EXISTENCE OF ABSOLUTELY CONTINUOUS SOLUTIONS FOR CONTINUITY EQUATIONS IN HILBERT SPACES

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Abstract. We prove existence of solutions to continuity equations in a separable Hilbert space. We look for solutions which are absolutely continuous with respect to a reference measure γ which is Fomin–differentiable with exponentially integrable partial logarithmic derivatives

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Contents

1. Introduction

We are given a separable Hilbert space H (norm $|\cdot|$, inner product $\langle\cdot,\cdot\rangle$), a Borel vector field $F:[0,T]\times H\to H$ and a Borel probability measure ζ on H. We are concerned with the following continuity equation,

$$\int_0^T\int_H [D_tu(t,x)+\langle D_xu(t,x),F(t,x)\rangle]\,\nu_t(dx)\,dt=-\int_H u(0,x)\,\zeta(dx),\quad \forall\, u\in\mathcal{F}C^1_{b,T}, \tag{1.1}$$

where the unknown $\nu=(\nu_t)_{t\in[0,T]}$ is a probability kernel such that $\nu_0=\zeta$. Moreover, D_x represents the gradient operator and $\mathcal{F}C^1_{b,T}$ is defined as follows: let $\mathcal{F}C_b^k$, for $k\in\mathbb{N}\cup\{\infty\}$, denote the set of all functions $f:H\to\mathbb{R}$ of the form

$$f(x)=\widetilde{f}(\langle h_1,x\rangle,\cdots,\langle h_N,x\rangle),\quad x\in H,$$

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where $N \in \mathbb{N}$, $\widetilde{f} \in C_b^k(\mathbb{R}^N)$ and $h_1, \cdots, h_N \in Y$, where Y is a dense linear subspace of H to be specified later. Then $\mathcal{F}C_{b,T}^k$ is defined to be the $\mathbb{R}$–linear span of all functions $u : [0,T] \times H \to \mathbb{R}$ of the form

$$u(t,x) = g(t)f(x), \quad (t,x) \in [0,T] \times H,$$

where $g \in C^1([0,T];\mathbb{R})$ with $g(T) = 0$ and $f \in \mathcal{F}C_b^k$. Correspondingly, let $\mathcal{V}\mathcal{F}C_{b,T}^k$ be the set of all maps $G : [0,T] \times H \to H$ of the form

$$G(t,x) = \sum_{i=1}^{N} u_i(t,x)h_i, \quad (t,x) \in [0,T] \times H, \tag{1.2}$$

where $N \in \mathbb{N}$, $u_1, \cdots, u_N \in \mathcal{F}C_{b,T}^k$ and $h_1, \cdots, h_N \in Y$. Clearly, $\mathcal{F}C_{b,T}^\infty$ is dense in $L^p([0,T] \times H, \nu)$ for all finite Borel measures ν on $[0,T] \times H$ and all $p \in [1,\infty)$. $\mathcal{V}\mathcal{F}C_b^k$ denotes the set of all G as in (1.2) with $u_i \in \mathcal{F}C_{b,T}^k$ replaced by $u_i \in \mathcal{F}C_b^k$.

It is well known that problem (1.1) in general admits several solutions even when H is finite dimensional. So, it is natural to look for well posedness of (1.1) within the special class of measures $(\nu_t)_{t\in[0,T]}$ which are absolutely continuous with respect to a given *reference measure* γ. In this case, denoting by $\rho(t,\cdot)$ the density of ν_t with respect to γ,

$$\nu_t(dx) = \rho(t,x)\gamma(dx), \quad t \in [0,T],$$

equation (1.1) becomes

$$\begin{aligned}&\int_0^T \int_H [D_t u(t,x) + \langle D_x u(t,x), F(t,x)\rangle]\ \rho(t,x)\,\gamma(dx)\,dt \\ &= -\int_H u(0,x)\,\rho_0(x)\gamma(dx), \quad \forall\, u \in \mathcal{F}C_{b,T}^1.\end{aligned} \tag{1.3}$$

Here $\rho_0 := \rho(0,\cdot)$ is given and $\rho(t,\cdot)$, $t \in [0,T]$, is the unknown.

In this paper we concentrate on existence of solutions to (1.3). Corresponding uniqueness results under somewhat stringent conditions are in preparation.

Our basic assumption on γ is the following

Hypothesis 1. *γ is a nonnegative measure on $(H, \mathcal{B}(H))$ with $\gamma(H) < \infty$ such that there exists a dense linear subspace $Y \subset T$ having the following properties:*

For all $y \in Y$ there exists $\beta_h : H \to \mathbb{R}$ Borel measurable such that for some $c_h > 0$

$$\int_H e^{c_h|\beta_h|}\,d\gamma < \infty$$

and

$$\int_H \partial_h u\,d\gamma = -\int_H u\beta_h\,d\gamma,$$

where $\partial_h u$ denotes the partial derivative of u in the direction h.

Assume from now on that γ satisfies Hypothesis 1.

Remark 1.1. *It is well known that the operator D_x = Fréchet–derivative with domain $\mathcal{F}C_b^1$ is closable in $L^p(H,\gamma)$ for all $p \in [1,\infty)$, see e.g.* [1]. *Its closure will again be denoted by D_x and its domain will be denoted by $W^{1,p}(H,\gamma)$.*

Let $D_x^* : dom(D_x^*) \subset L^2(H,\gamma;H) \to L^2(H,\gamma)$ denote the adjoint of D_x.

Lemma 1.2. *$\mathcal{VFC}_b^1 \subset dom(D_x^*)$ and for $G \in \mathcal{VFC}_b^1$, $G = \sum_{i=1}^N u_i h_i$ we have*

$$D_x^* G = -\sum_{i=1}^N (\partial_{h_i} u_i + \beta_{h_i} u_i).$$

Proof. For $v \in \mathcal{FC}_b^1$ we have

$$\int_H \langle D_x v, G\rangle_H \, d\gamma = \sum_{i=1}^N \int_H \partial_{h_i} v \, u_i \, d\gamma$$

$$= \sum_{i=1}^N \int_H \partial_{h_i}(v\, u_i)\, d\gamma - \sum_{i=1}^N \int_H v\, \partial_{h_i} u_i \, d\gamma$$

$$= -\int_H v \sum_{i=1}^N (\partial_{h_i} u_i + \beta_{h_i} u_i)\, d\gamma.$$

□

We stress that if H is infinite dimensional, β_h is typically not bounded and not continuous. Here are some examples. For G as in Lemma 1.2, below we sometimes use the notation

$$\text{div } G := \sum_{i=1}^N \partial_{h_i} u_i.$$

Example 1.3. (i) Let Q be a symmetric positive defined operator of trace class on H and $\gamma := N(0, Q)$, i.e. the centered Gaussian measure on H with covariance operator Q. Assume that $\ker Q = \{0\}$ and let Y be the linear span of all eigenvectors of Q. Then Hypothesis 1 is fulfilled with this Y and for $h \in Y$, $h = c_i h_1 + \cdots + c_N h_N$ with $Qh_i = \lambda_i^{-1} h_i$, we have

$$\beta_h(x) = -\sum_{i=1}^N c_i \lambda_i \langle h_i, x\rangle_H, \quad x \in H.$$

This, in particular, covers the case studied in [11], where only uniqueness of solutions to (1.3) was studied.

(ii) Let $H := L^2((0,1), d\xi)$ and $A := -\Delta$ with zero boundary conditions. Define

$$\gamma(dx) := \frac{1}{Z}\, e^{-\frac{1}{4}\int_0^1 |x(\xi)|^4 d\xi}\, N(0, -\tfrac{1}{2}\, A^{-1})(dx),$$

where

$$Z := \int_H e^{-\frac{1}{4}\int_0^1 |x(\xi)|^4 d\xi}\, N(0, -\tfrac{1}{2}\, A^{-1})(dx).$$

Then with Y as in (i) for $Q = -\frac{1}{2}\, A^{-1}$ we find for $h = c_i h_1 + \cdots + c_N h_N$ as in (i)

$$\beta_h(x) = -\sum_{i=1}^N c_i \left(\lambda_i \langle h_i, x\rangle_H - \int_0^1 h_i(\xi)\, x(\xi)^3\, d\xi \right), \quad \text{for } N(0, -\tfrac{1}{2}\, A^{-1})\text{–a.e. } x \in H$$

and obviously also the exponential integrability condition holds in Hypothesis 1.

(iii) Let H and A be as in (ii) and let γ be the invariant measure of the solution to

$$\begin{cases} dX(t) = [AX(t) + p(X(t))]dt + BdW(t), \\ X(0) = x, \quad x \in H, \end{cases} \tag{1.4}$$

where p is a decreasing polynomial of odd degree equal to $N > 1$, $B \in L(H)$ with a *bounded inverse* and W is an H–valued cylindrical Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t>0}, \mathbb{P})$ (see [10]). Then it was proved in [10, Proposition 3.5] that Hypothesis 1 holds with $Y := D(A)$, where A is as in (ii) above except that each β_h was only proved to be $L^p(L^2(0,1),\gamma)$ for every $p \geq 1$. More precisely, it was proved (see [10, eq. (3.17)]) that for all $h \in D(A)$

$$\left(\int_{L^2(0,1)} |\beta_h|^p \, d\gamma\right)^{\frac{1}{p}} \leq C_p |Ah|, \quad \forall\, p \geq 2,$$

where C_p is the constant of the Burkholder–Davis–Gundy inequality for $p \geq 2$ which (when proved by Itô's formula) can easily be seen to be smaller than $12\,p$ if $p \geq 4$. For the reader's convenience we include a proof in Appendix B below. Hence, because for all $n \in \mathbb{N}$ by Stirling's formula

$$\left(\frac{1}{n!} 12^n \, n^n\right)^{\frac{1}{n}} \leq 12n \left(\frac{1}{\sqrt{2\pi}}\, n^{-n-\frac{1}{2}}\, e^n\right)^{\frac{1}{n}} = 12e \left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{1}{n}} e^{-\frac{1}{2n}\ln n} \to 12e \quad \text{as } n \to \infty,$$

we have for all $\epsilon \in (0, (12e|Ah|)^{-1}), h \in D(A) \setminus \{0\}$,

$$\int_{L^2(0,1)} e^{\epsilon|\beta_h|} \, d\gamma \leq \sum_{n=0}^{\infty} \frac{1}{n!} \epsilon^n 12^n \, n^n |Ah|^n < \infty.$$

So, for any $c_h \in (0, (12e|Ah|)^{-1})$, exponential integrability holds for $|\beta_h|$ and Hypothesis 1 is satisfied.

Concerning F in (1.1) we assume:

Hypothesis 2. *(i) $F : [0,T] \times H \to H$ is Borel measurable and bounded.*

(ii) There exist $F_j \in \mathcal{VF}C^2_{b,T}$, $j \in \mathbb{N}$, such that each F_j satifies (2.8) *below and*

$$\begin{cases} \lim\limits_{j\to\infty} F_j = F \quad dt \otimes \gamma\text{-a.e.} \\[2ex] \sup\limits_{j\in\mathbb{N}} \|F_j\|_\infty < \infty, \\[2ex] M := \sup\limits_{j\in\mathbb{N}} C_{F_j} < \infty, \end{cases}$$

where C_{F_j} is defined in Lemma 2.4 below.

Definition 1.4. *Let $\rho_0 \in L^1(H,\gamma)$. A* solution *of the continuity equation* (1.3) *is a function $\rho \in L^1(0,T; L^1(H,\gamma))$ such that $\rho(0,\cdot) = \rho_0$ and* (1.3) *is fulfilled.*

If $\rho_0 \ln \rho_0 \in L^1(H,\gamma)$, in Section 2, we shall prove existence of a solution of (1.3) by introducing the following approximating equation, where F is replaced by (F_j) (fulfilling Hypothesis 2) and ρ_0 by $\rho_{j,0}$, where $(\rho_{j,0})$ is a sequence in $\mathcal{F}C^1_b$, converging to ρ_0 in $L^1(H,\gamma)$:

$$\begin{aligned} &\int_0^T \int_H \left[D_t u(t,x) + \langle D_x u(t,x), F_j(t,x)\rangle\right] \rho_j(t,x)\, \gamma(dx)\, dt \\ &= -\int_H u(0,x)\, \rho_0(x) \gamma(dx), \quad \forall\, u \in \mathcal{F}C^1_{b,T}, \end{aligned} \tag{1.5}$$

which has a solution ρ_j since F_j is regular. Then we shall show that a subsequence of (ρ_j) converges weakly to a solution of (1.3).

To our knowledge, earliest existence (and uniqueness) results for equation (1.3) concern the case where H is finite dimensional and the reference measure is the Lebesgue measure, see the seminal papers [14] and [3].

If H is infinite dimensional and γ is a Gaussian measure problem (1.1) has been studied in [4], [16], [11] and [17]. A very general approach in metric spaces has been presented in [6].

Our assumptions for getting existence of solutions are, however, weaker than the corresponding ones in these papers.

We finish this section with some notations and preliminaries. $\mathcal{B}(H)$ denotes the set of all Borel subsets and $\mathcal{P}(H)$ the set of all Borel probabilities on H. A *probability kernel* in $[0,T]$ is a mapping $[0,T] \to \mathcal{P}(H),\ t \mapsto \mu_t$, such that the mapping $[0,T] \to \mathbb{R},\ t \mapsto \mu_t(I)$ is measurable for any $I \in \mathcal{B}(H)$. $L(H)$ is the set of all linear bounded operators in H, $C_b(H)$, $C_b(H;H)$ the space of all real continuous and bounded mappings $\varphi\colon H \to \mathbb{R}$ and $\varphi\colon H \to H$ respectively, endowed with the sup norm

$$\|\varphi\|_\infty = \sup_{x\in H} |\varphi(x)|,$$

whereas $C_b^k(H)$, $k > 1$, will denote the space of all real functions which are continuous and bounded together with their derivatives of order less or equal to k. $B_b(H)$ will represent the space of all real, bounded and Borel mappings on H. Moreover, we shall denote by $\|\cdot\|_p$ the norm in $L^p(H,\gamma)$, $p \in [1,\infty]$. For any $x, y \in H$ we denote either by $\langle x, y\rangle$ or by $x \cdot y$ the scalar product between x and y. Finally, if (e_h) is an orthonormal basis in H we set $x_h = \langle x, e_h\rangle$ for all $x \in H$ and $G_h = \langle G, e_h\rangle$, $h \in \mathbb{N}$, for all $G \in L^2(H,\nu;H)$. Finally, we state a lemma, needed in what follows, whose straightforward proof is left to the reader.

Lemma 1.5. *Assume, besides Hypothesis 1, that $F \in$ dom (D_x^*) and $\varphi \in C_b^1(H)$. Then $\varphi F \in$ dom (D_x^*) and we have*

$$D_x^*(\varphi F) = \varphi\, D_x^*(F) - \langle D_x\varphi, F\rangle. \tag{1.6}$$

2. The main result

First we notice that if $F \in \mathrm{dom}\,(D_x^*)$ then a regular solution ρ to (1.3) solves the equation

$$\begin{cases} D_t\rho + \langle F, D_x\rho\rangle - D_x^*F\,\rho = 0,\\[2mm] \rho(0,\cdot) = \rho_0, \end{cases} \tag{2.1}$$

and vice versa. In fact, since for all $u \in \mathcal{VF}C^1_{b,T}$

$$\int_0^T D_t u(t,x)\,\rho(t,x)\,dt = -\int_0^T u(t,x)\,D_t\rho(t,x)\,dt - u(0,x)\rho_0(x), \quad x \in H, \tag{2.2}$$

and (thanks to Lemma 1.5)

$$\begin{aligned}
&\int_H \langle D_x u(t,x), F(t,x)\rangle\,\rho(t,x)\,\gamma(dx) = \int_H \langle D_x u(t,x), \rho(t,x)F(t,x)\rangle\,\gamma(dx)\\
&= \int_H u(t,x)\,D_x^*(\rho F)(t,x)\,\gamma(dx) = \int_H u(t,x)\,\rho(t,x)\,D_x^*F(t,x)\,\gamma(dx)\\
&- \int_H u(t,x)\,\langle D_x\rho(t,x), F(t,x)\rangle\,\gamma(dx).
\end{aligned} \tag{2.3}$$

Clearly (2.2) and (2.3) imply that (1.3) is equivalent to

$$\begin{cases} \displaystyle\int_0^T \int_H u(t,x)\,[-D_t\rho(t,x) + D_x^* F(t,x)\rho(t,x) - \langle D_x\rho(t,x), F(t,x)\rangle\,]\,\gamma(dx)\,dt = 0, \\ \rho(0,\cdot) = \rho_0, \end{cases} \tag{2.4}$$

for all $u \in \mathcal{VFC}_{b,T}^1$. By the density of $\mathcal{VFC}_{b,T}^1$ in $L^2([0,T]\times H, dt\otimes d\gamma)$ we obtain (2.1).

Theorem 2.1. *Assume that Hypotheses 1 and 2 hold. Let $\zeta := \rho_0 \cdot \gamma$ be a probability measure on $(H, \mathcal{B}(H))$ such that*

$$\int_H \rho_0 \,\ln \rho_0 \, d\gamma < \infty. \tag{2.5}$$

Then there exists $\rho : [0,T]\times H \to \mathbb{R}_+$, $\mathcal{B}([0,T]\times H)$–measurable such that $\nu_t(dx) = \rho(t,x)\gamma(dx)$, $t \in [0,T]$, are probability measures on $(H,\mathcal{B}(H))$ such that (1.1) *(equivalently* (1.3)*) holds. In addition,*

$$\int_0^T \int_H \rho(t,x)\,\ln \rho(t,x)\,\gamma(dx)\,dt < \infty. \tag{2.6}$$

Proof. By disintegration we shall reduce the proof to the case $H = \mathbb{R}^N$ and by regularization to Corollary A.2 in Appendix A.

Case 1. Suppose $F \in \mathcal{VFC}_{b,T}^2$, $\rho_0 \in \mathcal{FC}_b^1$, $\rho_0 \geq 0$ such that condition (2.8) below holds.

In this case we can find an orthonormal basis $\{e_i : i \in \mathbb{N}\}$ of H which consists of elements in Y such that for some $N \in \mathbb{N}$ (which we fix below)

$$F(t,x) = \sum_{i=1}^N g_i(t)\,f_i(x)\,e_i, \quad (t,x)\in[0,T]\times H, \tag{2.7}$$

where for $1 \leq i \leq N$, $g_i \in C^1([0,T];\mathbb{R})$ with $g_i(T) = 0$ and $f_i \in \mathcal{FC}_b^2$ such that for $x \in H$

$$f_i(x) = \widetilde{f}_i(\langle e_1, x\rangle, ..., \langle e_N, x\rangle)$$

and

$$\rho_0(x) = \widetilde{\rho}_0(\langle e_1, x\rangle, ..., \langle e_N, x\rangle)$$

with $\widetilde{f}_i \in C_b^2(\mathbb{R}^N)$, $\widetilde{\rho}_0 \in C_b^1(\mathbb{R}^N)$. Assume that

$$\textit{support of } \widetilde{f}_i \textit{ is compact}, \quad \forall\, 1 \leq i \leq N. \tag{2.8}$$

Define

$$H_N := \text{lin span}\,\{e_1, ..., e_N\}$$

and let $\Pi_N : H \to E$ be the orthogonal projection onto $E := H_N^\perp$, where $H_N^\perp$ is the orthogonal complement of H_N, i.e.

$$H = H_N \oplus E \equiv \mathbb{R}^N \times E, \tag{2.9}$$

hence, for $z \in H$, $z = (x,y)$ with unique $x \in \mathbb{R}^N$, $y \in E$.

Letting $\nu := \gamma \circ \Pi_N^{-1}$ be the image measure on $(E, \mathcal{B}(E))$ of γ under Π_N^{-1}. Then we have the following well known disintegration result for γ:

Lemma 2.2. *There exists $\Psi : \mathbb{R}^N \times E \to [0,\infty)$, $\mathcal{B}(\mathbb{R}^N\times E)$–measurable such that*

$$\gamma(dz) = \gamma(dx\,dy) = \Psi^2(x,y)dx\,\nu(dy), \tag{2.10}$$

where dx denotes Lebesgue measure on $\mathbb{R}^N$.

Furthermore, for every $y \in E$

$$\Psi(\cdot,y) \in H^{1,2}(\mathbb{R}^N, dx), \tag{2.11}$$

i.e. the Sobolev space of order 1 *in* $L^2(\mathbb{R}^N, dx)$.

Proof. See [2, Proposition 4.1]. $\square$

Now let us continue the proof of Theorem 2.1. We have by Hypothesis 1 that for all $1 \le i \le N$ there exists $c_i \in (0, \infty)$ such that

$$\begin{aligned} \infty &> \int_H e^{c_i|\beta_{e_i}|}\, d\gamma = \int_E \int_{\mathbb{R}^N} e^{c_i|\beta_{e_i}(x,y)|}\, \Psi^2(x,y)\, dx\, \nu(dy) \\ &= \int_E \int_{\mathbb{R}^N} \exp\left[c_i \left| \frac{\partial}{\partial x_i}\, \Psi^2(x,y)/\Psi^2(x,y) \right| \right] \Psi^2(x,y) dx\, \nu(dy), \end{aligned}$$

where we used that for $1 \le i \le N$

$$\beta_{e_i}(x,y) = \frac{\partial}{\partial x_i}\, \Psi^2(x,y)/\Psi^2(x,y), \quad (x,y) \in \mathbb{R}^N \times E = H, \tag{2.12}$$

which is an immediate consequence of the disintegration (2.10), and the right hand side of (2.12) is defined to be zero on $\{\Psi = 0\}$. Hence we can find $E_0 \subset \mathcal{B}(E)$ such that $\nu(E_0) = 1$ and

$$\int_{\mathbb{R}^N} \exp\left[c_i \left| \frac{\partial}{\partial x_i}\, \Psi^2(x,y)/\Psi^2(x,y) \right| \right] \Psi^2(x,y) dx < \infty \tag{2.13}$$

for $y \in E_0$. Below we fix $y \in E_0$.

Define for $M, l \in \mathbb{N}$ and $(x,y) \in \mathbb{R}^N \times E (= H)$

$$\Psi_M(x,y) := \left(\Psi^2(x,y) \wedge M \right)^{1/2}, \tag{2.14}$$

$$\Psi_{M,l}(x,y) := \left(\Psi_M^2(\cdot, y) * \delta_l \right)^{1/2}(x), \tag{2.15}$$

where $\delta_l(x) = l^N \eta(lx)$, $x \in \mathbb{R}^N$, $\eta \in \mathcal{S}(\mathbb{R}^N)$ (:= set of Schwartz test functions), $\eta > 0$, $\eta(x) = \eta(-x)$, $x \in \mathbb{R}^N$, and $\int_{\mathbb{R}^N} \eta\, dx = 1$.) We note that then clearly $\Psi_{M,l}(x,y) > 0$ for all $x \in \mathbb{R}^N$. Then by Corollary A.2 applied with the measure $\gamma_{M,l,y}(dx) := \Psi_{M,l}^2(x,y)dx$ replacing $\gamma(dx)$, we know that

$$\rho_{M,l}(t,(x,y)) := \rho_0(\xi(T, T-t, x))\, e^{\int_0^t D_{M,l}^* F(T-u, (\xi(T-u, T-t, x), y))\, du}, \quad (t,x) \in [0,T] \times \mathbb{R}^N, \tag{2.16}$$

where (see Lemma 1.2 and (2.7)

$$D_{M,l}^* F(r,(x,y)) := -\sum_{i=1}^N g_i(r) \left(\partial_{e_i} f_i(x) + f_i(x) \frac{\partial}{\partial x_i}\, \Psi_{M,l}^2(x,y)/\Psi_{M,l}^2(x,y) \right), \tag{2.17}$$

$r \in [0,T]$, $x \in \mathbb{R}^N$, solves

$$\begin{cases} D_t \rho_{M,l}(t,(x,y)) + \langle F(t,x), D_x \rho_{M,l}(t,(x,y)) \rangle - D_{M,l}^* F(t,(x,y)) \rho_{M,l}(t,(x,y)) = 0, \\ \rho_{M,l}(0,(x,y)) = \rho_0(x). \end{cases} \tag{2.18}$$

We need a few further lemmas of which the first is the most crucial.

Lemma 2.3. *Let $\epsilon > 0$. Then for all $1 \le i \le N$, $l, M \in \mathbb{N}$*

$$\begin{aligned} &\int_{\mathbb{R}^N} \exp\left[\epsilon\left|\left(\frac{\partial \Psi^2_{M,l}}{\partial x_i}/\Psi^2_{M,l}\right)(x,y)\right|\right]\,\Psi^2_{M,l}(x,y)\,dx \\ &\le \int_{\mathbb{R}^N} \exp\left[\epsilon\left|\left(\frac{\partial \Psi^2_{M}}{\partial x_i}/\Psi^2_{M}\right)(x,y)\right|\right]\,\Psi^2_{M}(x,y)\,dx \\ &\le \int_{\mathbb{R}^N} \exp\left[\epsilon\left|\beta_{e_i}(x,y)\right|\right]\,\Psi^2(x,y)\,dx. \end{aligned} \tag{2.19}$$

Proof. Obviously, the left hand side of (2.19) is dominated by

$$\int_{\mathbb{R}^N} \exp\left[\epsilon\int_{\mathbb{R}^N}\left(\left|\frac{\partial \Psi^2_M}{\partial x_i}\right|/\Psi^2_M\right)(\tilde{x},y)\,\Psi^2_M(\tilde{x},y)\,\delta_l(x-\tilde{x})\,d\tilde{x}(\Psi^2_{M,l}(x,y))^{-1}\right]\Psi^2_{M,l}(x,y))dx, \tag{2.20}$$

where we used that $\frac{\partial \Psi^2_M}{\partial x_i} = 0$ dx–a.e. on $\{\Psi^2_M = 0\}$.

Applying Jensen's inequality for fixed $x \in \mathbb{R}^N$ to the probability measure

$$\Psi^2_{M,l}(x,y))^{-1}\,\Psi^2_M(\tilde{x},y)\,\delta_l(x-\tilde{x})\,d\tilde{x}$$

and the convex function $r \to e^{\epsilon r}$, we obtain that the right hand side of (2.20) is dominated by

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \exp\left[\epsilon\left(\left|\frac{\partial \Psi^2_M}{\partial x_i}\right|/\Psi^2_M\right)(\tilde{x},y)\right]\,\Psi^2_M(\tilde{x},y)\,\delta_l(x-\tilde{x})\,d\tilde{x}\,dx.$$

By Young's inequality and since $\|\delta_l\|_{L^1(\mathbb{R}^N)} = 1$, the latter is dominated by

$$\int_{\mathbb{R}^N} \exp\left[\epsilon\left(\left|\frac{\partial \Psi^2_M}{\partial x_i}\right|/\Psi^2_M\right)(x,y)\right]\,\Psi^2_M(x,y)\,dx. \tag{2.21}$$

Hence the fist inequality in (2.19) is proved. To show the second we note that

$$\frac{\partial \Psi^2_M}{\partial x_i} = 1\!\!1_{\{\Psi^2 < M\}}\,\frac{\partial \Psi^2}{\partial x_i}, \quad dx\text{–a.s.}.$$

Hence the integral in (2.21) is dominated by

$$\int_{\mathbb{R}^N} \exp\left[\epsilon 1\!\!1_{\{\Psi^2<M\}}\left(\left|\frac{\partial \Psi^2}{\partial x_i}\right|/\Psi^2\right)(x,y)\right]\,\Psi^2(x,y)\,dx,$$

which in turn by (2.12) is dominated by the last integral in (2.19). □

Lemma 2.4. *Let*

$$\delta := \inf_{1\le i\le N} \frac{c_i}{N(\|g_i\|_\infty \|f_i\|_\infty + 1)}$$

Then

$$\begin{aligned} C_F := \sup_{M,l\in\mathbb{N}} \int_0^T \int_{\mathbb{R}^N} &\exp\left[-\delta\sum_{i=1}^N g_i(t)\partial_{e_i} f_i(x)\right]^+ \\ \times \exp&\left[\delta\sum_{i=1}^N \|g_i\|_\infty\|f_i\|_\infty\left(\left|\frac{\partial \Psi^2_{M,L}}{\partial x_i}\right|/\Psi^2_{M_L}\right)(x,y)\right]\,\Psi^2_{M_L}(x,y)\,dx\,dt < \infty. \end{aligned}$$

Proof. By (2.12), (2.13) and the generalized Hölder inequality this follows immediately from Lemma 2.3. □

Lemma 2.5. *There exist subsequences $(l_k)_{k\in\mathbb{N}}$, $(M_k)_{k\in\mathbb{N}}$ such that the following assertions hold:*

(i) For dx–a.e. $x\in\{\Psi>0\}(=\{\Psi_M>0\})$ we have for all $M\in\mathbb{N}$

$$\lim_{k\to\infty} D^*_{M,l_k}F(\cdot,(x,y)) = -\sum_{i=1}^N g_i\left[\partial_{e_i}f_i(x)+f_i(x)\left(\frac{\partial\Psi_M^2}{\partial x_i}/\Psi_M^2\right)(x,y)\right] = D^*_M F(\cdot,(x,y)),$$

and

$$\lim_{k\to\infty} D^*_{M_k}F(\cdot,(x,y)) = -\sum_{i=1}^N g_i\left[\partial_{e_i}f_i(x)+f_i(x)\beta_{e_i}(x,y)\right] = D^*_x F(\cdot,(x,y)),$$

in $L^1([0,T];dt)$.

(ii) Let ρ_M and ρ be defined as $\rho_{M,l}$ with $D^*_{M,l}F$ replaced by $D^*_M F$ and $D^*_x F$ respectively. Then (selecting a further subsequence if necessary) we have for dx–a.e. $x\in\{\Psi>0\}$ that for all $M\in N$

$$\lim_{k\to\infty}\rho_{M,l_k}(t,(x,y)) = \rho_M(t,(x,y)),\quad \forall\, t\in[0,T]$$

and

$$\lim_{k\to\infty}\rho_{M_k}(t,(x,y)) = \rho(t,(x,y)),\quad \forall\, t\in[0,T].$$

Proof. (i) Obviously, we can choose the subsequence $(l_k)_{k\in\mathbb{N}}$ such that for dx–a.e. $x\in\{\Psi>0\}$ and all $M\in\mathbb{N}$

$$\lim_{k\to\infty} D^*_{M,l_k}F(t,(x,y)) = D^*_M F(t,(x,y)),\quad \forall\, t\in[0,T].$$

By Lemma 2.4 the sequence $D^*_{M,l}F(\cdot,(\cdot,y))$, $l\in\mathbb{N}$, is uniformly integrable in $L^1([0,T]\times K, dt\,dx)$ for every compact $K\subset\mathbb{R}^N$. Consequently,

$$\lim_{k\to\infty} D^*_{M,l_k}F(\cdot,(\cdot,y)) = D^*_M F(\cdot,(\cdot,y))\qquad \text{in } L^1([0,T]\times K, dt\,dx).$$

Hence (selecting further sequences if necessary) the first assertion follows, since K was an arbitrary compact set in $\mathbb{R}^N$. The second assertion follows analogously, because

$$\left(\frac{\partial\Psi_M^2}{\partial x_i}/\Psi_M^2\right)(x,y) = 1\!\!1_{\{\Psi^2<M\}}\left(\frac{\partial\Psi^2}{\partial x_i}/\Psi^2\right)(x,y).$$

(ii) Clearly, for all $u,t\in[0,T]$, $x\mapsto\xi(T-u,T-t,x)$ is a C^1–diffeormorphism from $\mathbb{R}^N$ to $\mathbb{R}^N$ with strictly positive Jacobian which is bounded in $u,t\in[0,T]$ and x in any compact subset $K\subset\mathbb{R}^N$. So, the assertion follows by (i). □

Lemma 2.6. *Let $\chi\in C_0^1(\mathbb{R}^N)$, $\chi\ge 0$, $\chi=1$ on $\cup_{i=1}^N supp\ \widetilde{f}_i$ (see (2.8)) and let $l,M\in\mathbb{N}$. Then for all $t\in[0,T]$*

$$\begin{aligned}
&\int_{\mathbb{R}^N}\rho_{M,l}(t,(x,y))\left(\ln\rho_{M,l}(t,(x,y))-1\right)\Psi^2_{M,l}(x,y)\,\chi(x)\,dx\\
&\le e^{t/\delta}\Bigg[\int_{\mathbb{R}^N}\rho_0(x)|\ln\rho_0(x)-1|\Psi^2_{M,l}(x,y)\,\chi(x)\,dx \qquad (2.22)\\
&+C_F+\frac{t}{\delta}|\ln\delta|\int_{\mathbb{R}^N}\rho_0(x)\Psi^2_{M,l}(x,y)\,\chi(x)\,dx+\int_{\mathbb{R}^N}\Psi^2_{M,l}(x,y)\,\chi(x)\,dx,
\end{aligned}$$

where δ, C_F are the constants from Lemma 2.4.

Proof. By the regularity properties of $\rho_{M,L}$ stated in Corollary A.2 of Appendix A, all integrals below are well defined. Since $M, l \in \mathbb{N}$ and $y \in E_0$ are fixed, for simplicity of notation we denote the maps $x \mapsto \rho_{M,l}(t,(x,y))$ and $x \mapsto \Psi_{M,l}(x,y)$ by $\rho(t)$, Ψ respectively.

Then for $t \in [0,T]$,

$$\int_{\mathbb{R}^N} \rho(t)(\ln \rho(t) - 1)\, \chi\, \Psi^2 dx$$

$$= \int_{\mathbb{R}^N} \rho_0(\ln \rho_0 - 1)\, \chi\, \Psi^2 dx + \int_{\mathbb{R}^N} \int_0^t \tfrac{d}{ds}[\rho(s)(\ln \rho(s) - 1)]\, ds\, \chi\, \Psi^2 dx$$

$$= \int_{\mathbb{R}^N} \rho_0(\ln \rho_0 - 1)\, \chi\, \Psi^2 dx + \int_{\mathbb{R}^N} \int_0^t \ln \rho(s) D_s \rho(s)\, ds\, \chi\, \Psi^2 dx$$

$$= \int_{\mathbb{R}^N} \rho_0(\ln \rho_0 - 1)\, \chi\, \Psi^2 dx - \int_0^t \int_{\mathbb{R}^N} \langle F(s,x), D_x(\rho(s)(\ln \rho(s) - 1))\rangle\, \chi\, \Psi^2 dx\, ds$$

$$+ \int_0^t \int_{\mathbb{R}^N} D^*_{M,l}\, F(s,(\cdot,y))\rho(s)\, \ln \rho(s)\, \chi\, \Psi^2 dx\, ds$$

$$= \int_{\mathbb{R}^N} \rho_0(\ln \rho_0 - 1)\, \chi\, \Psi^2 dx + \int_0^t \int_{\mathbb{R}^N} D^*_{M,l}\, F(s,(\cdot,y))\rho(s)\, \chi\, \Psi^2 dx\, ds$$

$$\leq \int_{\mathbb{R}^N} \rho_0(\ln \rho_0 - 1)\, \chi\, \Psi^2 dx + \int_0^t \int_{\mathbb{R}^N} \left[e^{\delta (D^*_{M,l}\, F(s,(\cdot,y)))^+} + \tfrac{1}{\delta}\, \rho(s)\, (\ln \tfrac{1}{\delta}\, \rho(s) - 1) \right]\, \chi\, \Psi^2 dx\, ds,$$

where in the third inequality we used (2.18), in the fourth equality we used Fubini's theorem and the definition of $D^*_{M,l}$ as well as the fact that $\chi = 1$ on $\cup_{i=1}^N \operatorname{supp} \widetilde{f}_i$ and finally, in the last inequality we used that that for $a, b \geq 0$

$$ab \leq e^a + b(\ln b - 1).$$

Now the assertion follows by Lemma 2.4 and Gronwall's lemma, since by (2.18)

$$\int_{\mathbb{R}^N} \rho_{M,l}(t,(x,y))\, \Psi^2_{M,l}(x,y)\, \chi(x)\, dx = \int_{\mathbb{R}^N} \rho_0(x)\, \Psi^2_{M,l}(x,y)\, \chi(x)\, dx, \quad \forall \in [0,T] \tag{2.23}$$

and $r \ln r - r \geq -1$ for all $r \in [0,\infty)$. □

Lemma 2.7. *Let $M \in \mathbb{N}$, $\rho_{M,l,y}(t,x) := \rho_{M,l,y}(x)$, $t \in [0,T]$, $x \in \mathbb{R}^N$, and $\Psi_{M,l,y}(x) := \Psi_{M,l}(x,y)$, $x \in \mathbb{R}^N$. Then $\{\rho_{M,l,y} \cdot \Psi^2_{M,l,y} : l \in \mathbb{N}\}$ is uniformly integrable with respect to the measure $\chi(x)\, dx\, dt$, where χ is as in Lemma 2.6.*

Proof. Let $c \in (1,\infty)$. Then for all $l \in \mathbb{N}$ and $\rho_l := \rho_{M,l,y}$, $\Psi_l := \Psi_{M,l,y}$,

$$\int_0^T \int_{\mathbb{R}^N} \mathbb{1}_{\{\rho_l \Psi_l^2 \geq c\}}\, \rho_l \Psi_l^2\, \chi\, dx\, dt \leq \tfrac{1}{\ln c} \int_0^T \int_{\mathbb{R}^N} \mathbb{1}_{\{\rho_l \Psi_l^2 \geq c\}}\, (\ln \rho_l + \ln \Psi_l^2) \rho_l \Psi_l^2\, \chi\, dx\, dt$$

$$\leq \tfrac{1}{\ln c} \int_0^T \int_{\mathbb{R}^N} |\rho_l \ln \rho_l| \Psi_l^2\, \chi\, dx\, dt + \tfrac{\ln(M+1)}{\ln c} \int_0^T \int_{\mathbb{R}^N} \rho_l \Psi_l^2\, \chi\, dx\, dt.$$

Since $r \ln r - r \geq -1$, $r \in [0,\infty)$, and because of (2.23), it follows by Lemma 2.6, that both integrals on the right hand side of the last inequality are uniformly bounded in l and the assertion follows. □

Now we proceed with the proof of Case 1 of Theorem 2.1. It follows by (2.18) (analogously to (2.1)–(2.4) above) that for all

$$u(t,x) := g(t)f(x), \quad t \in [0,T],\, x \in \mathbb{R}^N, \tag{2.24}$$

$g \in C^1([0,T];\mathbb{R})$ with $g(T) = 0$ and $f \in C_0^1(\mathbb{R}^N)$ that

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} [D_t u(t,x) + \langle D_x u(t,x), F(t,x)\rangle]\, \rho_{M,l}(t,(x,y))\, \Psi_{M,l}^2(x,y)\, dx\, dt \\ &= -\int_{\mathbb{R}^N} u(0,x)\rho_0(x)\, \Psi_{M,l}^2(x,y)\, dx. \end{aligned} \tag{2.25}$$

By Lemma 2.5(ii) and Lemma 2.7 we can pass to the limit in (2.24) along the subsequence $(l_k)_{k\in\mathbb{N}}$ from Lemma 2.5 to conclude that for such u

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} [D_t u(t,x) + \langle D_x u(t,x), F(t,x)\rangle]\, \rho_M(t,(x,y))\, \Psi_M^2(x,y)\, dx\, dt \\ &= -\int_{\mathbb{R}^N} u(0,x)\rho_0(x)\, \Psi_M^2(x,y)\, dx. \end{aligned} \tag{2.26}$$

We can also pass to the limit in (2.23) to get

$$\int_{\mathbb{R}^N} \rho_M(t,(x,y)))\, \Psi_M^2(x,y)\, \chi(x)\, dx = \int_{\mathbb{R}^N} \rho_0(x)\, \Psi_M^2(x,y)\, \chi(x)\, dx, \quad \forall\, t \in [0,T]. \tag{2.27}$$

Furthermore, by Lemma 2.5(ii) and Lemma 2.6 we deduce from (2.22) by Fatou's lemma that for all $t \in [0,T]$

$$\begin{aligned} &\int_{\mathbb{R}^N} \rho_M(t,(x,y))(\ln \rho_M(t,(x,y)) - 1)\, \Psi_M^2(x,y)\, \chi(x)\, dx \\ &\le e^{t/\delta}\Big[\int_{\mathbb{R}^N} \rho_0(x)|\ln \rho_0(x) - 1|\, \Psi_M^2(x,y)\, \chi(x)\, dx + C_F + \tfrac{t}{\delta}|\ln\delta| \int_{\mathbb{R}^N} \rho_0(x)\, \Psi_M^2(x,y)\, \chi(x)\, dx \\ &+ \int_{\mathbb{R}^N} \Psi_M^2(x,y)\, \chi(x)\, dx\Big], \end{aligned} \tag{2.28}$$

where we used that $\Psi_M^2(x,y) \le M$ for all $l \in \mathbb{N}$ and δ, χ, C_F are as in Lemma 2.5.

Taking now the subsequence $(M_k)_{k\in\mathbb{N}}$ from Lemma 2.5 instead of M and using exactly analogous arguments as above, we can pass to the limit in (2.26), (2.27) and (2.28) to obtain that for all u as in (2.24)

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} [D_t u(t,x) + \langle D_x u(t,x), F(t,x)\rangle]\, \rho(t,(x,y))\, \Psi^2(x,y)\, dx\, dt \\ &= -\int_{\mathbb{R}^N} u(0,x)\rho_0(x)\, \Psi^2(x,y)\, dx, \end{aligned} \tag{2.29}$$

and for all $t \in [0,T]$

$$\int_{\mathbb{R}^N} \rho(t,(x,y))\, \Psi^2(x,y)\, \chi(x)\, dx = \int_{\mathbb{R}^N} \rho_0(x)\, \Psi^2(x,y)\, \chi(x)\, dx \tag{2.30}$$

and

$$\int_{\mathbb{R}^N} \rho(t,(x,y))(\ln \rho(t,(x,y)) - 1)\, \Psi^2(x,y)\, \chi(x)\, dx$$

$$\le e^{t/\delta} \Big[\int_{\mathbb{R}^N} \rho_0(x) |\ln \rho_0(x) - 1|\, \Psi^2(x,y)\, \chi(x)\, dx + C_F + \tfrac{t}{\delta} |\ln \delta| \int_{\mathbb{R}^N} \rho_0(x)\, \Psi^2(x,y)\, \chi(x)\, dx \tag{2.31}$$

$$+ \int_{\mathbb{R}^N} \Psi^2(x,y)\, \chi(x)\, dx \Big],$$

for δ, χ, C_F as in Lemma 2.5. Hence in the situation of Case 1 the assertion of Theorem 2.1 now follows easily from the disintegration formula (2.10) by approximating the functions u in (1.1) in the obvious way and letting $\chi \uparrow 1$ in (2.31) to get (2.6). □

Remark 2.8. (i) We here emphasize that in the situation of Case 1 we have an explicit formula for the solution density in (2.29) given by

$$\rho(t,(x,y)) = \rho_0(\xi(T, T-t, x)) e^{-\int_0^t D_x^* F(T-u, \xi(T-u, T-t, y)) du} \tag{2.32}$$

for $t \in [0,T]$ and dx–a.e. $x \in \mathbb{R}^N$ with ξ given as in Corollary A.2 of Appendix A.

(ii) Letting $\chi \uparrow 1$ in (2.31) and then integrating over $y \in E_0$ with respect to ν, from Lemma 2.2 we obtain that for all $t \in [0,T]$

$$\int_H \rho(t,x)(\ln \rho(t,x) - 1)\, \gamma(dx) \le e^{t/\delta} \Big[\int_H \rho_0 |\ln \rho_0 - 1|\, d\gamma + C_F + 1 + \tfrac{t}{\delta} |\ln \delta| \int_H \rho_0 d\gamma \Big] \tag{2.33}$$

and likewise from (2.30) that for all $t \in [0,T]$

$$\int_H \rho(t,x)\, \gamma(dx) = \int_H \rho_0(x)\, \gamma(dx) = 1. \tag{2.34}$$

Case 2. Let F_j, $j \in \mathbb{N}$, be as in Hypothesis 2. Choose nonnegative $\rho_{0,j} \in \mathcal{F}C_b^1$ such that

$$\lim_{j\to\infty} \rho_{0,j} = \rho_0 \quad \text{in } L^1(H,\gamma) \tag{2.35}$$

and

$$\sup_{j\in\mathbb{N}} \int_H \rho_{0,j} \, \ln \rho_{0,j} \, d\gamma < \infty. \tag{2.36}$$

For existence of such $\rho_{0,j}$, $j \in \mathbb{N}$, see Corollary C.3 in Appendix C below.

Let ρ_j be the corresponding solutions to (1.1) with F_j replacing F and $\zeta := \rho_0 \cdot \gamma$, which exist by Case 1. Then by (2.33) with $\rho_j, F_j, \rho_{0,j}$ replacing ρ, F and ρ_0 respectively, Hypothesis 2 and (2.36) imply that

$$\sup_{j\in\mathbb{N}} \sup_{t\in[0,T]} \int_H \rho_j(t,x) \, \ln \rho_j(t,x)\, \gamma(dx) < \infty. \tag{2.37}$$

This implies that a subsequence of $(\rho_j)_{j\in\mathbb{N}}$, again denoted by $(\rho_j)_{j\in\mathbb{N}}$ below, weakly* converges to some $\rho \in L^1([0,T]; L^1(H,\gamma))$.

By Case 1 we have for all $u \in \mathcal{F}C_{b,T}^1$

$$\int_0^T \int_H \left[\frac{d}{dt} u(t,x) + \langle D_x u(t,x), F_j(t,x) \rangle_H \right] \rho_j(t,x)\, \gamma(dx)\, dt$$
$$= - \int_H u(0,x)\, \rho_{0,j}(x) \gamma(dx). \tag{2.38}$$

So, by (2.35) we only have to consider the convergence of the left hand side of (2.38), more precisely only the part of it involving F_j. But

$$\begin{aligned} &\left|\int_0^T \int_H (\langle D_x u, F_j\rangle_H \,\rho_j - \langle D_x u, F\rangle_H \,\rho)\, d\gamma\, dt\right| \\ &\le \|Du\|_\infty \int_0^T \int_H |F_j - F|_H \,\rho_j \, d\gamma\, dt + \left|\int_0^T \int_H \langle F, Du\rangle\, (\rho_j - \rho)\, d\gamma\, dt\right| \end{aligned} \tag{2.39}$$

Because of the boundedness of $\langle F, Du\rangle$ the second term on the right hand side of (2.39) converges to 0 if $j \to \infty$. Let $\epsilon > 0$. Then, by Young's inequality, the first term on the right hand side of (2.39) is up to a constant dominated by

$$\int_0^T \int_H e^{\frac{1}{\epsilon}|F_j - F|_H}\, d\gamma\, dt + \epsilon \int_0^T \int_H \rho_j \ln(\epsilon \rho_j)\, d\gamma\, dt,$$

of which the first summand converges to zero as $j \to \infty$, since F_j, F are uniformly bounded, while the second summand is dominated by

$$\epsilon \int_0^T \int_H \rho_j \ln \rho_j \, d\gamma\, dt + \epsilon \ln \epsilon,$$

which can be made arbitrarily small uniformly in j because of (2.37). Hence putting all this together we conclude that the right hand side of (2.39) converges to 0 as $j \to \infty$.

It remains to prove (2.6). For this we are going to employ a result due to J. Komlos (see [18]). Namely, since $\rho_j,\ j \in \mathbb{N}$, is bounded in $L^1([0,T]\times H, dt \otimes dx)$ by [18, Theorem 1a] (selecting another subsequence if necessary) we may assume that

$$\rho^{(N)} := \frac{1}{N} \sum_{j=1}^N \rho_j \to \rho, \quad \gamma\text{–a.s..}$$

Hence, since $r \mapsto r \ln r$ is convex on $[0, \infty)$, by Fatou's lemma we obtain

$$\begin{aligned} &\int_0^T \int_H \rho \;\ln \rho\, d\gamma\, dt \le \liminf_{N\to\infty} \int_0^T \int_H \rho^{(N)} \;\ln \rho^{(N)}\, d\gamma\, dt \\ &\le \liminf_{N\to\infty} \frac{1}{N} \sum_{j=1}^N \int_0^T \int_H \rho_j \;\ln \rho_j \, d\gamma\, dt \\ &\le \sup_{j\in\mathbb{N}} \int_0^T \int_H \rho_j \;\ln \rho_j \, d\gamma\, dt, \end{aligned}$$

which is finite by (2.37). Finally from (2.34) and (2.35) it follows that $\nu_t(dx) := \rho(t,x)\,\gamma(dx)$ is a probability measure for all $t \in [0,T]$. Thus Theorem 2.1 is completely proved.

Remark 2.9. *Though the finite entropy condition on the initial measures ρ_0 is crucial in the proof of Theorem 2.1, it could be replaced by a corresponding assumption with $r \mapsto r \ln r$ replaced by another Young function (as in Appendix C below) and adjusting Hypothesis 2 (ii) accordingly.*

Appendix A. Deterministic Feynman–Kac formula and the solution of (2.1) for sufficiently regular F

Consider the equation

$$\begin{cases} \dfrac{d}{dt}\,\xi(t) = \widetilde{F}(t,\xi(t)), \\ \\ \xi(s) = x, \quad x \in \mathbb{R}^d, \end{cases} \tag{A.1}$$

with $\widetilde{F}$ regular. Let $V\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$ be also regular. We want to solve

$$\begin{cases} v_s(s,x) \;+\; \langle D_x v(s,x), \widetilde{F}(s,x)\rangle + V(s,x)v(s,x) = 0, \quad 0 \le s < T, \\ \\ v(T,x) \;=\; \varphi(x), \quad x \in H. \end{cases} \tag{A.2}$$

Proposition A.1. *Assume $\widetilde{F} \in C_b([0,T]\times\mathbb{R}^d;\mathbb{R}^d)$ such that $\widetilde{F}(t,\cdot) \in C^1(\mathbb{R}^d,\mathbb{R}^d)$ for all $t \in [0,T]$ and let $V \in C([0,T]\times\mathbb{R}^d)$ such that $V(t,\cdot) \in C^1(\mathbb{R}^d)$ for all $t \in [0,T]$ such that $D_xV : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ is continuous. Let $\varphi \in C^1(\mathbb{R}^d)$. Then the solution to* (A.2) *is given by*

$$v(s,x) = \varphi(\xi(T,s,x))e^{\int_s^T V(u,\xi(u,s,x))du}, \qquad (s,x) \in [0,T]\times\mathbb{R}^d, \tag{A.3}$$

where for $s \le t$, $\xi(t,s,x)$ denotes the solution to (A.1) *at time t when started at time s at $x \in \mathbb{R}^d$. In particular, $v(\cdot,x) \in C^1([0,T])$ for every $x \in \mathbb{R}^d$ and $D_t v \in C([0,T]\times\mathbb{R}^d)$.*

Proof. We only present the main steps. We shall check that v defined by (A.3) is a solution to (A.2).

For any decomposition $\{s = s_0 < s_1 < \cdots < s_n = T\}$ of $[s,T]$ we write

$$v(s,x) - \varphi(x) = -\sum_{k=1}^{n}[v(s_k,x) - v(s_{k-1},x)],$$

which is equivalent to,

$$\begin{aligned} v(s,x) - \varphi(x) &= -\sum_{k=1}^{n}[v(s_k,x) - v(s_k,\xi(s_k,s_{k-1},x))] \\ &-\sum_{k=1}^{n}[v(s_k,\xi(s_k,s_{k-1},x)) - v(s_{k-1},x)] =: J_1 - J_2. \end{aligned} \tag{A.4}$$

Concerning J_1 we write thanks to Taylor's formula

$$\begin{aligned} J_1 &\sim \sum_{k=1}^{n}\langle D_x v(s_k,x), \xi(s_k,s_{k-1},x) - x\rangle \sim \sum_{k=1}^{n}\langle D_x v(s_k,x), \widetilde{F}(s_k,x)\rangle(s_k - s_{k-1}) \\ &\to \int_s^T \langle D_x v(r,x), \widetilde{F}(r,x)\rangle dr. \end{aligned} \tag{A.5}$$

Concerning J_2 we write [1]

$$
\begin{aligned}
J_2 &= \sum_{k=1}^{n} v(s_k, \xi(s_k, s_{k-1}, x)) - v(s_{k-1}, x)) \\
&= \sum_{k=1}^{n} \varphi(\xi(T, s_k, \xi(s_k, s_{k-1}, x))) e^{\int_{s_k}^{T} V(u,\xi(u,s_k,\xi(s_k,s_{k-1},x)))du} \\
&- \sum_{k=1}^{n} \varphi(\xi(T, s_{k-1}, x)) e^{\int_{s_{k-1}}^{T} V(u,\xi(u,s_{k-1},x))du} \\
&= \sum_{k=1}^{n} \varphi(\xi(T, s_{k-1}, x)) \left[e^{\int_{s_k}^{T} V(u,\xi(u,s_{k-1},x))du} - e^{\int_{s_{k-1}}^{T} V(u,\xi(u,s_{k-1},x))du} \right] \\
&= \sum_{k=1}^{n} v(s_{k-1}, x)) \left(e^{-\int_{s_{k-1}}^{s_k} V(u,\xi(u,s_{k-1},x))du} - 1 \right) \\
&\sim -\sum_{k=1}^{n} v(s_{k-1}, x) V(s_{k-1}, x)(s_k - s_{k-1}) \to -\int_s^T v(r,x)V(r,x)dr.
\end{aligned}
\tag{A.6}
$$

Replacing J_1 and J_2 given by (A.5) and (A.6) respectively in (A.4), yields

$$
v(s,x) = \varphi(x) + \int_s^T \langle D_x v(r,x), \widetilde{F}(r,x) \rangle dr + \int_s^T v(r,x)V(r,x)dr
$$

and the claim is proved. □

As a trivial consequence we obtain

Corollary A.2. *Suppose $H = \mathbb{R}^d$ and γ satifies Hypothesis 1. Let $F \in C_b([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ such that $F(t,\cdot) \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ and $D_x^* F(t,\cdot) \in C^1(\mathbb{R}^d)$ for all $t \in [0,T]$, and $D_x^* F \in C([0,T] \times \mathbb{R}^d)$, $D_x D_x^* F \in C([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$. Then for every $\rho_0 \in C^1(\mathbb{R}^d)$, $\rho_0 \geq 0$,*

$$
\rho(t,x) := \rho_0(\xi(T, T-t, x)) e^{\int_0^t D_x^* F(T-u,\xi(T-u,T-t,x))du}
$$

is a solution of (2.1), where $\xi(\cdot, s, x)$ is the solution to (A.1) started at time s at $x \in \mathbb{R}^d$, with $\widetilde{F}(t,x) := -F(T-t,x)$, $(t,x) \in [0,T] \times \mathbb{R}^d$. Furthermore, $\rho(\cdot,x) \in C^1([0,T])$ for every $x \in \mathbb{R}^d$ and $D_t \rho \in C([0,T] \times \mathbb{R}^d)$.

Proof. Apply Proposition A.1 with $\widetilde{F}$ as in the assertion above,

$$
V(t,x) = D_x^* F(T-t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^d
$$

and $\varphi := \rho_0$.

□

[1] In the second line below we use that $\xi(T, s_k, \xi(s_k, s_{k-1}, x)) = \xi(T, s_{k-1}, x)$

Appendix B. A remark on the Burkholder–Davis–Gundy inequality

Our aim in this section is to prove the following proposition.

Proposition B.1. *Let $p \geq 4$. Then for every $t \geq 0$,*

$$\mathbb{E} \sup_{s\in[0,t]} \left| \int_0^t \Phi(s)dW(s) \right|^p \leq c_p \left[\mathbb{E} \left(\int_0^t \|\Phi(s)\|^2_{L_2^0} \, ds \right)^{p/2} \right], \tag{B.1}$$

where $c_p := 12^p \, p^p$.

Proof. Set

$$Z(t) = \int_0^t \Phi(s)dW(s), \quad t \geq 0,$$

and apply Itô's formula to $f(Z(\cdot))$ where $f(x) = |x|^p, \ x \in H$. Since

$$f_{xx}(x) = p(p-2)|x|^{p-4} x \otimes x + p|x|^{p-2} I, \quad x \in H,$$

we have

$$\|f_{xx}(x)\| \leq p(p-1)|x|^{p-2},$$

therefore

$$|\mathrm{Tr}\ \Phi^*(t) f_{xx}(Z(t))\Phi(t)Q| \leq p(p-1)|Z(t)|^{p-2} \|\Phi(t)\|^2_{L_2^0}.$$

By taking expectation in the identity

$$|Z(t)|^p = p \int_0^t |Z(s)|^{p-2} \langle Z(s), dZ(s) \rangle + \frac{1}{2} \int_0^t \mathrm{Tr}\ [\Phi^*(s) f_{xx}(Z(s))\Phi(s)Q] ds,$$

we obtain by the Burkholder–Davis–Gundy inequality for $p = 1$

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t]} |Z(s)|^p \leq \frac{p(p-1)}{2} \, \mathbb{E} \left(\int_0^t |Z(s)|^{p-2} \|\Phi(s)\|^2_{L_2^0} \, ds \right) \\
&+3p\mathbb{E} \left[\left(\int_0^t \|\Phi(s)\|^2_{L_2^0} \, |Z(s)|^{2p-2} ds \right)^{1/2} \right] \\
&\leq \frac{p(p-1)}{2} \, \mathbb{E} \left(\sup_{s\in[0,t]} |Z(s)|^{p-2} \int_0^t \|\Phi(s)\|^2_{L_2^0} ds \right) \\
&+3p\mathbb{E} \left[\sup_{s\in[0,t]} |Z(s)|^{p-1} \left(\int_0^t \|\Phi(s)\|^2_{L_2^0} ds \right)^{1/2} \right] \\
&\leq \frac{p(p-1)}{2} \left[\mathbb{E} \left(\sup_{s\in[0,t]} |Z(s)|^p \right) \right]^{\frac{p-2}{p}} \left[\mathbb{E} \left(\int_0^t \|\Phi(s)\|^2_{L_2^0} ds \right)^{\frac{p}{2}} \right]^{\frac{2}{p}} \\
&+3p\mathbb{E} \left[\sup_{s\in[0,t]} |Z(s)|^p \right]^{\frac{p-1}{p}} \left[\mathbb{E} \left(\int_0^t \|\Phi(s)\|^2_{L_2^0} ds \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} \\
&:= J_1 + J_2.
\end{aligned} \tag{B.2}$$

For J_1 we use Young's inequality with exponents $\frac{p}{p-2}$ and $\frac{p}{2}$ and find

$$J_1 \le \frac{1}{4}\,\mathbb{E}\left[\sup_{s\in[0,t]} |Z(s)|^p\right] + 2^{p-1}p^p\,\mathbb{E}\left(\int_0^t \|\Phi(s)\|^2_{L_2^0} ds\right)^{\frac{p}{2}}$$

For J_2 we use Young's inequality with exponents $\frac{p}{p-1}$ and p and find

$$J_2 \le \frac{1}{4}\,E\left[\sup_{s\in[0,t]} |Z(s)|^p\right] + \frac{1}{2}\,12^p\,p^p\,\mathbb{E}\left(\int_0^t \|\Phi(s)\|^2_{L_2^0} ds\right)^{\frac{p}{2}}.$$

Now (B.1) with $c_p := 12^p\,p^p$ follows. □

Appendix C. Density of $\mathcal{F}C_b^1$ in Orlicz spaces

Let $N : \mathbb{R} \to [0,\infty)$ be continuous and a Young function, i.e. convex, even and $N(0) = 0$.

Consider the measure space $(H, \mathcal{B}(H), \gamma)$, where H is as before a separable real Hilbert space with Borel σ–algebra $\mathcal{B}(H)$ and γ a nonnegative finite measure on $(H, \mathcal{B}(H))$. We recall that the Orlicz space L_N corresponding to N is defined as

$$L_N := L_N(H,\gamma) := \{f : H \to \mathbb{R} :\ f\ \mathcal{B}(H)\text{–}measurable\ and\ \int_H N(af)d\gamma < \infty\ for\ some\ a > 0\}$$

or equivalently

$$L_N := \{f : H \to \mathbb{R} :\ f\ \mathcal{B}(H)\text{–}measurable\ and\ \|f\|_{L_N} < \infty\},$$

where

$$\|f\|_{L_N} := \inf\left\{\lambda > 0 :\ \int_H N(f/\lambda) \le 1\right\}.$$

$(L_N, \|\cdot\|_{L_N})$ is a Banach space (see e.g. [20]).

Proposition C.1. *$\mathcal{F}C_b^1$ is dense in $((L_N, \|\cdot\|_{L_N})$, where $\mathcal{F}C_b^1$ is defined as in Section 1. Furthermore, if $f \in L_N$, $f \ge 0$, then there exist nonnegative $f_n \in \mathcal{F}C_b^1$, $n \in \mathbb{N}$, such that*

$$\lim_{n\to\infty} \|f - f_n\|_{L_N} = 0.$$

Proof. We need the following lemma whose proof is straightforward, see e.g. [19, Lemma 1.16]

Lemma C.2. *Let $f_n \in L_N$, $n \in \mathbb{N}$. Then the following assertions are equivalent:*

(i) $\lim\limits_{n\to\infty} \|f_n\|_{L_N} = 0$

(ii) For all $a \in (0,\infty)$

$$\limsup_{n\to\infty} \int_H N(af_n)\,d\gamma \le 1$$

(iii) For all $a \in (0,\infty)$

$$\lim_{n\to\infty} \int_H N(af_n)\,d\gamma = 0.$$

Proof of Proposition C.1.

We shall use a monotone class argument. Define

$$\mathcal{M} := \Big\{f : H \to \mathbb{R} :\ f\ bounded,\ \mathcal{B}(H)\text{–}measurable\ such\ that$$

$$\lim_{n\to\infty} \|f - f_n\|_{L_N} = 0,\ for\ some\ f_n \in \mathcal{F}C_b^1,\ n \in \mathbb{N}\Big\}.$$

Obviously, $\mathcal{M}$ is a linear space, $\mathcal{F}C_b^1 \subset \mathcal{M}$ and $\mathcal{F}C_b^1$ is closed under multiplication and contains the constant function 1. Furthermore, if $0 \le u_n \in \mathcal{M}$, $n \in \mathbb{N}$, such that $u_n \uparrow u$ as $n \to \infty$ for some bounded $u : H \to [0, \infty)$, then for each $n \in \mathbb{N}$ there exists $f_n \in \mathcal{F}C_b^1$ such that

$$\|u_n - f_n\|_{L_N} \le \frac{1}{n}. \tag{C.1}$$

But since N is continuous on $\mathbb{R}$, hence locally bounded, we have that for every $a \in (0, \infty)$, $N(a(u - u_n))$, $n \in \mathbb{N}$, are uniformly bounded. Consequently, by Lebesgue's dominated convergence theorem and Lemma C.2, we conclude that

$$\lim_{n\to\infty} \|u - u_n\|_{L_N} = 0. \tag{C.2}$$

(C.1) and (C.2) imply that $u \in \mathcal{M}$, and therefore $\mathcal{M}$ is a monotone vector space and thus by the monotone class theorem $\mathcal{M}$ is equal to the set of all bounded $\sigma(\mathcal{F}C_b^1)$–measurable functions on H. But $\sigma(\mathcal{F}C_b^1) = \mathcal{B}(H)$, since the weak and norm–Borel σ–algebra on a separable Banach space coincide. Hence $\mathcal{M}$ is equal to all bounded $\mathcal{B}(H)$–measurable functions on H. Since by Lemma C.2 and the same arguments as above every f in L_N can be approximated in the norm $\|\cdot\|_{L_N}$ by bounded $\mathcal{B}(H)$–measurable functions, the first assertion of the proposition is proved.

Now let $f \in L_N$, $f \ge 0$. By the argument above we may assume that f is bounded. Then by what we have just proved we can find $f_n \in \mathcal{F}C_b^1$ such that

$$\lim_{n\to\infty} \|f - f_n\|_{L_N} = 0.$$

Since $|f - f_n^+| = |f^+ - f_n^+| \le |f - f_n|$ for all $n \in \mathbb{N}$ and N is even and increasing (because N is convex and $N(0) = 0$), Lemma C.2 immediately implies that

$$\lim_{n\to\infty} \|f - f_n^+\|_{L_N} = 0.$$

Fix $n \in \mathbb{N}$ and for $\epsilon > 0$ take an increasing function $\chi_\epsilon \in C^1(\mathbb{R})$, $\chi_\epsilon(s) = s$, $\forall\, s \in [0, \infty)$ and $\chi_\epsilon(s) = -\epsilon$ if $s \in (-\infty, -2\epsilon)$. Then for each $n \in \mathbb{N}$

$$\lim_{m\to\infty} \left\| f_n^+ - \left(\chi_{\frac{1}{m}}(f_n) + \frac{1}{m} \right) \right\|_\infty = 0.$$

So, again by Lemma C.2 and Lebesgue's dominated convergence theorem it follows that

$$\lim_{m\to\infty} \left\| f_n^+ - \left(\chi_{\frac{1}{m}}(f_n) + \frac{1}{m} \right) \right\|_{L_N} = 0.$$

But obviously, $\chi_{\frac{1}{m}}(f_n) + \frac{1}{m} \in \mathcal{F}C_b^1$, $m \in \mathbb{N}$, and each such function is nonnegative. Hence the second part of the assertion follows. □

Corollary C.3. *Let $\rho \ge 0$, $\mathcal{B}(H)$–measurable such that*

$$\int_H \rho \log \rho \, d\gamma < \infty.$$

Then there exist nonnegative $\rho_n \in \mathcal{F}C_b^1$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \rho_n = \rho \quad \text{in } L^1(H, \gamma)$$

and

$$\sup_{n\in\mathbb{N}} \int_H \rho_n \log \rho_n \, d\gamma < \infty.$$

Proof. Let $N(s) := (|s|+1)\ \ln(|s|+1) - |s|$, $s \in \mathbb{R}$. Then it is easy to check that N is a continuous Young function. Hence by Proposition C.1 we can find $\rho_n \in \mathcal{F}C_b^1$, $\rho_n \geq 0$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \|\rho - \rho_n\|_{L_N} = 0. \tag{C.3}$$

Since $L_N \subset L^1(H,\gamma)$ continuously (see [19, Proposition 1.15]), the first assertion follows. Furthermore, we have for all $s \in (0,\infty)$

$$s \ln s - s \leq s \ln(s+1) \leq (s+1)\ln(s+1) - s = N(s)$$

and hence for $n \in \mathbb{N}$ by the convexity of N and every $a \in (0,\infty)$

$$\int_H \rho_n\ \ln \rho_n\, d\gamma = \frac{1}{a}\int_H a\rho_n\ \ln(a\rho_n)\ d\gamma - \ln a \int_H \rho_n\, d\gamma$$

$$\leq \frac{1}{a}\int_H N(a\rho_n)\ d\gamma + |1 - \ln a| \int_H \rho_n\, d\gamma$$

$$\leq \frac{1}{2a}\int_H N(2a(\rho_n - \rho))\ d\gamma + \frac{1}{2a}\int_H N(2a\rho)\ d\gamma + |1-\ln a| \int_H \rho_n\, d\gamma.$$

Hence by the first part of the assertion, (C.3) and Lemma C.2, it follows that

$$\limsup_{n\to\infty} \int_H \rho_n\ \ln \rho_n\, d\gamma \leq \frac{1}{2a}\int_H N(2a\rho)\ d\gamma + |1-\ln a| \int_H \rho\, d\gamma.$$

But since $\rho \in L_N$ we can find $a > 0$ such that the right hand side is finite. Hence the second part of the assertion also follows.

□

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