Infinite dimensional Langevin equations: Uniqueness and rate of convergence for finite dimensional approximations

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Abstract

The paper deals with the infinite dimensional stochastic equation $\mathrm{d}X = B(t,X)\,\mathrm{d}t + \mathrm{d}W$ driven by a Wiener process which may also cover stochastic partial differential equations. We study a certain finite dimensional approximation of $B(t,X)$ and give a qualitative bound for its rate of convergence to be high enough to ensure the weak uniqueness for solutions of our equation. Examples are given demonstrating the force of the new condition.

KEY WORDS stochastic partial differential equation, Girsanov theorem, weak uniqueness, martingale problem

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1 Introduction

One of the most studied stochastic differential equations is

$$(\star) \qquad \mathrm{d}X \;=\; B(t,X)\,\mathrm{d}t + \mathrm{d}W$$

which goes back to P. Langevin, C.R. Acad. Sci. Paris **146** (1908), S. 350ff. In our equation, B denotes a general measurable drift, for example $B(t,X) = b(X_t)$ where

$$b : E \to E,\ E \text{ locally convex space,}$$

while $\mathrm{d}W$ is the stochastic Itô-differential with respect to some Wiener process with a covariance given by the problem that the equation models.

In difference to the deterministic equation without the Itô-differential $\mathrm{d}W$, for singular drifts B, instaed of a strong solution X, one can often only construct a weak solution, that is a probability measure on some path space over E; and, the question of weak uniqueness arises: Is this probability measure unique for a given initial distribution? In the case $\dim E = d$ where equation $(\star)$ is usually assumed to be driven by a d-dimensional Wiener process W, a good answer (cf. Prop.5.3.10 in [10] for example) based on the celebrated Girsanov theorem [4] is the following: For a given initial distribution there is at most one probability measure $\mathbf{P}$ on the path space $C([0,\infty) \to \mathbb{R}^d)$ such that the coordinate process X satisfies both

$$(1) \qquad \mathbf{P}(\{\int_0^T \|B(t,X)\|^2_{\mathbb{R}^d}\,\mathrm{d}t < \infty\}) = 1,\ \forall T > 0,$$

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and

$$X_t - X_0 - \int_0^t B(s,X)\,\mathrm{d}s,\ t \geq 0, \text{ is a } d\text{-dimensional Wiener process.}$$

In the important infinite dimensional case $\dim E = \infty$ where the drift B may also cover differential operators on function spaces, an analogous result holds true. Indeed, if the driving Wiener process is white in time and has the inner product of a separable Hilbert space H as covariance in space where $E' \subseteq H' \equiv H \subseteq E$ densely and continuously, then one just has to replace (1) by

$$\mathbf{P}(\{\int_0^T \|B(t,X)\|_H^2\,\mathrm{d}t < \infty\}) = 1,\ \forall T > 0, \tag{2}$$

assuming that an appropriate path space is chosen.

Obviously, we can think about (1) as a version of (2) if we understand the inner product of $\mathbb{R}^d$ as the "covariance in space" of a d-dimensional Wiener process. But in infinite dimensions, the condition (2) is useless for important applications, because $B(t,X)$ takes values in a space different from the Hilbert space H. Let us explain this phenomenon by means of an example which has recently moved again into sight of the research (cf. [14],[8],[12]).

To start with, denote the Laplacian with fixed boundary conditions (Dirichlet or Neumann) in a bounded open rectangle $\mathcal{O} \subseteq \mathbb{R}^2$ by Δ and look at the simple equation

$$\mathrm{d}X = -\frac{1}{2}(1-\Delta)X\,\mathrm{d}t + \mathrm{d}W \tag{3}$$

driven by space-time white noise, that is $H = L^2(\mathcal{O})$. A good choice for E is the space of generalized functions $H^{-\infty} = \bigcup_{\alpha\in\mathbb{R}} H^\alpha$ where H^α is the scale of Sobolev spaces generated by $(1-\Delta)$ and $L^2(\mathcal{O})$. We know the unique explicit solution of the last equation given by

$$X_t = S(t)X_0 + \int_0^t S(t-s)\,\mathrm{d}W_s,\ t \geq 0,$$

where

$$S(t) = \exp\{-\frac{1}{2}t(1-\Delta)\},\ t \geq 0.$$

As a consequence, the solution above starting from $X_0 \equiv 0$ is an $H^{-\delta}$-valued process, $\delta > 0$ arbitrary but fixed, such that

$$\|X_t\|_H = \|X_t\|_{L^2(\mathcal{O})} = \infty,\ t \geq 0, \text{ a.s.},$$

because the semigroup of the Laplacian in 2 dimensions is not smoothing enough to define an H-valued stochastic integral with respect to W (cf. [15]).

Hence, for $b = -\frac{1}{2}(1-\Delta)$ we also have

$$\|B(t,X)\|_H = \|b(X_t)\|_H = \infty,\ t \geq 0, \text{ a.s.},$$

and, (2) is useless or is not satisfied. Fortunately, here we do not need the condition (2) to decide the question of uniqueness for the equation (3).

Even more, because of the outstanding properties of its explicit solution above, one can consider perturbations of the equation (3), that is

$$b = -\frac{1}{2}(1-\Delta) + \beta,$$

getting the result (cf. [7],[13]): For a given initial distribution, there is at most one probability measure $\mathbf{P}$ on the corresponding path space such that the coordinate process X satisfies both

$$\mathbf{P}(\{\int_0^T \|\beta(X_t)\|_H^2\,\mathrm{d}t < \infty\}) = 1,\ \forall T > 0, \tag{4}$$

and

$$X_t - X_0 - \int_0^t B(s,X)\,\mathrm{d}s,\ t \geq 0,\ \text{ is an } E\text{-valued Wiener process associated with } H.$$

The example announced above is

$$\beta(z) \ = \ -\lambda : z^3 :,\ z \in H^{-\delta},\ \lambda > 0 \text{ constant},$$

that is the third Wick power which models polynomial interaction in the two dimensional quantum field theory (cf. [5]).

From the mathematical point of view, β is highly singular since we only know that $\beta \in L^2(H^{-\delta} \to H^{-\delta}; \mu_0)$ and $\|\beta(z)\|_{L^2(\mathcal{O})} = \infty$ for μ_0-a.e. $z \in H^{-\delta}$ where μ_0 is the Gaussian measure on $H^{-\delta}$ with the characteristic functional

$$\int_{H^{-\delta}} \exp\{i_{H^\delta}\langle \phi, z\rangle_{H^{-\delta}}\}\,\mu_0(\mathrm{d}z) \ = \ \exp\{-\frac{1}{2}\|\phi\|^2_{H^{-1}}\},\ \phi \in H^\delta.$$

Now, there are two different methods to construct weak solutions of the equation

$$\mathrm{d}X \ = \ [-\frac{1}{2}(1-\Delta)X - \lambda : X^3 :]\,\mathrm{d}t + \mathrm{d}W.$$

By the first method, cf. [1], based on Dirichlet form techniques, a full Markov family related to the equation is constructed, while using the second method, cf. [14], based on tightness arguments, only a weak stationary solution starting from the equilibrium

$$X_0 \sim \mu,\ \mathrm{d}\mu = \exp\{-\frac{1}{2}\lambda : z^4 : (\mathcal{O})\}\,\mathrm{d}\mu_0,$$

is constructed. But the Markov family also gives a weak stationary solution starting from $X_0 \sim \mu$. Furthermore, in [14] is shown that any weak stationary solution starting from $X_0 \sim \mu$ does not satisfy

$$\mathbf{P}(\{\int_0^T \|: X_t^3 :\|^2_{L^2(\mathcal{O})}\,\mathrm{d}t < \infty\}) \ = \ 1$$

for any $T > 0$; and, because it lacks of another applicable criterion, we do not know whether the solutions starting from $X_0 \sim \mu$ constructed by the two different methods above are equal or not.

In this paper we establish a new condition which ensures, that a class of probability measures solving equation ($\star$) weakly, has at most one element. We also give examples showing that our condition works in situations where the condition (2) resp. (4) is hardly checkable or is even not satisfied.

The idea is to refine the condition (2) as follows. Additionally assume that E is a topological Souslin space and that there exist $e_k \in E'$, $k = 1, 2, ...$, separating the points of E such that $\{e_k\} \subseteq H' \equiv H$ forms an orthonormal basis of H. Then the condition (2) reads

$$\mathbf{P}(\{\int_0^T \sum_{k=1}^{\infty} {}_{E'}\langle e_k, B(t,X)\rangle_E^2\,\mathrm{d}t < \infty\}) \ = \ 1,\ \forall T > 0.$$

Remark: The condition ignores the impact of different structures of B corresponding to different sequences $\{e_k\}$ because the limes of the Fourier series is the same for all possible bases. In order to regard such different structures of B, we consider the σ-fields

$$\mathcal{F}_t^N \ = \ \sigma\{{}_{E'}\langle e_k, X_s\rangle_E,\ s \leq t,\ k = 1, ..., N\}$$

and investigate the rate of convergence for

$$\mathbf{E}\left(\left[\begin{array}{c} {}_{E'}\langle e_1, B(t,X)\rangle_E \\ \vdots \\ {}_{E'}\langle e_N, B(t,X)\rangle_E \end{array}\right] \middle| \mathcal{F}_t^N\right) \stackrel{N\to\infty}{\longrightarrow} B(t,X) \approx \left[\begin{array}{c} {}_{E'}\langle e_1, B(t,X)\rangle_E \\ \vdots \\ {}_{E'}\langle e_N, B(t,X)\rangle_E \\ \vdots \end{array}\right] \tag{5}$$

which from a heuristic point of view "measures", how fast the drift $B(t,X)$ is approximated by N-dimensional functionals of the first N components of the process X restricted to the time interval $[0,t]$, $t \geq 0$, if N goes to infinity. Our condition gives a qualitative bound (see Remark 2a) below) for the rate of convergence to be high enough to ensure the uniqueness for the weak solution $\mathbf{P}$ of $(\star)$. The rate of convergence for (5) is of course expected to depend on the special choice of the approximating finite dimensional subspaces. So it is not surprising that we easily found an example showing that our condition can be satisfied or not according to the choice of the sequence $\{e_k\}$.

Please note that up to now we could neither successfully check our condition nor prove that it is not satisfied in case of the open problem discussed above. But, we may easily reprove all known results about the weak uniqueness for solutions of the equations corresponding to the stochastic quantization in two dimensions (see Remark 5b)).

2 Notation and main result

Let E be a real locally convex space which is additionally assumed to be a topological Souslin space, i.e. the continuous image of a complete metric space. If E' is its topological dual, let $\langle\cdot,\cdot\rangle$ denote the dual pairing between E' and E. Suppose that there exist $e_k \in E'$, $k = 1, 2, ...,$ separating the points of E and introduce the σ-algebras

$$\mathcal{A}_N = \sigma\{\langle e_k, \cdot\rangle,\ k = 1, ..., N\},\ N \geq 1,$$

we need for finite dimensional approximations. It is well known in the theory of Souslin locally convex spaces that for the Borel-σ-algebra

$$\mathfrak{B}(E) = \bigvee_N \mathcal{A}_N \tag{6}$$

holds true (cf. [17]).

Let exist a separable Hilbert space H which is densely and continuously embedded into E. By identifying H with its dual H' we obtain

$$E' \subseteq H \subseteq E \quad \text{densely and continuously.}$$

Remark that this property is crucial for the existence of a Wiener process in E (cf. Prop. 7.2.2 in [3]). Furthermore assume that the sequence $\{e_k\}$ chosen above forms an orthonormal basis of H.

Now introduce the path space Ω to be the set of all E-valued continuous functions on $[0,\infty)$ and define for $t \geq 0$

$$X_t : \Omega \to E : \omega \mapsto \omega(t)$$

as well as

$$\mathcal{F}_t = \sigma\{X_s,\ s \leq t\},\ \mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t.$$

Of course, $X = (X_t)_{t\geq 0}$ is an adapted E-valued stochastic process, the so-called *coordinate process*, on the measure space $(\Omega, \mathcal{F})$ with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$.

Consider a $\mathfrak{B}([0,\infty)) \otimes \mathcal{F}/\mathfrak{B}(E)$-measurable function

$$B : [0,\infty) \times \Omega \to E$$

particularly satisfying that for every $t \geq 0$ the mapping

$$\omega \mapsto B(t,\omega) \text{ is } \mathcal{F}_t/\mathfrak{B}(E)\text{-measurable;}$$

let μ be a probability measure on $(E, \mathfrak{B}(E))$. We say that $(\mu, \{e_k\}, B)$ defines a martingale problem.

Definition 1 A probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F})$ is said to be a *solution of the martingale problem* $(\mu, \{e_k\}, B)$ if

(i) $\mathbf{P} \circ X_0^{-1} \sim \mu$;

(ii) For all $k \geq 1$ and all $T > 0$,

$$\int_0^T \mathbf{E}|\langle e_k, B(t,X)\rangle| \,\mathrm{d}t < \infty \quad \text{and} \quad \mathbf{P}(\{\int_0^T \langle e_k, B(t,X)\rangle^2 \,\mathrm{d}t < \infty\}) = 1;$$

(iii) For every $e_{k_1}, e_{k_2} \in \{e_k\}$, the processes

$$W_t^{k_i} := \langle e_{k_i}, X_t\rangle - \langle e_{k_i}, X_0\rangle - \int_0^t \langle e_{k_i}, B(s,X)\rangle \,\mathrm{d}s,\ t \geq 0,\ i = 1,2,$$

and

$$W_t^{k_1} W_t^{k_2} - t\,\delta_{k_1,k_2},\ t \geq 0,$$

are continuous local martingales on the stochastic basis $(\Omega, \mathcal{F}, \mathbf{P}, \mathbb{F})$.

The set of all solutions to $(\mu, \{e_k\}, B)$ is denoted by $\mathbf{S}(\mu, \{e_k\}, B)$.

Remark 1 a) The condition (ii) of the definition above assumes more than is necessary for the existence of the integrals in Definition 1,(iii). This condition also covers more or less "natural" assumptions related to the uniqueness of the martingale problem (see Section 3).

b) For $\mathbf{P} \in \mathbf{S}(\mu, \{e_k\}, B)$, the processes W^k, $k \in \mathbb{N}$, defined in Definition 1,(iii) are independent one-dimensional standard Wiener processes on $(\Omega, \mathcal{F}, \mathbf{P}, \mathbb{F})$.

c) A *weak solution of the equation* $(\star)$ starting from μ is a probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F})$ with $\mathbf{P} \circ X_0^{-1} \sim \mu$ such that the process

$$W_t := X_t - X_0 - \int_0^t B(s,X)\,\mathrm{d}s,\ t \geq 0,$$

is well-defined and presents an E-valued continuous Gaussian process on $(\Omega, \mathcal{F}, \mathbf{P})$ with covariance

$$\mathbf{E}\langle \phi, W_t\rangle\langle \psi, W_s\rangle \;=\; (t \wedge s)(\phi, \psi)_H,$$

i.e. an *E-valued Wiener process associated with H*. If $\mathbf{P}$ is a weak solution of $(\star)$ starting from μ additionally satisfying Definition 1,(ii), then $\mathbf{P} \in \mathbf{S}(\mu, \{e_k\}, B)$, obviously. But the converse is in general not true. Nevertheless, the uniqueness of the martingale problem implies that there is not more than one weak solution of $(\star)$ additionally satisfying Definition 1,(ii).

d) Each $\mathfrak{B}(E)/\mathfrak{B}(E)$ measurable function $b : E \to E$ naturally defines a drift function B by $B(t,\omega) = b(\omega(t))$. In what follows we use the small letter b to denote drift functions defined by this means.

As in the introduction, we introduce the σ-algebras

$$\mathcal{F}_t^N = \sigma\{X_s^{-1}(\mathcal{A}_N),\, s \le t\},\ N \ge 1,$$

and remark that for all $t \ge 0$

$$\mathcal{F}_t = \bigvee_N \mathcal{F}_t^N \tag{7}$$

follows from (6). We say that a *probability measure* $\mathbf{P}$ *on* $(\Omega, \mathcal{F})$ *satisfies the condition* $(C^B_{\{e_k\}})$ if

$$(C^B_{\{e_k\}}) \qquad \begin{cases} \text{The sequence} \\ \left(\int_0^T \sum_{k=1}^N \left[\mathbf{E}(\langle e_k, B(s,X)\rangle | \mathcal{F}_s^N) - \langle e_k, B(s,X)\rangle\right]^2 \mathrm{d}s \right)_{N=1}^{\infty} \\ \text{converges to zero in probability } \mathbf{P} \text{ for every } T > 0. \end{cases}$$

Remark 2 a) If $\mathbf{P} \in \mathbf{S}(\mu, \{e_k\}, B)$ then, from (7) and the properties of the conditional expectation follows, that each summand $[\mathbf{E}(\langle e_k, B(s,X)\rangle | \mathcal{F}_s^N) - \langle e_k, B(s,X)\rangle]$ converges to zero in $L^1(\mathbf{P})$, which we have also denoted by (5) above. But, only if the rate of convergence for (5), separately considered for each component, is high enough then the condition $(C^B_{\{e_k\}})$ can be satisfied.

b) Often, in concrete examples, one can explicitly determine the rate of convergence for the convergence demanded in $(C^B_{\{e_k\}})$ which gives a good quantity for the rate of convergence with respect to (5).

The main result of the paper is

Theorem 1 *There is not more than one solution of the martingale problem* $(\mu, \{e_k\}, B)$ *which satisfies the condition* $(C^B_{\{e_k\}})$.

Remark 3 The theorem only deals with the uniqueness problem, that means, its assumptions are not sufficient for the existence of a weak solution of $(\star)$ for which, possibly, further conditions are needed. But, in infinite dimensions many techniques apply to the construction of solutions, each of them having its own framework with special assumptions (also see the next section). Therefore, we decided to consider the uniqueness problem apart from the existence problem using a framework as wide as possible.

The theorem will be proven in the last section of the paper; in the following section we demonstrate how the theorem applies.

3 Corollaries and examples

First of all we treat the finite dimensional case.

Corollary 1 *Let* $E = H = \mathbb{R}^d$*, denote by* $\{e_k\}$ *the sequence* $\{e_1, ..., e_d, e_d, ...\}$ *where* $(e_1, ..., e_d)$ *is the canonical basis of* $\mathbb{R}^d$*, and introduce the corresponding martingale problem* $(\mu, \{e_k\}, B)$ *as above. Then, for each initial distribution* μ*, there is not more than one solution to* $(\mu, \{e_k\}, B)$.

PROOF. Of course, $E, H, \{e_k\}$ satisfy all assumptions made at the beginning of Section 2. Emphasize that

$$\mathcal{A}_N = \mathfrak{B}(E) \text{ for } N \ge d$$

and, thus,

$$\mathcal{F}_t^N = \mathcal{F}_t \text{ for } N \geq d.$$

As a consequence, if there is a probability measure $\mathbf{P} \in \mathbf{S}(\mu, \{e_k\}, B)$ then it satisfies the condition $(C^B_{\{e_k\}})$ proving the corollary by Theorem 1. ■

Remark 4 The proof above shows that in the finite dimensional case the conditional expectations with respect to the "approximating" σ-algebras $\mathcal{F}_t^N$ are not needed to decide the weak uniqueness, and so we do not need the first part of condition (ii) in Definition 1 ensuring the existence of them either. Hence, the only condition on the drift B is the second part of condition (ii) in Definition 1 which coincides with the well-known condition (1) discussed in the introduction.

Now, we give a first result to what extent the condition (2) resp. (4) is generalized by our theorem in infinite dimensions.

Corollary 2 *Let $A : [0,\infty) \times \Omega \to E$ be a $\mathfrak{B}([0,\infty)) \otimes \mathcal{F}/\mathfrak{B}(E)$-measurable function such that*

$$\exists\, N_0\, \forall\, N \geq N_0\, \forall\, k \leq N\, \forall\, t \geq 0 : \omega \mapsto \langle e_k, A(t,\omega)\rangle \text{ is } \mathcal{F}_t^N\text{-measurable.}$$

For an arbitrary but fixed initial distribution μ, consider the subset of solutions

$$\mathfrak{K} = \{\mathbf{P} \in \mathbf{S}(\mu, \{e_k\}, B) : \mathbf{E} \int_0^T \|(B-A)(s,X)\|_H^2 \, \mathrm{d}s < \infty, T > 0\}.$$

Then there is not more than one element in $\mathfrak{K}$.

Proof. First remark that our setting allows to extend the norm in H to E by

$$\|z\|_H^2 := \sum_k \langle e_k, z\rangle^2, \; z \in E,$$

being finite or possibly infinite.

Suppose $\mathfrak{K} \neq \emptyset$, if $\mathfrak{K} = \emptyset$ then there is nothing to show. Fix $T > 0$ and choose $\mathbf{P} \in \mathfrak{K}$, arbitrarily. In what follows, we verify the condition $(C^B_{\{e_k\}})$ for $\mathbf{P}$ proving the corollary.

Obviously, $(C^B_{\{e_k\}})$ is equivalent to $(C^{B-A}_{\{e_k\}})$ because of our assumption, and so it is sufficient to show that

$$\lim_{N\to\infty} \mathbf{E} \int_0^T \sum_{k=1}^N \left[\mathbf{E}(\langle e_k, (B-A)(s,X)\rangle | \mathcal{F}_s^N) - \langle e_k, (B-A)(s,X)\rangle\right]^2 \mathrm{d}s \;=\; 0$$

which particularly follows from

$$\lim_{N\to\infty} \mathbf{E} \int_0^T \sum_{k=1}^\infty \left[\mathbf{E}(\langle e_k, (B-A)(s,X)\rangle | \mathcal{F}_s^N) - \langle e_k, (B-A)(s,X)\rangle\right]^2 \mathrm{d}s \;=\; 0.$$

But, we have that for all N

$$\mathbf{E} \int_0^T \sum_{k=1}^\infty \left[\mathbf{E}(\langle e_k, (B-A)(s,X)\rangle | \mathcal{F}_s^N) - \langle e_k, (B-A)(s,X)\rangle\right]^2 \mathrm{d}s$$

$$\leq 2 \int_0^T \sum_{k=1}^\infty \mathbf{E}\langle e_k, (B-A)(s,X)\rangle^2 \, \mathrm{d}s$$

$$= 2\,\mathbf{E}\int_0^T \|(B-A)(s,X)\|_H^2\,\mathrm{d}s \;<\; \infty$$

since $\mathbf{P}\in\mathfrak{K}$. Therefore, applying Lebesgue's theorem, we get

$$\lim_{N\to\infty}\mathbf{E}\int_0^T\sum_{k=1}^{\infty}\left[\mathbf{E}(\langle e_k,(B-A)(s,X)\rangle|\mathcal{F}_s^N)-\langle e_k,(B-A)(s,X)\rangle\right]^2\,\mathrm{d}s$$

$$=\int_0^T\sum_{k=1}^{\infty}\lim_{N\to\infty}\mathbf{E}\left[\mathbf{E}(\langle e_k,(B-A)(s,X)\rangle|\mathcal{F}_s^N)-\langle e_k,(B-A)(s,X)\rangle\right]^2\,\mathrm{d}s\;=\;0$$

which converges to zero by (7). ■

Remark 5 a) If weak uniqueness holds for the solutions of the equation

$$\mathrm{d}X \;=\; A(t,X)\,\mathrm{d}t+\mathrm{d}W$$

then the set $\mathfrak{K}_0=\{\mathbf{P}:\mathbf{P}$ is a weak solution of $(\star)$ such that $\mathbf{E}\int_0^T\|(B-A)(s,X)\|_H^2\,\mathrm{d}s<\infty$, $T>0.\}$ is already known to contain not more than one element. The condition on the integral in $\mathfrak{K}_0$ to be finite can even be replaced by the weaker condition $\mathbf{P}(\{\int_0^T\|(B-A)(s,X)\|_H^2\,\mathrm{d}s<\infty\})=1$, $T>0$, if the solutions of the equation $(\star)$ with drift A are unique in a sense stronger than weak uniqueness (cf. [13]). But in infinite dimensions one can often only verify the condition $\mathbf{P}(\{\int_0^T\|(B-A)(s,X)\|_H^2\,\mathrm{d}s<\infty\})=1$, $T>0$, of which the condition (4) is a special case, for probability measures contained in a subclass of $\mathfrak{K}_0$. For a well-known example remind the open problem discussed in the introduction, and, consider the equation $(\star)$ with the drift

$$-\frac{1}{2}(1-\Delta)^{1-\alpha}z-(1-\Delta)^{-\alpha}\beta(z),\;\; z\in H^{-\infty},$$

driven by an $H^{-\infty}$-valued Wiener process associated with $H^{-\alpha}$ for some $\alpha>0$. Choose $A=-\frac{1}{2}(1-\Delta)^{1-\alpha}$. Then from the results in [14] follows that $\mathfrak{K}_0$ and its nonempty subclass $\{\mathbf{P}\in\mathfrak{K}_0:$ X is a stationary process on $(\Omega,\mathcal{F},\mathbf{P})\}$ coincide.

b) Applying Corollary 2, we obtain the same result for the example in the part a) above. Indeed, $-\frac{1}{2}(1-\Delta)^{1-\alpha}$ presents an easy possibility to fix A if $\{e_k\}$ is chosen to be the eigenbasis of the corresponding Laplacian.

We have already mentioned in the introduction that one feature of our condition $(C^B_{\{e_k\}})$ consists in including the special structure of the drift B with respect to an appropriate sequence $\{e_k\}$, and, the result below is the consequence.

Proposition 1 *Fix $E,H,\{e_k\}$ as in the previous section but additionally choose another sequence $\{\tilde{e}_k\}$ of the same kind as $\{e_k\}$. If $\mathbf{P}\in\mathbf{S}(\mu,\{e_k\},B)\cap\mathbf{S}(\mu,\{\tilde{e}_k\},B)$ satisfies the condition $(C^B_{\{e_k\}})$ then it does not follow that $\mathbf{P}$ also satisfies the condition $(C^B_{\{\tilde{e}_k\}})$, in general. Moreover, this implication even remains true if $\mathbf{P}$ is the only solution to $(\mu,\{\tilde{e}_k\},B)$.*

Proof. Consider the Laplacian Δ_N with Neumann boundary conditions in the interval $(0,1)$ of the real line. Let $\{e_k\}$ be its eigenbasis in $H=L^2((0,1))$ and set $E=H^{-\infty}=\bigcup_{\alpha\in\mathbb{R}}H^\alpha$ where H^α is the scale of Sobolev spaces generated by $1-\Delta_N$ and $L^2((0,1))$. The space E equipped with its canonical topology becomes a Souslin locally convex space of which its topological dual is $E'=H^\infty=\bigcap_{\alpha\in\mathbb{R}}H^\alpha$. It is well-known for $\{e_k\}$ to be a subset of E' as well as to separate the points of E, and Sobolev's embedding theorem particularly gives

$$E'\subseteq H\subseteq E\quad\text{densely and continuously.}$$

So, $E, H, \{e_k\}$ satisfy all assumptions made at the beginning of Section 2. Define the function

$$B : [0,\infty) \times \Omega \to E$$

by

$$\langle \phi, B(t,\omega)\rangle \;=\; \langle \phi'', \omega(t)\rangle,\ \phi \in E', \tag{8}$$

where ϕ'' denotes the second derivative of ϕ. Now, in E, consider the equation

$$X_t \;=\; \int_0^t B(s,X)\,\mathrm{d}s + W_t,\ t \geq 0, \tag{9}$$

that is

$$\langle \phi, X_t\rangle \;=\; \int_0^t \langle \phi'', X_s\rangle\,\mathrm{d}s + \langle \phi, W_t\rangle,\ t \geq 0,\ \phi \in E',$$

driven by an E-valued Wiener process associated with H (see Remark 1).

Equation (9) has already been studied in early papers about stochastic partial differential equations and we know that it has a unique strong solution with outstanding properties (cf. [18] for example). On the one hand, there is exactly one probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F})$ with $\mathbf{P} \circ X_0^{-1} = \delta_{\{0\}}$ solving the equation (9) weakly, and, on the other hand, $\mathbf{P} \in \mathbf{S}(\delta_{\{0\}}, \{e_k\}, B)$.

But, all elements of $\mathbf{S}(\delta_{\{0\}}, \{e_k\}, B)$ trivially satisfy the condition $(C^B_{\{e_k\}})$ since

$$\langle e_k, B(s,X)\rangle \;=\; -\lambda_k \langle e_k, X_s\rangle,\ s \geq 0,\ \forall\, k \geq 1,$$

where λ_k is the eigenvalue of $-\Delta_N$ corresponding to e_k. Applying Theorem 1, the only weak solution of the equation (9) is also the only solution to $(\delta_{\{0\}}, \{e_k\}, B)$.

Now involve the space $C_0^\infty((0,1))$ of all infinitely many differentiable functions having compact support in $(0,1)$. This space equipped with its canonical topology has the space of Schwartz distributions on $(0,1)$ as its topological dual, which we denote by $\tilde{E}$. Then it holds that

$$C_0^\infty((0,1)) \subseteq E' \subseteq H \subseteq E \subseteq \tilde{E} \quad \text{continuously}$$

as well as

$$C_0^\infty((0,1)) \subseteq H \subseteq \tilde{E} \quad \text{densely},$$

and we choose a sequence $\{\tilde{e}_k\} \subseteq C_0^\infty((0,1))$ separating the points of $\tilde{E}$ which also forms an orthonormal basis of H. Remark that such a sequence as a matter of fact exists in our situation.

Of course, $\{\tilde{e}_k\}$ is different from $\{e_k\}$ because $\{e_k\}$ is not included in $C_0^\infty((0,1))$. Furthermore, $\{\tilde{e}_k\}$ also separates the points of E. Again, $\mathbf{P} \in \mathbf{S}(\delta_{\{0\}}, \{\tilde{e}_k\}, B)$ holds true for the measure $\mathbf{P}$ on $(\Omega, \mathcal{F})$ mentioned above. Even more, $\mathbf{P}$ is the only solution to $(\delta_{\{0\}}, \{\tilde{e}_k\}, B)$. Indeed, if $\mathbf{P}_1$ is any solution to $(\delta_{\{0\}}, \{\tilde{e}_k\}, B)$ then it also solves equation (9) weakly, because $\{\tilde{e}_k\}$ separates the points of E and the series

$$\sum_{k=1}^\infty \left(\langle \tilde{e}_k, X_t\rangle - \int_0^t \langle \tilde{e}_k, B(s,X)\rangle\,\mathrm{d}s \right) \tilde{e}_k \;=\; \sum_{k=1}^\infty \left\langle \tilde{e}_k\,,\, X_t - \left(\int_0^t X_s\,\mathrm{d}s \right)'' \right\rangle \tilde{e}_k$$

converges in E $\mathbf{P}_1$-a.s. to an E-valued Wiener process associated with H as in the proof of Prop.7.2.3 in [3]. Hence, $\mathbf{P} = \mathbf{P}_1$ by the weak uniqueness for the solutions of (9). But we do not know whether $\mathbf{P}$ satisfies the condition $(C^B_{\{\tilde{e}_k\}})$ or not.

Denote path space, σ-algebras resp. coordinate process corresponding to $\tilde{E}, H, \{\tilde{e}_k\}$ by $\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathcal{F}}_t^N$ resp. $\tilde{X}$ and define the function

$$\tilde{B} : [0,\infty) \times \tilde{\Omega} \to \tilde{E}$$

as in (8) only replacing $\phi \in E'$ by $\phi \in C_0^\infty((0,1))$. Clearly, the process X is an $\mathcal{F}/\tilde{\mathcal{F}}$-measurable mapping

$$X : \Omega \to \tilde{\Omega},$$

and, the image measure $\tilde{\mathbf{P}} := \mathbf{P} \circ X^{-1}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ is a solution to $(\delta_{\{0\}}, \{\tilde{e}_k\}, \tilde{B})$. Moreover, $\mathbf{P}$ satisfies the condition $(C^B_{\{\tilde{e}_k\}})$ if and only if $\tilde{\mathbf{P}}$ satisfies $(C^{\tilde{B}}_{\{\tilde{e}_k\}})$.

To sum up, using the eigenbasis $\{e_k\}$ to the Neumann Laplacian Δ_N, we introduced the space E together with an E-valued function B and realized, that there is exactly one solution $\mathbf{P}$ to $(\delta_{\{0\}}, \{e_k\}, B)$ and that $\mathbf{P}$ particularly satisfies the condition $(C^B_{\{e_k\}})$. Then, for an appropriate sequence $\{\tilde{e}_k\}$ different from $\{e_k\}$ we saw, that the measure $\mathbf{P}$ induces a measure $\tilde{\mathbf{P}}$ on another measure space $(\tilde{\Omega}, \tilde{\mathcal{F}})$ solving the martingale problem $(\delta_{\{0\}}, \{\tilde{e}_k\}, \tilde{B})$.

Now, we repeat this procedure with respect to the Dirichlet Laplacian Δ_D knowing that another eigenbasis appears leading to a space different from E and so on. But, the same sequence $\{\tilde{e}_k\}$ can be chosen and the induced measure on $(\tilde{\Omega}, \tilde{\mathcal{F}})$, which we denote by $\tilde{\mathbf{Q}}$, also solves the martingale problem $(\delta_{\{0\}}, \{\tilde{e}_k\}, \tilde{B})$. In what follows, we use the symbol $\mathbf{Q}$ for the measure of which the image measure is $\tilde{\mathbf{Q}}$. Settle that if we speak about $\mathbf{Q}$, then the underlying structure is assumed to be defined by the eigenbasis to the Dirichlet Laplacian. In this sense, $\mathbf{Q}$ is the only solution to $(\delta_{\{0\}}, \{e_k\}, B)$, $\mathbf{Q}$ particularly satisfies $(C^B_{\{e_k\}})$ and $\mathbf{Q}$ satisfies the condition $(C^B_{\{\tilde{e}_k\}})$ if and only if $\tilde{\mathbf{Q}}$ satisfies $(C^{\tilde{B}}_{\{\tilde{e}_k\}})$.

Turn to the measures $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{Q}}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ both solving the martingale problem $(\delta_{\{0\}}, \{\tilde{e}_k\}, \tilde{B})$. It is well-known in the theory that $\tilde{X}$ presents a Gaussian process on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ resp. $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{Q}})$ with variance

$$\int_0^t \int_0^1 \left(\int_0^1 g_N(t-s,x,y)\phi(y)\,\mathrm{d}y \right)^2 \mathrm{d}x\mathrm{d}s,\ t \geq 0,\ \phi \in C_0^\infty((0,1)),$$

resp.

$$\int_0^t \int_0^1 \left(\int_0^1 g_D(t-s,x,y)\phi(y)\,\mathrm{d}y \right)^2 \mathrm{d}x\mathrm{d}s,\ t \geq 0,\ \phi \in C_0^\infty((0,1)),$$

where g_N resp. g_D denotes Green's function of the problem

$$\frac{\partial}{\partial t} - \Delta_N = 0 \quad \text{resp.} \quad \frac{\partial}{\partial t} - \Delta_D = 0.$$

As a consequence, the measures $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{Q}}$ are different from each other and, applying Theorem 1, one of them cannot satisfy the condition $(C^{\tilde{B}}_{\{\tilde{e}_k\}})$. Hence, one of the measures $\mathbf{P}$ or $\mathbf{Q}$ does not satisfy the condition $(C^B_{\{\tilde{e}_k\}})$ proving the proposition. ■

Remark 6 Consider the drift B in the one case above where $(C^B_{\{\tilde{e}_k\}})$ is not satisfied. Then B can also be given by the function (see Remark 1,d))

$$b(z) = \sum_{k=1}^{\infty} -\lambda_k \langle e_k, z \rangle e_k,\ z \in E,$$

but, with respect to $\{\tilde{e}_k\}$ such a "nice" structure fails. Manifestly, the rate of convergence for (5) with respect to $\{\tilde{e}_k\}$ is too low for $(C^B_{\{\tilde{e}_k\}})$ to be satisfied. Emphasize that the martingale problem $(\delta_{\{0\}}, \{\tilde{e}_k\}, B)$ is well-posed since it has only one solution. So, one cannot say that $\{\tilde{e}_k\}$ would be badly chosen.

In the following example we discuss a situation where (remind Remark 5a)) the condition $\mathbf{P}(\{\int_0^T \|(B-A)(s,X)\|_H^2 \, \mathrm{d}s < \infty\}) = 1$ is never satisfied, while the condition $(C^B_{\{e_k\}})$ applies easily.

Example Introduce $\Delta_N, \{e_k\}, H^\alpha, E, H, \{\lambda_k\}$ as in the proof of Proposition 1 but, in what follows, we abbreviate the symbol Δ_N by Δ because Neumann boundary conditions are only considered.

Let g be the continuous real function defined by

$$g(x) \;=\; \begin{cases} \sqrt{|x|} & : \quad |x| \le 1 \\ 1 & : \quad |x| \ge 1 \end{cases}$$

which is not lipschitz continuous. We use g to construct the drift function

$$\tilde{b}(z) \;=\; \Delta z + \sum_{k=1}^{\infty} g(\langle e_k, z\rangle) e_k, \; z \in E.$$

The corresponding equation $(\star)$ starting from zero has a pathwise unique strong solution. Indeed, choose a probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F})$ admitting a sequence of independent one-dimensional Wiener processes (W^k). Then for each $k \ge 1$ exists a pathwise unique solution to the one-dimensional stochastic equation

$$y_t^k \;=\; \int_0^t (-\lambda_k y_s^k + g(y_s^k)) \, \mathrm{d}s \;+\; W_t^k, \; t \ge 0,$$

and the process $(\sum_{k=1}^\infty y_t^k e_k)_{t \ge 0}$ is the only strong solution of the corresponding equation $(\star)$ starting from zero driven by the E-valued Wiener process $W_t = \sum_{k=1}^\infty W_t^k e_k$, $t \ge 0$, associated with H.

Of course, we can easily handle the equation $(\star)$ with drift $\tilde{b}$ since it leads to a separate system of one-dimensional stochastic equations. A simple idea to couple the components of a drift is presented by

$$b(z) \;=\; \Delta z + \sum_{k=1}^{\infty} g(\langle e_{k+1}, z\rangle) e_k, \; z \in E.$$

We will see that, although, the structure of b is in some sense close to the structure of $\tilde{b}$, the equation $(\star)$ corresponding to b is far off to be handled easily.

At first, in order to construct a solution, various techniques fail: Picard iteration does not work because g is not lipschitz continuous; Girsanov's method is not known to be applicable since the expectation of the density process cannot sufficiently be estimated; construction via comparison theorem as in [6] does not work in this situation either. However, a possible construction of a weak solution is the following one, where all solutions constructed below are supposed to start from zero.

For $N \ge 1$ define

$$b^N(z) \;=\; \Delta z + \sum_{k=1}^{N} g(\langle e_{k+1}, z\rangle) e_k, \; z \in E,$$

and let $\mathbf{Q}$ denote the unique probability measure on $(\Omega, \mathcal{F})$ weakly solving the equation

$$\mathrm{d}X \;=\; \Delta X \, \mathrm{d}t + \mathrm{d}W.$$

Then we have for all $T \ge 0$

$$\mathbf{E}_{\mathbf{Q}} \exp\{\frac{1}{2} \int_0^T \| \sum_{k=1}^{N} g(\langle e_{k+1}, z\rangle) e_k \|_H^2 \, \mathrm{d}t\} \;<\; \infty$$

since g is bounded, that is Novikov's condition is satisfied. Applying Girsanov's theorem, there exists a probability measure $\mathbf{P}^N$ on $(\Omega, \mathcal{F})$ weakly solving the equation $(\star)$ corresponding to b^N which is absolutely continuous with respect to $\mathbf{Q}$.

Fix $k \geq 1$ and define W^k as in Definition 1,(iii). Then the process $\langle e_k, X\rangle$ satisfies the stochastic differential inequality

$$\langle e_k, X_t\rangle \leq \langle e_k, X_s\rangle - \lambda_k \int_s^t \langle e_k, X_r\rangle \, \mathrm{d}r + (t-s) + W_t^k - W_s^k, \; s \leq t, \; \mathbf{P}^N\text{-a.s.},$$

resp.

$$\langle e_k, X_t\rangle \geq \langle e_k, X_s\rangle - \lambda_k \int_s^t \langle e_k, X_r\rangle \, \mathrm{d}r + W_t^k - W_s^k, \; s \leq t, \; \mathbf{P}^N\text{-a.s.}$$

Using [2], $\langle e_k, X\rangle$ can be estimated by the solution to the stochastic differential equation

$$\mathrm{d}o^k = (-\lambda_k o^k + 1)\,\mathrm{d}t + \mathrm{d}W^k, \; o_0^k = 0,$$

resp.

$$\mathrm{d}u^k = -\lambda_k u^k \,\mathrm{d}t + \mathrm{d}W^k, \; u_0^k = 0,$$

we explicitly know:

$$\int_0^t e^{-\lambda_k(t-s)} \,\mathrm{d}W_s^k \leq \langle e_k, X_t\rangle \leq \int_0^t e^{-\lambda_k(t-s)} \,\mathrm{d}s + \int_0^t e^{-\lambda_k(t-s)} \,\mathrm{d}W_s^k,$$

that is

$$(10) \qquad |\langle e_k, X_t\rangle| \leq \begin{cases} \dfrac{1}{\lambda_k} + \left|\displaystyle\int_0^t e^{-\lambda_k(t-s)} \,\mathrm{d}W_s^k\right| & \text{if } \lambda_k \neq 0 \\ t + |W_t^k| & \text{if } \lambda_k = 0 \end{cases}$$

for all $t \geq 0$ $\mathbf{P}^N$-a.s. As a consequence, we obtain for every $T \geq 0$

$$\begin{aligned} \sup_{t\in[0,T]} \mathbf{E}_{\mathbf{P}^N} \|X_t\|_{H^\alpha}^2 &= \sup_{t\in[0,T]} \sum_{k=1}^\infty (1+\lambda_k)^\alpha \mathbf{E}_{\mathbf{P}^N} \langle e_k, X_t\rangle^2 \\ (11) \qquad &\leq C\left(T^2 + \sum_{k=2}^\infty (1+\lambda_k)^\alpha \frac{1}{\lambda_k}\right), \; \alpha \in \mathbb{R}, \; \forall N \geq 1, \end{aligned}$$

which immediately implies (applay Aldous criterion in [9] for example) that the sequence $(\mathbf{P}^N)$ is tight. Let $\mathbf{P}$ denote the weak limit of some subsequence of $(\mathbf{P}^N)$. Then, applying (11) again, $\mathbf{P}$ is easily shown to be a weak solution of the equation $(\star)$ corresponding to b with $\mathbf{P}(\{X_0 = 0\}) = 1$.

Of course, we are interested in the question whether $\mathbf{P}$ is the only solution or not. The answer has proved to be an easy application of our main theorem, and, we do not know another method to decide this question.

At first we notice that if $\mathbf{P}$ is an arbitrary weak solution of the equation $(\star)$ corresponding to b starting from zero, then (10) also holds true for all $t \geq 0$ $\mathbf{P}$-a.s. simply copying the proof of (10) with respect to $\mathbf{P}^N$. On the one hand, from (10) follows for every $T \geq 0$ that

$$\sup_{t\in[0,T]} \mathbf{E}\langle e_k, X_t\rangle^2 \leq \begin{cases} 2\dfrac{1}{\lambda_k^2} + 2\dfrac{1}{2\lambda_k} & \text{if } \lambda_k \neq 0 \\ 2T^2 + 2T & \text{if } \lambda_k = 0 \end{cases}$$

and, hence (see Remark 1c)), $\mathbf{P}$ is a solution to the martingale problem $(\delta_{\{0\}}, \{e_k\}, b)$. On the other hand, additionally using the Burkholder-Davis-Gundy inequality, we get for $k \geq 2$

$$\sup_{t\geq 0} \mathbf{E}|\langle e_k, X_t\rangle| \leq \frac{1}{\lambda_k} + c_1\sqrt{\frac{1}{2\lambda_k}} \leq (1+c_1)\frac{1}{\sqrt{\lambda_k}}$$

why $\mathbf{P}$ even satisfies the condition $(C^b_{\{e_k\}})$. Indeed,

$$\begin{aligned}
& \mathbf{E}\int_0^T \sum_{k=1}^N \left[\mathbf{E}(\langle e_k, b(X_t)\rangle|\mathcal{F}_t^N) - \langle e_k, b(X_t)\rangle\right]^2 \mathrm{d}t \\
= \; & \mathbf{E}\int_0^T \left[\mathbf{E}(g(\langle e_{N+1}, X_t\rangle)|\mathcal{F}_t^N) - g(\langle e_{N+1}, X_t\rangle)\right]^2 \mathrm{d}t \\
\le \; & 2\int_0^T \mathbf{E}g(\langle e_{N+1}, X_t\rangle)^2 \,\mathrm{d}t \\
\le \; & 2\int_0^T \mathbf{E}|\langle e_{N+1}, X_t\rangle| \,\mathrm{d}t \\
\le \; & 2(1+c_1)T\frac{1}{\sqrt{\lambda_{N+1}}} \overset{N\to\infty}{\longrightarrow} 0, \; \forall T > 0.
\end{aligned} \tag{12}$$

Now, Theorem 1 applies showing that the sequence $(\mathbf{P}^N)$ of probability measures on $(\Omega, \mathcal{F})$ constructed above weakly converges to the only weak solution of the corresponding equation $(\star)$ starting from zero.

Emphasize that the same is true if we replace Δ by the d-dimensional Laplacian with Neumann or Dirichlet boundary conditions in a bounded cube $\mathcal{O} \subseteq \mathbb{R}^d$, $d \ge 1$. The proof is exactly the same. Only remark that, in the case $d = 1$, it is sufficient to apply (11) for $\alpha = 0$ while, in the case $d > 1$, an appropriate $\alpha < 0$ has to be chosen.

In a last step, return to the condition

$$\mathbf{P}\left(\{\int_0^T \|\sum_{k=1}^\infty g(\langle e_{k+1}, X_t\rangle)e_k\|_H^2 \,\mathrm{d}t < \infty\}\right) = 1, \; \forall T > 0, \tag{13}$$

being an alternative (see Remark 5a)) to our condition $(C^b_{\{e_k\}})$ used above for the weak uniqueness result. Consider again the Laplacian in different dimensions d and fix an arbitrary weak solution $\mathbf{P}$ of the equation $(\star)$ corresponding to b with $\mathbf{P}(\{X_0 = 0\}) = 1$. For $d = 1$ we are not able to verify (13) but we cannot exclude that (13) is satisfied either. Even so, already in the case $d = 2$, (13) is not satisfied, which we will show by leading the opposite to a contradiction.

Before doing this, remember the following properties of the sequence $\{\lambda_k\}$ in the case $d = 2$:

$$\sum_{k=2}^\infty \lambda_k^{-1} = \infty \quad \text{and} \quad \sum_{k=2}^\infty \lambda_k^{-1-\delta} < \infty, \; \forall \delta > 0.$$

Now, assume that the condition (13) would be satisfied for the arbitrary solution $\mathbf{P}$ above. Then, for a fixed time $T > 0$, we have

$$\begin{aligned}
\infty \; > \; & \int_0^T \sum_{k=1}^\infty g(\langle e_{k+1}, X_t\rangle)^2 \,\mathrm{d}t \\
\ge \; & \int_0^T \sum_{k=1}^\infty |\langle e_{k+1}, X_t\rangle| \mathbf{1}_{\{|\langle e_{k+1}, X_t\rangle| < 1\}} \,\mathrm{d}t \\
\ge \; & \int_0^T \sum_{k=1}^\infty \langle e_{k+1}, X_t\rangle^2 \mathbf{1}_{\{|\langle e_{k+1}, X_t\rangle| < 1\}} \,\mathrm{d}t \quad \mathbf{P}\text{-a.s.}
\end{aligned}$$

which implies

$$\int_0^T \sum_{k=1}^\infty \langle e_k, X_t\rangle^2 \,\mathrm{d}t < \infty \quad \mathbf{P}\text{-a.s.} \tag{14}$$

where we have used the following lemma proven later.

Lemma 1

$$\int_0^T \langle e_1, X_t\rangle^2 \, \mathrm{d}t + \int_0^T \sum_{k=1}^{\infty} \langle e_{k+1}, X_t\rangle^2 \mathbf{1}_{\{|\langle e_{k+1}, X_t\rangle| \geq 1\}} \, \mathrm{d}t < \infty \quad \mathbf{P}\text{-}a.s.$$

Define

$$\beta(z) = \sum_{k=1}^{\infty} g(\langle e_{k+1}, z\rangle) e_k, \; z \in E,$$
$$S(t) = \exp\{t\Delta\}, \; t \geq 0,$$

and

$$W_t = \sum_{k=1}^{\infty} W_t^k e_k, \; t \geq 0,$$

where the sequence (W^k) is introduced as in Definition 1,(iii). Certainly, the process W presents an E-valued Wiener process associated with H (cf. Prop.7.2.3 in [3]), and, it is well-known by stochastic convolution that the coordinate process X satisfies the equation

$$X_t = \int_0^t S(t-s)\beta(X_s) \, \mathrm{d}s + \int_0^t S(t-s) \, \mathrm{d}W_s, \; t \geq 0, \; \mathbf{P}\text{-a.s.},$$

for the weak solution $\mathbf{P}$ of the considered equation. Because of (13), we obtain

$$\sup_{t\in[0,T]} \| \int_0^t S(t-s)\beta(X_s) \, \mathrm{d}s\|_H^2 \leq T \int_0^T \|S(t-s)\beta(X_s)\|_H^2 \, \mathrm{d}s$$
$$\leq T \int_0^T \|\beta(X_s)\|_H^2 \, \mathrm{d}s < \infty \quad \mathbf{P}\text{-a.s.}$$

while (14) yields

$$\|X_t\|_H^2 < \infty \text{ for Lebsgue a.e. } t \in [0,T] \; \mathbf{P}\text{-a.s.}$$

Hence,

$$X_t - \int_0^t S(t-s)\beta(X_s) \, \mathrm{d}s = \int_0^t S(t-s) \, \mathrm{d}W_s \in H$$

for Lebsgue a.e. $t \in [0,T]$ $\mathbf{P}$-a.s. which is a contradiction. Indeed, the so-called Ornstein-Uhlenbeck process

$$\int_0^t S(t-s) \, \mathrm{d}W_s = \sum_{k=1}^{\infty} \left(\int_0^t e^{-\lambda_k (t-s)} \, \mathrm{d}W_s^k \right) e_k, \; t \geq 0,$$

is known to satisfy

$$\| \int_0^t S(t-s) \, \mathrm{d}W_s\|_H = \infty, \; t \geq 0, \; \mathbf{P}\text{-a.s.},$$

because of the divergence

$$\sum_{k=2}^{\infty} \lambda_k^{-1} = \infty.$$

Remark 7 We have considered the alternative condition $\mathbf{P}(\{\int_0^T \|(B-A)(s,X)\|_H^2 \, \mathrm{d}s < \infty\}) = 1$ for $A = \Delta$, which is the only relevant case in our example (see Remark 5a)).

We finish this section with the

PROOF of Lemma 1. It is sufficient to show

$$\left(\mathbf{E}\int_0^T\sum_{k=1}^\infty\langle e_{k+1},X_t\rangle^2\mathbf{1}_{\{|\langle e_{k+1},X_t\rangle|\geq 1\}}\,\mathrm{d}t\right)^2$$

$$=\left(\int_0^T\sum_{k=2}^\infty\mathbf{E}\,\lambda_k^{-\frac{\delta}{2}}\langle e_k,X_t\rangle^2\lambda_k^{\frac{\delta}{2}}\mathbf{1}_{\{|\langle e_k,X_t\rangle|\geq 1\}}\,\mathrm{d}t\right)^2<\infty$$

for some $\delta>0$. Applying Hölder's inequality, the last integral can be estimated by the product

$$\int_0^T\sum_{k=2}^\infty\mathbf{E}\,\lambda_k^{-\delta}\langle e_k,X_t\rangle^4\,\mathrm{d}t\cdot\int_0^T\sum_{k=2}^\infty\lambda_k^\delta\,\mathbf{P}(\{|\langle e_k,X_t\rangle|\geq 1\})\,\mathrm{d}t.$$

For $k\geq 2$, using (10), from the Burkholder-Davis-Gundy inequality follows that

$$\mathbf{E}\langle e_k,X_t\rangle^4\leq C\left(\frac{1}{\lambda_k^4}+t\frac{1}{4\lambda_k}\right)\leq T\tilde{C}\frac{1}{\lambda_k},\ t\in[0,T],$$

leading to

$$\int_0^T\sum_{k=2}^\infty\mathbf{E}\,\lambda_k^{-\delta}\langle e_k,X_t\rangle^4\,\mathrm{d}t\leq T^2\tilde{C}\sum_{k=2}^\infty\lambda_k^{-1-\delta}<\infty,$$

and,

$$\int_0^T\sum_{k=2}^\infty\lambda_k^\delta\,\mathbf{P}(\{|\langle e_k,X_t\rangle|\geq 1\})\,\mathrm{d}t<\infty$$

remains to show.

Remind the inequality before (10) which is also true for all $t\geq 0$ $\mathbf{P}$-a.s. by the same reason as (10) holds true with respect to $\mathbf{P}$ (see above). For $k\geq 2$, this inequality gives

$$\mathbf{P}(\{\langle e_k,X_t\rangle\geq 1\})\leq\mathbf{P}(\{\frac{1}{\lambda_k}+\int_0^t e^{-\lambda_k(t-s)}\,\mathrm{d}W_s^k\geq 1\})$$

resp.

$$\mathbf{P}(\{\langle e_k,X_t\rangle\leq -1\})\leq\mathbf{P}(\{\int_0^t e^{-\lambda_k(t-s)}\,\mathrm{d}W_s^k\leq -1\})$$

for all $t\geq 0$. Now, the random variable $\int_0^t e^{-\lambda_k(t-s)}\,\mathrm{d}W_s^k$ on $(\Omega,\mathcal{F},\mathbf{P})$ is normally distributed with zero mean and variance

$$\sigma_{t,k}^2=\frac{1}{2\lambda_k}(1-e^{-2\lambda_k t}).$$

Therefore, if $\lambda_k-1>0$,

$$\mathbf{P}(\{\langle e_k,X_t\rangle\geq 1\})\leq\int_{1-\frac{1}{\lambda_k}}^\infty\frac{1}{\sqrt{2\pi\sigma_{t,k}^2}}\exp\{-\frac{x^2}{2\sigma_{t,k}^2}\}\,\mathrm{d}x$$

$$\leq\int_{\frac{\sqrt{2}(\lambda_k-1)}{\sqrt{\lambda_k}}}^\infty\frac{1}{\sqrt{2\pi}}\exp\{-\frac{y^2}{2}\}\,\mathrm{d}y$$

and, analogously,

$$\mathbf{P}(\{\langle e_k,X_t\rangle\leq -1\})\leq\int_{\sqrt{2\lambda_k}}^\infty\frac{1}{\sqrt{2\pi}}\exp\{-\frac{y^2}{2}\}\,\mathrm{d}y$$

leading to

$$\mathbf{P}(\{|\langle e_k, X_t\rangle| \geq 1\}) \leq \frac{1}{\sqrt{\pi}}\left(\frac{\sqrt{\lambda_k}}{\lambda_k - 1}e^{-\frac{(\lambda_k-1)^2}{\lambda_k}} + \frac{1}{\sqrt{\lambda_k}}e^{-\lambda_k}\right),\ t \geq 0.$$

Choose $k_0 \geq 2$ such that $\lambda_k - 1 > 0$ for all $k \geq k_0$. Then, the last inequality implies

$$\int_0^T \sum_{k=k_0}^{\infty} \lambda_k^{\delta}\, \mathbf{P}(\{|\langle e_k, X_t\rangle| \geq 1\})\, \mathrm{d}t \;<\; \infty$$

ending the proof of the lemma. ■

Remark 8 In (12) we have got a concrete bound for the rate of convergence of the finite dimensional approximation considered in (5) with respect to b (see Remark 2). From the properties of the eigenvalues $\{\lambda_k\}$ follows that the rate of convergence becomes lower if the dimension of the considered Laplacian increases.

4 Proof of the theorem

Because otherwise there is nothing to show, we assume that two solutions $\mathbf{P}, \tilde{\mathbf{P}}$ to the martingale problem $(\mu, \{e_k\}, B)$ exist. We will show that $\mathbf{P}$ and $\tilde{\mathbf{P}}$ coincide if they satisfy the condition $(C^B_{\{e_k\}})$.

At first, we easily find that the measure $1/2(\mathbf{P} + \tilde{\mathbf{P}})$ satisfies both the martingale problem $(\mu, \{e_k\}, B)$ and the condition $(C^B_{\{e_k\}})$. Hence, without restricting the generality, we may assume that $\tilde{\mathbf{P}} \ll \mathbf{P}$ as well as $D = \mathrm{d}\tilde{\mathbf{P}}/\mathrm{d}\mathbf{P}$ is bounded, which we also write

$$\tilde{\mathbf{P}} \;=\; D\mathbf{P},\ D \text{ bounded.} \tag{15}$$

Fix $T > 0$. For $N \geq 1$, define

$$\mathbf{P}^N = \mathbf{P}_{|\mathcal{F}_T^N} \quad \text{and} \quad \tilde{\mathbf{P}}^N = \tilde{\mathbf{P}}_{|\mathcal{F}_T^N},$$

and, consider the N-dimensional process

$$X_t^N \;=\; (\langle e_1, X_t\rangle, ..., \langle e_N, X_t\rangle),\ t \in [0, T],$$

with values in $C([0,T] \to \mathbb{R}^N)$. We equip the space $C([0,T] \to \mathbb{R}^N)$ with the canonical filtration $(\mathfrak{B}_t)_{t\in[0,T]}$,

$$\mathfrak{B}_t \;=\; \sigma\left\{\{f \in C([0,T] \to \mathbb{R}^N) | f(s) \in A\},\ s \leq t,\ A \in \mathfrak{B}(\mathbb{R}^N)\right\},$$

and denote the corresponding right-continuous filtration by $(\mathfrak{B}_{t+})_{t\in[0,T]}$.

Then it is well-known (cf. for example Lemma 4.9 in [11]) that there exist measurable functionals $\alpha_{N,k}$ resp. $\tilde{\alpha}_{N,k}$, $k = 1, ..., N$, defined on $([0,T] \times C([0,T] \to \mathbb{R}^N),\ \mathfrak{B}([0,T]) \otimes \mathfrak{B}_t)$ which are $(\mathfrak{B}_{t+})$-adapted* such that

$$\alpha_{N,k}(t, X^N) \;=\; \mathbf{E}(\langle e_k, B(t,X)\rangle | \mathcal{F}_t^N) \text{ for Lebesgue-a.e. } t \in [0,T]\ \mathbf{P}\text{-a.s.}$$

resp.

$$\tilde{\alpha}_{N,k}(t, X^N) \;=\; \tilde{\mathbf{E}}(\langle e_k, B(t,X)\rangle | \mathcal{F}_t^N) \text{ for Lebesgue-a.e. } t \in [0,T]\ \tilde{\mathbf{P}}\text{-a.s.}$$

*$\alpha_{N,k}(t,\cdot)$ resp. $\tilde{\alpha}_{N,k}(t,\cdot)$ are $\mathfrak{B}_{t+}$-measurable for every $t \in [0,T]$.

Because of $\mathbf{P}, \tilde{\mathbf{P}} \in \mathbf{S}(\mu, \{e_k\}, B)$, applying the multidimensional version of Th.7.12 in [11], we obtain that the N-dimensional process W^N resp. $\tilde{W}^N$,

$$W_t^{N,k} := \langle e_k, X_t \rangle - \langle e_k, X_0 \rangle - \int_0^t \alpha_{N,k}(s, X^N)\, \mathrm{d}s,\ t \in [0,T],\ k = 1, ..., N,$$

resp.

$$\tilde{W}_t^{N,k} := \langle e_k, X_t \rangle - \langle e_k, X_0 \rangle - \int_0^t \tilde{\alpha}_{N,k}(s, X^N)\, \mathrm{d}s,\ t \in [0,T],\ k = 1, ..., N,$$

defined on $(\Omega, \mathcal{F}_T^N, \mathbf{P}^N)$ resp. $(\Omega, \mathcal{F}_T^N, \tilde{\mathbf{P}}^N)$ is a Wiener process. Remark that the processes $W^{N,k}$ and $\tilde{W}^{N,k}$ are different from the process W^k defined in Definition 1,(iii), $k = 1, ..., N$. The definition of W^N and $\tilde{W}^N$ immediately gives for $k = 1, ..., N$

$$W_t^{N,k} - \tilde{W}_t^{N,k} = \int_0^t [\tilde{\alpha}_{N,k}(s, X^N) - \alpha_{N,k}(s, X^N)]\, \mathrm{d}s,\ t \in [0,T]. \tag{16}$$

Now, introduce the process

$$\mathcal{D}_t^N = \mathbf{E}(D|\mathcal{F}_t^N),\ t \in [0,T],$$

which is a bounded martingale on the stochastic basis $(\Omega, \mathcal{F}_T^N, \mathbf{P}^N, (\mathcal{F}_t^N)_{t\in[0,T]})$. Particularly using the last condition of Definition 1,(ii), the second part of Th.7.12 in [11] implies that there is a continuous version D^N of $\mathcal{D}^N$ on $\left(\Omega, \overline{\mathcal{F}_T^N}^{\mathbf{P}^N}, \mathbf{P}^N, (\mathcal{F}_t^N \vee \mathcal{N}_{\mathbf{P}^N})_{t\in[0,T]}\right)$ where the subset $\mathcal{N}_{\mathbf{P}^N}$ of the completion $\overline{\mathcal{F}_T^N}^{\mathbf{P}^N}$ of $\mathcal{F}_T^N$ consists of all sets having $\mathbf{P}^N$-measure zero. Moreover, the filtration $(\mathcal{F}_t^N \vee \mathcal{N}_{\mathbf{P}^N})_{t\in[0,T]}$ is even right-continuous according to Th.5.19 in [11].

Because of

$$\tilde{\mathbf{P}}^N = D_T^N \mathbf{P}^N \quad \text{on} \quad (\Omega, \overline{\mathcal{F}_T^N}^{\mathbf{P}^N})$$

from the more general version of Girsanov's theorem (cf. [16] for example) follows that the N-dimensional process

$$W_t^{N,k} - \left[\frac{1}{D^N} \cdot D^N, W^{N,k}\right]_t,\ t \in [0,T],\ k = 1, ..., N,$$

is a Wiener process on $\left(\Omega, \overline{\mathcal{F}_T^N}^{\mathbf{P}^N}, \tilde{\mathbf{P}}^N, (\mathcal{F}_t^N \vee \mathcal{N}_{\mathbf{P}^N})_{t\in[0,T]}\right)$ and therewith also on $(\Omega, \tilde{\mathcal{F}}_T^N, \tilde{\mathbf{P}}^N, \tilde{\mathbb{F}}^N)$ where

$$\tilde{\mathcal{F}}_T^N = \overline{\mathcal{F}_T^N}^{\tilde{\mathbf{P}}^N},\ \tilde{\mathbb{F}}^N = (\tilde{\mathcal{F}}_t^N)_{t\in[0,T]},\ \tilde{\mathcal{F}}_t^N = \mathcal{F}_t^N \vee \mathcal{N}_{\tilde{\mathbf{P}}^N}.$$

Here $[\cdot\,,\cdot]$ denotes the quadratic covariation of continuous local martingales and $\frac{1}{D^N} \cdot D^N$ abbreviates the stochastic integral process $\int_0^\cdot 1/D_s^N\, \mathrm{d}D_s^N$.

As a consequence, (16) implies that for each $k = 1, ..., N$ the finite variation process

$$\int_0^\cdot [\tilde{\alpha}_{N,k}(s, X^N) - \alpha_{N,k}(s, X^N)]\, \mathrm{d}s - \left[\frac{1}{D^N} \cdot D^N, W^{N,k}\right]$$

is a continuous local martingale on $(\Omega, \tilde{\mathcal{F}}_T^N, \tilde{\mathbf{P}}^N, \tilde{\mathbb{F}}^N)$. Hence, we have

$$\int_0^t [\tilde{\alpha}_{N,k}(s, X^N) - \alpha_{N,k}(s, X^N)]\, \mathrm{d}s = \left[\frac{1}{D^N} \cdot D^N, W^{N,k}\right]_t,\ t \in [0,T],\ \tilde{\mathbf{P}}^N\text{-a.s.},\ k = 1, ..., N,$$

which together with (16) gives

$$\tilde{W}_t^{N,k} = W_t^{N,k} - \left[\frac{1}{D^N} \cdot D^N, W^{N,k}\right]_t,\ t \in [0,T],\ \tilde{\mathbf{P}}^N\text{-a.s.},\ k = 1, ..., N,$$

finally leading to

$$(17)\qquad \int_0^t [\tilde{\alpha}_{N,k}(s,X^N) - \alpha_{N,k}(s,X^N)]\,\mathrm{d}s \;=\; \left[\frac{1}{D^N}\cdot D^N, \tilde{W}^{N,k}\right]_t,$$

$$t \in [0,T],\ \tilde{\mathbf{P}}^N\text{-a.s.},\ k = 1,...,N,$$

since the quadratic covariation of a continuous local martingale and a finite variation process vanishes.

Applying Girsanov's theorem again, the process

$$\frac{1}{D^N}\cdot D^N - \left[\frac{1}{D^N}\cdot D^N, \frac{1}{D^N}\cdot D^N\right]$$

is a continuous local martingale on $(\Omega, \tilde{\mathcal{F}}_T^N, \tilde{\mathbf{P}}^N, \tilde{\mathbb{F}}^N)$ which, using Th.7.12 in [11] once more, can be represented by $\tilde{W}^N$, that is

$$(18)\qquad \frac{1}{D^N}\cdot D^N - \left[\frac{1}{D^N}\cdot D^N, \frac{1}{D^N}\cdot D^N\right] = \sum_{k=1}^N \Phi^{N,k}\cdot \tilde{W}^{N,k}$$

where the $\tilde{\mathbb{F}}^N$-adapted measurable processes $\Phi^{N,k}$ on $(\Omega, \tilde{\mathcal{F}}_T^N, \tilde{\mathbf{P}}^N)$ satisfy

$$\tilde{\mathbf{P}}^N\left(\{\int_0^T (\Phi_s^{N,k})^2\,\mathrm{d}s < \infty\}\right) \;=\; 1,\ k = 1,...,N.$$

On the one hand, (18) gives a formula for $\frac{1}{D^N}\cdot D^N$ we put into (17) getting

$$\int_0^t [\tilde{\alpha}_{N,k}(s,X^N) - \alpha_{N,k}(s,X^N)]\,\mathrm{d}s \;=\; \int_0^t \Phi_s^{N,k}\,\mathrm{d}s,\ t\in[0,T],\ \tilde{\mathbf{P}}^N\text{-a.s.},\ k = 1,...,N.$$

On the other hand, (18) establishes for D^N the linear stochastic differential equation

$$\begin{aligned}\mathrm{d}D_s^N \;&=\; D_s^N\,\mathrm{d}M_s^N \;+\; \frac{1}{D_s^N}\,\mathrm{d}[D^N, D^N]_s\\ &=\; D_s^N\,\mathrm{d}M_s^N \;+\; D_s^N\,\mathrm{d}[M^N, M^N]_s\end{aligned}$$

on $(\Omega, \tilde{\mathcal{F}}_T^N, \tilde{\mathbf{P}}^N, \tilde{\mathbb{F}}^N)$ driven by the continuous local martingale

$$M^N \;=\; \sum_{k=1}^N \Phi^{N,k}\cdot\tilde{W}^{N,k}$$

which has the representation

$$M_t^N \;=\sum_{k=1}^N \int_0^t [\tilde{\alpha}_{N,k}(s,X^N) - \alpha_{N,k}(s,X^N)]\,\mathrm{d}\tilde{W}_s^{N,k},\ t\in[0,T],\ \tilde{\mathbf{P}}^N\text{-a.s.},$$

using the equality before the stochastic differential equation above. Of course, the explicit solution of that equation is

$$(19)\qquad D_t^N \;=\; \exp\left\{M_t^N + \frac{1}{2}[M^N, M^N]_t\right\},\ t\in[0,T],\ \tilde{\mathbf{P}}^N\text{-a.s.}$$

Remark that $D_0^N = 1$ actually holds $\tilde{\mathbf{P}}^N$-a.s. true since we have $\mathbf{P}\circ X_0 = \tilde{\mathbf{P}}\circ X_0 = \mu$ by our assumption $\mathbf{P}, \tilde{\mathbf{P}} \in \mathbf{S}(\mu, \{e_k\}, B)$.

For the moment, we will show that the sequence $\left([M^N, M^N]_T\right)_{N=1}^{\infty}$ converges to zero in probability $\tilde{\mathbf{P}}$. Indeed, from the definition of $\alpha_{N,k}, \tilde{\alpha}_{N,k}$, using $\tilde{\mathbf{P}}^N = \tilde{\mathbf{P}}_{|\mathcal{F}_T^N}$, we $\tilde{\mathbf{P}}$-a.s. have that

$$
\begin{aligned}
[M^N, M^N]_T &= \int_0^T \sum_{k=1}^N \left[\tilde{\mathbf{E}}(\langle e_k, B(t,X)\rangle|\mathcal{F}_t^N) - \mathbf{E}(\langle e_k, B(t,X)\rangle|\mathcal{F}_t^N)\right]^2 \mathrm{d}t \\
&\le 2\int_0^T \sum_{k=1}^N \left[\tilde{\mathbf{E}}(\langle e_k, B(t,X)\rangle|\mathcal{F}_t^N) - \langle e_k, B(t,X)\rangle\right]^2 \mathrm{d}t \,+ \\
&\quad\; 2\int_0^T \sum_{k=1}^N \left[\mathbf{E}(\langle e_k, B(t,X)\rangle|\mathcal{F}_t^N) - \langle e_k, B(t,X)\rangle\right]^2 \mathrm{d}t \\
&= \tilde{S}^N + S^N.
\end{aligned}
$$

But, it is an assumption of the theorem that S^N, $N \to \infty$, converges to zero in probability $\mathbf{P}$, i.e. every subsequence has a subsequence converging to zero $\mathbf{P}$-a.s. and therewith also $\tilde{\mathbf{P}}$-a.s. because of $\tilde{\mathbf{P}} \ll \mathbf{P}$. Repeating this idea with respect to $(\tilde{S}^N)_{N=1}^{\infty}$ beginning with an appropriate subsequence, we obtain that every subsequence of $(\tilde{S}^N + S^N)_{N=1}^{\infty}$ has a subsequence converging to zero $\tilde{\mathbf{P}}$-a.s. leading to the same result for $\left([M^N, M^N]_T\right)_{N=1}^{\infty}$, i.e. $[M^N, M^N]_T$, $N \to \infty$, converges to zero in probability $\tilde{\mathbf{P}}$.

As a consequence, the law of the process $[M^N, M^N]$ considered on $(\Omega, \overline{\mathcal{F}}^{\tilde{\mathbf{P}}}, \tilde{\mathbf{P}})$ converges to zero, and so does the law of the continuous local martingale M^N for at least a subsequence by using Th.VI.4.13 in [9] (without loss of generality, we continue indexing with N possibly only meaning a subsequence). But, convergence in law to a constant here implies convergence in probability to this constant which finally even gives $M_T^N \to 0$, $N \to \infty$, in probability $\tilde{\mathbf{P}}$. Summing up, $M_T^N + \frac{1}{2}[M^N, M^N]_T \to 0$, $N \to \infty$, in probability $\tilde{\mathbf{P}}$ and, hence, a subsequence converges to zero $\tilde{\mathbf{P}}$-a.s. This gives $D_T^{N_m} \to 1$, $m \to \infty$, $\tilde{\mathbf{P}}$-a.s., by (19) since the exponential function is continuous. Remember that D^N is a $\mathbf{P}$-version of the martingale $\mathcal{D}^N$, that is $D_T^N = \mathcal{D}_T^N$ $\mathbf{P}$-a.s., in particular. Because of $\tilde{\mathbf{P}} \ll \mathbf{P}$, then $D_T^N = \mathcal{D}_T^N$ also holds $\tilde{\mathbf{P}}$-a.s. true and we get

$$\mathcal{D}_T^{N_m} \to 1,\ m \to \infty,\ \tilde{\mathbf{P}}\text{-a.s.}$$

Using (7), from the properties of the conditional expectation follows that

$$\mathcal{D}_T^{N_m} = \mathbf{E}(D|\mathcal{F}_T^{N_m}) \to \mathbf{E}(D|\mathcal{F}_T),\ m \to \infty, \text{ in } L^1(\mathbf{P}).$$

Thus, a subsequence of $(\mathcal{D}_T^{N_m})_{m=1}^{\infty}$ converges to $\mathbf{E}(D|\mathcal{F}_T)$ $\mathbf{P}$-a.s. and therewith also $\tilde{\mathbf{P}}$-a.s. applying $\tilde{\mathbf{P}} \ll \mathbf{P}$ again, why we may identify

$$\mathbf{E}(D|\mathcal{F}_T) = 1 \ \ \tilde{\mathbf{P}}\text{-a.s.}$$

But, the last assertion is also $\mathbf{P}$-a.s. true. Indeed,

$$
\begin{aligned}
\int_{\{\mathbf{E}(D|\mathcal{F}_T)=1\}} \mathrm{d}\mathbf{P} &= \int_{\{\mathbf{E}(D|\mathcal{F}_T)=1\}} \mathbf{E}(D|\mathcal{F}_T)\, \mathrm{d}\mathbf{P} \\
&= \int_{\{\mathbf{E}(D|\mathcal{F}_T)=1\}} D\, \mathrm{d}\mathbf{P} = \int_{\{\mathbf{E}(D|\mathcal{F}_T)=1\}} \mathrm{d}\tilde{\mathbf{P}} = 1.
\end{aligned}
$$

Since $\mathcal{F} = \bigvee_{T\ge 0} \mathcal{F}_T$ is satisfied by definition we get

$$1 = \mathbf{E}(D|\mathcal{F}_T) \overset{T\to\infty}{\longrightarrow} D = 1 \text{ in } L^1(\mathbf{P})$$

finishing the proof of the theorem by using (15). ■

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