Backward uniqueness of stochastic parabolic like equations driven by Gaussian multiplicative noise

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Abstract

One proves here the backward uniqueness of solutions to stochastic semilinear parabolic equations and also for the tamed Navier–Stokes equations driven by linearly multiplicative Gaussian noises. Applications to approximate controllability of nonlinear stochastic parabolic equations with initial controllers are given. The method of proof relies on the logarithmic convexity property known to hold for solutions to linear evolution equations in Hilbert spaces with self-adjoint principal part.

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1 Introduction

Consider the stochastic parabolic equation

$$
\begin{aligned}
&dX(t) - \sum_{i,j=1}^{d} \frac{\partial}{\partial \xi_i}\left(a_{ij}(t,\xi)\,\frac{\partial X(t)}{\partial \xi_j}\right) dt + b(t,\xi)\cdot \nabla X(t)\,dt \\
&\qquad +\psi(t,\xi,X(t))dt = X(t)dW(t) \text{ in } (0,T)\times \mathcal{O}, \\
&X(0,\xi) = x(\xi),\ \xi\in \mathcal{O}; \qquad X(t,\xi)=0 \text{ on } (0,T)\times \partial\mathcal{O},
\end{aligned}
\tag{1.1}
$$

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where $\mathcal{O} \subset \mathbb{R}^d$, $1 \le d < \infty$, is a bounded and open domain with the smooth boundary $\partial\mathcal{O}$ and W is a Wiener process of the form

$$W(t,\xi) = \sum_{j=1}^{\infty} \mu_j e_j(\xi)\beta_j(t),\ \xi \in \overline{\mathcal{O}},\ t \ge 0. \tag{1.2}$$

Here $\{e_j\}_{j=1}^N \subset C^2(\overline{\mathcal{O}})$ is an orthonormal basis in $L^2(\mathcal{O})$, $\{\beta_j\}_{j=1}^\infty$ is an independent system of real-valued Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration$(\mathcal{F}_t)_{t\ge 0}$, and $\{\mu_j\} \subset \mathbb{R}$ is such that

$$\sum_{j=1}^{\infty} \mu_j^2 \|e_j\|_{C_b^2}^2 < \infty, \tag{1.3}$$

where $\|\cdot\|_{C_b^2}$ denotes the supnorm of the functions and its first and second order derivatives.

As regards the functions $a_{ij} : [0,T] \times \overline{\mathcal{O}} \to \mathbb{R}$, $b : [0,T] \times \overline{\mathcal{O}} \to \mathbb{R}$ and $\psi : [0,T] \times \overline{\mathcal{O}} \to \mathbb{R}$, we assume that the following conditions hold

$$\begin{aligned} &a_{ij} \in C([0,T] \times \overline{\mathcal{O}}), \frac{\partial}{\partial t}\, a_{ij} \in C([0,T] \times \overline{\mathcal{O}}),\ \frac{\partial}{\partial \xi_j}\, a_{ij} \in C([0,T] \times \overline{\mathcal{O}}),\\ &a_{ij} = a_{ji},\ \forall i,j = 1,...,d,\ b \in C([0,T] \times \overline{\mathcal{O}}; \mathbb{R}^d),\ \mathrm{div}_\xi b \in C([0,T] \times \overline{\mathcal{O}}), \end{aligned} \tag{1.4}$$

$$\begin{aligned} &\sum_{i,j=1}^{d} a_{ij}(t,\xi) u_i u_j \ge \gamma |u|_d^2,\ \forall u = (u_1,...,u_d) \in \mathbb{R}^d,\\ &(t,\xi) \in [0,T] \times \overline{\mathcal{O}}, \end{aligned} \tag{1.5}$$

where $\gamma > 0$ and $|\cdot|_d$ is the Euclidean norm on $\mathbb{R}^d$.

$$\begin{aligned} &\psi, \psi_\varepsilon \in C([0,T] \times \overline{\mathcal{O}} \times \mathbb{R}),\ \psi(t,\xi,0) \equiv 0,\\ &|\psi_\varepsilon(t,\xi,r)| \le C(1 + |r|_d),\ \forall (t,\xi,r) \in [0,T] \times \overline{\mathcal{O}} \times R. \end{aligned} \tag{1.6}$$

Moreover, $r \to \psi(t,\xi,r)$ is monotonically nondecreasing and

$$\begin{aligned} |\psi(t,\xi,r_1) - \psi(t,\xi,r_2)| &\le L|r_1 - r_2||\psi_0(t,\xi,r_1,r_2)|,\\ &\forall r_1, r_2 \in \mathbb{R},\ (t,\xi) \in [0,T] \times \overline{\mathcal{O}}, \end{aligned} \tag{1.7}$$

where $\psi_0 \in C([0,T] \times \overline{\mathcal{O}} \times \mathbb{R} \times \mathbb{R})$ and $L > 0$,

$$|\psi_0(t,\xi,r_1,r_2)| \le C(|r_1|^q + |r_2|^q + 1),\ \forall r_1, r_2 \in \mathbb{R},\ (t,\xi) \in [0,T] \times \overline{\mathcal{O}}, \tag{1.8}$$

where

$$0 \le q < \frac{d+2}{d-2} \quad \text{if } d > 2, \qquad q \in (1, \infty) \quad \text{if } d = 2, \tag{1.9}$$

and no polynomial growth condition of the form (1.9) is necessary if $d = 1$.

In the following, we denote by $L^2(\mathcal{O})$ the space of Lebesgue square integrable functions on $\mathcal{O}$ with the norm denoted by $|\cdot|_2$ and the scalar product $\langle \cdot, \cdot \rangle$. We denote by $W^{m,p}(\mathcal{O})$, $H_0^1(\mathcal{O})$ and $H^{-1}(\mathcal{O})$ the standard Sobolev spaces on $\mathcal{O}$ with the usual norms $\|u\|_{m,p}$, $\|\cdot\|_1$ and $\|\cdot\|_{-1}$, respectively.

We note that, under assumptions (1.2)–(1.9), for each $x \in L^2(\mathcal{O})$, equation (1.1) has a unique solution X satisfying

$$X \in L^\infty(0, T; L^2(\Omega; L^2(\mathcal{O}))) \cap L^2([0, T) \times \Omega; H_0^1(\mathcal{O})) \tag{1.10}$$

$$\mathbb{E} \int_0^T \left\| e^{W(t)} \frac{d}{dt} (e^{-W(t)} X(t)) \right\|_{-1}^2 dt < \infty. \tag{1.11}$$

Moreover, if $x \in L^\infty(\mathcal{O})$, then $X \in L^\infty((0, T) \times \mathcal{O})$, $\mathbb{P}$-a.s. (See [3], Corollary 6.1 and [9], Theorem 2.1, p. 425.) By a solution to (1.1), we mean an $(\mathcal{F}_t)_{t \ge 0}$–adapted process $X : [0, T] \to L^2(\mathcal{O})$, with continuous sample paths, which satisfies the equation

$$\begin{aligned} X(t, \xi) &- \int_0^t \sum_{i,j=1}^d \frac{\partial}{\partial \xi_i} \left(a_{ij}(s, \xi) \frac{\partial X}{\partial \xi_j}(s, \xi) \right) ds \\ &+ \int_0^t (b(s, \xi) \cdot \nabla X(s, \xi) + \psi(s, \xi, X(s, \xi))) ds \\ &= x(\xi) + \int_0^t X(s, \xi) dW(s), \ t \in [0, T], \ \xi \in \mathcal{O}, \ \mathbb{P}\text{-a.s.} \end{aligned} \tag{1.12}$$

(Here, $\dfrac{\partial}{\partial \xi_i}$ are taken in sense of distributions.)

For deterministic linear parabolic equations of the form (1.1) (with $\psi \equiv 0$) and, more generally, for linear evolution equations in Hilbert spaces with principal part self-adjoint of class C^1 with respect to t, it is known that one has backward uniqueness of solutions X, that is, if $X_1(T) = X_2(T)$, then $X_1 \equiv X_2$. (See [1], [5], [11].) Here, we shall prove that such a result remains true in the stochastic case (1.1). A few consequences of this result to approximate controllability with respect to the initial data x are derived and, in particular, the approximate controllability of (1.1) with respect to the initial data x. In Section 4, we prove a similar result for tamed stochastic Navier–Stokes equations.

Other results in this direction were obtained recently in [8]. However, only for linear SPDE, the method used here is completely different. In particular, in contrast to [8], we obtain pathwise estimates, instead of estimates in expectation.

The idea of the proof in the parabolic case is to reduce equation (1.1) by a rescaling procedure to a random parabolic equation and apply to this equation the standard calculation to prove that $\log |y_1 - y_2|^2$ is quasi-concave in t, where $y_i = e^{-W} X_i$. This procedure allows us to obtain sharp estimates on $|X_1(t) - X_2(t)|_2$ as a function of $|X_1(T) - X_2(T)|_2$. The rescaling approach can also be done for $2D$ stochastic Navier–Stokes equations (see [4]), but only for (analytically) weak solutions which have not enough regularity to apply the above arguments. As a first step, we therefore consider stochastic tamed Navier–Stokes equations (see [15], [16], [17]) in this paper. But, in this case, we use a direct approach based on a computation of $d(\log |X_1(t) - X_2(t)|^2)$ via Itô's formula, which still leads to the backward uniqueness, but the obtained estimates are only in expectation. As a matter of fact, the advantage of a rescaling procedure is that it provides pathwise estimates. Its implementation is, however, much harder for stochastic tamed Navier–Stokes equations.

2 The first main result

Everywhere in the following, we assume that conditions (1.3)–(1.9) are satisfied.

Theorem 2.1 *Let X_1, X_2 be two solutions to* (1.1)*, such that $X_1(0), X_2(0) \in L^\infty(\mathcal{O})$, $\mathbb{P}$-a.s. Then, if $X_1(T) \equiv X_2(T)$, $\mathbb{P}$-a.s., we have $X_1 \equiv X_2$. Moreover, there is a random variable $\gamma^* : \Omega \to \mathbb{R}$, such that $\mathbb{P}$-a.s.,*

$$(2.1) \quad |X_1(t) - X_2(t)|_2 \le \exp\left(\frac{\gamma^* \|X_1(t_0) - X_2(t_0)\|_1^2}{|X_1(t_0) - X_2(t_0)|_2^2} \right) |X_1(T) - X_2(T)|_2,$$

for all $t \in [t_0, T]$, where t_0 is arbitrary in $(0, T)$.

If the function $r \to \psi(t, \xi, r)$ is Lipschitz (uniformly in (t, ξ)), then (2.1) *extends to all X_1, X_2 with $X_1(0), X_2(0) \in L^2(\mathcal{O})$.*

As will explicitly be seen in the proof, γ^* is given by (3.16), (3.17) and it depends on $(\|e^{-W}X_1\|^q_{L^\infty(\mathcal{Q})} + \|e^{-W}X_2\|^q_{L^\infty(\mathcal{Q})})$ and W. Here, $\mathcal{Q} = (0,T) \times \mathcal{O}$.

As a direct consequence of Theorem 2.1, we obtain the following approximate controllability result.

Theorem 2.2 *Assume further that $r \to \psi(t,\xi,r)$ is differentiable and that $\frac{\partial}{\partial r}\psi \in L^\infty((0,T)\times\mathcal{O}\times\mathbb{R})$. Then $\mathbb{P}$-a.s. the space $\{X^x(T);\ x \in L^2(\mathcal{O})\}$ is dense in $L^2(\mathcal{O})$. (Here, X^x is the solution to* (1.1).*)*

In the control theory literature, this property is called the *approximate controllability with respect to the initial data x which is viewed as a start controller* (see, e.g., [13]).

3 Proofs

Proof of Theorem 2.1

By the transformation $X = e^W y$, we reduce (1.1) to the random parabolic equation

$$\begin{aligned} &\frac{\partial y}{\partial t} - e^{-W}\sum_{i,j=1}^{d}\frac{\partial}{\partial \xi_i}\left(a_{ij}\frac{\partial}{\partial \xi_j}(e^W y)\right) + \mu y + e^{-W}\psi(t,\xi,e^W y) = 0 \\ &\hspace{20em}\text{in } (0,T)\times\mathcal{O}, \\ &y(0,\xi) = x(\xi),\ \xi\in\mathcal{O}; \qquad y = 0 \text{ on } (0,T)\times\partial\mathcal{O}, \end{aligned} \tag{3.1}$$

where

$$\mu(\xi) = \frac{1}{2}\sum_{j=1}^{\infty}\mu_j^2 e_j^2(\xi),\ \xi\in\overline{\mathcal{O}},\ t\in[0,T].$$

Equivalently,

$$\begin{aligned} &\frac{\partial y}{\partial t} - \sum_{i,j=1}^{d}\frac{\partial}{\partial \xi_i}\left(a_{ij}\frac{\partial y}{\partial \xi_j}\right) + a_0 y + a_1\cdot\nabla y + e^{-W}\psi(t,\xi,e^W y) = 0 \\ &\hspace{20em}\text{in } (0,T)\times\mathcal{O}, \\ &y(0,\xi) = x(\xi),\ \xi\in\mathcal{O};\ \ y = 0 \text{ on } (0,T)\times\partial\mathcal{O}, \end{aligned} \tag{3.2}$$

where $a_0 : [0,T]\times\overline{\mathcal{O}}\to\mathbb{R},\ a_1 : [0,T]\times\overline{\mathcal{O}}\to\mathbb{R}^d$ are given by

$$a_0 = \mu(\xi) + \sum_{i,j=1}^{d} a_{ij} \left(\frac{\partial^2 W}{\partial \xi_i \partial \xi_j} + \frac{\partial W}{\partial \xi_i} \frac{\partial W}{\partial \xi_j} \right) + \sum_{i,j=1}^{d} \frac{\partial a_{ij}}{\partial \xi_i} \frac{\partial W}{\partial \xi_j} \tag{3.3}$$

$$a_1 = \left\{ 2 \sum_{i=1}^{d} a_{ij} \frac{\partial W}{\partial \xi_j} \right\}_{j=1}^{d}. \tag{3.4}$$

We refer to [3] for a rigorous justification of this rescaling argument and for the equivalence of (1.1) and (3.1) as well as the precise formulation of the latter.

We set $H = L^2(\mathcal{O})$, $V = H_0^1(\mathcal{O})$, $V^* = H^{-1}(\mathcal{O})$ with the norms $|\cdot|_2$, $\|\cdot\|_1$ respectively $\|\cdot\|_{-1}$ and define the operators $A(t) : V \to V^*$, $B(t) : V \to H$ and $B_1(t) : V \to V^*$, $t \in [0, T]$, by

$$\begin{aligned}
\langle A(t)y, z\rangle &= \sum_{i,j=1}^{d} \int_{\mathcal{O}} a_{ij}(t,\xi) \frac{\partial y}{\partial \xi_i}(t,\xi) \frac{\partial z}{\partial \xi_j}(t,\xi) d\xi, \ \forall y, z \in V, \\
(B(t)y)(\xi) &= a_0(t,\xi) y(\xi) + a_1(t,\xi) \cdot \nabla y(\xi), \ \xi \in \mathcal{O}, \ y \in V, \\
(B_1(t)y)(\xi) &= e^{-W(t,\xi)} \psi(t, \xi, e^{W(t,\xi)} y(\xi)), \ \xi \in \mathcal{O}, y \in V.
\end{aligned}$$

Here, $\langle \cdot, \cdot \rangle$ is the pairing between V and V^* which coincides with the scalar product of H on $H \times V$.

We note that there exist $\alpha_1, \alpha_2 > 0$ such that for $t \in [0, T]$

$$\|A(t)z\|_{-1} \le \alpha_1 \|z\|_1, \ \forall z \in V, \tag{3.5}$$

$$\langle A(t)z, z\rangle \ge \alpha_2 \|z\|_1^2, \ \forall z \in V. \tag{3.6}$$

We rewrite (3.2) as

$$\begin{aligned}
&\frac{dy}{dt}(t) + A(t)y(t) + B(t)y(t) + B_1(t)y(t) = 0, \ t \in (0, T), \\
&y(0) = x.
\end{aligned} \tag{3.7}$$

For $x \in L^2(\mathcal{O})$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, equation (3.2) (equivalently (3.7)) has a unique solution

$$y \in C([0, T]; L^2(\mathcal{O})) \cap L^2(0, T; H_0^1(\mathcal{O})), \ \frac{dy}{dt} \in L^2(0, T; H^{-1}(\mathcal{O})).$$

(See, e.g., [12].) By the smoothing effect of solutions on initial data we have also that

$$Ay, \frac{dy}{dt} \in L^2(\delta, T; L^2(\mathcal{O})),\ y \in C([\delta, T]; H_0^1(\mathcal{O})), \tag{3.8}$$

for all $\delta \in (0, T)$. This follows by the following arguments.

Consider the approximating equation

$$\begin{aligned} &\frac{dy_\varepsilon(t)}{dt} + A(t)y_\varepsilon(t) + B(t)y_\varepsilon(t) + B_1^\varepsilon(t)y_\varepsilon(t) = 0, \\ &y_\varepsilon(t) = x, \end{aligned} \tag{3.9}$$

where $B_1^\varepsilon(t)y_\varepsilon(t) = e^{-W(t)}\psi_\varepsilon(t, \cdot, e^{W(t)}y_\varepsilon(t))$ and $\psi_\varepsilon(r) = \psi((1 + \varepsilon\psi)^{-1}r)$, $\forall r \in \mathbb{R}$, is the Yosida approximation of $r \to \psi(\cdot, r)$.

Multiplying (3.9) by $y_\varepsilon(t)$ and integrating over $(0, t) \times \mathcal{O}$, we get

$$|y_\varepsilon(t)|_2^2 + \int_0^t \|y_\varepsilon(s)\|_1^2 \le C(1 + |x|_2^2),\ t \in [0, T], \tag{3.10}$$

where C is independent of ε.

Clearly, $y_\varepsilon \to y$ in $C([0, T]; L^2(\mathcal{O})) \cap L^2(0, T; H_0^1(\mathcal{O}))$ for $\varepsilon \to 0$. Moreover, since $B(t)y_\varepsilon(t) + B_1^\varepsilon(t)y_\varepsilon(t) = f_\varepsilon \in L^2(0, T; H)$, we have that

$$\sqrt{t}\,\frac{dy_\varepsilon}{dt},\ \sqrt{t}\,A(t)y_\varepsilon \in L^2(0, T; H),\ \forall t \in (0, y).$$

Now, as easily seen by the definition of $A(t)$ and by (1.6), we have

$$\begin{aligned} &\langle A(t)y_\varepsilon(t), B(t)y_\varepsilon(t)\rangle + \langle A(t)y_\varepsilon(t), B_1^\varepsilon y_\varepsilon(t)\rangle \\ &\qquad \ge -C(\|y_\varepsilon(t)\|_1^2 + |A(t)y_\varepsilon(t)|_2\|y_\varepsilon(t)\|_1 + 1) \\ &\qquad \ge -\frac{1}{2}\,|A(t)y_\varepsilon(t)|_2^2 - C(\|y_\varepsilon(t)\|_1^2 + 1),\ \forall t \in [0, T]. \end{aligned} \tag{3.11}$$

where C is independent of ε.

Then, multiplying (3.9) by $tA(t)y_\varepsilon(t)$ and integrating over $(0, t) \times \mathcal{O}$, we get after some calculation involving (3.10), (3.11) that

$$t\,\langle Ay_\varepsilon(t), y_\varepsilon(t)\rangle + \int_0^t |A(s)y_\varepsilon(s)|^2 ds \le C\left(\int_0^t \|y_\varepsilon(s)\|_1^2 ds + 1\right) \le C(|x|_2^2 + 1), \quad \forall t \in [0, T].$$

This yields

$$\int_0^t s\left(|A(s)y_\varepsilon(s))^2+\left|\frac{dy_\varepsilon}{dt}(s)\right|^2\right)ds\le C(1+|x|^2),$$

where C is independent of ε. Then, (3.8) follows.

Moreover, if $x\in L^\infty(\mathcal{O})$, then $y\in L^\infty((0,T)\times\mathcal{O})$ (see [10], Theorem 2.1, p. 425). It follows also that the process $t\longrightarrow y(t)$ is $(\mathcal{F}_t)_{t\ge 0}$–adapted.

Let y_1,y_2 be two solutions to (1.1) with $y_i(0)\in L^\infty(\mathcal{O})$ and $y_i=e^{-W}X_i$, $i=1,2$. We set

$$g(t,\xi)=\begin{cases}\dfrac{B_1(t)y_1(t,\xi)-B_1(t)y_2(t,\xi)}{y_1(t,\xi)-y_2(t,\xi)} & \text{on } [y_1\ne y_2]\\ 0 & \text{on } [y_1=y_2].\end{cases}$$

We have by (1.7) (1.8), that

$$|g(t,\xi)|\le C(\|y_1\|^q_{L^\infty(\mathcal{Q})}+\|y_2\|^q_{L^\infty(\mathcal{Q})}+1),\ \forall(t,\xi)\in\mathcal{Q}:=[0,T]\times\mathcal{O}.$$

Hence $g\in L^\infty(0,T)\times\mathcal{O})$ and C is independent of ω.

We set $z=y_1-y_2$ and get by (3.7) that

$$\frac{dz}{dt}+A(t)z+B(t)z+g(t,\xi)z=0,\quad t\in(0,T).\tag{3.12}$$

We have that $z\in L^2(0,T;V)$, $g\in L^\infty((0,T)\times\mathcal{O})$, $B(t)z\in L^2(0,T;H)$, and $\frac{dz}{dt}, A(t)z\in L^2(\delta,T;H)$ for each $\delta\in(0,T)$. Moreover, by (1.4), we have that $\left(\frac{d}{dt}A(t)\right)z(t)\in C([0,T];V^*)$. It follows also that, by (1.4), we have

$$\left|\left(\frac{d}{dt}A(t)\right)z(t)\right|_{-1}\le C\|z(t)\|_1,\quad\forall t\in(0,T).$$

(Here and everywhere in the following, we shall denote by the same symbol C several positive constants independent of ω.)

Arguing as in [5], [11], we get by (3.12) that $\mathbb{P}$-a.s.

$$\begin{aligned}&\frac{d}{dt}\left(\langle A(t)z(t),z(t)\rangle\,|z(t)|_2^{-2}\right)\\ &=\left(2\left\langle A(t)z(t),\frac{dz}{dt}(t)\right\rangle+\left\langle\left(\frac{d}{dt}A(t)\right)z(t),z(t)\right\rangle\right)|z(t)|_2^{-2}\\ &-2\left\langle\frac{dz}{dt}(t),z(t)\right\rangle\langle A(t)z(t),z(t)\rangle\,|z(t)|_2^{-4},\ t\in[0,T].\end{aligned}\tag{3.13}$$

Of course, (3.13) holds on a maximal interval $[0, T^*]$, where $z(t) \neq 0$. By uniqueness of the solution to the linear Cauchy problem associated with (3.12), $z(t) = 0$ on $[T_0, T]$. Hence, replacing if necessary T by T^* we may assume without any loss of generality that $z(t) \neq 0$ for $t \in [0, T]$. We set $f(t) = B(t)z(t) + g(t)z$ and, by (3.13) we have, a.e., $t \in (0, T)$,

$$
\begin{aligned}
&\frac{d}{dt}\left(\langle A(t)z(t), z(t)\rangle\, |z(t)|_2^{-2}\right)\\
&\quad = -2\,\langle A(t)z(t), A(t)z(t) + f(t)\rangle\, |z(t)|_2^{-2}\\
&\quad + \left\langle \left(\frac{d}{dt} A(t)\right) z(t), z(t) \right\rangle |z(t)|_2^{-2}\\
&\quad + 2\,\langle A(t)z(t) + f(t), z(t)\rangle\, \langle A(t)z(t), z(t)\rangle\, |z(t)|_2^{-4}\\
&\quad \leq C\|z(t)\|_1^2 |z(t)|_2^{-2} - [2|A(t)z(t)|_2^2 + 2\,\langle A(t)z(t), f(t)\rangle]|z(t)|_2^{-2}\\
&\quad + 2[\langle A(t)z(t), z(t)\rangle^2 + \langle f(t), z(t)\rangle\, \langle A(t)z(t), z(t)\rangle]|z(t)|_2^{-4}\\
&\quad = C\|z(t)\|_1^2 |z(t)|_2^{-2} - 2\left[\left|A(t)z(t) + \frac{1}{2}\, f(t)\right|_2^2 - \frac{1}{4}\,|f(t)|_2^2\right] |z(t)|_2^{-2}\\
&\quad +2\left[\left\langle A(t)z(t) + \frac{1}{2}\, f(t), z(t)\right\rangle^2 - \frac{1}{4}\,\langle f(t), z(t)\rangle^2\right] |z(t)|_2^{-4}\\
&\quad \leq C\|z(t)\|_1^2 |z(t)|_2^{-2} + |f(t)|_2^2 |z(t)|_2^{-2}\\
&\quad \leq C\alpha_2^{-1}\,\langle A(t)z(t), z(t)\rangle\, |z(t)|_2^{-2} + |f(t)|_2^2 |z(t)|_2^{-2}.
\end{aligned}
\tag{3.14}
$$

On the other hand, by (3.3), (3.4) we have

$$
|f(t)|_2^2 \leq C_2(\nu_1 + \gamma_2)^2 \|z(t)\|_1^2,
$$

where

$$
\begin{aligned}
\nu_1 = \nu_1(\omega) &= C\left(\sup_{t\in[0,T]} \|W(t)\|_{C_b^2} + \sup_{(t,\xi)\in\mathcal{Q}} |\nabla W(t,\xi)|^2\right)\\
\gamma_2 = \gamma_2(\omega) &= \|y_1\|_{L^\infty(\mathcal{Q})}^q + \|y_2\|_{L^\infty(\mathcal{Q})}^q + 1\\
&= \|e^{-W} X_1\|_{L^\infty(\mathcal{Q})}^q + \|e^{-W} X_2\|_{L^\infty(\mathcal{Q})}^q + 1.
\end{aligned}
$$

Then, substituting into (3.14) yields, for $t \in (0, T)$,

$$
\frac{d}{dt}\left(\langle A(t)z(t), z(t)\rangle\, |z(t)|_2^{-2}\right) \leq (C_1 + C_2(\nu_1 + \gamma_2)^2 |\,\langle A(t)z(t), z(t)\rangle|\, |z(t)|_2^{-2},
$$

where C_1, C_2 are independent of ω. Hence

$$\langle A(t)z(t), z(t)\rangle |z(t)|_2^{-2} \le \exp(\gamma_1^*(t-t_0)) \langle A(t_0)z(t_0), z(t_0)\rangle |z(t_0)|_2^{-2}, \tag{3.15}$$

for $t_0 \le t \le T$. Here, γ_1^* is the random variable

$$\gamma_1^*(\omega) = C_1 + C_2(\nu_1(\omega) + \gamma_2(\omega))^2. \tag{3.16}$$

On the other hand, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\log(|z(t)|_2^2) &= -\langle A(t)z(t) + B(t)z(t) + g(t)z(t), z(t)\rangle |z(t)|_2^{-2} \\
&\ge -\langle A(t)z(t), z(t)\rangle |z(t)|_2^{-2} - C_2(\nu_1+\gamma_2)\|z(t)\|_1 |z(t)|_2^{-1} \\
&\ge -C_3(\nu_1+\gamma_2+1)\langle A(t)z(t), z(t)\rangle |z(t)|_2^{-2}, \ \forall t \in (0,T).
\end{aligned}$$

Then, by (3.15) we obtain

$$\frac{1}{2}\frac{d}{dt}(\log|z(t)|_2^2) \ge -C_3(\nu_1+\gamma_2+1)\exp(\gamma_1^*(t-t_0))\langle A(t_0)z(t_0), z(t_0)\rangle |z(t_0)|_2^{-2},$$

for all $0 < t_0 < t < T$.

Integrating from t to T, we obtain estimate (2.1), where

$$\gamma^* = C_4(\nu_1+\gamma_2+1)\frac{1}{\gamma_1^*}\exp(\gamma_1^*(T-t_0)), \tag{3.17}$$

where C_4 is independent of ω. If ψ is Lipschitz in r uniformly with respect to (t,ξ), then $g \in L^\infty(0,T) \times \mathcal{O})$ for all X_i with $X_i(0) \in L^2(\mathcal{O})$, $i = 1,2$, and so condition $X_i(0) \in L^\infty(\mathcal{O})$ is no longer necessary. This completes the proof. ∎

Proof of Theorem 2.2

Denote by $S(t) : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ the flow

$$S(t)x = y^x(t), \ t \in (0,T), \tag{3.18}$$

where y^x is the solution to (3.7) (equivalently (3.2)). It is easily seen that $S(T)$ is Fréchet differentiable on $L^2(\mathcal{O})$ and its Fréchet derivative at the origin $\Gamma : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ is given by $\Gamma u = DS(T)(0)u = v(T)$, where v is the solution to the equation

$$\begin{aligned} &\frac{dv}{dt} + A(t)v + B(t)v + e^{-W}\psi_r(t,\xi,e^W\widetilde{y})v = 0 \text{ in } (0,T),\\ &v(0) = u, \end{aligned} \tag{3.19}$$

where $\widetilde{y}$ is the solution to (3.7) with $\widetilde{y}(0) = 0$ and $\psi_r = \dfrac{\partial}{\partial r}\,\psi$.

Then, the dual operator $\Gamma^* : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ is given by $\Gamma^* p = z(0)$, $\forall p \in L^2(\mathcal{O})$, where z is the solution to backward dual equation

$$\begin{aligned} &\frac{dz}{dt} - A(t)z - B^*(t)z - e^{-W(t)}\psi_r(t,\xi,e^{W(t)}\widetilde{y}(t))z = 0,\ t \in (0,T),\\ &z(T) = p, \end{aligned} \tag{3.20}$$

which, clearly, is well posed for all $p \in L^2(\mathcal{O})$.

By Theorem 2.1 (applied to the backward equation (3.20)), the operator Γ^* is injective on $L^2(\mathcal{O})$ and, as well known (see e.g., Proposition IV.1 in [5]), this implies that the space $\{S(T)x;\ x \in L^2(\mathcal{O})\}$ is dense in $L^2(\mathcal{O})$, as claimed. ■

Remark 3.1 One might ask whether, under the assumptions of Theorem 2.2, the set $\{X(T,x);\ x \in L^2(\mathcal{O})\}$ is dense in $L^2(\Omega;\ L^2(\mathcal{O}))$, that is, in the mean square norm $(\mathbb{E}|\cdot|_2^2)^{\frac{1}{2}}$. Clearly, this happens if the stochastic backward dual equation associated with (1.1), that is,

$$\begin{cases} dp + \displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial \xi_i}\left(a_{ij}(t,\xi)\,\frac{\partial p}{\partial \xi_j}\right) - \operatorname{div}(bp)dt - \psi_r(t,\xi,0)pdt \\ \qquad + \displaystyle\sum_{j=1}^{\infty} \mu_j e_j q_j dt = \sum_{j=1}^{\infty} q_j(t)d\beta_j(t) \text{ in } (0,T)\times\mathcal{O}, \\ p = 0 \text{ on } (0,T)\times\partial\mathcal{O}, \end{cases} \tag{3.21}$$

has the forward uniqueness property, that is, $p(0) \equiv 0$ implies $p \equiv 0$. However, as far as we know, this is an open problem.

4 The second main result, the backward uniqueness for stochastic $3D$-tamed Navier–Stokes equations

Consider the stochastic equation

$$
\begin{aligned}
&dX - \nu\Delta X dt + (X\cdot\nabla)X dt + g_N(|X|_3^2)X dt = X dW + \nabla p dt \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,T)\times\mathcal{O}, \\
&\nabla\cdot X = 0 \text{ in } (0,T)\times\mathcal{O}; \quad X = 0 \text{ on } (0,T)\times\partial\mathcal{O}, \\
&X(0) = x \text{ in } \mathcal{O},
\end{aligned}
\tag{4.1}
$$

where $\mathcal{O}$ is a bounded and open subset of $\mathbb{R}^3$, with smooth boundary $\partial\mathcal{O}$ and $|\cdot|_3$ denotes the Euclidean norm on $\mathbb{R}^3$. W is the Wiener process on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ from the previous sections, i.e.,

$$
W(t,\xi) = \sum_{j=1}^{\infty} \mu_j e_j(\xi)\beta_j(t), \quad \xi\in\overline{\mathcal{O}}, \ t\ge 0,
$$

where $\{e_j\}\subset C^2(\overline{\mathcal{O}})$ is an orthonormal basis in $L^2(\mathcal{O})$, but with $\mu_j\in\mathbb{R}$ satisfying the stronger condition

$$
\sum_{j=1}^{\infty} \mu_j^2(|e_j|_\infty^2 + |\nabla e_j|_\infty^2) < \infty,
\tag{4.2}
$$

where $|\cdot|_\infty$ is the norm in $L^\infty(\mathcal{O})$. Here, $g_N\in C^1(\mathbb{R}^+)$, $N\in\mathbb{N}$, is a given function such that

$$
g_N(r) = \begin{cases} 0 & \text{for } r\in[0,N], \\ \dfrac{r-N-1}{\nu} & \text{for } r\ge N+1, \\ 0\le g_N'(r)\le C & \text{for } r\in\mathbb{R}. \end{cases}
\tag{4.3}
$$

Equation (4.1) is a modified version of the stochastic Navier–Stokes systems and was introduced by Röckner and Zhang [15] (see also [17], [16]). In the deterministic case, any bounded solution to the standard stochastic Navier–Stokes equations is a solution to (4.1) for sufficiently large N. However, in contrast to the case of the standard stochastic $3D$–Navier–Stokes equation, which in general has a (probabilistically) weak solution only (see, e.g., [7], [9], [14]), problem (4.1) is well posed in the (probabilistically) strong sense in an appropriate space, even in $3-D$.

Remark 4.1 In all what follows we could have taken a more general noise term than XdW, more precisely, the same type of noise as in [15]. All the arguments are exactly the same in this more general case. However, we restrict ourselves to XdW for simplicity and in order not to change the frame in comparison to Sections 2 and 3.

By strong solution to (4.1), we mean a pair of $(\mathcal{F}_t)_{t\geq 0}$–adapted processes $X : [0,T] \to H = \{y \in (L^2(\mathcal{O}))^3;\ \nabla \cdot y = 0,\ y \cdot \vec{n} = 0 \text{ on } \partial\mathcal{O}\}$, $p : [0,T] \to H^1(\mathcal{O})$, which has continuous sample paths and satisfies

$$\begin{aligned}
&X \in L^\infty(0,T;L^2(\Omega;H)) \cap L^2((0,T)\times\Omega;(H^1_0(\mathcal{O}))^3)\\
&X(t) = \nu\int_0^t \Delta X(s)ds - \int_0^t ((X(s)\cdot\nabla)X(s) + g_N(|X(s)|_3^2)X(s))ds\\
&\qquad\qquad + \int_0^t \nabla p(s)ds + \int_0^t X(s)dW(s),\ \forall t\in[0,T],\ \mathbb{P}\text{-a.s.}
\end{aligned}$$

For each $x \in (H^1_0(\mathcal{O}))^3 \cap H$, equation (4.1) has a unique strong solution X, which satisfies

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\|X(t)\|^2_{(H^1_0(\mathcal{O}))^3}\right] + \int_0^T \mathbb{E}\left[\|X(t)\|^2_{(H^2(\mathcal{O}))^3}\right]dt \leq C\|x\|^2_{(H^1_0(\mathcal{O}))^3}. \tag{4.4}$$

(See [16], Theorem 3.1.)

Furthermore, since our initial condition is not random, we also have (see [16, Formula (3.12)])

$$\int_0^T \mathbb{E}[\|X(t)\|^6_{(H^1(\mathcal{O}))^3}]dt < \infty. \tag{4.5}$$

In the following, we shall use the standard notations

$$V = (H^1_0(\mathcal{O}))^3 \cap H,\ A = -\Pi\,\Delta,\ D(A) = (H^2(\mathcal{O}))^3 \cap V,$$

where $\Pi : (L^2(\mathcal{O}))^3 \to H$ is the Leray projection (see [18]). We set, also,

$$b(y,z,\theta) = \int_{\mathcal{O}} y_i D_i z_j \theta_j d\xi,\quad \forall y,z,\theta \in V,$$

and denote by $B : V \to V^*$ the operator

$$\langle BX, \varphi \rangle = b(X, X, \varphi), \ \forall \varphi \in V.$$

The norm of V will be taken as

$$\|y\| = \langle Ay, y \rangle^{\frac{1}{2}},$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between V and its dual V^*. This norm is equivalent to $\| \cdot \|_{(H^1(\mathcal{O}))^3}$. On $V \times H$, this is just the scalar product of H. The norm of H is denoted in the following by $| \cdot |$. We recall that we have

$$|b(y, z, \theta)| \le C \|y\|_{m_1} \|z\|_{m_2+1} \|\theta\|_{m_3}, \tag{4.6}$$

where $m_1 + m_2 + m_3 \ge \frac{3}{2}$, if $m_i \ne \frac{3}{2}$, and $m_1 + m_2 + m_3 > \frac{3}{2}$, if one of the m_i is $\frac{3}{2}$. (Here, $\| \cdot \|_m$ is the norm of the Sobolev space $H^m(\mathcal{O})$.)

Then, we can rewrite (4.1) as the stochastic differential equation on the space H

$$\begin{aligned} & dX + \nu AX dt + BX dt + \Pi(g_N(|X|_3^2)X)dt = \sum_{j=1}^{\infty} \mu_j \Pi(Xe_j) d\beta_j, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in (0, T), \\ & X(0) = x. \end{aligned} \tag{4.7}$$

It is useful for the time being to mention that, as shown in [16, Theorem 3.1], the solution X to (4.7) is obtained as

$$\begin{aligned} X = \lim_{n \to \infty} u_n \quad & \text{weakly in } L^2(\Omega_T; (H^2(\mathcal{O}))^3) \\ & \text{weakly-star in } L^2(\Omega; L^\infty(0, T; (H_0^1(\mathcal{O}))^3)), \\ \Pi_n F(u_n) \to F \quad & \text{weakly in } L^2(\Omega_T; H), \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} \Omega_T &= [0, T] \times \Omega, \\ Fu &= -\nu Au - Bu - \Pi(g_N(|u|_3^2)u) \end{aligned}$$

and Π_n is the orthogonal projection of H onto $H_n = \text{span}(\widetilde{e}_1, \widetilde{e}_2, .., \widetilde{e}_n)$, $\{\widetilde{e}_i, \ i \ge 1\} \subset (H^2(\mathcal{O}))^3 \cap V$ being a fixed orthonormal basis in H consisting of eigenvectors of A. Moreover, u_n is the solution to the finite dimensional stochastic differential equation

$$\begin{aligned} du_n(t) &= \Pi_n F(u_n(t)) dt + \sum_{j=1}^{\infty} \mu_j \Pi_n(u_n e_j) d\beta_j, \\ u_n(0) &= \Pi_n x. \end{aligned} \tag{4.9}$$

Theorem 4.2 below is the backward uniqueness result for strong solutions to (4.1).

Theorem 4.2 *Let X_1, X_2 be two solutions to* (4.1)*, which satisfy* (4.4)*. Then, for any pair of solutions X_1, X_2 of* (4.1)*,*

$$\begin{aligned}\mathbb{E}\left[e^{-C\gamma(t)}\log(|X_1(t)-X_2(t)|^2)\right] &\le \mathbb{E}\left[e^{-C\gamma(T)}\log(|X_1(T)-X_2(T)|^2)\right]\\ &\quad + C + \|X_1(0)-X_2(0)\|\,|X_1(0)-X_2(0)|^{-1},\ \forall t\in(0,T),\end{aligned}\tag{4.10}$$

where

$$\begin{aligned}\gamma(t) = \int_0^t (&\|X_1(s)\|^2_{(W^{1,4}(\mathcal{O}))^3} + \|X_2(s)\|^2_{(W^{1,4}(\mathcal{O}))^3}\\ &+\|X_1(s)\|^4 + \|X(s)\|^4 + 1)ds,\ t\ge 0,\end{aligned}\tag{4.11}$$

which is finite $\mathbb{P}$-a.s. by (4.4)*,* (4.5)*, and C is a positive constant independent of $\omega\in\Omega$. Furthermore, the last summand in* (4.10) *is defined to be zero, if $X_1(0) = X_2(0)$. In particular, in the deterministic case, i.e. when the noise is zero, it follows that $X_1(T) = X_2(T)$ implies $X_1(t) = X_2(t)$ for all $t\in[0,T]$.*

Remark 4.3 The expectations in (4.10) are well defined because of (4.4), but maybe equal to $-\infty$, as happens in the case when $X_1(T) = X_2(T)$ $\mathbb{P}$-a.s.

Proof of Theorem 4.2. For simplicity, we shall take $\nu = 1$ in the following.

We set $Z = X_1 - X_2$ and, by (4.7), we get for Z the linear equation

$$dZ + AZdt + F_1dt + F_2dt = \Pi(ZdW) \text{ in } (0,T),\tag{4.12}$$

where $F_i : [0,T]\to H,\ i = 1,2$, are given by

$$F_1 = \Pi((Z\cdot\nabla)X_1 + (X_2\cdot\nabla)Z)\tag{4.13}$$

$$F_2 = \Pi(g_N(|X_1|_3^2)X_1 - g_N(|X_2|_3^2)X_2).\tag{4.14}$$

We have for dt-a.e. $t\in[0,T]$

$$\begin{aligned}|F_1(t)| &\le C_1\|Z(t)\|(\|\nabla X_1(t)\|_{(L^4(\mathcal{O}))^3} + \|X_2(t)\|_{(L^\infty(\mathcal{O}))^3})\\ &\le C_2\|Z(t)\|(\|X_1(t)\|_{(W^{1,4}(\mathcal{O}))^3} + \|X_2(t)\|_{(W^{1,4}(\mathcal{O}))^3}),\\ &\qquad\forall t\in[0,T],\end{aligned}\tag{4.15}$$

because, by the Rellich–Kondrachev theorem (see, e.g., [14, p. 285]), $(W^{1,4}(\mathcal{O}))^3 \subset L^\infty(\mathcal{O})$. By (4.3) it follows that

$$|g_N(|X_1|_3^2)X_1 - g_N(|X_2|_3^2)X_2|_3 \le C_3(|X_1(t)|_3^2 + |X_2(t)|_3^2)|Z(t)|_3 \quad \text{a.e. in } (0,T)\times\mathcal{O}\times\Omega,$$

which, by the Sobolev embedding, implies

$$|F_2(t)| \le C_4(\|X_1(t)\|^2 + \|X_2(t)\|^2)\|Z\|. \tag{4.16}$$

(Here and everywhere in the following, C_i, $i = 1, ...$, are positive constants independent of $\omega \in \Omega$.)

Also, in this case, we have (see (4.9))

$$Z = \lim_{n\to\infty} z_n \quad \begin{array}{l}\text{weakly in } L^2(\Omega_T;(H^2(\mathcal{O}))^3),\\ \text{weakly-star in } L^2(\Omega; L^\infty(0,T;(H_0^1(\mathcal{O}))^3)),\end{array} \tag{4.17}$$

where

$$\begin{aligned} dz_n + A_n z_n dt + F_1^n dt + F_2^n dt &= \sum_{j=1}^\infty \mu_j \Pi_n(z_n e_j)d\beta_j,\\ z_n(0) &= \Pi_n(Z(0)), \end{aligned} \tag{4.18}$$

where $F_i^n = \Pi_n F_i$, $i = 1,2$, $A_n = \Pi_n A$. Moreover, estimates (4.15)–(4) hold in this case for z_n, $u_n^1 = \Pi_n X_1$, $u_n^2 = \Pi_n X_2$ instead of Z, X_1 and X_2, respectively.

We consider the function

$$\varphi_\varepsilon(y) = \frac{\|y\|^2}{|y|^2+\varepsilon}, \quad y \in V,$$

where $\varepsilon > 0$ is arbitrary but fixed. We see that φ_ε is C^2 on V and its Gateaux derivative $D\varphi_\varepsilon \in V'$ is given by

$$D\varphi_\varepsilon(y) = 2[Ay(|y|^2+\varepsilon) - y\|y\|^2](|y|^2+\varepsilon)^{-2},\ y \in V. \tag{4.19}$$

Moreover, we have, for the second derivative D^2,

$$\begin{aligned} D^2\varphi_\varepsilon(y)(h) =& 2(|y|^2+\varepsilon)^{-2}[(|y|^2+\varepsilon)Ah - h\|y\|^2 + 2Ay\,\langle y,h\rangle - 2y\,\langle Ah, y\rangle]\\ &- 4\,\langle y,h\rangle\,[Ay(|y|^2+\varepsilon) - y\|y\|^2](|y|^2+\varepsilon)^{-3},\ \forall y,h \in V. \end{aligned} \tag{4.20}$$

If we heuristically apply Itô's formula to φ_ε in equation (4.12), we get

$$
\begin{aligned}
& d\varphi_\varepsilon(Z(t)) + 2|AZ(t)|^2(|Z(t)|^2+\varepsilon)^{-1}dt \\
& \quad - 2\|Z(t)\|^4(|Z(t)|^2+\varepsilon)^{-2}dt \\
& \quad + 2\left\langle AZ(t), F_1(t)+F_2(t)\right\rangle (|Z(t)|^2+\varepsilon)^{-1}dt \\
& \quad - 2\left\langle Z(t), F_1(t)+F_2(t)\right\rangle \|Z(t)\|^2(|Z(t)|^2+\varepsilon)^{-2}dt \\
& \quad = \frac{1}{2}\sum_{j=1}^{\infty}\mu_j^2\left\langle D^2\varphi_\varepsilon(Z(t))(Z(t)e_j), Z(t)e_j\right\rangle dt \\
& \quad + \left\langle D\varphi_\varepsilon(Z(t)), Z(t)dW(t)\right\rangle,
\end{aligned}
\tag{4.21}
$$

for $t \in [0,T]$. However, it should be said that, since Z is not a semimartingale in V, the Itô formula cannot be applied in (4.12) and so to get (4.21) we shall invoke a more sophisticated argument based on the approximating equation (4.17). Namely, we shall apply Itô's formula in (4.18) to the function

$$\varphi_\varepsilon(v) = \|v\|^2\rho_\varepsilon(|v|^2),\ \forall v \in V,$$

where $\rho_\varepsilon(r) = \frac{1}{r+\varepsilon}$, $\forall r \geq 0$. For φ_ε, (4.19) and (4.20) remain valid, and so we get

$$
\begin{aligned}
& d\varphi_\varepsilon(z_n(t)) + 2|A_n z_n(t)|^2(|z_n(t)|^2+\varepsilon)^{-1}dt \\
& \quad -2\|z_n(t)\|^4(|z_n(t)|^2+\varepsilon)^{-2}dt \\
& \quad +2\left\langle A_n z_n(t), F_1^n(t)+F_2^n(t)\right\rangle (|z_n(t)|^2+\varepsilon)^{-1}dt \\
& \quad -2\left\langle z_n(t), F_1^n(t)+F_2^n(t)\right\rangle \|z_n(t)\|^2(|z_n(t)|^2+\varepsilon)^{-2}dt \\
& \quad = \frac{1}{2}\sum_{j=1}^{\infty}\mu_j^2\left\langle D^2\varphi_\varepsilon(z_n(t))\right\rangle \left\langle z_n(t)e_j, z_n(t)e_j\right\rangle dt \\
& \quad +\sum_{j=1}^{\infty}\left\langle D\varphi_\varepsilon(z_n(t)), z_n(t)dW(t)\right\rangle.
\end{aligned}
\tag{4.22}
$$

Taking into account (4.17), we may pass to the limit in (4.22) and get for Z just formula (4.21).

We have

$$
\begin{aligned}
&|AZ|^2(|Z|^2+\varepsilon)^{-1} - \|Z\|^4(|Z|^2+\varepsilon)^{-2} \\
&\quad - \langle Z, F_1+F_2\rangle \|Z\|^2(|Z|^2+\varepsilon)^{-2} + \langle AZ, F_1+F_2\rangle (|Z|^2+\varepsilon)^{-1} \\
&\quad = (|Z|^2+\varepsilon)^{-2}[|AZ|^2(|Z|^2+\varepsilon) - \langle AZ, Z\rangle^2 \\
&\quad + \langle AZ, F_1+F_2\rangle (|Z|^2+\varepsilon) - \langle Z, F_1+F_2\rangle \|Z\|^2] \\
&\quad = (|Z|^2+\varepsilon)^{-2}\left[\left|AZ+\frac{1}{2}(F_1+F_2)\right|^2 |Z|^2 - \frac{1}{4}|F_1+F_2|^2|Z|^2\right. \\
&\quad \left. - \left(\langle AZ, Z\rangle + \left\langle \frac{1}{2}(F_1+F_2), Z\right\rangle\right)^2 + \frac{1}{4}\langle F_1+F_2, Z\rangle^2\right] \qquad (4.23) \\
&\quad +\varepsilon(|Z|^2+\varepsilon)^{-2}(|AZ|^2 + \langle AZ, F_1+F_2\rangle) \\
&\quad \geq -\frac{1}{4}(|Z|^2+\varepsilon)^{-2}|Z|^2|F_1+F_2|^2 \\
&\quad + \varepsilon(|Z|^2+\varepsilon)^{-2}(|AZ|^2 + \langle AZ, F_1+F_2\rangle) \\
&\quad \geq -\frac{1}{4}(|Z|^2+\varepsilon)^{-1}|F_1+F_2|^2 - \frac{\varepsilon}{4}|F_1+F_2|^2(|Z|^2+\varepsilon)^{-2} \\
&\quad \geq -\frac{1}{2}(|Z|^2+\varepsilon)^{-1}|F_1+F_2|^2.
\end{aligned}
$$

We have also by (4.2) and (4.20)

$$
\begin{aligned}
&\left|\sum_{j=1}^{\infty} \mu_j^2 \left\langle D^2\varphi_\varepsilon(Z)(Ze_j), Ze_j\right\rangle\right| \\
&\qquad \leq C_4(|Z|^2+\varepsilon)^{-1}\|Z\|^2 \sum_{j=1}^{\infty} \mu_j^2(|e_j|_\infty^2 + |\nabla e_j|_\infty^2) \leq C_5\varphi_\varepsilon(Z).
\end{aligned} \tag{4.24}
$$

On the other hand, by (4.15), (4) we see that

$$
\begin{aligned}
&|F_1+F_2|^2(|Z|^2+\varepsilon)^{-1} \\
&\leq C_6(\|X_1\|^2_{(W^{1,4}(\mathcal{O}))^3} + \|X_2\|^2_{(W^{1,4}(\mathcal{O}))^3} + \|X_1\|^4 + \|X_2\|^4)\|Z\|^2(|Z|^2+\varepsilon)^{-1} \\
&\leq C_6(\gamma'-1)\varphi_\varepsilon(Z),
\end{aligned}
$$

where γ' is the derivative of γ given in (4.11).

Substituting (4.23), (4.24) into (4.21), we obtain that

$$d\varphi_\varepsilon(Z(t)) \le C_7\gamma'(t)\varphi_\varepsilon(Z(t))dt + \langle D\varphi_\varepsilon(Z(t)), Z(t)dW(t)\rangle . \tag{4.25}$$

We note that (4.5) ensures the integrability of the integrands in the right hand side.

Integrating (4.25) from 0 to t, multiplying by $\exp(-C_7\gamma(t))$ and applying Itô's product rule, we obtain

$$e^{-C_7\gamma(t)}\varphi_\varepsilon(Z(t)) \le \varphi_\varepsilon(Z(0)) + \int_0^t e^{-C_7\gamma(s)} \langle D\varphi_\varepsilon(Z(s)), Z(s)dW(s)\rangle ,$$

and this yields

$$\mathbb{E}[\varphi_\varepsilon(Z(t))\exp(-C_7\gamma(t))] \le \varphi_\varepsilon(Z(0)),\ \forall t\in[0,T]. \tag{4.26}$$

Next, we apply the Itô formula to (4.12) and the function

$$\psi_\varepsilon(z) = \frac{1}{2}\log(|z|^2+\varepsilon),\ z\in V.$$

Taking into account that $D\psi_\varepsilon(z) = z(|z|^2+\varepsilon)^{-1}$, we obtain that

$$d\left(\frac{1}{2}\log(|Z(t)|^2+\varepsilon)\right) + \varphi_\varepsilon(Z(t))dt + \langle F_1(t)+F_2(t), Z(t)\rangle (|Z(t)|^2+\varepsilon)^{-1}dt$$
$$= \frac{1}{2}\sum_{j=1}^\infty \mu_j^2 \left\langle D^2\psi_\varepsilon(Z(t))(Z(t)e_j), Z(t)e_j\right\rangle dt + \langle D\psi_\varepsilon(Z(t)), Z(t)dW(t)\rangle .$$

By (4.15), (4.16), we get after some calculations that

$$\begin{aligned}
& d\left(\frac{1}{2}\log(|Z(t)|^2+\varepsilon)\right) + \varphi_\varepsilon(Z(t))dt \\
& \ge -C_8(\|X_1(t)\|_{(W^{1,4}(\mathcal{O}))^3} + \|X_2(t)\|_{(W^{1,4}(\mathcal{O}))^3} \\
& + \|X_1(t)\|^2 + \|X_2(t)\|^2)|Z(t)|\|Z(t)\|(|Z(t)|^2+\varepsilon)^{-1}dt \\
& -C_9dt + \langle D\psi_\varepsilon(Z(t)), Z(t)dW(t)\rangle \\
& \ge -\varphi_\varepsilon(Z(t))dt - C_{10}\gamma'(t)dt + \langle D\psi_\varepsilon(Z(t)), Z(t)dW(t)\rangle ,
\end{aligned}$$

because, by (4.2),

$$\sum_{j=1}^\infty \mu_j^2 \left\langle D^2\psi_\varepsilon(Z(t))(Z(t)e_j), Z(t)e_j\right\rangle \le C_9.$$

This yields

$$d(\log(|Z(t)|^2+\varepsilon)) \geq -4\varphi_\varepsilon(Z(t))dt - C_{11}\gamma'(t)dt + \langle D\psi_\varepsilon(Z(t)), Z(t)dW\rangle .$$

Letting $T \geq r > t$ and integrating, we obtain

$$\begin{aligned}\log(|Z(r)|^2+\varepsilon) \geq \log(|Z(t)|^2+\varepsilon) - 4\int_t^r \varphi_\varepsilon(Z(s))ds \\ - C_{11}\int_t^r \gamma'(s)ds + \int_t^r \langle D\psi_\varepsilon(Z(s)), Z(s)dW(s)\rangle .\end{aligned}$$

Then, multiplying by $\exp(-C_{11}\gamma(t))$ and using Itô's product formula, we get as above

$$\begin{aligned}&e^{-C_{11}\gamma(r)}\log(|Z(r)|^2+\varepsilon) = e^{-C_{11}\gamma(t)}\log(|Z(t)|^2+\varepsilon)\\ &+\int_t^r e^{-C_{11}\gamma(s)}d\log(|Z(s)|^2+\varepsilon) - C_{11}\int_t^r \log(|Z(s)|^2+\varepsilon)\gamma'(s)e^{-C_{11}\gamma(s)}ds\\ &\geq e^{-C_{11}\gamma(t)}\log(|Z(t)|^2+\varepsilon) - \int_t^r e^{-C_{11}\gamma(s)}(4\varphi_\varepsilon(Z(s)) + C_{11}\gamma'(s))ds\\ &+\int_t^r e^{-C_{11}\gamma(s)}\langle D\psi_\varepsilon(Z(s)), X(s)dW(s)\rangle\\ &-C_{11}\int_t^r \log(|Z(s)|^2+\varepsilon)\gamma'(s)e^{-C_{11}\gamma(s)}ds.\end{aligned}$$

Taking $r = T$, we get taking expectation

$$\begin{aligned}&\mathbb{E}[e^{-C_{11}\gamma(t)}\log(|Z(t)|^2+\varepsilon)] \leq \mathbb{E}[e^{-C_{11}\gamma(T)}\log(|Z(T)|^2+\varepsilon)]\\ &\quad + 4\mathbb{E}\int_t^T e^{-C_{11}\gamma(s)}\varphi_\varepsilon(Z(s))ds\\ &\quad +C_{11}\mathbb{E}\int_t^T e^{-C_{11}\gamma(s)}\gamma'(s)\log(|Z(s)|^2+\varepsilon)ds + e^{-C_{11}\gamma(t)} - e^{C_{11}\gamma(T)}.\end{aligned}$$

Then, by (4.26), we obtain (because we may assume that $C_{11} > C_7$ and $\varepsilon \leq 1$)

$$\begin{aligned}&\mathbb{E}[e^{-C_{11}\gamma(t)}\log(|Z(t)|^2+\varepsilon)] \leq \mathbb{E}[e^{-C_{11}\gamma(T)}\log(|Z(T)|^2+\varepsilon)]\\ &\quad +\varphi_\varepsilon(Z(0)) + e^{-C_{11}\gamma(t)} - e^{C_{11}\gamma(T)}\\ &\quad +C_{11}\mathbb{E}\int_t^T e^{-C_{11}\gamma(s)}\gamma'(s)\log(|Z(s)|^2+1)ds.\end{aligned} \tag{4.27}$$

On the other hand, we have

$$
\begin{aligned}
&\mathbb{E}\int_0^T \gamma'(s)\log(|Z(s)|^2+1)ds \\
&\quad\le C_{13}\mathbb{E}\int_0^T (\|X_1(t)\|^4+\|X_2(t)\|^4+\|X_1(t)\|^2_{(W^{1,4}(\mathcal{O}))^3} \\
&\quad+\|X_2(t)\|^2_{(W^{1,4}(\mathcal{O}))^3}+1)(1+\log(|X_1(t)|^2+|X_2(t)|^2+1))dt \\
&\quad=: C_{14}<\infty.
\end{aligned}
\tag{4.28}
$$

Here is the argument, to show that indeed $C_{14}<\infty$.

We have the interpolation inequality

$$
\|u\|_{(W^{1,4}(\mathcal{O}))^3}\le C\|u\|^{1-\alpha}_{(H^2(\mathcal{O}))^3}\|u\|^{\alpha}_{(L^2(\mathcal{O}))^3}, \tag{4.29}
$$

for some $\alpha\in(0,1)$. (The latter is a consequence of the fact that, for $\alpha\in(0,1)$ suitably chosen, we have $D(A^\alpha)=(H,D(A))_\alpha=(H^{2\alpha}(\mathcal{O}))^3\cap V\subset(W^{1,4}(\mathcal{O}))^3$, for all $\alpha\in(1/2,1)$.) This yields

$$
\begin{aligned}
&\|X_1(t)\|^2_{(W^{1,4}(\mathcal{O}))^3}+\|X_2(t)\|^2_{(W^{1,4}(\mathcal{O}))^3} \\
&\quad\le C(\|X_1(t)\|^{2(1-\alpha)}_{(H^2(\mathcal{O}))^3}+\|X_2(t)\|^{2(1-\alpha)}_{(H^2(\mathcal{O}))^3})(|X_1(t)|^{2\alpha}+|X_2(t)|^{2\alpha}), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall t\in(0,T).
\end{aligned}
$$

Taking into account that for an arbitrary $\beta\in(0,1)$

$$
|\log(|z|^2+1)|\le C_\beta(|z|^\beta+1),\ \forall z,
$$

we get by (4.4), (4.5) that $C_{14}<\infty$. Then, by (4.27), we obtain

$$
\begin{aligned}
&\mathbb{E}[e^{-C_{11}\gamma(t)}\log(|Z(t)|^2+\varepsilon)] \\
&\quad\le\mathbb{E}[e^{-C_{11}\gamma(T)}\log(|Z(T)|^2+\varepsilon)]+e^{-C_{11}\gamma(t)}-e^{-C_{11}\gamma(T)}+\varphi_\varepsilon(Z(0))+C_{14}.
\end{aligned}
$$

Letting $\varepsilon\to0$, we get (4.11), as desired.

Remark 4.4 One might suspect that a controllability result similar to Theorem 2.2 remains valid in this case too. However, this requires a forward uniqueness result for the linearized backward stochastic equation corresponding to (4.1) which, as in the case of equation (3.21), remains open.

Remark 4.5 By inspecting the previous proof, it is clear that Theorem 4.2 remains true for any pair of solutions X_1, X_2 to the stochastic Navier–Stokes equation

$$dX - \nu\Delta X dt + (X\cdot\nabla)X dt = X dW \text{ in } (0,T)\times\mathcal{O},$$
$$\nabla\cdot X = 0,\ X = 0 \text{ on } (0,T)\times\partial\mathcal{O}.$$

which satisfies condition (4.4) (if any). Anyway, Theorem 4.2 remains true for linear Oseen–Stokes equations of the form

$$dX - \nu\Delta X, dt + ((X\cdot\nabla)a + (b\cdot\nabla)X)dt = X\,dW$$
$$\nabla\cdot X = 0,\ X = 0 \ \text{ on } (0,T)\times\partial\mathcal{O}.$$

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References

[1] S. Agmon, L. Nirenberg, Lower bound and uniqueness theorems for solutions of differential erquations in Banach spaces, *Comm. Pure Appl. Math.,* 20 (1967), 207-229.

[2] V. Barbu, *Nonlinear Differential Equations of Monotone Type in Banach Spaces*, Springer, 2010.

[3] V. Barbu, M. Röckner, An operatorial approach to stochastic partial differential equations driven by linear multiplicative noise, *J. European Math. Soc.* (to appear).

[4] V. Barbu, M. Röckner, Existence of strong solutions to $2-D$ Navier–Stokes equations with linear multiplicative noise by rescaling (in preparation).

[5] C. Bardos, L.Tartar, Sur l'unicité retrograde des équations paraboliques et questions voisines, *Arch. Rat. Mech. Anal.*, 50 (1973), 10-25.

[6] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2011, New York. Heidelberg. London.

[7] Z. Brzezniak, E. Motyl, Existence and martingale solution of the stochastic Navier–Stokes euations in unbounded $2D$ and $3D$ domains, *J. Differential Equations*, 254 (2013), 1627-1642.

[8] Z. Brzezniak, M. Neklyudov, Backward uniqueness and the existence of the spectral limit for linear parabolic SPDEs, *Stoch. Processes and Their Appl.*, 123 (2013), 1851-1870.

[9] F. Flandoli, D. Gatarek, Martingale and stationary solutions for stochastic Navier–Stokes equations, *Probab. Theory Rel. Fields*, 102 (1995), 367-391.

[10] O.A. Ladyženskaja, V.A. Solonikov, N.N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, American Math. Soc., 1968.

[11] J.L. Lions, B. Malgrange, Sur l'unicité rétrograde dans les problèmes mixtes paraboliques, *Math. Scand.*, 8 (1960), 277-286.

[12] J.L. Lions, *Quelques méthodes de resolution des problèmes aux limites nonlinéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[13] J.L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer–Verlag, Berlin–New York–Heidelberg, 1978.

[14] R. Mikulevicius, B.L. Rozovskii, Global L_2–solutions of stochastic Navier–Stokes equations, *The Annals of Probability*, vol. 33 (2005), 137-176.

[15] M. Röckner, X. Zhang, Tamed $3D$ Navier–Stokes equation: existence, uniqueness and regularity, IDA-QP, no. 4 (2000), 525-549.

[16] M. Röckner, T.S. Zhang, Stochastic $3D$-tamed Navier–Stokes equations: existence, uniqueness and small time large deviation principle, *J. Diff. Equations*, 252 (2012), 716-744.

[17] M. Röckner, T.S. Zhang, X. Zhang, Large deviations for stochastic tamed $3D$ Navier–Stokes equations, *Appl. Math. Optim.*, 61 (2010), 267-285.

[18] R. Temam, *Navier–Stokes Equations*, North Holland, Amsterdam, New York, Oxford, 1979.