.

It is a well-known fact that every sequence of functionals on a second-countable space with values in $\overline{\mathbb{R}}$ has a Γ-convergent subsequence (see [DM93, Theorem 8.5]). So in our case we have the following (see [KS03, Theorem 2.3]):

Theorem 2.26. *Every sequence $\{\mathscr{E}^n\}$ of symmetric, non-negative quadratic forms (with values in $\overline{\mathbb{R}}$) has a Γ-convergent subsequence whose Γ-limit is a symmetric, non-negative, closed quadratic form on H.*

This just means that the space of symmetric, non-negative forms is relatively sequentially compact w.r.t. the Γ-topology.

Definition 2.27. We say that $\mathscr{E}^n \to \mathscr{E}$ *compactly* if $\mathscr{E}^n \to \mathscr{E}$ Mosco and if $\{\mathscr{E}^n\}$ is asymptotically compact.

Lemma 2.25 and Theorem 2.26 together imply:

Corollary 2.28. *If $\{\mathscr{E}^n\}$ is asymptotically compact, it has a compact convergent subsequence.*

With every non-negative symmetric closed form $\mathscr{E}$ we associate a non-negative self-adjoint operator $-A$ with $\mathscr{D}(A) \subset \mathscr{D}(\sqrt{-A}) = \mathscr{D}(\mathscr{E})$ and $\mathscr{E}(u,v) = (\sqrt{-A}u, \sqrt{-A}v)$,

$u, v \in \mathscr{D}(\mathscr{E})$. Let $T_t := e^{tA}$, $t \geq 0$ be the associated semigroup and $G_\alpha := (\alpha - A)^{-1}$, $\alpha > 0$ the associated resolvent (see [FOT94] for details).

For further reference for convergence problems of spectral structures (in the symmetric case) we would again like to mention [KS03, Section 2]. The next Theorem (whose proof has been reformulated for this paper and can be found in the Appendix) shows the essential power of Mosco convergence, which is - as a variational convergence - necessary and sufficient for strong convergence of resolvents and semigroups.

Theorem 2.29 (Mosco, Kuwae, Shioya). *Let $\{\mathscr{E}^n : H_n \to \overline{\mathbb{R}}\}$ be a sequence of non-negative, symmetric, closed forms and let $\mathscr{E}$ be a closed form on H. The following statements are equivalent:*

(1) $\{\mathscr{E}^n\}$ Mosco converges to $\mathscr{E}$,

(2) $\{G^n_\alpha\}$ strongly converges to G_α for all $\alpha > 0$,

(3) $\{T^n_t\}$ strongly converges to T_t for all $t \geq 0$.

Proof. See Appendix A. $\square$

Now we shall state some useful conditions on a sequence of forms $\{\mathscr{E}^n : H_n \to \overline{\mathbb{R}}\}$, which can easily be checked in many applications and give us a nice criterion for Mosco convergence. The idea is to restrict strong and weak convergence of vectors along the sequence of domains $\{\mathscr{D}(\mathscr{E}^n)\}$, which contains enough information to verify Mosco convergence.

Definition 2.30. Suppose that we are given a convergent sequence of Hilbert spaces $H_n \to H$ and a sequence of non-negative symmetric closed forms $\{\mathscr{E}^n : H_n \to \overline{\mathbb{R}}\}$. We say that *a sequence of pairs $\{(H_n, \mathscr{E}^n)\}$ converges to $(H, \mathscr{E})$* if the following conditions hold

(1) $\Phi_n(C) \subset \mathscr{D}(\mathscr{E}^n)$ for every $n \in \mathbb{N}$.

(2) $C \subset \mathscr{D}(\mathscr{E})$ is dense in $(\mathscr{D}(\mathscr{E}), \mathscr{E}_1^{1/2})$.

(3) $\lim_n \mathscr{E}^n(\Phi_n(\varphi)) = \mathscr{E}(\varphi)$ for every $\varphi \in C$.

If we have a convergent sequence $\{(H_n, \mathscr{E}^n)\}$ of pairs in the above sense, we have a convergent sequence of Hilbert spaces $\{(\mathscr{D}(\mathscr{E}^n), \mathscr{E}^n_1(\ ,\ ))\}$ which converges to $(\mathscr{D}(\mathscr{E}), \mathscr{E}_1(\ ,\ ))$ in the sense of Definition 2.3. Let us denote the space $\dot{\bigcup}_n \mathscr{D}(\mathscr{E}^n) \dot{\cup} \mathscr{D}(\mathscr{E})$ by $\mathscr{H}_{\mathscr{E}}$. Hence we can construct the corresponding isometries $\Psi^{\mathscr{E}}_n$, $n \in \mathbb{N}$ as in Theorem 2.10. Then for every $u \in \mathscr{D}(\mathscr{E})$, $\Psi^{\mathscr{E}}_n(u) \to u$ strongly in $\mathscr{H}_{\mathscr{E}}$.

Lemma 2.31. *Assume that $\{(H_n, \mathscr{E}^n)\}$ converges to $(H, \mathscr{E})$. If $u_n \to u$ strongly in $\mathscr{H}_{\mathscr{E}}$, then $u_n \to u$ strongly in $\mathscr{H}$.*

Proof. Let $u_n \to u$, $u_n \in \mathscr{D}(\mathscr{E}^n)$, $n \in \mathbb{N}$, $u \in \mathscr{D}(\mathscr{E})$ strongly in $\mathscr{H}_{\mathscr{E}}$. Then there exists a sequence $\{\varphi_m\} \subset C$ with

$$\|u - \varphi_m\|_H^2 \leq \text{const.}\mathscr{E}_1(u - \varphi_m) \to 0$$

as $m \to \infty$. Clearly,

$$\lim_m \overline{\lim_n} \|u_n - \Phi_n(\varphi_m)\|_{H_n}^2 \leq \text{const.} \lim_m \overline{\lim_n} \mathscr{E}_1^n(u_n - \Phi_n(\varphi_m)) = 0,$$

hence $u_n \to u$ strongly in $\mathscr{H}$. The case $\mathscr{D}(\mathscr{E}^{n_k}) \to \mathscr{D}(\mathscr{E}^{n_0})$ for some arbitrary sequence of natural numbers with $\lim_k n_k \to n_0$ is trivial. □

The same is not true for weak convergence. More precisely, we have the following

Proposition 2.32. *Assume that $\{(H_n, \mathscr{E}^n)\}$ converges to $(H, \mathscr{E})$. If for any sequence $u_n \to u$ weakly in $\mathscr{H}_{\mathscr{E}}$, $u_n \in \mathscr{D}(\mathscr{E}^n)$, $n \in \mathbb{N}$, $u \in \mathscr{D}(\mathscr{E})$ we have that $u_n \to u$ weakly in $\mathscr{H}$, too, then $\mathscr{E}^n \to \mathscr{E}$ Mosco.*

For the proof, we need the following

Lemma 2.33. *Assume that $\{(H_n, \mathscr{E}^n)\}$ converges to $(H, \mathscr{E})$ and (M1) holds. Then $\mathscr{E}^n \to \mathscr{E}$ Mosco (i.e., (M1) $\Rightarrow$ (M2) in this case).*

Proof. Let us take $u \in \mathscr{D}(\mathscr{E})$. Clearly, $\Psi_n^{\mathscr{E}}(u) \to u$ $\mathscr{H}_{\mathscr{E}}$-strongly and also $\mathscr{H}$-strongly by Lemma 2.31. By Lemma 2.12 we have

$$\lim_n \mathscr{E}_1^n(\Psi_n^{\mathscr{E}}(u)) = \mathscr{E}_1(u)$$

and

$$\lim_n \|\Psi_n^{\mathscr{E}}(u)\|_{H_n}^2 = \|u\|_H^2.$$

Therefore

$$\lim_n \mathscr{E}^n(\Psi_n^{\mathscr{E}}(u)) = \mathscr{E}(u)$$

which gives us (M2) for $u \in \mathscr{D}(\mathscr{E})$. If $u \in H \setminus \mathscr{D}(\mathscr{E})$ (M2) holds for any $\mathscr{H}$-strongly convergent sequence $u_n \to u$ as a consequence of (M1). □

Proof of Proposition 2.32. We only have to prove (M1) by the preceding Lemma. Let $u_n \in H_n$, $n \in \mathbb{N}$, $u \in H$ with $u_n \to u$ $\mathscr{H}$-weakly. If $\underline{\lim}_n \mathscr{E}^n(u_n, u_n) = +\infty$ there is nothing to prove. So assume that $\underline{\lim}_n \mathscr{E}^n(u_n, u_n) < +\infty$, which gives that for a subsequence $\{u_{n_k}\}$ we have $\sup_k \mathscr{E}^{n_k}(u_{n_k}) < +\infty$. But by Lemma 2.14 (2) $\sup_n \|u_n\|_{H_n} < +\infty$. By extracting another subsequence if necessary we can use Lemma 2.14 (1) to force that for some $\tilde{u} \in \mathscr{D}(\mathscr{E})$ we have $u_{n_k} \to \tilde{u}$ $\mathscr{H}_{\mathscr{E}}$-weakly and $\lim_k \mathscr{E}^{n_k}(u_{n_k}) = \underline{\lim}_n \mathscr{E}^n(u_n)$. By assertion $u_{n_k} \to \tilde{u}$ $\mathscr{H}$-weakly, too, which shows $\tilde{u} = u$. Now,

$$\lim_k \mathscr{E}_1^n(u_{n_k}, \Psi_k^{\mathscr{E}}(u)) = \mathscr{E}_1(u, u)$$

and by Lemma 2.31

$$\lim_k (u_{n_k}, \Psi_k^{\mathscr{E}}(u))_{H_n} = (u,u)_H.$$

Combining the last two equalities, we get

$$\lim_k \mathscr{E}(u_{n_k}, \Psi_k^{\mathscr{E}}(u)) = \mathscr{E}(u,u).$$

Therefore,

$$\begin{aligned}\mathscr{E}(u,u)^2 &= \lim_k |\mathscr{E}^{n_k}(u_{n_k}, \Psi_k^{\mathscr{E}}(u))|^2 \\ &\le \lim_k \mathscr{E}^{n_k}(u_{n_k}, u_{n_k})\mathscr{E}^{n_k}(\Psi_k^{\mathscr{E}}(u), \Psi_k^{\mathscr{E}}(u)) \le \varliminf_n \mathscr{E}(u_n, u_n)\mathscr{E}(u,u),\end{aligned}$$

which gives us (M1). □

2.3. Convergence of non-symmetric forms

2.3.1. Generalized forms

To analyze convergence of a sequence of non-symmetric forms defined on different Hilbert spaces, we will assume our forms to be so called generalized (Dirichlet) forms, following the framework of [Sta99, Section I]. In other words, we assume our form $\mathscr{E}^n$ to be associated with some coercive closed form $(\mathscr{A}^n, \mathscr{V}_n)$ and some properly chosen linear operator $(\Lambda^n, \mathscr{D}(\Lambda^n, H_n))$. We prove necessary and sufficient conditions on the forms $\{\mathscr{E}^n\}$ for strong convergence of the associated resolvents and semigroups. We also give conditions for the Mosco-convergence of the symmetric parts $\{\tilde{\mathscr{A}}^n\}$ of $\{\mathscr{E}^n\}$ to the symmetric part $\tilde{\mathscr{A}}$ of $\mathscr{E}$.

Now let $\mathscr{A}$ be a bilinear form on H with a domain $\mathscr{V} \subset H$. The *symmetric part* $\tilde{\mathscr{A}}$ of $\mathscr{A}$ is defined by

$$\tilde{\mathscr{A}}(u,v) := \frac{1}{2}\left[\mathscr{A}(u,v) + \mathscr{A}(v,u)\right], \quad u,v \in \mathscr{V}.$$

The *antisymmetric part* $\check{\mathscr{A}}$ of $\mathscr{A}$ is defined by

$$\check{\mathscr{A}}(u,v) := \frac{1}{2}\left[\mathscr{A}(u,v) - \mathscr{A}(v,u)\right], \quad u,v \in \mathscr{V}.$$

It is clear that $\mathscr{A} = \tilde{\mathscr{A}} + \check{\mathscr{A}}$. For $\alpha > 0$, set

$$\mathscr{A}_\alpha(u,v) := \mathscr{A}(u,v) + \alpha(u,v)_H, \quad u,v \in \mathscr{V}.$$

$\tilde{\mathscr{A}}_\alpha$ is defined similarly. We suppose that $(\mathscr{A}, \mathscr{V})$ is a *coercive closed form* with sector constant $K \ge 1$, that is,

(1) $(\tilde{\mathscr{A}}, \mathscr{V})$ is a non-negative definite, symmetric, closed form,

(2) $(\mathscr{A},\mathscr{V})$ satisfies the *weak sector condition*, i.e., there exists a sector constant $K \geq 1$ such that
$$|\mathscr{A}_1(u,v)| \leq K \mathscr{A}_1(u,u)^{1/2} \mathscr{A}_1(v,v)^{1/2} \quad \text{for all } u,v \in \mathscr{V}.$$

Equipped with the norm $\| \ \|_{\mathscr{V}} := \tilde{\mathscr{A}}_1^{1/2}(\ )$, $\mathscr{V}$ becomes a Hilbert space. Identifying H with its dual H' we obtain a dense and continuous embedding $\mathscr{V} \subset H \equiv H' \subset \mathscr{V}'$. The pairing between $\mathscr{V}$ and $\mathscr{V}'$ is expressed by ${}_{\mathscr{V}'}(\ ,\ )_{\mathscr{V}}$.

Let Λ be a linear operator on H with a linear domain $\mathscr{D}(\Lambda, H)$. We assume the following:

(1) Λ generates a C_0-semigroup of contractions $(U_t)_{t\geq 0}$ on H.

(2) $(U_t)_{t\geq 0}$ can be restricted to a C_0-semigroup of contractions on $\mathscr{V}$.

Denote the infinitesimal generator of the restricted semigroup by $(\Lambda, \mathscr{D}(\Lambda,\mathscr{V}))$. Note that the adjoint operator $(\hat{\Lambda}, \mathscr{D}(\hat{\Lambda},\mathscr{V}'))$ of $(\Lambda, \mathscr{D}(\Lambda,\mathscr{V}))$ also satisfies the conditions above. In particular, $\mathscr{D}(\Lambda, H) \cap \mathscr{V}$ is dense in $\mathscr{V}$. It follows from [Sta99, Lemma I.2.3] that $\Lambda : \mathscr{D}(\Lambda, H) \cap \mathscr{V} \to \mathscr{V}'$ is closable. Let us denote its closure by $(\Lambda, \mathscr{F})$. Then $\mathscr{F}$ is a Hilbert space with norm

$$\|\cdot\|_{\mathscr{F}} := \left(\|\cdot\|_{\mathscr{V}}^2 + \|\Lambda \cdot\|_{\mathscr{V}'}^2 \right)^{1/2}.$$

Furthermore, define $\hat{\mathscr{F}} := \mathscr{D}(\hat{\Lambda},\mathscr{V}') \cap \mathscr{V}$ with norm

$$\|\cdot\|_{\hat{\mathscr{F}}} := \left(\|\cdot\|_{\mathscr{V}}^2 + \|\hat{\Lambda} \cdot\|_{\mathscr{V}'}^2 \right)^{1/2}.$$

$\mathscr{F}$ and $\hat{\mathscr{F}}$ are dense in $\mathscr{V}$, ${}_{\mathscr{V}'}(\Lambda u, u)_{\mathscr{V}H} \leq 0$ for $u \in \mathscr{F}$, ${}_{\mathscr{V}'}(\hat{\Lambda} u, u)_{\mathscr{V}H} \leq 0$ for $u \in \hat{\mathscr{F}}$ and $\mathscr{D}(\Lambda,\mathscr{V})$ is dense in $\mathscr{F}$ (cf. [Sta99, Lemma I.2.5] and [Sta99, Lemma I.2.6]).

Now for given $\mathscr{A}$ and Λ, define the *bilinear form* $\mathscr{E}$ *associated with* $(\mathscr{A},\mathscr{V})$ *and* $(\Lambda, \mathscr{D}(\Lambda, H))$ on H by

$$\mathscr{E}(u,v) := \begin{cases} \mathscr{A}(u,v) - {}_{\mathscr{V}'}(\Lambda u, v)_{\mathscr{V}}, & \text{if } u \in \mathscr{F}, v \in \mathscr{V}, \\ \mathscr{A}(u,v) - {}_{\mathscr{V}'}(\hat{\Lambda} v, u)_{\mathscr{V}}, & \text{if } u \in \mathscr{V}, v \in \hat{\mathscr{F}}. \end{cases}$$

We extend $\mathscr{E}$ to a form defined on H and taking values in $\overline{\mathbb{R}}$ by setting $\mathscr{E}(u,v) = +\infty$ for every other case, even if $u \in H \setminus \mathscr{V}$ and $v = 0$.

We also define the *co-form* $\hat{\mathscr{E}}$ by

$$\hat{\mathscr{E}}(u,v) := \begin{cases} \mathscr{A}(v,u) - {}_{\mathscr{V}'}(\hat{\Lambda} u, v)_{\mathscr{V}}, & \text{if } u \in \hat{\mathscr{F}}, v \in \mathscr{V}, \\ \mathscr{A}(v,u) - {}_{\mathscr{V}'}(\Lambda v, u)_{\mathscr{V}}, & \text{if } u \in \mathscr{V}, v \in \mathscr{F}. \end{cases}$$

Remark 2.34. *Let $(\mathscr{A},\mathscr{V})$ be a coercive closed form and $\Lambda = 0$. Clearly $\mathscr{F} = \mathscr{V} = \hat{\mathscr{F}}$ and $\mathscr{E} = \mathscr{A}$ is a generalized form by* [Sta99, Example I.4.9 (i)]*. This is the case of (Dirichlet) forms as described in* [MR92].

Let us recall some useful facts.

As usually, we define for $\alpha > 0$

$$\mathscr{E}_\alpha(u,v) := \mathscr{E}(u,v) + \alpha(u,v)_H.$$

Proposition 2.35. *For all $\alpha > 0$ there exist continuous, linear bijections $W_\alpha : \mathscr{V}' \to \mathscr{F}$ and $\hat{W}_\alpha : \mathscr{V}' \to \hat{\mathscr{F}}$ such that*

$$\mathscr{E}_\alpha(W_\alpha f, v) = {}_{\mathscr{V}'}(f,v)_{\mathscr{V}} = \mathscr{E}_\alpha(v, \hat{W}_\alpha f)$$

for all $f \in \mathscr{V}'$, $v \in \mathscr{V}$. $(W_\alpha)_{\alpha>0}$ and $(\hat{W}_\alpha)_{\alpha>0}$ satisfy the resolvent equation (cf. [Sta99, Proposition I.3.4]).

Furthermore, there exists a unique C_0-resolvent $(G_\alpha)_{\alpha>0}$ and a unique C_0-coresolvent $(\hat{G}_\alpha)_{\alpha>0}$ on H (being the restrictions of W_α, $\hat{W}_\alpha$ resp. to H). such that for all $\alpha > 0$, $f \in H$ and $u \in \mathscr{V}$

$$\begin{aligned} &G_\alpha(H) \subset \mathscr{F}, \quad \hat{G}_\alpha(H) \subset \hat{\mathscr{F}}, \\ &\mathscr{E}_\alpha(G_\alpha f, u) = \mathscr{E}_\alpha(u, \hat{G}_\alpha f) = (f,u)_H. \end{aligned} \tag{2.17}$$

$\hat{G}_\alpha$ is the adjoint of G_α and αG_α, $\alpha\hat{G}_\alpha$ are contraction operators. Also, we have for $u \in \mathscr{V}$ that

$$\lim_{\alpha\to\infty} \alpha G_\alpha u = u$$

strongly in $\mathscr{V}$ and thus in H.

Proof. See [Sta99, Section I.3]. □

Note that the second line of (2.17) is equivalent with

$${}_{\mathscr{V}'}((M_\alpha - \Lambda)G_\alpha f, g)_{\mathscr{V}} = (f,g)_H = {}_{\mathscr{V}'}((\hat{M}_\alpha - \hat{\Lambda})\hat{G}_\alpha f, g)_{\mathscr{V}}, \quad f \in H,\ g \in \mathscr{V},$$

where for $\alpha > 0$ we set $M_\alpha : \mathscr{V} \to \mathscr{V}'$, ${}_{\mathscr{V}'}(M_\alpha u, \cdot)_{\mathscr{V}} := \mathscr{A}_\alpha(u,\cdot)$ and $\hat{M}_a : \mathscr{V} \to \mathscr{V}'$, ${}_{\mathscr{V}'}(\hat{M}_\alpha u, \cdot)_{\mathscr{V}} := \mathscr{A}_\alpha(\cdot, u)$.

Define approximate forms $\mathscr{E}^{(\beta)}$, $\beta > 0$ of $\mathscr{E}$ by

$$\mathscr{E}^{(\beta)}(u,v) = \beta(u - \beta G_\beta u, v)_H, \quad u,v \in H$$

and set $\mathscr{E}^{(\beta)}_\alpha(u,v) = \mathscr{E}^{(\beta)}(u,v) + \alpha(u,v)_H$.

Proposition 2.36. *(i) $\mathscr{E}^{(\beta)}(u,v) = \mathscr{E}(\beta G_\beta u, v)$ for $u \in H$, $v \in \mathscr{V}$.*

(ii) $\mathscr{E}^{(\beta)}(u,u) = \mathscr{E}(\beta G_\beta u, \beta G_\beta u) + \beta\|u - \beta G_\beta u\|_H^2$ *for* $u \in H$.

(iii) $\lim_{\beta\to\infty} \mathscr{E}^{(\beta)}(u,v) = \mathscr{E}(u,v)$ *for* $u \in \mathscr{F}$, $v \in \mathscr{V}$.

(iv) If $\sup_{\beta>0} \mathscr{E}_1^{(\beta)}(u,u) < \infty$, *then* $u \in \mathscr{V}$.

Proof. For (i)–(iii), see [MR92, Lemma I.2.11] and [Sta98, Proposition 2.7 (iii)].

(iv): Since $\mathscr{E}(v,v) \geq \mathscr{A}(v,v)$ for $v \in \mathscr{F}$ and βG_β is contractive, we have by (ii)

$$\begin{aligned}\mathscr{E}_1^{(\beta)}(u,u) &= \mathscr{E}(\beta G_\beta u, \beta G_\beta u) + \beta\|u - \beta G_\beta u\|_H^2 + \|u\|_H^2 \\ &\geq \mathscr{A}_1(\beta G_\beta u, \beta G_\beta u) + \beta\|u - \beta G_\beta u\|_H^2.\end{aligned}$$

Hence the assumption $\sup_{\beta>0} \mathscr{E}_1^{(\beta)}(u,u) < \infty$ implies that

$$\sup_{\beta>0} \mathscr{A}_1(\beta G_\beta u, \beta G_\beta u) < \infty, \tag{2.18}$$

$$\sup_{\beta>0} \beta\|u - \beta G_\beta u\|_H^2 < \infty. \tag{2.19}$$

From (2.19), $\beta G_\beta u \to u$ in H as $\beta \to \infty$. Combining this and (2.18), we have that $u \in \mathscr{V}$ by [MR92, Lemma I.2.12]. □

Let $(T_t)_{t\geq0}$, $(\hat{T}_t)_{t\geq0}$ resp. be the C_0-semigroup of contractions, the C_0-cosemigroup of contractions resp. associated with $(G_\alpha)_{\alpha>0}$, $(\hat{G}_\alpha)_{\alpha>0}$ resp.

2.3.2. Criteria of convergence

In this section we shall give necessary and sufficient conditions on a sequence of generalized forms $\{\mathscr{E}^n\}$ for the strong convergence of the associated resolvents $\{G_\alpha^n\}$ and equivalently the weak convergence of coresolvents $\{\hat{G}_\alpha^n\}$ (cf. Corollary 2.18). We point out, that in a natural way we have to introduce a condition on the asymmetry of the $\{\mathscr{E}^n\}$, which differs from (M1) and (M2), since clearly for symmetric forms all information is contained on the diagonal. To this end, much stronger assumptions have to be stated to obtain Mosco convergence of the symmetric parts $\{\tilde{\mathscr{A}}^n\}$.

In order to handle double indexed sequences we need the following Lemma, which is just an elegant way to apply standard diagonal arguments.

Lemma 2.37. *Let* X *be a metrizable space with some metric* d *and* $\{x_{n,m} \mid n,m \in \mathbb{N}\}$ *a double indexed sequence in* X, $\{x_m\}_{m\in\mathbb{N}} \subset X$, $x \in X$ *such that*

$$d - \lim_{n\to+\infty} x_{n,m} = x_m$$

and

$$d - \lim_{m\to+\infty} x_m = x.$$

Then there exists a mapping $n \mapsto m(n)$ increasing to $+\infty$ so that

$$d - \lim_{n\to\infty} x_{n,m(n)} = x.$$

Proof. See Appendix A. □

Corollary 2.38. *Suppose that double sequences $\{u_{i,j}\}_{i,j\in\mathbb{N}} \subset \mathscr{H}$, $\{a_{i,j}\}_{i,j\in\mathbb{N}} \subset \mathbb{R}$ and $u \in H$, $a \in \mathbb{R}$ satisfy that*

$$\begin{aligned} &d_{\mathscr{H}} - \lim_{j\to\infty} d_{\mathscr{H}} - \lim_{i\to\infty} u_{i,j} = u, \\ &\lim_{j\to\infty} \lim_{i\to\infty} a_{i,j} = a. \end{aligned}$$

Then there exists a mapping $i \mapsto \{j(i)\}$, increasing to $+\infty$, so that

$$\begin{aligned} &d_{\mathscr{H}} - \lim_{i\to\infty} u_{i,j(i)} = u, \\ &\lim_{j\to\infty} a_{i,j(i)} = a. \end{aligned}$$

Proof. Apply Lemma 2.37 to the product space $\mathscr{H} \times \mathbb{R}$ with the *(1-product) metric* $d_1((h_1,a_1),(h_2,a_2)) := d_{\mathscr{H}}(h_1,h_2) + |a_1 - a_2|$, $h_1,h_2 \in \mathscr{H}$, $a_1,a_2 \in \mathbb{R}$. □

Now we shall finally come to the criteria of convergence. First we define a functional, which measures the rate of asymmetry of our form $\mathscr{E}$, and is, indeed, an equivalent norm to $\| \ \|_{\hat{\mathscr{F}}}$ (cf. Lemma 2.39 below). So let

$$\Theta(u) := \sup_{\|v\|_{\mathscr{V}}=1} \mathscr{E}_1(v,u) = \|\mathscr{E}_1(\cdot,u)\|_{\mathscr{V}'}, \quad \text{for } u \in \hat{\mathscr{F}},$$

which is finite. If $u \in H \setminus \hat{\mathscr{F}}$, we extend Θ to a functional on H with values in $\overline{\mathbb{R}}$ by setting $\Theta(u) := +\infty$.

Lemma 2.39. *For $u \in \hat{\mathscr{F}}$, we have*

(i) $\Theta(u) \le K\|u\|_{\hat{\mathscr{F}}}$,

(ii) $\|u\|_{\mathscr{V}} \le \Theta(u)$,

(iii) $\|\hat{\Lambda}u\|_{\mathscr{V}'} \le (K+1)\Theta(u)$,

(iv) In particular, $\| \ \|_{\hat{\mathscr{F}}} \sim \Theta(\ )$.

Proof. (i):

$$\Theta(u) \le \sup_{\|v\|_{\mathscr{V}}=1} |\mathscr{A}_1(v,u)| + \sup_{\|v\|_{\mathscr{V}}=1} \left| {}_{\mathscr{V}'}(\hat{\Lambda}u, v)_{\mathscr{V}} \right| \le K\|u\|_{\mathscr{V}} + \|\hat{\Lambda}u\|_{\mathscr{V}'} \le K\|u\|_{\hat{\mathscr{F}}},$$

recalling that $K \ge 1$.

(ii):

$$\|u\|_{\mathscr{V}}^2 \le \mathscr{E}_1(u,u) = \|u\|_{\mathscr{V}} \mathscr{E}_1(\frac{u}{\|u\|_{\mathscr{V}}}, u) \le \|u\|_{\mathscr{V}} \sup_{\|v\|_{\mathscr{V}}=1} \mathscr{E}_1(v,u) = \|u\|_{\mathscr{V}} \Theta(u).$$

(iii): For $v \in \mathscr{V}$ we have by (ii)

$$\begin{aligned} {}_{\mathscr{V}'}(\hat{\Lambda} u, v)_{\mathscr{V}} &= \mathscr{A}_1(v,u) - \mathscr{E}_1(v,u) \\ &\le K\|v\|_{\mathscr{V}} \|u\|_{\mathscr{V}} + \|v\|_{\mathscr{V}} \Theta(u) \\ &\le (K+1)\|v\|_{\mathscr{V}} \Theta(u). \end{aligned}$$

Hence $\|\hat{\Lambda} u\|_{\mathscr{V}'} \le (K+1)\Theta(u)$.

(iv): Obvious by (i)–(iii).

□

We arrive at the main convergence Theorem of this section. It is a generalization of [Hin98, Theorem 3.1] by M. Hino. From now on, we consider that we are given forms $\{\mathscr{E}^n\}$, $\mathscr{E}$ resp. on H_n, H resp. The operators, spaces and norms related to $\mathscr{E}^n$ are denoted by supplementing a suffix n, such as G_α^n, Θ^n and $\mathscr{V}_n$. It is to be remarked, that the sector constants K_n of the $\mathscr{A}^n$'s are not necessarily assumed to be uniformly bounded.

Definition 2.40. Suppose $C \subset \mathscr{F}$ densely w.r.t. $\| \ \|_{\mathscr{F}}$. Consider the following conditions:

(F1) If a sequence $\{u_n\}$ weakly convergent to u in $\mathscr{H}$ satisfies $\underline{\lim}_n \Theta^n(u_n) < \infty$, then $u \in \mathscr{V}$.

(F2) For any $w \in C$, any $u \in \mathscr{V}$ and any sequence $\{u_n\}$ weakly convergent to u in $\mathscr{H}$, $u_n \in \mathscr{V}_n$, $n \in \mathbb{N}$, there exists $\{w_n\}$, $w_n \in H_n$, $n \in \mathbb{N}$ converging to w strongly in $\mathscr{H}$ such that $\lim_n \mathscr{E}^n(w_n, u_n) = \mathscr{E}(w,u)$.

(F2') There exists a linear subspace $\tilde{C}$ of C such that $\tilde{C} \subset \mathscr{F}$ densely w.r.t $\| \ \|_{\mathscr{F}}$ and for any sequence $n_k \uparrow \infty$ and every $w \in \tilde{C}$, $u \in \mathscr{V}$ and any sequence $\{u_k\}$, $u_k \in H_{n_k}$, $k \in \mathbb{N}$ converging weakly to u and satisfying $\sup_k \Theta^{n_k}(u_k) < \infty$ one has a sequence $\{w_k\}$, $w_k \in H_{n_k}$, $k \in \mathbb{N}$ converging $\mathscr{H}$-strongly to w with $\underline{\lim}_k \mathscr{E}^{n_k}(w_k, u_k) \le \mathscr{E}(w,u)$.

(R) $\{G_\alpha^n\}$ converges to G_α strongly for $\alpha > 0$.

(CR) $\{\hat{G}_\alpha^n\}$ converges to $\hat{G}_\alpha$ weakly for $\alpha > 0$.

Define also (F1a) (resp. (F1b)) by replacing $\Theta^n(u_n)$ by $\|u_n\|_{\hat{\mathscr{F}}_n}$ (resp. $\|u_n\|_{\mathscr{V}_n}$) in (F1) and (F2'a) (resp. (F2'b)) by replacing $\Theta^{n_k}(u_k)$ by $\|u_k\|_{\hat{\mathscr{F}}_{n_k}}$ (resp. $\|u_k\|_{\mathscr{V}_{n_k}}$) in (F2').

We have (F1b) $\Rightarrow$ (F1) and (F2'b) $\Rightarrow$ (F2') by Lemma 2.39 (ii).

Theorem 2.41. *Suppose that $C \subset \mathscr{F}$ densely w.r.t. $\| \ \|_{\mathscr{F}}$. Then*

$$\text{(F2)} \quad \Rightarrow \quad \text{(F2')},$$

$$\text{(F1)(F2')} \quad \Leftrightarrow \quad \text{(F1)(F2)} \quad \Leftrightarrow \quad \text{(R)} \quad \Leftrightarrow \quad \text{(CR)}.$$

Proof. The equivalence between (R) and (CR) follows immediately from Corollary 2.18 (1).

(F2) $\Rightarrow$ (F2'):
By letting $u_n = 0$ for every n in (F2), we know that for each $w \in C$ there exists $\{w_n\}$ converging to w strongly in $\mathscr{H}$ such that $w_n \in \mathscr{V}_n$ for every n. Since C is dense in $\mathscr{F}$ and therefore also dense in $\mathscr{V}$, we can find a sequence $\{w'_n\}$ converging to any $w \in \mathscr{V}$ with $w'_n \in \mathscr{V}_n$, $n \in \mathbb{N}$ by Corollary 2.38. Take an arbitrary sequence $n_k \uparrow \infty$ and $\{u_k\}$, $u_k \in H_{n_k}$, $k \in \mathbb{N}$ weakly convergent to $u \in H$ in $\mathscr{H}$ with $\sup_k \Theta^{n_k}(u_k) < \infty$. Then by Lemma 2.39 (iv) $u_k \in \hat{\mathscr{F}}_{n_k} \subset \mathscr{V}_{n_k}$, $u \in \mathscr{V}$. Now take u'_n weakly convergent to u with $u'_n \in \mathscr{V}_n$ and $u'_{n_k} = u_k$. Take an arbitrary $w \in \tilde{C} \subset C$. Then by the observations above there exists a sequence $\{w_k\}$, $w_k \in \mathscr{V}_{n_k}$ converging strongly to w such that

$$\varliminf_k \mathscr{E}^{n_k}(w_k, u_k) \leq \varlimsup_n \mathscr{E}^n(w_n, u'_n) = \mathscr{E}(w, u).$$

(F1),(F2') $\Rightarrow$ (CR):
We follow the argument of Röckner and Zhang in [RZ97]. Choose a sequence $\{f_n\}$ $\mathscr{H}$-weakly convergent to some $f \in H$. Let $\alpha > 0$. We shall prove that $\hat{G}^n_\alpha f_n$ converges $\mathscr{H}$-weakly to $\hat{G}_\alpha f$. It suffices to prove that for any sequence $n_k \uparrow \infty$ we can extract a subsequence $\{n_{k_l}\}$ such that $\hat{G}^{n_{k_l}}_\alpha f_{n_{k_l}}$ converges to $\hat{G}_\alpha f$ weakly in $\mathscr{H}$. Set $u_n := \hat{G}^n_\alpha f_n$. Since $\|\hat{G}^n_\alpha\|_{\mathscr{L}(H_n)} \leq \alpha^{-1}$ and $\{f_n\}$ is uniformly bounded by Lemma 2.14 (2) one can extract a subsequence u_{n_k} converging $\mathscr{H}$-weakly to some $u \in H$ using Lemma 2.14 (1).

Note that for some constant $C_\alpha > 0$ depending only on α by Cauchy's inequality and the uniform boundedness of $\{f_n\}$ and since $u_{n_k} = \hat{G}^{n_k}_\alpha f_{n_k} \in \hat{\mathscr{F}}_{n_k}$

$$\begin{aligned}
\sup_k \Theta^{n_k}(u_{n_k}) &= \sup_k \sup_{\|w\|_{\mathscr{V}_{n_k}}=1} \mathscr{E}^{n_k}_1(w, u_{n_k}) \\
&\leq \sup_k \sup_{\|w\|_{\mathscr{V}_{n_k}}=1} \left[|\mathscr{E}^{n_k}_\alpha(w, u_{n_k})| + |1-\alpha||(w, u_{n_k})_{H_{n_k}}| \right] \\
&\leq \sup_k \sup_{\|w\|_{\mathscr{V}_{n_k}}=1} \left[|(f_{n_k}, w)_{H_{n_k}}| + \frac{|1-\alpha|}{\alpha} \|w\|_{H_{n_k}} \|f_{n_k}\|_{H_{n_k}} \right] \\
&\leq C_\alpha \sup_k \sup_{\|w\|_{\mathscr{V}_{n_k}}=1} \|f_{n_k}\|_{H_{n_k}} \|w\|_{H_{n_k}} = C_\alpha \sup_k \|f_{n_k}\|_{H_{n_k}} < \infty
\end{aligned}$$

which yields $u \in \mathscr{V}$ by (F1).

For any $w \in \tilde{C}$, by extracting a subsequence if necessary, we can choose $\{w_k\}$ strongly convergent to w in $\mathscr{H}$ such that $w_k \in \mathscr{V}_{n_k}$ and

$$\underline{\lim}_{k} \mathscr{E}^{n_k}(w_k, u_{n_k}) \le \mathscr{E}(w,u)$$

by (F2'). Since $\mathscr{E}^{n_k}_\alpha(w_k, u_{n_k}) = (w_k, f_{n_k})_{H_{n_k}}$, it follows that

$$0 = \underline{\lim}_{k} \left[\mathscr{E}^{n_k}_\alpha(w_k, u_{n_k}) - (w_k, f_{n_k})_{H_{n_k}}\right] \le \mathscr{E}_\alpha(w,u) - (w,f)_H.$$

Hence $\mathscr{E}_\alpha(w,u) \ge (w,f)_H$. By substituting $-w$ for w, this becomes an equality. Now we have by Proposition 2.35, using that $\tilde{C} \subset C \subset \mathscr{F}$ densely and continuously, that for every $w \in \mathscr{F}$

$$\mathscr{E}_\alpha(w,u) = (w,f)_H = \mathscr{E}_\alpha(w, \hat{G}_\alpha f).$$

Hence ${}_{\mathscr{V}'}((M_\alpha - \Lambda)w, u - \hat{G}_\alpha f)_{\mathscr{V}} = 0$, but $(M_\alpha - \Lambda)(\mathscr{F}) = \mathscr{V}'$ by [Sta99, Section I.3], hence $u = \hat{G}_\alpha f$.

This yields $\hat{G}^n_\alpha f_n \to \hat{G}_\alpha f$ weakly in $\mathscr{H}$. Since f_n was an arbitrary weakly convergent sequence, we have $\hat{G}^n_\alpha \to \hat{G}_\alpha$ weakly in the sense of convergence of bounded operators.

(R) $\Rightarrow$ (F1):
Let $u_n \to u$ weakly in $\mathscr{H}$ and $M := \underline{\lim}_n \Theta^n(u_n) < \infty$. Choose a sequence $\{v_n\}$ converging strongly to u in $\mathscr{H}$. From Proposition 2.36 (i),(ii), for $\alpha > 0$,

$$\begin{aligned}
&\alpha(v_n - \alpha G^n_\alpha v_n, u_n)_{H_n} + (\alpha G^n_\alpha v_n, u_n)_{H_n} \\
=&\mathscr{E}^n_1(\alpha G^n_\alpha v_n, u_n) \\
\le&\Theta^n(u_n)\|\alpha G^n_\alpha v_n\|_{\mathscr{V}_n} \\
\le&\Theta^n(u_n)\left[\mathscr{E}^n_1(\alpha G^n_\alpha v_n, \alpha G^n_\alpha v_n)\right]^{1/2} \\
\le&\Theta^n(u_n)\mathscr{E}^{n,(\alpha)}_1(v_n, v_n)^{1/2} \\
=&\Theta^n(u_n)\left[\alpha(v_n - \alpha G^n_\alpha v_n, v_n)_{H_n} + \|v_n\|^2_{H_n}\right]^{1/2}
\end{aligned}$$

Taking $\underline{\lim}_n$ on both sides, we have

$$\mathscr{E}^{(\alpha)}_1(u,u) - \|u\|^2_H + (\alpha G_\alpha u, u)_H \le M\mathscr{E}^{(\alpha)}_1(u,u)^{1/2}.$$

Hence

$$\mathscr{E}^{(\alpha)}_1(u,u)^{1/2} \le \frac{1}{2}\left[M + \sqrt{M^2 + 4\left(\|u\|^2_H - (\alpha G_\alpha u, u)_H\right)}\right],$$

which implies that

$$\sup_{\alpha>0} \mathscr{E}^{(\alpha)}_1(u,u) \le \frac{1}{2}\left[M + \sqrt{M^2 + 8\|u\|^2_H}\right] < \infty.$$

From Proposition 2.36 (iv), we obtain that $u \in \mathscr{V}$.

(R) $\Rightarrow$ (F2):
Let $u_n \to u$ weakly in $\mathscr{H}$, $u_n \in \mathscr{V}_n$, $u \in \mathscr{V}$ and $w \in C$. Since $\Phi_n(w) \to w$ $\mathscr{H}$-strongly, we have

$$d_{\mathscr{H}} - \lim_{\alpha\to\infty} d_{\mathscr{H}} - \lim_{n\to\infty} \alpha G^n_\alpha \Phi_n(w) = w,$$

and

$$\begin{aligned}\lim_{\alpha\to\infty}\lim_{n\to\infty} \mathscr{E}^{n,(\alpha)}(\Phi_n(w), u_n) &= \lim_{\alpha\to\infty}\lim_{n\to\infty} \alpha(\Phi_n(w) - \alpha G^n_\alpha\Phi_n(w), u_n)_{H_n}\\ &= \lim_{\alpha\to\infty} \mathscr{E}^{(\alpha)}(w,u) = \mathscr{E}(w,u).\end{aligned}$$

Due to Corollary 2.38, we can take a nondecreasing sequence $\{\alpha_n\}$, $\alpha_n \to \infty$ such that

$$d_{\mathscr{H}} - \lim_{n\to\infty} \alpha_n G^n_{\alpha_n}\Phi_n(w) = w, \quad \lim_{n\to\infty} \mathscr{E}^{n,(\alpha_n)}(\Phi_n(w), u_n) = \mathscr{E}(w,u).$$

Recall that by Proposition 2.35 $G^n_\alpha(H) \subset \mathscr{F}_n$ for any $\alpha > 0$ and $n \in \mathbb{N}$. Setting $w_n := \alpha_n G^n_{\alpha_n}\Phi_n(w)$, we hence have $w_n \to w$ $\mathscr{H}$-strongly and by Proposition 2.36 (i) that

$$\mathscr{E}^n(w_n, u_n) = \mathscr{E}^{n,(\alpha_n)}(\Phi_n(w), u_n) \to \mathscr{E}(w,u)$$

as $n \to \infty$ and (F2) is proved. □

Corollary 2.42. *Suppose that $C \subset \mathscr{F}$ densely.*

(i) (F1b),(F2'b) $\Rightarrow$ (R),

(ii) If the sector constants K_n of the $\mathscr{A}^n$'s are uniformly bounded, then (F1a),(F2'a) $\Leftrightarrow$ (R).

Proof. (i): This is trivial, since clearly (F1b) $\Rightarrow$ (F1) and (F2'b) $\Rightarrow$ (F2') by Lemma 2.39 (ii).

(ii): This is an consequence of Theorem 2.41 and Lemma 2.39.

□

Definition 2.43. We say that a sequence of generalized forms $\{\mathscr{E}^n\}$ *converges in the generalized sense* to a generalized form $\mathscr{E}$, if $C \subset \mathscr{F}$ densely and (F1) and (F2) (or equivalently (F1) and (F2')) hold. We shall also use this notion, if (F1a) (or (F1b)) and (F2) (or (F2'a) or (F2'b)) hold, provided the sector constants K_n of the $\mathscr{E}^n$'s are uniformly bounded.

Remark 2.44. *According to Theorem 2.21, corresponding statements to (R), (CR) hold also for the associated semigroups $(T^n_t)_{t\geq 0}$, $n \in \mathbb{N}$, $(T_t)_{t\geq 0}$ resp. and the associated cosemigroups $(\hat{T}^n_t)_{t\geq 0}$, $n \in \mathbb{N}$, $(\hat{T}_t)_{t\geq 0}$ resp.*

Now we would like to prove some properties of convergence associated with the above Theorem. First we want to point out that if $C \subset \mathscr{F}$ densely, C contains enough elements to determine the behavior of a sequence $\mathscr{E}^n$ along particular strongly convergent sequences, so that (F2) turns out to be not that restrictive in comparison with (M2) as it might seem. This shall be illustrated by the following Proposition.

Proposition 2.45. *Let $\mathscr{E}^n$, $n \in \mathbb{N}$, $\mathscr{E}$ be as in Theorem 2.41 and $C \subset \mathscr{F}$ densely. Assume that (F2) holds. Then the following stronger version of (F2) holds:*

For any $w \in \mathscr{F}$ and any $u_n \in \mathscr{V}_n$, $n \in \mathbb{N}$, $u \in \mathscr{V}$ with $u_n \to u$ $\mathscr{H}$-weakly there exists a sequence $\{w_n\}$, $w_n \in H_n$, $n \in \mathbb{N}$, $w_n \to w$ $\mathscr{H}$-strongly such that

$$\lim_n \mathscr{E}^n(w_n, u_n) = \mathscr{E}(w, u).$$

This result also extends to (F2') in an obvious way.

Proof. Let $w \in \mathscr{F}$ and $u_n \in \mathscr{V}_n$, $n \in \mathbb{N}$, $u \in \mathscr{V}$ with $u_n \to u$ $\mathscr{H}$-weakly. Then there exists a sequence $\{w^m\} \subset C$ such that $w^m \to w$ strongly in $\| \ \|_{\mathscr{F}}$ (and therefore in $\| \ \|_H$). Now pick by (F2) for every w^m a sequence $\{w_n^m\} \subset \mathscr{H}$, $w_n^m \in H_n$, $n \in \mathbb{N}$ such that $w_n^m \to w^m$ $\mathscr{H}$-strongly and

$$\lim_n \mathscr{E}^n(w_n^m, u_n) = \mathscr{E}(w^m, u). \tag{2.20}$$

Clearly,

$$\lim_m \mathscr{E}(w^m, u) = \mathscr{E}(w, u).$$

By Theorem 2.10 and Corollary 2.38 there exists an increasing sequence of natural numbers $m(n) \uparrow \infty$ with

$$w_n^{m(n)} \to w \quad \mathscr{H}\text{-strongly}$$

and

$$\lim_n \mathscr{E}^n(w_n^{m(n)}, u_n) = \mathscr{E}(w, u).$$

The proof is complete. □

In the case of coercive closed forms generalized convergence implies an even stronger version of (F2) than in the above Proposition. More precisely, we have:

Proposition 2.46. *Let $\mathscr{E}^n = \mathscr{A}^n$, $n \in \mathbb{N}$, $\mathscr{E} = \mathscr{A}$ be coercive closed forms. Assume that $C \subset \mathscr{V}$ densely and for convenience that $C \subset D(L)$ (here L is the infinitesimal generator). Then (F1),(F2) are equivalent with (F1) and the following condition:*

(FF2) *For every $w \in D(L) \subset \mathscr{V}$ there exists a $\mathscr{H}$-strongly convergent sequence $\{w_n\}$, $w_n \in H_n$, $n \in \mathbb{N}$ such that*

$$\lim_n \mathscr{E}^n(w_n, u_n) = \mathscr{E}(w, u)$$

for any $\{u_n\}$, $u_n \in \mathscr{V}_n$, $n \in \mathbb{N}$, $u \in \mathscr{V}$ with $u_n \to u$ $\mathscr{H}$-weakly.

Proof. It is clear that (F1),(FF2) $\Rightarrow$ (F1),(F2). Let us prove the converse. Assuming that (F1),(F2) hold, Theorem 2.41 tells us that (R) holds. Take $w \in D(L)$ and define $v := (1-L)w \in H$. We have $G_1^n \Psi_n v \to G_1 v$ $\mathscr{H}$-strongly by (R). Set $w_n := G_1^n \Psi_n v \in D(L^n)$. We have that

$$(1-L^n)w_n = \Psi_n v \to v = (1-L)w$$

$\mathscr{H}$-strongly. Hence $L^n w_n \to Lw$ $\mathscr{H}$-strongly. But for every $n \in \mathbb{N}$

$$\mathscr{E}^n(w_n, u_n) = (-L^n w_n, u_n)_{H_n}$$

making sense for every $u_n \in \mathscr{V}_n$.

Now take any $u_n \in \mathscr{V}_n$, $n \in \mathbb{N}$, $u \in \mathscr{V}$ with $u_n \to u$ $\mathscr{H}$-weakly. Then,

$$\lim_n \mathscr{E}^n(w_n, u_n) = \lim_n (-L^n w_n, u_n)_{H_n} = (-Lw, u)_H = \mathscr{E}(w, u)$$

by strong convergence of $L^n w_n \to Lw$. □

Dealing with generalized convergence we are interested in the question, whether the convergence $\mathscr{E}^n \to \mathscr{E}$ in the sense of Theorem 2.41 is sufficient for the Mosco- or Γ-convegence of the symmetric parts $\tilde{\mathscr{A}}^n \to \tilde{\mathscr{A}}$. Actually, to establish this, much stronger assumptions on the forms have to be stated; in general we need to assume the sector constants being uniformly bounded and conditions similar to (F1) and (F2) on the dual forms $\hat{\mathscr{E}}^n$, $\hat{\mathscr{E}}$ to hold.

Remark 2.47. *It is clear by Theorem 2.29 and Theorem 2.41 that in case of symmetric forms and provided $C \subset \mathscr{V}$ densely (F1) and (F2) are just another characterization of Mosco convergence.*

If we compare Mosco convergence and generalized convergence, the main question is: can we benefit from the fact, that the symmetric parts $\tilde{\mathscr{A}}^n$ of given forms $\mathscr{E}^n$ converge Mosco to a symmetric form $\tilde{\mathscr{A}}$ being the symmetric part of a form $\mathscr{E}$? As we will see later, the difficult part is to prove (M1) since (M2) can be obtained easily in the most applications using Lemma 2.33. Accordingly, assuming Mosco convergence of the symmetric parts, we end up verifying (F2), as the following Proposition shows.

Proposition 2.48. *Let $\mathscr{E}^n$, $n \in \mathbb{N}$, $\mathscr{E}$ be as in Theorem 2.41. Assume that $C \subset \mathscr{F}$ densely. If the symmetric parts $\tilde{\mathscr{A}}^n$, $n \in \mathbb{N}$, $\tilde{\mathscr{A}}$ resp. associated with $\mathscr{E}^n$, $n \in \mathbb{N}$, $\mathscr{E}$ resp. fulfill (M1), then (F1) holds.*

Proof. Let $u_n \to u$ $\mathscr{H}$-weakly with $\underline{\lim}_n \Theta^n(u_n) < \infty$. Using Lemma 2.39 (ii) and (M1)

$$\tilde{\mathscr{A}}(u,u) \leq \underline{\lim}_n \tilde{\mathscr{A}}^n(u_n, u_n) \leq \underline{\lim}_n \|u_n\|_{\mathscr{V}_n}^2 \leq \underline{\lim}_n (\Theta^n(u_n))^2 < \infty,$$

which yields $u \in \mathscr{V}$. □

Remark 2.49. *Obviously, if* $\mathscr{A}^n \equiv 0$, $\mathscr{V}_n = H_n$, $n \in \mathbb{N}$, $\mathscr{A} \equiv 0$, $\mathscr{V} = H$, *i.e., our generalized forms* $\{\mathscr{E}^n\}$ *depend only on operators* $(\Lambda^n, D(\Lambda^n, H_n)) = (L^n, D(L^n))$, $n \in \mathbb{N}$ *(see* [Sta99, Remark I.4.10] *for details), condition (F1) can be omitted.*

Remark 2.50. *One can redo all of the steps above, if* $(\mathscr{A}^n, \mathscr{V}_n)$, $n \in \mathbb{N}$ *and* $(\mathscr{A}, \mathscr{V})$ *are* coercive closed forms in a wide sense*, that is, for some bound constant* $\lambda \in \mathbb{R}$, *independent from* n,

(1) $(\tilde{\mathscr{A}_\lambda}, \mathscr{V})$ *is a non-negative, symmetric, closed form.*

(2) $(\mathscr{A}_\lambda, \mathscr{V})$ *satisfies the* weak sector condition*, i.e., there exists a sector constant* $K \geq 1$ *such that*

$$|\mathscr{A}_{\lambda+1}(u,v)| \leq K \mathscr{A}_{\lambda+1}(u,u)^{1/2} \mathscr{A}_{\lambda+1}(v,v)^{1/2} \quad \textit{for all}\, u, v \in \mathscr{V}.$$

And similarly for the $\mathscr{A}^n$*'s (where the sector constants can depend on* n *but* λ *not). In this case the resolvents* G_α^n, $n \in \mathbb{N}$ *and* G_α *are only defined for* $\alpha > \lambda$. *One can easily verify that all the proofs above apply to this slight generalization.*

2.4. Strong graph limits and generalized forms

Definition 2.51. For each $n \in \mathbb{N}$ let A_n be a closed linear operator on H_n with dense linear domain $D(A_n)$. $\{A_n\}$ is said to be *convergent in the strong graph sense*, if for each sequence $\{u_n\}$, $u_n \in D(A_n)$, such that $u_n \to 0 \in H$ $\mathscr{H}$-strongly and the $\mathscr{H}$-strong limit of $\{A_n u_n\}$ exists, we have that $A_n u_n \to 0 \in H$ strongly in $\mathscr{H}$.

If $\{A_n\}$ converges in the strong graph sense, the following linear operator $(A, D(A))$ is well-defined:

$$\begin{aligned} D(A) := \big\{ u \in H \;\mid\; &\exists \{u_n\},\ u_n \in D(A_n),\ u_n \to u \in H\ \mathscr{H}\text{-strongly}, \\ &A_n u_n \text{ converges } \mathscr{H}\text{-strongly} \big\}, \end{aligned}$$

and for $u \in D(A)$

$$\begin{aligned} Au := \lim_n A_n u_n \quad &\text{with } \{u_n\} \text{ such that } u_n \in D(A_n),\ u_n \to u\ \mathscr{H}\text{-strongly} \\ &\text{and } A_n u_n \text{ converges } \mathscr{H}\text{-strongly}. \end{aligned}$$

$(A, D(A))$ is called *strong graph limit* (as an operator) of $\{A_n\}$ and we say that $\{A_n\}$ *converges to* A *in the strong graph sense.*

Definition 2.52. For each $n \in \mathbb{N}$ let A_n be a closed linear operator on H_n with dense linear domain $D(A_n)$. The *strong graph limit* (as a linear space) Γ_∞ of $\{A_n\}$ is defined to be the set of pairs $(u,v) \in H \times H$ such that there exists a sequence of vectors $\{u_n\}$, $u_n \in D(A_n)$ with $u_n \to u$ and $A_n u_n \to v$ strongly in $\mathscr{H}$.

We immediately obtain: $\{A_n\}$ converges to some densely defined closed linear operator A on H in the strong graph sense, if and only if Γ_∞ coincides with the *graph* $\Gamma(A) := \{(u, Au) \subset H \times H \mid u \in D(A)\}$ of A, that is, $\Gamma_\infty = \Gamma(A)$.

We remark that the conditions "closed" and "densely defined" can be relaxed, but turn out to be reasonable in the following considerations.

The following Theorem, which has not been proved in this setting before, ensures that strong graph convergence is powerful enough to characterize convergence of forms and resolvents (even in the non-symmetric case). Unfortunately, this notion of convergence is yet hard to handle, since in many cases the domains of the generators are difficult to specify explicitly.

Theorem 2.53. *Let $\mathscr{E}^n$, $n \in \mathbb{N}$, $\mathscr{E}$ resp. be generalized forms on H_n, $n \in \mathbb{N}$, H resp. Suppose $C \subset \mathscr{F}$ densely. Denote by $(G_\alpha^n)_{\alpha>0}$, $n \in \mathbb{N}$, $(G_\alpha)_{\alpha>0}$ resp. the associated C_0-contraction resolvents, by $(T_t^n)_{t\geq 0}$, $n \in \mathbb{N}$, $(T_t)_{t\geq 0}$ resp. the associated C_0-contraction semigroups and by A^n, $n \in \mathbb{N}$, A resp. the associated infinitesimal generators. Then the following statements are equivalent:*

(1) $G_\alpha^n \to G_\alpha$ strongly for $\alpha > 0$.

(2) $T_t^n \to T_t$ strongly for $t \geq 0$.

(3) $A_n \to A$ in the strong graph sense.

(4) $\mathscr{E}^n \to \mathscr{E}$ in the generalized sense.

Proof. (1) $\Leftrightarrow$ (2) follows from Theorem 2.21.

(1) $\Leftrightarrow$ (4) follows from Theorem 2.41.

Note that in our setting $(0, \infty) \subset \rho(A)$ (where $\rho(\cdot)$ denotes the resolvent set) and A is closed and densely defined. The same holds for A^n, $n \in \mathbb{N}$.

Let us prove (1) $\Rightarrow$ (3):
Assume that $G_\alpha^n \to G_\alpha$ strongly for any $\alpha > 0$. We would like to prove $\Gamma(A) = \Gamma_\infty$. Let $u \in D(A)$ and set $u_n := G_1^n \Psi_n (1 - A)u$. Then $u_n \in D(A^n)$ for every n and

$$\|u_n - \Psi_n u\|_{H_n} = \|G_1^n \Psi_n (1 - A)u - \Psi_n u\|_{H_n}.$$

$\Psi_n u \to u$ and $\Psi_n(1-A)u \to (1-A)u$ $\mathscr{H}$-strongly and by assumption $G_1^n \Psi_n(1-A)u \to G_1(1-A)u = u$ $\mathscr{H}$-strongly. Thus by Lemma 2.8 $u_n \to u$ $\mathscr{H}$-strongly. Now,

$$\begin{aligned}
&\|A^n u_n - \Psi_n A u\|_{H_n} \\
\leq &\|A^n G_1^n \Psi_n (1 - A)u - \Psi_n A u\|_{H_n} \\
= &\|-(1 - A^n) G_1^n \Psi_n (1 - A)u + G_1^n \Psi_n (1 - A)u - \Psi_n A u\|_{H_n} \\
\leq &\|\Psi_n A u - \Psi_n A u\|_{H_n} + \|G_1^n \Psi_n (1 - A)u - \Psi_n u\|_{H_n},
\end{aligned}$$

which clearly tends to 0 as $n \to \infty$. Therefore $(u, Au) = \lim_n(u_n, A^n u_n)$ in $\mathscr{H} \times \mathscr{H}$, hence $\Gamma(A) \subset \Gamma_\infty$.

Now let $(u, v) \in \Gamma_\infty$. Then there exist $u_n \in D(A^n)$, $n \in \mathbb{N}$, $u_n \to u$ $\mathscr{H}$-strongly and $A^n u_n \to v$ $\mathscr{H}$-strongly. Furthermore $(1 - A^n)u_n \to u - v$ $\mathscr{H}$-strongly and

$$u_n = G_1^n(1 - A^n)u_n \to G_1(u - v) =: w \in D(A)$$

by strong convergence of resolvents. But since $\mathscr{H}$ is Hausdorff (e.g. by Theorem 2.10), we must have $u = w \in D(A)$. Furthermore,

$$Au = Aw = AG_1(u - v) = -(1 - A)G_1(u - v) + G_1(u - v) = v - u + \underbrace{G_1(u - v)}_{=w=u} = v.$$

Hence $\Gamma_\infty \subset \Gamma(A)$. The assertion is proved.

Let us now prove (3) $\Rightarrow$ (1):
Let $A_n \to A$ in the strong graph sense. Let $\alpha > 0$ and $v \in H$. We have that $v = (\alpha - A)u$ with $u := G_\alpha v \in D(A)$. By assumption there exist $u_n \in D(A^n)$, $u_n \to u$ $\mathscr{H}$-strongly and $A^n u_n \to Au$. We would like to apply Lemma 2.20:

$$\begin{aligned}
&\|G_\alpha^n \Psi_n v - \Psi_n G_\alpha v\|_{H_n} \\
=&\|G_\alpha^n \Psi_n(\alpha - A)u - \Psi_n G_\alpha(\alpha - A)u\|_{H_n} \\
\leq&\|G_\alpha^n \Psi_n(\alpha - A)u - u_n\|_{H_n} + \|u_n - \Psi_n u\|_{H_n} \\
=&\|G_\alpha^n \Psi_n(\alpha - A)u - G_\alpha^n(\alpha - A^n)u_n\|_{H_n} + \|u_n - \Psi_n u\|_{H_n} \\
\leq&\frac{1}{\alpha}\|\Psi_n(\alpha - A)u - (\alpha - A^n)u_n\|_{H_n} + \|u_n - \Psi_n u\|_{H_n},
\end{aligned}$$

which clearly tends to 0 as $n \to \infty$, since $(\alpha - A^n)u_n \to (\alpha - A)u$ strongly and $\Psi_n u \to u$ strongly. The proof is complete. $\square$

3. Examples of Mosco convergence

3.1. Finite dimensional symmetric case

We would like to shortly recall the results of A.V. Kolesnikov from [Kol05a], since the convergence of elliptic non-symmetric forms can be proved using the Mosco convergence of symmetric $a_{i,j}$-forms, in particular, if the considered forms have Mosco-convergent symmetric parts, we can apply Proposition 2.48.

Let $d \geq 1$ and let dx be the Lebesgue measure on $\mathbb{R}^d$ and $\mathscr{B}(\mathbb{R}^d)$ the Borel σ-algebra of $\mathbb{R}^d$. Let $|\cdot|$ denote the d-dimensional Euclidean norm and $\langle \cdot, \cdot \rangle$ the d-dimensional Euclidean inner product. Denote by $C_0^\infty(\mathbb{R}^d)$ the set of all infinitely differentiable continuous (real valued) functions with compact support.

Assumption 2 (Convergence of speed measures). *$\sigma_n > 0$ dx-a.e., $\sigma_n \in L^1_{\text{loc}}(dx)$, $n \in \mathbb{N}$ and there exists a function σ such that $\sigma > 0$ dx-a.e., $\sigma \in L^1_{\text{loc}}(\mathbb{R}^d; dx)$ and $\{\mu_n = \sigma_n dx\}$ tends to $\mu = \sigma dx$ vaguely, i.e.,*

$$\lim_n \int_{\mathbb{R}^d} \psi \, \sigma_n dx = \int_{\mathbb{R}^d} \psi \, \sigma dx$$

for every continuous function ψ with compact support.

Let $H_n := L^2(\mathbb{R}^d; \mu_n)$, $n \in \mathbb{N}$, $H := L^2(\mathbb{R}^d; \mu)$, $C = C_0^\infty(\mathbb{R}^d)$ and Φ_n the identity operators on C. Then $H_n \to H$ in the sense of Defintion 2.3. As above $\mathscr{H} = \bigcup_n H_n \dot{\cup} H$. Note that Φ_n is well-defined since the measures μ_n have full support.

Lemma 3.1. *Consider the following statements:*

(1) $f_n \to f$ strongly (weakly) in $L^2(\mathbb{R}^d; dx)$.

(2) $\frac{f_n}{\sqrt{\sigma_n}} \to \frac{f}{\sqrt{\sigma}}$ strongly (weakly) in $\mathscr{H}$.

Assume that $\sqrt{\sigma_n} \to \sqrt{\sigma}$ weakly in $L^2_{\text{loc}}(\mathbb{R}^d; dx)$. Then (1) implies (2) for the strong convergence and (2) implies (1) for the weak convergence. If, in addition, $\sqrt{\sigma_n} \to \sqrt{\sigma}$ strongly in $L^2_{\text{loc}}(\mathbb{R}^d; dx)$, then (1) and (2) are equivalent.

Proof. The proof is taken from [Kol05a, Lemma 3.1]. Suppose first that $\sqrt{\sigma_n} \to \sqrt{\sigma}$ weakly in $L^2_{\text{loc}}(\mathbb{R}^d; dx)$. Let us prove (1) $\Rightarrow$ (2) for the strong convergence. Therefore, let $f_n \to f$ in $L^2(\mathbb{R}^d; dx)$. Let us find a sequence $\varphi_n \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi_n\sqrt{\sigma} \to f$ in $L^2(\mathbb{R}^d; dx)$. Then $\varphi_n \to \frac{f}{\sqrt{\sigma}}$ in $L^2(\mathbb{R}^d; \sigma dx)$. Hence

$$\lim_m \overline{\lim_n} \int_{\mathbb{R}^d} \left(\varphi_m - \frac{f_n}{\sqrt{\sigma_n}}\right)^2 \sigma_n dx$$
$$= \lim_m \overline{\lim_n} \int_{\mathbb{R}^d} (\varphi_m\sqrt{\sigma_n} - f_n)^2 dx = \lim_m \int_{\mathbb{R}^d} (\varphi_m\sqrt{\sigma} - f)^2 dx = 0,$$

by weak convergence of $\sqrt{\sigma_n} \to \sqrt{\sigma}$ in $L^2_{\text{loc}}(\mathbb{R}^d; dx)$ and Assumption 2.

Let us prove (2) $\Rightarrow$ (1) for the weak convergence. Therefore, let $\frac{f_n}{\sqrt{\sigma_n}} \to \frac{f}{\sqrt{\sigma}}$ weakly in $\mathscr{H}$. Then for every $\mathscr{H}$-strongly convergent sequence $u_n \to u$ we have that

$$\lim_n \int_{\mathbb{R}^d} \frac{f_n}{\sqrt{\sigma_n}} u_n \sigma_n dx = \int_{\mathbb{R}^d} \frac{f}{\sqrt{\sigma}} u \, \sigma dx.$$

Now let $v_n \to v$ strongly in $L^2(\mathbb{R}^d; dx)$. Then by the assertion proved before we have that $\frac{v_n}{\sqrt{\sigma_n}} \to \frac{v}{\sqrt{\sigma}}$ $\mathscr{H}$-strongly. Clearly,

$$\lim_n \int_{\mathbb{R}^d} f_n v_n dx = \lim_n \int_{\mathbb{R}^d} \frac{f_n}{\sqrt{\sigma_n}} \frac{v_n}{\sqrt{\sigma_n}} \sigma_n dx = \int_{\mathbb{R}^d} \frac{f}{\sqrt{\sigma}} \frac{v}{\sqrt{\sigma}} \sigma dx = \int_{\mathbb{R}^d} fv \, dx.$$

Now suppose that $\sqrt{\sigma_n} \to \sqrt{\sigma}$ strongly in $L^2_{\text{loc}}(\mathbb{R}^d; dx)$. We would like to prove (2) $\Rightarrow$ (1) for the strong convergence. Therefore, let $\frac{f_n}{\sqrt{\sigma_n}} \to \frac{f}{\sqrt{\sigma}}$ strongly in $\mathscr{H}$. Then there exists a sequence of $C_0^\infty(\mathbb{R}^d)$-functions $\{\varphi_m\}$ such that $\varphi_m\sqrt{\sigma} \to f$ in $L^2(\mathbb{R}^d; dx)$ and

$$\lim_m \overline{\lim_n} \int_{\mathbb{R}^d} \left(\varphi_m - \frac{f_n}{\sqrt{\sigma_n}}\right)^2 \sigma_n dx = \lim_m \overline{\lim_n} \int_{\mathbb{R}^d} \left(\varphi_m\sqrt{\sigma_n} - f_n\right)^2 dx = 0.$$

Since $\lim_m \lim_n \varphi_m\sqrt{\sigma_n} = f$ (all the limits are $L^2(\mathbb{R}^d; dx)$-limits), one can find a subsequence n_k such that $\varphi_k\sqrt{\sigma_{n_k}}$ tends to f and

$$\lim_k \|\varphi_k\sqrt{\sigma_{n_k}} - f_{n_k}\|_{L^2(\mathbb{R}^d;dx)} = 0.$$

Hence $f_{n_k} \to f$ in $L^2(\mathbb{R}^d; dx)$. Since we can do the same with every subsequence of $\{f_n\}$, we get that $f_n \to f$ in $L^2(\mathbb{R}^d; dx)$.

Let us prove (1) $\Rightarrow$ (2) for the weak convergence. Therefore, let $f_n \to f$ weakly in $L^2(\mathbb{R}^d; dx)$. Then for every $u_n \to u$ strongly in $L^2(\mathbb{R}^d; dx)$ we have

$$\lim_n \int_{\mathbb{R}^d} f_n u_n dx = \int_{\mathbb{R}^d} fu \, dx.$$

Now let $v_n \to v$ strongly in $\mathscr{H}$. By the assertion proved before we have that $v_n\sqrt{\sigma_n} \to v\sqrt{\sigma}$ strongly in $L^2(\mathbb{R}^d; dx)$. Then

$$\lim_n \int_{\mathbb{R}^d} \frac{f_n}{\sqrt{\sigma_n}} v_n \sigma_n dx = \lim_n \int_{\mathbb{R}^d} f_n v_n \sqrt{\sigma_n} dx = \int_{\mathbb{R}^d} fv\sqrt{\sigma} dx = \int_{\mathbb{R}^d} \frac{f}{\sqrt{\sigma}} v \, \sigma dx.$$

The proof is complete. $\square$

For each $n \in \mathbb{N}$ consider the following symmetric form:

$$\mathscr{E}^n(f,g) = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a^n_{i,j}(x) \frac{\partial f(x)}{\partial x_i} \frac{\partial g(x)}{\partial x_j} dx, \quad f,g \in C_0^\infty(\mathbb{R}^d).$$

Here $a^n_{i,j}$, $1 \le i,j \le d$, are Borel locally integrable functions and $a^n_{i,j} = a^n_{j,i}$ for $1 \le i,j \le d$. We will denote by A^n the symmetric $d \times d$-matrix $(A^n(x))_{i,j} := a^n_{i,j}(x)$, $x \in \mathbb{R}^d$. Then our forms can be written as

$$\mathscr{E}^n(f,g) = \int_{\mathbb{R}^d} \langle A^n \nabla f, \nabla g \rangle \, dx.$$

We suppose that the A^n's are dx-a.e. positive definite. We denote the elements of the inverse matrix $(A^n)^{-1}$ by $(a^{-1})^n_{i,j}$, $((A^n)^{-1}(x))_{i,j} = (a^{-1})^n_{i,j}(x)$, $x \in \mathbb{R}^d$.

For an arbitrary Borel function $f : \mathbb{R}^d \to \mathbb{R}$ let us define the set

$$R(f) := \left\{ x \;\middle|\; \exists \varepsilon > 0, \int_{|x-y| \le \varepsilon} \frac{dy}{|f(y)|} < \infty \right\},$$

where we adopt the convention $\frac{1}{0} := +\infty$. Evidently, $R(f)$ is the largest open set V such that $\frac{1}{f} \in L^1_{\text{loc}}(V; dx)$. Let $R(A)$ be the largest open set V such that $(a^{-1})_{i,j} \in L^1_{\text{loc}}(V; dx)$ for $1 \le i,j \le d$ and define $R(A^n)$ similarly.

We say that an arbitrary Borel function f satisfies *Hamza's condition* if for dx-a.e. $x \in \mathbb{R}^d$ $f(x) > 0$ implies $x \in R(f)$. This is equivalent to $dx(\mathbb{R}^d \setminus R(f)) = 0$.

The following Assumption ensures the closability of $(\mathscr{E}^n, C_0^\infty(\mathbb{R}^d))$ (this is weaker than the standard Assumption made in [MR92, Section II.2.b)]).

Assumption 3. *$R(A) \subset R(\sigma)$, $R(A^n) \subset R(\sigma_n)$, $dx(\mathbb{R}^d \setminus R(A^n)) = dx(\mathbb{R}^d \setminus R(A)) = 0$.*

Lemma 3.2. *The form $(\mathscr{E}_+, \mathscr{D}(\mathscr{E}_+))$ defined by*

$$\mathscr{D}(\mathscr{E}_+) = \Bigg\{ f \in L^2(\mathbb{R}^d; \sigma dx) \;\Bigg|\; f \text{ admits weak derivatives } \partial_i f \text{ in } R(A)$$
$$\text{for every } i \in \{1, \ldots, d\}, \; \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a_{i,j}(x) \frac{\partial f(x)}{\partial x_i} \frac{\partial f(x)}{\partial x_j} dx < \infty \Bigg\},$$

$$\mathscr{E}_+(f,g) = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a_{i,j}(x) \frac{\partial f(x)}{\partial x_i} \frac{\partial f(x)}{\partial x_j} dx, \quad f,g \in \mathscr{D}(\mathscr{E}_+),$$

is closed.

Proof. See [Kol05a, Lemma 3.4]. □

$(\mathscr{E}_+, \mathscr{D}(\mathscr{E}_+))$ is the so called "maximal" extension of $(\mathscr{E}, C_0^\infty(\mathbb{R}^d))$. Let us denote by $\mathscr{D}(\mathscr{E}_0)$ the completion of $C_0^\infty(\mathbb{R}^d)$ w.r.t. $\mathscr{E}_1^{1/2}(u) = \big(\mathscr{E}(u) + \|u\|_{L^2(\mathbb{R}^d;\sigma dx)}\big)^{1/2}$. The extension of $(\mathscr{E}, C_0^\infty(\mathbb{R}^d))$ to this space, denoted by $(\mathscr{E}_0, \mathscr{D}(\mathscr{E}_0))$, is a closed form on $L^2(\mathbb{R}^d;\sigma dx)$ and called the "minimal" extension. Let us extend our notation to $(\mathscr{E}_0^n, \mathscr{D}(\mathscr{E}_0^n))$ and $(\mathscr{E}_+^n, \mathscr{D}(\mathscr{E}_+^n))$ in the obvious way. A.V. Kolesnikov has proved in [Kol05a] that the Mosco topology is generally not Hausdorff (not even in the one dimensional case), i.e., additional geometric assumptions have to be stated to identify a unique Mosco limit. Particularly, we need to assume that the minimal and maxmial extensions coincide.

Theorem 3.3. *Let $\sqrt{\sigma_n} \to \sqrt{\sigma}$ weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^d;dx)$,*

$$\sup_n \int_\Omega \frac{dx}{\sigma_n} < \infty, \quad \sup_n \int_\Omega |(a^{-1})^n_{i,j}| dx < \infty, \quad 1 \le i,j \le d$$

for every bounded domain $\Omega \in \mathscr{B}(\mathbb{R}^d)$ and there exists a $d \times d$ matrix A with Borel locally integrable coefficients, which is symmetric and dx-a.e. positive definite. Assume that $(a^{-1})_{i,j} \in L^1_{\mathrm{loc}}(\mathbb{R}^d;dx)$ and

$$a^n_{i,j} dx \to a_{i,j} dx, \quad (a^{-1})^n_{i,j} dx \to (a^{-1})_{i,j} dx$$

vaguely.

Let

$$\mathscr{E}(\varphi,\psi) = \int_{\mathbb{R}^d} \langle A\nabla\varphi, \nabla\psi\rangle \, dx, \quad \varphi,\psi \in C_0^\infty(\mathbb{R}^d).$$

Suppose that the minimal and maximal extensions of $(\mathscr{E}, C_0^\infty(\mathbb{R}^d))$ in $L^2(\mathbb{R}^d;\sigma dx)$ coincide. Then $\mathscr{E}_0^n \to \mathscr{E}$ and $\mathscr{E}_+^n \to \mathscr{E}$ Mosco.

Proof. See [Kol05a, Theorem 1.1]. □

3.2. Infinite dimensional symmetric case

The following results are mainly based on the work of A.V. Kolesnikov in [Kol06]. We reduce the general case to the case of a "Gelfand triplet" which shall also be used in the results on generalized convergence in Chapter 4.2.

Throughout this section we fix a separable real Hilbert space X with inner product $\langle\ ,\ \rangle_X$ and norm $\|\ \|_X = \langle\ ,\ \rangle_X^{1/2}$ which is densely and continuously embedded into some separable real Banach space E. With E' we denote the topological dual of E and by ${}_{E'}\langle\ ,\ \rangle_E : E' \times E \to \mathbb{R}$ the corresponding dualization. Denote by $\mathscr{B}(E)$ the Borel σ-field of E. Identifying X with its dual X' we have that

$$E' \subset X' \equiv X \subset E \quad \text{densely and continuously}$$

and ${}_{E'}\langle\ ,\ \rangle_E$ restricted to $E' \times X$ coincides with $\langle\ ,\ \rangle_X$. Here E' is endowed with the operator norm $\|\ \|_{E'} := \|\ \|_{\mathscr{L}(E,\mathbb{R})}$.

Define the linear space of cylindrical test functions on E by

$$\mathscr{F}C_0^\infty(E) := \{f(l_1, \ldots, l_m) \mid m \in \mathbb{N},\ f \in C_0^\infty(\mathbb{R}^m),\ l_1, \ldots, l_m \in E'\}.$$

Here $C_0^\infty(\mathbb{R}^m)$ denotes the set of all infinitely differentiable (real-valued) functions on $\mathbb{R}^m$ with compact support and all partial derivatives continuous. Define for $u \in \mathscr{F}C_0^\infty(E)$ and $k \in E$ the following Gâteaux-type derivative

$$\frac{\partial u}{\partial k}(z) := \frac{d}{ds} u(z + sk)\Big|_{s=0},\ z \in E.$$

Observe that if $u = f(l_1, \ldots, l_m)$, then

$$\frac{\partial u}{\partial k} = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(l_1, \ldots, l_m)\, {}_{E'}\langle l_i, k\rangle_E \in \mathscr{F}C_0^\infty(E).$$

This shows that for fixed $u \in \mathscr{F}C_0^\infty(E)$ and $z \in E$, $h \mapsto \frac{\partial u}{\partial h}(z)$ is a continuous linear functional on X. Define $\nabla_X u(z) \in X$ by

$$\langle \nabla_X u(z), h\rangle_X = \frac{\partial u}{\partial h}(z),\ h \in X. \tag{3.1}$$

To apply our results on Mosco convergence we fix a sequence $\{\mu_n\}$ of Borel probability measures on E and a Borel probability measure μ on E.

Assumption 4. *μ_n converges weakly to μ.*

Note that by Prokhorov's theorem $\{\mu_n\}$ is tight.

Assumption 5. *μ is h-quasi-invariant for every $h \in X$, i.e., $\mu \circ T_h^{-1}$ is absolutely continuous with respect to μ where $T_h(z) := z - h$.*

This implies that $\operatorname{supp}\mu = E$.

Now consider following sequence of Hilbert spaces $\{H_n\} = \{L^2(E; \mu_n)\}$. Set $C := \mathscr{F}C_0^\infty(E)$ and $H := L^2(E; \mu)$. Note that $C \subset H$ densely by the Hahn-Banach Theorem and a monotone class argument (cf. [MR92, Section II.3.a)] or [AR90, Remark 3.1]). Let Φ_n be the identity operator. It is well-defined since μ has full support. Set $\mathscr{H} := \dot{\bigcup}_n H_n \dot{\cup} H$. $\{H_n\}$ converges to H by weak convergence of measures.

Now consider a weakly convergent sequence $h_n \to h$ of vectors from X, i.e.,

$$\langle h_n, g\rangle_X \to \langle h, g\rangle_X,$$

for every $g \in X$. Suppose additionally $\|h_n\|_X = 1$ for every n, $\|h\|_X = 1$. Fix some $l \in E'$ such that $\langle l, h_n\rangle_X \neq 0$ for every n, which exists by the Hahn-Banach Theorem. Denote by P_{h_n} the projection $P_{h_n} : E \to E_0 := \{x \in E \mid l(x) = 0\}$

$$P_{h_n}(z) := z - \frac{{}_{E'}\langle l, z\rangle_E}{\langle l, h_n\rangle_X} h_n,\ z \in E.$$

It is well known that every probability measure μ_n has a conditional distribution $\rho_n(x,\cdot)$ on the real line such that letting $\nu_n := \mu_n \circ (P_{h_n})^{-1}$ one has following disintegration formula

$$\int_E u(z)\mu_n(dz) = \int_{E_0}\int_{\mathbb{R}} u(x+sh_n)\rho_n(x,ds)\nu_n(dx), \tag{3.2}$$

(see [DM78]).

Assumption 6. *Every μ_n is h_n-quasi invariant and $h_n \to h$ weakly.*

It follows from this Assumption that the $\rho_n(x,ds)$ have densities w.r.t. the Lebesgue measure, i.e. $\rho_n(x,ds) = \rho_n(x+sh_n)ds$ for ν_n-a.e. $x \in E_0$.

Assumption 7. *The following sequence of Borel measures*

$$\tilde{\mu}_n^{h_n} := \frac{ds}{\rho_n(x+sh_n)}\nu_n(dx)$$

is uniformly bounded on all sets of the type $E_0^N := \{z \in E \mid |l(z)| \leq N\}$. That is, the sequence

$$\int_E u(z)d\tilde{\mu}_n^{h_n} := \int_{E_0}\int_{\mathbb{R}} u(x+sh_n)\frac{ds}{\rho_n(x+sh_n)}\nu_n(dx)$$

is bounded for every bounded Borel function $u : E \to \mathbb{R}$ with support in E_0^N.

In particular, for ν_n-almost every x (hence, $P_{h_n}(\mu_n)$-a.e.) the function $(\rho_n(\cdot,x))^{-1}$ is locally integrable.

For every h_n consider following partial form

$$\mathscr{D}(\mathscr{E}^{h_n}_{\mu_n,0}) := \mathscr{F}C_0^\infty(E),$$

and

$$\mathscr{E}^{h_n}_{\mu_n,0}(u,v) := \int_E \frac{\partial u}{\partial h_n}\frac{\partial v}{\partial h_n}d\mu_n, \quad u,v \in \mathscr{F}C_0^\infty(E).$$

A sufficient condition for the closability of $\mathscr{E}^{h_n}_{\mu_n,0}$ is that μ_n *admits a logarithmic derivative along* h_n, i.e. there exists a measurable function $\beta^{\mu_n}_{h_n} \in L^2(E;\mu_n)$ such that

$$\int_E \frac{\partial \varphi}{\partial h_n}d\mu_n = -\int_E \varphi\beta^{\mu_n}_{h_n}d\mu_n$$

for every $\varphi \in \mathscr{F}C_0^\infty(E)$ (cf. [MR92, Section II.3.a)]).

Denote the "minimal closed extension" by $(\mathscr{E}^{h_n}_{\mu,0}, \mathscr{D}(\mathscr{E}^{h_n}_{\mu,0}))$, which is by definition the closure of $(\mathscr{E}^{h_n}_{\mu_n,0}, \mathscr{F}C_0^\infty(E))$ in $L^2(E;\mu_n)$. Furthermore, define a "maximal" partial form by

$$\mathscr{D}(\mathscr{E}^{h_n}_{\mu_n}) := \left\{ u \in L^2(E;\mu_n) \,\middle|\, \begin{array}{l}\text{for } \nu_n\text{-a.e. } x \in E_0,\ s \mapsto u(x+sh_n) \text{ has an absolutely} \\ \text{continuous } (ds)\text{-version } \tilde{u}_x \text{ and } \dfrac{\partial u}{\partial h_n} := \left(\dfrac{d\tilde{u}(x+sh_n)}{ds}\right) \in L^2(E;\mu_n)\end{array} \right\},$$

and

$$\mathscr{E}^{h_n}_{\mu_n}(u,v) := \int_E \frac{\partial u}{\partial h_n}\frac{\partial v}{\partial h_n} d\mu_n, \quad u,v \in \mathscr{D}(\mathscr{E}^{h_n}_{\mu_n}).$$

It follows from the local integrability of $(\rho_n(\cdot,x))^{-1}$, that $\mathscr{E}^{h_n}_{\mu_n}$ is closed (cf. [AR90]).

Lemma 3.4. *For fixed n set $\mu := \mu_n$, $\nu := \nu_n$, $\rho := \rho_n$ and $h := h_n$. For the minimal and maximal partial forms defined as above following statement holds*

$$(\mathscr{E}^h_{\mu,0}, \mathscr{D}(\mathscr{E}^h_{\mu,0})) = (\mathscr{E}^h_{\mu}, \mathscr{D}(\mathscr{E}^h_{\mu})).$$

Proof. See [Kol06, Lemma 3.3] □

Now we formulate the main result for the Mosco convergence of partial Dirichlet forms.

Theorem 3.5. *Let Assumptions 4-7 hold. Suppose that there exist disintegrations $\mu_n = \rho_n(x+sh_n)ds\, \nu_n(dx)$ such that*

(1) $\mu_n \to \mu$ weakly,

(2) $\nu_n \to \nu$ weakly,

(3) there exists an increasing sequence of numbers $\{n_i\}$, $n_i \uparrow \infty$ such that $\{\mathbb{1}_{E_0^{n_i}}\tilde{\mu}_n^{h_n}\}$ is tight and moreover,

$$\mathbb{1}_{E_0^{n_i}}\tilde{\mu}_n^{h_n} \to \mathbb{1}_{E_0^{n_i}}\tilde{\mu}^{h}$$

weakly for every n_i as $n \to \infty$.

Then $\mathscr{E}^{h_n}_{\mu_n} \to \mathscr{E}^h_\mu$ Mosco.

Proof. See [Kol06, Theorem 3.4]. □

We shall now deal with the gradient case. For this purpose we fix an orthonormal basis $\{e_i \mid i \in \mathbb{N}\}$ of the separable Hilbert space X. Furthermore we fix a sequence of vectors $h_n \in X$, $\|h_n\|_X = 1$ weakly converging to e_1. Now construct a two index sequence $\{e_n^i\}$ such that

(1) for fixed n every $\{e_n^i \mid i \in \mathbb{N}\}$ is an orthonormal basis of X,

(2) for fixed i every $\{e_n^i\}$ converges weakly to e_i.

(To do so one can easily check that the following construction has the wanted properties. Just set

$$P^i \equiv \text{ The unique orthogonal projection from } X \text{ to } \operatorname{lin}(e^i),$$

and

$$e_n^1 := P^1(h_n), \quad e_n^i := P^i(h_n) + e_i,\ i \geq 2.)$$

Recall that a sum of closed forms is closed (cf. [MR92, Proposition I.3.7]). We will consider a sequence of forms

$$\mathscr{E}_{\mu_n} = \sum_{i=1}^{\infty} \mathscr{E}_{\mu_n}^{e_n^i}$$

with the domain of definition

$$\mathscr{D}(\mathscr{E}_{\mu_n}) = \bigcap_{i=1}^{n} \mathscr{D}(\mathscr{E}_{\mu_n}^{e_n^i}),$$

which is the "maximal extension" of a gradient form. The "minmal extension" of $(\mathscr{E}_{\mu_n}, \mathscr{F}C_0^\infty(E))$ denoted by $(\mathscr{E}_{\mu_n,0}, \mathscr{D}(\mathscr{E}_{\mu_n,0}))$ is the closure of $(\mathscr{E}_{\mu_n}, \mathscr{F}C_0^\infty(E))$ in $L^2(E;\mu_n)$. Note that $\mathscr{F}C_0^\infty(E) \subset \mathscr{D}(\mathscr{E}_{\mu_n})$ since

$$\sum_{i=1}^{\infty} \langle h, e_n^i \rangle_X^2 = \|h\|_X < \infty$$

for every $h \in X$ (cf. [MR92, Proof of Proposition II.3.4]).

This type of form coincides with the infinite dimensional Dirichlet integral in the sense of the gradient introduced in (3.1), since by Parseval's identity

$$\begin{aligned}\mathscr{E}_{\mu_n}(u,v) &= \sum_{i=1}^{\infty} \int_E \frac{\partial u}{\partial e_n^i} \frac{\partial v}{\partial e_n^i} d\mu_n = \int_E \sum_{i=1}^{\infty} \langle \nabla_X u, e_n^i \rangle_X \langle e_n^i, \nabla_X v \rangle_X \, d\mu_n \\ &= \int_E \langle \nabla_X u, \nabla_X v \rangle_X \, d\mu_n, \quad u,v \in \mathscr{D}(\mathscr{E}_{\mu_n}).\end{aligned}$$

Theorem 3.6. *Let $\{\mu_n\}$ and $\{e_n^i\}$ satisfy conditions (1)-(3) of Theorem 3.5 for every i. Assume additionally, that μ_n is e_n^i-quasi invariant for every i. Suppose that $(\mathscr{E}_{\mu,0}, \mathscr{D}(\mathscr{E}_{\mu,0})) = (\mathscr{E}_\mu, \mathscr{D}(\mathscr{E}_\mu))$, i.e., $\mathscr{F}C_0^\infty(E)$ is dense in $\mathscr{D}(\mathscr{E}_\mu)$ w.r.t. the norm $(\mathscr{E}_\mu)_1^{1/2}$. Then $\mathscr{E}_{\mu_n} \to \mathscr{E}_\mu$ Mosco.*

Proof. Condition (M1) follows from the fact that (M1) is fulfilled for every sequence of partial forms $\{\mathscr{E}_{\mu_n}^{e_n^i}\}$. Let us verify (M2). Since $\mathscr{F}C_0^\infty(E)$ is dense in $(\mathscr{D}(\mathscr{E}_\mu), (\mathscr{E}_\mu)_1^{1/2})$, Lemma 2.33 implies that it suffices to show that $\mathscr{E}_{\mu_n}(f) \to \mathscr{E}_\mu(f)$ for every $f \in \mathscr{F}C_0^\infty(E)$. It is clear that

$$\sup_n \sum_{i=1}^{\infty} \langle h, e_n^i \rangle_X^2 = \sup_n \|h\|_X = \|h\|_X < \infty \tag{3.3}$$

for all $h \in X$. Take $f = \varphi(l_1, \dots, l_d)$, $\varphi \in C_0^\infty(\mathbb{R}^d)$, $l_1, \dots, l_d \in E'$, then

$$\mathscr{E}_{\mu_n}(f) = \sum_{i=1}^{\infty} \int_E \sum_{j_1,j_2=1}^{d} \frac{\partial \varphi}{\partial x_{j_1}}(l_1,\dots,l_d) \frac{\partial \varphi}{\partial x_{j_2}}(l_1,\dots,l_d) \langle l_{j_1}, e_n^i \rangle_X \langle l_{j_2}, e_n^i \rangle_X \, d\mu_n.$$

The claim follows form the Cauchy's inequality, weak convergence $\mu_n \to \mu$ and (3.3). □

3.2.1. Logarithmic derivatives

This Subsection is taken from [Kol06] with slight changes, since it turns out that this result will be used to prove convergence in the non-symmetric case.

Now consider the same situation as above, but assume only that $\mu_n \to \mu$ weakly. It is clear by Prokhorov's theorem that $\{\mu_n\}$ is tight. Recall that the well-known fact, that L^2-convergence of the logarithmic derivatives of measures implies strong convergence of the corresponding semigroups. We fix some $h \in X$ such that every μ_n has the logarithmic derivative $\beta_h^{\mu_n} \in L^2(E;\mu_n)$ along h and consider the sequence of partial forms $\{\mathscr{E}_{\mu_n}^h\}$ defined by

$$\mathscr{E}_{\mu_n}^h(f,g) = \int_E \frac{\partial f}{\partial h}\frac{\partial g}{\partial h} d\mu_n$$

for $f, g \in \mathscr{F}C_0^\infty(E)$. The condition $\beta_h^{\mu_n} \in L^2(E;\mu_n)$ implies the closability of these forms (in this case h is called *well-μ_n-admissible*, see [MR92, Section II.3.a)]). As usually, the maximal closure of $(\mathscr{E}_{\mu_n}^h, \mathscr{F}C_0^\infty(E))$ is considered. It was proved in [RZ92] that $\mathscr{F}C_0^\infty(E)$ is dense in $(\mathscr{D}(\mathscr{E}_\mu^h), (\mathscr{E}_\mu^h)_1^{1/2})$ for every partial form $\mathscr{E}_\mu^h$ if μ admits a logarithmic derivative along h.

Proposition 3.7. *Let* $\sup_n \|\beta_h^{\mu_n}\|_{L^2(E;\mu_n)} < \infty$. *Then μ possesses a logarithmic derivative along h and $\{\mathscr{E}_{\mu_n}^h\}$ Γ-converges to $\mathscr{E}_\mu^h$. If, in addition, $\|\beta_h^{\mu_n}\|_{L^2(E;\mu_n)} \to \|\beta_h\|_{L^2(E;\mu)}$ for some $\beta_h \in L^2(E;\mu)$, then $\mathscr{E}_{\mu_n}^h \to \mathscr{E}_\mu^h$ Mosco.*

Proof. Condition (M2) resp. (G2) can be verified as in Lemma 2.33. Let us verify condition (G1). Extract from $\{\beta_h^{\mu_n}\}$ an $\mathscr{H}$-weakly convergent subsequence, denoted in the following again by $\{\beta_h^{\mu_n}\}$, such that $\beta_h^{\mu_n} \to \beta_h \in L^2(E;\mu)$ $\mathscr{H}$-weakly. Then by the properties of weak convergence in $\mathscr{H}$

$$\int_E \varphi\beta \, d\mu = \lim_n \int_E \varphi\beta_h^{\mu_n} d\mu_n = -\lim_n \int_E \frac{\partial\varphi}{\partial h} d\mu_n = -\int_E \frac{\partial\varphi}{\partial h} d\mu$$

for every $\varphi \in \mathscr{F}C_0^\infty(E)$. Hence μ has the logarithmic derivative $\beta_h^\mu := \beta_h \in L^2(E;\mu)$ and, moreover, $\beta_h^{\mu_n} \to \beta_h^\mu$ $\mathscr{H}$-weakly. Now let $f_n \to f$ strongly in $\mathscr{H}$ with $\sup_n \mathscr{E}_{\mu_n}^h(f_n) < \infty$ (since otherwise (G1) is trivial). W.l.o.g. $f_n \in \mathscr{D}(\mathscr{E}_{\mu_n}^h)$. Let $K \subset E$ be a compact set. Obviously the tightness of measures $\{\mu_n\}$ and Cauchy's inequality

$$\left(\int_{E\setminus K} \left|\frac{\partial f_n}{\partial h}\right| d\mu_n\right)^2 \le \mu_n(E\setminus K)\int_E \left(\frac{\partial f_n}{\partial h}\right)^2 d\mu_n$$

imply that the sequence of measures $\{\nu_n\} = \left\{\frac{\partial f_n}{\partial h}\mu_n\right\}$ is tight. Extract a weakly convergent sequence (denoted in the following again by $\{\nu_n\}$) $\nu_n \to \nu$. The following relations show that ν is absolutely continuous w.r.t. μ:

$$\begin{aligned}\int_E \varphi \, d\nu = \lim_n \int_E \varphi \frac{\partial f_n}{\partial h} d\mu_n \\ = \lim_n \left[-\int_E \frac{\partial \varphi}{\partial h} f_n d\mu_n - \int_E \varphi f_n \beta_h^{\mu_n} d\mu_n \right] \\ = -\int_E \frac{\partial \varphi}{\partial h} f \, d\mu - \int_E \varphi f \beta_h^{\mu} d\mu,\end{aligned}$$

$\varphi \in \mathscr{F}C_0^\infty(E)$. We also observe that f admits a weak derivative along h and, moreover, $\dfrac{d\nu}{d\mu} = \dfrac{\partial f}{\partial h}$. Hence by Cauchy's inequality

$$\int_E \varphi \frac{\partial f}{\partial h} d\mu = \lim_n \int_E \varphi \frac{\partial f_n}{\partial h} d\mu_n \le \left(\underline{\lim}_n \int_E \left(\frac{\partial f_n}{\partial h} \right)^2 d\mu_n \right)^{1/2} \cdot \left(\int_E \varphi^2 d\mu \right)^{1/2}.$$

Choosing a sequence $\varphi_n \to \frac{\partial f}{\partial h}$ in $L^2(E;\mu)$ one can easily complete the proof.

It can be easily seen form the proof that the stronger assumption $\|\beta_h^{\mu_n}\|_{L^2(E;\mu_n)} \to \|\beta_h\|_{L^2(E;\mu)}$ implies that $\beta_h^{\mu_n} \to \beta_h =: \beta_h^{\mu}$ $\mathscr{H}$-strongly and (M1) is fulfilled so that $\mathscr{E}_{\mu_n}^h \to \mathscr{E}_{\mu}^h$ Mosco. □

Remark 3.8. *Note that Theorem 3.6 works now in the case of one common orthonormal basis $\{e_i\}$, if we assume that each μ_n, $n \in \mathbb{N}$, μ admits a logarithmic derivative along each e_i, $i \in \mathbb{N}$ denoted by $\beta_i^{\mu_n}$ and that $\beta_i^{\mu_n} \to \beta_i^{\mu}$ $\mathscr{H}$-strongly. The proof is essentially the same as above (see also* [MR92, Proof of Proposition II.3.4]*).*

4. Examples of general convergence

4.1. Finite dimensional $a_{i,j}$-forms

Let $\sigma_n dx = \mu_n$, $n \in \mathbb{N}$, $\sigma dx = \mu$ as in Chapter 3.1. Let Assumption 2 hold.

Let $d \geq 3$. For $n \in \mathbb{N}$ let $a^n_{i,j}, b^n_i, d^n_i, c^n \in L^1_{\text{loc}}(\mathbb{R}^d; dx)$, $1 \leq i,j \leq d$ and define for $u, v \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned}\mathscr{E}^n(u,v) := \sum_{i,j=1}^d \int a^n_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \sum_{i=1}^d \int b^n_i \frac{\partial u}{\partial x_i} v \, dx \\ + \sum_{i=1}^d \int d^n_i u \frac{\partial v}{\partial x_i} dx + \int c^n uv \, dx.\end{aligned}$$

Then for each n $(\mathscr{E}^n, C_0^\infty(\mathbb{R}^d))$ is a densely defined bilinear form on $L^2(\mathbb{R}^d; \sigma_n dx)$. Set $\tilde{a}^n_{i,j} := \frac{1}{2}(a^n_{i,j} + a^n_{j,i})$, $\check{a}^n_{i,j} := \frac{1}{2}(a^n_{i,j} - a^n_{j,i})$, $b^n := (b^n_1, \ldots, b^n_d)$ and $d^n := (d^n_1, \ldots, d^n_d)$.

Theorem 4.1. *Suppose that*

(i) we have

$$\sum_{i,j=1}^d \tilde{a}^n_{i,j} \xi_i \xi_j \geq |\xi|^2$$

dx-a.e. for all $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$ and for every n,

(ii) for some uniform constant $M > 0$ we have $|\check{a}^n_{i,j}| \leq M$ for all $1 \leq i,j \leq d$ and for every n,

(iii) $|b^n + d^n| \in L^d_{\text{loc}}(\mathbb{R}^d; dx)$, $c^n \in L^{d/2}_{\text{loc}}(\mathbb{R}^d; dx)$ for every n,

(iv) $|b^n - d^n| \in L^d(\mathbb{R}^d; dx)$ or $\frac{|b^n - d^n|}{\sqrt{\sigma_n}} \in L^\infty(\mathbb{R}^d; dx)$ for every n,

(v) for some uniform $\alpha_0 > 0$ we have $(c^n + \alpha_0 \sigma_n) dx - \sum_{i=1}^d \frac{\partial d^n_i}{\partial x_i}$ is a positive measure on $\mathscr{B}(\mathbb{R}^d)$ for every n,

(vi) $b^n = \beta^n + \gamma^n$ such that $|\beta^n|, |\gamma^n| \in L^1_{\text{loc}}(\mathbb{R}^d; dx)$ and $(c^n + \alpha_0 \sigma_n) dx - \sum_{i=1}^d \frac{\partial \gamma^n_i}{\partial x_i}$ is a positive measure on $\mathscr{B}(\mathbb{R}^d)$ and $|\beta^n| \in L^d(\mathbb{R}^d; dx)$ or $\frac{\beta^n}{\sqrt{\sigma_n}} \in L^\infty(\mathbb{R}^d; dx)$ for every n.

Then there exists $\alpha \in (0,\infty)$ (independent of n) such that each $(\mathscr{E}^n_\alpha, C_0^\infty(\mathbb{R}^d))$ is closable on $L^2(\mathbb{R}^d;\sigma_n dx)$ and its closure $(\mathscr{E}^n_\alpha, \mathscr{D}(\mathscr{E}^n_\alpha))$ is a semi-Dirichlet form.

Proof. See the proof of [RS95, Theorem 1.2] and note the fact that all conditions above are taken uniformly in n. □

Let $a_{i,j}, b_i, d_i, c \in L^1_{\text{loc}}(\mathbb{R}^d;dx)$, $1 \le i,j \le d$ satisfy the conditions of Theorem 4.1. For $u, v \in C_0^\infty(\mathbb{R}^d)$ define

$$\begin{aligned}\mathscr{E}(u,v) := &\sum_{i,j=1}^d \int a_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx + \sum_{i=1}^d \int b_i \frac{\partial u}{\partial x_i} v\, dx \\ &+ \sum_{i=1}^d \int d_i u \frac{\partial v}{\partial x_i} dx + \int cuv\, dx.\end{aligned}$$

By Theorem 4.1 there exists a closed extension of $(\mathscr{E}_\alpha, C_0^\infty(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d;\sigma dx)$ which we shall denote by $(\mathscr{E}_\alpha, \mathscr{D}(\mathscr{E}_\alpha))$, for some $\alpha \in (0,\infty)$ as in Theorem 4.1.

Theorem 4.2. *Let for each $1 \le i,j \le d$*

$$\frac{a^n_{i,j}}{\sqrt{\sigma_n}} \to \frac{a_{i,j}}{\sqrt{\sigma}},\quad \frac{\partial a^n_{i,j}}{\partial x_j}(\sqrt{\sigma_n})^{-1} \to \frac{\partial a_{i,j}}{\partial x_j}(\sqrt{\sigma})^{-1} \quad \textit{strongly in } L^2_{\text{loc}}(\mathbb{R}^d;dx),$$

$$\frac{b^n_i}{\sqrt{\sigma_n}} \to \frac{b_i}{\sqrt{\sigma}} \quad \textit{strongly in } L^2_{\text{loc}}(\mathbb{R}^d;dx),$$

$$\frac{d^n_i}{\sqrt{\sigma_n}} \to \frac{d_i}{\sqrt{\sigma}},\quad \frac{\partial d^n_i}{\partial x_i}(\sqrt{\sigma_n})^{-1} \to \frac{\partial d_i}{\partial x_i}(\sqrt{\sigma})^{-1} \quad \textit{strongly in } L^2_{\text{loc}}(\mathbb{R}^d;dx),$$

$$\frac{c^n}{\sqrt{\sigma_n}} \to \frac{c}{\sqrt{\sigma}} \quad \textit{strongly in } L^2_{\text{loc}}(\mathbb{R}^d;dx),$$

where the derivatives are taken in the distributional sense.

Assume that $\sqrt{\sigma_n} \to \sqrt{\sigma}$ weakly in $L^2_{\text{loc}}(\mathbb{R}^d;dx)$.

Also assume that $C_0^\infty(\mathbb{R}^d)$ is dense in $\mathscr{D}(\mathscr{E}_\alpha)$ w.r.t. $\tilde{\mathscr{E}}^{1/2}_{\alpha+1}$.

Then $\mathscr{E}^n_\alpha \to \mathscr{E}_\alpha$ in the generalized sense, more precisely, conditions (F1b) and (F2) of Definition 2.40 hold.

Before we prove Theorem 4.2, we shall consider two Lemmas, which will firstly provide a useful observation about “local” strong and weak convergence, and secondly illuminate the underlying structure of approximation of our forms.

Lemma 4.3. *Let $\varphi \in C_0^\infty(\mathbb{R}^d)$, $u_n \in L^2(\mathbb{R}^d;\sigma_n dx)$, $n \in \mathbb{N}$, $u \in L^2(\mathbb{R}^d;\sigma dx)$. If $u_n \to u$ strongly (weakly) in $\mathscr{H}$, then $\varphi u_n \to \varphi u$ strongly (weakly) in $\mathscr{H}$, too.*

Proof. Note that clearly $\varphi \in L^2(\mathbb{R}^d;\sigma_n dx)$ for any n and $\varphi \in L^2(\mathbb{R}^d;\sigma dx)$ (with corresponding dx-classes respected), if $\varphi \in C_0^\infty(\mathbb{R}^d)$. Let $\{\varphi_m\} \subset C_0^\infty(\mathbb{R}^d)$ such that $\varphi_m \to u$ in $L^2(\mathbb{R}^d;\sigma dx)$ and

$$\lim_m \overline{\lim_n} \|\varphi_m - u_n\|_{L^2(\mathbb{R}^d;\sigma_n dx)} = 0.$$

Then by Hölder inequality

$$\|\varphi_m\varphi - u\varphi\|_{L^2(\mathbb{R}^d;\sigma dx)} \le \|\varphi\|_{L^\infty(\mathbb{R}^d;dx)}\|\varphi_m - u\|_{L^2(\mathbb{R}^d;\sigma dx)} \to 0$$

as $m \to \infty$ and

$$\|\varphi_m\varphi - u_n\varphi\|_{L^2(\mathbb{R}^d;\sigma_n dx)} \le \|\varphi\|_{L^\infty(\mathbb{R}^d;dx)}\|\varphi_m - u_n\|_{L^2(\mathbb{R}^d;\sigma_n dx)} \to 0$$

as $m, n \to \infty$. The case of weak convergence follows from Lemma 2.13 since $\varphi \cdot \psi \in C_0^\infty(\mathbb{R}^d)$ for any $\varphi, \psi \in C_0^\infty(\mathbb{R}^d)$. $\square$

Lemma 4.4. *Consider $(\mathscr{E}_\alpha, \mathscr{D}(\mathscr{E}_\alpha))$ as above. Suppose that the conditions of Theorem 4.2 hold. Then $(L^2(\mathbb{R}^d;\sigma_n dx), \tilde{\mathscr{E}}_\alpha^n) \to (L^2(\mathbb{R}^d;\sigma dx), \tilde{\mathscr{E}}_\alpha)$ in the sense of Definition 2.30.*

Proof. (1) is trivial. (2) follows from the assumption. For (3) just observe that for $\varphi \in C_0^\infty(\mathbb{R}^d)$, $\varphi\sqrt{\sigma_n} \to \varphi\sqrt{\sigma}$ weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^d;dx)$ by Lemma 3.1 and the assumption that $\sqrt{\sigma_n} \to \sqrt{\sigma}$ weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^d;dx)$. Also note that $\frac{\tilde{a}^n_{i,j}}{\sqrt{\sigma_n}} \to \frac{\tilde{a}_{i,j}}{\sqrt{\sigma_n}}$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^d;dx)$ for $1 \le i,j \le d$ by linearity. Then clearly (using the product rule)

$$\begin{aligned}
\tilde{\mathscr{E}}_\alpha^n(\varphi,\varphi) =& \sum_{i,j=1}^d \int \tilde{a}^n_{i,j}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}dx + \frac{1}{2}\sum_{i=1}^d \int b_i^n \frac{\partial\varphi^2}{\partial x_i}dx + \frac{1}{2}\sum_{i=1}^d \int d_i^n \frac{\partial\varphi^2}{\partial x_i}dx \\
&+ \int c^n\varphi^2 dx + \alpha\int \varphi^2\sigma_n dx \\
=& \sum_{i,j=1}^d \int \frac{\tilde{a}^n_{i,j}}{\sqrt{\sigma_n}}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\sqrt{\sigma_n}dx + \frac{1}{2}\sum_{i=1}^d \int \frac{b_i^n}{\sqrt{\sigma_n}}\frac{\partial\varphi^2}{\partial x_i}\sqrt{\sigma_n}dx \\
&+ \frac{1}{2}\sum_{i=1}^d \int \frac{d_i^n}{\sqrt{\sigma_n}}\frac{\partial\varphi^2}{\partial x_i}\sqrt{\sigma_n}dx + \int \frac{c^n}{\sqrt{\sigma_n}}\varphi^2\sqrt{\sigma_n}dx + \alpha\int\varphi^2\sigma_n dx \\
\xrightarrow[n\to\infty]{}& \sum_{i,j=1}^d \int \frac{\tilde{a}_{i,j}}{\sqrt{\sigma}}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\sqrt{\sigma}dx + \frac{1}{2}\sum_{i=1}^d \int \frac{b_i}{\sqrt{\sigma}}\frac{\partial\varphi^2}{\partial x_i}\sqrt{\sigma}dx \\
&+ \frac{1}{2}\sum_{i=1}^d \int \frac{d_i}{\sqrt{\sigma}}\frac{\partial\varphi^2}{\partial x_i}\sqrt{\sigma}dx + \int \frac{c}{\sqrt{\sigma}}\varphi^2\sqrt{\sigma}dx + \alpha\int\varphi^2\sigma dx \\
=& \tilde{\mathscr{E}}_\alpha(\varphi,\varphi).
\end{aligned}$$

The proof is complete. $\square$

Proof of Theorem 4.2. First note that we are in the situation of Remark 2.50 and that the sector constants K_n of the $\mathscr{E}^n$'s are uniformly bounded (since all closability conditions are taken uniformly in n) and thus Corollary 2.42 applies. Let us first prove (F1b). Let $u_n \in H_n$, $u \in H$, $u_n \to u$ $\mathscr{H}$-weakly with $\underline{\lim}_n (\tilde{\mathscr{E}}^n_{\alpha+1})^{1/2}(u_n) < \infty$. By Lemma 4.4 we can consider the sequence of Hilbert spaces $\mathscr{H}_{\tilde{\mathscr{E}}_\alpha} = \dot{\bigcup}_{n\in\mathbb{N}} \mathscr{D}(\tilde{\mathscr{E}}^n_\alpha) \dot{\cup} \mathscr{D}(\tilde{\mathscr{E}}_\alpha)$. We can extract a subsequence $\{u_{n_k}\}$ converging weakly in $\mathscr{H}_{\tilde{\mathscr{E}}_\alpha}$ to some $\tilde{u} \in \mathscr{D}(\tilde{\mathscr{E}}_\alpha)$ with $\underline{\lim}_k \tilde{\mathscr{E}}^{n_k}_{\alpha+1}(u_{n_k}) \geq \tilde{\mathscr{E}}_{\alpha+1}(\tilde{u})$. But clearly $\tilde{u} = u$, thus we obtain (F1b) (here we have used Lemma 2.14).

Now we would like to prove (F2). Let $\varphi \in C_0^\infty(\mathbb{R}^d)$. We choose $\Phi_n(\varphi) = \varphi \to \varphi$ as the desired $\mathscr{H}$-strongly convergent sequence. Then, if $u_n \in \mathscr{D}(\mathscr{E}^n_\alpha)$, $u \in \mathscr{D}(\mathscr{E}_\alpha)$, $u_n \to u$ $\mathscr{H}$-weakly with $\sup_n (\tilde{\mathscr{E}}^n_{\alpha+1})^{1/2}(u_n) < \infty$, we have by the assumption that $\sqrt{\sigma_n} \to \sqrt{\sigma}$ weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^d; dx)$ and Lemma 3.1 that $u_n\sqrt{\sigma_n} \to u\sqrt{\sigma}$ weakly in $L^2(\mathbb{R}^d; dx)$ and ("locally") via partial integration:

$$
\begin{aligned}
\mathscr{E}^n_\alpha(\varphi, u_n) = &\sum_{i,j=1}^d \int a^n_{i,j} \frac{\partial \varphi}{\partial x_i}\frac{\partial u_n}{\partial x_j} dx + \sum_{i=1}^d \int b^n_i \frac{\partial \varphi}{\partial x_i} u_n dx \\
&+ \sum_{i=1}^d \int d^n_i \varphi \frac{\partial u_n}{\partial x_i} dx + \int c^n \varphi u_n dx + \alpha \int \varphi u_n \sigma_n dx \\
= &- \sum_{i,j=1}^d \left[\int \frac{\partial a^n_{i,j}}{\partial x_j}\frac{\partial \varphi}{\partial x_i} u_n dx + \int a^n_{i,j} \frac{\partial^2 \varphi}{x_i x_j} u_n dx \right] + \sum_{i=1}^d \int b^n_i \frac{\partial \varphi}{\partial x_i} u_n dx \\
&- \sum_{i=1}^d \left[\int \frac{\partial d^n_i}{\partial x_i} \varphi u_n dx + \int d^n_i \frac{\partial \varphi}{x_i} u_n dx \right] + \int c^n \varphi u_n dx + \alpha \int \varphi u_n \sigma_n dx \\
= &- \sum_{i,j=1}^d \left[\int \frac{\partial a^n_{i,j}}{\partial x_j} (\sqrt{\sigma_n})^{-1} \frac{\partial \varphi}{\partial x_i} u_n \sqrt{\sigma_n} dx + \int \frac{a^n_{i,j}}{\sqrt{\sigma_n}} \frac{\partial^2 \varphi}{x_i x_j} u_n \sqrt{\sigma_n} dx \right] \\
&+ \sum_{i=1}^d \int \frac{b^n_i}{\sqrt{\sigma_n}} \frac{\partial \varphi}{\partial x_i} u_n \sqrt{\sigma_n} dx \\
&- \sum_{i=1}^d \left[\int \frac{\partial d^n_i}{\partial x_i} (\sqrt{\sigma_n})^{-1} \varphi u_n \sqrt{\sigma_n} dx + \int \frac{d^n_i}{\sqrt{\sigma_n}} \frac{\partial \varphi}{x_i} u_n \sqrt{\sigma_n} dx \right] \\
&+ \int \frac{c^n}{\sqrt{\sigma_n}} \varphi u_n \sqrt{\sigma_n} dx + \alpha \int \varphi u_n \sigma_n dx \\
\underset{n\to\infty}{\longrightarrow} = &- \sum_{i,j=1}^d \left[\int \frac{\partial a_{i,j}}{\partial x_j} (\sqrt{\sigma})^{-1} \frac{\partial \varphi}{\partial x_i} u \sqrt{\sigma} dx + \int \frac{a_{i,j}}{\sqrt{\sigma}} \frac{\partial^2 \varphi}{x_i x_j} u \sqrt{\sigma} dx \right] \\
&+ \sum_{i=1}^d \int \frac{b_i}{\sqrt{\sigma}} \frac{\partial \varphi}{\partial x_i} u \sqrt{\sigma} dx - \sum_{i=1}^d \left[\int \frac{\partial d_i}{\partial x_i} (\sqrt{\sigma})^{-1} \varphi u \sqrt{\sigma} dx + \int \frac{d_i}{\sqrt{\sigma}} \frac{\partial \varphi}{x_i} u \sqrt{\sigma} dx \right] \\
&+ \int \frac{c}{\sqrt{\sigma}} \varphi u \sqrt{\sigma} dx + \alpha \int \varphi u \, \sigma dx
\end{aligned}
$$

$$
\begin{aligned}
&= - \sum_{i,j=1}^{d} \left[\int \frac{\partial a_{i,j}}{\partial x_j} \frac{\partial \varphi}{\partial x_i} u \, dx + \int a_{i,j} \frac{\partial^2 \varphi}{x_i x_j} u \, dx \right] + \sum_{i=1}^{d} \int b_i \frac{\partial \varphi}{\partial x_i} u \, dx \\
&\quad - \sum_{i=1}^{d} \left[\int \frac{\partial d_i}{\partial x_i} \varphi u \, dx + \int d_i \frac{\partial \varphi}{x_i} u \, dx \right] + \int c \varphi u \, dx + \alpha \int \varphi u \, \sigma dx \\
&= \sum_{i,j=1}^{d} \int a_{i,j} \frac{\partial \varphi}{\partial x_i} \frac{\partial u}{\partial x_j} dx + \sum_{i=1}^{d} \int b_i \frac{\partial \varphi}{\partial x_i} u \, dx \\
&\quad + \sum_{i=1}^{d} \int d_i \varphi \frac{\partial u}{\partial x_i} dx + \int c \varphi u \, dx + \alpha \int \varphi u \, \sigma dx \\
&= \mathscr{E}_\alpha(\varphi, u)
\end{aligned}
$$

The proof is complete. $\square$

4.2. Infinite dimensional $a_{i,j}$-forms

Let $E, X, \{\mu_n\}, \mu$ be as in Chapter 3.2, i.e.,

$$E' \subset X' \equiv X \subset E \quad \text{densely and continuously.}$$

Considering this situation we would like to make the slight change that we consider $\mathscr{F}C_b^\infty(E)$ instead of $\mathscr{F}C_0^\infty(E)$. $\mathscr{F}C_b^\infty(E)$ is defined to be the linear space

$$\mathscr{F}C_b^\infty(E) := \{ f(l_1, \ldots, l_m) \mid m \in \mathbb{N},\ f \in C_b^\infty(\mathbb{R}^m),\ l_1, \ldots, l_m \in E' \},$$

where $C_b^\infty(\mathbb{R}^m)$ denotes the space of all infinitely differentiable continuous and bounded (real-valued) functions on $\mathbb{R}^m$ with all partial derivatives bounded. This does not affect the general setting, since this choice is standard. Nevertheless note that we need $\mathscr{F}C_0^\infty(E)$ functions for the results of A.V. Kolesnikov in [Kol06]. Let Assumptions 4 and 5 hold. Then $H_n := L^2(E; \mu_n) \to L^2(E; \mu) =: H$ in the sense of convergent Hilbert spaces (here we set $C := \mathscr{F}C_b^\infty(E)$). As usually set $\mathscr{H} := \dot{\bigcup}_n H_n \dot{\cup} H$.

Assume that there exists an orthonormal basis $K_0 := \{e_i \mid i \in \mathbb{N}\}$ of X whose elements are well-μ_n-admissible for each $n \in \mathbb{N}$ (see Chapter 3.2.1). Denote by $\beta_i^{\mu_n}$ the logarithmic derivative of μ_n along e_i, $n \in \mathbb{N}, i \in \mathbb{N}$.

Then next Lemma is analog to Lemma 4.3.

Lemma 4.5. *Let $\varphi \in \mathscr{F}C_b^\infty(E)$, $u_n \in L^2(E; \mu_n)$, $n \in \mathbb{N}$, $u \in L^2(E; \mu)$. If $u_n \to u$ strongly (weakly) in $\mathscr{H}$, then $\varphi u_n \to \varphi u$ strongly (weakly) in $\mathscr{H}$, too.*

Proof. Note that clearly $\varphi \in L^2(E; \mu_n)$ for any n and $\varphi \in L^2(E; \mu)$ (with corresponding L^2-classes respected), if $\varphi \in \mathscr{F}C_b^\infty(E)$. Let $\{\varphi_m\} \subset \mathscr{F}C_b^\infty(E)$ such that $\varphi_m \to u$ in $L^2(E; \mu)$ and

$$\lim_m \overline{\lim_n} \|\varphi_m - u_n\|_{L^2(E;\mu_n)} = 0.$$

Then by Hölder inequality

$$\|\varphi_m\varphi - u\varphi\|_{L^2(E;\mu)} \le \|\varphi\|_{L^\infty(E;\mu)}\|\varphi_m - u\|_{L^2(E;\mu)} \to 0$$

as $m \to \infty$ and

$$\|\varphi_m\varphi - u_n\varphi\|_{L^2(E;\mu_n)} \le \|\varphi\|_{L^\infty(E;\mu_n)}\|\varphi_m - u_n\|_{L^2(E;\mu_n)} \to 0$$

as $m, n \to \infty$ (note that $\sup_n \|\varphi\|_{L^\infty(E;\mu_n)} = \sup_n \operatorname{ess\,sup}^{(\mu_n)}_{z\in E}\varphi(z) < \infty$). The case of weak convergence follows from Lemma 2.13 since $\varphi \cdot \psi \in \mathscr{F}C_b^\infty(E)$ for any $\varphi, \psi \in \mathscr{F}C_b^\infty(E)$. □

Fix $n \in \mathbb{N}$. Let $A^{(n)} := (a_{i,j}^{(n)})_{i,j\in\mathbb{N}}$, where each $a_{i,j}^{(n)}$, $i,j \in \mathbb{N}$ is a $\mathscr{B}(E)$-measurable function on E. For each $z \in E$ define a linear operator on X by

$$A^{(n)}(z)h := \sum_{i=1}^\infty \sum_{j=1}^\infty a_{i,j}^{(n)}(z)\,\langle h, e_j\rangle_X\, e_i, \quad h \in X.$$

Assume that $\sum_{i,j=1}^\infty |a_{i,j}^{(n)}|(z) < \infty$, $z \in E$, so that each $A^{(n)}(z)$ is a bounded (even trace class) operator. Assume that $z \mapsto \left\langle A^{(n)}(z)h_1, h_2\right\rangle_X$ is $\mathscr{B}(E)$-measurable for all $h_1, h_2 \in X$. Furthermore, assume that there exists a uniform ellipticity constant $c > 0$ not depending on n with

$$\left\langle A^{(n)}(z)h, h\right\rangle_X = \sum_{i=1}^\infty \sum_{j=1}^\infty a_{i,j}^{(n)}(z)\,\langle h, e_i\rangle_X\,\langle h, e_j\rangle_X \ge c\|h\|_X^2 \tag{4.1}$$

for all $h \in X$.

Set $\tilde{A}^{(n)} := \frac{1}{2}(A^{(n)} + \hat{A}^{(n)})$, $\breve{A}^{(n)} := \frac{1}{2}(A^{(n)} - \hat{A}^{(n)})$ where $\hat{A}^{(n)}(z)$ denotes the adjoint of $A^{(n)}(z)$, $z \in E$. One easily observes that $\tilde{A}$ and $\breve{A}$ resp. can be constructed as in (4.1) with $\tilde{a}_{i,j}^{(n)} := \frac{1}{2}(a_{i,j}^{(n)} + a_{j,i}^{(n)})$ and $\breve{a}_{i,j}^{(n)} := \frac{1}{2}(a_{i,j}^{(n)} - a_{j,i}^{(n)})$ resp. Assume that $\|\tilde{A}^{(n)}\|_{\mathscr{L}(X)} \in L^1(E;\mu_n)$, $\|\breve{A}^{(n)}\|_{\mathscr{L}(X)} \in L^\infty(E;\mu_n)$.

Let $c^{(n)} \in L^\infty(E;\mu_n)$ and $b^{(n)}, d^{(n)} \in L^\infty(E \to X;\mu_n)$ such that

$$\int_E \left(\left\langle b^{(n)}, \nabla_X u\right\rangle_X + c^{(n)}u\right) d\mu_n \ge 0,$$
$$\int_E \left(\left\langle d^{(n)}, \nabla_X u\right\rangle_X + c^{(n)}u\right) d\mu_n \ge 0$$

for all $u \in \mathscr{F}C_b^\infty(E)$, $u \ge 0$.

Define for $n \in \mathbb{N}$, $u, v \in \mathscr{F}C_b^\infty(E)$

$$\begin{aligned}\mathscr{E}^n(u,v) = &\int_E \left\langle A^{(n)}(z)\nabla_X u(z), \nabla_X v(z)\right\rangle_X \mu_n(dz)\\ &+ \int_E \left\langle b^{(n)}(z), \nabla_X u(z)\right\rangle_X v(z)\mu_n(dz)\\ &+ \int_E u(z)\left\langle d^{(n)}(z), \nabla_X v(z)\right\rangle_X \mu_n(dz)\\ &+ \int_E c^{(n)}(z)u(z)v(z)\mu_n(dz).\end{aligned}$$

Then for $n \in \mathbb{N}$ each $(\mathscr{E}^n, \mathscr{F}C_b^\infty(E))$ is a densely defined positive bilinear form on $L^2(E;\mu_n)$ which is closable and whose closure $(\mathscr{E}^n, \mathscr{D}(\mathscr{E}^n))$ is a Dirichlet form. Each $\mathscr{E}^n$ has a sector constant bounded by

$$K_n := \sup_{z\in E}\left[\|\check{A}^{(n)}(z)\|_{\mathscr{L}(X)} \vee \|d^{(n)}(z) - b^{(n)}(z)\|_X\right]$$

by the results of [MR92, Section II.3.e)] (and also fulfills the (semi-)Dirichlet property).

Define

$$Q^n(u,v) := \int_E \langle \nabla_X u(z), \nabla_X v(z)\rangle_X \mu_n(dz), \quad u, v \in \mathscr{F}C_b^\infty(E).$$

Since each e_i, $i \in \mathbb{N}$ is well-μ_n-admissible, $(Q^n, \mathscr{F}C_b^\infty(E))$ is closable and its closure $(Q^n, \mathscr{D}(Q^n))$ is a symmetric Dirichlet form (cf. [MR92, Sections II.3.a) and II.3.b)]. Let us define $(Q, \mathscr{D}(Q))$ in the same way.

For each $n \in \mathbb{N}$, $i, j \in \mathbb{N}$ set $b_i^{(n)} := \left\langle b^{(n)}, e_i\right\rangle_X$, $d_i^{(n)} := \left\langle d^{(n)}, e_i\right\rangle_X$ and assume that

$$a_{i,j}^{(n)}, b_i^{(n)}, d_i^{(n)}, c^{(n)} \in L^2(E;\mu_n).$$

Theorem 4.6. *First assume that the sector constants of the $\mathscr{E}^n$'s are uniformly bounded, i.e., $\sup_n K_n < \infty$.*

Then suppose that for $i, j \in \mathbb{N}$ there are given $a_{i,j}, b_i, d_i, c \in L^2(E;\mu)$ such that

$$\begin{aligned}a_{i,j}^{(n)} &\to a_{i,j} \quad \mathscr{H}\text{-strongly},\\ b_i^{(n)} &\to b_i \quad \mathscr{H}\text{-strongly},\\ d_i^{(n)} &\to d_i \quad \mathscr{H}\text{-strongly},\\ c^{(n)} &\to c \quad \mathscr{H}\text{-strongly},\end{aligned}$$

for $i, j \in \mathbb{N}$ and that the limiting coefficients fulfill the same conditions as above.

Assume also that for $i \in \mathbb{N}$ there exists a $\beta_i \in L^2(E;\mu)$ such that

$$\|\beta_i^{\mu_n}\|_{L^2(E;\mu_n)} \to \|\beta_i\|_{L^2(E;\mu)}.$$

Also assume that there exists a uniform constant $c_0 > 0$ such that

$$c_0 Q_1^n(u,u) \le \tilde{\mathscr{E}}_1^n(u,u)$$

for every $u \in \mathscr{D}(\mathscr{E}^n) \subset \mathscr{D}(Q^n)$, $n \in \mathbb{N}$.
Assume that $(\mathscr{E}, \mathscr{F}C_b^\infty(E))$ defined as follows

$$\begin{aligned}\mathscr{E}(u,v) := \sum_{i,j=1}^{\infty} \int_E a_{i,j} \frac{\partial u}{\partial e_i} \frac{\partial v}{\partial e_j} d\mu + \sum_{i=1}^{\infty} \int_E b_i \frac{\partial u}{\partial e_i} v \, d\mu \\ + \sum_{i=1}^{\infty} \int_E u d_i \frac{\partial v}{\partial e_i} d\mu + \int_E c u v \, d\mu, \quad u, v \in \mathscr{F}C_b^\infty(E),\end{aligned}$$

is closable with closure $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ and that $\mathscr{F}C_b^\infty(E)$ is dense in $\mathscr{D}(\mathscr{E})$ w.r.t. $\tilde{\mathscr{E}}_1^{1/2}$.
Then $\mathscr{E}^n \to \mathscr{E}$ in the generalized sense, in particular, (F1b) and (F2'b) hold and $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a (semi-)Dirichlet form.

Lemma 4.7. *Let the conditions of Theorem 4.6 hold. Then $(L^2(E;\mu_n), \tilde{\mathscr{E}}^n) \to (L^2(E;\mu), \tilde{\mathscr{E}})$ in the sense of Definition 2.30.*

Proof. (1) is trivial. (2) is a condition on $\mathscr{E}$. To prove (3) let $\varphi \in \mathscr{F}C_b^\infty(E)$. Then clearly

$$\begin{aligned}\tilde{\mathscr{E}}^n(\varphi,\varphi) &= \sum_{i,j=1}^{\infty} \int_E \tilde{a}_{i,j}^{(n)} \frac{\partial \varphi}{\partial e_i} \frac{\partial \varphi}{\partial e_j} d\mu_n + \frac{1}{2} \sum_{i=1}^{\infty} \int_E b_i^{(n)} \frac{\partial \varphi^2}{\partial e_i} d\mu_n \\ &\quad + \frac{1}{2} \sum_{i=1}^{\infty} \int_E d_i^{(n)} \frac{\partial \varphi^2}{\partial e_i} d\mu_n + \int_E c^{(n)} \varphi^2 d\mu_n \\ &\xrightarrow[n\to\infty]{} \sum_{i,j=1}^{\infty} \int_E \tilde{a}_{i,j} \frac{\partial \varphi}{\partial e_i} \frac{\partial \varphi}{\partial e_j} d\mu + \frac{1}{2} \sum_{i=1}^{\infty} \int_E b_i \frac{\partial \varphi^2}{\partial e_i} d\mu \\ &\quad + \frac{1}{2} \sum_{i=1}^{\infty} \int_E d_i \frac{\partial \varphi^2}{\partial e_i} d\mu + \int_E c \varphi^2 d\mu \\ &= \tilde{\mathscr{E}}(\varphi,\varphi),\end{aligned}$$

where we have used Lemma 4.5 and the product rule for Gâteaux derivatives. Note that all sums are finite by the above conditions on the coefficients (cf. [MR92, Section II.3.a)]). □

Proof of Theorem 4.6. We would first like to prove (F2'b). Note that by assumption the sector constants K_n are uniformly bounded. Set $\tilde{C} := C = \mathscr{F}C_b^\infty(E)$. By assumption $C \subset \mathscr{D}(\mathscr{E})$ densely w.r.t. $\tilde{\mathscr{E}}_1^{1/2}$. Let $n_k \in \mathbb{N}$, $n_k \uparrow \infty$, $n_{k+1} > n_k$. Let $u_k \in L^2(E;\mu_{n_k})$, $u \in \mathscr{D}(\mathscr{E})$ such that $u_k \to u$ $\mathscr{H}$-weakly and $\sup_k (\tilde{\mathscr{E}}_1^{n_k})^{1/2}(u_k) < \infty$. W.l.o.g $u_k \in \mathscr{D}(\mathscr{E}^{n_k})$. By

$c_0 Q_1^n(u,u) \le \mathscr{E}_1^n(u,u)$ for each $u \in \mathscr{D}(\mathscr{E}^n) \subset \mathscr{D}(Q^n)$ we have that $\sup_k (Q_1^{n_k})^{1/2}(u_k) < \infty$. Clearly each u_k admits a weak derivative in each direction e_i denoted by $\frac{\partial u_k}{\partial e_i}$. Hence by the proof of Proposition 3.7 and Remark 3.8 $Q^n \to Q$ Mosco (since a subsequence of $\beta_i^{\mu_n}$ converges $\mathscr{H}$-strongly to $\beta_i^{\mu} := \beta_i$ being the logarithmic derivative of μ in direction e_i) and we can extract a subsequence $\{u_{k_l}\}$, $u_{k_l} \in \mathscr{D}(\mathscr{E}^{n_{k_l}})$ such that $u_{k_l} \to u$ $\mathscr{H}$-weakly and u admits a weak derivative in each direction e_i denoted by $\frac{\partial u}{\partial e_i}$ such that $\frac{\partial u_{k_l}}{\partial e_i} \to \frac{\partial u}{\partial e_i}$ weakly in $\mathscr{H}$. Clearly for $\varphi \in \mathscr{F}C_b^\infty(E)$ by Lemma 4.5

$$
\begin{aligned}
\mathscr{E}^{n_{k_l}}(\varphi, u_{k_l}) = & \sum_{i,j=1}^{\infty} \int_E a_{i,j}^{(n_{k_l})} \frac{\partial \varphi}{\partial e_i} \frac{\partial u_{k_l}}{\partial e_j} d\mu_{n_{k_l}} + \sum_{i=1}^{\infty} \int_E b_i^{(n_{k_l})} \frac{\partial \varphi}{\partial e_i} u_{k_l} d\mu_{n_{k_l}} \\
& + \sum_{i=1}^{\infty} \int_E d_i^{(n_{k_l})} \varphi \frac{\partial u_{k_l}}{\partial e_i} d\mu_{n_{k_l}} + \int_E c^{(n_{k_l})} \varphi u_{k_l} d\mu_{n_{k_l}} \\
\xrightarrow[l\to\infty]{} & \sum_{i,j=1}^{\infty} \int_E a_{i,j} \frac{\partial \varphi}{\partial e_i} \frac{\partial u}{\partial e_j} d\mu + \sum_{i=1}^{\infty} \int_E b_i \frac{\partial \varphi}{\partial e_i} u \, d\mu \\
& + \sum_{i=1}^{\infty} \int_E d_i \varphi \frac{\partial u}{\partial e_i} d\mu + \int_E c \varphi u \, d\mu \\
= & \mathscr{E}(\varphi, u).
\end{aligned}
$$

Since this is the limit of a subsequence we immediately get

$$
\underline{\lim}_k \, \mathscr{E}^{n_k}(\varphi, u_k) \le \mathscr{E}(\varphi, u).
$$

(F2'b) is proved.

To prove (F1b) let $u_n \in L^2(E;\mu_n)$, $u \in L^2(E;\mu)$, $u_n \to u$ $\mathscr{H}$-weakly with $\underline{\lim}_n (\tilde{\mathscr{E}}_1^n)^{1/2}(u_n) < \infty$. By Lemma 4.7 we can consider the sequence of Hilbert spaces $\mathscr{H}_{\tilde{\mathscr{E}}} = \bigcup_{n\in\mathbb{N}} \mathscr{D}(\tilde{\mathscr{E}}^n) \dot{\cup} \mathscr{D}(\tilde{\mathscr{E}})$. We can extract a subsequence $\{u_{n_k}\}$ converging weakly in $\mathscr{H}_{\tilde{\mathscr{E}}}$ to some $\tilde{u} \in \mathscr{D}(\tilde{\mathscr{E}})$ with $\underline{\lim}_k \tilde{\mathscr{E}}_1^{n_k}(u_{n_k}) \ge \tilde{\mathscr{E}}_1(\tilde{u})$ (here we have used Lemma 2.14). But clearly $\tilde{u} = u$, thus we obtain (F1b).

The proof is complete. $\square$

5. Applications to stochastics

5.1. Convergence of laws

Let E be an infinite dimensional separable complete metric space (i.e., a *Polish* space) resp. a locally compact space if finite dimensional. Define $\mathscr{F}C_b^\infty(E)$ as in the previous Chapter. Denote by $\mathscr{B}_b(E)$ the set of all bounded Borel-measurable (real-valued) functions on E and by $C_b(E)$ the set of all bounded and continuous (real-valued) functions on E.

Let μ be a Borel probability measure on the Borel σ-algebra $\mathscr{B}(E)$ of E. Let

$$\mathbb{M} = (\Omega, \mathscr{M}, (X_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in E})$$

be a right process with lifetime $\zeta \equiv +\infty$ (i.e., the process is *conservative*). Define a probability measure $\mathbb{P}$ on $(\Omega, \mathscr{M})$ by

$$\mathbb{P}(\Gamma) := \int_E \mathbb{P}_z(\Gamma)\mu(dz), \quad \Gamma \in \mathscr{M}.$$

Let $J := \{t_1, \ldots, t_k\} \subset [0, \infty)$, $0 \leq t_1 \leq \cdots \leq t_k$, $k \in \mathbb{N}$. The finite dimensional distribution $\mathbb{P}^J$ of $\mathbb{P}$ on $(E^J, \mathscr{B}(E^J))$ is defined by

$$\mathbb{P}^J(A) := \mathbb{P}((X_{t_1}, \ldots, X_{t_k}) \in A), \quad A \in \mathscr{B}(E^J).$$

Let m_n, $n \in \mathbb{N}$, m resp. be fully supported Borel measures on E such that $m_n \to m$ weakly (resp. vaguely if E is locally compact). Then

$$L^2(E; m_n) =: H_n \to H := L^2(E; m)$$

in the sense of convergent Hilbert spaces, where $C := \mathscr{F}C_b^\infty(E) \subset L^2(E; m)$ is dense (see e.g. [MR92, Section II.3.a)]). As usually set $\mathscr{H} := \dot{\bigcup}_n H_n \dot{\cup} H$. (If E is locally compact, we take $C := C_0^\infty(E)$).

Assume that for every $n \in \mathbb{N}$ there exists a conservative right process

$$\mathbb{M}^{(n)} = (\Omega, \mathscr{M}, (X_t^{(n)})_{t\geq 0}, (\mathbb{P}_z^{(n)})_{z\in E})$$

on a common measure space $(\Omega, \mathscr{M})$ and with common state space E. Define $\mathbb{P}^{(n)}$, $\mathbb{P}^{(n),J}$, similarly w.r.t. some Borel probability measure μ_n, $n \in \mathbb{N}$ on $\mathscr{B}(E)$.

Assumption 8. *(1) $\mu_n \to \mu$ weakly.*

(2) Each μ_n, $n \in \mathbb{N}$, μ resp. is absolutely continuous w.r.t. m_n, $n \in \mathbb{N}$, m resp. with positive densities h_n, $n \in \mathbb{N}$, h resp.

(3) $h_n \in L^2(E; m_n)$, $n \in \mathbb{N}$, $h \in L^2(E; m)$ resp. and $\sup_n \|h_n\|_{L^2(E;m_n)} < \infty$.

Clearly by weak convergence,

$$\lim_n \int_E \varphi h_n \, dm_n = \lim_n \int_E \varphi \, d\mu_n = \int_E \varphi \, d\mu = \int_E \varphi h \, dm$$

for any $\varphi \in \mathscr{F}C_b^\infty(E) \subset C_b(E)$. Hence we get by Lemma 2.13 that $h_n \to h$ weakly in $\mathscr{H}$. (The argument is similar in the finite dimensional case with vague convergence and $C = C_0^\infty(E) \subset C_0(E)$).

Assumption 9. *(1) $\{\mathbb{P}^{(n)}\}_{n \in \mathbb{N}}$ is tight.*

(2) The (common) path space Ω of the $\mathbb{M}^{(n)}$'s is Polish (provided the processes are constructed canonically).

(1) can be verified in many cases with help of the so called Lyons-Zheng decomposition (cf. [FOT94], [LZ93], [LZ94], [LZ96], [RZ96] and [Tak89]). For (2) we can use the so called Skorokhod metric, since our considered processes are $\mathbb{P}^{(n)}$-a.s. right continuous.

Theorem 5.1. *Let $\mathbb{M}^{(n)}$, $n \in \mathbb{N}$, $\mathbb{M}$ resp. be right processes as above associated with (generalized) Dirichlet forms $\mathscr{E}^{(n)}$, $n \in \mathbb{N}$, $\mathscr{E}$ resp. on $L^2(E; m_n)$, $n \in \mathbb{N}$, $L^2(E; m)$ resp. Denote by $(T_t^{(n)})_{t \geq 0}$, $n \in \mathbb{N}$, $(T_t)_{t \geq 0}$ resp. the associated L^2-semigroups. Let Assumptions 8 and 9 hold. If $T_t^{(n)} \to T_t$ strongly in $\mathscr{H}$ for any $t \geq 0$, then $\mathbb{P}^{(n)} \to \mathbb{P}$ weakly.*

Proof. Let $J = \{t_1, \ldots, t_k\}$, $0 \leq t_1 \leq \cdots \leq t_k$, $k \in \mathbb{N}$ be a subset of $[0, \infty)$. Denote by $p_t(\cdot, dx)$, $t \geq 0$ the transition semigroup of $(X_t)_{t \geq 0}$. Let $\mathscr{F}C_b^\infty(E^J) \ni g(x) = g_1(x_1) \cdots g_k(x_k)$, $g_i \in \mathscr{F}C_b^\infty(E)$, $1 \leq i \leq k$. Then

$$\begin{aligned}
\int g \, d\mathbb{P}^J &= \int g_1(X_{t_1}) \cdots g_k(X_{t_k}) d\mathbb{P} \\
&= \iint g_1(X_{t_1}) \cdots g_k(X_{t_k}) d\mathbb{P}_z \, \mu(dz) \\
&= \int \int \cdots \int g_1(x_1) p_{t_1}(z, dx_1) g_2(x_2) p_{t_2 - t_1}(x_1, dx_2) \cdots \\
&\quad \cdots g_k(x_k) p_{t_k - t_{k-1}}(x_{k-1}, dx_k) \mu(dz) \\
&= \int p_{t_1}(g_1 p_{t_2 - t_1}(g_2 \cdots p_{t_k - t_{k-1}} g_k \cdots))(z) \mu(dz).
\end{aligned}$$

$p_t f$ is a m-version of $T_t f$ for every $f \in \mathscr{B}_b(E) \cap L^2(E; m)$. Then clearly

$$p_{t_{k-1}-t_{k-2}}(g_{k-1} p_{t_k - t_{k-1}} g_k)$$

is a m-version of

$$T_{t_{k-1}-t_{k-2}}(g_{k-1} T_{t_k - t_{k-1}} g_k)$$

since $g_{k-1} p_{t_k - t_{k-1}} g_k$ is bounded and contained in L^2-class of $g_{k-1} T_{t_k - t_{k-1}} g_k$. By an induction argument

$$p_{t_1}(g_1 p_{t_2 - t_1}(g_2 \cdots p_{t_k - t_{k-1}} g_k \cdots))$$

is a m-version of

$$T_{t_1}(g_1 T_{t_2 - t_1}(g_2 \cdots T_{t_k - t_{k-1}} g_k \cdots)).$$

A similar statement holds for every $n \in \mathbb{N}$. Since $h_n \to h$ $\mathscr{H}$-weakly as a consequence of Assumption 8 and $T_t^{(n)} \to T_t$, $t \geq 0$ strongly in the sense of Definition 2.15 we get

$$\begin{aligned}
&\lim_n \int g \, d\mathbb{P}^{(n),J} \\
=&\lim_n \int T_{t_1}^{(n)}(g_1 T_{t_2-t_1}^{(n)}(g_2 \cdots T_{t_k-t_{k-1}}^{(n)} g_k \cdots)) d\mu_n \\
=&\lim_n \int T_{t_1}^{(n)}(g_1 T_{t_2-t_1}^{(n)}(g_2 \cdots T_{t_k-t_{k-1}}^{(n)} g_k \cdots)) h_n \, dm_n \\
=&\int T_{t_1}(g_1 T_{t_2-t_1}(g_2 \cdots T_{t_k-t_{k-1}} g_k \cdots)) h \, dm \\
=&\int T_{t_1}(g_1 T_{t_2-t_1}(g_2 \cdots T_{t_k-t_{k-1}} g_k \cdots)) d\mu
\end{aligned}$$

by an induction argument, since

$$T_{t_k-t_{k-1}}^{(n)} g_k \to T_{t_k-t_{k-1}} g_k$$

$\mathscr{H}$-strongly by strong convergence of semigroups and

$$g_{k-1} T_{t_k-t_{k-1}}^{(n)} g_k \to g_{k-1} T_{t_k-t_{k-1}} g_k$$

$\mathscr{H}$-strongly by Lemma 4.5 (Lemma 4.3 resp.).

Hence we obtain

$$\lim_n \int g \, d\mathbb{P}^{(n),J} = \int g \, d\mathbb{P}^J$$

for every $g \in \mathscr{P} := \mathrm{lin}\{g \mid g(x) = g_1(x_1) \cdots g_k(x_k),\ g_i \in \mathscr{F}C_b^\infty(E),\ 1 \leq i \leq k\}$.

Note that Assumption 9 (1) claims the tightness of $\{\mathbb{P}^{(n)}\}$. Now assume that $\mathbb{P}^{(n)} \not\to \mathbb{P}$ weakly. Then by Prokhorov's Theorem (which can be applied by Assumption 9 (2)) there exists a subsequence $\{\mathbb{P}^{(n_k)}\}$ of $\{\mathbb{P}^{(n)}\}$ which weakly converges to some probability measure $\mathbb{P}_1$ on $(\Omega, \mathscr{M})$, $\mathbb{P}_1 \neq \mathbb{P}$. But

$$\lim_n \int f \, d\mathbb{P}^{(n),J} = \int f \, d\mathbb{P}^J$$

for every $f \in \mathscr{P}$. On the other hand, by weak convergence $\mathbb{P}^{(n_k)} \to \mathbb{P}_1$ we have

$$\lim_k \int f \, d\mathbb{P}^{(n_k),J} = \int f \, d\mathbb{P}_1^J$$

for every $f \in C_b(E^J)$, J as above. Clearly

$$\int f \, d\mathbb{P}_1^J = \int f \, d\mathbb{P}^J$$

for every $f \in \mathscr{P}$. We would like to show that $\sigma(\mathscr{P}) = \mathscr{B}(E^J)$. We follow an argument found in [MR92, Chapter IV.4.b)]. Therefore note that by the Hahn-Banach Theorem (cf. [RS72, Theorem III.5]) $\{f \mid f(x) = \sin l_1(x_1) \cdots \sin l_k(x_k),\ l_i \in E',\ 1 \le i \le k,\ x = (x_1, \ldots, x_k) \in E^J\}$ separates the points of E^J, i.e.,

$$(E^J \times E^J) \setminus d = \bigcup_{\substack{l_i \in E' \\ 1 \le i \le k}} (\sin l_1 \cdots \sin l_k, \sin l_1 \cdots \sin l_k)^{-1}((\mathbb{R} \times \mathbb{R}) \setminus d')$$

where d, d' denotes the diagonal in $E^J \times E^J$, $\mathbb{R} \times \mathbb{R}$ resp. Since $E^J \times E^J$ is strongly Lindelöf as a separable metric space the above open cover of $(E^J \times E^J) \setminus d$ has a countable subcover, i.e., there exist $l_{i_n} \in E'$, $1 \le i \le k$, $n \in \mathbb{N}$, such that $K := \{\sin l_{1_n} \cdots \sin l_{k_n} \mid n \in \mathbb{N}\}$ separates the points of E^J. Now by [Sch73, Lemma 18, p. 108] it follows that $\sigma(\mathscr{P}) = \sigma(K) = \sigma((E^J)') = \mathscr{B}(E^J)$. Clearly $\mathscr{P}$ is an algebra. Now by monotone class arguments (by setting

$$H_0 := \left\{ f \;\middle|\; f \in \mathscr{B}_b(E^J) \text{ such that } \int f \, d\mathbb{P}_1^J = \int f \, d\mathbb{P}^J \right\}$$

as our monotone vector space) we get that

$$\int f \, d\mathbb{P}_1^J = \int f \, d\mathbb{P}^J$$

for every $f \in \mathscr{B}_b(E^J)$, J as above. Since the finite dimensional distributions of the process determine the probability measure $\mathbb{P}_1$ uniquely, we get $\mathbb{P}_1 = \mathbb{P}$ on $(\Omega, \mathscr{M})$, which creates a contradiction. Thus $\mathbb{P}^{(n)} \to \mathbb{P}$ weakly. $\square$

Remark 5.2. *If we had assumed instead, that $h_n \to h$ $\mathscr{H}$-strongly, which in our case would have followed from $\|h_n\|_{L^2(E;m_n)} \to \|h\|_{L^2(E;m)}$ by Lemma 2.12, weak convergence of semigroups $T_t^{(n)} \to T_t$, $t \ge 0$ would be sufficient for the same result (using Lemma 4.5). This completes the above result in terms of Theorem 2.41, which grants the weak convergence of co-semigroups and thus we get a sufficiently strong type of convergence of the dual processes $\hat{\mathbb{M}}^{(n)}$, $n \in \mathbb{N}$ to $\hat{\mathbb{M}}$.*

A. Remaining proofs from Chapter 2

In this chapter we want to repeat some of the proofs from [KS03]. Some proofs might also be taken from [Kol05a] or [Kol06]. Some of the proofs have been rewritten for this paper, some even completely redone. All proofs that cannot be found in this form in literature are left to read already in Chapter 2.

Please note that we still use the convention $H := H_\infty$, $\Phi_n := \Phi_{\infty,n}$ and $C := \mathscr{C}_\infty$.

Proof of Lemma 2.7. (1): For the "if"-part assume that $\|u_n\|_{H_n} \to 0$. Set $\varphi_m = 0 \in C$ for every $m \in \mathbb{N}$. Then

$$\lim_m \overline{\lim_n} \|u_n - \Phi_n(0)\|_{H_n} = \lim_m \overline{\lim_n} \|u_n\|_{H_n} = 0,$$

hence clearly $u_n \to 0$ $\mathscr{H}$-strongly.

For the "only if"-part assume that $u_n \to 0 \in H$ strongly in $\mathscr{H}$. There exists a sequence $\{\varphi_m\} \subset C$ with $\varphi_m \to 0$ in H and

$$\lim_m \overline{\lim_n} \|u_n - \Phi_n(\varphi_m)\|_{H_n} = 0.$$

Clearly,

$$\overline{\lim_n} \|u_n\|_{H_n} \le \overline{\lim_n} \|u_n - \Phi_n(\varphi_m)\|_{H_n} + \overline{\lim_n} \|\Phi_n(\varphi_m)\|_{H_n} \to 0$$

as $m \to \infty$.

(2): Obvious.

(3): Let $u_n \in H_n$, $n \in \mathbb{N}$, $u \in H$ such that $u_n \to u$ $\mathscr{H}$-strongly. Choose $\{\varphi_m\} \subset C$ with $\lim_m \|\varphi_m - u\|_H = 0$ and $\lim_m \overline{\lim}_n \|\Phi_n(\varphi_m) - u_n\|_{H_n} = 0$. Evidently,

$$\big|\|u_n\|_{H_n} - \|u\|_H\big| \le \|u_n - \Phi_n(\varphi_m)\|_{H_n} + \big|\|\Phi_n(\varphi_m)\|_{H_n} - \|u\|_H\big|.$$

The first term tends to zero as $m, n \to \infty$. Taking $n \to \infty$ in the second term, one obtains

$$\big|\|\varphi_m\|_H - \|u\|_H\big| \le \|\varphi_m - u\|_H,$$

which tends to 0 as $m \to \infty$.

(4): Let $\{\tilde{u}_m\} \subset C$ with $\tilde{u}_m \to u$ in H and $\{\tilde{v}_m\} \subset C$ with $\tilde{v}_m \to v$ in H. Then by Cauchy's inequality

$$\begin{aligned}
&|(u_n, v_n)_{H_n} - (u, v)_H| \\
&= |(u_n - \Phi_n(\tilde{u}_m), v_n)_{H_n} + (\Phi_n(\tilde{u}_m), v_n - \Phi_n(\tilde{v}_m))_{H_n} + (\Phi_n(\tilde{u}_m), \Phi_n(\tilde{v}_m))_{H_n} - (u, v)_H| \\
&\qquad \le \|v_n\|_{H_n} \|u_n - \Phi_n(\tilde{u}_m)\|_{H_n} + \|\Phi_n(\tilde{u}_m)\|_{H_n} \|v_n - \Phi_n(\tilde{v}_m)\|_{H_n} \\
&\qquad\qquad + |(\Phi_n(\tilde{u}_m), \Phi_n(\tilde{v}_m))_{H_n} - (u, v)_H| \,. \qquad \text{(A.1)}
\end{aligned}$$

By (3) $\{\|v_n\|_{H_n}\}$ is uniformly bounded in n (and in m) as clearly is $\{\|\Phi_n(\tilde{u}_m)\|_{H_n}\}$, too. So taking the limit $m, n \to \infty$ the third line of (A.1) tends to 0. But by the polarization identity and linearity of each Φ_n

$$\begin{aligned}
(\Phi_n(\tilde{u}_m), \Phi_n(\tilde{v}_m))_{H_n} &= \frac{1}{4}\left[\|\Phi_n(\tilde{u}_m) + \Phi_n(\tilde{v}_m)\|_{H_n}^2 - \|\Phi_n(\tilde{u}_m) - \Phi_n(\tilde{v}_m)\|_{H_n}^2\right] \\
&\longrightarrow \frac{1}{4}\left[\|u + v\|_H^2 - \|u - v\|_H^2\right] = (u, v)_H
\end{aligned}$$

as $m, n \to \infty$, thus the fourth line of (A.1) tends to 0 as well.

□

Proof of Lemma 2.12. The "only if"-part follows from Lemma 2.7 (3) and (4) combined with the fact that $\Phi_n(\varphi) \to \varphi$ strongly for every $\varphi \in C$.

To prove the "if"-part, let $\varphi_m \to u$ in H, $\{\varphi_m\} \subset C$.

$$\begin{aligned}
\lim_m \overline{\lim_n} \|u_n - \Phi_n(\varphi_m)\|_{H_n} &= \lim_m \overline{\lim_n} \left|\|u_n\|_{H_n}^2 - 2(u_n, \Phi_n(\varphi_m))_{H_n} + \|\Phi_n(\varphi_m)\|_{H_n}^2\right|^{1/2} \\
&= \lim_m \left|\|u\|_H^2 - 2(u, \varphi_m)_H + \|\varphi_m\|_H^2\right|^{1/2} = \lim_m \|u - \varphi_m\|_H = 0.
\end{aligned}$$

□

Proof of Lemma 2.13. The "only if" part follows from the fact that $\Phi_n(\varphi) \to \varphi$ strongly.

To prove the "if" part, let $\{u_n\}$, u as in the assertion and $(u_n, \Phi_n(\varphi))_{H_n} \to (u, \varphi)_H$ for all $\varphi \in C$. Take $v_n \to v$ strongly in $\mathscr{H}$. By strong convergence there exists a sequence $\{\varphi_m\}$, $\varphi_m \in C$ with $\|\varphi_m - v\|_H \to 0$. We have to prove $(u_n, v_n)_{H_n} \to (u, v)_H$. By Cauchy's inequality one obtains

$$\begin{aligned}
&|(u_n, v_n)_{H_n} - (u, v)_H| \\
\le\, &|(u_n, v_n - \Phi_n(\varphi_m))_{H_n}| + |(u_n, \Phi_n(\varphi_m))_{H_n} - (u, v)_H| \\
\le\, &\|u_n\|_{H_n} \|v_n - \Phi_n(\varphi_m)\|_{H_n} + |(u_n, \Phi_n(\varphi_m)_{H_n} - (u, v)_H| \,.
\end{aligned}$$

$\{\|u_n\|_{H_n}\}$ is uniformly bounded in n by assumption, $\|v_n - \Phi_n(\varphi_m)\|_{H_n}$ tends to zero as $m, n \to \infty$ by strong convergence. Furthermore, by assumption, $(u_n, \Phi_n(\varphi_m))_{H_n} \to (u, \varphi_m)_H$ as $n \to \infty$, so the last expression yields by Cauchy's inequality

$$|(u, \varphi_m)_H - (u, v)_H| = |(u, \varphi_m - v)_H| \le \|u\|_H \|v - \varphi_m\|_H \to 0$$

as $m \to \infty$. □

Proof of Lemma 2.14. (1): Pick a complete orthonormal basis $\{e_k\}$ of H. By uniform boundedness of $\{u_n\}$ we can assume that

$$\overline{\lim_n}(u_n, \Psi_n(e_1))_{H_n} =: a_1 \in \mathbb{R}$$

exists. Similarly,

$$\overline{\lim_n}(u_n, \Psi_n(e_k))_{H_n} =: a_k \in \mathbb{R}.$$

By a diagonal argument we can find a common subsequence $n_l \uparrow +\infty$ such that

$$\lim_l (u_{n_l}, \Psi_{n_l}(e_k))_{H_{n_l}} = a_k$$

for every $k \in \mathbb{N}$.

Fix $N \in \mathbb{N}$. Let $\mathscr{L}_n^N := \operatorname{lin}\{\Psi_n(e_k) \mid k = 1, \ldots, N\} \subset H_n$ and $P_{\mathscr{L}_n^N} : H_n \to \mathscr{L}_n^N$ be the orthogonal projection on the (finite-dimensional) linear subspace $\mathscr{L}_n^N$ of H_n. $\{\Psi_n(e_k) \mid k = 1, \ldots, N\}$ clearly is an orthonormal basis of $\mathscr{L}_n^N$ (recalling that the Ψ_n's are unitary operators). Clearly for every $n \in \mathbb{N}$,

$$P_{\mathscr{L}_n^N}(u_n) = \sum_{k=1}^{N} (u_n, \Psi_n(e_k))_{H_n} \Psi_n(e_k),$$

and therefore (using the orthonormality of the $\Psi_n(e_k)$'s),

$$\sum_{k=1}^{N} |a_k|^2 = \lim_l \sum_{k=1}^{N} \left|(u_{n_l}, \Psi_{n_l}(e_k))_{H_{n_l}}\right|^2 \le \overline{\lim_n} \|P_{\mathscr{L}_n^N}(u_n)\|_{H_n}^2 \le \overline{\lim_n} \|u_n\|_{H_n}^2 < \infty$$

for any $N \in \mathbb{N}$. This gives us the existence of

$$u := \sum_{k=1}^{\infty} a_k e_k \in H.$$

Let $\varphi \in \bigcup_{N \ge 1} \operatorname{lin}\{e_k \mid k = 1, \ldots, N\} =: C_0 \subset H$. It suffices to prove that

$$\lim_l (u_{n_l}, \Psi_{n_l}(\varphi))_{H_{n_l}} = (u, \varphi)_H$$

for every such φ, noting that $C_0 \subset H$ is dense.

So, let $\varphi = \sum_{k=1}^{N} (\varphi, e_k)_H e_k$ (i.e., φ depends only on the first N coordinates). Then,

$$\begin{aligned}
(u, \varphi)_H &= \sum_{k=1}^{N} a_k (\varphi, e_k)_H \\
&= \lim_l \sum_{k=1}^{N} (u_{n_l}, \Psi_{n_l}(e_k))_{H_{n_l}} (\Psi_{n_l}(e_k), \Psi_{n_l}(\varphi))_{H_{n_l}} \\
&= \lim_l (P_{\mathscr{L}_{n_l}^N}(u_{n_l}), \Psi_{n_l}(\varphi))_{H_{n_l}} \\
&= \lim_l (u_{n_l}, \Psi_{n_l}(\varphi))_{H_{n_l}}.
\end{aligned}$$

(2): Suppose $\sup_n \|u_n\|_{H_n} = \infty$. Then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\|u_{n_k}\|_{H_{n_k}} \geq k$. Setting

$$v_k := \frac{1}{k} \cdot \frac{u_{n_k}}{\|u_{n_k}\|_{H_{n_k}}},$$

one has $\|v_k\|_{H_{n_k}} = 1/k \to 0$ and hence by Lemma 2.7 (1) $v_k \to 0$ in $\mathscr{H}$, which implies $(u_{n_k}, v_k)_{H_{n_k}} \to (u, 0)_H = 0$. On the other hand,

$$(u_{n_k}, v_k)_{H_{n_k}} = \frac{1}{k} \|u_{n_k}\|_{H_{n_k}} \geq 1.$$

This is a contradiction and thus we obtain $\sup_n \|u_n\|_{H_n} < \infty$.

Let $\{v_n\}$ be a sequence with $v_n \in H_n$ which strongly converges to u (which exists by Corollary 2.11). Then, $(u_n, v_n)_{H_n} \to (u, u)_H$. Hence,

$$\begin{aligned} 0 &\leq \underline{\lim}_n \|u_n - v_n\|_{H_n}^2 \\ &= \underline{\lim}_n (\|u_n\|_{H_n}^2 - 2(u_n, v_n)_{H_n} + \|v_n\|_{H_n}^2) \\ &= \underline{\lim}_n \|u_n\|_{H_n}^2 - \|u\|_H^2. \end{aligned}$$

This completes the proof of the first assertion. The second follows from Lemma 2.7 (3) and Lemma 2.12.

(3): The "only if"-part is trivial. We prove the "if" part. The assumption implies that $u_n \to u$ weakly. Setting $v_n := u_n$ and $v := u$ in the assumption, we have $\|u_n\|_{H_n} \to \|u\|_H$, which proves the assertion by statement (2) of this Lemma.

□

Proof of Lemma 2.17. (1): For any $\varepsilon > 0$ there is a unit vector $u \in H$ such that $\|Bu\|_H > \|B\|_{\mathscr{L}(H)} - \varepsilon$. Let $u_n \in H_n$ be strongly converging to u. Note that $\|u_n\|_{H_n} \to 1$. Since $B_n \to B$ strongly, we have $\|B_n u_n\|_{H_n} \to \|Bu\|_H$ and therefore,

$$\underline{\lim}_n \|B_n\|_{\mathscr{L}(H_n)} \geq \lim_n \frac{\|B_n u_n\|_{H_n}}{\|u_n\|_{H_n}} = \|Bu\|_H > \|B\|_{\mathscr{L}(H)} - \varepsilon,$$

which gives the desired statement.

(2): There is a sequence of unit vectors $u_n \in H_n$ such that $|\|B_n\|_{\mathscr{L}(H_n)} - \|B_n u_n\|_{H_n}| \to 0$. Replacing with a subsequence, we assume that u_n weakly converges to a vector $u \in H$ with $\|u\|_H \leq 1$. Since $B_n u_n \to Bu$ strongly by the assumption, we have

$$\|B\|_{\mathscr{L}(H)} \geq \|Bu\|_H = \lim_n \|B_n u_n\|_{H_n} = \lim_n \|B_n\|_{\mathscr{L}(H_n)},$$

which together with (1) completes the proof.

□

Proof of Lemma 2.19. Note that the compactness of B is equivalent to that of $\hat{B}$. Let $\{u_m\} \subset H$, be a sequence of vectors in H weakly converging to a vector $u \in H$. It suffices to prove that $\hat{B}u_m \to \hat{B}u$ H-strongly. We easily see that $\hat{B}u_m \to \hat{B}u$ H-weakly. For each $m \in \mathbb{N}$, there is a sequence $u_{m,n}$ with $u_{m,n} \in H_n$ such that $\lim_n u_{m,n} = u_m$ $\mathscr{H}$-strongly. Since $\hat{B}_n \to \hat{B}$ strongly, we have $\lim_n \hat{B}_n u_{m,n} = \hat{B}u_m$ $\mathscr{H}$-strongly. A diagonal argument yields (see Corollary 2.37) that there is a sequence of natural numbers $n_m \uparrow \infty$ such that

$$\lim_m u_{m,n_m} = u \quad \mathscr{H}\text{-weakly}, \tag{A.2}$$

$$\lim_m \left| \|\hat{B}_{n_m} u_{m,n_m}\|_{H_{n_m}} - \|\hat{B}u_m\|_H \right| = 0. \tag{A.3}$$

The compact convergence $\hat{B}_n \to \hat{B}$ and (A.2) together show that $\lim_m \hat{B}_{n_m} u_{m,n_m} = \hat{B}u$ $\mathscr{H}$-strongly. Hence, by (A.3), $\|\hat{B}u_m\|_H \to \|\hat{B}u\|_H$ and so $\hat{B}u_m \to \hat{B}u$ H-strongly. This completes the proof. □

Proof of Theorem 2.29. The proof uses only our notions of convergence and basic facts about symmetric closed forms, which can be found e.g. in [MR92] or [FOT94], in particular, if we refer to Proposition 2.36 (which is stated after this Theorem), we shall remark that the proof does not depend on the progress of this paper and can be found in [MR92, Chapter I.2]. We also use Lemma 2.37, whose proof is self-contained and can be found later in this Appendix.

(2) ⇔ (3) is a special case of Theorem 2.21.

Let us prove (1) ⇒ (2). Let $\alpha > 0$, $\{u_n\}$, $u_n \in H_n$, $u_n \to u \in H$ strongly. Define $z_n := G^n_\alpha u_n$, $z := G_\alpha u$. The vector z_n is characterized as the unique minimizer of $v \mapsto \mathscr{E}^n(v,v) + \alpha(v,v)_{H_n} - 2(u_n,v)_{H_n}$ over H_n (cf. [MR92, Proof of Theorem I.2.6] and also [MR92, Theorem I.2.8]). Since $\sup_n \|G^n_\alpha\|_{\mathscr{L}(H_n)} \le \alpha^{-1}$, we can extract a subsequence of $\{z_n\}$ by Lemma 2.14 (1), still denoted by $\{z_n\}$, which converges weakly to some $\tilde{z} \in H$. For an arbitrary given $v \in H$ by condition (M2) we can find a sequence $\{v_n\}$, $v_n \in H_n$, $v_n \to v$ strongly such that $\lim_n \mathscr{E}^n(v_n,v_n) = \mathscr{E}(v,v)$. Since for every n,

$$\mathscr{E}^n(z_n,z_n) + \alpha(z_n,z_n)_{H_n} - 2(u_n,z_n)_{H_n} \le \mathscr{E}^n(v_n,v_n) + \alpha(v_n,v_n)_{H_n} - 2(u_n,v_n)_{H_n}, \tag{A.4}$$

by taking condition (M1), Lemma 2.7 (3) and Lemma 2.14 (2) into account, we find in the limit

$$\mathscr{E}(\tilde{z},\tilde{z}) + \alpha(\tilde{z},\tilde{z})_H - 2(u,\tilde{z})_H \le \mathscr{E}(v,v) + \alpha(v,v)_H - 2(u,v)_H.$$

Therefore $\tilde{z} = G_\alpha u$. By the uniqueness of such $\tilde{z}$ it proves that $z_n \to z$ weakly. We now prove that $(z_n,z_n)_{H_n} \to (z,z)_H$. By (M2) we choose $\{v_n\}$, $v_n \in H_n$, $v_n \to z$ strongly such that $\lim_n \mathscr{E}^n(v_n,v_n) = \mathscr{E}(z,z)$, therefore, by rewriting (A.4) as

$$\mathscr{E}^n(z_n,z_n) + \alpha \left\| z_n - \frac{u_n}{\alpha} \right\|^2_{H_n} \le \mathscr{E}^n(v_n,v_n) + \alpha \left\| v_n - \frac{u_n}{\alpha} \right\|^2_{H_n}$$

we get in the limit again by condition (M1) and Lemma 2.7 (3)

$$\overline{\lim_n} \left\| z_n - \frac{u_n}{\alpha} \right\|_{H_n}^2 \leq \left\| z - \frac{u}{\alpha} \right\|_H^2,$$

hence $\left\| z_n - \frac{u_n}{\alpha} \right\|_{H_n}^2 \to \left\| z - \frac{u}{\alpha} \right\|_H^2$, and this together with Lemma 2.7 (3) and Lemma 2.14 (2) concludes the proof.

Let us now prove (2) $\Rightarrow$ (1). Suppose G_α^n converges strongly to G_α for every $\alpha > 0$. We first want to prove (M1). For $\alpha > 0$, $n \in \mathbb{N}$ define approximate forms

$$\mathscr{E}^{n,(\alpha)}(u,v) := \alpha(u - \alpha G_\alpha^n u, v)_{H_n}, \quad u, v \in H_n$$

(cf. Proposition 2.36 and [MR92, Chapter I.2]). Now let $\{v_n\}$, $v_n \in H_n$, $v_n \to u \in H$ weakly. By the strong convergence of G_α^n we have

$$\lim_n \mathscr{E}^{n,(\alpha)}(u_n, u_n) = \mathscr{E}^{(\alpha)}(u,u)$$

for every $u_n \to u$ strongly. First observe that for every n, $\alpha > 0$,

$$\begin{aligned} \mathscr{E}^{n,(\alpha)}(v_n, v_n) - \mathscr{E}^{n,(\alpha)}(u_n, u_n) - 2\mathscr{E}^{n,(\alpha)}(u_n, v_n - u_n) \\ = \mathscr{E}^{n,(\alpha)}(v_n, v_n) + \mathscr{E}^{n,(\alpha)}(u_n, u_n) - 2\mathscr{E}^{n,(\alpha)}(u_n, v_n) \\ = \mathscr{E}^{n,(\alpha)}(u_n - v_n, u_n - v_n) \geq 0. \end{aligned}$$

So for every n, $\alpha > 0$,

$$\mathscr{E}^n(v_n, v_n) \geq \mathscr{E}^{n,(\alpha)}(v_n, v_n) \geq \mathscr{E}^{n,(\alpha)}(u_n, u_n) + 2\mathscr{E}^{n,(\alpha)}(u_n, v_n - u_n).$$

It is easy to see that $v_n - u_n \to 0$ weakly, thus, by strong convergence of $\{G_\alpha^n\}$ we get $2\mathscr{E}^{n,(\alpha)}(u_n, v_n - u_n) = 2\alpha(u_n - \alpha G_\alpha^n u_n, v_n - u_n)_{H_n} \to 0$. Taking $\underline{\lim}_n$ we get for every $\alpha > 0$

$$\underline{\lim_n} \mathscr{E}(v_n, v_n) \geq \mathscr{E}^{(\alpha)}(u,u).$$

Since this holds for any $\alpha > 0$, we conclude

$$\underline{\lim_n} \mathscr{E}(v_n, v_n) \geq \mathscr{E}(u,u).$$

(M1) is proved.
To prove (M2) let $u_n \to u$ strongly. By strong convergence of $\{G_\alpha^n\}$ we have

$$\mathscr{E}(u,u) = \lim_{\alpha \to \infty} \lim_n \mathscr{E}^{n,(\alpha)}(u_n, u_n)$$

and

$$\lim_{\alpha \to \infty} \lim_n \alpha G_\alpha^n u_n = u.$$

By Lemma 2.38 (for $\mathscr{H}$ is metrizable) pick a sequence of natural numbers $\alpha_n \uparrow \infty$ with

$$\mathscr{E}(u,u) = \lim_n \mathscr{E}^{n,(\alpha_n)}(u_n, u_n) \quad \text{and} \quad \lim_n \alpha_n G_{\alpha_n}^n u_n = u.$$

Set $w_n := \alpha_n G^n_{\alpha_n} u_n$. Observe that $w_n \to u$ stongly. Using Proposition 2.36 (ii) we get

$$\mathscr{E}^{n,(\alpha_n)}(u_n, u_n) = \mathscr{E}^n(w_n, w_n) + \alpha_n \|u_n - w_n\|^2_{H_n} \geq \mathscr{E}^n(w_n, w_n)$$

for every n. Taking $\overline{\lim}_n$ and using (M1) proves the assertion. $\square$

Proof of Lemma 2.37. The proof is taken from [Att84, Lemma 1.15 and Corollary 1.18]. First, let $\{a_{n,m}\} \subset \overline{\mathbb{R}}$, $n \in \mathbb{N}$, $m \in \mathbb{N}$ be a double indexed sequence. We will prove that there exists a mapping $n \mapsto m(n)$ increasing to $+\infty$, such that

$$\underline{\lim}_{n\to+\infty} a_{n,m(n)} > \underline{\lim}_{m\to+\infty} \underline{\lim}_{n\to+\infty} a_{n,m}. \tag{A.5}$$

Let $a_m := \underline{\lim}_{n\to+\infty} a_{n,m}$ and $a := \underline{\lim}_{m\to+\infty} a_m$. If $a = -\infty$, there is nothing to prove. So, assume $a > -\infty$ and take $(\tilde{a}_p)_{p\in\mathbb{N}}$ a sequence of real numbers defined as follows:
If $a < +\infty$, take $\tilde{a}_p := a - 2^{-p}$, if $a = +\infty$, take $\tilde{a}_p := p$.
By definition of a, there exists an increasing sequence $(m_p)_{p\in\mathbb{N}}$, $m_p \uparrow +\infty$ such that

$$a_m > \tilde{a}_p \quad \text{for all} \quad m \geq m_p.$$

This can be condensed in

$$a_m > (a - 2^{-p}) \wedge p \quad \text{for all} \quad m \geq m_p.$$

In the same way, there exists an increasing sequence $(n_p)_{p\in\mathbb{N}}$, $n_p \uparrow +\infty$ such that

$$a_{n,m_p} > (a_{m_p} - 2^{-p}) \wedge p \quad \text{for all} \quad n \geq n_p.$$

Set $m(n) := m_p$ if $n_p \leq n < n_{p+1}$ and verify that (A.5) is satisfied: when $n_p \leq n < n_{p+1}$, we get by the above

$$a_{n,m(n)} > (a_{m_p} - 2^{-p}) \wedge p > (((a - 2^{-p}) \wedge p) - 2^{-p}) \wedge p.$$

Thus, for all $n \geq n_p$

$$a_{n,m(n)} > (((a - 2^{-p}) \wedge p) - 2^{-p}) \wedge p.$$

It follows that

$$\underline{\lim}_{n\to+\infty} a_{n,m(n)} > (((a - 2^{-p}) \wedge p) - 2^{-p}) \wedge p.$$

This being true for any $p \in \mathbb{N}$, using the fact that for any $a \in \overline{\mathbb{R}}$,

$$(((a - 2^{-p}) \wedge p) - 2^{-p}) \wedge p$$

increases to a as p goes to $+\infty$, we get:

$$\underline{\lim}_{n\to+\infty} a_{n,m(n)} > a = \underline{\lim}_{m\to+\infty} \underline{\lim}_{n\to+\infty} a_{n,m}.$$

This proves our first assertion. Clearly, replacing $a_{n,m}$ by $-a_{n,m}$ we obtain

$$\overline{\lim_{n\to+\infty}} a_{n,m(n)} < \overline{\lim_{m\to+\infty}}\, \overline{\lim_{n\to+\infty}} a_{n,m}. \tag{A.6}$$

Now let us prove the Lemma. Therefore, let $\{x_{n,m}\}$, $\{x_m\}$, x as above. Set $a_{n,m} := d(x_{n,m}, x) \subset \mathbb{R}$. By the above there exists an increasing map $n \mapsto m(n)$ such that (A.6) holds. By definition of $a_{n,m}$,

$$\overline{\lim_{n\to+\infty}} a_{n,m} = \lim_{n\to+\infty} d(x_{n,m}, x) = d(x_m, x)$$

and

$$\overline{\lim_{m\to+\infty}}\, \overline{\lim_{n\to+\infty}} a_{n,m} = \lim_{m\to+\infty} d(x_m, x) = 0.$$

So,

$$\overline{\lim_{n\to+\infty}} d(x_{n,m(n)}, x) = 0,$$

which proves the assertion. $\square$

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