Weak-coupling limit for ergodic environments

Martin Friesen*
Yuri Kondratiev†

August 11, 2016

Abstract: The main aim of this work is to establish an averaging principle for a wide class of interacting particle systems of birth-and-death type in the continuum. This principle is an important step in the analysis of Markov evolutions and is usually applied for the associated semigroups related to backward Kolmogorov equations, c.f. [Kur73]. Our approach is based on the study of forward Kolmogorov equations (a.k.a. Fokker-Planck equations). We describe a system evolving as a Markov process on the space of finite configurations, whereas its rates depend on the actual state of another (equilibrium) process on the space of locally finite configurations. We will show that ergodicity of the environment process implies the averaging principle for the solutions of the coupled Fokker-Planck equations.

AMS Subject Classification: 37N25, 46N30, 46N55, 47N30, 92D1
Keywords: Averaging principle; Fokker-Planck equation; Interacting particle systems; Weak-coupling; Random evolution

1 Introduction

Several models of dynamics for interacting particle systems are described by Markov evolutions of finitely or infinitely many indistinguishable particles in a location space, lets say for simplicity in $\mathbb{R}^d$. Therefore, the natural state space for such Markov evolutions is given by

$$\Gamma = \{\gamma \subset \mathbb{R}^d \mid \ |\gamma \cap K| < \infty, \ \ \forall K \subset \mathbb{R}^d \text{ compact } \},$$

where $|\gamma \cap K|$ is the number of points of the configuration γ inside the volume K, cf. [AKR98a]. Stochastic dynamics of interacting particle systems include elementary events

*Department of Mathematics, Bielefeld University, Germany, mfriesen@math.uni-bielefeld.de
†Department of Mathematics, Bielefeld University, Germany, kondrat@math.uni-bielefeld.de

such as birth ($\gamma \to \gamma \cup x,\ x \in \mathbb{R}^d \backslash \gamma$), death ($\gamma \to \gamma \backslash x,\ x \in \gamma$) and jumps of individual particles ($\gamma \to \gamma \backslash x \cup y,\ x \in \gamma,\ y \in \mathbb{R}^d \backslash \gamma$). Here and in the following we will always write $\gamma \cup x$ instead of $\gamma \cup \{x\}$ and likewise for $\gamma \backslash x, \gamma \backslash x \cup y, \mathbb{R}^d \backslash \gamma$, etc. It is also possible (at least for Markov evolutions with the constraint $|\gamma| < \infty$) to include other types of elementary events such as collisions, fragmentation, etc., see e.g. [BK03, Kol06, EW03].
In this work we will use as environmental processes so-called birth-and-death Markov evolutions on Γ which are described by their Markov generators L^E. These operators are given by means the following heuristic form

$$(L^E F)(\gamma) = \sum_{x \in \gamma} d(x, \gamma \backslash x)(F(\gamma \backslash x) - F(\gamma)) + \int_{\mathbb{R}^d} b(x, \gamma)(F(\gamma \cup x) - F(\gamma))\mathrm{d}x. \tag{1.1}$$

With a heuristic generator is associated a backward Kolmogorov equation

$$\frac{\partial F_t}{\partial t} = LF_t, \quad F_t|_{t=0} = F_0, \tag{1.2}$$

that leads to the related semigroup (provided, of course, that a solution exists). The function $d(x, \gamma \backslash x) \geq 0$ is called the death intensity and $b(x, \gamma) \geq 0$ the birth intensity. The corresponding processes were constructed and studied for several special classes of intensities d and b. Equilibrium dynamics can be treated by the theory of Dirichlet forms. Such approach was first used in [AKR98a, AKR98b] to construct the equilibrium gradient diffusion process on Γ and to study their ergodicity. For equilibrium dynamics of birth-and-death type in [KL05] the Glauber dynamics were constructed and the existence of a spectral gap was shown. For both examples grand canonical Gibbs measures are invariant measures.
Non-equilibrium dynamics with particular intensities were considered in [GK06]. There the authors have used stochastic differential equations for the construction. For the contact model, the corresponding non-equilibrium process was constructed by a suitable approximation scheme in [KS06]. For other classes of intensities one tries to study instead the evolution of states in the weak form, i.e. the Fokker-Planck equation dual to (1.2), given by

$$\frac{\partial}{\partial t}\langle F, \mu_t \rangle = \langle LF, \mu_t \rangle, \quad \langle F, \mu_t \rangle|_{t=0} = \langle F, \mu_0 \rangle. \tag{1.3}$$

The pairing between functions $F : \Gamma \longrightarrow \mathbb{R}$ and probability measures μ on Γ is simply

$$\langle F, \mu \rangle = \int_{\Gamma} F(\gamma)\mu(\mathrm{d}\gamma).$$

It was developed an approach to the study of Fokker-Planck equations in terms of factorial moments of states a.k.a. correlation functions [KK02, FKO09]. For many specific models there exists already a number of works (see e.g. [FKK09, FKK12, FKK13, FKKZ14, FKKO15, FKKK15]) for pure birth-and-death dynamics. For dynamics of jumping particles we refer to [FKO12, BKK15]. See also [FKK15] for a review on the developed techniques so far.

Another class of interacting particle systems is formed by models where the number of particles is finite for each fixed moment of time. Such models are also under active investigation, see e.g. [BK03, EW03, FM04, Che04, Kol06, Fri16, Bez15]. In our notions such systems evolve on the state space

$$\Gamma_0 = \{\eta \subset \mathbb{R}^d \mid \ |\eta| < \infty\}$$

and the general form of the pre-generator is given by

$$(L^S F)(\eta) = \sum_{\xi \subset \eta} \int_{\Gamma_0} (F(\eta \backslash \xi \cup \zeta) - F(\eta)) K(\xi, \eta, \zeta) \mathrm{d}\lambda(\zeta), \tag{1.4}$$

where $\mathrm{d}\lambda(\zeta)$ denotes the Lebesgue-Poisson measure on Γ_0, see below. In this form the operator L^S includes birth, death and jumps of groups of particles. It is worth noting that it is also possible to consider general kernels $K(\xi, \eta, \mathrm{d}\zeta)$, but since we want to investigate here the evolution of densities we restrict ourselves to (1.4). Under suitable conditions on K this operator can be rewritten as

$$(L^S F)(\eta) = \int_{\Gamma_0} (F(\eta) - F(\xi)) Q(\eta, \mathrm{d}\xi),$$

where the transition kernel is given by

$$Q(\eta, \mathrm{d}\omega) = \sum_{\xi \subset \eta} \int_{\Gamma_0} \delta_{\eta \backslash \xi \cup \zeta}(\mathrm{d}\omega) K(\xi, \eta, \zeta) \mathrm{d}\lambda(\zeta).$$

Hence, under some general conditions, it determines a pure jump Markov process. For pure jump Markov processes equation (1.2) has been first analyzed in [Fel40] and the theory of this kind of stochastic processes is quite well developed. Between recent works on this subject we would like to mention [FMS14]. There the authors constructed (even in the time-inhomogeneous case) associated (sub-)Markov transition functions and gave a characterization for the related minimal process to be conservative. In [Che04] is given a complete summary for jump processes and considered applications to interacting particle systems on a lattice. In [Bez15] was constructed a Markov process with generator (1.1) for specific intensities as the strong solution to the related stochastic differential equations. Recently for the more general generator L^S in [Fri16] the corresponding conservative Feller evolution system was been constructed in the time-inhomogeneous case. We will use essentially the results of the latter work.

The main aim of this work is to describe the behaviour of a system with state space Γ_0 evolving in the presence of an equilibrium, ergodic environment, which is described by a Markov process with the state space Γ and an associated invariant measure μ. This situation is a particular case of so-called random evolution framework, see e.g. [Pin91, SHS02]. Examples for such environments have been constructed e.g. in [AKR98a, AKR98b, KL05]. There (via the Dirichlet forms technique) the existence of a Markov semigroup $T^E(t)$ on $L^2(\Gamma,\mu)$ has been shown, where μ is the unique invariant measure and $T^E(t)$ is symmetric on $L^2(\Gamma,\mu)$. As a consequence this semigroup can be extended to all $L^p(\Gamma,\mu)$ with $1 \leq p < \infty$ and for $p = 1$ this extension, also denoted by $T^E(t)$, gives the evolution of densities. More precisely, if $R \in L^1(\Gamma,\mu)$ and the environment is in the initial state $R\mathrm{d}\mu$, then the time evolution is given by $R_t\mathrm{d}\mu$, where $R_t = T^E(t)R$. Assume that $T^E(t)$ is ergodic on $L^1(\Gamma,\mu)$, i.e., $T^E(t)R \to \int\limits_{\Gamma} R(\gamma)\mathrm{d}\mu(\gamma),\ t \to \infty$ in $L^1(\Gamma,\mu)$ and denote by L^E its generator. We will study an evolution of a system described by the Kolmogorov operator L^S, cf. (1.4), with kernel $K(\gamma,\xi,\eta,\zeta)$, which additionally depends on the present microscopic state $\gamma \in \Gamma$ of the environment. Therefore, solutions to the Fokker-Planck equation

$$\frac{\partial \rho_t}{\partial t} = (L^S)^*\rho_t + L^E\rho_t, \quad \rho_t|_{t=0} = \rho_0,$$

on the space $L^1(\Gamma \times \Gamma_0, \mu \otimes \lambda)$ describe the evolution of densities of the joint Markov process for the system and environment. Here $(L^S)^*$ stands for the adjoint operator on densities $\rho(\eta,\gamma)$, which depends on γ as a parameter but acts only on the first variable η. Similarly, L^E acts only on the second variable γ. The weak-coupling limit is obtained via an approximation ρ_t^ε, where ρ_t^ε solves the rescaled version of the Fokker-Planck equation

$$\frac{\partial \rho_t^\varepsilon}{\partial t} = (L^S)^*\rho_t^\varepsilon + \frac{1}{\varepsilon}L^E\rho_t^\varepsilon, \quad \rho_t^\varepsilon|_{t=0} = \rho_0 \in L^1(\Gamma_0,\lambda).$$

Thus we will seek for the limit $\rho_t^\varepsilon \longrightarrow \overline{\rho}_t$ when $\varepsilon \to 0$. In such a case we will prove that $\overline{\rho}_t$ solves the Fokker-Planck equation for a finite system determined by the averaged generator

$$\overline{L}F(\eta) = \sum_{\xi\subset\eta} \int\limits_{\Gamma_0} (F(\eta\backslash\xi \cup \zeta) - F(\eta))\overline{K}(\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta),$$

where $\overline{K}(\xi,\eta,\zeta) = \int\limits_{\Gamma} K(\gamma,\xi,\eta,\zeta)\mathrm{d}\mu(\gamma)$. The aim of this work is to realize this approach and show for one specific example how this can be applied.

The paper is organized as follows. In the first section we collect general properties for the spaces Γ and Γ_0. The second section is devoted to the main result and is divided into four parts. In the first part we clarify the assumptions on the environment and extend the semigroup for the environment to the space $L^1(\Gamma \times \Gamma_0, \mu \otimes \lambda)$ of joint densities. Afterwards we will deal with general finite systems and related solutions to the Fokker-Planck equations. We will show that such solutions will leave the space $L^1(\Gamma_0,\lambda)$ of densities invariant and give a characterization when they will preserve the convex cone of probability measures. The third part will extend the considerations of the second part to systems depending on γ as a parameter. Finally, in the last part of the main section we will state and prove the main result about the averaging scheme for our specific situation. In the last section we will give an example for an interacting particle system in continuum for which our averaging results can be applied successfully.

2 Preliminaries

We are going to collect certain properties for the space of finite configurations, which shall be used later on, for more details see [AKR98a, KK02] and references therein. Let Γ_0 be the space of all finite subsets of $\mathbb{R}^d$, i.e.

$$\Gamma_0 = \{\eta \subset \mathbb{R}^d \mid \ |\eta| < \infty\},$$

where $|\eta|$ denotes the number of elements in the set η. This space has a natural decomposition into n-particle subspaces, $\Gamma_0 = \bigsqcup_{n=0}^{\infty} \Gamma_0^{(n)}$, where

$$\Gamma_0^{(n)} = \{\eta \subset \mathbb{R}^d \mid \ |\eta| = n\}, \quad n \geq 1$$

and in the case $n = 0$ we set $\Gamma_0^{(0)} = \{\emptyset\}$. For a compact $\Lambda \subset \mathbb{R}^d$ let

$$\Gamma_\Lambda = \{\eta \in \Gamma_0 \mid \ \eta \subset \Lambda\}.$$

and

$$\Gamma_\Lambda^{(n)} = \{\eta \in \Gamma_0^{(n)} \mid \ \eta \subset \Lambda\}.$$

Let $\widetilde{(\mathbb{R}^d)^n}$ be the space of all sequences $(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$ with $x_i \neq x_j$ for $i \neq j$. Denote by $\mathrm{sym}_n : \widetilde{(\mathbb{R}^d)^n} \to \Gamma_0^{(n)}$, $(x_1, \ldots, x_n) \longmapsto \{x_1, \ldots, x_n\}$ the symmetrization map and define a topology on $\Gamma_0^{(n)}$ via $A \subset \Gamma_0^{(n)}$ is open if $\mathrm{sym}_n^{-1}(A) \subset \widetilde{(\mathbb{R}^d)^n}$ is open. On Γ_0 we define the topology of disjoint unions, i.e., a set $A \subset \Gamma_0$ is open if and only if $A \cap \Gamma_0^{(n)}$ is open in $\Gamma_0^{(n)}$ for all $n \in \mathbb{N}$. Γ_0 equipped with this topology is a locally compact Polish space. We let $\mathcal{B}(\Gamma_0)$ stand for the Borel$-\sigma-$algebra on Γ_0. In this topology a sequence $(\eta_n)_{n \in \mathbb{N}} \subset \Gamma_0$ converges to $\eta \in \Gamma_0$ if and only if

$$\langle \varphi, \eta_n \rangle \longrightarrow \langle \varphi, \eta \rangle, \quad n \to \infty$$

for all continuous bounded functions φ, i.e., $\varphi \in C_b(\mathbb{R}^d)$. Here the pairing $\langle \cdot, \cdot \rangle$ is simply the pairing of the point measure $\eta = \sum_{x \in \eta} \delta_x$ with the function φ, i.e. $\langle \varphi, \eta \rangle := \sum_{x \in \eta} \varphi(x)$. Therefore the convergence $\eta_n \longrightarrow \eta$ can be equivalently reformulated to: $\eta = \{x_1, \ldots, x_l\}$

and for sufficiently large n there is a numeration $\eta_n = \{x_1^{(n)}, \dots, x_l^{(n)}\}$ such that $x_k^{(n)} \longrightarrow x_k$ as $n \to \infty$. Denote by $\mathrm{d}x$ the Lebesgue measure on $\mathbb{R}^d$ and by $\mathrm{d}^{\otimes n}x$ the product measure on $(\mathbb{R}^d)^n$. Let $\mathrm{d}^{(n)}x$ be the image measure of $\mathrm{d}^{\otimes n}x$ on $\Gamma_0^{(n)}$ via sym_n and set for $z > 0$

$$\lambda_z = \delta_\emptyset + \sum_{n=1}^{\infty} \frac{z^n}{n!} \mathrm{d}^{(n)}x.$$

This measure is called the Lebesgue-Poisson measure on Γ_0 and for $z = 1$ set $\lambda := \lambda_1$.

3 Main results

3.1 Environment

Let us start with the main assumption on the environment process on Γ:

(E) There exists a probability measure μ on Γ and a positive semigroup of contractions $T^E(t)$ on $L^1(\Gamma, \mu)$, which is assumed to be L^1-ergodic, i.e., for each $R \in L^1(\Gamma, \mu)$

$$\int_\Gamma |T^E(t)R - \langle R \rangle_\mu| \mathrm{d}\mu \longrightarrow 0, \quad t \to \infty.$$

Here $\langle R \rangle_\mu = \int_\Gamma R \mathrm{d}\mu$ denotes the average of R with respect to μ.

Denote by $(L^E, D(L^E))$ its generator. As a first step we will extend this semigroup to the space of Bochner integrable functions $L^1(\Gamma \to L^1(\Gamma_0, \lambda), \mu) =: \mathcal{L}_\mu$. It is well-known that $L^1(\Gamma \to L^1(\Gamma_0, \lambda), \mu) \cong L^1(\Gamma \times \Gamma_0, \mu \otimes \lambda)$. The subspace

$$D = \left\{ f = \sum_{k=1}^{n} R_k \rho_k \;\middle|\; n \in \mathbb{N}, R_k \in L^1(\Gamma, \mu), \rho_k \in L^1(\Gamma_0, \lambda) \right\} \subset \mathcal{L}_\mu$$

is dense and since $T^E(t)$ is positive it can be uniquely extended to $\mathcal{L}_\mu$, cf. [Gra04], such that for $f \in D$

$$T^E(t)f = \sum_{k=1}^{n} (T^E(t)R_k)\rho_k.$$

One has $\|(T^E(t)f)(\cdot, \gamma)\|_{L^1(\Gamma_0,\lambda)} \le T^E(t)\|f(\cdot, \gamma)\|_{L^1(\Gamma_0,\lambda)}$ for all $f \in D$, thus this extension will be a positive strongly continuous semigroup of contractions which shall be denoted again by $T^E(t)$. We will denote the generator of the extended semigroup also by $(L^E, D(L^E))$. The generator $(L^E, D(L^E))$ can be characterized by the relation

$$L^E f = \sum_{k=1}^{n} (L^E R_k)\rho_k,$$

where $f \in D$ with $R_k \in D(L^E)$. We obtain for $f \in D$

$$\|T^E(t)f - \langle f \rangle_\mu\|_{\mathcal{L}_\mu} \leq \sum_{k=1}^{n} \|T^E(t)R_k - \langle R_k \rangle_\mu\|_{L^1(\Gamma,\mu)} \|\rho_k\|_{L^1(\Gamma_0,\lambda)} \longrightarrow 0, \quad t \to \infty$$

and since $T^E(t)$ is a semigroup of contractions and D dense this implies for each $f \in \mathcal{L}_\mu$: $\|T^E(t)f - \langle f \rangle_\mu\|_{\mathcal{L}_\mu} \longrightarrow 0, \quad t \to \infty$. Note that $\langle f \rangle_\mu(\eta) := \int\limits_\Gamma f(\gamma,\eta)\mathrm{d}\mu(\gamma)$ is simply the projection of $\mathcal{L}_\mu$ onto $L^1(\Gamma_0, \lambda)$.

3.2 Isolated system

The class of Markov jump dynamics we want to describe in this section are given by a Markov pre-generator on bounded measurable functions $F : \Gamma_0 \longrightarrow \mathbb{R}$ as

$$(L^S F)(\eta) = \sum_{\xi \subset \eta} \int\limits_{\Gamma_0} (F(\eta \backslash \xi \cup \zeta) - F(\eta)) K(\xi, \eta, \zeta) \mathrm{d}\lambda(\zeta) \tag{3.1}$$

for $\eta \in \Gamma_0$. Such Kolomogorov operator includes death, birth and jumps of groups of particles. In order to give rigorous meaning to this expression we have to assume for the kernel $K : \Gamma_0 \times \Gamma_0 \times \Gamma_0 \longrightarrow \mathbb{R}_+$ the following:

(K) The map $(\xi, \eta, \zeta) \longmapsto K(\xi, \eta, \zeta)$ is measurable and for all $\xi, \eta \in \Gamma_0$

$$\int\limits_{\Gamma_0} K(\xi, \eta, \zeta)\mathrm{d}\lambda(\zeta) < \infty.$$

For $\eta \in \Gamma_0$ and $A \in \mathcal{B}(\Gamma_0)$ we define a transition kernel

$$Q(\eta, A) := \sum_{\xi \subset \eta} \int\limits_{\Gamma_0} \mathbb{1}_A(\eta \backslash \xi \cup \zeta) K(\xi, \eta, \zeta) \mathrm{d}\lambda(\zeta).$$

Then $q(\eta) := Q(\eta, \Gamma_0) = \sum\limits_{\xi \subset \eta} \int\limits_{\Gamma_0} K(\xi, \eta, \zeta)\mathrm{d}\lambda(\eta)$ is finite for each $\eta \in \Gamma_0$. The operator L^S is for any bounded measurable function given by

$$(L^S F)(\eta) = -q(\eta)F(\eta) + \int\limits_{\Gamma_0} F(\xi)Q(\eta, \mathrm{d}\xi) = \int\limits_{\Gamma_0} (F(\xi) - F(\eta))Q(\eta, \mathrm{d}\xi).$$

Hence there exists a (sub-)Markov transition function $P : \mathbb{R}_+ \times \Gamma_0 \times \mathcal{B}(\Gamma_0) \longrightarrow [0, 1]$ such that for any bounded and measurable function $F : \Gamma_0 \longrightarrow \mathbb{R}$

$$T^S(t)F(\eta) := \int\limits_{\Gamma_0} F(\xi)P(t, \eta, \mathrm{d}\xi), \quad t \geq 0 \tag{3.2}$$

is a positive semigroup of contractions, cf. [Fel40, FMS14, Che04, Kol06]. Moreover, this semigroup is continuous w.r.t. $t \geq 0$ in the sense that

$$T^S(t)F(\eta) \longrightarrow F(\eta), \quad t \to 0, \ \eta \in \Gamma_0.$$

For any bounded measurable function and any $\eta \in \Gamma_0$

$$\frac{T^S(t)F(\eta) - F(\eta)}{t} \longrightarrow L^S F(\eta), \quad t \to 0 \tag{3.3}$$

holds. Hence for any bounded measurable function F_0 and any fixed $\eta \in \Gamma_0$ the action of the semigroup $T^S(t)F(\eta)$ satisfies (1.2). Note that formula (3.1) is well-defined for any $\eta \in \Gamma_0$ and bounded measurable function F, but $L^S F$ does not need to be bounded in η. Using the same (sub-)Markov function, we can also define the evolution of states

$$T^S(t)^*\nu(\mathrm{d}\eta) = \int\limits_{\Gamma_0} P(t,\xi,\mathrm{d}\eta)\nu(\mathrm{d}\xi), \quad \nu \in \mathcal{M}(\Gamma_0), \tag{3.4}$$

where $\mathcal{M}(\Gamma_0)$ is the space of signed finite Borel measures on Γ_0 equipped with the total variation norm

$$\|\nu\| = |\nu|(\Gamma_0) = \nu_+(\Gamma_0) + \nu_-(\Gamma_0).$$

Therefore, for any probability measure ν_0 on Γ_0, $T^S(t)^*\nu_0 =: \nu_t$ provides a solution to the Fokker-Planck equation

$$\frac{\partial}{\partial t}\int\limits_{\Gamma_0} F(\eta)\nu_t(\mathrm{d}\eta) = \int\limits_{\Gamma_0} L^S F(\eta)\nu_t(\mathrm{d}\eta).$$

Note that $P(t,\xi,\cdot)$ is, in general, only a sub-probability function, i.e. $0 \leq P(t,\xi,\Gamma_0) \leq 1$ holds. The system is called conservative if $P(t,\xi,\Gamma_0) = 1$ holds for all $t \geq 0$ and $\xi \in \Gamma_0$. In such a case $P(t,\xi,\cdot)$ is the unique Markov transition probability function corresponding to the operator (3.1). That is $T^S(t)F(\eta)$ is the unique solution to (1.2) and $T^S(t)^*\mu_0$ is the unique solution to the Fokker-Planck equation.
In the following we give an alternative construction of $T^S(t)^*$ and deduce that it leaves the space $L^1(\Gamma_0,\lambda)$ invariant. Let

$$\mathrm{Dom} = \left\{ \nu \in \mathcal{M}(\Gamma_0) \;\middle|\; \int\limits_{\Gamma_0} q(\eta)|\nu|(\mathrm{d}\eta) < \infty \right\},$$

where $q(\eta) = \sum\limits_{\xi \subset \eta} \int\limits_{\Gamma_0} K(\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta)$ and define a transition kernel

$$Q(\eta, A) = \sum_{\xi \subset \eta} \int\limits_{\Gamma_0} \mathbb{1}_A(\eta\backslash\xi \cup \zeta)K(\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta), \quad A \in \mathcal{B}(\Gamma_0).$$

This kernel defines via

$$(BF)(\eta) = \int\limits_{\Gamma_0} F(\xi)Q(\eta, \mathrm{d}\xi) = \sum_{\xi \subset \eta} \int\limits_{\Gamma_0} F(\eta \backslash \xi \cup \zeta) K(\xi, \eta, \zeta) \mathrm{d}\lambda(\zeta)$$

an operator on bounded measurable functions F and via

$$(B^*\nu)(A) = \int\limits_{\Gamma_0} Q(\eta, A)\nu(\mathrm{d}\eta) = \int\limits_{\Gamma_0} \sum_{\xi \subset \eta} \int\limits_{\Gamma_0} \mathbb{1}_A(\eta \backslash \xi \cup \zeta) K(\xi, \eta, \zeta) \mathrm{d}\lambda(\zeta)\nu(\mathrm{d}\eta)$$

on measures $\nu \in \mathcal{M}(\Gamma_0)$. Both operators (B^*, Dom) as well as $(L^S)^* = -q + B^*$ given by

$$(L^S)^*\nu(A) = -\int\limits_A q(\eta)\nu(\mathrm{d}\eta) + (B^*\nu)(A) \tag{3.5}$$

are well-defined. The multiplicative part $(-q, \mathrm{Dom})$ generates a positive analytic semigroup of contractions such that

$$(e^{-tq}\nu)(A) = \int\limits_A e^{-tq(\eta)}\nu(\mathrm{d}\eta).$$

The resolvent of $(-q, \mathrm{Dom})$ exists for $\lambda > 0$ and satisfies

$$R(\lambda; -q)\nu(A) = \int\limits_A \frac{1}{\lambda + q(\eta)}\nu(\mathrm{d}\eta), \quad A \in \mathcal{B}(\Gamma_0).$$

Since (B^*, Dom) is a positive operator and for $0 \le \nu \in \mathrm{Dom}$ it holds that

$$B^*\nu(\Gamma_0) = \int\limits_{\Gamma_0} Q(\eta, \Gamma_0)\nu(\mathrm{d}\eta) = \int\limits_{\Gamma_0} q(\eta)\nu(\mathrm{d}\eta),$$

there exist an extension $(G, D(G))$ of $((L^S)^*, \mathrm{Dom})$ which is the generator of a sub-stochastic semigroup $\widetilde{T}(t)^*$, cf. Theorem 2.1 [TV06, ALMK11]. This semigroup is minimal, i.e., for any other sub-stochastic semigroup $S(t)^*$ on $\mathcal{M}(\Gamma_0)$ with generator being an extension of $((L^S)^*, \mathrm{Dom})$ one has $\widetilde{T}(t)^* \le S(t)^*$.

Lemma 3.1. *The semigroup $\widetilde{T}(t)^*$ coincides with $T^S(t)^*$, where $T^S(t)^*$ is defined in* (3.4). *Moreover $\widetilde{T}(t)^*$ leaves $L^1(\Gamma_0, \lambda)$ invariant.*

For convenience of the reader the proof can be found in the appendix. We should emphasize that without further assumptions the semigroup $T^S(t)^*$ might be not stochastic, i.e. $P(t, \xi, \Gamma_0) < 1$ may happen. Sufficient conditions for $T^S(t)^*$ being stochastic (and hence $P(t, \xi, \Gamma_0) = 1$) can be found in [Che04, TV06], and in the time-inhomogeneous case in [Fri16]. In the following we will need the following characterization for $T^S(t)^*$ being stochastic.

Theorem 3.2. *The semigroup $T^S(t)^*$ is stochastic if and only if its generator $(G, D(G))$ is the closure of $(-q + B^*, \mathrm{Dom})$.*

Proof. Assume that $(G, D(G))$ is the closure of $(-q + B^*, \mathrm{Dom})$, then it is well-known that $\widetilde{T}(t)^*$ is stochastic, cf. [TV06]. By Lemma 3.1 also $T^S(t)^*$ is stochastic. Conversely, assume that $T^S(t)^*$ is stochastic, then $\|T^S(t)^*\nu\| = \|\nu\|$ for any $0 \leq \nu \in \mathcal{M}(\Gamma_0)$. Hence Corollary 3.6 [ALMK11] implies in this case the assertion. $\square$

3.3 System in the presence of stationary environment

In this section we assume that K depends on another parameter $\gamma \in \Gamma$, i.e. $K(\gamma, \xi, \eta, \zeta)$. Let μ be a probability measure on Γ as in assumption (E). Since we work only with solutions to the Fokker-Planck equation on densities we have to assume that K is measurable with respect to all variables and

$$\int\limits_{\Gamma}\int\limits_{\Gamma_0} K(\gamma, \xi, \eta, \zeta)\mathrm{d}\lambda(\zeta)\mathrm{d}\mu(\gamma) < \infty, \quad \forall \xi, \eta \in \Gamma_0. \tag{3.6}$$

Let us outline the construction of the evolution of densities on $\mathcal{L}_\mu = L^1(\Gamma \times \Gamma_0, \mu \otimes \lambda)$. First of all, the Markov pre-generator L^S is of the form

$$L^S F(\gamma, \eta) = \sum_{\xi \subset \eta} \int\limits_{\Gamma_0} (F(\gamma, \eta \backslash \xi \cup \zeta) - F(\gamma, \eta)) K(\gamma, \xi, \eta, \zeta)\mathrm{d}\lambda(\zeta). \tag{3.7}$$

It can be rewritten as

$$L^S F(\gamma, \eta) = \int\limits_{\Gamma_0} (F(\gamma, \omega) - F(\gamma, \eta)) Q(\gamma, \eta, \mathrm{d}\omega),$$

where

$$Q(\gamma, \eta, A) = \sum_{\xi \subset \eta} \int\limits_{\Gamma_0} \mathbb{1}_A(\eta \backslash \xi \cup \zeta) K(\gamma, \xi, \eta, \zeta)\mathrm{d}\lambda(\zeta).$$

Define $q(\gamma, \eta) = Q(\gamma, \eta, \Gamma_0) = \sum\limits_{\xi \subset \eta} \int\limits_{\Gamma_0} K(\gamma, \xi, \eta, \zeta)\mathrm{d}\lambda(\zeta)$, then the adjoint operator on densities $\rho \in \mathcal{L}_\mu$ is given by

$$(L^S)^* \rho(\gamma, \eta) = -q(\gamma, \eta)\rho(\gamma, \eta) + (B^*\rho)(\gamma, \eta),$$

where

$$(B^*\rho)(\gamma, \eta) = \sum_{\xi \subset \eta} \int\limits_{\Gamma_0} \rho(\gamma, \eta \backslash \xi \cup \zeta) K(\gamma, \zeta, \eta \backslash \xi \cup \zeta, \xi)\mathrm{d}\lambda(\zeta). \tag{3.8}$$

Proceeding as in the case without γ, define the domain

$$\mathrm{Dom} = \left\{ \rho \in \mathcal{L}_\mu \;\middle|\; \int\limits_{\Gamma\times\Gamma_0} q(\gamma,\eta)|\rho(\gamma,\eta)|\mathrm{d}\mu(\gamma)\mathrm{d}\lambda(\eta) < \infty \right\}.$$

Then the multiplication operator $(-q, \mathrm{Dom})$ is the generator of an analytic positive semigroup. Moreover (B^*, Dom) is well-defined, and for any $0 \leq \rho \in \mathrm{Dom}$

$$\int\limits_{\Gamma}\int\limits_{\Gamma_0} B^*\rho(\gamma,\eta)\mathrm{d}\lambda(\eta)\mathrm{d}\mu(\gamma) = \int\limits_{\Gamma}\int\limits_{\Gamma_0} q(\gamma,\eta)\rho(\gamma,\eta)\mathrm{d}\lambda(\eta)\mathrm{d}\mu(\gamma).$$

Again by Theorem 2.1 [TV06] there exists an extension $(G^S, D(G^S))$ of $((L^S)^*, \mathrm{Dom})$, which is the generator of a sub-stochastic semigroup $T^S(t)^*$ on $\mathcal{L}_\mu$. This semigroup is the minimal sub-stochastic semigroup with generator being an extension of $((L^S)^*, \mathrm{Dom})$.

3.4 Weak-coupling limit

As it is already stated in the introduction, we are interested in the asymptotic regime $\varepsilon \to 0$ for solutions ρ_t^ε to the Cauchy problems

$$\frac{\partial \rho_t^\varepsilon}{\partial t} = (L^S)^*\rho_t^\varepsilon + \frac{1}{\varepsilon}L^E\rho_t^\varepsilon, \quad \rho_t^\varepsilon|_{t=0} = \rho_0 \in L^1(\Gamma_0,\lambda) \subset \mathcal{L}_\mu \tag{3.9}$$

on $\mathcal{L}_\mu$. Typically, it is hard to construct solutions to (3.9) in this generality. Let us define approximations $(L_\delta^S)^*$ by setting $K_\delta(\gamma,\xi,\eta,\zeta) := e^{-\delta q(\gamma,\eta)}K(\gamma,\xi,\eta,\zeta)$. Then L_δ^S is defined by (3.7) with K replaced by K_δ and $(L_\delta^S)^*$ is its adjoint given by

$$(L_\delta^S)^*\rho(\gamma,\eta) = -q(\gamma,\eta)e^{-\delta q(\gamma,\eta)}\rho(\gamma,\eta) + (B_\delta^*\rho)(\gamma,\eta).$$

The operator B_δ^* is simply given by (cf. (3.8))

$$(B_\delta^*\rho) = \sum_{\xi\subset\eta}\int\limits_{\Gamma_0} \rho(\gamma,\eta\backslash\xi\cup\zeta)e^{-\delta q(\gamma,\eta\backslash\xi\cup\zeta)}K(\gamma,\zeta,\eta\backslash\xi\cup\zeta,\xi)\mathrm{d}\lambda(\zeta).$$

Because of

$$\begin{aligned}
\|B_\delta^*\rho\|_{\mathcal{L}_\mu} &\leq \int\limits_{\Gamma}\int\limits_{\Gamma_0}\sum_{\xi\subset\eta}|\rho(\gamma,\eta\backslash\xi\cup\zeta)|e^{-\delta q(\gamma,\eta\backslash\xi\cup\zeta)}K(\gamma,\zeta,\eta\backslash\xi\cup\zeta,\xi)\mathrm{d}\lambda(\zeta)\mathrm{d}\lambda(\eta)\mathrm{d}\mu(\gamma) \\
&= \int\limits_{\Gamma}\int\limits_{\Gamma_0}|\rho(\gamma,\eta)|e^{-\delta q(\gamma,\eta)}q(\gamma,\eta)\mathrm{d}\lambda(\eta)\mathrm{d}\mu(\gamma) \\
&\leq \frac{1}{\delta}\|\rho\|_{\mathcal{L}_\mu}
\end{aligned}$$

the operator B_δ^* is bounded on $\mathcal{L}_\mu$ and hence so is $(L_\delta^S)^*$. Let us fix the notation for the limiting objects when $\varepsilon \to 0$ and $\delta \to 0$. Define the averaged functions $\overline{K}$ and $\overline{K}_\delta$ by

$$\overline{K}(\xi,\eta,\zeta) := \int\limits_\Gamma K(\gamma,\xi,\eta,\zeta)\mathrm{d}\mu(\gamma) \tag{3.10}$$

and

$$\overline{K}_\delta(\xi,\eta,\zeta) := \int\limits_\Gamma e^{-\delta q(\gamma,\eta)} K(\gamma,\xi,\eta,\zeta)\mathrm{d}\mu(\gamma). \tag{3.11}$$

Both functions are measurable and hence satisfy condition (K). Consequently, there exist semigroups $\overline{T}(t)$ and $\overline{T}_\delta(t)$ given by the associated (sub-)Markov functions $\overline{P}$ and $\overline{P}_\delta$ which are determined by

$$\overline{L}F(\eta) = \sum_{\xi\subset\eta} \int\limits_{\Gamma_0} (F(\eta\backslash\xi\cup\zeta) - F(\eta))\overline{K}(\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta)$$

and

$$\overline{L}_\delta F(\eta) = \sum_{\xi\subset\eta} \int\limits_{\Gamma_0} (F(\eta\backslash\xi\cup\zeta) - F(\eta))\overline{K}_\delta(\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta),$$

cf. (3.2) and (3.3). The adjoint semigroups on $L^1(\Gamma_0,\lambda)$ are denoted by $\overline{T}(t)^*$ and $\overline{T}_\delta(t)^*$ respectively. The corresponding generators are simply given by

$$(\overline{L}_\delta^*\rho)(\eta) = \sum_{\xi\subset\eta} \int\limits_{\Gamma_0} (\rho(\eta\backslash\xi\cup\zeta) - \rho(\eta))\overline{K}_\delta(\zeta,\eta\backslash\xi\cup\zeta,\xi)\mathrm{d}\lambda(\zeta)$$

and likewise for $\overline{L}^*$ with $\overline{K}_\delta$ replaced by $\overline{K}$.

Theorem 3.3. *Assume that condition (3.6) satisfied. Then for any $\varepsilon > 0$ the operator $(L_\delta^S)^* + \frac{1}{\varepsilon}L^E$ is the generator of a sub-stochastic semigroup $T_{\varepsilon,\delta}(t)$ on $\mathcal{L}_\mu$. For any $\delta > 0$ and any $\rho \in L^1(\Gamma_0,\lambda)$*

$$\lim_{\varepsilon\to 0} T_{\varepsilon,\delta}(t)\rho = \overline{T}_\delta(t)^*\rho \tag{3.12}$$

holds uniformly on compacts in $t \geq 0$. Assume that $\overline{T}(t)^*$ is stochastic, then for any $\rho \in L^1(\Gamma_0,\lambda)$

$$\lim_{\delta\to 0} \overline{T}_\delta(t)^*\rho = \overline{T}(t)^*\rho \tag{3.13}$$

holds uniformly on compacts in $t \geq 0$.

Proof. The operator $\frac{1}{\varepsilon}L^E$ is for any $\varepsilon > 0$ the generator of the semigroup $T^E(\frac{t}{\varepsilon})$ on $\mathcal{L}_\mu$. Since $(L^S_\delta)^*$ is bounded on $\mathcal{L}_\mu$ also the sum $(L^S_\delta)^* + \frac{1}{\varepsilon}L^E$ is the generator of a semigroup $T_{\varepsilon,\delta}(t)$. Due to the Trotter product formula this semigroup is sub-stochastic. So let us show (3.12), which holds true if we can apply [Kur73, Theorem 2.1]. Therefore observe that for $\rho \in \mathcal{L}_\mu$ and $\lambda > 0$

$$\left\| \lambda \int_0^\infty e^{-\lambda t} T^E(t)\rho \mathrm{d}t - \langle \rho \rangle_\mu \right\|_{\mathcal{L}_\mu} \leq \int_0^\infty e^{-s} \left\| T^E\left(\frac{s}{\lambda}\right)\rho - \langle \rho \rangle_\mu \right\|_{\mathcal{L}_\mu} \mathrm{d}s.$$

Since $T^E(t)$ is ergodic on $\mathcal{L}_\mu$ it follows that for fixed $s \geq 0$ the integrand tends to zero as $\lambda \to 0$. Due to $\|\langle \rho \rangle_\mu\|_{\mathcal{L}_\mu} \leq \|\rho\|_{\mathcal{L}_\mu}$ and the contraction property of $T^E(t)$ the integrand is bounded by $2\|\rho\|_{\mathcal{L}_\mu} e^{-s}$ and hence dominated convergence implies for all $\rho \in \mathcal{L}_\mu$

$$P\rho := \lim_{\lambda \to 0} \lambda \int_0^\infty e^{-\lambda t} T^E(t)\rho \mathrm{d}t = \langle \rho \rangle_\mu.$$

The operator P is a projection on $\mathcal{L}_\mu$ with range $\mathrm{Ran}(P) \cong L^1(\Gamma_0, \mathrm{d}\lambda)$. Following the notion of [Kur73] $C\rho := P(L^S_\delta)^*\rho = \overline{L}^*_\delta \rho$ is defined on $L^1(\Gamma_0, \mathrm{d}\lambda)$ and is additionally bounded, which implies (3.12). For the second assertion observe that by Theorem 3.2

$$\overline{\mathrm{Dom}} := \left\{ \rho \in L^1(\Gamma_0, \mathrm{d}\lambda) \;\middle|\; \int_{\Gamma_0} \overline{q}(\eta)|\rho(\eta)| \mathrm{d}\lambda(\eta) < \infty \right\}$$

is a core for $\overline{T}(t)^*$, since $\overline{T}(t)^*$ is stochastic. For any $\rho \in \overline{\mathrm{Dom}}$ it holds

$$\begin{aligned} &\|\overline{L}^*_\delta \rho - \overline{L}^* \rho\| \\ &\leq \int_{\Gamma_0} |\rho(\eta)||\overline{q}_\delta(\eta) - \overline{q}(\eta)| \mathrm{d}\lambda(\eta) \\ &\quad + \int_{\Gamma_0} \sum_{\xi \subset \eta} \int_{\Gamma_0} |\rho(\eta \backslash \xi \cup \zeta)||\overline{K_\delta}(\zeta, \eta \backslash \xi \cup \zeta, \zeta) - \overline{K}(\zeta, \eta \backslash \xi \cup \zeta, \xi)| \mathrm{d}\lambda(\zeta) \mathrm{d}\lambda(\eta) \end{aligned}$$

and by (3.10) and (3.11) for any $\delta > 0$ we obtain

$$|\overline{K_\delta}(\zeta, \eta \backslash \xi \cup \zeta, \xi) - \overline{K}(\zeta, \eta \backslash \xi \cup \zeta, \zeta)| \leq \int_\Gamma |1 - e^{-\delta q(\gamma, \eta \backslash \xi \cup \zeta)}| K(\gamma, \zeta, \eta \backslash \xi \cup \zeta, \xi) \mathrm{d}\mu(\gamma).$$

Since the integrand is bounded by $2K(\gamma, \zeta, \eta \backslash \xi \cup \zeta, \xi)$ and tends to zero for any $\gamma \in \Gamma$, dominated convergence yields that $|\overline{K_\delta}(\zeta, \eta \backslash \xi \cup \zeta, \xi) - \overline{K}(\zeta, \eta \backslash \xi \cup \zeta, \zeta)| \longrightarrow 0$ as $\delta \to 0$

for any $\eta \in \Gamma_0$, $\xi \subset \eta$ and $\zeta \in \Gamma_0$. Finally due to $|\overline{K_\delta}(\zeta, \eta\backslash\xi \cup \zeta, \xi) - \overline{K}(\zeta, \eta\backslash\xi \cup \zeta, \zeta)| \leq 2\overline{K}(\zeta, \eta\backslash\xi \cup \zeta, \xi)$ the second term tend to zero as $\delta \to 0$. For the first term observe

$$|\overline{q}_\delta(\eta) - \overline{q}(\eta)| \leq \sum_{\xi \subset \eta} \int_{\Gamma_0} |\overline{K}_\delta(\xi, \eta, \zeta) - \overline{K}(\xi, \eta, \zeta)| \mathrm{d}\lambda(\zeta),$$

then above argument implies $\overline{q}_\delta(\eta) \longrightarrow \overline{q}(\eta)$ for all $\eta \in \Gamma_0$ as $\delta \to 0$. The assertion follows from $\overline{q}_\delta \leq \overline{q}$ and dominated convergence. $\square$

4 Examples

Consider equilibrium diffusions or Glauber birth-and-death Markov dynamics on Γ for a given invariant (Gibbs) measure μ. For the construction of equilibrium diffusions and ergodicity see [AKR98a, AKR98b] and concerning equilibrium Glauber dynamics see [KL05]. We focus on one example and show how to apply our result to concrete interacting particle systems on Γ_0. Let us consider the spatial logistic model, i.e.,

$$\begin{aligned}(L^S F)(\gamma, \eta) = &\sum_{x \in \eta} \left(m(x, \gamma) + \sum_{y \in \eta\backslash x} a^-(x-y) \right) (F(\gamma, \eta\backslash x) - F(\gamma, \eta)) \\ &+ \sum_{x \in \eta} \lambda(x, \gamma) \int_{\mathbb{R}^d} a^+(x-y)(F(\gamma, \eta \cup y) - F(\gamma, \eta)) \mathrm{d}y.\end{aligned}$$

The statistical dynamics for such model (without the presence of an environment) has been analyzed, e.g., in [FKKK15, FKK09]. Here $m \geq 0$ is the intensity of the death of particles and $\lambda \geq 0$ describes fecundity effects caused by the environment in the state γ. Finally $a^- \geq 0$ is assumed to be symmetric. It describes the competition of particles from the configuration $\eta \in \Gamma_0$. The distribution of new particles is described by a symmetric probability density a^+ on $\mathbb{R}^d$. After scaling the averaged dynamics will be given by the generator

$$\begin{aligned}(\overline{L}F)(\gamma, \eta) = &\sum_{x \in \eta} \left(\overline{m}(x) + \sum_{y \in \eta\backslash x} a^-(x-y) \right) (F(\eta\backslash x) - F(\eta)) \\ &+ \sum_{x \in \eta} \overline{\lambda}(x) \int_{\mathbb{R}^d} a^+(x-y)(F(\eta \cup y) - F(\eta)) \mathrm{d}y,\end{aligned}$$

where $\overline{m}(x) = \int_\Gamma m(x, \gamma) \mathrm{d}\mu(\gamma)$ and $\overline{\lambda}(x) = \int_\Gamma \lambda(x, \gamma) \mathrm{d}\mu(\gamma)$ are the averaged intensities. Proceeding as in the previous section denote by $T_{\varepsilon,\delta}(t)$ the scaled semigroup on densities $\mathcal{L}_\mu$ and by $\overline{T}(t)^*$ and $\overline{T}_\delta(t)^*$ the semigroups on $L^1(\Gamma_0, \lambda)$ defined by the adjoint operator

$\overline{L}^*$ of $\overline{L}$ respectively their counterparts scaled by $\delta > 0$. The next result states conditions for which these semigroups exist and (3.12) holds.

Theorem 4.1. *Assume that all intensities $a^{\pm}, m, \lambda$ are non-negative, measurable, that a^+ is a probability density and that $m(x,\cdot), \lambda(x,\cdot)$ are integrable with respect to μ for any $x \in \mathbb{R}^d$. Then the semigroups $T_{\varepsilon,\delta}(t)$, $\overline{T}_\delta(t)^*$ and $\overline{T}(t)^*$ exist and (3.12) holds.*

Proof. First of all

$$\begin{aligned} q(\gamma,\eta) &= \sum_{x\in\eta} m(x,\gamma) + \sum_{x\in\eta}\sum_{y\in\eta\setminus x} a^-(x-y) + \sum_{x\in\eta}\lambda(x,\gamma) \\ &= \sum_{\xi\subset\eta}\int_{\Gamma_0} K(\gamma,\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta) \geq \int_{\Gamma_0} K(\gamma,\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta) \end{aligned}$$

for any $\eta \in \Gamma_0$ and fixed $\xi \subset \eta$. Hence

$$\int_{\Gamma}\int_{\Gamma_0} K(\gamma,\xi,\eta,\zeta)\mathrm{d}\lambda(\zeta)\mathrm{d}\mu(\gamma) \leq \int_{\Gamma} q(\gamma,\eta)\mathrm{d}\mu(\gamma) < \infty$$

and (3.6) holds. Condition (K) is obvious for $\overline{K}$ and $\overline{K}_\delta$. The considerations of the previous sections imply the existence of the semigroups and property (3.12) follows from Theorem 3.3. □

The reader may wonder why such weak assumptions are sufficient for existence and convergence of the semigroups. The crucial point here is that we consider an approximation by bounded linear operators and hence for each $\delta > 0$ no additional conditions are needed. In order to pass to the limit $\delta \to 0$ additional assumptions are necessary, which are given below.

Theorem 4.2. *Assume that the conditions of previous theorem are fulfilled. If $\overline{m}, \overline{\lambda}, a^-$ are bounded, then $\overline{T}(t)^*$ is stochastic and hence (3.13) holds. If $\overline{m}, \overline{\lambda}, a^-$ are locally bounded, then $\overline{T}(t)^*$ is still stochastic, provided there exists a continuous function $\varphi : \mathbb{R}^d \longrightarrow [1,\infty)$ with $\varphi(x) \longrightarrow \infty$ when $|x| \to \infty$ and $c > 0$ such that*

$$\overline{\lambda}(x)(a^+ * \varphi)(x) \leq c\varphi(x) + \varphi(x)\overline{m}(x), \quad x \in \mathbb{R}^d \tag{4.1}$$

holds.

Proof. In the first case set $E_n = \{\eta \in \Gamma_0 \mid |\eta| \leq n\}$, then $E_n \subset E_{n+1}$, $\bigcup\limits_{n\geq 1} E_n = \Gamma_0$, $\overline{q}(\eta) = \sum\limits_{x\in\eta}\overline{m}(x) + \sum\limits_{x\in\eta}\overline{\lambda}(x) + \sum\limits_{x\in\eta}\sum\limits_{y\in\eta\setminus x} a^-(x-y)$ is bounded on any E_n. Moreover, for $V(\eta) = |\eta|$ we obtain $\inf\limits_{\eta\notin E_n} V(\eta) \geq n+1 \to \infty$, $n \to \infty$ and hence the assertion follows from [Che04]. For the second case take $E_n = \{\eta \in \Gamma_0 \mid |\eta| \leq n,\ \eta \subset B_n\}$, where $B_n \subset \mathbb{R}^d$ is a ball centered at zero of radius n. Hence due to (4.1) we see that the Lyapunov function $V(\eta) = \sum_{x\in\eta} \varphi(x)$ satisfies

$$(\overline{L}V)(\eta) \leq cV(\eta), \quad \eta \in \Gamma_0.$$

The assertion follows again by [Che04]. $\square$

As a concrete case we can take $\mu = \pi_z$, that is the Poisson measure with intensity $z > 0$. Let us take for the interactions

$$m(x, \gamma) = m_0 + \sum_{y\in\gamma} \kappa(x - y)$$

and

$$\lambda(x, \gamma) = \lambda_0 + \sum_{y\in\gamma} \psi(x - y)$$

with $\lambda_0 > m_0$, $0 \leq \kappa, \psi \in L^1(\mathbb{R}^d)$ and $\langle\psi\rangle < \langle\kappa\rangle$. Then $\overline{m} = m_0 + z \int_{\mathbb{R}^d} \kappa(y)\mathrm{d}y = m_0 + z\langle\kappa\rangle$ and $\overline{\lambda} = \lambda_0 + z \int_{\mathbb{R}^d} \psi(y)\mathrm{d}y = \lambda_0 + \langle\psi\rangle$. Define

$$\beta(z) = (\lambda_0 + z\langle\psi\rangle - m_0 - z\langle\kappa\rangle),$$

then for the function $V(\eta) = 1 + |\eta|$ a short computation yields

$$(\overline{L}V)(\eta) \leq \beta(z)|\eta|$$

and therefore an a priori estimate on the evolution of densities, provided a^- is bounded. More precisely, let $0 \leq \rho \in L^1(\Gamma_0, \mathrm{d}\lambda)$ with $\int_{\Gamma_0} (1+|\eta|)\rho(\eta)\mathrm{d}\lambda(\eta) < \infty$ and $\int_{\Gamma_0} \rho(\eta)\mathrm{d}\lambda(\eta) = 1$, then the evolution of densities for the averaged system is given by $\rho_t = \overline{T}(t)^*\rho$ and by the Gronwall inequality we have

$$\int_{\Gamma_0} |\eta|\rho_t(\eta)\mathrm{d}\lambda(\eta) \leq e^{\beta(z)t} \int_{\Gamma_0} |\eta|\rho(\eta)\mathrm{d}\lambda(\eta), \quad t \geq 0.$$

Without the presence of an environment, i.e. $z = 0$, the number of particles within the system will grow exponentially in time. But due to the influence of the environment, such growth may be prevented or even exponential decay may be observed.

Appendix

In order to prove above Lemma 3.1 some preparation is needed. Denote by $BM(\Gamma_0)$ the space of all bounded measurable functions and by $\langle F, \nu \rangle := \int_{\Gamma_0} F(\eta)\nu(\mathrm{d}\eta)$ the dual pairing between $BM(\Gamma_0)$ and $\mathcal{M}(\Gamma_0)$. Now let C be a bounded linear operator on $BM(\Gamma_0)$ and C^* a bounded linear operator on $\mathcal{M}(\Gamma_0)$ such that

$$\langle F, C^*\nu \rangle = \langle CF, \nu \rangle, \quad F \in BM(\Gamma_0),\ \nu \in \mathcal{M}(\Gamma_0).$$

Then for any $\eta \in \Gamma_0$, $\nu \in \mathcal{M}(\Gamma_0)$ and $A \in \mathcal{B}(\Gamma_0)$

$$(C^*\delta_\eta)(A) = \langle \mathbb{1}_A, C^*\delta_\eta \rangle = \langle C\mathbb{1}_A, \delta_\eta \rangle = (C\mathbb{1}_A)(\eta). \tag{4.2}$$

The operator $R(\lambda, -q)$ can be realized on $BM(\Gamma_0)$ and likewise on $\mathcal{M}(\Gamma_0)$ as a bounded linear operator. For simplicity we will preserve the notation $R(\lambda, -q)$ for both realizations. Clearly $R(\lambda, -q)$ and $B^*R(\lambda, -q)$ are examples for C^* and $R(\lambda, -q)$ and $R(\lambda, -q)B$ for C, i.e it holds that $\langle F, R(\lambda, -q)\nu \rangle = \langle R(\lambda, -q)F, \nu \rangle$ and $\langle F, B^*R(\lambda, -q)\nu \rangle = \langle R(\lambda, -q)BF, \nu \rangle$, where $R(\lambda, -q)F(\eta) = \frac{1}{\lambda + q(\eta)}F(\eta)$ and

$$R(\lambda, -q)BF(\eta) = \frac{1}{\lambda + q(\eta)} \int_{\Gamma_0} F(\xi)Q(\eta, \mathrm{d}\xi), \quad \eta \in \Gamma_0.$$

Proof. Lemma 3.1

The construction of $\widetilde{T}(t)^*$, cf. Theorem 2.1 [ALMK11], shows that $(G, D(G))$ satisfies for any $\mu \in \mathcal{M}(\Gamma_0)$ and $\lambda > 0$

$$R(\lambda; G)\mu = \lim_{n\to\infty} R(\lambda, -q)\sum_{k=0}^{n}(QR(\lambda; -q))^k\mu \tag{4.3}$$

in the total variation norm. Fix $\lambda > 0$ and define on $\mathcal{M}(\Gamma_0)$ a bounded linear operator by

$$R(\lambda)\mu = \int_0^\infty e^{-\lambda t}T(t)^*\mu \mathrm{d}t.$$

The semigroup $T(t)^*$ is continuous w.r.t. the topology $\sigma(\mathcal{M}(\Gamma_0), BM(\Gamma_0))$ and hence the integral is well-defined w.r.t. this topology. Then (3.4) yields

$$R(\lambda)\mu = \int_{\Gamma_0} \widehat{P}(\lambda, \xi, \cdot)\mu(\mathrm{d}\xi), \tag{4.4}$$

where $\widehat{P}(\lambda, \xi, \cdot) = \int\limits_0^\infty e^{-\lambda t} P(t, \xi, \cdot) \mathrm{d}t$. Due to Theorem 2.16 [Che04] $\widehat{P}$ is the unique minimal solution to the equation

$$\widehat{P}(\lambda, \eta, A) = \frac{1}{\lambda + q(\eta)} \delta_\eta(A) + \frac{1}{\lambda + q(\eta)} \int\limits_{\Gamma_0} \widehat{P}(\lambda, \xi, A) Q(\eta, \mathrm{d}\xi).$$

Such a minimal solution can be constructed as follows, cf. Theorem 2.21 [Che04]. Set $\widehat{P}^{(0)}(\lambda, \eta, A) = \frac{1}{\lambda + q(\eta)} \delta_\eta(A)$ and for $n \geq 0$

$$\widehat{P}^{(n+1)}(\lambda, \eta, A) = \frac{1}{\lambda + q(\eta)} \int\limits_{\Gamma_0} \widehat{P}^{(n)}(\lambda, \xi, A) Q(\eta, \mathrm{d}\xi). \tag{4.5}$$

Then $\widehat{P}(\lambda, \eta, A)$ is given by $\widehat{P}(\lambda, \eta, A) = \sum\limits_{n=0}^\infty \widehat{P}^{(n)}(\lambda, \eta, A)$. Hence by (4.4) we get

$$R(\lambda)\mu(A) = \sum_{n=0}^\infty \int\limits_{\Gamma_0} \widehat{P}^{(n)}(\lambda, \eta, A) \mu(\mathrm{d}\eta) = \sum_{n=0}^\infty R^{(n)}(\lambda)\mu(A),$$

where $R^{(n)}(\lambda)\mu(A) = \int\limits_{\Gamma_0} \widehat{P}^{(n)}(\lambda, \eta, A)\mu(\mathrm{d}\eta)$. Therefore, in view of (4.3), it suffices to show for any $n \geq 0$, $\mu \in \mathcal{M}(\Gamma_0)$ and $A \in \mathcal{B}(\Gamma_0)$ that

$$R^{(n)}(\lambda)\mu(A) = R(\lambda; -q)(QR(\lambda; -q))^n \mu(A)$$

holds. For $n = 0$ this follows from

$$R^{(0)}(\lambda)\mu(A) = \int\limits_{\Gamma_0} \frac{1}{\lambda + q(\eta)} \mathbb{1}_A(\eta) \mu(\mathrm{d}\eta) = R(\lambda; -q)\mu(A).$$

Assume that this assertion holds for some $n \geq 0$. The induction hypothesis and (4.2) imply the relation

$$\begin{aligned} \widehat{P}^{(n)}(\lambda, \eta, A) &= \int\limits_{\Gamma_0} \widehat{P}^{(n)}(\lambda, \xi, A) \delta_\eta(\mathrm{d}\xi) = (R^{(n)}(\lambda)\delta_\eta)(A) \\ &= R(\lambda; -q)(QR(\lambda; -q))^n \delta_\eta(A) = (R(\lambda; -q)Q)^n R(\lambda; -q) \mathbb{1}_A(\eta). \end{aligned}$$

Finally by (4.2) and (4.5) this yields

$$\begin{aligned} R^{(n+1)}(\lambda)\mu(A) &= \int\limits_{\Gamma_0} \frac{1}{\lambda + q(\eta)} \int\limits_{\Gamma_0} \widehat{P}^{(n)}(\lambda, \xi, A) Q(\eta, \mathrm{d}\xi)\mu(\mathrm{d}\eta) \\ &= \int\limits_{\Gamma_0} \frac{1}{\lambda + q(\eta)} \int\limits_{\Gamma_0} (R(\lambda; -q)Q)^n R(\lambda; -q)\mathbb{1}_A(\xi) Q(\eta, \mathrm{d}\xi)\mu(\mathrm{d}\eta) \\ &= \int\limits_{\Gamma_0} (R(\lambda; -q)Q)^{n+1} R(\lambda; -q)\mathbb{1}_A(\eta)\mu(\mathrm{d}\eta) \\ &= R(\lambda; -q)(QR(\lambda; -q))^{n+1}\mu(A). \end{aligned}$$

□

References

[AKR98a] S. Albeverio, Y. Kondratiev, and M. Röckner. Analysis and geometry on configuration spaces. *J. Funct. Anal.*, 154(2):444–500, 1998.

[AKR98b] S. Albeverio, Yu. Kondratiev, and M. Röckner. Analysis and geometry on configuration spaces: the Gibbsian case. *J. Funct. Anal.*, 157(1):242–291, 1998.

[ALMK11] L. Arlotti, B. Lods, and M. Mokhtar-Kharroubi. On perturbed substochastic semigroups in abstract state spaces. *Z. Anal. Anwend.*, 30(4):457–495, 2011.

[Bez15] V. Bezborodov. Spatial birth-and-death markov dynamics of finite particle systems. *arXiv:1507.05804 [math.PR]*, 2015.

[BK03] V. P. Belavkin and V. N. Kolokoltsov. On a general kinetic equation for many-particle systems with interaction, fragmentation and coagulation. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 459(2031):727–748, 2003.

[BKK15] C. Berns, Y. Kondratiev, and O. Kutoviy. Markov Jump Dynamics with Additive Intensities in Continuum: State Evolution and Mesoscopic Scaling. *J. Stat. Phys.*, 161(4):876–901, 2015.

[Che04] M. Chen. *From Markov chains to non-equilibrium particle systems.* World Scientific Publishing Co., Inc., River Edge, NJ, second edition, 2004.

[EW03] A. Eibeck and W. Wagner. Stochastic interacting particle systems and nonlinear kinetic equations. *Ann. Appl. Probab.*, 13(3):845–889, 2003.

[Fel40] W. Feller. On the integro-differential equations of purely discontinuous Markoff processes. *Trans. Amer. Math. Soc.*, 48:488–515, 1940.

[FKK09] D. Finkelshtein, Y. Kondratiev, and O. Kutoviy. Individual based model with competition in spatial ecology. *SIAM J. Math. Anal.*, 41(1):297–317, 2009.

[FKK12] D. Finkelshtein, Y. Kondratiev, and O. Kutoviy. Semigroup approach to birth-and-death stochastic dynamics in continuum. *J. Funct. Anal.*, 262(3):1274–1308, 2012.

[FKK13] D. Finkelshtein, Y. Kondratiev, and O. Kutoviy. Establishment and fecundity in spatial ecological models: statistical approach and kinetic equations. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 16(2):1350014, 24, 2013.

[FKK15] D. Finkelshtein, Y. Kondratiev, and O. Kutoviy. Statistical dynamics of continuous systems: perturbative and approximative approaches. *Arab. J. Math. (Springer)*, 4(4):255–300, 2015.

[FKKK15] D. Finkelshtein, Y. Kondratiev, Y. Kozitsky, and O. Kutoviy. The statistical dynamics of a spatial logistic model and the related kinetic equation. *Math. Models Methods Appl. Sci.*, 25(2):343–370, 2015.

[FKKO15] D. Finkelshtein, Y. Kondratiev, O. Kutoviy, and M. J. Oliveira. Dynamical Widom-Rowlinson model and its mesoscopic limit. *J. Stat. Phys.*, 158(1):57–86, 2015.

[FKKZ14] D. Finkelshtein, Y. Kondratiev, O. Kutoviy, and E. Zhizhina. On an aggregation in birth-and-death stochastic dynamics. *Nonlinearity*, 27(6):1105–1133, 2014.

[FKO09] D. Finkelshtein, Y. Kondratiev, and M. J. Oliveira. Markov evolutions and hierarchical equations in the continuum. I. One-component systems. *J. Evol. Equ.*, 9(2):197–233, 2009.

[FKO12] D. Finkelshtein, Y. Kondratiev, and M. J. Oliveira. Kawasaki dynamics in the continuum via generating functionals evolution. *Methods Funct. Anal. Topology*, 18(1):55–67, 2012.

[FM04] N. Fournier and S. Méléard. A microscopic probabilistic description of a locally regulated population and macroscopic approximations. *Ann. Appl. Probab.*, 14(4):1880–1919, 2004.

[FMS14] E. A. Feinberg, M. Mandava, and A. N. Shiryaev. On solutions of Kolmogorov's equations for nonhomogeneous jump Markov processes. *J. Math. Anal. Appl.*, 411(1):261–270, 2014.

[Fri16] M. Friesen. Non-autonomous interacting particle systems in continuum. *to appear*, 2016.

[GK06] N. L. Garcia and T. G. Kurtz. Spatial birth and death processes as solutions of stochastic equations. *ALEA Lat. Am. J. Probab. Math. Stat.*, 1:281–303, 2006.

[Gra04] L. Grafakos. *Classical and modern Fourier analysis.* Pearson Education, Inc., Upper Saddle River, NJ, 2004.

[KK02] Y. Kondratiev and T. Kuna. Harmonic analysis on configuration space. I. General theory. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 5(2):201–233, 2002.

[KL05] Y. Kondratiev and E. Lytvynov. Glauber dynamics of continuous particle systems. *Ann. Inst. H. Poincaré Probab. Statist.*, 41(4):685–702, 2005.

[Kol06] V. N. Kolokoltsov. Kinetic equations for the pure jump models of k-nary interacting particle systems. *Markov Process. Related Fields*, 12(1):95–138, 2006.

[KS06] Y. Kondratiev and A. Skorokhod. On contact processes in continuum. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 9(2):187–198, 2006.

[Kur73] T. G. Kurtz. A limit theorem for perturbed operator semigroups with applications to random evolutions. *J. Functional Analysis*, 12:55–67, 1973.

[Pin91] M. Pinsky. *Lectures on Random Evolution.* World Scientific Publishing Co. Pte. Ltd., 1991.

[SHS02] A. Skorokhod, F. Hoppensteadt, and H. Salehi. *Random Perturbation Methods with Applications in Science and Engineering.* Springer, 2002.

[TV06] H. R. Thieme and J. Voigt. Stochastic semigroups: their construction by perturbation and approximation. In *Positivity IV—theory and applications*, pages 135–146. Tech. Univ. Dresden, Dresden, 2006.