Stochastic Equations in Hilbert Space with Lévy Noise and their Applications in Finance

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Introduction

In this diploma thesis we will solve a stochastic partial differential equation with Lipschitz nonlinearities where the noise is a Hilbert space valued Lévy process. We will also present applications of Lévy processes in Finance, in particular in stochastic volatility models. The solution of the mentioned equation may be useful to further improve such models which are up to now mainly based on Ornstein-Uhlenbeck type processes.

Lévy process have in recent time become more and more important in Finance applications. They have for example been used to model the volatility which is implicitly given in financial time series. In a model suggested by O. E. Barndorff-Nielsen and N. Shepard [BN-S] the stochastic volatility process is an Ornstein-Uhlenbeck type process, i.e. a solution to

$$dy(t) = ay(t)dt + bdx(t)$$

on $\mathbb{R}^1$ (or finite combinations of those). Here a, b are real constants and x is a Lévy process on $\mathbb{R}^1$. Due to the large number of influencing factors it seems to be more appropriate to construct such processes on an infinite dimensional state space. We will therefore construct a weak solution of the analogous equation on a separable Hilbert space H, i.e.

$$\begin{aligned} dY(t) &= JY(t)dt + CdX(t) \\ Y(0) &= Y_0. \end{aligned}$$

Here X is Lévy process on H, C a bounded linear operator on H and J the, possibly unbounded, generator of a C_0-semigroup on H. For a better fit to observed implied volatility data it might be useful to add a nonlinear term in this equation (see the discussion in section 5.4). Therefore we will extend the equation to

$$\begin{aligned} dY(t) &= (JY(t) + F(Y(t)))dt + CdX(t) \\ Y(0) &= Y_0, \end{aligned}$$

where $F : H \to H$ is a function that fulfills a Lipschitz condition. In Theorem 4.3.3 we will show that under condition (4.1) (cf. chapter 4) this equation has a unique mild solution and furthermore that this mild solution is also a weak solution. This result is, as far as we know, new for noises that are Lévy processes with values in a separable Hilbert space. It is also the central result in this diploma thesis.

In the following chapters we will first give an overview of the theory of Lévy process and infinitely divisible distributions. Afterwards we present the construction of the stochastic integral with respect to a martingale measure. Then we turn to stochastic partial differential equations and prove the above described existence and uniqueness result. Finally we give an insight to applications of Lévy processes in Finance. A more detailed description of the single chapters and sections will follow further below.

The stochastic integral will be constructed with respect to a Hilbert space-valued *martingale measure*. Lévy processes as integrators are an important special case. Real-valued martingale measures were introduced by J. B. Walsh [Wal] in order to solve stochastic partial differential equations. A martingale measure is a mapping $M : [0,t] \times S \times \Omega \to \mathbb{R}$ where $[0,t]$ is a time interval, $(\Omega, \mathcal{F}, P)$ some probability space and S a Lusin topological space. Simply speaking M is a martingale in the time component and a σ-finite measure in the S-component. N. El Karoui and S. Méléard [ElKMel] also studied vector-valued martingale measures. In [App b] by D. Applebaum a corresponding definition of martingale measures with values in a separable Hilbert space was given (therein called martingale-valued measures). Of special interest are martingale measures that fulfill a certain orthogonality condition and have a covariance structure that can be described by a family of trace class operators. (For further details and definitions see section 2.5). The *Lévy martingale measure*, which corresponds to a Lévy process, has all of these properties. We will focus on this example and show all important results for the Lévy martingale measure. Moreover we give a detailed review of martingale measures in general (see section 2.5).

Real valued stochastic integrals with Hilbert space-valued martingales as integrators were constructed by H. Kunita [Kun]. We will refer to this as the *weak stochastic integral*, since for simple functions it is a sum of inner products of integrands with increments of the martingale. Consequently there is also a *strong stochastic integral*: This integral takes values in the same Hilbert space as the integrator. The integrands are mappings with images in a certain class of linear operators. For the case where the martingales are continuous this kind of integral was constructed by B.L. Rozovskii [Roz]. More general is the construction of M. Métivier [Mét]. Here the integrators are allowed to be semimartingales and discontinuous. Both types of stochastic integrals, the weak and the strong one, are Itô integrals in the sense that they are defined as L^2-limits of the integrals of simple functions which approximate the integrand in a proper L^2-space. Another approach is to define the stochastic integral as a limit in distribution. This was done by A. Chojnowska-Michalik [ChM] for the semimartingales being Lévy processes and with deterministic integrands. A stochastic integral with respect to martingale measures has already been developed in [Wal] for the real-valued case. In [App b] this construction was carried forward to Hilbert spaces, i.e. martingale measures and integrands are both Hilbert space-valued.

We turn back to the equation from above and add a time dependent drift:

$$\begin{aligned} dY(t) &= (JY(t) + f(t))dt + CdX(t) \\ Y(0) &= Y_0. \end{aligned} \tag{1}$$

Again X is an H-valued Lévy process, where H shall denote a Hilbert space which will be our state space. For some $\varepsilon > 0$ we impose the condition

$$\sup_{0 \leq t \leq \tilde{T}} \|\Delta X(t)\| \in L^{2+\varepsilon}(\mathbb{R}) = L^{2+\varepsilon}(\Omega \to \mathbb{R})$$

on X, which is thereby in $L^2(H) = L^2(\Omega \to H)$ at any time $0 \leq t \leq T$. J is the, possibly unbounded generator, of a C_0-semigroup $(S(t), t \geq 0)$ on H and C a bounded linear operator on H. $f : [0,T] \to L^2(H)$ is an adapted stochastic process, which is Bochner integrable on $[0,T]$, and $Y_0 \in L^2(H)$ is a random initial value. We will construct a weak solution of this equation. $Y : [0,T] \to L^2(H)$ is a *weak solution* of (1) if Y is Bochner integrable on $[0,T]$ and for each $y \in D(J^*)$ and every $t \in [0,T]$

$$\begin{aligned} & (Y(t) - Y_0, y)_H \\ = & \int_0^t ((Y(s), J^* y)_H + (f(s), y)_H) ds + (X(t), C^* y)_H. \end{aligned}$$

(Here $(\cdot,\cdot)_H$ denotes the inner product in H).

In the case that the integrator X is a Brownian motion this equation was studied by G. Da Prato and J. Zabczyk in [DaPrZa], Chapter 5. They show existence and uniqueness of a weak solution. For $f \equiv 0$ the solution of (1) is called *Ornstein-Uhlenbeck process*. A. Chojnowska-Michalik [ChM] has first studied this equation for an integrator X being a Hilbert space-valued Lévy process. A unique weak solution (therein called mild solution) is constructed. The same result was obtained by D. Applebaum [App b] using alternative methods. As mentioned above the stochastic integral with respect to martingale measures is constructed there. The solution of equation (1) is then found based on this integration theory. It is worth to mention that this approach has the advantage that the Lévy Itô decomposition is preserved within the structure of the solution. We will pick up this construction in this diploma thesis and fully work out the argumentation. Furthermore in our equation we will also allow the additional drift term f.

Now we discuss the fundamentally new result. Compared with equation (1) the situation is more complicated if non-linear drift terms are introduced.

We consider the following equation with a Lipschitz drift:

$$\begin{aligned} dY(t) &= (JY(t) + F(Y(t)))dt + CdX(t) \\ Y(0) &= Y_0. \end{aligned} \tag{2}$$

The mapping $F : H \to H$ is measurable and Lipschitz on $[0, T]$. Again we are interested in a weak solution, i.e. a stochastic process $Y : [0, T] \to L^2(H)$, Bochner integrable on $[0, T]$, such that for all $y \in D(J^*)$ and all $t \in [0, T]$

$$\begin{aligned} & (Y(t) - Y_0, y)_H \\ = & \int_0^t ((Y(s), J^*y)_H + (F(Y(s)), y)_H)ds + (X(t), C^*y)_H. \end{aligned}$$

The idea is, to show existence of a mild solution, i.e. a stochastic process Y with

$$Y(t) = S(t)Y_0 + \int_0^t S(t-s)F(Y(s))ds + \int_0^t S(t-s)CdX(s). \tag{3}$$

It is then obtained that in our case a mild solution is also a weak solution. For a Gaussian noise X problem (2) was also examined in [DaPrZa] (see Chapter 7). For non-Gaussian noise these equations have recently been studied, but mainly in the real-valued case. S. Albeverio, J.-L. Wu and T.-S. Zhang [AlWuZh] give the solution to an equation of the type (2) (also allowing a non-constant diffusion coefficient) driven by a pure Poisson noise. Poisson noise usually is a compensated Poisson random measure (which is an example for a real-valued martingale measure; for the definition see section 2.3). One should also mention C. Müller [Mül] and D. Applebaum and J.-L. Wu [AppWu], where the latter also allow for a Gaussian part in the noise. C. Knoche [Kno a] studies (2) (with non-constant diffusion coefficient) on an infinite dimensional Hilbert space where the integrator is a compensated Poisson random measure (see also [Kno b]).

To our knowledge there are no results on the equation (2) on infinite dimensional Hilbert spaces where the integrator is a general Lévy process with values in the same Hilbert space. We will show in this diploma thesis that this equation has a unique weak solution, which is the central result. This is done in the framework of [App b], where the stochastic integral is constructed with respect to Hilbert space-valued martingale measures. We are able to show existence and uniqueness of a weak solution to (2) under certain condition. This condition is the already mentioned condition on the jumps of the Lévy process X: For some $\varepsilon > 0$

$$\sup_{0 \le t \le \tilde{T}} \|\Delta X(t)\| \in L^{2+\varepsilon}(\mathbb{R}) = L^{2+\varepsilon}(\Omega \to \mathbb{R}). \tag{4}$$

Our proof (see section 4.3) uses a fixed point argument.

We will now give an overview over the single chapters of this diploma thesis with special emphasis on contributions which are new compared to the literature cited above. Apart from the description here we also refer to the more detailed introductions at the beginning of every chapter. In *Chapter 1* the theory of *infinitely divisible measures* is developed. An understanding of infinitely divisible measures is important in the study of Lévy processes as such measures are the distributions of the increments of Lévy processes. The results presented here are mainly taken from [Lin]. The most important one is the *Lévy Khinchin Representation* (section 1.4) by which every infinitely divisible measure can be decomposed into a Dirac measure, a Gaussian measure and a "jump"-part given by the generalized exponent of a Lévy measure.

Lévy processes on a separable Hilbert space H will be studied in *Chapter 2*. We will first give some useful properties of H-valued martingales and mention some results for the Gaussian case before we come to general Lévy processes in section 2.3. Given a Lévy process X we introduce the corresponding *Poisson random measure* and the *compensated Poisson random measure* which in the time component is a martingale. Following the approach of S. Albeverio and B. Rüdiger [AlRü] we construct an integral with respect to the compensated Poisson random measure with deterministic functions as integrands. We will also cite their main result in the mentioned paper: the *Lévy Itô decomposition* on a separable Hilbert space (therein shown for general separable Banach spaces). In the final section of this chapter we will extend the definition of *martingale measures* to the Hilbert space case. As mentioned this extension was first done in [App b] and therefore we have taken most of the definitions from there. Note that we will use a slightly different definition of orthogonality for martingale measures (compare the introduction to the chapter). A very important example - and we will show that it really is an example - is the *Lévy martingale measure*, since it describes the martingale part of a Lévy process. Of special interest are *nuclear* martingale measures which in the martingale component have covariance operators that are non-negative self-adjoint and trace class. Each Lévy martingale measure has this property which is also proved in detail.

In *Chapter 3* stochastic integrals with respect to nuclear martingale measures are introduced. For the real-valued case this was also done in [Wal]. The construction in the H-valued situation follows the ideas of [App b]. However some details were left out there and we will give the precise argumentation here. First we construct the *strong stochastic integral*, where the integrands are certain operator-valued mappings, by an isometry. Next we will also introduce the real-valued *weak stochastic integral* (in section 3.2) with H-valued integrands. The construction is quite similar to the one of the strong stochastic integral. However we are able to extend the result from [App b] and obtain a larger class of integrands. The weak stochastic integral is very useful to solve the stochastic equations in Chapter 4, since one can easily “switch” between weak stochastic integrals and an inner product of some Hilbert space element with the strong stochastic integral. A Fubini theorem was given in [App b] for a deterministic integral and a stochastic integral with respect to a (real-valued) compensated Poisson random measure. We allow for arbitrary Hilbert space-valued martingale measures as integrators for the stochastic integral in our version of the theorem.

Stochastic partial differential equations with Lévy process are then studied in *Chapter 4*. We introduce the *stochastic convolution* corresponding to the C_0-semigroup $(S(t), t \geq 0)$. In section 4.2 existence and uniqueness of a weak solution to (1) is shown. As mentioned the result was obtained in [App b] with methods similar to the ones in [DaPrZa]. We give a detailed proof of the existence and the uniqueness guided by the arguments of [DaPrZa] for the Gaussian case. In section 4.3 we come to the main result of this diploma thesis which is stated in Theorem 4.3.3. We will prove the existence of a unique weak solution to the following equation with Lipschitz nonlinearity term $F(Y)$:

$$\begin{aligned} dY(t) &= (JY(t) + F(Y(t)))dt + CdX(t) \\ Y(0) &= Y_0. \end{aligned}$$

(Compare equation (2) above). As mentioned the result can be established under condition (4). The solution is a fixed point of the contraction $Y \mapsto \psi(Y)$ given by

$$\psi(Y)(t) = \int_0^t S(t-s)F(Y(s))ds + S(t)Y_0 + X_{J,C}(t)$$

which is uniquely determined by Banach's fixed point theorem.

Lévy processes have played an important role in Finance in recent time. Whereas traditionally financial models use Brownian motions to model financial time series many newer models use Lévy processes since they allow for jumps. It is an observation that financial data, such as for example stock prices, often contain bigger changes which can hardly be explained by a normal distribution. These large changes can be seen as the result of external shocks that force economic agents to sudden reactions at the marketplace. *Chapter 5* is therefore devoted to the use of Lévy processes, which may have both a Gaussian and a jump part, in Finance. We will present an option pricing model with pure jump Lévy processes which was developed by E. Eberlein and U. Keller in [Ebe] and [EbeKel] and give an overview over stochastic volatility modelling as it was carried out by O. E. Barndorff-Nielsen and N. Shepard [BN-S] as well as by E. Nicolato and E. Venardos [NicVen]. We will start with a review of the classical *Black Scholes model* which was set up by F. Black and M. Scholes [BlaSch]. A motivation for the use of Lévy processes follows which takes into account empirical and theoretical results. In section 5.3 we will then present the model of E. Eberlein and U. Keller which is based on *generalized hyperbolic Lévy motions*. The final section deals with the stochastic volatility model from [BN-S]. As for the generalized hyperbolic Lévy motions we will give an option pricing formula, a result from [NicVen]. Special focus is on the modelling of stochastic volatility itself by a stochastic process. O. E. Barndorff-Nielsen and N. Shepard suggest to represent stochastic volatility by an Ornstein-Uhlenbeck type process. Since changes in volatility are interpreted as the result of external shocks and these shocks can have various sources there are strong arguments to consider processes in infinite dimensional state spaces. Here the results of section 4.2 turn out to be useful. Empirical results from G. Bakshi, N. Ju and H. Ou-Yang [BaJuYa] may give the motivation to introduce nonlinearities in the dynamics of stochastic volatility processes. The stochastic equations studied in section 4.3 involve an additional Lipschitz drift term compared to Ornstein-Uhlenbeck type processes in infinite dimension. This may give rise for further progress in the modelling of stochastic volatility on the basis of Lévy processes.

I would like to thank Prof. Dr. Michael Röckner who encouraged me to study stochastic partial differential equations with Lévy noise. His lectures, which I have visited since my first semester in Bielefeld, gave me a deep insight into modern mathematics. I have learnt very much about the concepts and methods of Stochastic Analysis from him. Moreover I am greatful for many helpful discussions and proposals in connection with my diploma thesis.

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Chapter 1

Infinitely Divisible Measures

In this chapter we review the theory of *infinitely divisible measures*. The results presented here are mainly taken from [Lin]. Since we will introduce infinitely divisible measures on general separable Banach spaces we will shortly review the construction and some useful properties of the Bochner integral in section 1.1. In the following section an infinitely divisible measure will be defined as a probability measure which can for any $n \in \mathbb{N}$ be written as the n-fold convolution of another measure, the n-th root. Degenerated measures and Gaussian measures are simple examples for infinitely divisible measures. Section 1.3 gives the construction of exponents and generalized exponents of so-called Lévy measures. With this construction infinitely divisible measures can be generated by a certain class of σ-finite measures under a one-to-one mapping. Such a σ-finite measure, the *Lévy measure*, can be interpreted as the intensity of jumps for a process whose increments are distributed with the corresponding probability measure. The main result of this chapter is then the *Lévy Khinchin Representation* (section 1.4) by which every infinitely divisible measure can be decomposed in a degenerated measure, a Gaussian measure and a "jump"-part given by the generalized exponent of a Lévy measure.

1.1 The Bochner Integral

Let $(E, \|\cdot\|)$ be a Banach space over $\mathbb{R}$ (or $\mathbb{C}$). By E' we will denote the (topological) dual of E, i.e. the set of all linear continuous mappings from E into $\mathbb{R}$ (or $\mathbb{C}$). Let $\langle\, , \rangle : E \times E' \to \mathbb{R}$ (or $\mathbb{C}$) denote the corresponding dualization. Let $\mathfrak{B}(E)$ be the σ-algebra of Borel sets of E, i.e. the σ-algebra generated by the open subsets of E.

We will shortly repeat the construction and some basic properties of the Bochner integral. Let $(X, \mathcal{A}, \mu)$ be a measure space. A function $f : X \to E$ is said to be *strongly measurable* if it is measurable with respect to $\mathcal{A}$ and $\mathfrak{B}(E)$ (for short: measurable) and has separable range $f(X) \subset E$. Note that if E is separable every E-valued measurable function is strongly measurable.

Lemma 1.1.1. *Let $(X, \mathcal{A})$ be a measurable space and E be a Banach space. Then*

- *the collection of all measurable functions from X to E is closed under the formation of pointwise limits.*
- *the collection of all strongly measurable functions from X to E is closed under the formation of pointwise limits.*

Proof. (cf. [Coh] Prop. E.1.). □

A function $f : X \to E$ is *Bochner integrable* iff f is strongly measurable and

$$\int_X \|f(x)\| \mathrm{d}\mu < \infty.$$

Then there exist simple functions f_n, i.e. strongly measurable functions with finitely many values (cf. [Coh] Prop. E.2.), which are Bochner integrable such that

$$f(x) = \lim_{n \to \infty} f_n(x) \quad \text{for all } x \in X$$

and

$$\|f_n(x)\| \leq \|f(x)\| \quad \text{for all } x \in X.$$

For such a simple function $g = \sum_{i=1}^n a_i 1_{A_i}$ the *Bochner integral* can be defined in the usual way by

$$\int_X g \mathrm{d}\mu = \sum_{i=1}^n a_i \mu(A_i).$$

Then for the approximating sequence $\{f_n\}$ of f

$$\left\| \int_X f_n \mathrm{d}\mu - \int_X f_m \mathrm{d}\mu \right\| \leq \int_X \|f_n - f\| \mathrm{d}\mu + \int_X \|f_m - f\| \mathrm{d}\mu$$

and by the dominated convergence theorem for real functions this tends to 0 for $m, n \to \infty$. Hence $\{\int_X f_n \mathrm{d}\mu\}$ is a Cauchy sequence in E and we can define

$$\int_X f \mathrm{d}\mu = \lim_{n \to \infty} \int_X f_n \mathrm{d}\mu$$

independent of the chosen sequence.

The Bochner integral has the following basic properties

Proposition 1.1.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to E$ Bochner integrable, then*

$$\left\| \int_X f \, d\mu \right\| \leq \int_X \|f\| \, d\mu.$$

Proof. (cf. [Coh] Prop. E.5.). □

For $1 \le p < \infty$ we define

$$\|f\|_p := \left(\int_X \|f\|^p d\mu\right)^{\frac{1}{p}}$$

and for $p = \infty$

$$\|f\|_\infty := \inf_{\mu(N)=0} \sup_{x \in X \setminus N} \|f(x)\|.$$

One can show that $\|f\|_p = 0$ implies that $f = 0$ μ-a.s. (compare for example [Alt], 1.12 on p.45/46). Then we define for $1 \le p \le \infty$

$$L^p((X, \mathcal{A}, \mu) \to E) := \{f : X \to E; f \text{ strongly measurable and } \|f\|_p < \infty\}$$

where by definition $f = g$ in $L^p((X, \mathcal{A}, \mu) \to E)$ iff $f = g$ μ-a.e.

Proposition 1.1.3 (Riesz-Fischer). *$L^p((X, \mathcal{A}, \mu) \to E)$ is a Banach space for every $1 \le p \le \infty$. Let $\{f_n\} \subset L^p((X, \mathcal{A}, \mu) \to E)$ with $f_n \to f$ in $L^p((X, \mathcal{A}, \mu) \to E)$ for some f. Then there exists a subsequence $\{f_{n'}\}$ with $f_{n'} \to f$ μ-a.e.*

Proof. The first assertion is the statement of [Alt] Lemma 1.13 and Proposition (Satz) 1.17. The second assertion follows directly from the proofs of these propositions. □

Lemma 1.1.4 (Hölder-Inequality). *Let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$ ($1/\infty := 0$). Then for $f \in L^p((X, \mathcal{A}, \mu) \to E)$ and $g \in L^q((X, \mathcal{A}, \mu) \to E)$ we have $fg \in L^1((X, \mathcal{A}, \mu) \to E)$ and*

$$\|fg\|_1 \le \|f\|_p \|g\|_q.$$

Proof. (cf. [Alt] Lemma 1.14). □

Lemma 1.1.5 (Minkowski-Inequality). *Let $f, g \in L^p((X, \mathcal{A}, \mu) \to E)$. Then $f + g \in L^p((X, \mathcal{A}, \mu) \to E)$ and*

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

Proof. (cf. [Alt] Lemma 1.16). □

Proposition 1.1.6 (Lebesgue's dominated convergence Theorem). *Let $(X, \mathcal{A}, \mu)$ be a measure space and g be a real valued integrable function on X. Suppose f and $f_1, f_2, \ldots$ are strongly measurable E-valued functions on X such that*

$$f(x) = \lim_{n \to \infty} f_n(x) \quad \text{for } \mu\text{-a.e. } x \in X$$

and $\|f_n(x)\| \le g(x)$ μ-a.e. Then f and $f_1, f_2, \ldots$ are Bochner integrable and

$$\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu.$$

Proof. (cf. [Coh] Thm. E.6.). □

Proposition 1.1.7. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to E$ Bochner integrable. Then for each $a \in E'$*

$$\left\langle \int_X f \, d\mu, a \right\rangle = \int_X \langle f(x), a \rangle \, d\mu(x).$$

Proof. (cf. [Coh] Prop. E.11.). □

This can be generalized to closed linear operators:

Proposition 1.1.8. *Let E be separable, A be a closed linear operator on E and $f : X \to E$ Bochner integrable. If Af is also Bochner integrable then*

$$A \int_X f(x) d\mu(x) = \int_X Af(x) d\mu(x).$$

Proof. (cf. [DaPrZa], Prop. 1.6). □

1.2 Definition of Infinitely Divisible Measures

Now let $(E, \|\cdot\|)$ be a separable real Banach space, and E' its topological dual.

By $\mathfrak{M}(E)$ we will denote the set of all finite measures on E (i.e. on $\mathfrak{B}(E)$) and by $\mathfrak{P}(E)$ the set of all probability measures in $\mathfrak{M}(E)$. For measures μ, ν on E the *convolution* $\mu * \nu$ is defined by

$$(\mu * \nu)(B) = \int_E \mu(B - x) d\nu(x) \quad \text{for all } B \in \mathfrak{B}(E).$$

Recall that the *characteristic function* of μ is the function $\hat{\mu}$ from E' to $\mathbb{C}$ given by $\hat{\mu}(a) = \int_E \exp(i\langle x, a \rangle) d\mu(x)$.

Lemma 1.2.1. *Let $\mu, \nu \in \mathfrak{M}(E)$. Then $\mu = \nu$ iff $\hat{\mu}(a) = \hat{\nu}(a)$ for all $a \in E'$.*

Proof. (cf. [Lin] Prop. 1.7.1.). □

Lemma 1.2.2. *Let μ, ν be measures on E. Then for each $a \in E'$ we have*

$$(\widehat{\mu * \nu})(a) = \hat{\mu}(a)\hat{\nu}(a).$$

Proof.

$$\begin{aligned} (\widehat{\mu * \nu})(a) &= \int_E \exp(i\langle x, a \rangle) d(\mu * \nu)(x) \\ &= \int_E \int_E \exp(i\langle x + y, a \rangle) d\mu(x) d\nu(y) \\ &= \int_E \exp(i\langle x, a \rangle) d\mu(x) \int_E \exp(i\langle y, a \rangle) d\nu(y) = \hat{\mu}(a)\hat{\nu}(a). \end{aligned}$$

□

A probability measure $\mu \in \mathfrak{P}(E)$ is said to be *infinitely divisible* provided that for each natural number $n \in \mathbb{N}$ there exists a measure $\mu_n \in \mathfrak{P}(E)$ such that

$$\mu = (\mu_n)^n \tag{1.1}$$

where μ^n is the n-fold convolution of μ with itself.

Lemma 1.2.3. *$\mu \in \mathfrak{P}(E)$ is infinitely divisible iff for each $n \in \mathbb{N}$ there exists a measure $\mu_n \in \mathfrak{P}(E)$ such that*

$$\hat{\mu}(a) = (\hat{\mu}_n(a))^n \quad \textit{for all } a \in E'.$$

Proof. Follows by Lemma 1.2.2. □

Recall the definition of the *weak topology* on $\mathfrak{M}(E)$. A basis of neighborhoods of $\mu \in \mathfrak{M}(E)$ in the weak topology is given by

$$\left\{\nu \in \mathfrak{M}(E); \left|\int_E f_i d\mu - \int_E f_i d\nu\right| < \varepsilon, 1 \leq i \leq n\right\},$$

where $f_1, \ldots, f_n \in C_b(E)$ (the set of all bounded continuous functions from E to $\mathbb{R}$). If a sequence of measures $\{\mu_n\} \subset \mathfrak{M}(E)$ converges with respect to this topology, which is equivalent to

$$\int_E f d\mu_n \to \int_E f d\mu \quad \text{for all } f \in C_b(E),$$

we write $\mu_n \Rightarrow \mu$.

A set $K \subset \mathfrak{M}(E)$ is *weakly relatively compact* if for every sequence $\{\mu_n\} \subset K$ there exist a subsequence $\{\mu_{n'}\}$ and a measure μ such that $\mu_{n'} \Rightarrow \mu$.

This concept can be moderated: We will call a set $K \subset \mathfrak{M}(E)$ *relatively shift compact* if there exist elements $\{x_\mu\}_{\mu \in K} \subset E$ such that $\{\mu * \delta_{x_\mu}; \mu \in K\}$ is weakly relatively compact. If a sequence $\{\mu_n\}$ is relatively shift compact then we call a sequence $\{x_n\} \subset E$ *centralizing* if $\{\mu_n * \delta_{x_n}\}$ is weakly relatively compact. In particular, every set K which is weakly relatively compact is relatively shift compact since for any sequence in K the sequence which is constantly equal to 0 is a centralizing sequence.

Lemma 1.2.4. *A sequence $\{\mu_n\}$ in $\mathfrak{P}(E)$ converges weakly iff $\{\mu_n\}$ is weakly relatively compact and for each $a \in E'$ the limit* $\lim_{n\to\infty} \hat{\mu}_n(a)$ *exists in $\mathbb{C}$. Moreover if $\mu_n \Rightarrow \mu$ then $\hat{\mu}(a) = \lim_{n\to\infty} \hat{\mu}_n(a)$ for every $a \in E'$.*

Proof. (cf. [Lin] Prop. 1.8.2.). □

Lemma 1.2.5. *Let μ be an infinitely divisible measure. Then $\hat{\mu}(a) \neq 0$ for all $a \in E'$ and there exists a uniquely determined function h from E' into $\mathbb{C}$ with the following properties:*

- $h(0) = 0$.
- *h is continuous.*
- $\hat{\mu}(a) = \exp(h(a))$ *for all* $a \in E'$.

Moreover, the measures μ_n *with* $(\mu_n)^n = \mu$ *are uniquely determined and their characteristic functions are given by*

$$\hat{\mu}_n(a) = \exp(h(a)/n), \quad a \in E'.$$

Proof. $\hat{\mu}(a) \neq 0$ for all $a \in E'$ by [Lin] Prop. 5.1.1. Then by [Lin] Prop.1.9.1. h exists and is uniquely determined. Finally the uniqueness of μ and $\hat{\mu}$ is shown in [Lin] Prop. 5.1.4. □

Now since the measure μ_n in (1.1) is uniquely determined we write $\mu^{1/n}$ instead of μ_n and call it the *n-th root* of μ. Then μ^α can be easily defined for every nonnegative rational α by $\mu^{m/n} := (\mu^{1/n})^m$ if $\alpha = m/n$. This definition can be extended for any $\alpha \geq 0$ by approximation:

Lemma 1.2.6. *Let* μ *be an infinitely divisible measure. Then for every* $\alpha \geq 0$ *there exists a measure* μ^α *with characteristic function* $\widehat{\mu^\alpha}(a) = (\hat{\mu}(a))^\alpha$. *For any sequence* $\{\alpha_n\} \subset \mathbb{Q}_+$ *which converges to* α *we have* $\mu^{\alpha_n} \Rightarrow \mu^\alpha$. μ^α *has the following properties:*

$$\begin{aligned} \mu^{\alpha+\beta} &= \mu^\alpha * \mu^\beta \quad \text{for all } \alpha, \beta \geq 0, \\ \mu^0 &= \delta_0. \end{aligned}$$

Proof. (see [Lin] p.60/61: Remark and Prop. 5.1.7. and Corollary 5.1.8.). □

A measure $\mu \in \mathfrak{M}(E)$ is said to be *degenerated* if there exists an element $x \in E$ with $\mu(\{x\}) = \mu(E)$. Then we also write δ_x instead of μ.

Proposition 1.2.7. *Let* μ *be a degenerated measure. Then* μ *is infinitely divisible.*

Proof. If $\mu = \delta_x$ then choose $\mu_n := \delta_{(1/n)\,x}$. □

Let Q be a bounded linear operator from E' into E, i.e. $Q \in L(E', E)$. It is said to be *symmetric*, if for all $a_1, a_2 \in E'$

$$\langle Qa_1, a_2 \rangle = \langle Qa_2, a_1 \rangle.$$

Q is called *non-negative* provided that $\langle Qa, a \rangle \geq 0$ for all $a \in E'$.

A probability measure $\varrho \in \mathfrak{P}(E)$ is called *Gaussian symmetric* if there exists a non-negative and symmetric operator $Q \in L(E', E)$ such that

$$\hat{\varrho}(a) = \exp(-\langle Qa, a \rangle / 2) \quad \text{for all } a \in E'.$$

Q is then called the *covariance operator* of ϱ. By $\mathfrak{G}(E)$ denote the set of all Gaussian symmetric measures defined on E. In section 2.2 we will give a more intuitive definition for Gaussian measures on Hilbert spaces. For the moment it may be helpful to observe that in the finite dimensional case of $\mathbb{R}^n$ the characteristic function of a Gaussian measure is formally given in the same way as above, where Q can be represented by an $n \times n$-matrix. We have:

Lemma 1.2.8. *$\mathfrak{G}(E)$ is closed under convolutions.*

Proof. Let $\varrho, \sigma \in \mathfrak{G}(E)$ and R, S be the corresponding covariance operators. Then we have

$$\hat{\varrho}(a)\hat{\sigma}(a) = \exp(-(\langle Ra, a\rangle + \langle Sa, a\rangle)/2) = \exp(-\langle (R+S)a, a\rangle/2).$$

As $R+S$ is again non-negative and symmetric the result follows immediately from Lemma 1.2.2. □

Proposition 1.2.9. *Each Gaussian symmetric measure is infinitely divisible.*

Proof. (cf. [Lin] Prop. 5.2.2.). □

1.3 Exponents of Lévy Measures

Now for any finite measure $\lambda \in \mathfrak{M}(E)$ we define a mapping $e(\lambda)$ on $\mathfrak{B}(E)$ by

$$e(\lambda)(B) := e^{-\lambda(E)} \sum_{k=0}^{\infty} \lambda^k(B)/k! \quad \text{for each } B \in \mathfrak{B}(E).$$

We call $e(\lambda)$ the *exponent* of λ.

Lemma 1.3.1. *For each $\lambda \in \mathfrak{M}(E)$ the exponent $e(\lambda)$ is a probability measure on E. Moreover its characteristic function is given by*

$$\widehat{e(\lambda)}(a) = \exp(\hat{\lambda}(a) - \hat{\lambda}(0)) = \exp\left(\int_E (e^{i\langle x,a\rangle} - 1)\, d\lambda(x)\right).$$

Proof. Define μ_n by

$$\mu_n(B) := \left(\sum_{k=0}^{n} \lambda^k(E)/k!\right)^{-1} \sum_{k=0}^{n} \lambda^k(B)/k! \quad \text{for all } B \in \mathfrak{B}(E).$$

Note that $\lambda^k(E) = (\lambda(E))^k$, therefore we have $\lim_{n\to\infty} \mu_n(B) = e(\lambda)(B)$ for all $B \in \mathfrak{B}(E)$. So, $\mu_n \Rightarrow e(\lambda)$ by Portmanteau's theorem, hence for all

$a \in E'$

$$\begin{aligned}
\widehat{e(\lambda)}(a) &= \lim_{n\to\infty} \widehat{\mu_n}(a) = e^{-\lambda(E)} \lim_{n\to\infty} \left(\sum_{k=0}^{n} \int_E e^{i\langle x,a\rangle} d(\lambda^k/k!)(x) \right) \\
&= e^{-\lambda(E)} \sum_{k=0}^{\infty} \widehat{\lambda^k}(a)/k! = e^{-\lambda(E)} \sum_{k=0}^{\infty} (\hat{\lambda}(a))^k/k! \\
&= \exp(\hat{\lambda}(a) - \hat{\lambda}(0)) = \exp\left(\int_E (e^{i\langle x,a\rangle} - 1) d\lambda(x) \right).
\end{aligned}$$

□

Proposition 1.3.2. *The probability measure $e(\lambda)$ is infinitely divisible.*

Proof. (cf. [Lin] Cor. 5.3.3.). □

With degenerated measures, Gaussian symmetric measures and the exponent of a measure constructed above we have already seen three examples of infinitely divisible measures. As the infinitely divisible measures are by definition closed under convolution we get the following

Corollary 1.3.3. *For $x \in E$, $\varrho \in \mathfrak{G}(E)$ and $\lambda \in \mathfrak{M}(E)$ the convolution $\delta_x * \varrho * e(\lambda)$ is infinitely divisible.*

We will now study limits of finite measures and their relation to limits of the corresponding exponents. This will lead us to the central concept of Lévy measures introduced at the end of this section.

Proposition 1.3.4. *If $\{\lambda_n\} \subset \mathfrak{M}(E)$ is weakly relatively compact then so is $\{e(\lambda_n)\} \subset \mathfrak{P}(E)$. Furthermore, if $\lambda_n \Rightarrow \lambda$ in $\mathfrak{M}(E)$ then $e(\lambda_n) \Rightarrow e(\lambda)$ in $\mathfrak{P}(E)$.*

Proof. (cf. [Lin] Prop. 5.3.4. and Cor. 5.3.5.). □

Proposition 1.3.5. *Let $\{\lambda_n\} \subset \mathfrak{M}(E)$ be a sequence of finite measures such that $\{e(\lambda_n)\}$ is relatively shift compact. Then there exist a subsequence n' and a σ-finite measure λ with the following properties:*

1. *For each $\delta > 0$ we have $\lambda\{\|x\| > \delta\} < \infty$.*
2. *$\lambda(\{0\}) = 0$.*
3. *For each $\delta > 0$ such that $\lambda\{\|x\| = \delta\} = 0$ it follows that*

$$\lambda_{n'}\big|_{\{x\in E;\, \|x\|>\delta\}} \Rightarrow \lambda\big|_{\{x\in E;\, \|x\|>\delta\}}.$$

Proof. (cf. [Lin] Prop. 5.3.9.). □

Later on, λ will be interpreted as the intensity of jumps of a stochastic process. Notice that property 2. above is accompanied with the intuition that there might be no jumps of size zero. Property 1. means that "big" jumps have a certain finite intensity and therefore only occur "from time to time". If we restrict the intensity measure λ to these jumps, property 3. gives us that $\lambda\big|_{\{x\in E;\, \|x\|>\delta\}}$ is a weak limit of finite measures.

In order to find a centralizing sequence for $\{\lambda_n\}$ this gives the idea that one has to "compensate" small jumps. Usually this is done by setting $\delta = 1$ as an upper boundary. Then for $\lambda \in \mathfrak{M}(E)$ we define

$$x(\lambda) := -\int_{\{\|x\|\leq 1\}} x d\lambda(x)$$

(where the integral is a Bochner integral).

Defining the *generalized exponent* $e_s(\lambda)$ of $\lambda \in \mathfrak{M}(E)$ by

$$e_s(\lambda) := e(\lambda) * \delta_{x(\lambda)}$$

we have the following

Proposition 1.3.6. *Let $\{\lambda_n\} \subset \mathfrak{M}(E)$ be such that $\{e(\lambda_n)\}$ is relatively shift compact. Then the centralizing sequence for $\{e(\lambda_n)\}$ is $\{x(\lambda_n)\}$, i.e. $\{e(\lambda_n) * \delta_{x(\lambda_n)}\} = \{e_s(\lambda_n)\}$ is weakly relatively compact.*

Proof. (cf. [Lin] Prop. 5.3.11.). □

Lemma 1.3.7. *For $\lambda \in \mathfrak{M}(E)$*

$$\widehat{e_s(\lambda)}(a) = \exp\left(\int_E K(x,a) d\lambda(x)\right),$$

where $K(x,a)$ is defined by

$$K(x,a) := e^{i\langle x,a\rangle} - 1 - i\langle x,a\rangle 1_{\{\|x\|\leq 1\}}(x) \quad x \in E, a \in E'.$$

Proof. By Lemma 1.2.2 it is sufficient to prove that

$$\widehat{e(\lambda)}(a) = \exp\left(\int_E (e^{i\langle x,a\rangle} - 1) d\lambda(x)\right)$$

and

$$\widehat{\delta_{x(\lambda)}}(a) = \exp\left(\int_{\{\|x\|\leq 1\}} -i\langle x,a\rangle d\lambda(x)\right).$$

But the first statement follows by Lemma 1.3.1 and the second one by

$$\begin{aligned}\int_E \exp(i\langle y,a\rangle)d\delta_{x(\lambda)}(y) &= \exp(i\langle x(\lambda),a\rangle)\\ &= \exp\left(-i\left\langle \int_{\{\|x\|\le 1\}} x d\lambda(x), a\right\rangle\right)\\ &= \exp\left(\int_{\{\|x\|\le 1\}} -i\langle x,a\rangle d\lambda(x)\right)\end{aligned}$$

(where the last step is justified by Lemma 1.1.7). □

Proposition 1.3.8. *If $\lambda_n \Rightarrow \lambda$ and $\lambda\{\|x\|=1\}=0$ then $e_s(\lambda_n)\Rightarrow e_s(\lambda)$.*

Proof. (cf. [Lin] Prop. 5.3.15.). □

Now we also want to define exponents of σ-finite measures. Therefore we will define the following class of measures: A symmetric (i.e. $\lambda(B)=\lambda(-B)$ for all $B\in\mathfrak{B}(E)$) σ-finite measure λ on E is called a *Lévy measure* iff

1. $\lambda(\{0\})=0$ and
2. the function $a\mapsto \exp\left(\int_E(\cos\langle x,a\rangle-1)\mathrm{d}\lambda(x)\right)$ is the characteristic function of a probability measure ν.

For a measure μ define $\bar\mu$ by $\bar\mu(B):=\mu(-B)$. Then it is easy to check that, if μ is finite, the characteristic function of $\bar\mu$ is the complex conjugated characteristic function of μ. So, for symmetric measures, i.e. $\mu=\bar\mu$, the characteristic function only takes real values. Then the characteristic function $a\mapsto\exp\left(\int_E(\cos\langle x,a\rangle-1)\mathrm{d}\lambda(x)\right)$ is of the usual form for exponents of finite symmetric measures and therefore we will denote the corresponding measure ν by $e(\lambda)$ and call it the exponent of λ. An arbitrary σ-finite measure λ is called a Lévy measure iff the symmetric measure $\lambda+\bar\lambda$ is a Lévy measure. $\mathfrak{L}(E)$ denotes the set of all Lévy measures on E.

Proposition 1.3.9. *Let λ be a σ-finite measure on E with $\lambda\{0\}=0$. Then the following statements are equivalent:*

1. *$\lambda\in\mathfrak{L}(E)$.*
2. *For each $a\in E'$ the integral $\int_E|K(x,a)|d\lambda(x)$ is finite and the mapping*
$$a\mapsto\exp\left(\int_E K(x,a)d\lambda(x)\right)$$
is the characteristic function of a probability measure.
3. *There exists a sequence $\{\lambda_n\}\subset\mathfrak{M}(E)$ such that $\lambda_n\uparrow\lambda$ and $\{e_s(\lambda_n)\}$ is weakly relatively compact.*

4. *For each* $\delta > 0$ *we have* $\lambda\{\|x\| > \delta\} < \infty$ *and for some (for each) sequence* $\delta_n \downarrow 0$ *the sequence* $\{e_s(\lambda|_{\{x\in E;\, \|x\|>\delta_n\}})\}$ *is weakly relatively compact.*

Proof. (cf. [Lin] Prop. 5.4.8.). □

So, for any $\lambda \in \mathfrak{L}(E)$ by Prop. 1.3.9 (2) we may define

$$e_s(\lambda)(a) := \exp\left(\int_E K(x,a)d\lambda(x)\right).$$

Proposition 1.3.10. *1. If* $\lambda_1, \lambda_2 \in \mathfrak{L}(E)$ *then* $\lambda_1 + \lambda_2 \in \mathfrak{L}(E)$ *and* $e_s(\lambda_1 + \lambda_2) = e_s(\lambda_1) * e_s(\lambda_2)$

2. *If* $\lambda \in \mathfrak{L}(E)$ *then* $\bar{\lambda} \in \mathfrak{L}(E)$ *and* $e_s(\bar{\lambda}) = \overline{e_s(\lambda)}$.

3. *For each* $\lambda \in \mathfrak{L}(E)$ *the generalized exponent is an infinitely divisible measure and* $e_s(\lambda)^\alpha = e_s(\alpha\lambda)$ *for all* $\alpha \geq 0$.

Proof. 1. We have to show that $\tilde{\lambda} := \lambda_1 + \lambda_2 + \overline{\lambda_1 + \lambda_2}$ is a Lévy measure. But $\tilde{\lambda}(\{0\}) = 0$ and

$$\begin{aligned} & \exp\left(\int_E (\cos\langle x,a\rangle - 1)d\tilde{\lambda}(x)\right) \\ = \; & \exp\left(\int_E (\cos\langle x,a\rangle - 1)d(\lambda_1 + \bar{\lambda}_1)(x)\right) \\ & \exp\left(\int_E (\cos\langle x,a\rangle - 1)d(\lambda_2 + \bar{\lambda}_2)(x)\right) \\ = \; & e\widehat{(\lambda_1 + \bar{\lambda}_1)}(a)e\widehat{(\lambda_2 + \bar{\lambda}_2)}(a) = (e(\lambda_1 + \bar{\lambda}_1) * e(\lambda_2 + \bar{\lambda}_2))\widehat{\,}(a). \end{aligned}$$

And then

$$\begin{aligned} & (e_s(\lambda_1) * e_s(\lambda_2))\widehat{\,}(a) = \widehat{e_s(\lambda_1)}(a)\widehat{e_s(\lambda_2)}(a) \\ = \; & \exp\left(\int_E K(x,a)\mathrm{d}\lambda_1(x) + \int_E K(x,a)\mathrm{d}\lambda_2(x)\right) \\ = \; & [e_s(\lambda_1 + \lambda_2)]\widehat{\,}(a) \quad \text{for all } a \in E'. \end{aligned}$$

2. Follows by taking the complex conjugation of the characteristic function.

3. Let $\alpha \geq 0$ then with Lemma 1.2.6 we have

$$\begin{aligned} \widehat{e_s(\lambda)}^{\alpha}(a) & = \left(\exp\left(\int_E K(x,a)d\lambda(x)\right)\right)^\alpha \\ & = \exp\left(\int_E K(x,a)\alpha d\lambda(x)\right) = \widehat{e_s(\alpha\lambda)}(a) \quad \text{for all } a \in E'. \end{aligned}$$

Particularly $e_s(\lambda)$ is infinitely divisible, since $e_s(\lambda) = e_s(1/n\,\lambda)^n$. □

1.4 Lévy-Khinchin Representation

In order to show that every infinitely divisible measure can be represented by the convolution of measures of the above mentioned types we first need a uniqueness statement:

Proposition 1.4.1. *Let $\lambda, \sigma \in \mathfrak{L}(E)$ and $x \in E$ with*

$$e_s(\lambda) = e_s(\sigma) * \delta_x$$

then $\lambda = \sigma$ and $x = 0$.
Moreover if for $\varrho_1, \varrho_2 \in \mathfrak{G}(E)$, $\lambda_1, \lambda_2 \in \mathfrak{L}(E)$ and $x_1, x_2 \in E$

$$\varrho_1 * e_s(\lambda_1) * \delta_{x_1} = \varrho_2 * e_s(\lambda_2) * \delta_{x_2} \tag{1.2}$$

then $\varrho_1 = \varrho_2$, $\lambda_1 = \lambda_2$ and $x_1 = x_2$.

Proof. (cf. [Lin] Thms. 5.5.3. and 5.5.5.). □

Note that in [Lin] Lemma 5.5.2., which is used to prove Thm. 5.5.3. the condition $\lambda(\{0\}) = 0$ is necessary to obtain uniqueness. Therefore, we have included this condition in our definition for Lévy measures. The existence of representation (1.2) is a consequence of the next two propositions.

Proposition 1.4.2. *Suppose for $\mu \in \mathfrak{P}(E)$ there exists a sequence $\{\lambda_n\} \subset \mathfrak{M}(E)$ with $e(\lambda_n) \Rightarrow \mu$. Then there are $\lambda \in \mathfrak{L}(E)$, $\varrho \in \mathfrak{G}(E)$ and $x \in E$ such that*

$$\mu = e_s(\lambda) * \varrho * \delta_x.$$

Proof. (cf. [Lin] Prop. 5.6.3.). □

Proposition 1.4.3. *Let μ be an infinitely divisible measure then*

$$e(n\mu^{1/n}) \Rightarrow \mu.$$

Proof. (cf. [Lin] Prop. 5.7.2.). □

Now we obtain the Lévy-Khinchin representation for infinitely divisible measures:

Theorem 1.4.4. *A measure $\mu \in \mathfrak{P}(E)$ is infinitely divisible iff there exist a Lévy measure λ, a Gaussian symmetric measure ϱ and an element $x_0 \in E$, each uniquely determined, such that*

$$\mu = e_s(\lambda) * \varrho * \delta_{x_0}$$

or equivalently

$$\widehat{\mu}(a) = \exp\left(\int_E K(x,a)\,d\lambda(x) - \frac{1}{2}\langle Qa, a\rangle + i\langle x_0, a\rangle\right),$$

where $Q \in L(E', E)$ is the covariance operator of the Gaussian measure ϱ.

Proof. Follows from propositions 1.4.2, 1.4.3 and 1.4.1. □

With notations as above one often writes $\mu = [x_0, \varrho, \lambda]$. Then $[x_0, \varrho, \lambda]$ or equivalently $[x_0, Q, \lambda]$ is called the *generating triplet* of μ.

Corollary 1.4.5. *Let $\mu \in \mathfrak{P}(E)$ be an infinitely divisible measure with generating triplet $[x_0, Q, \lambda]$. Then for any $t \geq 0$ the probability measure μ^t is infinitely divisible and has generating triplet $[tx_0, tQ, t\lambda]$.*

Proof. Let ϱ be the Gaussian symmetric measure with covariance operator Q. By Proposition 1.3.10 we have

$$\mu^t = (e_s(\lambda) * \varrho * \delta_{x_0})^t = e_s(\lambda)^t * \varrho^t * (\delta_{x_0})^t = e_s(t\lambda) * \varrho^t * \delta_{tx_0}.$$

But

$$\widehat{\varrho^t} = \hat{\varrho}^t = \exp\left(-\frac{1}{2}\langle Qa, a\rangle\right)^t = \exp\left(-\frac{1}{2}\langle tQa, a\rangle\right),$$

hence ϱ^t is also Gaussian symmetric with covariance operator tQ. □

The following Proposition is useful to calculate the Lévy measure λ.

Proposition 1.4.6. *Let $\mu = [x_0, \varrho, \lambda]$ and $\delta > 0$ with $\lambda\{\|x\| = \delta\} = 0$. Then*

$$(n\mu^{1/n})|_{\{x \in E; \|x\| > \delta\}} \Rightarrow \lambda|_{\{x \in E; \|x\| > \delta\}}.$$

Proof. Together with 1.4.3 and part (i) of Thm. 5.6.2. from [Lin] the assertion follows. □

Remark 1.4.7. Note that such a δ always exists. Let λ be a σ-finite measure on E and set

$$C(\lambda) = \{\delta > 0; \lambda\{\|x\| = \delta\} = 0\}.$$

Then, according to [Lin] p. 65, we even have that $\mathbb{R}_+ \setminus C(\lambda)$ is at most countable.

There is also the following form of the Lévy-Khinchin representation, which is valid if E is a Hilbert space.

Theorem 1.4.8. *Let H be a separable Hilbert space with inner product $(\cdot, \cdot)_H$ and norm $\|\cdot\|$ and $\mu \in \mathfrak{P}(H)$. Then μ is infinitely divisible iff its characteristic function has the unique representation*

$$\widehat{\mu}(a) = \exp\left(\int_E L(x, a)\, d\lambda(x) - \frac{1}{2}(Qa, a)_H + i(x_0, a)_H\right)$$

where $x_0 \in H$, $Q \in L(H)$ is the covariance operator of a Gaussian symmetric measure and λ is a σ-finite measure with finite mass outside every neighborhood of the origin and

$$\int_{\{\|x\| \leq 1\}} \|x\|^2\, d\lambda(x) < \infty.$$

Here $L(x,a)$ is given by

$$L(x,a) := e^{i(x,a)_H} - 1 - \frac{i(x,a)_H}{1+\|x\|^2} \quad x,a \in H.$$

Proof. (cf. [Par], section VI, Thm. 4.10). □

Below we will need a characterization for Lévy measures on separable Hilbert spaces. Therefore we introduce the following definitions (cf. [ArGi], p. 158, 186): A separable Banach space E is *of type* p if there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$ and any centered independent random variables $X_1, \ldots, X_n$ in $L^p(E) = L^p((\Omega, \mathcal{F}, P) \to E)$

$$\mathbb{E}\left\|\sum_{i=1}^{n} X_i\right\|^p \leq c\sum_{i=1}^{n} \mathbb{E}\|X_i\|^p.$$

E is *of cotype* p if there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$ and any centered independent random variables $X_1, \ldots, X_n$ in $L^p(E)$

$$\mathbb{E}\left\|\sum_{i=1}^{n} X_i\right\|^p \geq c\sum_{i=1}^{n} \mathbb{E}\|X_i\|^p.$$

Lemma 1.4.9. *Let H be a separable Hilbert space. Then H is of type 2 and cotype 2.*

Proof. First observe that for any independent H-valued random variables $X, Y \in L^2(H)$ we have $\mathbb{E}((X,Y)_H) = (\mathbb{E}(X), \mathbb{E}(Y))_H$ since for independent elementary random variables $X = \sum_{i=1}^n \alpha_i 1_{A_i}, Y = \sum_{j=1}^m \beta_j 1_{B_j}$ we have

$$\begin{aligned}
\mathbb{E}((X,Y)_H) &= \mathbb{E}\left(\sum_{i,j}(\alpha_i,\beta_j)_H 1_{A_i\cap B_j}\right) = \sum_{i,j}(\alpha_i,\beta_j)_H P(A_i \cap B_j) \\
&= \sum_{i,j}(P(A_i)\alpha_i, P(B_j)\beta_j)_H = (\mathbb{E}(X), \mathbb{E}(Y))_H
\end{aligned}$$

and by approximation in $L^2(H)$ this follows for general X, Y. But then if X, Y are also centered we obtain

$$\begin{aligned}
\mathbb{E}\|X+Y\|^2 &= \mathbb{E}\|X\|^2 + 2\mathbb{E}((X,Y)_H) + \mathbb{E}\|Y\|^2 \\
&= \mathbb{E}\|X\|^2 + 2(\mathbb{E}(X),\mathbb{E}(Y))_H + \mathbb{E}\|Y\|^2 \\
&= \mathbb{E}\|X\|^2 + \mathbb{E}\|Y\|^2.
\end{aligned}$$

By induction we have

$$\mathbb{E}\left\|\sum_{i=1}^{n} X_i\right\|^2 = \sum_{i=1}^{n} \mathbb{E}\|X_i\|^2$$

for any centered independent random variables $X_1, \dots, X_n \in L^2(H)$ and hence the assertion is shown. □

Proposition 1.4.10. *A separable Banach space E is of type 2 iff every σ-finite measure λ on E with $\int_E (1 \wedge \|x\|^2) d\lambda(x) < \infty$ and $\lambda(\{0\}) = 0$ is a Lévy measure.*

Proof. (cf. [ArGi], chapter 3, Thm. 7.6.). □

Note that the condition $\lambda(\{0\}) = 0$ was added to be consistent with the definition of Lévy measures given above, which gives a one-to-one correspondence of Lévy measures with their generalized exponents (compare Prop. 1.4.1).

Proposition 1.4.11. *A separable Banach space E is of cotype 2 iff for every Lévy measure λ on E the integral $\int_E (1 \wedge \|x\|^2) d\lambda(x)$ is finite.*

Proof. (cf. [ArGi], chapter 3, Thm. 8.16.). □

So we obtain the following property of Lévy measures on separable Hilbert spaces:

Corollary 1.4.12. *Let H be a separable Hilbert space. Then a σ-finite measure λ with $\lambda(\{0\}) = 0$ is in $\mathfrak{L}(H)$ iff*

$$\int_H (1 \wedge \|x\|^2) d\lambda(x) < \infty. \tag{1.3}$$

Proof. The assertion follows directly from Lemma 1.4.9 and Propositions 1.4.10 and 1.4.11. □

Chapter 2

Lévy Processes in Hilbert Space

Lévy processes on a separable Hilbert space H will be studied in the following sections. We will first give some useful properties of H-valued martingales and mention some results for the Gaussian case before we come to general Lévy processes in section 2.3. Here, for a Lévy process X, the *Poisson random measure* $N(t, dx)$ is introduced which is for any $A \in \mathfrak{B}(H \setminus \{0\})$ with $0 \notin \bar{A}$ given by

$$N(t, A) = \#\{0 < s \leq t; \Delta X(s) \in A\} = \sum_{0<s\leq t} 1_A(\Delta X(s))$$

where $\Delta X(s)$ is the "jump" of the process X at time s. By centralization one obtains the *compensated Poisson random measure* $\tilde{N}$ with respect to which we will construct an integral

$$\int_A f(x)\tilde{N}(t, dx)$$

of H-valued functions. This construction is taken from [AlRü] and can be found in section 2.4. Any Lévy process can then (similar to the Lévy Khinchin representation for distributions) be written as the sum of a deterministic part, a Brownian motion part, an integral with respect to the compensated Poisson random measure and an integral with respect to the Poisson random measure. This is the *Lévy Itô decomposition* which was shown in [AlRü] on general separable Banach spaces. In the final section of this chapter we will extend the definition of *martingale measures* to the Hilbert space case. Martingale measures were first introduced in [Wal] as real-valued set functions depending on a time component, a Borel set describing a certain type of jumps and a random component. A martingale measure is in the time component a stochastic process and in the set component locally a measure. For a fixed Borel set the corresponding stochastic process is then a martingale. In [App b] this concept was generalized to H-valued martingales (where it was called martingale-valued measure; nevertheless we will keep the denomination martingale measure). Most of the definitions in this section are therefore taken from [App b]. As the central example we will treat the *Lévy martingale measure* M given by

$$M(t,A) = B_Q(t)\delta_0(A) + \int_{A\setminus\{0\}} x\tilde{N}(t,dx)$$

for $t \geq 0$ and A a Borel subset of the unity ball in H which "stays away" from zero. B_Q denotes a Brownian motion with covariance operator Q. The example was already mentioned in [App b], but we will give a detailed proof that this is in fact a martingale measure. The Lévy martingale measure corresponds to the martingale part of a Lévy process as represented in the Lévy Itô decomposition. In [App b], section 2.2., an *orthogonal* martingale(-valued) measure is defined as a martingale measure where $((M(t,A), M(t,B))_H, t \geq 0)$ for any disjoint A, B is a martingale. However it seems that the stronger definition of orthogonality as given in section 2.5 here is necessary to construct stochastic integrals with respect to martingale measures in sections 3.1 and 3.2. See in particular the proofs of Proposition 3.1.3 and Proposition 3.2.4. Of special interest are *nuclear* martingale measures which in the martingale component have covariance operators that are non-negative, self-adjoint and trace class. Each Lévy martingale measure has this property which is also proved in detail.

2.1 Martingales in Hilbert Space

Let $(\Omega, \mathcal{F}, P)$ be a *complete* probability space, i.e. for each $N_1 \subset \Omega$ such that there exists some $N \in \mathcal{F}$ with $N_1 \subset N$ and $P(N) = 0$ it follows that $N_1 \in \mathcal{F}$. We will take H as a real separable Hilbert space with inner product $(\cdot,\cdot)_H$ and associated norm $\|\cdot\|$. From now on if not otherwise stated all random variables and stochastic processes shall be H-valued. (An H-valued random variable is a strongly measurable function from Ω to H). Let $\mathcal{G}$ be a σ-field contained in $\mathcal{F}$. Then we can define the *conditional expectation* with respect to $\mathcal{G}$ using the following

Proposition 2.1.1. *Let X be a Bochner integrable H-valued random variable. Then there exists a P-a.s. unique Bochner integrable random variable Z, measurable with respect to $\mathcal{G}$ such that*

$$\int_A X\,dP = \int_A Z\,dP \quad \forall A \in \mathcal{G}.$$

Proof. A proof for random variables on general separable Banach spaces is given in [DaPrZa], Prop. 1.10. □

We will then write $\mathbb{E}(X|\mathcal{G})$ instead of Z.

Lemma 2.1.2. *Let X, Y be Bochner integrable H-valued random variables with $\mathbb{E}(\|X\| \cdot \|Y\|) < \infty$ and X is $\mathcal{G}$-measurable. Then*

$$\mathbb{E}((X,Y)_H|\mathcal{G}) = (X, \mathbb{E}(Y|\mathcal{G}))_H \quad P\text{-a.s.}.$$

Proof. Let $\{e_n\}$ be an orthonormal basis of H. With Lemma 1.1.7 for every $A \in \mathcal{G}$

$$\begin{aligned} & \int_A \mathbb{E}((e_n, Y)_H|\mathcal{G}) \mathrm{d}P = \int_A (e_n, Y)_H \mathrm{d}P = \Big(e_n, \int_A Y \mathrm{d}P\Big)_H \\ = \quad & \Big(e_n, \int_A \mathbb{E}(Y|\mathcal{G}) \mathrm{d}P\Big)_H = \int_A (e_n, \mathbb{E}(Y|\mathcal{G}))_H \mathrm{d}P. \end{aligned}$$

Hence $\mathbb{E}((e_n, Y)_H|\mathcal{G}) = (e_n, \mathbb{E}(Y|\mathcal{G}))_H$. Then by Lebesgue's dominated convergence theorem

$$\begin{aligned} & \mathbb{E}((X,Y)_H|\mathcal{G}) = \mathbb{E}\Big(\sum_{n=1}^{\infty}(X,e_n)_H(e_n,Y)_H \Big| \mathcal{G}\Big) \\ = \quad & \sum_{n=1}^{\infty} \mathbb{E}((X,e_n)_H(e_n,Y)_H|\mathcal{G}) = \sum_{n=1}^{\infty}(X,e_n)_H \mathbb{E}((e_n,Y)_H|\mathcal{G}) \\ = \quad & \sum_{n=1}^{\infty}(X,e_n)_H(e_n, \mathbb{E}(Y|\mathcal{G}))_H = (X, \mathbb{E}(Y|\mathcal{G}))_H. \end{aligned}$$

□

Now let $(\mathcal{F}_t, t \geq 0)$ be a filtration that is complete and *rightcontinuous*, i.e. $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$. The *predictable σ-algebra* $\mathcal{P} \subset \mathfrak{B}(\mathbb{R}_+) \otimes \mathcal{F}$ is the σ-algebra generated by the strongly left continuous and adapted processes on H.

A process $X = (X(t), t \geq 0)$, adapted to $(\mathcal{F}_t, t \geq 0)$ is a *martingale* if $\mathbb{E}(\|X(t)\|) < \infty$ for each $t \geq 0$ and $\mathbb{E}(X(t)|\mathcal{F}_s) = X(s)$ P-a.s. for all $0 \leq s \leq t < \infty$. We need the following

Proposition 2.1.3 (cf. [DaPrZa] Prop. 3.7.). *Let M be a martingale then:*

1. *$(\|M(t)\|, t \geq 0)$ is a submartingale.*
2. *For each increasing convex function g from $\mathbb{R}^+$ to $\mathbb{R}^+$ with*

$$\mathbb{E}(g(\|M(t)\|)) < \infty \quad \text{for each } t \geq 0$$

   $(g(\|M(t)\|), t \geq 0)$ is a submartingale.

Proof. Let $\{x_i; i \in \mathbb{N}\}$ be a countably dense subset of $\partial S := \{x \in H; \|x\| = 1\}$. Then $\sup_{i\in\mathbb{N}}(x, x_i)_H = (x, x)_H = 1$ for every $x \in \partial S$, because of the continuity of $y \mapsto (x, y)_H$ and since $\sup_{i\in\mathbb{N}}(x, x_i)_H \leq 1$. Hence for general $x \in H$

$$\sup_{i\in\mathbb{N}}(x, x_i)_H = \|x\| \sup_{i\in\mathbb{N}} \left(\frac{x}{\|x\|}, x_i\right)_H = \|x\|.$$

Let $s < t$, then by Lemma 2.1.2

$$\begin{aligned} \mathbb{E}(\|M(t)\||\mathcal{F}_s) &= \mathbb{E}\left(\sup_{i\in\mathbb{N}}(M(t), x_i)_H \middle| \mathcal{F}_s\right) \geq \sup_{i\in\mathbb{N}} \mathbb{E}((M(t), x_i)_H|\mathcal{F}_s) \\ &= \sup_{i\in\mathbb{N}}(\mathbb{E}(M(t)|\mathcal{F}_s), x_i)_H = \sup_{i\in\mathbb{N}}(M(s), x_i)_H = \|M(s)\|. \end{aligned}$$

The second assertion follows easily from the first one by Jensen's inequality. □

Hence if X is square-integrable, i.e. $\mathbb{E}(\|X(t)\|^2) < \infty$, then the real-valued process $(\|X(t)\|^2, t \geq 0)$ is a submartingale. Let X be strongly càdlàg. Then the Doob-Meyer decomposition gives us a unique adapted, increasing, predictable and rightcontinuous process $(\langle X\rangle(t), t \geq 0)$ with $\langle X\rangle(0) = 0$ such that $(\|X(t)\|^2 - \langle X\rangle(t), t \geq 0)$ is a real-valued martingale.

If $Y = (Y(t), t \geq 0)$ is another strongly càdlàg square-integrable martingale we will define $\langle X, Y\rangle$ by polarization:

$$\langle X, Y\rangle(t) = \frac{1}{4}(\langle X + Y\rangle(t) - \langle X - Y\rangle(t)).$$

By definition $\langle X, Y\rangle$ is also rightcontinuous and since $\langle X + Y\rangle$ and $\langle X - Y\rangle$ are both increasing it is of bounded variation.

Lemma 2.1.4. *If X and Y are strongly càdlàg square-integrable martingales then $\langle X, Y\rangle(0) = 0$ and $(X, Y)_H - \langle X, Y\rangle$ is a martingale. And $\langle X, Y\rangle$ is the unique predictable and rightcontinuous process of bounded variation with this property. In particular, $(X, Y) \mapsto \langle X, Y\rangle$ is bilinear.*

Proof. The first part follows as

$$\begin{aligned} & (X(t), Y(t))_H - \langle X, Y\rangle(t) \\ = & \frac{1}{4}\big((X(t) + Y(t), X(t) + Y(t))_H - (X(t) - Y(t), X(t) - Y(t))_H\big) \\ - & \frac{1}{4}\big(\langle X + Y\rangle(t) - \langle X - Y\rangle(t)\big) \\ = & \frac{1}{4}\big(\|X(t) + Y(t)\|^2 - \langle X + Y\rangle(t) - (\|X(t) - Y(t)\|^2 - \langle X - Y\rangle(t))\big). \end{aligned}$$

The uniqueness can be seen as follows. Let Z be another process with the same properties. Then $\langle X, Y\rangle - Z$ is a real-valued rightcontinuous and predictable martingale. Hence by [vWeWin] Theorem 6.4.11 it is P-a.s. continuous. Since $\langle X, Y\rangle$ and Z are both of bounded variation the same is valid for $\langle X, Y\rangle - Z$ and it is thereby constant. Hence for any $t \geq 0$

$$\langle X, Y\rangle(t) - Z(t) = \langle X, Y\rangle(0) - Z(0) = 0.$$

□

Lemma 2.1.5. *If X and Y are strongly càdlàg square-integrable martingales then*

$$\mathbb{E}((X(t) - X(s), Y(t) - Y(s))_H|\mathcal{F}_s) = \mathbb{E}(\langle X, Y\rangle(t) - \langle X, Y\rangle(s)|\mathcal{F}_s).$$

Proof. By the martingale properties of X, Y and $(X, Y)_H - \langle X, Y\rangle$ we get

$$\begin{aligned} & \mathbb{E}((X(t) - X(s), Y(t) - Y(s))_H|\mathcal{F}_s) \\ = \ & \mathbb{E}((X(t), Y(t))_H - (X(s), Y(s))_H|\mathcal{F}_s) \\ = \ & \mathbb{E}(\langle X, Y\rangle(t) - \langle X, Y\rangle(s)|\mathcal{F}_s). \end{aligned}$$

□

2.2 Gaussian Measures on Hilbert Space

A probability measure μ on H is called *Gaussian* if for any $h \in H$ there exist $m \in \mathbb{R}$ and $q \geq 0$ such that

$$\mu\{x \in H; (h, x)_H \in A\} = \mathcal{N}(m, q)(A) \quad \text{for all } A \in \mathfrak{B}(\mathbb{R})$$

where $\mathcal{N}(m, q)$ denotes the normal distribution on $\mathbb{R}$, with mean m and variance q.

Proposition 2.2.1. *For each Gaussian measure μ on H there exist $m \in H$ and a self-adjoint non-negative bounded linear operator Q such that for all $h, h_1, h_2 \in H$*

$$\begin{aligned} & \int_H (h, x)_H \mu(dx) = (h, m)_H \\ & \int_H (h_1, x)_H (h_2, x)_H \mu(dx) - (h_1, m)_H (h_2, m)_H = (Qh_1, h_2)_H. \end{aligned}$$

Then the characteristic function of μ is given by

$$\hat{\mu}(a) = \int_H \exp(i(a, x)_H)\mu(dx) = \exp\left(i(a, m)_H - \frac{1}{2}(Qa, a)_H\right).$$

Proof. (cf. [DaPrZa] p.53-55). □

In the Proposition above m is called the *mean* and Q is called the *covariance operator* of μ. As m and Q are uniquely determined one may also write $\mu = \mathcal{N}(m, Q)$. In the case that $m = 0$ we obtain

$$(Qh_1, h_2)_H = \int_H (h_1, x)_H (h_2, x)_H \mu(dx)$$

$$\text{and} \qquad \hat{\mu}(a) = \exp\left(-\frac{1}{2}(Qa, a)_H\right).$$

Hence this definition for Gaussian measures on Hilbert spaces is consistent with the rather abstract definition of Gaussian symmetric measures which was made in section 1.2. Therefore we will also refer to measures of the kind $\mathcal{N}(0, Q)$ as Gaussian symmetric measures and denote the set of all Gaussian symmetric measures by $\mathfrak{G}(H)$.

We will need *trace class* operators and some of their properties. We refer to [ReeSim], p.207-210. Let $\{e_k\}$ be an orthonormal basis of H. Then define the *trace* of some bounded linear operator Q on H as

$$tr(Q) = \sum_{k=1}^{\infty} (e_k, Qe_k)_H$$

if the series converges. By Theorem VI.18 in [ReeSim] this definition is independent of the chosen orthonormal basis. Let Q^* be the adjoint operator of Q then Q^*Q is non-negative since

$$(Q^*Qx, x)_H = \|Qx\|^2 \geq 0.$$

By the square root lemma (compare [ReeSim] Theorem VI.9) for every non-negative bounded linear operator R on H the *square root* $\sqrt{R}$ exists, i.e. a unique non-negative bounded linear operator S with $S^2 = R$. Then for any bounded linear operator Q we can define

$$|Q| = \sqrt{Q^*Q}.$$

A bounded linear operator Q on H is called *trace class* iff

$$tr|Q| < \infty.$$

Proposition 2.2.2. *Let $\mu \in \mathfrak{G}(H)$ with covariance operator Q. Then Q is trace class.*

Proof. (cf. [DaPrZa] Prop. 2.15). □

2.3 Lévy Processes and Poisson Random Measure

Now we will introduce the concept of Lévy processes. A stochastic process X is said to be *stochastically continuous* if for every $t \geq 0$ and $\varepsilon > 0$

$$\lim_{s \to t} P(\|X(s) - X(t)\| > \varepsilon) = 0.$$

We will call a stochastic process $X = (X(t), t \geq 0)$ with values in H and adapted to $(\mathcal{F}_t, t \geq 0)$ a *Lévy process* iff

- $X(0) = 0$.
- X has increments independent of the past, i.e. $X(t) - X(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t < \infty$.
- X has stationary increments, i.e. $X(t) - X(s)$ has the same distribution as $X(t-s)$ for all $0 \leq s < t < \infty$.
- X is stochastically continuous.
- X has strongly càdlàg paths.

The following proposition shows the relation between Lévy processes and infinitely divisible measures.

Proposition 2.3.1. *Let $X = (X(t), t \geq 0)$ be a Lévy process and μ^t be the distribution of $X(t)$. Then μ^t is infinitely divisible.*

Proof. For any natural number n let μ_n be the distribution of the increment $X(\frac{1}{n}t) - X(0)$. As the increments are stationary and independent we can write $X(t)$ as sum of i.i.d. random variables:

$$X(t) = \left(X\left(\frac{1}{n}t\right) - X(0) \right) + \cdots + \left(X(t) - X\left(\frac{n-1}{n}t\right) \right).$$

Hence for the distribution of $X(t)$ we have $\mu^t = (\mu_n)^n$. □

Corollary 2.3.2. *Let $X = (X(t), t \geq 0)$ be a Lévy process. Then there exists a unique exponent*

$$b(a) = i(x_0, a)_H - \frac{1}{2}(Qa, a)_H + \int_H e^{i(x,a)_H} - 1 - i(x,a)_H 1_{\{\|x\| \leq 1\}}(x)\, d\lambda(x)$$

with $x_0 \in H$, λ a Lévy measure and Q the covariance operator of a Gaussian symmetric measure, such that for every $t \geq 0$, $a \in H$

$$\mathbb{E}(\exp(i(a, X(t))_H)) = e^{tb(a)}.$$

Or equivalently the distribution of $X(t)$ has generating triplet $[tx_0, tQ, t\lambda]$.

Proof. Denote the distribution of $X(1)$ by μ and let $a \mapsto b(a)$ be such that $\hat{\mu}(a) = \exp(b(a))$ for each $a \in E'$. Then by the proof of Proposition 2.3.1 $\mu^{1/n}$ is the distribution of $X(1/n)$ and hence for any $\alpha \in \mathbb{Q}_+$ the distribution of $X(\alpha)$ is μ^α with characteristic function $a \mapsto \exp(\alpha b(a))$. Since X is stochastically continuous $\{\mu^{\alpha_n}\}$ converges for each sequence $\{\alpha_n\} \subset \mathbb{Q}_+$ converging in $\mathbb{R}$. But then by Lemma 1.2.6 the distribution of $X(t)$ is μ^t for every $t \geq 0$. Hence the assertion follows from Corollary 1.4.5. □

Example 2.3.3. For $H = \mathbb{R}$ let $N = (N(t), t \geq 0)$ be a Poisson process with intensity c, i.e. $N(0) = 0$ and

$$P(N(t) - N(s) = n) = e^{-c(t-s)} \frac{(c(t-s))^n}{n!} \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Then N is a Lévy process adapted to $(\mathcal{F}_t = \sigma(N(t)), t \geq 0)$: The independent and stationary increments property holds by definition. Also N has càdlàg paths and N is stochastically continuous as for every $m \in \mathbb{N} \cup \{0\}$

$$P(|N(t) - N(s)| > m) = 1 - e^{-c(t-s)} \sum_{n=0}^{m} \frac{(c(t-s))^n}{n!}$$

which converges to 0 for $t \to s$. Note that Poisson processes have the property $Var(N(t)) = tVar(N(1))$.

For a Lévy process X the "jump" at time t is given by $\Delta X(t) := X(t) - X(t-)$. Define for all $t > 0$ and $A \in \mathfrak{B}(H \setminus \{0\})$ with $0 \notin \bar{A}$

$$N(t, A) = \#\{0 < s \leq t; \Delta X(s) \in A\} = \sum_{0<s\leq t} 1_A(\Delta X(s)). \tag{2.1}$$

Proposition 2.3.4. *Let $A \in \mathfrak{B}(H \setminus \{0\})$ with $0 \notin \bar{A}$ be fixed. Then the process $N(\cdot, A) = (N(t, A), t \geq 0)$ is a Poisson process. In particular, $\sum_{0<s\leq t} 1_A(\Delta X(s))$ is a finite sum for every $t \geq 0$ P-a.s..*

Proof. (cf. [AlRü] Thm. 2.7, where the result is shown for arbitrary separable Banach spaces). □

Proposition 2.3.5. *Let $\omega \in \Omega$ and $t \geq 0$ be fixed. Then $N(t, \cdot)(\omega)$ can be interpreted as a set function from $\{A \in \mathfrak{B}(H \setminus \{0\}); 0 \notin \bar{A}\}$ to $\mathbb{R}_+ \cup \{+\infty\}$. Then for P-a.e. ω there exists a unique σ-finite measure ν_t on $\mathfrak{B}(H \setminus \{0\})$ or equivalently a σ-finite measure ν_t on $\mathfrak{B}(H)$ with $\nu_t(\{0\}) = 0$ which extends this set function.*

Proof. (cf. [AlRü] Thm. 2.13, Cor. 2.14). □

We will write $N(t, dx)$ instead of $\nu_t(dx)$.

Proposition 2.3.6. *If we define $\tilde{\nu}(A) := \mathbb{E}(N(1,A))$ for all $A \in \mathfrak{B}(H \setminus \{0\})$ with $0 \notin \bar{A}$ then $\tilde{\nu}$ can be uniquely extended to a σ-finite measure ν on $\mathfrak{B}(H \setminus \{0\})$ or equivalently a σ-finite measure ν on $B(H)$ with $\nu(\{0\}) = 0$.*

Proof. (cf. [AlRü] Thm. 2.17, Cor. 2.18). □

Proposition 2.3.7. *Let μ be the distribution of $X(1)$ and $[x_0, Q, \lambda]$ be the generating triplet of μ (like in Corollary 2.3.2). Then the measure ν on H from Proposition 2.3.6 is equal to the Lévy measure λ.*

Proof. Let $\delta > 0$ with $\lambda\{\|x\| = \delta\} = 0$. Then $B_\delta := \{x \in H; \|x\| > \delta\} \in \mathfrak{B}(H \setminus \{0\})$ and by Proposition 1.4.6

$$(n\mu^{1/n})|_{B_\delta} \Rightarrow \lambda|_{B_\delta}.$$

Let A be an open subset of B_δ. Then

$$\begin{aligned}
&\nu(A \cap B_\delta) = \mathbb{E}(N(1, A \cap B_\delta)) \\
= \;& \mathbb{E}\left(\sum_{0<s\leq 1} 1_{A\cap B_\delta}(\Delta X(s))\right) \\
= \;& \mathbb{E}\left(\lim_{n\to\infty}\sum_{k=0}^{n-1} 1_{A\cap B_\delta}\left(X\left(\frac{k+1}{n}\right) - X\left(\frac{k}{n}\right)\right)\right) \\
\leq \;& \liminf_{n\to\infty}\sum_{k=0}^{n-1} \mathbb{E}\left(1_{A\cap B_\delta}\left(X\left(\frac{k+1}{n}\right) - X\left(\frac{k}{n}\right)\right)\right) \\
= \;& \liminf_{n\to\infty} nP\left(X\left(\frac{1}{n}\right) \in A \cap B_\delta\right) \\
= \;& \liminf_{n\to\infty}(n\mu^{1/n})(A \cap B_\delta).
\end{aligned}$$

Hence by Portmanteau's theorem $(n\mu^{1/n})|_{B_\delta} \Rightarrow \nu|_{B_\delta}$, thus $\lambda|_{B_\delta} = \nu|_{B_\delta}$. As λ is σ-finite by Remark 1.4.7 there always exists a sequence $\{\delta_m\}$ fulfilling $\lambda\{\|x\| = \delta_m\} = 0$ and $\delta_m \downarrow 0$, i.e. $B_{\delta_m} \uparrow H \setminus \{0\}$. Then we obtain identity of λ and ν on $\mathfrak{B}(H \setminus \{0\})$, hence by $\lambda(\{0\}) = 0$ on $\mathfrak{B}(H)$. □

Adopting the terminology of [AlRü] we will call N a *Poisson random measure*. In the usual terminology a Poisson random measure is more than we have shown above (cf. for example [Kno b], p.5): Let $(U,\mathcal{U})$ be a measurable space. Let $\mathbb{M}$ be the space of non-negative (possibly infinite) integer-valued measures on $(U,\mathcal{U})$ and

$$\mathcal{B}_{\mathbb{M}} := \sigma(\mathbb{M} \to \mathbb{Z}_+ \cup \{+\infty\}, \mu \mapsto \mu(B) | B \in \mathcal{U}).$$

Then a random variable $\Pi : (\Omega, \mathcal{F}) \to (\mathbb{M}, \mathcal{B}_{\mathbb{M}})$ is called *Poisson random measure* on $(U,\mathcal{U})$ if the following conditions hold:

- For all $B \in \mathcal{U}$: $\Pi(B) : \Omega \to \mathbb{Z}_+ \cup \{+\infty\}$ is Poisson distributed with parameter $\mathbb{E}(\Pi(B))$, i.e.:

$$P(\Pi(B) = n) = \exp(-\mathbb{E}(\Pi(B)))(\mathbb{E}(\Pi(B)))^n/n!, \quad n \in \mathbb{N} \cup \{0\}.$$

  If $\mathbb{E}(\Pi(B)) = +\infty$ then $\Pi(B) = +\infty$ P-a.s.

- If $B_1, \dots, B_m \in \mathcal{U}$ are pairwise disjoint then $\Pi(B_1), \dots \Pi(B_m)$ are independent.

However we will keep our first definition for the remainder of the text. Set $\tilde{N}(t, A) = N(t, A) - t\lambda(A)$. Then $\tilde{N}$ is called *compensated Poisson random measure.*

Lemma 2.3.8. *For every $A \in \mathfrak{B}(H \setminus \{0\})$ with $0 \notin \bar{A}$ the process $\tilde{N}(\cdot, A) = (\tilde{N}(t, A), t \geq 0)$ is a martingale and $\mathbb{E}(\tilde{N}(t, A)) = 0$.*

Proof. By Proposition 2.3.4 $N(\cdot, A)$ is a Poisson process and for $s < t$

$$\begin{aligned} & \mathbb{E}(\tilde{N}(t, A)|\mathcal{F}_s) = \mathbb{E}(N(t, A) - t\lambda(A)|\mathcal{F}_s) \\ = \; & N(s, A) - s\lambda(A) + \mathbb{E}(N(t, A) - N(s, A) - (t - s)\lambda(A)|\mathcal{F}_s) \\ = \; & \tilde{N}(s, A) + \mathbb{E}(N(t - s, A)) - (t - s)\lambda(A) = \tilde{N}(s, A). \end{aligned}$$

As $\mathbb{E}(N(0, A)) = 0$ the second part follows. □

Lemma 2.3.9. *Let $A, B \in \mathfrak{B}(H \setminus \{0\})$ be disjoint with $0 \notin \bar{A}$ and $0 \notin \bar{B}$. Then*

$$\mathbb{E}(\tilde{N}(t, A)\tilde{N}(t, B)) = 0 \quad \textit{for every } t \geq 0.$$

Proof. The proof follows the idea of the proof for Theorem 38 (p.28/29) in [Pro]. Let $\{\tau_n\}$ be a sequence of partitions $0 = t_0^{(n)} < \dots < t_n^{(n)} = t$ of $[0, t]$ for which the mesh tends to 0. Then by the martingale property of $\tilde{N}(\cdot, A)$ and $\tilde{N}(\cdot, B)$

$$\begin{aligned} & \mathbb{E}(\tilde{N}(t, A)\tilde{N}(t, B)) \\ = \; & \mathbb{E}\left(\sum_{t_j^{(n)} \in \tau_n, j<n} (\tilde{N}(t_{j+1}^{(n)}, A) - \tilde{N}(t_j^{(n)}, A)) \right. \\ & \left. \sum_{t_k^{(n)} \in \tau_n, k<n} (\tilde{N}(t_{k+1}^{(n)}, B) - \tilde{N}(t_k^{(n)}, B)) \right) \\ = \; & \mathbb{E}\left(\sum_{t_j^{(n)} \in \tau_n, j<n} (\tilde{N}(t_{j+1}^{(n)}, A) - \tilde{N}(t_j^{(n)}, A))(\tilde{N}(t_{j+1}^{(n)}, B) - \tilde{N}(t_j^{(n)}, B)) \right). \end{aligned}$$

We have

$$\begin{aligned}
&\sum_{t_j^{(n)}\in\tau_n, j<n} \left|(\tilde N(t_{j+1}^{(n)},A)-\tilde N(t_j^{(n)},A))(\tilde N(t_{j+1}^{(n)},B)-\tilde N(t_j^{(n)},B))\right| \\
\leq\ & \left(\sum_{t_j^{(n)}\in\tau_n, j<n} (\tilde N(t_{j+1}^{(n)},A)-\tilde N(t_j^{(n)},A))^2\right)^{\frac12} \\
& \left(\sum_{t_k^{(n)}\in\tau_n, k<n} (\tilde N(t_{k+1}^{(n)},B)-\tilde N(t_k^{(n)},B))^2\right)^{\frac12}.
\end{aligned}$$

Since $N(\cdot,A)$ and $t\mapsto t\lambda(A)$ (and the same for B) are increasing it is easy to deduce that

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{t_j^{(n)}\in\tau_n, j<n} (\tilde N(t_{j+1}^{(n)},A)-\tilde N(t_j^{(n)},A))(\tilde N(t_{j+1}^{(n)},B)-\tilde N(t_j^{(n)},B)) \\
=\ & \lim_{n\to\infty}\sum_{t_j^{(n)}\in\tau_n, j<n} (N(t_{j+1}^{(n)},A)-N(t_j^{(n)},A))(N(t_{j+1}^{(n)},B)-N(t_j^{(n)},B)) \\
& +\lim_{n\to\infty}\sum_{t_j^{(n)}\in\tau_n, j<n} (N(t_{j+1}^{(n)},A)-N(t_j^{(n)},A))(t_{j+1}^{(n)}-t_j^{(n)})\lambda(B) \\
& +\lim_{n\to\infty}\sum_{t_j^{(n)}\in\tau_n, j<n} (t_{j+1}^{(n)}-t_j^{(n)})\lambda(A)(N(t_{j+1}^{(n)},B)-N(t_j^{(n)},B)) \\
& +\lim_{n\to\infty}\sum_{t_j^{(n)}\in\tau_n, j<n} (t_{j+1}^{(n)}-t_j^{(n)})\lambda(A)(t_{j+1}^{(n)}-t_j^{(n)})\lambda(B) \\
=\ & \sum_{0<s\leq t}\Delta N(s,A)\Delta N(s,B) \\
=\ & 0,
\end{aligned}$$

where the last equation is valid, because A and B are disjoint. But since

$$\sum_{t_j^{(n)}\in\tau_n, j<n} (\tilde N(t_{j+1}^{(n)},A)-\tilde N(t_j^{(n)},A))^2 \leq 3(\tilde N(t,A)^2+t^2\lambda(A)^2)$$

and the same for $\tilde N(t,B)$ we can apply Lebesgue's dominated convergence theorem and obtain

$$\mathbb{E}(\tilde N(t,A)\tilde N(t,B)) = \mathbb{E}\left(\sum_{0<s\leq t}\Delta N(s,A)\Delta N(s,B)\right)=0.$$

□

2.4 The Lévy Itô Decomposition

The goal of this section is to define integrals with respect to Poisson random measures and compensated Poisson random measures and to give a representation of the non-Gaussian part of a Lévy process by such integrals. This representation will give us the so-called Lévy Itô decomposition. Let X be a Lévy process and N the corresponding Poisson random measure, $\tilde{N}$ the corresponding compensated Poisson random measure.

Let $A \in \mathfrak{B}(H \setminus \{0\})$ with $0 \notin \bar{A}$ and $f : A \to H$ measurable. Define the following integral

$$\int_A f(x) N(t, dx) = \sum_{0<s\leq t} f(\Delta X(s)) 1_A(\Delta X(s)) \tag{2.2}$$

This is a finite sum P-a.s. since the number of summands is finite P-a.s.

For $f \in L^1(A, \lambda|_A; H) = L^1((A, \mathfrak{B}(A), \lambda|_A) \to H)$ one could define the integral with respect to the compensated Poisson random measure as follows

$$\int_A f(x) \tilde{N}(t, dx) = \int_A f(x) N(t, dx) - t \int_A f(x) \lambda(dx). \tag{2.3}$$

However we will choose a different definition which allows us to show some useful properties for the integral. We will then observe that this definition is compatible with the one given in (2.3). This construction of the integral is taken from [AlRü].

Let $\mathcal{S}(H)$ be the set of all simple functions f of the form

$$f(x) = \sum_{k=1}^{N} a_k 1_{A_k}(x), \tag{2.4}$$

where $a_k \in H$ and the A_k are disjoint sets in $\mathfrak{B}(H \setminus \{0\})$ with $0 \notin \overline{A_k}$.

The integral with respect to $\tilde{N}(t, dx)$ on any set $A \in \mathfrak{B}(H \setminus \{0\})$ can then by Proposition 2.3.5 be defined for $f \in \mathcal{S}(H)$, given by (2.4), as

$$\int_A f(x) \tilde{N}(t, dx) := \sum_{k=1}^{N} a_k \tilde{N}(t, A_k \cap A).$$

Note that this is a strongly càdlàg process.

Now let f be in $L^2_\lambda := L^2(H \setminus \{0\}, \lambda|_{H\setminus\{0\}}; H)$. Then it is said to be *strongly 2-integrable* on some set $A \in \mathfrak{B}(H \setminus \{0\})$ with respect to $\tilde{N}(t, dx)$ if for any sequence $\{f_n\} \subset \mathcal{S}(H)$ with $f_n \to f$ in L^2_λ the limes

$$\int_A f(x) \tilde{N}(t, dx) := \lim_{n\to\infty} \int_A f_n(x) \tilde{N}(t, dx) \tag{2.5}$$

exists in $L^2(H) = L^2((\Omega, \mathcal{F}, P); H)$ and does not depend on the chosen sequence $\{f_n\}$. Note that by Doob's inequality and the Borel-Cantelli Lemma it follows that this process has a strongly càdlàg modification.

Lemma 2.4.1. *Let $f \in L^2_\lambda$. Then there exists a sequence $\{f_n\} \subset \mathcal{S}(H)$ approximating f in L^2_λ.*

Proof. (cf. [AlRü] Proposition 3.8.). □

Proposition 2.4.2. *If $f \in L^1_\lambda \cap L^2_\lambda$ then f is strongly 2-integrable on all $A \in \mathfrak{B}(H \setminus \{0\})$ with respect to $\tilde{N}(t, dx)$.*

Proof. (cf. [AlRü] Theorem 3.22). □

The next Proposition shows the consistency with the "natural" definition of the integral:

Proposition 2.4.3. *Let f be strongly 2-integrable with respect to $\tilde{N}(t, dx)$ and $f \in L^2_\lambda$. Then for any $A \in \mathfrak{B}(H \setminus \{0\})$ with $0 \notin \bar{A}$ the strong integral from (2.5) coincides with the "natural" definition from (2.3), i.e. P-a.s.*

$$\int_A f(x)\tilde{N}(t, dx) = \sum_{0<s\leq t} f(\Delta X(s))1_A(\Delta X(s)) - t\int_A f(x)\lambda(dx). \qquad (2.6)$$

Proof. (cf. [AlRü] Prop. 3.26). □

Proposition 2.4.4. *Let $f \in L^2_\lambda$ be strongly 2-integrable on some $A \in \mathfrak{B}(H \setminus \{0\})$ and $t \geq 0$ then for any $y_1, y_2 \in H$*

$$\begin{aligned} & \mathbb{E}\left(\left(\int_A f(x)\tilde{N}(t, dx), y_1\right)_H \left(\int_A f(x)\tilde{N}(t, dx), y_2\right)_H\right) \\ = \; & t\int_A (f(x), y_1)_H (f(x), y_2)_H \lambda(dx). \end{aligned}$$

Proof. First we will prove the assertion for $f \in \mathcal{S}(H)$. So let f be given by

$$f(x) = \sum_{k=1}^{N} a_k 1_{A_k}(x)$$

with disjoint A_k, $k \in \{1, \ldots, N\}$. We will need the fact that $N(\cdot, A)$ for given $A \in \mathfrak{B}(H \setminus \{0\})$ with $0 \notin \bar{A}$ is a Poisson process (Prop. 2.3.4). From this it is easy to deduce that

$$\begin{aligned} \mathbb{E}(\tilde{N}(t, A)^2) &= Var(\tilde{N}(t, A)) + \mathbb{E}(\tilde{N}(t, A))^2 = Var(N(t, A)) \\ &= tVar(N(1, A)) = t\lambda(A). \end{aligned}$$

But then by Lemma 2.3.9 for any $y_1, y_2 \in H$

$$\begin{aligned}
&\mathbb{E}\left(\left(\int_A f(x)\tilde{N}(t,dx), y_1\right)_H \left(\int_A f(x)\tilde{N}(t,dx), y_2\right)_H\right) \\
= \;& \mathbb{E}\left(\left(\sum_{k=1}^N a_k \tilde{N}(t, A\cap A_k), y_1\right)_H \left(\sum_{k=1}^N a_k \tilde{N}(t, A\cap A_k), y_2\right)_H\right) \\
= \;& \sum_{j,k=1}^N (a_j, y_1)_H (a_k, y_2)_H \mathbb{E}(\tilde{N}(t, A\cap A_j)\tilde{N}(t, A\cap A_k)) \\
= \;& \sum_{k=1}^N (a_k, y_1)_H (a_k, y_2)_H \mathbb{E}(\tilde{N}(t, A\cap A_k)^2) \\
= \;& \sum_{k=1}^N (a_k, y_1)_H (a_k, y_2)_H t\lambda(A\cap A_k) \\
= \;& t\sum_{j,k=1}^N (a_j, y_1)_H (a_k, y_2)_H \int_A 1_{A_j}(x) 1_{A_k}(x)\lambda(dx) \\
= \;& t\int_A \left(\sum_{j=1}^N (a_j, y_1)_H 1_{A_j}(x)\right)\left(\sum_{k=1}^N (a_k, y_2)_H 1_{A_k}(x)\right)\lambda(dx) \\
= \;& t\int_A \left(\sum_{j=1}^N a_j 1_{A_j}(x), y_1\right)_H \left(\sum_{k=1}^N a_k 1_{A_k}(x), y_2\right)_H \lambda(dx) \\
= \;& t\int_A (f(x), y_1)_H (f(x), y_2)_H \lambda(dx).
\end{aligned}$$

Now let $f \in L^2_\lambda$ be arbitrary. There exists a sequence $\{f_n\} \subset \mathcal{S}(H)$ with $f_n \to f \in L^2_\lambda$. Set $X_n := \int_A f_n(x)\tilde{N}(t,dx)$ and $X := \int_A f(x)\tilde{N}(t,dx)$. Then $\{X_n\}$ converges to X in $L^2(H)$. Hence by the Cauchy-Schwartz and Hölder's inequality we obtain

$$\begin{aligned}
&\lim_{n\to\infty} |\mathbb{E}((X_n, y_1)_H (X_n, y_2)_H) - \mathbb{E}((X, y_1)_H (X, y_2)_H)| \\
\leq \;& \lim_{n\to\infty} (\mathbb{E}(|(X_n - X, y_1)||(X_n, y_2)_H|) + \mathbb{E}(|(X, y_1)||(X_n - X, y_2)|)) \\
\leq \;& \|y_1\|\|y_2\| \lim_{n\to\infty} (\mathbb{E}(\|X_n - X\|\|X_n\|) + \mathbb{E}(\|X_n - X\|\|X\|)) \\
\leq \;& \|y_1\|\|y_2\| \lim_{n\to\infty} \mathbb{E}(\|X_n - X\|^2)^{\frac{1}{2}} (\mathbb{E}(\|X_n\|^2)^{\frac{1}{2}} + \mathbb{E}(\|X\|^2)^{\frac{1}{2}}) \\
= \;& 0
\end{aligned}$$

and by the same arguments

$$\lim_{n\to\infty} \left| \int_A (f_n(x), y_1)_H (f_n(x), y_2)_H - (f(x), y_1)_H (f(x), y_2)_H \lambda(dx) \right|$$

$$
\begin{aligned}
&\leq \|y_1\|\|y_2\| \lim_{n\to\infty} \left(\int_A \|f_n(x) - f(x)\|^2 \lambda(dx) \right)^{\frac{1}{2}} \\
&\left(\left(\int_A \|f_n(x)\|^2 \lambda(dx) \right)^{\frac{1}{2}} + \left(\int_A \|f(x)\|^2 \lambda(dx) \right)^{\frac{1}{2}} \right) \\
&= 0.
\end{aligned}
$$

So, we have shown the equation for general strongly 2-integrable $f \in L^2_\lambda$. □

Proposition 2.4.5 (cf. [AlRü] Theorem 3.25). *Let $f \in L^2_\lambda$ then for any $A \in \mathfrak{B}(H \setminus \{0\})$ the integral*

$$\int_A f(x)\tilde{N}(t, dx)$$

exists and

$$\mathbb{E}\left(\left\| \int_A f(x)\tilde{N}(t, dx) \right\|^2 \right) = t \int_A \|f(x)\|^2 \lambda(dx) < \infty.$$

Proof. Let $f \in \mathcal{S}(H)$ then by definition f is strongly 2-integrable with respect to $\tilde{N}(t, dx)$. Let $\{e_k\}$ be an orthonormal basis of H then with Prop. 2.4.4 it follows

$$
\begin{aligned}
&\mathbb{E}\left(\left\| \int_A f(x)\tilde{N}(t, dx) \right\|^2 \right) = \mathbb{E}\left(\sum_{k=1}^{\infty} \left| \left(\int_A f(x)\tilde{N}(t, dx), e_k \right)_H \right|^2 \right) \\
= &\sum_{k=1}^{\infty} \mathbb{E}\left(\left| \left(\int_A f(x)\tilde{N}(t, dx), e_k \right)_H \right|^2 \right) = \sum_{k=1}^{\infty} t \int_A |(f(x), e_k)_H|^2 \lambda(dx) \\
= &t \int_A \sum_{k=1}^{\infty} |(f(x), e_k)_H|^2 \lambda(dx) = t \int_A \|f(x)\|^2 \lambda(dx).
\end{aligned}
$$

Now let $f \in L^2_\lambda$ be arbitrary and $\{f_n\} \subset \mathcal{S}(H)$ be a sequence with $f_n \to f \in L^2_\lambda$. Then

$$
\begin{aligned}
\mathbb{E}\left(\left\| \int_A (f_n(x) - f_m(x))\tilde{N}(t, dx) \right\|^2 \right) &= t \int_A \|f_n(x) - f_m(x)\|^2 \lambda(dx). \\
&\xrightarrow[n,m\to\infty]{} 0
\end{aligned}
$$

Hence the sequence of the integrals $\int_A f_n(x)\tilde{N}(t, dx)$ forms a Cauchy sequence in $L^2(H)$ and we can thereby define $\int_A f(x)\tilde{N}(t, dx)$ as the limit in $L^2(H)$. It can be checked that this limit does not depend on the chosen sequence. Therefore every $f \in L^2_\lambda$ is strongly 2-integrable and the integral fulfills the given equation. □

Let $S = \{x \in H; \|x\| < 1\}$ and $\mathcal{A}_0 = \{A \in \mathfrak{B}(S); 0 \notin \bar{A}\}$, $\mathcal{A} = \mathcal{A}_0 \cup \{A \cup \{0\}; A \in \mathcal{A}_0\}$. Note that $id|_S \in L^2_\lambda$ by Corollary 1.4.12.

Proposition 2.4.6. *Let $f \in L^2_\lambda$ and $A \in \mathcal{A}_0$ then $M = (M(t), t \geq 0)$ with*

$$M(t) = \int_A f(x)\tilde{N}(t, dx)$$

is a strongly càdlàg square integrable martingale with $M(0) = 0$.

Proof. By Prop. 2.4.5 $M(t)$ is square-integrable and obviously $M(0) = 0$. Let $s < t$. There exists a sequence $\{f_n\} \subset S(H)$ which converges to f in L^2_λ. For f_n like in (2.4) we have

$$\int_A f(x)\tilde{N}(t, dx) = \lim_{n\to\infty} \sum_{k=1}^{N_n} a_k^n \tilde{N}(t, A \cap A_k^n)$$

and

$$\int_A f(x)\tilde{N}(s, dx) = \lim_{n\to\infty} \sum_{k=1}^{N_n} a_k^n \tilde{N}(s, A \cap A_k^n),$$

where the limits are in $L^2(H)$. Then by Lemma 2.3.8 for every $B \in \mathcal{F}_s$

$$\mathbb{E}(1_B M(t)) = \lim_{n\to\infty} \sum_{k=1}^{N_n} a_k^n \int_B (\tilde{N}(t, A \cap A_k^n))\mathrm{d}P$$

$$= \lim_{n\to\infty} \sum_{k=1}^{N_n} a_k^n \int_B (\tilde{N}(s, A \cap A_k^n))\mathrm{d}P = \mathbb{E}(1_B M(s)).$$

By [Kun] Proposition 3 every square-integrable martingale is strongly càdlàg P-a.s.. □

A very important representation for Lévy processes is the so-called Lévy-Itô decomposition:

Theorem 2.4.7. *Let $X = (X(t), t \geq 0)$ be a Lévy process (on H) where the distribution of $X(t)$ has generating triplet $[tx_0, tQ, t\lambda]$ for each $t \geq 0$ (compare Corollary 2.3.2). Then for every $t \geq 0$*

$$X(t) = tx_0 + B_Q(t) + \int_{\{\|x\|<1\}} x\tilde{N}(t, dx) + \int_{\{\|x\|\geq 1\}} xN(t, dx), \qquad (2.7)$$

where $B_Q = (B_Q(t), t \geq 0)$ is a Brownian motion with covariance operator Q independent of $N(\cdot, A)$ for all $A \in \mathfrak{B}(H \setminus \{0\})$ with $0 \notin \bar{A}$.

Proof. This follows from [AlRü] Thm. 4.1 where the result is shown for any separable Banach space of type 2 and Lévy measures for which (1.3) is fulfilled. Notice that a Hilbert space is also of cotype 2 and therefore this condition is fulfilled by every Lévy measure. □

Corollary 2.4.8. *For each $A \in \mathcal{A}$ the process $M(\cdot, A)$ given by*

$$M(t, A) = B_Q(t)\delta_0(A) + \int_{A\setminus\{0\}} x\tilde{N}(t, dx)$$

is a strongly càdlàg square-integrable martingale.

Proof. Follows from Prop. 2.4.6. □

2.5 Martingale Measures

Let (S, Σ) be a Lusin topological space, i.e. a continuous one-to-one image of a Polish space and Σ a sub-σ-algebra of $\mathfrak{B}(S)$. Furthermore, let $\mathcal{A} \subset \Sigma$ be a ring and $\{S_n, n \in \mathbb{N}\} \subset \mathcal{A}$ with $S_n \uparrow S$ and $\Sigma_n := \Sigma|_{S_n} \subset \mathcal{A}$ for all $n \in \mathbb{N}$.

The definition given now is generalizing the definition of real-valued *martingale measures* in [Wal], p. 287, to such with values in Hilbert space. A *martingale measure* is a set function $M : \mathbb{R}_+ \times \mathcal{A} \times \Omega \to H$ fulfilling the following properties: $M(0, A) = M(t, \emptyset) = 0$ a.s., for all $A \in \mathcal{A}, t \geq 0$. For $t > 0$ $M(t, \cdot)$ is

1. finitely additive, i.e. $M(t, A \cup B) = M(t, A) + M(t, B)$ a.s., for all disjoint $A, B \in \mathcal{A}$,
2. σ-finite, i.e. $\sup\{\mathbb{E}(\|M(t, A)\|^2), A \in \Sigma_n\} < \infty$ for all $n \in \mathbb{N}$,
3. countably additive on each Σ_n, $n \in \mathbb{N}$, i.e. if $\{A_j\} \subset \Sigma_n$ is a sequence decreasing to the empty set then $\lim_{j\to\infty} \mathbb{E}(\|M(t, A_j)\|^2) = 0$.

Finally $M(\cdot, A) = (M(t, A), t \geq 0)$ is a strongly càdlàg square-integrable martingale for each $A \in \mathcal{A}$. In particular, the zero set in 1. is independent of t.

We call a martingale measure M *orthogonal* (compare the definition in [Wal], p. 288) iff for any disjoint $A, B \in \mathcal{A}$ and any orthonormal basis $\{e_n\}$ of H the process

$$((M(t, A), e_n)_H (M(t, B), e_m)_H, t \geq 0)$$

is a martingale for all $n, m \in \mathbb{N}$. In particular $((M(t, A), M(t, B))_H, t \geq 0)$ then is a martingale.

Now for each $t \geq 0$ the set function $\langle M\rangle(t, \cdot)$ given by $\langle M\rangle(t, A) = \langle M(\cdot, A)\rangle(t)$ is well defined and for disjoint sets $A, B \in \mathcal{A}$ by Lemma 2.1.4 and since M is orthogonal

$$\begin{aligned} \langle M\rangle(t, A \cup B) &= \langle M(\cdot, A \cup B)\rangle(t) = \langle M(\cdot, A) + M(\cdot, B)\rangle(t) \\ &= 2\langle M(\cdot, A), M(\cdot, B)\rangle(t) + \langle M\rangle(t, A) + \langle M\rangle(t, B) \\ &= \langle M\rangle(t, A) + \langle M\rangle(t, B). \end{aligned}$$

A martingale measure M is said to have *independent increments* iff $M((s,t],A)$ is independent of $\mathcal{F}_s$ for all $A \in \mathcal{A}, 0 \leq s < t < \infty$, where $M((s,t],\cdot) := M(t,\cdot) - M(s,\cdot)$.

The following definitions are all taken from [App b], section 2.2.

Let $T = \{T_A, A \in \mathcal{A}\}$ be a family of bounded non-negative self-adjoint operators on H. We call T a *positive-operator valued measure* on (S,Σ) iff

- $T_\emptyset = 0$,
- $T_{A\cup B} = T_A + T_B$ for all disjoint $A, B \in \mathcal{A}$.

If every $T_A, A \in \mathcal{A}$, is trace class, we will say that T is *trace class*.

T is said to be *decomposable* iff there exist a σ-finite measure ν on (S,Σ) and a family $\{T_x, x \in S\}$ of bounded non-negative self-adjoint operators on H such that for all $y \in H$ $x \mapsto T_x y$ is measurable and

$$T_A y = \int_A T_x y \nu(dx) \quad \text{for all } A \in \mathcal{A}, y \in H.$$

Let M be a martingale measure and $T = \{T_A, A \in \mathcal{A}\}$ a positive-operator valued measure which is trace class. Moreover let ϱ be a Radon measure on $(0,\infty)$. M is *nuclear* with (T,ϱ) iff

$$\mathbb{E}((M((s,t],A),x)_H(M((s,t],A),y)_H) = (x, T_A y)_H \varrho((s,t]) \tag{2.8}$$

for all $0 \leq s < t < \infty, A \in \mathcal{A}, x, y \in H$.

If T is decomposable then we call M *decomposable*.

Proposition 2.5.1 (cf. [App b] Proposition 2.1). *Let M be an orthogonal martingale measure, nuclear with (T,ϱ). Then*

$$\mathbb{E}(\langle M\rangle(t,A)) = tr(T_A)\varrho((0,t]) \quad \textit{for all } t \geq 0, A \in \mathcal{A}.$$

Proof. Let $\{e_n\}$ be an orthonormal basis of H. Then

$$\begin{aligned}
\mathbb{E}(\langle M\rangle(t,A)) &= \mathbb{E}(\|M(t,A)\|^2) = \mathbb{E}\left(\sum_{n=1}^{\infty} |(M(t,A),e_n)_H|^2\right) \\
&= \sum_{n=1}^{\infty} \mathbb{E}(|(M((0,t],A),e_n)_H|^2) = \sum_{n=1}^{\infty} (e_n, T_A e_n)_H \varrho((0,t]) \\
&= tr(T_A)\varrho((0,t]).
\end{aligned}$$

□

Now as above let $S = \{x \in H; \|x\| < 1\}$ and $\mathcal{A}_0 = \{A \in \mathfrak{B}(S); 0 \notin \bar{A}\}$, $\mathcal{A} = \mathcal{A}_0 \cup \{A \cup \{0\}; A \in \mathcal{A}_0\}$. Note that (S,Σ) with $\Sigma := \mathfrak{B}(S)$ is a Lusin topological space and for $S_n := \{x \in S; \frac{1}{n} \leq \|x\| < 1\}$ we have

$$S = \bigcup_{n\in\mathbb{N}} S_n$$

and

$$\Sigma_n := \mathfrak{B}(S_n) \subset \mathcal{A}.$$

Theorem 2.5.2. *Let X be a Lévy process and consider its Lévy Itô decomposition (2.7). The set function M given by*

$$M(t, A) = B_Q(t)\delta_0(A) + \int_{A\setminus\{0\}} x\tilde{N}(t, dx) \tag{2.9}$$

for all $t \geq 0, A \in \mathcal{A}$ is an orthogonal martingale measure with independent increments.

For the proof we need the following Lemma

Lemma 2.5.3. *Let $A, B \in \mathcal{A}$ be disjoint and X, Y be given by*

$$X(t) = B_Q(t)\delta_0(A) + \int_{A\setminus\{0\}} f(x)\tilde{N}(t, dx)$$
$$Y(t) = B_Q(t)\delta_0(B) + \int_{B\setminus\{0\}} g(x)\tilde{N}(t, dx)$$

for some $f, g \in L^2_\lambda$ and B_Q, $\tilde{N}$ as given in Theorem 2.5.2. Then for all $t \geq 0$ and an orthonormal basis $\{e_n\}$ of H

$$\mathbb{E}((X(t), e_n)_H (Y(t), e_m)_H) = 0 \quad \textit{for all } n, m \in \mathbb{N}.$$

In particular,

$$\mathbb{E}((X(t), Y(t))_H) = 0.$$

Proof. Fix $n, m \in \mathbb{N}$. Define

$$X_1(t) := B_Q(t)\delta_0(A)$$
$$X_2(t) := \int_{A\setminus\{0\}} f(x)\tilde{N}(t, dx)$$

and in the same way Y_1, Y_2. First let $f, g \in \mathcal{S}(H)$. Then X_1 is independent of Y_2 and Y_1 independent of X_2. But then since A, B are disjoint, $(X_1(t), e_n)_H (Y_1(t), e_m)_H = 0$ and thereby

$$\begin{aligned}
& \mathbb{E}((X(t), e_n)_H (Y(t), e_m)_H) \\
= \; & \mathbb{E}((X_1(t) + X_2(t), e_n)_H (Y_1(t) + Y_2(t), e_m)_H) \\
= \; & \mathbb{E}((X_1(t), e_n)_H)\mathbb{E}((Y_2(t), e_m)_H) + \mathbb{E}((X_2(t), e_n)_H)\mathbb{E}((Y_1(t), e_m)_H) \\
+ \; & \mathbb{E}((X_2(t), e_n)_H (Y_2(t), e_m)_H) \\
= \; & \mathbb{E}((X_2(t), e_n)_H (Y_2(t), e_m)_H).
\end{aligned}$$

But for f and g of the form (2.4) by Lemma 2.3.9 we have

$$\begin{aligned}
&\mathbb{E}((X_2(t), e_n)_H (Y_2(t), e_m)_H) \\
= \;& \mathbb{E}\left(\sum_{j=1}^{N}(a_j, e_n)_H \tilde{N}(t, A_j \cap A) \sum_{k=1}^{N}(a_k, e_m)_H \tilde{N}(t, A_k \cap B)\right) \\
= \;& \sum_{j,k=1}^{N}(a_j, e_n)_H (a_k, e_m)_H \mathbb{E}(\tilde{N}(t, A_j \cap A)\tilde{N}(t, A_k \cap B)) = 0.
\end{aligned}$$

Now let $f, g \in L^2_\lambda$ be arbitrary. Then there exist sequences $\{f_N\} \subset \mathcal{S}(H)$ and $\{g_N\} \subset \mathcal{S}(H)$ with $f_N \to f$ and $g_N \to g$ in L^2_λ. Hence if we write

$$\begin{aligned}
X_2^N(t) &:= \int_{A\setminus\{0\}} f_N(x)\tilde{N}(t, dx) \\
Y_2^N(t) &:= \int_{B\setminus\{0\}} g_N(x)\tilde{N}(t, dx),
\end{aligned}$$

then $X_2^N(t)$ converges to $X_2(t)$ and $Y_2^N(t)$ to $Y_2(t)$ in $L^2(H)$. With the same arguments as in the proof of Prop. 2.4.4 (Cauchy-Schwartz and Hölder's inequality) we obtain

$$\mathbb{E}((X_2(t), e_n)_H (Y_2(t), e_m)_H) = \lim_{N\to\infty} \mathbb{E}((X_2^N(t), e_n)_H (Y_2^N(t), e_m)_H) = 0$$

and also

$$\begin{aligned}
&\mathbb{E}((X(t), e_n)_H (Y(t), e_m)_H) \\
= \;& \mathbb{E}((X_1(t), e_n)_H)\mathbb{E}((Y_2(t), e_m)_H) + \mathbb{E}((X_2(t), e_n)_H)\mathbb{E}((Y_1(t), e_m)_H) \\
= \;& 0.
\end{aligned}$$

□

Proof of Theorem 2.5.2. By Corollary 1.4.12 $\lambda|_{A\setminus\{0\}}$ is again a Lévy measure and the Lévy process with generating triplet $[0, \delta_0(A)tQ, t\lambda|_{A\setminus\{0\}}]$ can by the Lévy Itô decomposition be written as $M(\cdot, A)$. Hence $M(\cdot, A)$ has independent increments.

We are going to show that M is a martingale measure:

1. Clearly $M(0, A) = M(t, \emptyset) = 0$ for all $A \in \mathcal{A}, t \geq 0$.
2. M is finitely additive since for disjoint sets $A, B \in \mathcal{A}$ one obtains

$$\begin{aligned}
&M(t, A) + M(t, B) \\
= \;& B_Q(t)\delta_0(A \cup B) + \int_{A\setminus\{0\}} x\tilde{N}(t, dx) + \int_{B\setminus\{0\}} x\tilde{N}(t, dx) \\
= \;& M(t, A \cup B).
\end{aligned}$$

3. Next we will show that M is σ-finite. Simply because $A \setminus \{0\}$ and $\{0\}$ are disjoint we can apply Lemma 2.5.3 to

$$\begin{aligned} M_1(t,A) &:= B_Q(t)\delta_0(A) \\ M_2(t,A) &:= \int_{A\setminus\{0\}} x\tilde{N}(t,dx). \end{aligned}$$

We have

$$\begin{aligned} & \mathbb{E}(\|M(t,A)\|^2) = \mathbb{E}(\|M_1(t,A) + M_2(t,A)\|^2) \\ = \; & \mathbb{E}(\|M_1(t,A)\|^2) + 2\mathbb{E}((M_1(t,A), M_2(t,A))_H) + \mathbb{E}(\|M_2(t,A)\|^2) \\ = \; & \mathbb{E}(\|M_1(t,A)\|^2) + \mathbb{E}(\|M_2(t,A)\|^2). \end{aligned}$$

But by Proposition 2.2.1

$$\begin{aligned} & \sup\{\mathbb{E}(\|B_Q(t)\delta_0(A)\|^2), A \in \Sigma_n\} \leq \mathbb{E}(\|B_Q(t)\|^2) \\ = \; & \sum_{k=1}^{\infty} (tQe_k, e_k)_H = tr(tQ) < \infty. \end{aligned}$$

And by Proposition 2.4.5 for each $A \in \mathcal{A}$

$$\begin{aligned} & \mathbb{E}\left(\left\|\int_{A\setminus\{0\}} x\tilde{N}(t,dx)\right\|^2\right) = t\int_{A\setminus\{0\}} \|x\|^2 \lambda(dx) \\ \leq \; & t\int_S \|x\|^2\lambda(dx) < \infty. \end{aligned}$$

4. M is also countably additive. Let $n \in \mathbb{N}$ and $\{A_j\} \subset \Sigma_n$ be a sequence decreasing to $\emptyset$. Then

$$\begin{aligned} & \lim_{j\to\infty} \mathbb{E}(\|M(t,A_j)\|^2) \\ = \; & \lim_{j\to\infty} (\mathbb{E}(\|M_1(t,A_j)\|^2) + \mathbb{E}(\|M_2(t,A_j)\|^2)) \\ = \; & \lim_{j\to\infty} \left(\delta_0(A_j)\, tr(tQ) + t\int_{A_j\setminus\{0\}} \|x\|^2\lambda(dx)\right) \\ \leq \; & \lim_{j\to\infty} (\delta_0(A_j)\, tr(tQ) + t\lambda(A_j \setminus \{0\})) = 0. \end{aligned}$$

5. By Corollary 2.4.8 $M(\cdot, A)$ is a strongly càdlàg square-integrable martingale for every $A \in \mathcal{A}$.

Let $\{e_n\}$ be an orthonormal basis of H. With $M(\cdot,A)$ also $(M(\cdot,A), e_n)_H$ is a martingale for every $n \in \mathbb{N}$. As a Lévy process $M(\cdot, A)$ has stationary and independent increments and therefore $(M(\cdot, A), e_n)_H$ has the same property. Then M is also orthogonal since by Lemma 2.5.3

$$\begin{aligned}
& \mathbb{E}((M(t,A),e_n)_H(M(t,B),e_m)_H - (M(s,A),e_n)_H(M(s,B),e_m)_H|\mathcal{F}_s) \\
= \; & \mathbb{E}((M(t,A)-M(s,A),e_n)_H(M(t,B)-M(s,B),e_m)_H|\mathcal{F}_s) \\
= \; & \mathbb{E}((M(t,A)-M(s,A),e_n)_H(M(t,B)-M(s,B),e_m)_H) \\
= \; & \mathbb{E}((M(t-s,A),e_n)_H(M(t-s,B),e_m)_H) = 0,
\end{aligned}$$

if $A, B \in \mathcal{A}$ disjoint. □

The orthogonal martingale measure in (2.9) is called *Lévy martingale measure.*

Proposition 2.5.4. *Let M be a Lévy martingale measure given by*

$$M(t,A) = B_Q(t)\delta_0(A) + \int_{A\setminus\{0\}} x\tilde{N}(t,dx) \quad \text{for all } t \geq 0, A \in \mathcal{A}.$$

Then M is nuclear with (T, ϱ) where ϱ is the Lebesgue measure on $\mathbb{R}_+$ and $T = \{T_A; A \in \mathcal{A}\}$ with

$$T_A y = Qy\delta_0(A) + \int_{A\setminus\{0\}} (x,y)_H x\lambda(dx).$$

In particular, T is decomposable with $\nu = \lambda + \delta_0$ and

$$T_x = \begin{cases} Q & \text{if } x = 0 \\ (x,\cdot)_H x & \text{if } x \neq 0. \end{cases}$$

Proof. Let the family $F = \{F_A; A \in \mathcal{A}_0\}$ of linear operators on H be given by

$$F_A y = \int_A (x,y)_H x\lambda(dx).$$

First we will show that F with $F_\emptyset = 0$ and $F_{\{0\}} = 0$ is a positive-operator valued measure on (S, Σ). For any $A \in \mathcal{A}_0$ the operator F_A is bounded since

$$\begin{aligned}
\|F_A\| &= \sup_{\|y\|\leq 1}\left\|\int_A (x,y)_H x\lambda(dx)\right\| \leq \sup_{\|y\|\leq 1}\int_A |(x,y)_H|\|x\|\lambda(dx) \\
&\leq \int_A \left|\left(x, \frac{x}{\|x\|}\right)_H\right| \|x\|\lambda(dx) = \int_A \|x\|^2\lambda(dx) < \infty.
\end{aligned}$$

Moreover, it is non-negative:

$$(F_A y, y)_H = \left(\int_A (x,y)_H x\lambda(dx), y\right)_H = \int_A |(x,y)_H|^2\lambda(dx) \geq 0.$$

And for any $y, z \in H$

$$\begin{aligned}(F_A y, z)_H &= \left(\int_A (x,y)_H x\lambda(dx), z\right)_H = \int_A (x,y)_H (x,z)_H \lambda(dx) \\ &= \left(y, \int_A (x,z)_H x\lambda(dx)\right)_H = (y, F_A z)_H,\end{aligned}$$

hence F_A is self-adjoint. Clearly $F_{A\cup B} = F_A + F_B$ for disjoint $A, B \in \mathcal{A}_0$.

By Propositions 2.2.1 and 2.2.2 Q is non-negative, self-adjoint and trace class.

Observing that

$$T_A = Q\delta_0(A) + F_{A\setminus\{0\}} \quad \text{for all } A \in \mathcal{A},$$

we obtain that T is a positive-operator valued measure.

We will now find that T is trace class. Since we already know that Q is trace class we only have to show this for F_A. Let $\{e_n\}$ be an orthonormal basis of H then by Corollary 1.4.12

$$\begin{aligned}tr(F_A) &= \sum_{n=1}^{\infty}(e_n, F_A e_n)_H = \sum_{n=1}^{\infty}\int_A |(x,e_n)_H|^2 \lambda(dx) \\ &= \int_A \|x\|^2 \lambda(dx) < \infty.\end{aligned}$$

The decomposability of T follows as $T_0 = Q$ and

$$F_{A\setminus\{0\}} y = \int_{A\setminus\{0\}} (x,y)_H x\lambda(dx) = \int_{A\setminus\{0\}} T_x y\lambda(dx).$$

It remains to show that M fulfills (2.8), i.e. for every $y_1, y_2 \in H$ and $A \in \mathcal{A}$

$$\mathbb{E}((M((s,t],A),y_1)_H (M((s,t],A),y_2)_H) = (t-s)(y_1, T_A y_2)_H.$$

As $M(\cdot, A)$ has stationary increments it is sufficient to show this for $s = 0$. By Lemma 2.5.3 for $M_1(t) := B_Q(t)\delta_0(A)$ and $M_2(t) := \int_{A\setminus\{0\}} x\tilde{N}(t,dx)$

$$\begin{aligned}&\mathbb{E}((M(t,A),y_1)_H (M(t,A),y_2)_H) \\ =& \sum_{i,j=1,2} \mathbb{E}((M_i(t,A),y_1)_H (M_j(t,A),y_2)_H) \\ =& \ \mathbb{E}((M_1(t,A),y_1)_H (M_1(t,A),y_2)_H) \\ +& \ \mathbb{E}((M_2(t,A),y_1)_H (M_2(t,A),y_2)_H).\end{aligned}$$

But by Prop. 2.2.1 and Corollary 2.3.2

$$\mathbb{E}((B_Q(t),y_1)_H (B_Q(t),y_2)_H) = t(y_1, Qy_2)_H,$$

and with Prop. 2.4.4 for every $A \in \mathcal{A}_0$

$$\begin{aligned}
&\mathbb{E}\left(\left(\int_A x\tilde{N}(t,dx), y_1\right)_H \left(\int_A x\tilde{N}(t,dx), y_2\right)_H\right) \\
= \quad & t\int_A (x,y_1)_H (x,y_2)_H \lambda(dx) \\
= \quad & t\left(y_1, \int_A (x,y_2)_H x\lambda(dx)\right)_H = t(y_1, F_A y_2)_H.
\end{aligned}$$

Hence (2.8) follows. □

Chapter 3

Stochastic Integration

We will now introduce stochastic integrals with respect to nuclear martingale measures. For the real-valued case this was done in [Wal]. The construction in the H-valued situation follows the ideas of [App b]. However, not all the proofs are given there which we will therefore state in this diploma thesis. In section 3.1 we begin with the *strong stochastic integral*. This is an H-valued stochastic integral where the integrands are operator-valued mappings. As usual the stochastic integral is first defined for simple functions by an isometry. These simple functions are dense in a certain space $\mathcal{N}_2(T)$ on which the stochastic integral can be defined by L^2-limits. The proof that these simple functions are dense in $\mathcal{N}_2(T)$ is not given in [App b] and therefore a proof is given here. We are able to show this by a Hilbert space argument. By a linear extension of the isometry the integral is then defined on the whole space. Finally, we show that the obtained stochastic integral process is a strongly càdlàg martingale.

In section 3.2 the *weak stochastic integral* is defined as a real valued process. The construction is quite similar to the one for the strong integral. The integrands are H-valued functions and for simple functions the integral is an inner product in the Hilbert space H. We show that the weak stochastic integral can also be defined by an isometry and thereby obtain a larger class of integrands than in [App b]. There the integral was constructed with a contracting mapping into a smaller space of integrands instead of the isometry. The last section of this chapter is devoted to properties of the stochastic integral. Very central is the connection between weak and strong stochastic integrals which will be very useful in Chapter 4. Moreover, we show a Fubini theorem for strong stochastic and deterministic integrals. If the integrands are compensated Poisson random measures, such a theorem can also be found in [App b]. We prove the result more generally for arbitrary martingale measures.

In the whole chapter let M be an orthogonal martingale measure with independent increments, nuclear with (T, ϱ) and decomposable with $T_A y =$

$\int_A T_x y\nu(dx)$. (One could for example keep in mind the Lévy martingale measure). Note that then (S, Σ) is some Lusin topological space. Let the time $\tilde{T} > 0$ be fixed.

3.1 Strong Stochastic Integrals

Next we will define the H-valued *strong stochastic integral* of some operator-valued mapping with respect to the martingale measure M.

By $L(H)$ we denote the space of all bounded linear operators on H with operator norm $\|T\| := \sup_{\|x\|\le 1} \|Tx\|$. $(L(H), \|\cdot\|)$ is a Banach space (compare [ReeSim] Thm. III.2.). However the *norm topology* generated by $\|\cdot\|$ is too strong for our purposes. Therefore we will consider the *strong topology* on $L(H)$, i.e.

$$R_n \xrightarrow{s} R \quad \text{iff} \quad R_n h \to Rh \text{ for all } h \in H.$$

The corresponding Borel σ-algebra $\mathcal{L}$ is generated by sets of the form

$$\{R \in L(H); Rh \in A\} \quad \text{with } h \in H, A \in \mathfrak{B}(H).$$

Let $(G, \mathcal{G})$ be a measurable space. Then an operator-valued mapping $R : G \to L(H)$ is $\mathcal{G}/\mathcal{L}$-measurable or *strongly measurable* iff $R(\cdot)h$ is $\mathcal{G}/\mathfrak{B}(H)$-measurable for all $h \in H$. Given a measure μ on $(G, \mathcal{G})$ we can construct a Bochner integral for such operator-valued mappings: (compare [DaPrZa], p.24) $R : G \to L(H)$ is *Bochner integrable* iff for any $h \in H$ the function $R(\cdot)h$ is Bochner integrable and there exists an operator $S \in L(H)$ such that

$$\int_G Rh d\mu = Sh \quad \text{for all } h \in H.$$

Then we write

$$\int_G R d\mu := S.$$

Since H is separable we have (with $\{e_k\}$ being an orthonormal basis of H)

$$\left\| \int_G R d\mu \right\| = \sup_{k\in\mathbb{N}} \left\| \int_G Re_k d\mu \right\| \le \int_G \sup_{k\in\mathbb{N}} \|Re_k\| d\mu = \int_G \|R\| d\mu.$$

We will also consider the following class of operators: An operator S is *Hilbert-Schmidt* iff $tr(S^*S) < \infty$. The space $L_2(H)$ of all Hilbert-Schmidt operators with inner product $(S_1, S_2)_2 := tr(S_1^* S_2)$ and induced norm $\|S\|_2 = \sqrt{tr(S^*S)}$ is a separable Hilbert space and a two sided $L(H)$-ideal (cf. [Wei] p.133/34): Let $C \in L_2(H)$, $S_1, S_2 \in L(H)$ then

$$\|S_1 C S_2\|_2 \le \|S_1\| \|S_2\| \|C\|_2.$$

Moreover $\|C\| \leq \|C\|_2$. $L_2(H)$ is a strongly measurable subset of $L(H)$ (cf. [DaPrZa] p.25). For simplicity we will also denote the trace σ-algebra of $\mathcal{L}$ on $L_2(H)$ by $\mathcal{L}$. For $1 \leq p < \infty$ we can now define

$$\begin{aligned} & L^p((G,\mathcal{G},\mu) \to (L_2(H),\mathcal{L})) \\ := \; & \{R : G \to L_2(H); R \text{ is } \mathcal{L}\text{-measurable and } \|R\|_p < \infty\}, \end{aligned}$$

where

$$\|R\|_p := \left(\int_G \|R\|_2^p d\mu \right)^{\frac{1}{p}}$$

and $R_1 = R_2$ in $L^p((G,\mathcal{G},\mu) \to (L_2(H),\mathcal{L}))$ iff $R_1 h = R_2 h$ for all $h \in H$ μ-a.s. Of course the Riesz-Fischer Theorem is also valid for $L^p((G,\mathcal{G},\mu) \to (L_2(H),\mathcal{L}))$.

For $0 \leq t \leq \tilde{T}$ define $\mathcal{N}_2(T;t) := \mathcal{N}_2(T;\nu,\varrho;t)$ as the space of all operator valued mappings R on $[0,t] \times \Omega \times S$ such that $(s,\omega,x) \mapsto R(s,\omega,x)y$ is $\mathcal{P}\otimes\Sigma$-measurable for every $y \in H$ and

$$\|R\|_{\mathcal{N}_2(T;t)} := \mathbb{E}\left(\int_0^t \int_S \left\| R(s,x)\sqrt{T_x} \right\|_2^2 \nu(dx)\varrho(ds) \right)^{\frac{1}{2}} < \infty.$$

(The square root of the bounded non-negative and self-adjoint operators T_x exists as described on page 30 and is also selfadjoint, cf. [Wei] p.186, Satz 7.20a). Two mappings $R_1, R_2 \in \mathcal{N}_2(T;t)$ are defined to be equal in $\mathcal{N}_2(T;t)$ if

$$\left\| (R_1(s,x) - R_2(s,x))\sqrt{T_x} \right\|_2 = 0 \quad \varrho \otimes P \otimes \nu\text{-a.e..}$$

For $\mathcal{N}_2(T;\tilde{T})$ just write $\mathcal{N}_2(T)$.

Lemma 3.1.1. $\mathcal{N}_2(T)$ *with inner product*

$$(R_1,R_2)_{\mathcal{N}_2(T)} := \mathbb{E}\left(\int_0^{\tilde{T}} \int_S tr(R_1(s,x)T_x R_2(s,x)^*)\nu(dx)\varrho(ds) \right)$$

is a Hilbert space.

Proof. Let $\{R_n\}$ be a Cauchy sequence in $\mathcal{N}_2(T)$. Then by the Riesz-Fischer Theorem there exist some $S \in L^2(([0,\tilde{T}] \times \Omega \times S, \mathcal{P}\otimes\Sigma, \varrho\otimes P\otimes\nu) \to (L_2(H),\mathcal{L}))$ and a subsequence $\{R_{n'}\}$ such that $R_{n'}(s,\omega,x)\sqrt{T_x}$ converges to $S(s,\omega,x)$ in $\|\cdot\|_2$ $\varrho\otimes P\otimes\nu$-a.e. For (s,ω,x) fixed choose an orthonormal basis $\{e_k\}$ of H where each e_k is either in $ker(\sqrt{T_x})$ or in its orthogonal complement $(ker(\sqrt{T_x}))^\perp$ (where $ker(\sqrt{T_x}) := \{y \in H; \sqrt{T_x}y = 0\}$). Now define

$$R(s,\omega,x)h := \begin{cases} S(s,\omega,x)\sqrt{T_x}^{-1}h & \text{if } h \in \sqrt{T_x}(H) \\ 0 & \text{if } h \in \left(\sqrt{T_x}(H)\right)^\perp, \end{cases}$$

where

$$\sqrt{T_x}^{-1} : \sqrt{T_x}\left(\left(ker(\sqrt{T_x})\right)^{\perp}\right) = \sqrt{T_x}(H) \to \left(ker\left(\sqrt{T_x}\right)\right)^{\perp}$$

is the pseudo inverse of $\sqrt{T_x}$. But then

$$\begin{aligned}
&\left\|(R_n(s,\omega,x) - R(s,\omega,x))\sqrt{T_x}\right\|_2^2 \\
= &\sum_{k=1}^{\infty}\left\|(R_n(s,\omega,x) - R(s,\omega,x))\sqrt{T_x}e_k\right\|^2 \\
= &\sum_{k=1}^{\infty}\left\|R_n(s,\omega,x)\sqrt{T_x}e_k - S(s,\omega,x)\sqrt{T_x}^{-1}\sqrt{T_x}e_k\right\|^2 \\
= &\sum_{e_k\in(ker(\sqrt{T_x}))^{\perp}}\left\|R_n(s,\omega,x)\sqrt{T_x}e_k - S(s,\omega,x)e_k\right\|^2 \\
\leq &\sum_{k=1}^{\infty}\left\|(R_n(s,\omega,x)\sqrt{T_x} - S(s,\omega,x))e_k\right\|^2 \\
= &\left\|R_n(s,\omega,x)\sqrt{T_x} - S(s,\omega,x)\right\|_2^2.
\end{aligned}$$

Hence $\{R_{n'}(s,\omega,x)\sqrt{T_x}\}$ converges to $R(s,\omega,x)\sqrt{T_x}$ in $\|\cdot\|_2$ $\varrho\otimes P\otimes\nu$-a.e. and therefore $\{R_n\}$ converges to R in $\mathcal{N}_2(T)$. □

The construction of the integral is started from simple functions. Let $\mathcal{S}_2(T) := \mathcal{S}_2(T;\nu,\varrho)$ be the subspace of all $R \in \mathcal{N}_2(T)$ which are of the form

$$R = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} R_{ij}1_{(t_i,t_{i+1}]}1_{A_j}, \tag{3.1}$$

where $N_1, N_2 \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_{N_1+1} = \tilde{T}$. Here $A_0, \ldots, A_{N_2+1} \in \mathcal{A}$ are disjoint sets (which by being in $\mathcal{A}$ have finite measure) and each R_{ij} is an $\mathcal{F}_{t_i}/\mathcal{L}$-measurable random variable taking values in $L(H)$ (equivalently $R_{ij}h$ is $\mathcal{F}_{t_i}$-measurable for each $h \in H$).

Lemma 3.1.2. *The subspace $\mathcal{S}_2(T)$ is dense in $\mathcal{N}_2(T)$.*

Proof. Let $\{e_n\}$ be an orthonormal basis of H and for $k, l \in \mathbb{N}$ the operator $S_{kl} \in L(H)$ shall be defined by

$$S_{kl}e_n = \begin{cases} e_l & \text{if } n = k \\ 0 & \text{if } n \neq k. \end{cases}$$

Then the adjoint of S_{kl} is $S_{kl}^* = S_{lk}$. Note that

$$tr(S_{kl}T_xS_{lk}) \leq tr(T_x)$$

and hence the mapping with constant value S_{kl} is in $\mathcal{N}_2(T)$. Consider the simple function $S \in \mathcal{S}_2(T)$ given by

$$S(s,\omega,x) = 1_B(s)1_F(\omega)1_A(x)S_{kl},$$

where $B = (t_1, t_2]$ with $t_1, t_2 \in [0, \tilde{T}]$, $A \in \mathcal{A}$ and $F \in \mathcal{F}_{t_i}$. Now choose a mapping $R \in (\mathcal{S}_2(T))^\perp$ (which shall denote the orthogonal complement of $\mathcal{S}_2(T)$ in $\mathcal{N}_2(T)$). Then $(R,S)_{\mathcal{N}_2(T)} = 0$ and thereby

$$\begin{aligned}
& \mathbb{E}\left(\int_0^{\tilde{T}} \int_S tr(R(s,x)T_x S(s,x)^*)\nu(dx)\varrho(ds)\right) \\
= & \mathbb{E}\left(1_F \int_B \int_A \sum_{n=1}^{\infty} (R(s,x)T_x S_{lk} e_n, e_n)_H \nu(dx)\varrho(ds)\right) \\
= & \mathbb{E}\left(1_F \int_B \int_A (R(s,x)T_x e_k, e_l)_H \nu(dx)\varrho(ds)\right) \\
= & 0.
\end{aligned}$$

For any $G \in \mathcal{P} \otimes \Sigma$ set

$$\mu(G) := \int_G (R(s,x)T_x e_k, e_l)_H d(\varrho \otimes P \otimes \nu).$$

Then this defines a signed measure μ on $\mathcal{P} \otimes \Sigma$. Now $\mu(G) = 0$ for every $G \in \mathcal{P} \otimes \Sigma$ of the type $B \times F \times A$. Since the system of all sets of this type is closed against intersection and generates the σ-algebra $\mathcal{P} \otimes \Sigma$ we obtain $\mu = 0$ on $\mathcal{P} \otimes \Sigma$. Hence $(R(s,x)T_x e_k, e_l)_H = 0$ $\varrho \otimes P \otimes \nu$-a.e. for any $k, l \in \mathbb{N}$ and therefore $R(s,x)T_x = 0$ $\varrho \otimes P \otimes \nu$-a.e. But then

$$\begin{aligned}
\|R\|^2_{\mathcal{N}_2(T)} &= \mathbb{E}\left(\int_0^{\tilde{T}} \int_S \|R(s,x)\sqrt{T_x}\|_2^2 \nu(dx)\varrho(ds)\right) \\
&= \mathbb{E}\left(\int_0^{\tilde{T}} \int_S tr(R(s,x)T_x R(s,x)^*)\nu(dx)\varrho(ds)\right) \\
&= 0
\end{aligned}$$

and $\mathcal{S}_2(T)^\perp = \{0\}$. □

Now define for $0 \le t \le \tilde{T}$ and every $R \in \mathcal{S}_2(T)$ (like in (3.1))

$$J_t(R) = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} R_{ij} M((t \wedge t_i, t \wedge t_{i+1}], A_j). \tag{3.2}$$

It is easy to show that $J_t(R)$ is independent of the representation of R.

Obviously, all results from above are also valid for $\mathcal{N}_2(T;t)$ for $0 \le t \le \tilde{T}$ and $R_{|(0,t] \times S \times \Omega} \in \mathcal{N}_2(T;t)$ for every $R \in \mathcal{N}_2(T)$.

Proposition 3.1.3 (cf. [App b] p. 11/12). *J_t, given by (3.2) for every $R \in \mathcal{S}_2(T)$, can be extended to an isometry from $\mathcal{N}_2(T;t)$ to $L^2(H) := L^2((\Omega, \mathcal{F}, P) \to H)$.*

Proof. Let $R \in \mathcal{S}_2(T)$ be given by (3.1). Notice that with R_{ij} also R_{ij}^* is $\mathcal{F}_{t_i}/\mathcal{L}$-measurable. Since

$$\mathbb{E}(\|J_t(R)\|^2) = \mathbb{E}\left(\left\|\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} R_{ij}M((t\wedge t_i, t\wedge t_{i+1}], A_j)\right\|^2\right),$$

we will study the individual terms under the sum. For $i < k$ such that $t_i, t_k < t$ we obtain

$$\begin{aligned}
&\mathbb{E}((R_{ij}M((t_i, t\wedge t_{i+1}], A_j), R_{kl}M((t_k, t\wedge t_{k+1}], A_l))_H) \\
=\ &\mathbb{E}((R_{kl}^* R_{ij}M((t_i, t\wedge t_{i+1}], A_j), \mathbb{E}(M((t_k, t\wedge t_{k+1}], A_l)|\mathcal{F}_{t_k}))_H) \\
=\ &0
\end{aligned}$$

by the martingale property of $M(\cdot, A_l)$. (Notice that this expression exists by Hölder's inequality). And for $j \neq l$ and $N_{ij} := M((t_i, t\wedge t_{i+1}], A_j)$, since M is orthogonal, $(N_{ij}, e_n)_H(e_m, N_{il})_H$ is a martingale for every $n, m \in \mathbb{N}$ and

$$\begin{aligned}
&\mathbb{E}((R_{ij}M((t_i, t\wedge t_{i+1}], A_j), R_{il}M((t_i, t\wedge t_{i+1}], A_l))_H) \\
=\ &\mathbb{E}\left(\sum_{n=1}^{\infty}(R_{ij}N_{ij}, e_n)_H(e_n, R_{il}N_{il})_H\right) \\
=\ &\mathbb{E}\left(\sum_{n=1}^{\infty}(N_{ij}, R_{ij}^* e_n)_H(R_{il}^* e_n, N_{il})_H\right) \\
=\ &\sum_{n,m,r=1}^{\infty}\mathbb{E}((N_{ij}, e_m)_H(e_m, R_{ij}^* e_n)_H(R_{il}^* e_n, e_r)_H(e_r, N_{il})_H) \\
=\ &\sum_{n,m,r=1}^{\infty}\mathbb{E}((e_m, R_{ij}^* e_n)_H(R_{il}^* e_n, e_r)_H\mathbb{E}((N_{ij}, e_m)_H(e_r, N_{il})_H|\mathcal{F}_{t_i})) \\
=\ &0.
\end{aligned}$$

By the independent increments property of M and since M is nuclear we have

$$
\begin{aligned}
& \mathbb{E}(\|R_{ij}M((t_i, t\wedge t_{i+1}], A_j)\|^2) \\
= & \sum_{n,m,r=1}^{\infty} \mathbb{E}((e_m, R_{ij}^* e_n)_H (R_{ij}^* e_n, e_r)_H)\mathbb{E}((N_{ij}, e_m)_H (e_r, N_{ij})_H) \\
= & \sum_{n,m,r=1}^{\infty} \mathbb{E}((e_m, R_{ij}^* e_n)_H (R_{ij}^* e_n, e_r)_H)(e_r, T_{A_j} e_m)_H \varrho((t_i, t\wedge t_{i+1}]) \\
= & \sum_{m=1}^{\infty} \mathbb{E}((R_{ij}e_m, R_{ij}T_{A_j}e_m)_H)\varrho((t_i, t\wedge t_{i+1}]) \\
= & \mathbb{E}(tr(R_{ij}^* R_{ij} T_{A_j}))\varrho((t_i, t\wedge t_{i+1}]).
\end{aligned}
$$

Moreover, for every $A \in \mathcal{A}$ and some operator $S \in L(H)$

$$
\begin{aligned}
tr(ST_A) &= \sum_{n=1}^{\infty}(e_n, ST_A e_n)_H = \sum_{n=1}^{\infty}\left(S^* e_n, \int_A T_x e_n \nu(dx)\right)_H \\
&= \int_A tr(ST_x)\nu(dx).
\end{aligned}
$$

Using these calculations we find

$$
\begin{aligned}
& \mathbb{E}(\|J_t(R)\|^2) = \mathbb{E}\left(\left\|\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} R_{ij}M((t\wedge t_i, t\wedge t_{i+1}], A_j)\right\|^2\right) \\
= & \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\mathbb{E}(\|R_{ij}M((t\wedge t_i, t\wedge t_{i+1}], A_j)\|^2) \\
= & \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\mathbb{E}(tr(R_{ij}^* R_{ij}T_{A_j}))\varrho((t\wedge t_i, t\wedge t_{i+1}]) \\
= & \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\mathbb{E}\left(\int_{A_j} tr(R_{ij}^* R_{ij}T_x)\nu(dx)\right)\varrho((t\wedge t_i, t\wedge t_{i+1}]) \\
= & \mathbb{E}\left(\int_0^{\tilde{T}}\int_S \|1_{(0,t]}(s)R(s,x)\sqrt{T_x}\|_2^2 \nu(dx)\varrho(ds)\right) \\
= & \|R_{|(0,t]\times\Omega\times S}\|^2_{\mathcal{N}_2(T;t)}.
\end{aligned}
$$

Now let $R \in \mathcal{N}_2(T)$. Then R can be approximated in $\mathcal{N}_2(T)$ by a sequence $\{R_n\} \subset \mathcal{S}_2(T)$. Hence

$$
\lim_{n,m\to\infty}\mathbb{E}(\|J_t(R_n) - J_t(R_m)\|^2) = \lim_{n,m\to\infty}\|(R_n - R_m)_{|(0,t]\times\Omega\times S}\|^2_{\mathcal{N}_2(T;t)} = 0,
$$

and therefore $\{J_t(R_n)\}$ is a Cauchy sequence in $L^2(H)$. Now we can define $J_t(R)$ as the limit of $J_t(R_n)$ in $L^2(H)$. □

Now for any $R \in \mathcal{N}_2(T) = \mathcal{N}_2(T; \tilde{T})$ the *strong stochastic integral* of R with respect to the orthogonal and nuclear martingale measure M is defined as

$$\int_0^t \int_S R(s,x) M(ds,dx) := J_t(R)$$

for any $0 \le t \le \tilde{T}$.

Remark 3.1.4. A sufficient condition for R with $(s,\omega,x) \mapsto R(s,\omega,x)y$ is $\mathcal{P} \otimes \Sigma$-measurable for every $y \in H$ to be in $\mathcal{N}_2(T)$ is

$$\begin{aligned} \|R\|_{\mathcal{N}_2(T)} &= \mathbb{E}\left(\int_0^{\tilde{T}} \int_S \left\| R(s,x)\sqrt{T_x} \right\|_2^2 \nu(dx)\varrho(ds) \right)^{\frac{1}{2}} \\ &\le \mathbb{E}\left(\int_0^{\tilde{T}} \int_S \|R(s,x)\|^2 \, tr(T_x)\nu(dx)\varrho(ds) \right)^{\frac{1}{2}} < \infty. \end{aligned}$$

Theorem 3.1.5. *The process* $(\int_0^t \int_S R(s,x)M(ds,dx), t \ge 0)$ *is an* H*-valued strongly càdlàg square-integrable martingale. Furthermore*

$$\begin{aligned} & \mathbb{E}\left(\left\| \int_0^t \int_S R(s,x)M(ds,dx) \right\|^2 \right) \qquad (3.3) \\ = \; & \mathbb{E}\left(\int_0^{\tilde{T}} \int_S 1_{(0,t]}(s) R(s,x)\sqrt{T_x}\|_2^2 \nu(dx)\varrho(ds) \right). \end{aligned}$$

Proof. Let $R \in \mathcal{S}_2(T)$ be given by (3.1). For $s \le t$ set $i_0 := \max\{i; t_i \le s\}$. Without loss of generality we can assume that $t_{N_1} < t$ and have

$$\begin{aligned} & \mathbb{E}\left(\int_0^t \int_S R(s,x)M(ds,dx) \middle| \mathcal{F}_s \right) \\ = \; & \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} \mathbb{E}(R_{ij}M((t_i, t\wedge t_{i+1}], A_j)|\mathcal{F}_s) \\ = \; & \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\sum_{n=1}^{\infty} \mathbb{E}((M((t_i, t\wedge t_{i+1}], A_j), R_{ij}^* e_n)_H e_n|\mathcal{F}_s) \\ = \; & \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\sum_{n=1}^{\infty} \mathbb{E}((M((t_i, t\wedge t_{i+1}], A_j), R_{ij}^* e_n)_H|\mathcal{F}_s) e_n \\ = \; & \sum_{j=0}^{N_2}\sum_{i=0}^{i_0-1}\sum_{n=1}^{\infty} (M((t_i, t_{i+1}], A_j), R_{ij}^* e_n)_H e_n \\ + \; & \sum_{j=0}^{N_2}\sum_{n=1}^{\infty} (\mathbb{E}(M((t_{i_0}, t\wedge t_{i_0+1}], A_j)|\mathcal{F}_s), R_{i_0 j}^* e_n)_H e_n \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=0}^{N_2} \sum_{i=i_0+1}^{N_1} \sum_{n=1}^{\infty} \mathbb{E}((\mathbb{E}(M((t_i, t \wedge t_{i+1}], A_j)|\mathcal{F}_{t_i}), R_{ij}^* e_n)_H | \mathcal{F}_s) e_n \\
&= \sum_{j=0}^{N_2} \left(\sum_{i=0}^{i_0-1} R_{ij} M((t_i, t_{i+1}], A_j) + R_{i_0 j} M((t_{i_0}, s], A_j) \right) \\
&= \int_0^s \int_S (F(s,x), M(ds, dx))_H.
\end{aligned}
$$

Now the result can be extended to any $R \in \mathcal{N}_2(T)$: R can be written as the limit of some $\{R_n\} \subset \mathcal{S}_2(T)$ in $\mathcal{N}_2(T)$. Then the corresponding integral can by Prop. 3.1.3 be written as a limes of martingales in $L^2(H)$, hence it is also a martingale. Again by Proposition 3 in [Kun] every square-integrable martingale is strongly càdlàg P-a.s.. □

3.2 Weak Stochastic Integrals

Apart from the strong stochastic integral another (real-valued) stochastic integral can be constructed. The *weak stochastic integral* of an H-valued function F with respect to M will be written as

$$\int_0^t \int_S (F(s,x), M(ds, dx))_H.$$

Remark 3.2.1. Alternatively to the construction which we will give in this section, the weak stochastic integral can also be obtained as a special case of the strong stochastic integral. In the last section we have constructed the strong stochastic integral for integrands R for which $R(s,\omega,x)$ is a linear operator from H into the same space. If the construction is generalized to integrands for which $R(s,\omega,x)$ is a linear operator from H to another Hilbert space $\tilde{H}$ then in particular, one can choose $\tilde{H} = \mathbb{R}$. Hence by the Riesz isomorphism one can construct a stochastic integral for H-valued functions which takes values in $\mathbb{R}$. For such a function $F : [0, \tilde{T}] \times \Omega \times S \to H$ we can identify $F(s,\omega,x) \in H$ with the operator $S_{F(s,\omega,x)}$ defined by

$$S_{F(s,\omega,x)} y = (F(s,\omega,x), y)_H$$

for any $y \in H$. Let $L_2(H,\mathbb{R})$ denote the space of all Hilbert-Schmidt operators from H to $\mathbb{R}$ and $\|\cdot\|_{L_2(H,\mathbb{R})}$ the corresponding Hilbert-Schmidt norm. Then for an orthonormal basis $\{e_k\}$ of H we have

$$\|S_{F(s,\omega,x)} \sqrt{T_x}\|^2_{L_2(H,\mathbb{R})} = \sum_{k=1}^{\infty} \left|(F(s,\omega,x), \sqrt{T_x} e_k)_H\right|^2$$

$$= \sum_{k=1}^{\infty} \left|(\sqrt{T_x} F(s,\omega,x), e_k)_H\right|^2 = \|\sqrt{T_x} F(s,\omega,x)\|^2.$$

Hence

$$\mathbb{E}\left(\int_0^{\tilde{T}}\int_S \left\|S_{F(s,x)}\sqrt{T_x}\right\|^2_{L_2(H,\mathbb{R})}\nu(dx)\varrho(ds)\right)^{\frac{1}{2}}$$

is equal to $\|F\|_{\mathcal{N}_2^w(T)}$ as it is defined in equation (3.4) below. Moreover, it can easily be seen that for simple functions F the definition of $I_t(F)$ in (3.6), i.e.

$$I_t(F) = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}(F_{ij}, M((t\wedge t_i, t\wedge t_{i+1}], A_j))_H,$$

coincides with the one for $J_t(R)$ in (3.2) if $R(s,\omega,x) = S_{F(s,\omega,x)}$. Taking L^2-limits yields that for the weak stochastic integral as constructed below and general integrands F we have

$$\int_0^t\int_S (F(s,x), M(ds,dx))_H = \int_0^t\int_S S_{F(s,x)}M(ds,dx).$$

We will now give an own construction for the weak stochastic integral.

The space of weakly stochastically integrable functions will be denoted by $N_2^w(T) := N_2^w(T;\nu,\varrho)$. It is the collection of all $F : [0,\tilde{T}]\times\Omega\times S\to H$ which are $\mathcal{P}\otimes\Sigma$-measurable and for which

$$\|F\|_{\mathcal{N}_2^w(T)} := \mathbb{E}\left(\int_0^{\tilde{T}}\int_S \left\|\sqrt{T_x}F(s,x)\right\|^2\nu(dx)\varrho(ds)\right)^{\frac{1}{2}} < \infty. \tag{3.4}$$

$F = G$ in $\mathcal{N}_2^w(T)$ iff $\|\sqrt{T_x}F(s,x) - \sqrt{T_x}G(s,x)\| = 0$ $\varrho\otimes P\otimes\nu$-a.e. As in section 3.1 $\mathcal{N}_2^w(T;t)$ is defined as the space of the restrictions on $[0,t]\times\Omega\times S$.

Lemma 3.2.2. *$\mathcal{N}_2^w(T)$ with inner product*

$$(F_1,F_2)_{\mathcal{N}_2^w(T)} := \mathbb{E}\left(\int_0^{\tilde{T}}\int_S \left(\sqrt{T_x}F_1(s,x), \sqrt{T_x}F_2(s,x)\right)_H\nu(dx)\varrho(ds)\right)$$

is a Hilbert space.

Proof. Let $\{F_n\}$ be a Cauchy sequence in $\mathcal{N}_2^w(T)$. Then with the Riesz-Fischer Theorem there exists a function $G\in L^2(\varrho\otimes P\otimes\nu) := L^2(([0,\tilde{T}]\times\Omega\times S,\mathcal{P}\otimes\Sigma,\varrho\otimes P\otimes\nu)\to H)$ such that $\sqrt{T.}F_n\to G$ in $L^2(\varrho\otimes P\otimes\nu)$. Since for fixed (s,x) $\sqrt{T_x}F_n(s,x)$ only takes on values in $\sqrt{T_x}(H)$, which is a closed subspace of H, this is also the case for $G(s,x)$. Hence we can define

$F(s,x) := \sqrt{T_x}^{-1} G(s,x)$. But then

$$\begin{aligned}
&\mathbb{E}\left(\int_0^{\tilde{T}}\int_S \left\|\sqrt{T_x}(F_n(s,x)-F(s,x))\right\|^2 \nu(dx)\varrho(ds)\right)\\
=\ &\mathbb{E}\left(\int_0^{\tilde{T}}\int_S \left\|\sqrt{T_x}F_n(s,x)-\sqrt{T_x}\sqrt{T_x}^{-1}G(s,x)\right\|^2 \nu(dx)\varrho(ds)\right)\\
=\ &\mathbb{E}\left(\int_0^{\tilde{T}}\int_S \left\|\sqrt{T_x}F_n(s,x)-G(s,x)\right\|^2 \nu(dx)\varrho(ds)\right).
\end{aligned}$$

Hence $F_n \to F$ in $\mathcal{N}_2^w(T)$. □

As before the construction of the stochastic integral will be done for simple functions first. As $\mathcal{S}_2^w(T)$ define the subspace of all $F \in \mathcal{N}_2^w(T)$ which are of the form

$$F = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} F_{ij} 1_{(t_i,t_{i+1}]} 1_{A_j} \tag{3.5}$$

with $N_1, N_2 \in \mathbb{N}, 0 = t_0 < t_1 < \cdots < t_{N_1+1} = \tilde{T}$, each F_{ij} being a bounded $\mathcal{F}_{t_i}$-measurable random variable and $A_0, \ldots, A_{N_2+1} \in \mathcal{A}$ disjoint.

Lemma 3.2.3. *The subspace $\mathcal{S}_2^w(T)$ is dense in $\mathcal{N}_2^w(T)$.*

Proof. The proof can be given along the lines of the proof of Lemma 3.1.2. Let $\{e_n\}$ be an orthonormal basis of H. For some $k \in \mathbb{N}$ define $G(s,\omega,x) := e_k 1_B(s) 1_C(\omega) 1_A(x)$ where $B = (t_1,t_2]$ with $t_1,t_2 \in [0,\tilde{T}]$, $A \in \mathcal{A}$ and $C \in \mathcal{F}_{t_i}$. Choose some $F \in (\mathcal{S}_2^w(T))^\perp$. Then

$$\begin{aligned}
0 &= (F,G)_{\mathcal{N}_2^w(T)}\\
&= \mathbb{E}\left(\int_0^{\tilde{T}}\int_S \left(\sqrt{T_x}F(s,x), \sqrt{T_x}G(s,x)\right)_H \nu(dx)\varrho(ds)\right)\\
&= \mathbb{E}\left(1_C \int_B\int_A (T_xF(s,x), e_k)_H \nu(dx)\varrho(ds)\right).
\end{aligned}$$

Again

$$\mu(G) := \int_G (T_xF(s,x), e_k)_H d(\varrho \otimes P \otimes \nu), \quad G \in \mathcal{P} \otimes \Sigma$$

defines a signed measure on $\mathcal{P} \otimes \Sigma$. As in Lemma 3.1.2 $(T_xF(s,x), e_k)_H = 0$ for all $k \in \mathbb{N}$ $\varrho \otimes P \otimes \nu$-a.s. Then

$$\begin{aligned}
\|F\|^2_{\mathcal{N}_2^w(T)} &= \mathbb{E}\left(\int_0^{\tilde{T}}\int_S (T_xF(s,x), F(s,x))_H \nu(dx)\varrho(ds)\right)\\
&= \mathbb{E}\left(\int_0^{\tilde{T}}\int_S \sum_{k=1}^{\infty} (T_xF(s,x), e_k)_H (e_k, F(s,x))_H \nu(dx)\varrho(ds)\right)\\
&= 0,
\end{aligned}$$

hence $(\mathcal{S}_2^w(T))^\perp = \{0\}$. □

For $F \in \mathcal{S}_2^w(T)$ and $0 \le t \le \tilde{T}$ define

$$I_t(F) = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} (F_{ij}, M((t \wedge t_i, t \wedge t_{i+1}], A_j))_H. \tag{3.6}$$

It is again easy to show that this is independent of the representation of F.

Proposition 3.2.4. *I_t, for $F \in \mathcal{S}_2^w(T)$ given by (3.6), can be extended to an isometry from $\mathcal{N}_2^w(T;t)$ to $L^2(\Omega, \mathcal{F}, P)$.*

Proof. Let $F \in \mathcal{S}_2^w(T)$ be like in (3.5), i.e.

$$F = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} F_{ij} 1_{(t_i, t_{i+1}]} 1_{A_j}.$$

Then

$$\mathbb{E}(|I_t(F)|^2) = \mathbb{E}\left(\left| \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (F_{ij}, M((t_i \wedge t, t_{i+1} \wedge t], A_j))_H \right|^2 \right),$$

and for $i < k$ with $t_i, t_k < t$ we have

$$\begin{aligned} & \mathbb{E}((F_{ij}, M((t_i, t_{i+1} \wedge t], A_j))_H (F_{kl}, M((t_k, t_{k+1} \wedge t], A_l))_H) \\ = \; & \mathbb{E}((F_{ij}, M((t_i, t_{i+1} \wedge t], A_j))_H \mathbb{E}((F_{kl}, M((t_k, t_{k+1} \wedge t], A_l))_H | \mathcal{F}_{t_k})) \\ = \; & \mathbb{E}((F_{ij}, M((t_i, t_{i+1} \wedge t], A_j))_H (F_{kl}, \mathbb{E}(M((t_k, t_{k+1} \wedge t], , A_l) | \mathcal{F}_{t_k}))_H) \\ = \; & 0. \end{aligned}$$

(Notice that this expression exists by Hölder's inequality). Take an orthonormal basis $\{e_n\}$ of H and set $N_{ij} := M((t_i, t_{i+1} \wedge t], A_j)$. Now let $j \neq l$, then since M is orthogonal $(N_{ij}, e_n)_H (e_m, N_{il})_H$ is a martingale and

$$\begin{aligned} & \mathbb{E}((F_{ij}, M((t_i, t_{i+1} \wedge t], A_j))_H (F_{il}, M((t_i, t_{i+1} \wedge t], A_l))_H) \\ = & \sum_{n,m=1}^{\infty} \mathbb{E}((F_{ij}, e_n)_H (e_n, N_{ij})_H (F_{il}, e_m)_H (e_m N_{il})_H) \\ = & \sum_{n,m=1}^{\infty} \mathbb{E}((F_{ij}, e_n)_H (F_{il}, e_m)_H \mathbb{E}((e_n, N_{ij})_H (e_m N_{il})_H | \mathcal{F}_{t_i})) \\ = & \; 0. \end{aligned}$$

Since M has independent increments

$$\begin{aligned}
&\mathbb{E}(|(F_{ij}, M((t_i, t_{i+1}\wedge t], A_j))_H|^2)\\
=&\sum_{n,m=1}^{\infty} \mathbb{E}((F_{ij}, e_n)_H (F_{ij}, e_m)_H)\mathbb{E}((e_n, N_{ij})_H (e_m N_{ij})_H)\\
=&\sum_{n,m=1}^{\infty} \mathbb{E}((F_{ij}, e_n)_H (F_{ij}, e_m)_H)(e_n, T_{A_j} e_m)_H \varrho((t_i, t_{i+1}\wedge t])\\
=&\sum_{m=1}^{\infty} \mathbb{E}((F_{ij}, T_{A_j} e_m)_H (F_{ij}, e_m)_H)\varrho((t_i, t_{i+1}\wedge t])\\
=&\ \mathbb{E}((T_{A_j} F_{ij}, F_{ij})_H)\varrho((t_i, t_{i+1}\wedge t])\\
=&\ \mathbb{E}\left(\int_{A_j} (T_x F_{ij}, F_{ij})_H \nu(dx)\right)\varrho((t_i, t_{i+1}\wedge t]).
\end{aligned}$$

Hence with the calculations from above

$$\begin{aligned}
&\mathbb{E}(|I_t(F)|^2)\\
=&\ \mathbb{E}\left(\left|\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}(F_{ij}, M((t_i\wedge t, t_{i+1}\wedge t], A_j))_H\right|^2\right)\\
=&\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\mathbb{E}\left(|(F_{ij}, M((t_i\wedge t, t_{i+1}\wedge t], A_j))_H|^2\right)\\
=&\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\mathbb{E}\left(\int_{A_j}(T_x F_{ij}, F_{ij})_H \nu(dx)\right)\varrho((t_i\wedge t, t_{i+1}\wedge t])\\
=&\ \mathbb{E}\left(\int_0^{\tilde{T}}\int_S \left\|1_{(0,t]}(s)\sqrt{T_x}F(s,x)\right\|^2 \nu(dx)\varrho(ds)\right)\\
=&\ \|F_{|(0,t]\times\Omega\times S}\|^2_{\mathcal{N}_2^w(T;t)}.
\end{aligned}$$

But then for arbitrary $F \in \mathcal{N}_2^w(T)$ there exists a sequence $\{F_n\} \subset \mathcal{S}_2^w(T)$ which converges to F in $\mathcal{N}_2^w(T)$ and

$$\lim_{n,m\to\infty}\mathbb{E}(|I_t(F_n) - I_t(F_m)|^2) = \lim_{n,m\to\infty}\|(F_n - F_m)_{|(0,t]\times\Omega\times S}\|^2_{\mathcal{N}_2^w(T;t)} = 0.$$

So we can define $I_t(F)$ to be the limit of $I_t(F_n)$ in $L^2(\Omega, \mathcal{F}, P)$ and of course the equality still holds. □

Let $F \in \mathcal{N}_2^w(T)$. Then define the *weak stochastic integral* of F with respect to the orthogonal martingale measure M as

$$\int_0^t \int_S (F(s,x), M(ds,dx))_H := I_t(F). \tag{3.7}$$

Proposition 3.2.5. *Let $F \in \mathcal{N}_2^w(T)$. Then $(\int_0^t \int_S (F(s,x), M(ds,dx))_H$, $t \geq 0)$ is a centered square-integrable martingale.*

Proof. First let $F \in \mathcal{S}_2^w(T)$ be like in (3.5). Let $s \leq t$ and set $i_0 := \max\{i; t_i \leq s\}$. Without loss of generality assume $t_{N_1} < t$. Then

$$
\begin{aligned}
& \mathbb{E}\left(\int_0^t \int_S (F(s,x), M(ds,dx))_H \middle| \mathcal{F}_s\right) \\
= & \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \mathbb{E}((F_{ij}, M((t_i, t \wedge t_{i+1}], A_j))_H | \mathcal{F}_s) \\
= & \sum_{j=0}^{N_2} \sum_{i=0}^{i_0-1} (F_{ij}, M((t_i, t_{i+1}], A_j))_H \\
+ & \sum_{j=0}^{N_2} (F_{i_0 j}, \mathbb{E}(M((t_{i_0}, t \wedge t_{i_0+1}], A_j) | \mathcal{F}_s))_H \\
+ & \sum_{j=0}^{N_2} \sum_{i=i_0+1}^{N_1} \mathbb{E}((F_{ij}, \mathbb{E}(M((t_i, t \wedge t_{i+1}], A_j) | \mathcal{F}_{t_i}))_H | \mathcal{F}_s) \\
= & \sum_{j=0}^{N_2} \left(\sum_{i=0}^{i_0-1} (F_{ij}, M((t_i, t_{i+1}], A_j))_H + (F_{i_0 j}, M((t_{i_0}, s], A_j))_H \right) \\
= & \int_0^s \int_S (F(s,x), M(ds,dx))_H.
\end{aligned}
$$

Now if $F \in \mathcal{N}_2^w(T)$ then it can be approximated by a sequence $\{F_n\} \subset \mathcal{S}_2^w(T)$ in $\mathcal{N}_2^w(T)$. Then by Proposition 3.2.4 the weak stochastic integral $\int_0^t \int_S (F(s,x), M(ds,dx))_H$ can be approximated by the weak integrals $\int_0^t \int_S (F_n(s,x), M(ds,dx))_H$ in $L^2(\Omega, \mathcal{F}, P)$. Hence it is also a martingale. It is obvious that this martingale is centered and square-integrable. □

3.3 Properties of the Stochastic Integral

In this section we will show some useful properties of the weak and the strong stochastic integral which were constructed above.

Proposition 3.3.1 (cf. [App b] Theorem 3). *Let $C \in L(H)$ and $R \in \mathcal{N}_2(T)$. Then $CR \in \mathcal{N}_2(T)$ and for every $0 \leq t \leq \tilde{T}$*

$$C \int_0^t \int_S R(s,x) M(ds,dx) = \int_0^t \int_S CR(s,x) M(ds,dx) \quad P\text{-a.s.}.$$

Proof. $CR \in \mathcal{N}_2(T)$ since

$$\begin{aligned}
& \mathbb{E}\left(\int_0^{\tilde{T}}\int_S \|CR(t,x)\sqrt{T_x}\|_2^2\nu(dx)\varrho(dt)\right)^{\frac{1}{2}} \\
\leq\ & \mathbb{E}\left(\int_0^{\tilde{T}}\int_S \|C\|^2\|R(t,x)\sqrt{T_x}\|_2^2\nu(dx)\varrho(dt)\right)^{\frac{1}{2}} = \|C\|\|R\|_{\mathcal{N}_2(T)}.
\end{aligned}$$

Moreover, for $R \in \mathcal{S}_2(T)$ like in (3.1) we have

$$\begin{aligned}
& C\int_0^t\int_S R(s,x)M(ds,dx) = C\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} R_{ij}M((t\wedge t_i, t\wedge t_{i+1}], A_j) \\
=\ & \int_0^t\int_S CR(s,x)M(ds,dx).
\end{aligned}$$

Now let $\{R_n\} \subset \mathcal{S}_2(T)$ with $R_n \to R$ in $\mathcal{N}_2(T)$. Then by Proposition 3.1.3

$$\begin{aligned}
& \mathbb{E}\left(\int_0^{\tilde{T}}\int_S \|1_{(0,t]}(s)C(R(s,x)-R_n(s,x))\sqrt{T_x}\|_2^2\nu(dx)\varrho(dt)\right) \\
\leq\ & \|C\|^2\mathbb{E}\left(\int_0^{\tilde{T}}\int_S \|1_{(0,t]}(s)(R(s,x)-R_n(s,x))\sqrt{T_x}\|_2^2\nu(dx)\varrho(dt)\right),
\end{aligned}$$

and by definition the sequence of the integrals $\int_0^t\int_S CR_n(s,x)M(ds,dx)$ converges to $\int_0^t\int_S CR(s,x)M(ds,dx)$ in $L^2(H)$. Also

$$\begin{aligned}
& \mathbb{E}\left(\left\|C\int_0^t\int_S (R_n(s,x)-R(s,x))M(ds,dx)\right\|^2\right) \\
\leq\ & \|C\|^2\mathbb{E}\left(\int_0^{\tilde{T}}\int_S \|1_{(0,t]}(s)(R_n(s,x)-R(s,x))\sqrt{T_x}\|_2^2\nu(dx)\varrho(dt)\right).
\end{aligned}$$

Hence

$$\begin{aligned}
& \mathbb{E}\left(\left\|\int_0^t\int_S CR(s,x)M(ds,dx) - C\int_0^t\int_S R(s,x)M(ds,dx)\right\|^2\right)^{\frac{1}{2}} \\
\leq\ & \lim_{n\to\infty}\mathbb{E}\left(\left\|\int_0^t\int_S C(R(s,x)-R_n(s,x))M(ds,dx)\right\|^2\right)^{\frac{1}{2}} \\
+\ & \lim_{n\to\infty}\mathbb{E}\left(\left\|C\int_0^t\int_S (R_n(s,x)-R(s,x))M(ds,dx)\right\|^2\right)^{\frac{1}{2}} \\
=\ & 0.
\end{aligned}$$

□

Proposition 3.3.2 (Compatibility of weak and strong integration; cf. Thm. 4 in [App b]). *Let $R \in \mathcal{N}_2(T)$. Then $R^*y \in \mathcal{N}_2^w(T)$ for every $y \in H$ and for each $0 \le t \le \tilde{T}$*

$$\left(y, \int_0^t \int_S R(s,x) M(ds,dx)\right)_H = \int_0^t \int_S (R(s,x)^* y, M(ds,dx))_H \qquad (3.8)$$

P-a.s..

Proof. Let $R \in \mathcal{N}_2(T)$ and $y \in H$, then

$$\begin{aligned} & \|R^*y\|^2_{\mathcal{N}_2^w(T)} \\ = \;& \mathbb{E}\left(\int_0^{\tilde{T}} \int_S \left\|\sqrt{T_x} R(s,x)^* y\right\|^2 \nu(dx)\varrho(ds)\right) \\ \le \;& \|y\|^2 \mathbb{E}\left(\int_0^{\tilde{T}} \int_S \left\|\sqrt{T_x} R(s,x)^*\right\|^2 \nu(dx)\varrho(ds)\right) \\ \le \;& \|y\|^2 \mathbb{E}\left(\int_0^{\tilde{T}} \int_S \left\|R(s,x)\sqrt{T_x}\right\|_2^2 \nu(dx)\varrho(ds)\right) \\ = \;& \|y\|^2 \|R\|^2_{\mathcal{N}_2(T)} < \infty. \end{aligned}$$

Hence both integrals exist. Now (3.8) can be shown for every $R \in \mathcal{S}_2(T)$:

$$\begin{aligned} & \left(y, \int_0^t \int_S R(s,x) M(ds,dx)\right)_H \\ = \;& \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} (y, R_{ij} M((t \wedge t_i, t \wedge t_{i+1}], A_j))_H \\ = \;& \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} (R_{ij}^* y, M((t \wedge t_i, t \wedge t_{i+1}], A_j))_H \\ = \;& \int_0^t \int_S (R(s,x)^* y, M(ds,dx))_H. \end{aligned}$$

But for $R \in \mathcal{N}_2(T)$ there can be chosen a sequence $\{R_n\} \subset \mathcal{S}_2(T)$ with $R_n \to R$ in $\mathcal{N}_2(T)$. Obviously $\{R_n^* y\} \subset \mathcal{S}_2^w(T)$. Then for

$$I_t(R^*y) = \int_0^t \int_S (R(s,x)^* y, M(ds,dx))_H$$

and

$$J_t(R) = \int_0^t \int_S R(s,x) M(ds,dx),$$

we have

$$\begin{aligned}
& \mathbb{E}\left(|(y, J_t(R))_H - I_t(R^*y)|^2\right)^{\frac{1}{2}} \\
\leq\ & \lim_{n\to\infty}\left(\mathbb{E}\left(|(y, J_t(R) - J_t(R_n))_H|^2\right)^{\frac{1}{2}} + \mathbb{E}\left(|I_t(R_n^*y) - I_t(R^*y)|^2\right)^{\frac{1}{2}}\right) \\
\leq\ & \|y\| \lim_{n\to\infty} \mathbb{E}\left(\|J_t(R) - J_t(R_n)\|^2\right)^{\frac{1}{2}} + \lim_{n\to\infty} \|R^*y - R_n^*y\|_{\mathcal{N}_2^w(T)} \\
\leq\ & 2\|y\| \lim_{n\to\infty} \|R - R_n\|_{\mathcal{N}_2(T)} = 0.
\end{aligned}$$

□

Remark 3.3.3. Using a slightly generalized version of Prop. 3.3.1 and with the notations from Remark 3.2.1 we can give an alternative proof:

$$\begin{aligned}
& \left(y, \int_0^t \int_S R(s,x)M(ds,dx)\right)_H = S_y\left(\int_0^t \int_S R(s,x)M(ds,dx)\right) \\
=\ & \int_0^t \int_S S_y R(s,x)M(ds,dx) = \int_0^t \int_S S_{R(s,x)^*y} M(ds,dx) \\
=\ & \int_0^t \int_S (R(s,x)^*y, M(ds,dx))_H.
\end{aligned}$$

Now a useful stochastic version of the Fubini theorem will be shown. Introduce a measure space $(W, \mathcal{W}, \mu)$ with μ finite. By $\mathcal{G}_2(W)$ denote the space of all operator valued mappings R on $[0,\tilde{T}] \times \Omega \times S \times W$ such that $(s,\omega,x,w) \mapsto R(s,\omega,x,w)y$ is $\mathcal{P} \otimes \Sigma \otimes \mathcal{W}$-measurable for each $y \in H$ and

$$\|R\|_{\mathcal{G}_2(W)} := \left[\mathbb{E}\left(\int_W \int_0^{\tilde{T}} \int_S \|R(s,x,w)\sqrt{T_x}\|_2^2 \nu(dx)\varrho(ds)\mu(dw)\right)\right]^{\frac{1}{2}} < \infty.$$

Again two mappings are identified in $\mathcal{G}_2(W)$ if they are equal $\varrho \otimes P \otimes \nu \otimes \mu$-a.e..

Theorem 3.3.4 (Stochastic Fubini). *Let $R \in \mathcal{G}_2(W)$. Then for each $0 \leq t \leq \tilde{T}$*

$$\begin{aligned}
& \int_W \left(\int_0^t \int_S R(s,x,w)M(ds,dx)\right)\mu(dw) \\
=\ & \int_0^t \int_S \left(\int_W R(s,x,w)\mu(dw)\right) M(ds,dx)
\end{aligned}$$

where the left hand side is meant as an $L^2(H)$-valued Bochner integral.

Proof. Note that by Jensen's inequality

$$\left\|\int_W R(s,x,w)\mu(W)^{-1}\mu(dw)\right\|^2 \leq \int_W \|R(s,x,w)\|^2 \mu(W)^{-1}\mu(dw).$$

$R(\cdot,\cdot,w)$ is in $\mathcal{N}_2(T)$ for μ-a.e. $w \in W$ and then by Prop. 3.1.3

$$\begin{aligned}
&\mathbb{E}\left(\int_W \left\|\int_0^t \int_S R(s,x,w)M(ds,dx)\right\|^2 \mu(dw)\right) \\
= &\int_W \mathbb{E}\left(\left\|\int_0^t \int_S R(s,x,w)M(ds,dx)\right\|^2\right) \mu(dw) \\
= &\int_W \mathbb{E}\left(\int_0^{\tilde{T}} \int_S \|1_{(0,t]}(s)R(s,x,w)\sqrt{T_x}\|_2^2 \nu(dx)\varrho(ds)\right) \mu(dw) < \infty.
\end{aligned}$$

Hence $\int_0^t \int_S R(s,x,\cdot)M(ds,dx)$ is Bochner integrable with respect to μ. For an orthonormal basis $\{e_k\}$ of H

$$\begin{aligned}
&\left\|\left(\int_W R(s,x,w)\mu(dw)\right)\sqrt{T_x}\right\|_2^2 \\
= &\sum_{k=1}^{\infty} \left\|\int_W R(s,x,w)\sqrt{T_x}e_k \mu(dw)\right\|^2 \\
\leq &\mu(W)\sum_{k=1}^{\infty} \int_W \|R(s,x,w)\sqrt{T_x}e_k\|^2 \mu(dw) \\
= &\mu(W)\int_W \|R(s,x,w)\sqrt{T_x}\|_2^2 \mu(dw),
\end{aligned}$$

and then $\int_W R(s,x,w)\mu(dw) \in \mathcal{N}_2(T)$, since

$$\begin{aligned}
&\mathbb{E}\left(\int_0^{\tilde{T}} \int_S \left\|\left(\int_W R(s,x,w)\mu(dw)\right)\sqrt{T_x}\right\|_2^2 \nu(dx)\varrho(ds)\right) \\
\leq &\mu(W)\mathbb{E}\left(\int_0^{\tilde{T}} \int_S \int_W \|R(s,x,w)\sqrt{T_x}\|_2^2 \mu(dw)\nu(dx)\varrho(ds)\right) < \infty.
\end{aligned}$$

Thereby, the second integral also exists by definition of the stochastic integral. By $\mathcal{S}_2(W)$ denote the space of all R of the form

$$R = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\sum_{k=0}^{N_3} R_{ijk} 1_{(t_i,t_{i+1}]} 1_{A_j} 1_{B_k},$$

where $N_1, N_2, N_3 \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_{N_1+1} = \tilde{T}$, $A_0, \ldots, A_{N_2+1}$ are disjoint sets in $\mathcal{A}$ and $B_0, \ldots, B_{N_3+1} \in \mathcal{W}$ are disjoint and each R_{ijk} is an $\mathcal{F}_{t_i}/\mathcal{L}$-measurable random variable with values in $L(H)$. Since $\mathcal{S}_2(T)$ is dense in $\mathcal{N}_2(T)$ it follows that $\mathcal{S}_2(W)$ is dense in $\mathcal{G}_2(W)$. But for $R \in \mathcal{S}_2(W)$ the left and the right side of the equation to show are equal to

$$\sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\sum_{k=0}^{N_3} R_{ijk} M((t \wedge t_i, t \wedge t_{i+1}], A_j)\mu(B_k).$$

If $R \in \mathcal{G}_2(W)$ is arbitrary, it can be approximated by a sequence $\{R_n\} \subset \mathcal{S}_2(W)$. But then

$$\begin{aligned} & \mathbb{E}\left(\left\|\int_W \left(\int_0^t \int_S (R(s,x,w) - R_n(s,x,w))M(ds,dx)\right)\mu(dw)\right\|^2\right) \\ \leq \;& \mu(W)\|R - R_n\|^2_{\mathcal{G}_2(W)} \to 0 \end{aligned}$$

for $n \to \infty$ and

$$\begin{aligned} & \mathbb{E}\left(\left\|\int_0^t \int_S \left(\int_W (R(s,x,w) - R_n(s,x,w))\mu(dw)\right) M(ds,dx)\right\|^2\right) \\ \leq \;& \mu(W)\|R - R_n\|^2_{\mathcal{G}_2(W)} \to 0 \end{aligned}$$

for $n \to \infty$. □

Chapter 4

Stochastic Equations with Lévy Noise

Now we apply the developed integration theory from the last chapter and consider stochastic partial differential equations driven by a Lévy process. If $(S(t), t \geq 0)$ is a C_0-semigroup $(S(t), t \geq 0)$ with generator J and C a bounded linear operator acting on H then we can define the *stochastic convolution*

$$X_{J,C}(t) := \int_0^t S(t-s)CdX(s)$$

for any H-valued Lévy process X (see section 4.1). In the subsequent section 4.2 we will find a weak solution to the following Ornstein-Uhlenbeck type stochastic differential equation

$$\begin{aligned} dY(t) &= (JY(t) + f(t))dt + CdX(t) \\ Y(0) &= Y_0. \end{aligned}$$

Here f is an adapted process and Y_0 some initial value. In [DaPrZa] existence and uniqueness is shown for the case where X is a Brownian motion with drift. [App b] has generalized the result (in the case $f \equiv 0$) to Lévy processes with similar methods. We show existence of a weak solution following the approach of [DaPrZa]. The compatibility of weak and strong integration as well as the stochastic Fubini theorem, which were shown before, are very useful tools.

In section 4.3 a non-linear drift term is added to the Ornstein-Uhlenbeck equation. For a function $F : H \to H$ fulfilling a Lipschitz condition we have the equation

$$\begin{aligned} dY(t) &= (JY(t) + F(Y(t)))dt + CdX(t) \\ Y(0) &= Y_0. \end{aligned}$$

The idea of section 4.3 is to prove existence and uniqueness of a weak solution by a fixed point argument as in Chapter 7 of [DaPrZa]. However, this is not straightforward. Since for a Lévy process first and second moments may not exist we have to impose an additional condition on the jumps of the Lévy process X: For some $\varepsilon > 0$

$$\sup_{0\leq t\leq \tilde{T}} \|\Delta X(t)\| \in L^{2+\varepsilon}(\mathbb{R}) = L^{2+\varepsilon}(\Omega \to \mathbb{R}).$$

We find that a stochastic process Y is a weak solution of the considered equation if and only if Y is a fixed point under the mapping $Y \mapsto \psi(Y)$ with

$$\psi(Y)(t) = \int_0^t S(t-s)F(Y(s))ds + S(t)Y_0 + X_{J,C}(t)).$$

Existence and uniqueness is then proved by an application of Banach's fixed point theorem on a suitable space. Different from [DaPrZa] we do not have to put together the solution from solutions on single subintervals of $[0,\tilde{T}]$ by using a weighted norm on the corresponding Banach space, as in [Kno b].

Assumption 4.0.1. In the whole chapter we take the following objects and properties as given:

- Let X be a Lévy process with Lévy Itô decomposition (2.7) and for some $\varepsilon > 0$

$$\sup_{0\leq t\leq \tilde{T}} \|\Delta X(t)\| \in L^{2+\varepsilon}(\mathbb{R}) = L^{2+\varepsilon}(\Omega \to \mathbb{R}). \tag{4.1}$$

- $(S(t), t \geq 0)$ shall be a C_0-semigroup with infinitesimal generator J.
- $C \in L(H)$.
- $f : [0,\tilde{T}] \to L^2(H) = L^2(\Omega \to H)$ a stochastic process, adapted to $(\mathcal{F}_t; t \geq 0)$ which is (as $L^2(H)$-valued function) Bochner integrable on $[0,\tilde{T}]$.
- $Y_0 \in L^2(H) = L^2(\Omega \to H)$ and $\mathcal{F}_0$-measurable.

Remark 4.0.2. Let $N = (N(t), t \geq 0)$ be a real-valued Poisson process with intensity $c > 0$. Then for any $1 \leq p < \infty$ we have $N(t) \in L^p(\mathbb{R})$ for $t \geq 0$ and $t \mapsto N(t)$ is continuous in $L^p(\mathbb{R})$: Since for $n \in \mathbb{N}$ with $n \geq p$

$$\mathbb{E}(|N(t) - N(s)|^p) \leq \mathbb{E}(|N(t) - N(s)|^n)$$

it is sufficient to consider $p = n \in \mathbb{N}$. But the moment generating function of a random variable Y which is Poisson distributed with parameter $\alpha > 0$ is

$$\varphi_\alpha(t) = \mathbb{E}\left(e^{tY}\right) = \sum_{k=0}^{\infty} e^{-\alpha}\frac{\alpha^k e^{tk}}{k!} = \exp(\alpha(e^t - 1)).$$

Since $\varphi_\alpha \in C^\infty(\mathbb{R})$ all moments $\mathbb{E}(Y^n) = \varphi_\alpha^{(n)}(0)$, $n \in \mathbb{N}$ exist. Moreover for any $n \in \mathbb{N}$ the n-th derivative of φ_α is of the form

$$\varphi_\alpha^{(n)}(t) = \alpha g_n(\alpha, t)$$

where $\alpha \mapsto g_n(\alpha, t)$ is continuous and therefore bounded in α on each compactum $[-\varepsilon, \varepsilon]$. Hence

$$\lim_{\alpha \to 0} \varphi_\alpha^{(n)}(0) = \lim_{\alpha \to 0} \alpha g_n(\alpha, 0) = 0.$$

Now the continuity of N in $L^n(\mathbb{R})$ for every $n \in \mathbb{N}$ follows, because

$$\lim_{s \to t} \mathbb{E}((N(t) - N(s))^n) = \lim_{s \to t} \mathbb{E}((N(t-s))^n) = \lim_{s \to t} (t-s) g_n(t-s, 0) = 0.$$

We obtain that by (4.1) $X(t) \in L^2(H)$ for any $0 \leq t \leq \tilde{T}$: Let $p > 1$ be given with $1/p + 1/(2+\varepsilon) = 1/2$. Then by the generalized Hölder's inequality

$$\begin{aligned}
& \mathbb{E}\left(\left\| \sum_{0<s\leq t} \Delta X(s) 1_{\{\|x\|\geq 1\}}(\Delta X(s)) \right\|^2 \right)^{\frac{1}{2}} \\
\leq \; & \mathbb{E}\left(\left(\sup_{0\leq t\leq \tilde{T}} \|\Delta X(t)\| \right)^2 N(t, \{\|x\| \geq 1\})^2 \right)^{\frac{1}{2}} \\
\leq \; & \mathbb{E}\left(\left(\sup_{0\leq t\leq \tilde{T}} \|\Delta X(t)\| \right)^{2+\varepsilon} \right)^{\frac{1}{2+\varepsilon}} \mathbb{E}\left(N(t, \{\|x\| \geq 1\})^p \right)^{\frac{1}{p}} \\
< \; & \infty
\end{aligned}$$

since $N(\cdot, \{\|x\| \geq 1\})$ is a Poisson process and therefore in $L^p(\mathbb{R})$. But the martingale part in the Lévy Itô decomposition is in $L^2(H)$ by construction. Therefore we will treat X as a mapping $X : [0, \tilde{T}] \to L^2(H)$.

4.1 The Stochastic Convolution

First we will define the *stochastic convolution.* Some preparations on C_0-semigroups are needed (compare [Paz] sections 1.1. and 1.2.):

A *strongly continuous semigroup of bounded linear operators* or a C_0-*semigroup* is a one parameter family $(S(t), t \geq 0)$ of bounded linear operators $S(t) \in L(H)$ with

- $S(0) = Id$, where Id is the identity operator on H,
- $S(s+t) = S(t)S(s)$ for every $s, t \geq 0$ (*semigroup property*),

- $\lim_{t\downarrow 0} S(t)x = x$ for every $x \in H$ (*strong continuity*).

Lemma 4.1.1. *Let $(S(t), t \geq 0)$ be a C_0-semigroup. Then there exist constants $\beta \geq 0$ and $M \geq 1$ such that*

$$\|S(t)\| \leq Me^{\beta t} \quad \textit{for all } t \geq 0. \tag{4.2}$$

Proof. (cf. [Paz] Ch. 1, Theorem 2.2.). □

Lemma 4.1.2. *If $(S(t), t \geq 0)$ is a C_0-semigroup then for every $x \in H$, $t \mapsto S(t)x$ is a strongly continuous function from $\mathbb{R}_+$ into H.*

Proof. (cf. [Paz] Ch. 1, Cor. 2.3.). □

Define the *infinitesimal generator* of a C_0-semigroup $(S(t), t \geq 0)$ as the operator J with domain

$$D(J) = \left\{ x \in H; \lim_{t\downarrow 0} \frac{S(t)x - x}{t} \text{ exists} \right\}$$

and

$$Jx = \lim_{t\downarrow 0} \frac{S(t)x - x}{t} \quad \text{for all } x \in D(J).$$

Proposition 4.1.3. *Let $(S(t), t \geq 0)$ be a C_0-semigroup with infinitesimal generator J. Then*

- *for every $x \in H$ and $t \geq 0$*

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} S(s)x ds = S(t)x,$$

- *for every $x \in H$ and $t \geq 0$*

$$\int_0^t S(s)x ds \in D(J)$$

  and

$$J\left(\int_0^t S(s)x ds \right) = S(t)x - x,$$

- *for every $x \in D(J)$ and $t \geq 0$*

$$S(t)x \in D(J)$$

  and

$$\frac{d}{dt}S(t)x = JS(t)x = S(t)Jx,$$

- *for every $x \in D(J)$ and $s, t \geq 0$*

$$S(t)x - S(s)x = \int_s^t S(\tau)Jxd\tau = \int_s^t JS(\tau)xd\tau.$$

Proof. (cf. [Paz] Ch. 1, Thm. 2.4.). □

Proposition 4.1.4 (Properties of the infinitesimal generator). *The infinitesimal generator J of a C_0-semigroup $(S(t), t \geq 0)$ is densely defined, i.e. $D(J)$ is dense in H, and J is a closed linear operator.*

Proof. (cf. [Paz] Ch. 1, Cor. 2.5.). □

Proposition 4.1.5 (The dual semigroup and its generator). *Given a C_0-semigroup $(S(t), t \geq 0)$ with infinitesimal generator J, $(S(t)^*, t \geq 0)$ is a C_0-semigroup with infinitesimal generator J^* which therefore is again densely defined and closed.*

Proof. (cf. [Paz] Ch. 1, Cor. 10.6.). □

Proposition 4.1.6. *Let $(S(t), t \geq 0)$ and $(T(t), t \geq 0)$ be C_0-semigroups with infinitesimal generators J and K respectively. If $J = K$ then $S(t) = T(t)$ for each $t \geq 0$.*

Proof. (cf. [Paz] Ch. 1, Thm. 2.6.). □

Given the objects from Assumption 4.0.1 we are going to define the *stochastic convolution* $X_{J,C}$ as

$$X_{J,C}(t) := \int_0^t S(t-s)CdX(s) \quad \text{for every } t \geq 0. \tag{4.3}$$

If $X_{J,C}$ shall be well-defined we have to give sense to the stochastic integral in (4.3). By the Lévy Itô decomposition (Theorem 2.4.7) the Lévy process X can be written as

$$X(t) = tx_0 + B_Q(t) + \int_{\{\|x\|<1\}} x\tilde{N}(t, dx) + \int_{\{\|x\|\geq 1\}} xN(t, dx) \tag{4.4}$$

Recall that here $S = \{x \in H; \|x\| < 1\}$, $\mathcal{A}_0 = \{A \in \mathfrak{B}(S); 0 \notin \bar{A}\}$ and $\mathcal{A} = \mathcal{A}_0 \cup \{A \cup \{0\}; A \in \mathcal{A}_0\}$. We will define the stochastic integral in (4.3) by

$$\int_0^t S(t-s)CdX(s) := \int_0^t S(t-s)Cx_0 ds \tag{4.5}$$

$$+ \int_0^t S(t-s)CdB_Q(s) \tag{4.6}$$

$$+ \int_0^t \int_{\{\|x\|<1\}} S(t-s)Cx\tilde{N}(ds, dx) \tag{4.7}$$

$$+ \int_0^t \int_{\{\|x\|\geq 1\}} S(t-s)CxN(ds, dx), \tag{4.8}$$

where the individual terms in (4.5), (4.6), (4.7) and (4.8) are defined as follows:

We can define the integral in (4.5) as a Bochner integral. By (4.2) this exists since

$$\begin{aligned}\int_0^t \|S(t-s)Cx_0\| ds &\leq \|Cx_0\| \int_0^t \|S(t-s)\| ds \\ &\leq \|Cx_0\| M \int_0^t e^{\beta(t-s)} ds < \infty.\end{aligned}$$

The integrators in (4.6) and (4.7) are Lévy martingale measures like in (2.9). By Theorem 2.5.2 and Proposition 2.5.4 a Lévy martingale measure is an orthogonal martingale measure with independent increments which is nuclear and decomposable. Hence one can construct the strong stochastic integral with respect to such a Lévy martingale measure. Set for $A \in \mathcal{A}$

$$M_1(t, A) = B_Q(t)\delta_0(A)$$

and

$$M_2(t, A) = \int_{A \setminus \{0\}} x\tilde{N}(t, dx).$$

Now define

$$\int_0^t S(t-s)C dB_Q(s) := \int_0^t \int_{\{\|x\|<1\}} S(t-s)CM_1(ds, dx). \tag{4.9}$$

Usually the stochastic integral with respect to a Brownian motion is defined different from (4.9). See for example [DaPrZa], section 4.2., for the construction. We believe that for our class of integrands (i.e. the ones from $\mathcal{N}_2(T)$) both integrals coincide. In any case in this text we will always use the definition from (4.9). This is well defined since by Proposition 2.5.4 (where $\nu = \delta_0, T_0 = Q$ and ϱ the Lebesgue measure on $\mathbb{R}_+$)

$$\begin{aligned}\|S(t-\cdot)C\|^2_{\mathcal{N}_2(T;t)} &= \int_0^t \int_{\{\|x\|<1\}} \|S(t-s)C\sqrt{T_x}\|_2^2 \delta_0(dx) ds \\ &\leq tr(Q) \int_0^t \|S(t-s)C\|^2 ds \\ &\leq \|C\|^2 M^2 tr(Q) \int_0^t e^{2\beta(t-s)} ds < \infty.\end{aligned}$$

By Lemma 4.1.2 $t \mapsto S(t)y$ is strongly continuous for every $y \in H$. Hence $t \mapsto S(t)y$ is $\mathcal{P}$-measurable. So $S(t-\cdot)C \in \mathcal{N}_2(T; \delta_0, ds; t)$.

In the same way set

$$\int_0^t \int_{\{\|x\|<1\}} S(t-s)Cx\tilde{N}(ds, dx) := \int_0^t \int_{\{\|x\|<1\}} S(t-s)CM_2(ds, dx),$$

where, with $\nu = \lambda, T_x = (x,\cdot)_H x$ and ϱ the Lebesgue measure on $\mathbb{R}_+$,

$$\begin{aligned}\|S(t-\cdot)C\|^2_{\mathcal{N}_2(T;t)} &= \int_0^t \int_{\{\|x\|<1\}} \|S(t-s)C\sqrt{T_x}\|_2^2 \lambda(dx)ds \\ &\leq \int_0^t \int_{\{\|x\|<1\}} \|S(t-s)C\|^2 \|x\|^2 \lambda(dx)ds \\ &\leq \|C\|^2 M^2 \int_0^t e^{2\beta(t-s)} \int_{\{\|x\|<1\}} \|x\|^2 \lambda(dx)ds < \infty,\end{aligned}$$

hence $S(t-\cdot)C \in \mathcal{N}_2(T;\lambda, ds; t)$.

Finally, the term in (4.8) will be defined as a random sum:

$$\begin{aligned}&\int_0^t \int_{\{\|x\|\geq 1\}} S(t-s)CxN(ds,dx) \\ :=\ &\sum_{0<s\leq t} S(t-s)C\Delta X(s)1_{\{\|x\|\geq 1\}}(\Delta X(s)).\end{aligned}$$

This is in $L^2(H)$ since

$$\begin{aligned}&\mathbb{E}\left(\left\|\sum_{0<s\leq t} S(t-s)C\Delta X(s)1_{\{\|x\|\geq 1\}}(\Delta X(s))\right\|^2\right) \\ \leq\ & M^2 e^{2\beta\tilde{T}} \|C\|^2 \mathbb{E}\left(\left(\sup_{0\leq t\leq \tilde{T}} \|\Delta X(t)\|\right)^2 N(t, \{\|x\|\geq 1\})^2\right) \\ <\ & \infty.\end{aligned}$$

Theorem 4.1.7 (cf. [App b] Thm. 6). *For a Lévy process X, $C \in L(H)$ and $(S(t), t \geq 0)$ a C_0-semigroup with infinitesimal generator J the stochastic convolution $X_{J,C} : [0,\tilde{T}] \to L^2(H)$ given by*

$$X_{J,C}(t) := \int_0^t S(t-s)CdX(s) \quad \text{for every } t \geq 0$$

exists. Furthermore under Assumption 4.0.1 $t \mapsto X_{J,C}(t)$ is (as an $L^2(H)$-valued mapping) continuous.

Proof. The existence follows from the construction above. We will investigate the continuity for the individual integrals from equations (4.5) to (4.8). First the continuity of $t \mapsto \int_0^t S(t-s)Cx_0 ds$ is obvious since this is continuous as an H-valued Bochner integral. Consider $t \mapsto \int_0^t \int_{\{\|x\|\geq 1\}} S(t-$

$s)CxN(ds, dx)$. We have

$$
\begin{aligned}
&\mathbb{E}\left(\left\|\int_0^t \int_{\{\|x\|\geq 1\}} S(t-r)CxN(dr,dx)\right.\right.\\
&\left.\left. - \int_0^s \int_{\{\|x\|\geq 1\}} S(s-r)CxN(dr,dx)\right\|^2\right)^{\frac{1}{2}}\\
\leq\ &\mathbb{E}\left(\left\|\sum_{0<r\leq t}(S(t-r)-S(s-r\wedge s))C\Delta X(r)1_{\{\|x\|\geq 1\}}(\Delta X(r))\right\|^2\right)^{\frac{1}{2}}\\
&+\mathbb{E}\left(\left\|\sum_{t\wedge s<r\leq t\vee s} S(s-r\wedge s)C\Delta X(r)1_{\{\|x\|\geq 1\}}(\Delta X(r))\right\|^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

But $t\mapsto S(t)y$ is strongly continuous for any $y\in H$ and

$$
\begin{aligned}
&\left\|\sum_{0<r\leq t}(S(t-r)-S(s-r\wedge s))C\Delta X(r)1_{\{\|x\|\geq 1\}}(\Delta X(r))\right\|\\
\leq\ & 2Me^{\beta\tilde{T}}\|C\|\sup_{0\leq t\leq\tilde{T}}\|\Delta X(t)\|\left(N(t,\{\|x\|\geq 1\})\right)\in L^2(\mathbb{R})
\end{aligned}
$$

which can be seen as in Remark 4.0.2 by the generalized Hölder's inequality ($\sup_{0\leq t\leq\tilde{T}}\|\Delta X(t)\|\in L^{2+\varepsilon}(\mathbb{R})$). Hence we can apply Lebesgue's dominated convergence theorem and obtain that the first summand vanishes for $s\to t$. The second summand can be estimated by

$$
\begin{aligned}
&Me^{\beta\tilde{T}}\|C\|\mathbb{E}\left(\left(\sup_{0\leq t\leq\tilde{T}}\|\Delta X(t)\|\right)^{2+\varepsilon}\right)^{\frac{1}{2+\varepsilon}}\\
&\mathbb{E}((N(t\vee s,\{\|x\|\geq 1\})-N(t\wedge s,\{\|x\|\geq 1\}))^p)^{\frac{1}{p}}
\end{aligned}
$$

for p with $1/p+1/(2+\varepsilon)=1/2$. But the Poisson process $N(\cdot,\{\|x\|\geq 1\})$ is (as an $L^p(\mathbb{R})$-valued mapping) continuous.

Set

$$M(t,A)=B_Q(t)\delta_0(A)+\int_{A\setminus\{0\}} x\tilde{N}(t,dx)$$

for any $A\in\mathcal{A}$. We will show the continuity of $t\mapsto\int_0^t\int_{\{\|x\|<1\}} S(t-$

$r)CM(dr,dx)$ in $L^2(H)$. For $0 \le s,t \le \tilde{T}$ and $\nu = \lambda + \delta_0$ we have

$$\begin{aligned}
&\mathbb{E}\left(\left\|\int_0^t \int_{\{\|x\|<1\}} S(t-r)CM(dr,dx)\right.\right.\\
&\left.\left.-\int_0^s \int_{\{\|x\|<1\}} S(s-r)CM(dr,dx)\right\|^2\right)^{\frac{1}{2}}\\
\le\ & \mathbb{E}\left(\left\|\int_0^t \int_{\{\|x\|<1\}} (S(t-r)-S(s-r\wedge s))CM(dr,dx)\right\|^2\right)^{\frac{1}{2}}\\
&+\mathbb{E}\left(\left\|\int_{t\wedge s}^{t\vee s} \int_{\{\|x\|<1\}} S(s-r\wedge s)CM(dr,dx)\right\|^2\right)^{\frac{1}{2}}\\
=\ & \left(\int_0^t \int_{\{\|x\|<1\}} \|(S(t-r)-S(s-r\wedge s))C\sqrt{T_x}\|_2^2 \nu(dx)dr\right)^{\frac{1}{2}}\\
&+\left(\int_{t\wedge s}^{t\vee s} \int_{\{\|x\|<1\}} \|S(s-r\wedge s)C\sqrt{T_x}\|_2^2 \nu(dx)dr\right)^{\frac{1}{2}}.
\end{aligned}$$

But

$$\begin{aligned}
&\lim_{s\to t} \int_{t\wedge s}^{t\vee s} \int_{\{\|x\|<1\}} \|S(s-r\wedge s)C\sqrt{T_x}\|_2^2 \nu(dx)dr\\
\le\ & \lim_{s\to t} M^2 e^{2\beta\tilde{T}}\|C\|^2 \int_{t\wedge s}^{t\vee s}\left(trQ + \int_{\{\|x\|<1\}} \|x\|^2\lambda(dx)\right)dr = 0.
\end{aligned}$$

For fixed $x \in \{\|x\|<1\}$ let $\{e_k\}$ be an eigenbasis of the Hilbert-Schmidt operator $\sqrt{T_x}$ with eigenvalues α_k. Then we obtain

$$\begin{aligned}
&\|(S(t-r)-S(s-r\wedge s))C\sqrt{T_x}\|_2^2\\
=\ & \sum_{k=1}^{\infty} \alpha_k^2 \|(S(t-r)-S(s-r\wedge s))Ce_k\|^2.
\end{aligned}$$

Since $\sum_{k=1}^{\infty}\alpha_k^2\delta_{e_k}$ is a finite measure on $\{e_k\} \subset H$ and for any $k \in \mathbb{N}$ $\|(S(t-r\wedge t)-S(s-r\wedge s))Ce_k\| \le Me^{\beta\tilde{T}}\|C\|$ we can apply Lebesgue's dominated convergence theorem:

$$\begin{aligned}
&\lim_{s\to t} \int_0^t \int_{\{\|x\|<1\}} \|(S(t-r)-S(s-r\wedge s))C\sqrt{T_x}\|_2^2 \nu(dx)dr\\
\le\ & \lim_{s\to t} \int_0^{\tilde{T}} \int_{\{\|x\|<1\}} \sum_{k=1}^{\infty} \alpha_k^2 \|(S(t-r\wedge t)-S(s-r\wedge s))Ce_k\|^2 \nu(dx)dr
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\tilde{T}} \int_{\{\|x\|<1\}} \sum_{k=1}^{\infty} \alpha_k^2 \lim_{s\to t} \|(S(t-r\wedge t) - S(s-r\wedge s))Ce_k\|^2 \nu(dx)dr \\
&= 0
\end{aligned}
$$

by the strong continuity of the semigroup $(S(t), t \geq 0)$. □

Proposition 4.1.8. *Given property (4.1) we have*

$$
\sup_{0\leq t\leq \tilde{T}} \mathbb{E}(\|X_{J,C}(t)\|^2) < \infty.
$$

In particular, with Theorem 4.1.7 it follows that $X_{J,C}$ is Bochner integrable on $[0, \tilde{T}]$.

Proof. Again consider the Lévy Itô decomposition and set

$$
M(t, A) = B_Q(t)\delta_0(A) + \int_{A\setminus\{0\}} x\tilde{N}(t, dx)
$$

for any $A \in \mathcal{A}$. Then

$$
\begin{aligned}
&\mathbb{E}\left(\left\| \int_0^t \int_{\{\|x\|<1\}} S(t-s)CM(ds, dx)\right\|^2\right)^{\frac{1}{2}} \\
&= \|1_{(0,t]}S(t-s)C\|_{\mathcal{N}_2(T)} \leq Me^{\beta\tilde{T}}\|C\|_{\mathcal{N}_2(T)} < \infty.
\end{aligned}
$$

For the deterministic part we have

$$
\left\|\int_0^t S(t-s)Cx_0 ds\right\| \leq Me^{\beta\tilde{T}}\tilde{T}\|Cx_0\| < \infty.
$$

Finally, again by the generalized Hölder's inequality for p with $1/p + 1/(2 + \varepsilon) = 1/2$

$$
\begin{aligned}
&\mathbb{E}\left(\left\| \int_0^t \int_{\{\|x\|\geq 1\}} S(t-s)CxN(ds, dx)\right\|^2\right)^{\frac{1}{2}} \\
&\leq Me^{\beta\tilde{T}}\|C\|\mathbb{E}\left(\left(\sup_{0\leq t\leq\tilde{T}} \|\Delta X(t)\|\right)^{2+\varepsilon}\right)^{\frac{1}{2+\varepsilon}} \mathbb{E}(N(t, \{\|x\|\geq 1\})^p)^{\frac{1}{p}} \\
&\leq Me^{\beta\tilde{T}}\|C\|\mathbb{E}\left(\left(\sup_{0\leq t\leq\tilde{T}} \|\Delta X(t)\|\right)^{2+\varepsilon}\right)^{\frac{1}{2+\varepsilon}} \mathbb{E}(N(\tilde{T}, \{\|x\|\geq 1\})^p)^{\frac{1}{p}} \\
&< \infty.
\end{aligned}
$$

□

4.2 Ornstein-Uhlenbeck Type Processes

Given Assumption 4.0.1 the following stochastic differential equation will be solved:

$$\begin{aligned} dY(t) &= (JY(t) + f(t))dt + CdX(t) \\ Y(0) &= Y_0. \end{aligned} \tag{4.10}$$

A stochastic process $Y : [0, \tilde{T}] \to L^2(H)$, adapted to $(\mathcal{F}_t; t \geq 0)$ with $Y(0) = Y_0$ is a *weak solution* of (4.10) iff Y (as $L^2(H)$-valued function) is continuous and Bochner integrable on $[0, \tilde{T}]$ and for every $y \in D(J^*)$ and every $t \in [0, \tilde{T}]$

$$\begin{aligned} & (Y(t) - Y_0, y)_H \\ = & \int_0^t ((Y(s), J^* y)_H + (f(s), y)_H) ds + (X(t), C^* y)_H. \end{aligned} \tag{4.11}$$

(This equation is meant to be in $L^2(\mathbb{R}) = L^2(\Omega \to \mathbb{R})$).

Theorem 4.2.1 (Existence and uniqueness of a weak solution). *Under Assumption 4.0.1 the stochastic differential equation (4.10) has a unique weak solution Y given by*

$$Y(t) = S(t)Y_0 + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)CdX(s). \tag{4.12}$$

The result will be shown with methods similar to the ones in [DaPrZa] Theorem 5.4, which deals with the case where X is Gaussian with drift. For X being a Lévy process and $f \equiv 0$ this result was obtained in [App b], Theorem 7.

For the proof we will use the following proposition on differential equations on the Hilbert space $L^2(H)$:

Proposition 4.2.2. *For each $U_0 \in L^2(H)$ there exits a unique weak solution to*

$$\begin{aligned} dU(t) &= (JU(t) + f(t))dt \\ U(0) &= U_0 \end{aligned} \tag{4.13}$$

given by

$$U(t) = S(t)U_0 + \int_0^t S(t-s)f(s)ds.$$

Proof. (cf. [Bal] where the result is shown on general Banach spaces). □

Proposition 4.2.3. *The stochastic convolution $X_{J,C}$, i.e. the process given by*

$$X_{J,C}(t) = \int_0^t S(t-s)CdX(s)$$

is a weak solution of (4.10) for $Y_0 = 0$ and $f \equiv 0$.

Proof. Let $y \in D(J^*)$. We will find an expression for $(y, X_{J,C}(t))_H - (C^*y, X(t))_H$ which will give us that $X_{J,C}$ is a weak solution. By the Lévy Itô decomposition we can write X as the sum of two processes X_1 and X_2 with

$$X_1(t) = tx_0 + \int_{\{\|x\|\geq 1\}} xN(t,dx)$$

and

$$X_2(t) = B_Q(t) + \int_{\{\|x\|<1\}} x\tilde{N}(t,dx).$$

Define a Lévy martingale measure M by

$$M(t,A) = B_Q(t)\delta_0(A) + \int_{A\setminus\{0\}} x\tilde{N}(t,dx), \quad A \in \mathcal{A}.$$

Then the stochastic convolution can be written as

$$\begin{aligned} X_{J,C}(t) &= \int_0^t S(t-s)CdX_1(s) + \int_0^t S(t-s)CdX_2(s) \\ &= \int_0^t S(t-s)Cx_0 ds + \sum_{0<s\leq t} S(t-s)C\Delta X(s)1_{\{\|x\|\geq 1\}}(\Delta X(s)) \\ &+ \int_0^t \int_{\{\|x\|<1\}} S(t-s)CM(ds,dx). \end{aligned}$$

We will also use that

$$\begin{aligned} &(S(t-r)C - C)^*y = C^*(S(t-r)^*y - y) = C^* \int_0^{t-r} S(s)^*J^*yds \\ =\ & C^* \int_r^t S(s-r)^*J^*yds = \int_0^t 1_{[0,s]}(r)C^*S(s-r)^*J^*yds \\ =\ & \left(\int_0^t 1_{[0,s]}(r)S(s-r)Cds\right)^* J^*y. \end{aligned}$$

Now by the compatibility of weak and strong stochastic integration (Prop.

3.3.2) and by the stochastic Fubini Theorem (cf. Thm. 3.3.4) we obtain

$$
\begin{aligned}
&\left(y, \int_0^t S(t-r) C dX_2(r)\right)_H - (C^* y, X_2(t))_H \\
= &\left(y, \int_0^t S(t-r) C dX_2(r)\right)_H - \left(y, C \int_0^t \int_{\{\|x\|<1\}} M(dr, dx)\right)_H \\
= &\left(y, \int_0^t \int_{\{\|x\|<1\}} (S(t-r)C - C) M(dr, dx)\right)_H \\
= &\int_0^t \int_{\{\|x\|<1\}} ((S(t-r)C - C)^* y, M(dr, dx))_H \\
= &\int_0^t \int_{\{\|x\|<1\}} \left(\left(\int_0^t 1_{[0,s]}(r) S(s-r) C ds\right)^* J^* y, M(dr, dx)\right)_H \\
= &\left(J^* y, \int_0^t \int_{\{\|x\|<1\}} \left(\int_0^t 1_{[0,s]}(r) S(s-r) C ds\right) M(dr, dx)\right)_H \\
= &\left(J^* y, \int_0^t \left(\int_0^t \int_{\{\|x\|<1\}} 1_{[0,s]}(r) S(s-r) C M(dr, dx)\right) ds\right)_H \\
= &\int_0^t \left(J^* y, \int_0^s S(s-r) C dX_2(r)\right)_H ds.
\end{aligned}
$$

Using once again that

$$
(S(t-r)C - C)^* y = \left(\int_0^t 1_{[0,s]}(r) S(s-r) C ds\right)^* J^* y
$$

and by the (deterministic) Fubini Theorem we have

$$
\begin{aligned}
&\left(y, \int_0^t S(t-r) C dX_1(r)\right)_H - (C^* y, X_1(t))_H \\
= &\int_0^t ((S(t-r)C - C)^* y, x_0)_H \, dr \\
+ &\sum_{0<r\le t} \big((S(t-r)C - C)^* y, \Delta X(r) 1_{\{\|x\|\ge 1\}}(\Delta X(r))\big)_H \\
= &\int_0^t \left(\left(\int_0^t 1_{[0,s]}(r) S(s-r) C ds\right)^* J^* y, x_0\right)_H dr \\
+ &\sum_{0<r\le t} \left(\left(\int_0^t 1_{[0,s]}(r) S(s-r) C ds\right)^* J^* y, \Delta X(r) 1_{\{\|x\|\ge 1\}}(\Delta X(r))\right)_H \\
= &\int_0^t \int_0^t 1_{[0,s]}(r) (J^* y, S(s-r) C x_0)_H ds dr
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(J^*y, \sum_{0<r\le t}\left(\int_0^t 1_{[0,s]}(r)S(s-r)Cds\right)\Delta X(r)1_{\{\|x\|\ge 1\}}(\Delta X(r))\right)_H \\
&= \int_0^t\int_0^t 1_{[0,s]}(r)(J^*y, S(s-r)Cx_0)_H drds \\
&+ \left(J^*y, \int_0^t\left(\sum_{0<r\le t} 1_{[0,s]}(r)S(s-r)C\Delta X(r)1_{\{\|x\|\ge 1\}}(\Delta X(r))\right)ds\right)_H \\
&= \int_0^t \left(J^*y, \int_0^s S(s-r)CdX_1(r)\right)_H ds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&(y, X_{J,C}(t))_H - (C^*y, X(t))_H \\
&= \left(y, \int_0^t S(t-s)CdX_1(s) + \int_0^t S(t-s)CdX_2(s)\right)_H - (C^*y, X(t))_H \\
&= \int_0^t \left(J^*y, \int_0^s S(s-r)CdX_1(r) + \int_0^s S(s-r)CdX_2(r)\right)_H ds \\
&= \int_0^t (J^*y, X_{J,C}(s))_H ds,
\end{aligned}
$$

and thereby $X_{J,C}$ is a weak solution of (4.10) with $Y_0 = 0$ and $f \equiv 0$. □

Proof of Theorem 4.2.1. Let $y \in D(J^*)$ and Y be given as in (4.12), i.e.

$$Y(t) = S(t)Y_0 + \int_0^t S(t-s)f(s)ds + X_{J,C}(t).$$

By Prop. 4.2.3 $X_{J,C}$ is a weak solution of

$$
\begin{aligned}
d\tilde{Y}(t) &= J\tilde{Y}(t)dt + CdX(t) \\
\tilde{Y}(0) &= 0.
\end{aligned}
$$

But

$$Y(t) - X_{J,C}(t) = S(t)Y_0 + \int_0^t S(t-s)f(s)ds$$

is the unique weak solution of (4.13) for $U_0 = Y_0$. Hence

$$
\begin{aligned}
&(Y(t) - X_{J,C}(t), y)_H - (Y_0, y)_H \\
&= \int_0^t ((Y(s) - X_{J,C}(s), J^*y)_H + (f(s), y)_H)ds,
\end{aligned}
$$

and therefore by Proposition 4.2.3 and (4.11)

$$\begin{aligned}
& (Y(t), y)_H - (Y_0, y)_H \\
= & \int_0^t ((Y(s) - X_{J,C}(s), J^* y)_H + (f(s), y)_H) ds \\
+ & (X_{J,C}(t), y)_H \\
= & \int_0^t ((Y(s) - X_{J,C}(s), J^* y)_H + (f(s), y)_H) ds \\
+ & \int_0^t (X_{J,C}(s), J^* y)_H ds + (X(t), C^* y)_H \\
= & \int_0^t ((Y(s), J^* y)_H + (f(s), y)_H) ds + (X(t), C^* y)_H.
\end{aligned}$$

Hence Y as given in (4.12) is a weak solution of (4.10) and it is unique: For any solutions Y, Z and $0 \le t \le \tilde{T}$ we have by (4.11)

$$(Y(t) - Z(t), y)_H = \int_0^t (Y(s) - Z(s), J^* y)_H ds.$$

Hence $Y - Z$ is the weak solution of (4.13) for $U_0 = 0$ and $f \equiv 0$, but then $Y - Z \equiv 0$. □

The weak solution of (4.10) with $f \equiv 0$, i.e. Y given by

$$Y(t) = S(t) Y_0 + \int_0^t S(t-s) C dX(s)$$

is called *Ornstein-Uhlenbeck process.*

4.3 Lipschitz Nonlinearities

We will now solve a type of stochastic differential equation which has an additional nonlinear term. In section 7.1 of [DaPrZa] these equations with Lipschitz nonlinearities are treated in the case where the integrator is a Brownian motion.

Again given Assumption 4.0.1 we introduce the following equation:

$$\begin{aligned}
dY(t) &= (JY(t) + F(Y(t))) dt + C dX(t) \\
Y(0) &= Y_0,
\end{aligned} \tag{4.14}$$

where $F : H \to H$ is measurable and Lipschitz, i.e. there exists a constant $L > 0$ such that

$$\|F(x) - F(y)\| \le L \|x - y\| \quad \text{for all } x, y \in H.$$

A stochastic process $Y : [0,\tilde{T}] \to L^2(H)$, adapted to $(\mathcal{F}_t; t \geq 0)$ with $Y(0) = Y_0$ is a *weak solution* of (4.14) iff Y (as $L^2(H)$-valued function) is continuous and Bochner integrable on $[0,\tilde{T}]$ and for all $y \in D(J^*)$ and all $t \in [0,\tilde{T}]$

$$\begin{aligned} & (Y(t) - Y_0, y)_H \\ = & \int_0^t ((Y(s), J^*y)_H + (F(Y(s)), y)_H)ds + (X(t), C^*y)_H. \end{aligned} \tag{4.15}$$

(Again we interpret this equation as an equation in $L^2(\mathbb{R}) = L^2(\Omega \to \mathbb{R})$).

Proposition 4.3.1. *Let Y with $Y(0) = Y_0$ (as $L^2(H)$-valued function) be continuous and Bochner integrable on $[0,\tilde{T}]$. Then Y is a weak solution of (4.14) iff it is a* mild solution*, i.e. Y fulfills*

$$Y(t) = S(t)Y_0 + \int_0^t S(t-s)F(Y(s))ds + X_{J,C}(t) \tag{4.16}$$

for every $0 \leq t \leq \tilde{T}$.

Proof. Set

$$\gamma(Y)(t) = \int_0^t S(t-s)F(Y(s))ds.$$

Let Y satisfy (4.16) and $0 \leq t \leq \tilde{T}$. Then

$$Y(t) - \gamma(Y)(t) = S(t)Y_0 + X_{J,C}(t).$$

Hence by Theorem 4.2.1 $Y - \gamma(Y)$ is a weak solution of (4.10) with $f \equiv 0$. Now by (4.11) for each $y \in D(J^*)$:

$$\begin{aligned} & (Y(t) - \gamma(Y)(t), y)_H - (Y_0, y)_H \\ = & \int_0^t (Y(s) - \gamma(Y)(s), J^*y)_H ds + (X(t), C^*y)_H. \end{aligned}$$

And

$$\begin{aligned} & \int_0^t (\gamma(Y)(s), J^*y)_H ds \\ = & \int_0^t \int_0^t (1_{[0,s]}(r)S(s-r)F(Y(r)), J^*y)_H dr ds \\ = & \int_0^t \left(\left(\int_0^{t-r} S(s)ds\right) F(Y(r)), J^*y\right)_H dr \\ = & \int_0^t \left(F(Y(r)), \int_0^{t-r} S(s)^* J^*y ds\right)_H dr \\ = & (\gamma(Y)(t), y)_H - \left(\int_0^t F(Y(r))dr, y\right)_H. \end{aligned}$$

Hence for every $y \in D(J^*)$

$$\begin{aligned}
& (Y(t) - Y_0, y)_H \\
= \;& (\gamma(Y)(t), y)_H + \int_0^t (Y(s) - \gamma(Y)(s), J^* y)_H ds + (X(t), C^* y)_H \\
= \;& \int_0^t (Y(s), J^* y)_H ds + \left(\int_0^t F(Y(s)) ds, y \right)_H + (X(t), C^* y)_H,
\end{aligned}$$

thus Y is a weak solution of (4.14).

Now let Y be a weak solution. Then for every $y \in D(J^*)$:

$$\begin{aligned}
& (Y(t), y)_H - (Y_0, y)_H \\
= \;& \int_0^t (Y(s), J^* y)_H ds + \int_0^t (F(Y(s)), y)_H ds + (X(t), C^* y)_H \\
= \;& (\gamma(Y)(t), y)_H + \int_0^t (Y(s) - \gamma(Y)(s), J^* y)_H ds + (X(t), C^* y)_H.
\end{aligned}$$

Hence $(Y - \gamma(Y))$ fulfills (4.11) with $f \equiv 0$. Thereby we obtain that $(Y - \gamma(Y))$ is a weak solution of (4.10) with $f \equiv 0$. But then Theorem 4.2.1 implies (4.16). □

To find a weak solution of (4.14) we will construct a contraction on the following space

$$\begin{aligned}
\mathcal{Z} \;:=\; & \{Z : [0, \tilde{T}] \to L^2(H) \text{ continuous}; \\
& Z(t)\, \mathcal{F}_t\text{-measurable for all } t \in [0, \tilde{T}], \|Z\|_{\mathcal{Z}} < \infty\}
\end{aligned}$$

with

$$\|Z\|_{\mathcal{Z}} := \left(\sup_{0 \le t \le \tilde{T}} e^{-ct} \mathbb{E}(\|Z(t)\|^2) \right)^{\frac{1}{2}}$$

and $c := 2(M^2 L^2 \tilde{T} + \beta)$.

Proposition 4.3.2. *$(\mathcal{Z}, \|\cdot\|_{\mathcal{Z}})$ is a Banach space.*

Proof. Note that $\|\cdot\|_{\mathcal{Z}}$ is equivalent to the supremum norm on the space of all bounded mappings from $[0, \tilde{T}]$ to $L^2(H) = L^2((\Omega, \mathcal{F}, P) \to H)$. This space is complete (cf. for example [Die] (7.1.3.)). Therefore, for the limit Z of a Cauchy sequence $\{Z_n\} \subset \mathcal{Z}$ it remains to show that Z is adapted. But since for every $t \in [0, \tilde{T}]$ $Z_n(t)$ converges in $L^2(H)$ it converges P-a.s. for a subsequence, hence the limit $Z(t)$ is $\mathcal{F}_t$-measurable again. □

Theorem 4.3.3 (Weak solution of SDE with Lipschitz nonlinearities). *Under Assumption 4.0.1 the stochastic equation (4.14) has exactly one weak solution Y.*

Proof. For $Z \in \mathcal{Z}$ set

$$\psi(Z)(t) := \int_0^t S(t-s)F(Z(s))ds + S(t)Y_0 + X_{J,C}(t).$$

Now $\psi(Z)$ is adapted and ψ maps $\mathcal{Z}$ into itself: By Proposition 4.1.8 we have $\|X_{J,C}\|_{\mathcal{Z}} < \infty$. Also $\|S(\cdot)Y_0\|_{\mathcal{Z}} < \infty$, since $\|S(t)\| \leq M\exp(\beta\tilde{T})$ and $Y_0 \in L^2(H)$. It remains to show that $\|\gamma(Y)\|_{\mathcal{Z}} < \infty$ for

$$\gamma(Z)(t) = \int_0^t S(t-s)F(Z(s))ds.$$

But

$$\begin{aligned}
\|\gamma(Y)\|_{\mathcal{Z}}^2 &= \sup_{0\leq t\leq\tilde{T}} e^{-ct}\mathbb{E}(\|\gamma(Z)(t)\|^2) \\
&\leq \sup_{0\leq t\leq\tilde{T}} e^{-ct}\mathbb{E}\left(t\int_0^t \|S(t-s)\|^2\|F(Z(s))\|^2 ds\right) \\
&\leq \sup_{0\leq t\leq\tilde{T}} e^{-ct}\mathbb{E}\left(t\int_0^t M^2e^{2\beta(t-s)}\|F(Z(s))\|^2 ds\right) \\
&= M^2\sup_{0\leq t\leq\tilde{T}} te^{(2\beta-c)t}\int_0^t e^{-2\beta s}\mathbb{E}(\|F(Z(s))\|^2)ds \\
&\leq M^2\sup_{0\leq t\leq\tilde{T}} te^{(2\beta-c)t}\int_0^t e^{-2\beta s}\mathbb{E}((L\|(Z(s))\| + \|F(0)\|)^2)ds \\
&\leq M^2\sup_{0\leq t\leq\tilde{T}} te^{(2\beta-c)t}\left(\int_0^t e^{(c-2\beta)s}3L^2\|Z\|_{\mathcal{Z}}^2 ds + 3\|F(0)\|^2\int_0^t e^{-2\beta s}ds\right) \\
&< \infty.
\end{aligned}$$

Now we will show that ψ is strongly contracting: Let $Z_1, Z_2 \in \mathcal{Z}$. Then

$$\begin{aligned}
\|\psi(Z_1)-\psi(Z_2)\|_{\mathcal{Z}}^2 &= \sup_{0\leq t\leq\tilde{T}} e^{-ct}\mathbb{E}(\|\psi(Z_1)(t)-\psi(Z_2)(t)\|^2) \\
&\leq \sup_{0\leq t\leq\tilde{T}} e^{-ct}\mathbb{E}\left(t\int_0^t \|S(t-s)\|^2\|F(Z_1(s))-F(Z_2(s))\|^2 ds\right) \\
&\leq \sup_{0\leq t\leq\tilde{T}} e^{-ct}\mathbb{E}\left(t\int_0^t M^2e^{2\beta(t-s)}L^2\|Z_1(s)-Z_2(s)\|^2 ds\right) \\
&= M^2L^2\sup_{0\leq t\leq\tilde{T}} te^{(2\beta-c)t}\int_0^t e^{-2\beta s}\mathbb{E}(\|Z_1(s)-Z_2(s)\|^2)ds \\
&\leq M^2L^2\sup_{0\leq t\leq\tilde{T}} te^{(2\beta-c)t}\int_0^t e^{(c-2\beta)s}\|Z_1-Z_2\|_{\mathcal{Z}}^2 ds \\
&= M^2L^2\left(\sup_{0\leq t\leq\tilde{T}} \frac{t}{c-2\beta}\left(1-e^{(2\beta-c)t}\right)\right)\|Z_1-Z_2\|_{\mathcal{Z}}^2
\end{aligned}$$

$$\leq \frac{1}{2}\|Z_1 - Z_2\|^2_{\mathcal{Z}}.$$

By Banach's Fixed Point Theorem there exists one and only one adapted process $Y \in \mathcal{Z}$ which is (as $L^2(H)$-valued function) continuous such that $\psi(Y)(t) = Y(t)$ for all $t \in [0, \tilde{T}]$. Hence we obtain a process which fulfills (4.16). Then by Proposition 4.3.1 Y is a weak solution of (4.14). □

Chapter 5

Applications in Finance

This final chapter presents some applications of Lévy processes in Finance. In particular, we will see that the results of Chapter 4 and especially section 4.3 can be useful. First we give a review of the classical *Black Scholes model* which is the basis of all option pricing formulas that are presented afterwards. In section 5.2 a motivation for the use of Lévy processes in financial modelling is given. While the imagination that prices are the result of a large number of market actions is an argument for the normality of log price returns, empirical findings do not support this. The results of [Fam] suggest to model log prices by a Lévy process. It is then the idea of [Gem] we pick up and point out that a Lévy process can be written as a time changed Brownian motion and the stochastic time change can be interpreted as a measure of economic activity. A certain class of Lévy processes, the *generalized hyperbolic Lévy motion* is studied in [Ebe] and [EbeKel]. Some of the main results of these papers are given here, including an option pricing formula. The final section introduces *stochastic volatility* in the modelling. While in the Black Scholes model and also in the model of [Ebe] the volatility of the stock was constant it is now described itself by a stochastic process. We present the model of [BN-S] where the log-price dynamics are given by

$$dX(t) = (\mu + \beta\sigma^2(t))dt + \sigma(t)dW(t) + \varrho dZ(t)$$

with W being a standard Brownian motion and Z a (pure jump) Lévy process. Here the volatility σ^2 is the solution of an Ornstein-Uhlenbeck type stochastic differential equation:

$$d\sigma^2(t) = -\lambda\sigma^2(t)dt + dZ(\lambda t).$$

Since the volatility may be influenced by a large number of factors we will now go over to the infinite dimensional case. This leads to the well-known equation

$$dY(t) \quad = \quad JY(t)dt + CdZ(t)$$

and we can construct an Ornstein-Uhlenbeck process with the methods of Chapter 4. The stochastic volatility process can now be obtained by choosing $L^2([0,1], dx)$ as state space and then simply integrating over $[0,1]$. The empirical findings of [BaJuYa] on the dynamics of implied volatility suggest to introduce an additional non-linear term in the drift. In our context this gives the equation

$$dY(t) \quad = \quad (JY(t) + F(Y(t)))dt + CdZ(t).$$

For non-linear functions F that fulfill a Lipschitz condition we can now apply the new results from section 4.3 where we have shown existence and uniqueness of a weak solution to this equation.

5.1 The classical Black-Scholes Model

Before we turn to applications of general Lévy processes in Finance Theory we will (heuristically) review the classical option pricing formula by Fischer Black and Myron Scholes which was first introduced in [BlaSch]. Apart from [BlaSch] which gives the microeconomic foundation of the formula we also refer to [Bjö] and [Irl].

Consider the stock of a firm. We will derive a pricing formula for a *European call option.* This option gives the owner of the option the right to buy one share of the stock at a certain time T, the so-called *maturity date*, for a fixed price K, the *striking price.* Of course the holder does not need to exercise the option, so he or she will only do this if the stock price at time T is greater than K.

Again let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a complete and rightcontinuous filtration $(\mathcal{F}_t, t \geq 0)$. In the sequel all processes shall be adapted to $(\mathcal{F}_t, t \geq 0)$. The price of the stock, at different times $t \geq 0$ shall be modelled as a real-valued stochastic process $S = (S(t), t \geq 0)$. In the set-up of the Black Scholes model "the distribution of stock prices at the end of any finite interval is log-normal". Hence it is convenient to define S by

$$S(t) = S(0)\exp(X(t)) \quad \text{for all } t \geq 0 \tag{5.1}$$

where $S(0)$ is some initial value and X is a Brownian motion with drift and start in $X(0) = 0$. If we set $X(t) = \mu t - \frac{\sigma^2}{2}t + \sigma W(t)$ (with W being a standard Brownian motion) then S is the solution of the stochastic differential equation

$$dS(t) = S(t)(\mu dt + \sigma dW(t)) \tag{5.2}$$

which is also called *stochastic exponential.* (For the relation between ordinary and stochastic exponential see also [ConTan] section 8.4). The parameter μ, the *drift*, is the expected return on the stock, σ is the *volatility*, which measures "how risky" an investment in this stock is.

Later on we will extend this model to price processes for which X with $X(0) = 0$ is a Lévy process. Since for Lévy processes the stochastic exponential might be negative we will usually define the stock price process by (5.1) and refer to X as the *log-price* (the logarithm of the price).

The short-term interest rate r shall for simplicity be constant over time. In order to derive a price for the call option we can calculate what the net present value of the corresponding claim would be if it could be observed at time $t = 0$. Since the option will only be exercised if $S(T) > K$ and the stock could immediately be sold for $S(T)$ by the owner of the option the discounted value of the claim (with interest rate r) is $\exp(-rT)(S(T) - K) \wedge 0$.

So far we have investigated the price process S on the probability space $(\Omega, \mathcal{F}, P)$ where P is the "real-world" probability measure. Suppose all economic agents where *risk neutral*: Then they would not distinguish between a safe (deterministic) payment of some amount c and an uncertain (stochastic) payment which has the expectation c. In this case the price of the option would just be given by the expectation of the net present value $\exp(-rT)(S(T) - K) \wedge 0$. But for most agents it can not be expected that they are risk-neutral - many of them might be risk-averse, some of them might be risk-loving.

However we can find a probability measure Q equivalent to P under which the option price can be derived as if agents were risk-neutral. This is not the "real-world" probability measure, but an implicit probability measure that one would obtain if it was derived from the prices of financial assets. Usually risk-adjustments will be made in the prices of financial assets, in our case the stock and the option. If these adjustments are already made with the measure Q, which has by the Radon-Nykodym Theorem a density with respect to P, all agents should behave risk-neutral under this measure. But in this case the price of the option should be the expectation of the net present value of the claim (under Q), i.e.

$$\mathbb{E}_Q(\exp(-rT)(S(T) - K) \wedge 0).$$

Otherwise speculators could gain arbitrage profits: If the price of the option would for example be lower than this expectation then they could buy the option from borrowed money and the expected payout at time T would be larger than the amount they will have to pay back at time T. Vice versa they could short-sale the option and put the money on a savings account with interest rate r and in expectation the claim at time T will be lower than the deposit on the account.

Absence of such arbitrage strategies under Q can only be achieved if $(\exp(-rt)S(t); t \geq 0)$ is a martingale under Q, since then the expectation with respect to Q is constant over time. As we shall see now it is a consequence of Girsanov's theorem that such a probability measure Q which is equivalent to P exists. Q is then often called the *risk-neutral measure* or a *martingale measure* (the latter name we will not use to avoid confusion with the martingale measure as defined in section 2.5). For S given by (5.2) the risk-neutral measure Q has Radon-Nykodym density

$$\frac{dQ}{dP} = \exp\left(\frac{r-\mu}{\sigma} W(T) - \frac{(r-\mu)^2}{2\sigma^2} T\right).$$

We obtain

$$S(t) = S(0)\exp\left(\sigma W(t) + \left(\mu - \frac{\sigma^2}{2}\right)t\right) \quad P\text{-a.s.}$$

and

$$S(t) = S(0)\exp\left(\sigma \bar{W}(t) + \left(r - \frac{\sigma^2}{2}\right)t\right) \quad Q\text{-a.s.}$$

where $\bar{W} = (\bar{W}(t); t \geq 0) = (W(t) - \frac{r-\mu}{\sigma}t; t \geq 0)$ is a standard Brownian motion under Q. This means that S solves (5.2) under Q for $\mu = r$, hence it behaves like a stock with drift equal to the interest rate. Now we calculate that the price of the option is

$$\begin{aligned} & \mathbb{E}_Q(\exp(-rT)(S(T)-K)\wedge 0) \\ = \; & \exp(-rT)\mathbb{E}_Q\left(\left(S(0)\exp\left(\sigma\bar{W}(T) + \left(r - \frac{\sigma^2}{2}\right)T\right) - K\right)\wedge 0\right). \end{aligned}$$

But for Z with distribution $N(a, \gamma^2)$ one can calculate (cf. for example [Irl], p.155):

$$\begin{aligned} & \mathbb{E}((b\exp(Z) - K)\wedge 0) \\ = \; & b\exp\left(a + \frac{\gamma^2}{2}\right)\Phi\left(\frac{\log\left(\frac{b}{K}\right) + a + \gamma^2}{\gamma}\right) - K\Phi\left(\frac{\log\left(\frac{b}{K}\right) + a}{\gamma}\right) \end{aligned}$$

where Φ is the distribution function of $N(0,1)$. Hence for $b = S(0)$, $a = (r - \frac{\sigma^2}{2})T$ and $\gamma = \sigma\sqrt{T}$ we obtain that the price of the option is

$$\begin{aligned} & p_{BS}(S(0), 0, \sigma^2) \\ := \; & e^{-rT}\mathbb{E}_Q\left(\left(S(0)\exp\left(\sigma\bar{W}(T) + \left(r - \frac{\sigma^2}{2}\right)T\right) - K\right)\wedge 0\right) \\ = \; & S(0)\Phi\left(\frac{\log\left(\frac{S(0)}{K}\right) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \qquad (5.3) \\ & -Ke^{-rT}\Phi\left(\frac{\log\left(\frac{S(0)}{K}\right) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right). \end{aligned}$$

This formula is the famous Black-Scholes formula (which is equation (13) in [BlaSch]). Notice that the price of the option is independent of the expected return on the stock μ. The parameters are the initial value of the stock $S(0)$, its volatility σ, the striking price K, the maturity date T of the option and the constant short-term interest rate r. The price of the European call option on S is then at any time $0 \leq t \leq T$ given by

$$\begin{aligned} p(S(t),t) &:= p_{call}(S(t),t) := \mathbb{E}_Q\left(e^{-r(T-t)}(S(T)-K)\wedge 0 \Big| \mathcal{F}_t\right) \\ &= S(t)\Phi\left(\frac{\log\left(\frac{S(t)}{K}\right)+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \qquad (5.4) \\ &\quad -Ke^{-r(T-t)}\Phi\left(\frac{\log\left(\frac{S(t)}{K}\right)+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right). \end{aligned}$$

A *European put option* gives its holder the right to sell one share of the stock at maturity date T for the striking price K. Similar arguments as above or just the put-call-parity

$$S(t)+p_{put}(S(t),t) = p_{call}(S(t),t) + Ke^{-r(T-t)}$$

lead to the pricing formula

$$\begin{aligned} p_{put}(S(t),t) &:= \mathbb{E}_Q\left(e^{-r(T-t)}(K-S(T))\wedge 0 \Big| \mathcal{F}_t\right) \\ &= Ke^{-r(T-t)}\Phi\left(-\frac{\log\left(\frac{S(t)}{K}\right)+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \qquad (5.5) \\ &\quad -S(t)\Phi\left(-\frac{\log\left(\frac{S(t)}{K}\right)+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right). \end{aligned}$$

More generally, one may derive a pricing formula for claims that can be expressed by a measurable function H applied to $S(T)$:

$$p_{claim}(S(t),t) := \mathbb{E}_Q\left(e^{-r(T-t)}H(S(T)) \Big| \mathcal{F}_t\right).$$

5.2 Asset Price Models with Lévy Processes

One of the assumptions in the Black-Scholes model is that the log-prices of stocks shall be described by a Brownian motion. It can be asked if this is the proper class of processes to model such prices or if it might be more appropriate to represent log-prices by a Lévy process. There are several empirical studies about the distributions of the increments of log-prices.

Many of these conclude that not the normal distribution but others lead to a much better representation.

In [Fam] time series of daily prices for the thirty titles in the Dow-Jones Industrial Average of some five years were taken for the research. The frequency of changes of certain sizes in log-prices were compared to the frequency predicted by a normal distribution. Also the independence of such increments was tested. The result was that the assumption of independent increments can be justified, but the log-prices do not seem to behave like a Brownian motion. For each single stock the empirical distributions "contained more relative frequency in the central bell than would be expected under a normality hypothesis" and "in every case the extreme tails of the distributions contained more relative frequency than would be expected under the Gaussian hypothesis." This means that very small and very large changes in the log-prices are more likely to occur than it would be under a normal distribution. Hence the results are a strong argument to departure from the normal distribution and to use other infinitely divisible distributions, i.e. log-prices should be modelled by a Lévy process.

Very soon after the release of [BlaSch] a pricing formula for discontinuous processes with a certain kind of jumps was developed in [Mer]. Many other publications from various authors which described log-prices by Lévy processes followed, especially in recent years. It has to be emphasized that this does not decrease the importance of the findings in [BlaSch]. The formula given therein was the first pricing formula for derivatives of stocks and the basis for further developments later on.

Normal distributed log-price returns by the central limit theorem can be interpreted as the result of infinitely many - or a large number of - independent market transactions. But this intuitive argument for the normality assumption in the Black Scholes model does not stand in contradiction to the empirical results mentioned above: Every local semimartingale can in distribution be described by a time changed Brownian motion as was shown in [Mon]. It is pointed out in [Gem] that the existence of a risk-neutral measure under which the discounted stock prices are martingales implies that these stock prices and also the log-prices have to be semimartingales under the real world probability measure P. Log-prices can therefore be described by a time changed Brownian motion. One may write

$$\log S(t) = X(t) = W(T(t)) \tag{5.6}$$

where W is a Brownian motion. The stochastic time change can be interpreted as a measure of economic activity. [Gem] writes that (5.6) "illuminates how asset prices respond to the arrival of information. Some days, very little news, good or bad is released; trading is typically slow and prices barely fluctuate. In contrast, when new information arrives and traders adjust their expectations accordingly, trading becomes brisk and the price evolution accelerates." Two classes of pure jump Lévy processes are investigated in [Gem], the generalized hyperbolic Lévy motion and the CGMY process. A nice special case is the *normal inverse Gaussian process* (NIG) which can be written as

$$X(t) = W(T(t))$$

where W is a Brownian motion and T is an *inverse Gaussian process* independent of W, i.e. $T(t)$ is the first arrival time of a Brownian motion independent of W at the level t. The class of generalized hyperbolic Lévy motions, where the normal inverse Gaussian process belongs to, was investigated in detail in [Ebe] which we will review in the next section.

5.3 Generalized Hyperbolic Lévy Motions

The results of this section are taken from [Ebe]. A *generalized hyperbolic distribution* is a probability distribution on $\mathbb{R}$ which has a density with respect to the Lebesgue measure - depending on parameters $\lambda, \alpha, \beta, \delta, \mu$ - given by

$$\begin{aligned} & d_{GH}(x) := d_{GH}(x; \lambda, \alpha, \beta, \delta, \mu) \\ := \quad & a(\lambda, \alpha, \beta, \delta) \left(\delta^2 + (x-\mu)^2\right)^{(\lambda - \frac{1}{2})/2} K_{\lambda - \frac{1}{2}} \left(\alpha \sqrt{\delta^2 + (x-\mu)^2}\right) e^{\beta(x-\mu)} \end{aligned}$$

where

$$a(\lambda, \alpha, \beta, \delta) = \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda - \frac{1}{2}} \delta^\lambda K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}$$

is a normalizing constant and K_ν is a so-called *modified Bessel function* with index ν which has integral representation

$$K_\nu(z) = \frac{1}{2} \int_0^\infty y^{\nu - 1} \exp\left(-\frac{1}{2} z \left(y + y^{-1}\right)\right) dy.$$

Note that K_ν cannot always be expressed explicitly and therefore has to be treated numerically. (For the exact definition of modified Bessel functions, representations and approximation results see [AbrSte], section 9.6). The parameters of d_{GH} have the following properties: $\alpha > 0$ determines the shape, β with $0 \leq |\beta| < \alpha$ the skewness and $\mu \in \mathbb{R}$ the location. $\delta > 0$ is a scaling parameter, $\lambda \in \mathbb{R}$ characterizes certain subclasses. μ and δ can be compared to the parameters μ, σ of $N(\mu, \sigma)$.

[Ebe] defines a *generalized hyperbolic Lévy motion* as the Lévy process $X = (X(t), t \geq 0)$ generated by a generalized hyperbolic distribution (which is infinitely divisible) for which the distribution of $X(1)$ has density d_{GH}.

A *normal inverse Gaussian* distribution (NIG) is a distribution belonging to the subclass for $\lambda = -1/2$. The density of an NIG can also be expressed as

$$d_{NIG}(x) = \frac{\alpha}{\pi} \exp\left(\delta\sqrt{\alpha^2-\beta^2} + \beta(x-\mu)\right) \frac{K_1\left(\alpha\delta\sqrt{1+\left(\frac{x-\mu}{\delta}\right)^2}\right)}{\sqrt{1+\left(\frac{x-\mu}{\delta}\right)^2}}.$$

In [Gem] it is shown that the generated process can be written as a time changed Brownian motion as described in section 5.2. The characteristic function φ_{NIG} is derived in [Ebe]:

$$\varphi_{NIG}(u) = \exp(i\mu u)\frac{\exp(\delta\sqrt{\alpha^2-\beta^2})}{\exp(\delta\sqrt{\alpha^2-(\beta+iu)^2})}.$$

It follows that for any $t \geq 0$ the distribution of $X(t)$ is also an NIG with parameters $\alpha, \beta, t\delta, t\mu$.

Now we will describe the asset pricing model. Let $X = (X(t), t \geq 0)$ be a generalized hyperbolic Lévy motion. The price process is then given by

$$S(t) = S(0)\exp(X(t)) \tag{5.7}$$

for some initial value $S(0) = s \in \mathbb{R}$ and $X(0) = 0$. In general the risk-neutral measure for this model is not unique, but in [EbeKel] at least the existence of such a measure is shown. We will shortly repeat the construction here.

With d_{GH}^{*t} denote the density of the distribution of $X(t)$ (which is the density of the t-fold convolution of the distribution with density d_{GH}). For any $\theta \in \mathbb{R}$ with

$$\mathbb{E}(\exp(\theta X(t))) < \infty \quad \text{for all } t \geq 0$$

define a new density $d_{GH}^{*t}(\cdot;\theta)$ by

$$d_{GH}^{*t}(x;\theta) = \frac{\exp(\theta x) d_{GH}^{*t}(x)}{\int_{-\infty}^{\infty} \exp(\theta y) d_{GH}^{*t}(y)dy}. \tag{5.8}$$

Under the corresponding probability measure P^θ, for which the distribution of $X(t)$ has density $d_{GH}^{*t}(\cdot;\theta)$, the process X is again a Lévy process, the so-called *Esscher transform* of X. Let the short term interest rate be $r > 0$. If we choose θ such that

$$S(0) = \exp(-rt)\mathbb{E}_{P^\theta}(S(t)) \tag{5.9}$$

for every $t \geq 0$ then the discounted price process $(\exp(-rt)S(t); t \geq 0)$ is a martingale under P^θ. Since for any independent random variables $Y_1, \ldots, Y_n$

$$\mathbb{E}\left(\exp\left(\sum_{k=1}^n Y_k\right)\right) = \prod_{k=1}^n \mathbb{E}(\exp(Y_k))$$

and with similar arguments like in [Lin] Corollary 5.1.8. (compare Lemma 1.2.6) we have by the independence and stationarity of the increments of X

$$\begin{aligned} & \int_{-\infty}^{\infty} \exp(ux) d_{GH}^{*t}(x;\theta) dx = \mathbb{E}_{P_\theta}(\exp(uX(t))) \\ = \;& \mathbb{E}_{P_\theta}(\exp(uX(1)))^t = \left(\int_{-\infty}^{\infty} \exp(ux) d_{GH}(x;\theta) dx \right)^t . \end{aligned}$$

Hence (5.9) translates to

$$\begin{aligned} & \exp(r) = \mathbb{E}_{P^\theta}(\exp(X(1))) = \int_{-\infty}^{\infty} \exp(x) d_{GH}(x;\theta) dx \\ = \;& \left(\int_{-\infty}^{\infty} \exp(\theta y) d_{GH}(y) dy \right)^{-1} \int_{-\infty}^{\infty} \exp(x) \exp(\theta x) d_{GH}(x) dx \end{aligned}$$

and

$$\begin{aligned} r \;=\;& \log \left(\int_{-\infty}^{\infty} \exp((\theta+1)x) d_{GH}(x) dx \right) - \log \left(\int_{-\infty}^{\infty} \exp(\theta x) d_{GH}(x) dx \right) \\ =\;& \log \mathbb{E}\left(e^{(\theta+1)X(1)} \right) - \log \mathbb{E}\left(e^{\theta X(1)} \right). \end{aligned} \tag{5.10}$$

According to [EbeKel] there exists a unique solution θ^* for θ in (5.10) which can be derived by numerical methods.

By [Ebe] equation (3.5), we know that

$$\begin{aligned} & \int_{-\infty}^{\infty} \exp(ux) d_{GH}(x) dx \\ = \;& \exp(\mu u) \left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta+u)^2} \right)^{\lambda/2} \frac{K_\lambda\left(\delta\sqrt{\alpha^2 - (\beta+u)^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}. \end{aligned}$$

In the case where X is an NIG process this can be reduced to a simple form. By equations 9.6.2 and 9.6.10 in [AbrSte] it follows that

$$\begin{aligned} & K_{-1/2}(z) = \frac{\pi}{2} \left(\sum_{k=0}^{\infty} \frac{(z/2)^{2k+1/2}}{k! + \Gamma(k+3/2)} - \sum_{k=0}^{\infty} \frac{(z/2)^{2k-1/2}}{k! + \Gamma(k+1/2)} \right) \\ = \;& \frac{\pi}{2}\sqrt{\frac{2}{z}} \left(\sum_{k=0}^{\infty} \frac{(z/2)^{2k+1}}{\Gamma(2k+2)\sqrt{\pi}/(2^{2k+1})} - \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{\Gamma(2k+1)\sqrt{\pi}/(2^{2k})} \right) \\ = \;& \sqrt{\frac{\pi}{2z}} (\sinh z - \cosh z) = -\sqrt{\frac{\pi}{2z}} \exp(-z). \end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} \exp(ux) d_{GH}(x) dx = \exp\left(\mu u - \delta \left(\sqrt{\alpha^2 - (\beta+u)^2} - \sqrt{\alpha^2-\beta^2} \right) \right)$$

and

$$r = \mu + \delta \left(\sqrt{\alpha^2 - (\beta+\theta)^2} - \sqrt{\alpha^2 - (\beta+\theta+1)^2} \right)$$

if X is an NIG.

Now for any generalized hyperbolic Lévy motion and corresponding θ^* we can derive a pricing formula for a European call option with maturity date T and striking price K. Set $\gamma := \log(\frac{K}{S(0)})$. Then with (5.10) we obtain

$$\begin{aligned}
& p(S(0)) := \mathbb{E}_{P^{\theta^*}} \left(e^{-rT}(S(T) - K) \wedge 0 \right) \\
= \;& \mathbb{E}_{P^{\theta^*}} \left(e^{-rT} \left(S(0)e^{X(T)} - K \right) 1_{\{X(T) \geq \gamma\}} \right) \\
= \;& e^{-rT} S(0) \int_\gamma^\infty e^x d_{GH}^{*T}(x; \theta^*) dx - e^{-rT} K \int_\gamma^\infty d_{GH}^{*T}(x; \theta^*) dx \\
= \;& e^{-rT} S(0) \mathbb{E} \left(e^{\theta^* X(T)} \right)^{-1} \int_\gamma^\infty e^{(\theta^*+1)x} d_{GH}^{*T}(x) dx \\
& -e^{-rT} K \int_\gamma^\infty d_{GH}^{*T}(x; \theta^*) dx \\
= \;& \exp \left(-T \left(r + \log \mathbb{E} \left(e^{\theta^* X(1)} \right) \right) \right) S(0) \int_\gamma^\infty e^{(\theta^*+1)x} d_{GH}^{*T}(x) dx \\
& -e^{-rT} K \int_\gamma^\infty d_{GH}^{*T}(x; \theta^*) dx \\
= \;& \exp \left(-T \log \mathbb{E} \left(e^{(\theta^*+1) X(1)} \right) \right) S(0) \int_\gamma^\infty e^{(\theta^*+1)x} d_{GH}^{*T}(x) dx \\
& -e^{-rT} K \int_\gamma^\infty d_{GH}^{*T}(x; \theta^*) dx \\
= \;& \mathbb{E} \left(e^{(\theta^*+1) X(T)} \right)^{-1} S(0) \int_\gamma^\infty e^{(\theta^*+1)x} d_{GH}^{*T}(x) dx \\
& -e^{-rT} K \int_\gamma^\infty d_{GH}^{*T}(x; \theta^*) dx
\end{aligned}$$

and the pricing formula is

$$\begin{aligned}
& p(S(0)) \\
= \;& S(0) \int_\gamma^\infty d_{GH}^{*T}(x; \theta^* + 1) dx - e^{-rT} K \int_\gamma^\infty d_{GH}^{*T}(x; \theta^*) dx. \qquad (5.11)
\end{aligned}$$

In [EbeKel] the fit of a certain class of generalized hyperbolic Lévy motions to financial data is tested. Ten titles from the German DAX index were investigated. The hypothesis of a hyperbolic distribution could be accepted for all stocks at the level 0.01. An estimation of parameters $\alpha, \beta, \delta, \mu$ for theses stocks can also be found there.

The NIG process also gives a good fit to financial data as it is pointed out in [Ryd]. An algorithm for the simulation of NIG processes and an approximation procedure is given.

5.4 Stochastic Volatility

So far we have dealt with models in which the volatility of the stock, i.e. σ is constant over time. From an economical point of view this is a very restricting assumption. It seems more realistic that the volatility may change over time depending on several factors of influence. The crucial point will then be to find an adequate stochastic process representing the volatility.

We will assume that Q is a given risk-neutral measure. This could for example be obtained by the construction via an Esscher transform as it was described in section 5.3.

Again consider a European call option on a given stock with striking price K and maturity date T. A method to derive a price for this option in the case that the stock price process has stochastic volatility is developed in [HulWhi]. The stock price S is there described by

$$dS(t) = \mu S(t)dt + \sigma(t)S(t)dW(t)$$

where $\sigma(t) = \sqrt{\sigma^2(t)}$ and σ^2 is a certain non-negative stochastic process modelling the volatility, which we will therefore call the *stochastic volatility process*. If σ was constant we would be back in the context of section 5.1. The *mean variance* over the time period $[0, T]$ is defined as

$$\bar{V} = \frac{1}{T}\int_0^T \sigma^2(s)ds. \tag{5.12}$$

Now in the model of [HulWhi] the price of the option is given by

$$p(S(0)) = \mathbb{E}_Q(p_{BS}(S(0), 0, \bar{V})) \tag{5.13}$$

where $p_{BS}(S(0), 0, \bar{V})$ is the Black Scholes price of the option as derived in (5.3) for $\sigma^2 = \bar{V}$. Hence the option price under stochastic volatility may be derived as a mixture of Black Scholes prices.

Ole E. Barndorff-Nielsen and Neil Shepard have introduced a model where the stochastic volatility is the solution of a stochastic differential equation driven by a Lévy process. We will describe the so-called *BN-S model* following the ideas of [BN-S]. The log-price X shall be given by

$$dX(t) = (\mu + \beta\sigma^2(t))dt + \sigma(t)dW(t) + \varrho d\bar{Z}(t) \tag{5.14}$$

and the stochastic volatility process shall be the solution of

$$d\sigma^2(t) = -\lambda\sigma^2(t)dt + dZ(\lambda t). \tag{5.15}$$

Here W is a standard Brownian motion. The parameter μ can be interpreted as the drift of the stock if $\beta = -1/2$ like in the Black Scholes model. To obtain more flexibility β can be any real value. By Z we denote the so-called *background driving Lévy process* (BDLP) and set $\bar{Z}(t) = Z(t) - \mathbb{E}(Z(t))$ as a centered version of Z. We choose $\lambda > 0$ and ϱ any real number.

Equation (5.15) is of Ornstein-Uhlenbeck type, i.e. the same type of stochastic differential equation as (4.10) for $f \equiv 0$ where the Hilbert space H is in this case $\mathbb{R}$. The solution σ^2 is therefore given by

$$\sigma^2(t) = e^{-\lambda t}\sigma^2(0) + \int_0^t e^{-\lambda(t-s)} dZ(\lambda s).$$

In order to have that σ^2 is non-negative we assume that Z is a non-Gaussian Lévy process with non-negative increments, a so-called *subordinator*.

Now (5.15) can be interpreted as follows. Caused by external shocks the volatility of the stock increases suddenly. The proportion of the increase is given by a jump of the BDLP. This can be seen as an immediate reaction of the market on new relevant information. Depending on how important investors valuate the information the jump may be of different sizes. When time is elapsing the activity of the market becomes lower again. Therefore volatility decays exponentially, where the rate is determined by the parameter λ.

Following the approach of [HulWhi] a pricing formula for European call options in the context of the BN-S model is constructed in [NicVen]. Here equation (5.14) was modified: Now $\bar{Z}$ is equal to the subordinator Z and $\varrho \leq 0$. By $\mathcal{M}'$ denote the set of all risk-neutral measures Q, such that the log-price and the stochastic volatility may under Q again be described by (5.14) and (5.15), perhaps with different parameters and another law for Z.

Let $Q \in \mathcal{M}'$. We define the *effective log-price* by

$$S_{eff} = S(0)\exp(X_T + \varrho Z(\lambda T) - \lambda T \kappa^Q(\varrho))$$

where κ^Q is given by

$$\kappa^Q(\theta) = \int_{\mathbb{R}_+} (e^{\theta x} - 1)\lambda^Q(dx)$$

with λ^Q denoting the Lévy measure of Z under Q. According to [NicVen] the price of the option can now be derived as

$$p(S(0)) = \mathbb{E}_Q(p_{BS}(S_{eff}), 0, \bar{V})). \tag{5.16}$$

[NicVen] also determine the range of prices depending on the risk-neutral measure Q.

It is suggested in [BN-S] and [NicVen] to use stochastic volatility processes given by (5.15) as "building blocks" for more general processes.

The approach taken there is to define the stochastic volatility process σ^2 as a weighted sum of independent Ornstein-Uhlenbeck processes where each of these may represent the volatility changes caused by a single influencing factor:

$$\begin{aligned}\sigma^2(t) &= \sum_{j=1}^{N} \sigma_j^2(t) \\ d\sigma_j^2(t) &= -\lambda_j \sigma_j^2(t)dt + dZ_j(\lambda_j t).\end{aligned}$$

By this set-up the stochastic volatility is the result of external shocks coming from different sources. As the number of such sources, which may be macroeconomic developments or actions of individual agents, can be assumed to be very large it seems appropriate to choose an infinite dimensional model.

In the situation described in section 4.2, i.e. on any separable Hilbert space H (and under Assumption 4.0.1), Theorem 4.2.1 gives the unique weak solution of the Ornstein-Uhlenbeck type stochastic differential equation

$$dY(t) = JY(t)dt + CdZ(t) \tag{5.17}$$

where Z is an H-valued Lévy process. The solution is given by

$$Y(t) = S(t)Y_0 + \int_0^t S(t-s)CdZ(s). \tag{5.18}$$

(Recall that $(S(t), t \geq 0)$ is a C_0-semigroup with infinitesimal generator J and $C \in L(H)$). Now much richer stochastic volatility processes can be constructed by choosing for example $H = L^2([0,1], dx)$ (dx denoting the Lebesgue measure). If Y is an $L^2([0,1], dx)$-valued Ornstein-Uhlenbeck process we can define a real valued process σ^2 by

$$\sigma^2(t) = \int_0^1 Y(t)(x)dx \tag{5.19}$$

for any $t \geq 0$.

In [BN] stochastic volatility processes are constructed by integration over a continuum of real-valued Ornstein-Uhlenbeck processes. Hence the approach there is similar to the one suggested above. However the single Ornstein-Uhlenbeck processes there have to fulfil a certain independence condition. This indicates that the model described here might be more flexible as the influencing factors are allowed to interact with each other.

Recent empirical results indicate that a non-linear drift term should be added in the dynamics of the stochastic volatility process. In [BaJuYa] it has been tried to estimate the dynamics of stochastic volatility. The authors examined the Chicago Board Option Exchange Market Volatility Index (VIX).

This index represents *implied volatilities*, i.e. volatilities derived from realized option prices and prices of the corresponding stocks by the Black Scholes formula. Underlyings are eight stocks from the S&P 100 index and corresponding options. The VIX is the average of the eight implied volatilities. [BaJuYa] use the index as a proxy for stochastic volatility. Different models for the dynamics of stochastic volatility are ranked by their goodness of fit to the data of the VIX in an empirical survey. One of the results is that models which include non-linear drift terms are preferred against models from the same model class without any non-linear term in the drift.

If we incorporate these results in the infinite dimensional context from above this leads to equations of the type

$$dY(t) = (JY(t) + F(Y(t)))dt + CdZ(t) \tag{5.20}$$

for a non-linear function $F : H \to H$.

If in addition F fulfills a Lipschitz condition then we are in the context of section 4.3. For a Lévy process Z such that condition (4.1) is valid there exists a solution to (5.20) by Theorem 4.3.3. From the solution Y one may now obtain the stochastic volatility process σ^2 like in (5.19).

Symbols

$\mathfrak{B}(E)$	Borel σ-algebra of a Banach space E	p.9
$L^p((X, \mathcal{A}, \mu) \to E)$	L^2-space of E-valued functions	p.11
$\mathfrak{M}(E)$	set of all finite measures on E	p.12
$\mathfrak{P}(E)$	set of all probability measures on E	p.12
$L(E, F)$	space of bounded linear operators from E to F	p.14
$\mathfrak{G}(E)$	set of all Gaussian symmetric probability measures on E	p.15
$e(\lambda)$	the exponent of a finite measure λ	p.15
$e_s(\lambda)$	the generalized exponent of a finite measure λ	p.17
λ	a Lévy measure	p.18
$\mathfrak{L}(E)$	set of all Lévy measures on E	p.18
$[x_0, Q, \lambda]$, $[x_0, \varrho, \lambda]$	generating triplets of an infinitely divisible measure	p.21
$K(x, a)$	kernel	p.17
$L(x, a)$	classical kernel	p.22
$L^p(E)$	$L^p((\Omega, \mathcal{F}, P) \to E)$	p.22
$\mathbb{E}(X\|\mathcal{G})$	conditional expectation of a real- or H-valued random variable X with respect to $\mathcal{G}$	p.27
$\mathcal{P}$	the predictable σ-algebra on $\mathbb{R}_+ \times \Omega$	p.27
trQ	trace of a bounded linear operator Q	p.30
$\sqrt{R}$	square root of a non-negative bounded linear operator R	p.30
$N(t, A)$	Poisson random measure	p.33
$\tilde{N}(t, A)$	compensated Poisson random measure	p.34
L^2_λ	$L^2(H \setminus \{0\}, \lambda\|_{H\setminus\{0\}}; H)$	p.36
B_Q	Brownian motion with covariance operator Q	p.40
(S, Σ)	a Lusin topological space	p.41
$\mathcal{A}_0, \mathcal{A}$	rings on (S, Σ)	p.41
$\mathcal{A}_0, \mathcal{A}$	in the case $S = \{x \in H; \\|x\\| < 1\}$ see	p.40
$M(t, A)$	martingale measure	p.41

T	a positive-operator valued measure	p.42
$L(H)$	$L(H,H)$	p.50
$\mathcal{L}$	Borel σ-algebra generated by the strong topology on $L(H)$	p.50
$\mathcal{N}_2(T)$	space of strongly integrable mappings	p.51
$\mathcal{S}_2(T)$	subspace of the simple functions in $\mathcal{N}_2(T)$	p.52
$\mathcal{N}_2^w(T)$	space of weakly integrable functions	p.58
$\mathcal{S}_2^w(T)$	subspace of the simple functions in $\mathcal{N}_2^w(T)$	p.59
$\mathcal{G}_2(W)$	space of functions integrable with respect to "$M(ds,dx) \otimes \mu(dw)$"	p.65
$D(J)$	domain of an operator J	p.72
$X_{J,C}$	stochastic convolution	p.73

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