Differentiability of solutions of stationary Fokker–Planck–Kolmogorov equations with respect to a parameter

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We obtain sufficient conditions for the differentiability of solutions to stationary Fokker–Planck–Kolmogorov equations with respect to a parameter. In particular, this gives conditions for the differentiability of stationary distributions of diffusion processes with respect to a parameter.

Keywords Stationary Fokker–Planck–Kolmogorov equation; Differentiability with respect to a parameter.

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1. Introduction and main results

The goal of this paper is to give broad sufficient conditions for the differentiability of solutions to stationary Fokker–Planck–Kolmogorov equations

$$\partial_{x_i} \partial_{x_j} (a_{ij}^x \mu_\alpha) - \partial_{x_i} (b_i^x \mu_\alpha) = 0$$

with respect to a parameter. In particular, we obtain sufficient conditions for the differentiability of invariant measures of diffusion processes with respect to a parameter. Our conditions are expressed in terms of Lyapunov functions and apply to unbounded coefficients. The results of [18] and [23], where the problem was first studied, are generalized and reinforced in the case of one-fold differentiability: substantially broader assumptions about the coefficients are considered, the main novelty is that rapidly growing coefficients are allowed. Dependence of solutions on parameters, in particular, differentiability and continuity with respect to parameters, obviously belongs to questions of general interest, which are important both for the theory and diverse applications such as control theory (see, e.g., [1] and [17]). However, the case of equations on the whole space has not been studied in sufficient generality so far (except for the already cited pioneering papers [18] and [23], where the case of bounded coefficients was examined). The results of this paper are new even in the one-dimensional case. Our conditions become especially simple in the case where $a_{ij}^x$ and $\partial_x a_{ij}^x$ are uniformly bounded and $\partial_x a_{ij}^0, \partial_x, \partial_x a_{ij}^0, b_i^0, \partial_x b_i^0$ have at most polynomial growth: just the relation $\lim_{|x| \to \infty} \sup_{\alpha} \langle b_\alpha(x), x \rangle = -\infty$ for the drift coefficient $b_\alpha$ is needed. Some auxiliary results obtained below on solvability of non-homogeneous Fokker–Planck–Kolmogorov equations and related a priori estimates can be useful in other problems such as discrete approximations.

Let us explain our framework. Suppose first that we are given a single second order elliptic operator

$$L\varphi = a^{ij} \partial_{x_i} \partial_{x_j} \varphi + b^i \partial_{x_i} \varphi,$$

where the usual summation with respect to repeated indices is meant, $a^{ij}$ and $b^i$ are real Borel functions on $\mathbb{R}^d$, and the matrix $A(x) = (a^{ij}(x))_{i,j \leq d}$ is positive-definite for
each \( x \). We say that a bounded Borel measure \( \mu \) satisfies the stationary Fokker–Planck–Kolmogorov equation

\[
\partial_{x_i} \partial_{x_j} (a^{ij} \mu) - \partial_{x_i} (b^i \mu) = 0,
\]

or, in a shorter form,

\[
L^* \mu = 0 \quad (1.1)
\]
on a domain \( \Omega \) in \( \mathbb{R}^d \) (in our main results \( \Omega = \mathbb{R}^d \)) if the coefficients \( a^{ij} \) and \( b^i \) are locally integrable in \( \Omega \) with respect to the measure \( |\mu| \) (which holds automatically for locally bounded coefficients) and we have the integral identity

\[
\int L \varphi \, d\mu = 0 \quad \forall \varphi \in C^\infty_0(\Omega).
\]
For example, this equation holds for stationary probabilities of the diffusion process governed by the stochastic equation

\[
d\xi_t = \sqrt{2A(\xi_t)} dw_t + b(\xi_t) dt.
\]
Suppose now that for every \( \alpha \in [0,1] \) we are given a second order elliptic operator

\[
L_\alpha \varphi = a^{ij} \partial_{x_i} \partial_{x_j} \varphi + b^i \partial_{x_i} \varphi
\]
with coefficients satisfying certain conditions specified below. Suppose also that for each \( \alpha \) there is a unique probability measure \( \mu_\alpha \) satisfying the stationary Fokker–Planck–Kolmogorov equation

\[
L_\alpha^* \mu_\alpha = 0 \quad (1.2)
\]
in the sense explained above. The goal of this paper is to provide broad sufficient conditions for the continuity and differentiability of \( \mu_\alpha \) and its density \( \varrho_\alpha \) with respect to the parameter \( \alpha \). In particular, if there is a diffusion \( \xi_{\alpha,t} \) with generator \( L_\alpha \) and a stationary distribution \( \mu_\alpha \), our results provide broad conditions for the continuity and differentiability of the density of \( \mu_\alpha \) with respect to the parameter \( \alpha \).

Recall that the Sobolev class \( W^{p,1}(U) \) on a domain \( U \) in \( \mathbb{R}^d \) consists of all functions \( f \in L^p(U) \) having generalized derivatives \( \partial_{x_i} f \in L^p(U) \) and is equipped with the Sobolev norm

\[
\|f\|_{p,1} = \|f\|_p + \|\partial_{x_1} f\|_p + \cdots + \|\partial_{x_d} f\|_p,
\]
where \( \| \cdot \|_p \) denotes the \( L^p \)-norm. The class \( C^k_b(\Omega) \) consists of functions on \( \Omega \) with \( k \) bounded continuous derivatives and \( C^\infty_b(\Omega) \) is the intersection of these classes.

It is known (see [4], [6]) that if for every ball \( U \) in \( \Omega \) there exists a number \( p = p(U) > d \) such that \( a^{ij}|_U \in W^{p,1}(U) \), \( b^i|_U \in L^p(U) \) and \( \inf_U \det A > 0 \), then any solution \( \mu \) to equation (1.1) has a continuous density \( \varrho \) whose restriction to every ball \( U \) belongs to the Sobolev class \( W^{p,1}(U) \) with the corresponding \( p = p(U) > d \). Moreover, if \( \mu \geq 0 \) is not identically zero and \( \Omega \) is connected, then \( \varrho > 0 \).

In this case the equation \( L^* \mu = 0 \) can be written as the equation

\[
\partial_{x_i} \partial_{x_j} (a^{ij} \varrho) - \partial_{x_i} (b^i \varrho) = 0
\]
for \( \varrho \) (understood in the sense of distributions) and further transformed into the divergence form equation

\[
\text{div} (A \nabla \varrho - (b - \text{div} A) \varrho) = 0, \quad \text{div} A = (\partial_{x_j} a^{ij}, \ldots, \partial_{x_i} a^{ij}).
\]
There is a vast literature devoted to the theory of such equations, see, e.g., [16], [21], [22], and references in [6].

A sufficient condition for the existence of a probability solution to (1.1) on the whole space under the local assumptions mentioned above is the existence of a Lyapunov function
$V \in C^2(\mathbb{R}^d)$ such that $V(x) \to +\infty$ and $LV(x) \leq -\kappa < 0$ outside of a compact set, see [10] or a somewhat weaker result in [7].

A sufficient condition for the uniqueness of a probability solution to (1.1) under the same local assumptions is the existence of a Lyapunov function $V \in C^2(\mathbb{R}^d)$ such that $V(x) \to +\infty$ and $LV(x) \leq qV(x)$ for some number $q \geq 0$, see [6], [9], and [12]. In particular, the existence condition above ensures also the uniqueness.

In case of coefficients depending on a parameter $\alpha \in [0,1]$, we need uniformity in $\alpha$ of the above conditions. Namely, we assume throughout that we deal with real coefficients $a^{ij}_\alpha$ and $b^i_\alpha$ on $\mathbb{R}^d$, Borel measurable in $(x,\alpha)$ and satisfying the following conditions:

the matrices $A_\alpha(x) = (a^{ij}_\alpha(x))_{i,j \leq d}$ are symmetric and for every ball $U \subset \mathbb{R}^d$ we have

$$
sup_\alpha \| a^{ij}_\alpha \|_{W^{p,1}(U)} \leq M_1(U) < \infty, \quad sup_\alpha \| b^i_\alpha \|_{L^p(U)} \leq M_2(U) < \infty, \quad (1.3)
$$

where $p = p(U) > d$, and for all $x$ we have

$$
A_\alpha(x) \geq c_0 I, \quad c_0 > 0, \quad (1.4)
$$

where $I$ is the unit operator and $c_0$ is a constant (independent of $U$).

Unlike the case of a boundary value problem on a bounded domain with a nice boundary, where the differentiability of solutions with respect to a parameter under our basic assumptions follows relatively easily from suitable a priori estimates and compactness of embeddings (see, e.g., [14, Chapter X, Section 5, Theorem 15, Chapter III, Section 6]), the case of the whole space is more subtle and much less studied. Already in the one-dimensional case with $A_\alpha = 1$ (where a probability solution is unique) and smooth $b_\alpha(x)$ the continuity of the density in $\alpha$ can fail (see Example 1.8).

The lack of compactness will be compensated by suitable Lyapunov functions. The concept of uniform tightness of families of measures will be useful.

Recall that a family $\mathcal{M}$ of probability measures is uniformly tight if, for each $r > 0$, there is a compact set $K$ such that $\mu(\mathbb{R}^d \setminus K) \leq r$ for all $\mu \in \mathcal{M}$. A necessary and sufficient condition for the uniform tightness is the existence of a locally bounded Borel function $W \geq 0$ such that $\lim_{|x| \to \infty} W(x) = +\infty$ and

$$
\sup_{\mu \in \mathcal{M}} \int_{\mathbb{R}^d} W \, d\mu < \infty.
$$

The case of continuity is much easier and here we have the following result (in which (1.4) is replaced by a local bound).

**Proposition 1.1.** Suppose that (1.3) holds, $\inf_{\alpha \in U} \det A_\alpha(x) > 0$ for every ball $U$ and that the family of measures $\mu_\alpha$ (that are unique probability solutions to the corresponding equations (1.2)) is uniformly tight. Assume also that, for every ball $U$, the restrictions of $a^{ij}_\alpha$ and $b^i_\alpha$ to $U$ are continuous in $\alpha$ in the space $L^1(U)$. Then, one can choose densities $\varrho_\alpha$ of $\mu_\alpha$ such that the function $\varrho_\alpha(x)$ will be jointly continuous. In addition, the mapping $\alpha \mapsto \varrho_\alpha$ with values in $L^1(\mathbb{R}^d)$ is continuous, i.e., the mapping $\alpha \mapsto \mu_\alpha$ is continuous in the variation norm.

A sufficient condition for the uniform tightness of the measures $\mu_\alpha$ is the existence of a single Lyapunov function $V$ such that $V(x) \to +\infty$ and $\sup_\alpha L_\alpha V(x) = -\infty$ as $|x| \to \infty$. Certainly, this condition ensures also the existence and uniqueness of solutions.

In order to have the local continuity in $L^1$ it is enough to have the usual continuity of the coefficients in $\alpha$ along with their uniform integrability on balls.

The case of differentiability is much harder and requires some auxiliary results presented in the next section.
Recall that a mapping $\alpha \mapsto f_\alpha$ from $(0, 1)$ to $L^p(U)$ is differentiable if there is a mapping $\alpha \mapsto g_\alpha$ from $(0, 1)$ to $L^p(U)$ such that $(f_{\alpha+s} - f_\alpha)/s \to g_\alpha$ in $L^p(U)$ as $s \to 0$ for each fixed $\alpha \in (0, 1)$. If $\alpha \mapsto g_\alpha$ is continuous, then $f_\alpha$ is said to be continuously differentiable in $L^p$.

Suppose that for every ball $U$ there is a number $p_0 = p_0(U) > d$ such that the mappings $\alpha \mapsto a^{ij}_\alpha|_U$ and $\alpha \mapsto b^i_\alpha|_U$ are continuously differentiable in $L^{p_0}(U)$. 

Note that this condition is fulfilled if, in addition to (1.3), the functions $a^{ij}_\alpha$, $\partial_x a^{ij}_\alpha$ and $b^i_\alpha$ are differentiable in $\alpha$ and their derivatives in $\alpha$ are continuous in $\alpha$ and locally bounded in both variables.

Set
\[
A_\alpha := (a^{1i}_\alpha, \ldots, a^{di}_\alpha), \quad \text{div} \, A_\alpha := (\text{div} \, A^{1i}_\alpha, \ldots, \text{div} \, A^{di}_\alpha),
\]
\[
B_\alpha := \partial_x a_\alpha = (\partial_x a^{1i}_\alpha, \ldots, \partial_x a^{di}_\alpha),
\]
\[
S_\alpha := \partial_\alpha A_\alpha = (\partial_\alpha a^{1i}_\alpha), j \leq d, \quad R^i_\alpha = \partial_\alpha \partial_x a^{ij}_\alpha, \quad R_\alpha := (R^1_\alpha, \ldots, R^d_\alpha) = \text{div} \, S_\alpha.
\]
We assume that
\[
\sup_{\alpha, x} \|S_\alpha(x)\| \leq \lambda_0 < \infty. \quad (1.6)
\]
This condition is obviously fulfilled if $A_\alpha(x)$ does not depend on $\alpha$ or is uniformly Lipschitzian in $\alpha$.

Condition (1.6) implies that in (1.5) actually a stronger condition on the diffusion coefficient is fulfilled: the functions $\alpha \mapsto a^{ij}_\alpha|_U$ are continuously differentiable in every $L^p(U)$ with $p_1 < \infty$. In particular, we can take $p_1 = p_1(U) > \frac{dp}{p-d}$, where $p = p(U)$ is the number from (1.3).

Our main theorem is this.

**Theorem 1.2.** Let (1.3), (1.4), (1.5), and (1.6) hold. Suppose that $V \in C^2(\mathbb{R}^d)$ and $W$ is a locally integrable function such that
\[
\lim_{|x| \to \infty} V(x) = +\infty, \quad \lim_{|x| \to \infty} W(x) = +\infty, \quad \sup_{\alpha} L_\alpha V(x) \leq -W(x) \quad \text{if} \quad |x| \geq R \quad (1.7)
\]
for some $R > 0$. Assume also that for some numbers $C_V > 0, m \geq 1$ we have
\[
\sup_{\alpha} \left( |A^{-1/2}_\alpha (b_\alpha - \text{div} \, A_\alpha)|^2 + |A^{-1/2}_\alpha (\partial_\alpha b_\alpha - \partial_\alpha \text{div} \, A_\alpha)|^2 + |L_\alpha V|^2 \right) \leq C_V + C_V V^m W, \quad (1.8)
\]
Finally, assume that for some $\varepsilon < 1/(4m + 1)$ there is a ball outside of which
\[
\sup_{\alpha} \langle A_\alpha \nabla V, \nabla V \rangle \leq \varepsilon VW. \quad (1.9)
\]
Then $\partial_\alpha \varrho_\alpha$ exists and for each $\alpha \in (0, 1)$ satisfies the equation
\[
L^*_\alpha \partial_\alpha \varrho_\alpha = \text{div} \, (B_\alpha \varrho_\alpha - R_\alpha \varrho_\alpha - S_\alpha \nabla \varrho_\alpha). \quad (1.10)
\]
In addition, the mapping $\alpha \mapsto \varrho_\alpha$ with values in $L^1(\mathbb{R}^d)$ is differentiable.

Finally, if the diffusion matrix $A$ does not depend on $\alpha$, then (1.8) can be replaced by the simpler condition $\sup_{\alpha} (|A^{-1/2}_\alpha \partial_\alpha b_\alpha|^2 + |L_\alpha V|) \leq C_V + C_V V^m W$.

Thus, the theorem employs seven conditions (1.3)–(1.9) (or four global conditions (1.6)–(1.9) once we fix our local assumptions), but if $A_\alpha = 1$, $|b_\alpha|$ and $|\partial_\alpha b_\alpha|$ have polynomial bounds, then, by taking $V(x) = |x|^2$, it suffices to have only one condition that
\[
\lim_{|x| \to \infty} \sup_{\alpha} \langle b_\alpha(x), x \rangle = -\infty.
\]
Let us briefly comment on the hypotheses of this theorem.

**Remark 1.3.** (i) As explained above, condition (1.7) ensures the existence and uniqueness of probability solutions to (1.2) for each \( \alpha \). It also ensures the uniform boundedness of the integrals of \( W \) with respect to the measures \( \mu_\alpha \); moreover, in Lemma 2.2 we shall see that for each \( k < 4m + 1 \) the integrals of \( V^k W \) against \( \mu_\alpha \) are uniformly bounded. It is worth noting that, as shown in [11], the existence of a certain Lyapunov function of class \( W^{d,2}_\text{loc}(\mathbb{R}^d) \) is necessary for the existence of a probability solution \( \mu \) to (1.1) such that \( |a^{ij}(x)|/(1 + |x|^2), |b^i(x)|/(1 + |x|) \) are \( \mu \)-integrable.

(ii) Note also that if \( A \) is constant (independent of \( \alpha \)) and nondegenerate, then (1.4) and (1.6) are fulfilled (along with the first condition in (1.3)) and \( R_\alpha = S_\alpha = 0 \).

(iii) If \( A_\alpha \) is Lipschitzian in \( \alpha \), then (1.8) implicitly yields that \( b_\alpha \) is locally bounded outside of some ball, since, on every bounded set where \( \sup_\alpha L_\alpha V \leq -W \), the right-hand side of (1.8) is dominated by \( C + C \sup_\alpha |b_\alpha| \), while the left-hand side dominates a multiple of \( \sup_\alpha |b_\alpha|^2 \). However, the last assertion of the theorem allows locally unbounded drifts in the case of the diffusion matrix independent of \( \alpha \).

(iv) It follows from (1.4) that (1.8) is ensured by the estimate

\[
\sup_\alpha \left( |b_\alpha - \text{div} A_\alpha|^2 + |\partial_\alpha b_\alpha - \partial_\alpha \text{div} A_\alpha|^2 + |L_\alpha V|^2 \right) \leq C_V + C_V V^m W.
\]

However, for growing diffusion coefficients the operators \( A_\alpha^{-1/2} \) in (1.8) can help. Certainly, for uniformly bounded \( A_\alpha \) both estimates are equivalent.

Let us briefly explain the idea of our proof. Given a sequence \( h_k \to 0 \), we consider the differences \( \delta_k \varrho = (\varrho_\alpha - \varrho_{\alpha-h_k})/h_k \) and observe that they satisfy non-homogeneous equations

\[
L_\alpha^* \delta_k \varrho = \text{div} F_k
\]

with certain vector fields \( F_k \). It would be nice to obtain some uniform bounds on these solutions and their appropriate convergence. It turns out that our rather general assumptions about the coefficients do not allow to justify this procedure directly (at least, we have not managed to do this), which leads to an additional technical step at which the above plan is realized for less general coefficients. However, an appropriate approximation brings our proof to the end. This plan requires a preliminary study of the above non-homogeneous equation, which has already been investigated in [5], however, here we obtain new existence results for this equation along with certain a priori estimates that can be useful for other purposes.

Immediate examples are cases with uniformly elliptic diffusion matrices and polynomial or exponential bounds on the drift coefficients possessing a sufficient dissipativity. In these examples, rather technical conditions (1.8) and (1.9) are easily verified.

**Corollary 1.4.** Suppose that \( A_\alpha \), \( A_\alpha^{-1} \) and \( \partial_\alpha A_\alpha \) are uniformly bounded, (1.5) holds and that

\[
|\partial_{x_k} a_\alpha^{ij}(x)| + |\partial_\alpha \partial_{x_k} a_\alpha^{ij}(x)| + |b_\alpha^i(x)| + |\partial_\alpha b_\alpha^i(x)| \leq C + C|x|^k \quad \forall \alpha, \ x
\]

for some constants \( C \) and \( k \). Assume also that

\[
\lim_{|x| \to \infty} \sup_\alpha \langle b_\alpha(x), x \rangle = -\infty.
\]

Then \( \varrho_\alpha(x) \) is differentiable in \( \alpha \) and \( \partial_\alpha \varrho_\alpha(x) \) satisfies the equation indicated in the theorem.
Proof. Let us take $V(x) = |x|^2$ and $W(x) = -\sup_\alpha \langle x, b_\alpha(x) \rangle$. Then

$$L_\alpha V(x) = 2\text{trace } A_\alpha(x) + 2\langle x, b_\alpha(x) \rangle \leq -W(x)$$

outside of some ball. Clearly, for each $\varepsilon > 0$ outside of some ball we also have

$$\langle A_\alpha(x) \nabla V(x), \nabla V(x) \rangle = 4\langle A_\alpha(x)x, x \rangle \leq \varepsilon |x|^2 W(x).$$

In addition, there is a number $C_1$ such that

$$|L_\alpha V(x)| \leq C_1 + C_1 |x|^{2k+1}.$$ 

Therefore, all hypotheses of the theorem are satisfied (with $m = k + 1/2$). \qed

**Corollary 1.5.** Suppose that the operator norms of $A_\alpha$, $A_\alpha^{-1}$ and $\partial_\alpha A_\alpha$ are uniformly bounded, condition (1.5) holds and that

$$|\partial_\alpha a_\alpha^j(x)| + |\partial_\alpha \partial_\alpha a_\alpha^j(x)| + |b_\alpha^j(x)| + |\partial_\alpha b_\alpha^j(x)| \leq C \exp(q|x|^\beta) \quad \forall x, \alpha$$

for some positive numbers $C$, $q$, and $\beta$. Assume also that there is a number

$$\gamma > (9 \sup_{\alpha,x} \|A_\alpha(x)\| + 1/4)q\beta$$

such that outside of some ball we have

$$\sup_{\alpha} \langle b_\alpha(x), x \rangle \leq -\gamma |x|^3.$$ 

Then $g_\alpha(x)$ is differentiable in $\alpha$ and $\partial_\alpha g_\alpha(x)$ satisfies the equation indicated in the theorem.

**Proof.** Let us take $V(x) = \exp(q|x|^2)$, $s = \beta/2$, $c = \sup_{\alpha,x} \|A_\alpha(x)\|$. We have $V(x) = f(V_0(x))$, where $V_0(x) = |x|^2$, $f(u) = \exp(qu^2)$. Hence

$$f'(u) = qsu^{s-1}f(u), \quad f''(u) = qs(s-1)u^{s-2}f(u) + q^2s^2u^{2s-2}f(u),$$

which gives the equality

$$L_\alpha V(x) = qs\langle x, x \rangle^{s-1}V(x)L_\alpha V_0(x) + 4\left(4s(s-1)\langle x, x \rangle^{s-2}V(x) + q^2s^2\langle x, x \rangle^{2s-2}V(x)\right)\langle A_\alpha(x)x, x \rangle$$

$$= qs\langle x, x \rangle^{s-1}V(x)\left(L_\alpha V_0(x) + 4(s-1)\langle A_\alpha(x)x/|x|, x/|x| \rangle + qs\langle x, x \rangle^s\right).$$

Therefore, once $qs < 2\gamma$, the right-hand side is dominated outside of some ball by the function

$$W(x) := -\kappa \langle x, x \rangle^{2s-1}V(x) = -\kappa \langle x, x \rangle^{\beta-1}V(x),$$

where $\kappa = qs(2\gamma - qs)$. On the other hand, for each $\delta > 0$ there is $C_\delta > 0$ such that

$$|L_\alpha V(x)| \leq C_\delta + C_\delta V^{2+\delta}(x),$$

since $|L_\alpha V_0(x)| \leq 2\text{trace } A_\alpha(x) + 2C|x|V(x)$. Finally,

$$|A_\alpha^{1/2} \nabla V(x)|^2 \leq 4q^2s^2|x|^{2s-2}V(x) \leq \varepsilon V(x)W(x)$$

outside of a sufficiently large ball depending on a given $\varepsilon < (4m+1)^{-1}$, where $m = 2 + \delta$ and $\delta > 0$ is small enough so that $4qs < \varepsilon(2\gamma - qs)$; such a choice is possible, since $4qs < (2\gamma - qs)/9$ due to the estimate $\gamma > (9c + 1/4)q\beta$. \qed

**Example 1.6.** Let $A = I$, $b(x) = -x + h_\alpha(x)$, where $\sup_{\alpha,x} |h_\alpha(x)| < \infty$, $h_\alpha(x)$ is continuously differentiable in $\alpha$, and $|\nabla h_\alpha(x)| \leq C \exp(q|x|^2)$, $q < 1/20$. Then probability solutions $\mu_\alpha$ to the corresponding equations (1.2) exist, are unique and have densities $g_\alpha$ differentiable in $\alpha$. 
Example 1.7. (The case considered in [18] and [23].) Let the coefficients $a^{ij}_\alpha(x)$ and $b^i_\alpha(x)$ be of class $C^1_\beta$ in both variables, let $A^{-1}_\alpha(x)$ be uniformly bounded, and let

$$\sup_\alpha (b^i_\alpha(x), x) \to -\infty \quad \text{as} \ |x| \to \infty.$$ 

Then $g_\alpha(x)$ is continuously differentiable in both variables.

Note that in applications of these results to stationary distributions of diffusions governed by stochastic equations $d\xi_{\alpha,t} = \sigma_\alpha(\xi_{\alpha,t})dw_t + b_\alpha(\xi_{\alpha,t})dt$ the hypotheses must be checked for the matrices $A_\alpha = \sigma_\alpha \sigma_\alpha^*/2$.

Let us consider examples showing that certain additional assumptions, besides smoothness of the coefficients, are needed to guarantee even the continuity of densities with respect to the parameter.

Example 1.8. One can find a bounded function $b_\alpha(x)$, $(x, \alpha) \in \mathbb{R} \times \mathbb{R}$, of class $C^\infty$ in both variables such that the integral

$$J_\alpha = \int_{-\infty}^{+\infty} \exp \int_0^x b_\alpha(y) \, dy \, dx$$

exists, but is not continuous at $\alpha = 0$. It is not difficult to give explicit examples of such functions; it suffices to take a positive integrable smooth function $g$ such that $g'/g$ is bounded (say, $(1 + x^2)^{-1}$) and set $g(\alpha, x) = g(x) + \alpha g(ax)$; in this case the integral in $x$ is not continuous in $\alpha$ at the origin. Then the probability density

$$g_\alpha(x) = J_\alpha^{-1} \exp \int_0^x b_\alpha(y) \, dy, \quad b_\alpha(x) = \partial_x g(\alpha, x)/g(\alpha, x),$$

satisfies the equation $g''_\alpha - (b_\alpha g)_\alpha = 0$, but $g_\alpha(x)$ is discontinuous in $\alpha$ at $\alpha = 0$ for all $x$. A bit more involved example (see the next example) provides bounded $b_\alpha(x)$ that is Lipschitzian in $\alpha$. It is also worth noting that if we consider our equation with a parameter as an equation with an extra variable (or pass to a system of equations), then we obtain a degenerate equation.

Example 1.9. Let us give an explicit example (suggested by I.S. Yaroslavtsev) of a uniformly bounded function $b_\alpha(x)$ with bounded $\partial_\alpha b_\alpha(x)$ such that the probability solution $g_\alpha(x)$ to the corresponding equation $J_\alpha^*(g_\alpha dx) = 0$ is not continuous in $\alpha$.

Set $b_\alpha(x) = -x - 1$ if $x < 0$, $\alpha \in (-1, 1)$ and define $b_\alpha(x)$ for $x \geq 0$, $\alpha \in (-1, 1)$ as follows. Let $b_0(x) = -(x+1)^{-1/2}$. We now construct three functions $\varphi_1$, $\varphi_2$ and $\varphi_3$ on $\mathbb{R}^2$.

Set $\varphi_1(x) = 2(x+1)^{-1/2}$. Next, on the domain $\alpha \leq \varphi_1(x)$ we set $b_\alpha(x) = -(x+1)^{-1/2} + \alpha$.

Clearly, on the curve $(x, \varphi_1(x))$ our function equals $(x+1)^{-1/2}$. Next, we take a decreasing function $\varphi_2$ on $(0, +\infty)$ such that for all $\alpha > 0$

$$\int_{0}^{\varphi_2^{-1}(\alpha)} \exp \int_0^x b_\alpha(y) \, dy \, dx = 2 + \int_{0}^{\infty} \exp \int_0^x b_\alpha(y) \, dy \, dx.$$ 

Note that $\varphi_2^{-1}(\alpha)$ is determined by the previously defined values of $b$. The values $\varphi_2^{-1}(\alpha)$ are well-defined. Indeed, once $\alpha > 0$ is fixed, we have $b_\alpha(x) = -(x+1)^{-1/2} + \alpha$ if $x < 4\alpha^{-2} - 1$, $b_\alpha(x) = (x+1)^{-1/2}$ if $4\alpha^{-2} - 1 \leq x$, hence the integral in the left-hand side of the expression above taken over $[0, +\infty)$ is infinite and this integral equals $2 - \alpha < 2$ when taken over $[0, 4\alpha^{-2} - 1]$, which enables one to find suitable $\varphi_2^{-1}(\alpha) > 4\alpha^{-2} - 1$. Note that $\varphi_2$ is continuous decreasing and $\varphi_2(x) \to 0$ as $x \to \infty$.

On the domain $\varphi_1(x) < \alpha \leq \varphi_2(x)$ we set $b_\alpha(x) = (x+1)^{-1/2}$. Finally, we take

$$\varphi_3(x) = \varphi_2(x) + 2(x+1)^{-1/2}.$$
and on the domain $\varphi_2(x) < \alpha < \varphi_3(x)$ we set $b_\alpha(x) = (x+1)^{-1/2} - (\alpha - \varphi_2(x))$ and on the domain $\alpha \geq \varphi_3(x)$ we set $b_\alpha(x) = -(x+1)^{-1/2}$. It is clear that $|b_\alpha(x)| \leq 1$, $b$ is continuous and $|\partial_\alpha b_\alpha(x)| \leq 1$, more precisely, in the interiors of the domains bounded by the three curves defined above $\partial_\alpha b_\alpha(x)$ is 1, 0 and $-1$ and again 0, respectively. The corresponding solution $g_\alpha(x)$ is discontinuous at $\alpha = 0$. This property can be retained by smoothing $b$ and making it differentiable in $\alpha$ everywhere with a uniformly bounded derivative in $\alpha$.

It is instructive to see which conditions of the theorem cannot be ensured in this example. Here Corollary 1.4 almost applies with $V(x) = x^2$ and for any fixed $\alpha$ we have $b_\alpha(x) < x^{-1/2}/2$ outside of some interval, but there is no uniformity in $\alpha$.

2. Auxiliary results

A useful fact employed below is that in the case where $LV(x) \leq -1$ outside of a ball and $\mu$ is a probability solution to the equation $L^\ast \mu = 0$, we have $|LV| \in L^1(\mu)$. Actually, the following is true (see [6]): if

$$LV \leq \Psi - \Phi,$$

where $\Psi$ and $\Phi$ are Borel functions such that $\Psi \in L^1(\mu)$ and $\Phi \geq 0$, then

$$\int_{\mathbb{R}^d} \Phi d\mu \leq \int_{\mathbb{R}^d} \Psi d\mu. \quad (2.1)$$

It will be important below that if a function $u$ on a domain $\Omega$ satisfies the equation

$$\partial_{x_i}(a^{ij}\partial_{x_j}u) + \partial_{x_i}(b^i u) = \text{div} G,$$

where $G = (G^i)$ is a measurable vector field and

$$\|a^{ij}\|_{W^{p,1}(\Omega)} + \|b^i\|_{L^p(\Omega)} + \sup_{\Omega} |\text{det}(a^{ij})|^{-1} \leq K$$

with some $p > d$, then for every ball $U$ with compact closure in $\Omega$ there is a constant $C(p, K, U, \Omega)$ that depends only on $p, K, U$ and the distance from $U$ to the boundary of $\Omega$ such that

$$\|u\|_{W^{p,1}(U)} \leq C(p, K, U, \Omega) \left( \|u\|_{L^1(\Omega)} + \|G\|_{L^p(\Omega)} \right). \quad (2.2)$$

The Sobolev embedding theorem yields also a bound

$$\sup_U |u| \leq C'(p, K, U, \Omega) \left( \|u\|_{L^1(\Omega)} + \|G\|_{L^p(\Omega)} \right), \quad (2.3)$$

where $C'$ depends on the same objects as $C$. In particular, having a family of solutions to different equations with a common bound $K$, we obtain the uniform boundedness in the Sobolev norm on any inner ball, provided we have their uniform boundedness in $L^1$ on a slightly larger ball along with a common bound for the $L^p$-norms of the right-hand sides on that larger ball. A detailed proof can be found, e.g., in [20].

If $G = 0$ and $u \geq 0$ in $\Omega$, then, according to Harnack’s inequality,

$$\sup_{x \in U} u(x) \leq H(K, U, \Omega) \inf_{x \in U} u(x), \quad (2.4)$$

where the number $H(K, U, \Omega)$ depends only on $p, K, U$ and the distance from $U$ to the boundary of $\Omega$.

Lemma 2.1. Suppose that (1.3) holds, the family $\{\mu_\alpha\}$ is uniformly tight, and, for each closed ball $U$, we have $\inf_{\alpha, x \in U} A_\alpha(x) > 0$ and the mappings $\alpha \mapsto a_\alpha^{ij}|_U$ and $\alpha \mapsto b_\alpha^i|_U$
with values in $L^1(U)$ are continuous. Then, for every ball $U$, the continuous versions of the densities $\varrho_\alpha$ satisfy the estimate
\[
\inf \min_{\alpha} \varrho_\alpha(x) \geq m(U) > 0,
\]
where $m(U)$ does not depend on $\alpha$.

Proof. Suppose that there is a sequence $\alpha_n \to \alpha$ in $[0, 1]$ for which $\min_{x \in U} \varrho_\alpha(x) \to 0$. It follows by (2.4) and (2.2) that passing to a subsequence we can assume that the functions $\varrho_{\alpha_n}$ converge locally uniformly to some function $\varrho$. By the uniform tightness, we have also convergence in $L^1(\mathbb{R})$ and $\varrho$ is a probability density. It is readily seen that $L^*_\alpha \varrho = 0$, since $L_{\alpha_n} \varphi \to L_\alpha \varphi$ in $L^1(\mathbb{R}^d)$ for each smooth $\varphi$ with compact support. Hence $\varrho$ is positive by Harnack’s inequality, which leads to a contradiction. \qed

We need also the following a priori estimate for a probability solution $\mu$ of the equation $L^* \mu = 0$.

Lemma 2.2. Let $k \geq 1$. Suppose that
\[
LV \leq -W \quad \text{and} \quad (A\nabla V, \nabla V) \leq \varepsilon VW
\]
outside of some compact set $S_0$, where $0 \leq \varepsilon < k^{-1}$. Then
\[
\int_{\mathbb{R}^d \setminus S_0} V^{k+1} W \, d\mu \leq (k+1)^{-1}(1-k\varepsilon)^{-1} \int_{S_0} |LV^{k+1}| \, d\mu. \tag{2.5}
\]

Proof. Let us consider the function $V_0 = V^{k+1}$. We have
\[
LV_0 = (k+1)V^k LV + k(k+1)V^{k-1}(A\nabla V, \nabla V) \leq -(k+1)(1-k\varepsilon)V^k W
\]
outside of $S_0$. Hence we can apply estimate (2.1) with functions $\Psi = |LV^{k+1}I_{S_0}$ and $\Phi = (k+1)(1-k\varepsilon)V^k W I_{\mathbb{R}^d \setminus S_0}$.

Once a probability solution $\mu$ to the equation $L^* \mu = 0$ exists, it satisfies (under our local assumptions about $A$ and $b$, see (1.3) and (1.4)) the following estimate (see [5]):
\[
\int_{\mathbb{R}^d} \left| \frac{A^{1/2} \nabla \varrho}{\varrho} \right|^2 \, d\mu \leq \int_{\mathbb{R}^d} |A^{-1/2}(b - \text{div } A)|^2 \, d\mu, \quad \text{div } A := (\partial_{x_i} a^{ij}, \ldots, \partial_{x_i} a^{ij}), \tag{2.6}
\]
provided the right-hand side is finite and
\[
\liminf_{r \to \infty} \int_{r \leq |x| \leq 2r} \left| r^{-2}|a^{ij}| + r^{-1}|\partial_{x_k} a^{ij}| \right| \, d\mu = 0. \tag{2.7}
\]
The last assumption is fulfilled, e.g., if the mapping $A$ is Lipschitzian or, more generally, if the functions $|a^{ij}(x)|/(1+|x|^2)$ and $|\partial_{x_k} a^{ij}(x)|/(1+|x|)$ are $\mu$-integrable. In particular, this condition is satisfied if $a^{ij}$ and $\partial_{x_k} a^{ij}$ are $\mu$-integrable on the whole space. However, it is not known whether (2.6) is satisfied for all solutions without the extra assumption (2.7). For this reason, we show in the next lemma that in the presence of a suitable Lyapunov function even without (2.7) there is a unique probability solution satisfying (2.6).

Lemma 2.3. Suppose that the coefficients $a^{ij}$ and $b^i$ satisfy our local assumptions (see (1.3) and (1.4)) and there is a function $V \in C^2(\mathbb{R}^d)$ such that $\lim_{|x| \to \infty} V(x) = +\infty$ and outside of some ball
\[
LV(x) \leq -1, \quad \psi(|x|)|A^{-1/2}(b - \text{div } A)(x)|^2 \leq |LV(x)|,
\]
where \( \psi \) is a locally bounded Borel function on \([0, +\infty)\) with \(\lim_{t \to +\infty} \psi(t) = +\infty\). Then there is a unique probability solution \( \mu \) to the equation \( L^* \mu = 0 \) such that (2.6) holds provided the right-hand side is finite.

**Proof.** It is known that for almost every \( t \in \mathbb{R} \) the compact set \( U_t = \{ V \leq t \} \) has boundary of finite perimeter (see [15, Section 5.5] and [24, Chapter 5]); certainly, if we had \( V \in C^d(\mathbb{R}^d) \), then by Sard’s theorem \( V^{-1}(t) \) would be a \( C^1 \)-surface for almost each \( t \), but we do not assume such a regularity of \( V \). Hence there is an increasing sequence \( t_n \to +\infty \) of points with this property. Set \( U_n = U_{t_n} \). Let

\[
 f^i = b^i - \partial_{x^i} a^i, \quad h = \partial_{x^i} f^i.
\]

According to [21] (see also [22]), for each \( n \), there is a solution \( w_n \in W^{2,1}_0(U_n) \) to the Dirichlet problem

\[
 \partial_{x^i}(a^{ij}\partial_{x^j} w_n) - \partial_{x^i}(f^i w_n) = h,
\]

where \( W^{2,1}_0(U_n) \) is the closure of \( C_0^\infty(U_n) \) in \( W^{2,1}(U_n) \). Therefore, the function

\[
 \varrho_n := w_n + 1
\]

satisfies the homogeneous equation \( L^* \varrho_n = 0 \) in \( U_n \) and the boundary condition \( \varrho_n|_{\partial U_n} = 1 \) in the sense that \( \varrho_n - 1 \in W^{2,1}_0(U_n) \). Let us observe that it follows from [22] (Theorem 2 applies with \( \gamma = 0 \)) that \( \varrho_n \geq 0 \) and consequently by Harnack’s inequality \( \varrho_n > 0 \) in \( U_n \). Indeed, the hypotheses of [22] are satisfied due to our choice of \( t_n \) which makes possible to use the Gauss–Green formula for \( U_n \) (see [24, Section 5.8]). Let us normalize our solutions in such a way that \( \varrho_n \) becomes a probability density on \( U_n \) for all \( n \). Then, due to the existence of a Lyapunov function, by a standard procedure (see, e.g., [6] and [7]), one can select a subsequence in \( \{ \varrho_n \} \) that locally uniformly converges to a probability solution \( \varrho \) of the equation \( L^* \varrho = 0 \). It is also known that in this situation there is a number \( M \) such that

\[
 \int_{U_n} |LV|\varrho_n \, dx \leq M \quad \forall \, n. \tag{2.8}
\]

This can be derived from (2.1) applied to \( \Psi = |LV|I_{U_n} \) and \( \Phi = |LV|I_{\mathbb{R}^d \setminus U_n} \) for a suitable number \( n_1 \).

Finally, we verify (2.6) for this particular solution. To this end, we multiply the equation for \( \varrho_n \) by \( \log \varrho_n - c_n \), where \( c_n \) is the constant boundary value of \( \varrho_n \) (obtained after normalization, so that it need not be 1 anymore), and integrate by parts (which is possible due to the above choice of \( U_n \)) obtaining the equality

\[
 \int_{U_n} \langle A \nabla \varrho_n, \nabla \varrho_n \rangle \, dx = \int_{U_n} \langle b - \text{div} \, A, \nabla \varrho_n \rangle \, dx = \int_{U_n} \langle A^{-1/2}(b - \text{div} \, A), A^{1/2} \nabla \varrho_n \rangle \, dx.
\]

Applying the Cauchy inequality to the right-hand side, we arrive at the uniform estimate

\[
 \int_{U_n} \left| A^{1/2} \nabla \varrho_n \right|^2 \varrho_n \, dx \leq \int_{U_n} |A^{-1/2}(b - \text{div} \, A)|^2 \varrho_n \, dx. \tag{2.9}
\]

Let us show that

\[
 \lim_{n \to \infty} \int_{U_n} |A^{-1/2}(b - \text{div} \, A)|^2 \varrho_n \, dx = \int_{\mathbb{R}^d} |A^{-1/2}(b - \text{div} \, A)|^2 \varrho \, dx. \tag{2.10}
\]

Let \( \varepsilon > 0 \). Take \( R > 0 \) such that \( |A^{-1/2}(b - \text{div} \, A)(x)|^2 \leq \varepsilon |LV(x)| \) whenever \( |x| > R \). Then

\[
 \int_{|x| > R} |A^{-1/2}(b - \text{div} \, A)(x)|^2 \varrho_n(x) \, dx \leq \varepsilon M.
\]
By Fatou’s theorem the same is true for $\varrho$ in place of $\varrho_n$. Since $\varrho_n \to \varrho$ locally uniformly, we obtain equality (2.10). For every smooth compactly supported vector field $v$ we have

$$\int_{\mathbb{R}^d} \left< \frac{\nabla \varrho}{\varrho}, v \right> \varrho dx = \lim_{n \to \infty} \int_{\mathcal{U}_n} \left< \frac{\nabla \varrho_n}{\varrho_n}, v \right> \varrho_n dx,$$

since the left-hand side is the integral of $-\varrho \text{div} v$, which is the limit of the integrals of $-\varrho_n \text{div} v$. Combined with (2.9) and (2.10) this yields (2.6). Finally, as noted above, the uniqueness of a probability solution follows from the estimate $LV \leq -1$ outside of a ball. $\square$

Having an operator $L$ satisfying the same local assumptions as $L_{\alpha}$ (see (1.3) and (1.4)), let us consider the equation

$$L^* w = \text{div} (\varrho F), \quad (2.11)$$

where $\varrho$ is a probability solution of the equation $L^* \varrho = 0$ and $F$ is a Borel vector field such that $|A^{-1/2}F|^2 \varrho \in L^1(\mathbb{R}^d)$. We arrive at this equation by formally differentiating (1.2) in $\alpha$.

Writing $w = v \varrho$, we obtain the following equation on $v$:

$$\text{div} (\varrho A \nabla v) + \text{div} (vb_0) = \text{div} (\varrho F), \quad b_0^i = \partial_{x_j} (a^i_{ij} \varrho) - b^i \varrho. \quad (2.12)$$

Let us observe that

$$\text{div} (vb_0) = \langle \nabla v, b_0 \rangle.$$

Indeed,

$$\text{div} b_0 = 0$$

due to the equality $\partial_{x_i} \partial_{x_j} (a^i_{ij} \varrho) - \partial_{x_i} (b^i \varrho) = 0$. Therefore, (2.12) can be rewritten as

$$\text{div} (\varrho A \nabla v) + \langle \nabla v, b_0 \rangle = \text{div} (\varrho F). \quad (2.13)$$

In the next section we use the results of this section on equation (2.11) in the situation where $\varrho = \varrho_\alpha$ and

$$F = B_\alpha - R_\alpha - S_\alpha \frac{\nabla \varrho_\alpha}{\varrho_\alpha},$$

$$B_\alpha = (\partial_\alpha b_1^i, \ldots, \partial_\alpha b_d^i), \quad R_\alpha = (R_1^i, \ldots, R_d^i), \quad R_\alpha^i = \partial_{x_j} (a^i_{ij}), \quad S_\alpha = (\partial_{x_j} a^i_{ij}).$$

Note that vector fields $F$ of such a form appear in the equations satisfied by the derivatives $\partial_{x_i} \varrho_\alpha$.

**Proposition 2.4.** Suppose that $v$ is a solution to (2.13) on the domain $\Omega = \{ V < R \}$, where $V \in C^2(\Omega)$ is a nonnegative function such that there exist a measurable function $W \geq 1$, a measurable function $\Psi \geq 0$ and a number $R_0 \in (0, R)$ such that

$$LV(x) \leq \Psi(x) - W(x) \quad \text{if} \ V(x) \geq R_0.$$

Then

$$\int_{R_0 < V < R} v^2 W \varrho dx \leq 2 \int_{V < R_0} v^2 |LV| \varrho dx + 6R \int_{V < R} |A^{1/2} \nabla v|^2 \varrho dx$$

$$+ 2 \int_{R_0 < V < R} v^2 \Psi \varrho dx + 2R \int_{V < R} |A^{-1/2} F|^2 \varrho dx$$

$$+ 4 \int_{V < R} |A^{-1/2} F|^2 |A^{1/2} \nabla V|^2 W^{-1} \varrho dx. \quad (2.14)$$
If $|A^{1/2}\nabla V|^2 \leq C_V VW + C_V$ with some number $C_V \geq 1$, then

$$
\int_{R_0 < V < R} v^2 W \varrho \, dx \leq 2 \int_{V < R_0} v^2 |L V| \varrho \, dx + 6R \int_{V < R} |A^{1/2} \nabla v|^2 \varrho \, dx + 2 \int_{R_0 < V < R} v^2 \Psi \varrho \, dx + C_V (6R + 1) \int_{V < R} |A^{-1/2} F|^2 \varrho \, dx. \quad (2.15)
$$

**Proof.** We multiply equation (2.13) by $v\psi$, where $\psi \in C_0^\infty(\Omega)$, integrate by parts (which is possible due to our assumptions about the coefficients yielding the Sobolev regularity of all solutions) and obtain the equality

$$
\int_{\Omega} |A^{1/2} \nabla v|^2 \psi \varrho \, dx = \frac{1}{2} \int_{\Omega} v^2 L v \varrho \, dx + \int_{\Omega} \langle F, \nabla \psi \rangle v \varrho \, dx + \int_{\Omega} \langle F, \nabla v \rangle \psi \varrho \, dx, \quad (2.16)
$$

in which we used the intermediate equalities

$$
\int_{\Omega} \text{div} (v \varrho \nabla v) \psi \varrho \, dx = - \int_{\Omega} \langle A \nabla v, \nabla \psi \rangle \psi \varrho \, dx - \int_{\Omega} \langle A \varphi, \nabla \psi \rangle \psi \varrho \, dx
$$

$$
= - \int_{\Omega} \langle A \nabla v, v \nabla \psi \rangle \psi \varrho \, dx + \int_{\Omega} \frac{v^2}{2} \text{div} (A \nabla \psi) \varrho \, dx + \int_{\Omega} \frac{v^2}{2} \langle A \nabla \psi, \nabla \varrho \rangle \, dx
$$

$$
= - \int_{\Omega} \langle A \nabla v, v \nabla \psi \rangle \psi \varrho \, dx + \int_{\Omega} \frac{v^2}{2} \text{div} (A \nabla \psi) \varrho \, dx + \int_{\Omega} \frac{v^2}{2} \langle \nabla \psi, A \nabla \varrho \rangle \, dx,
$$

where in the latter identity we used the condition that $\text{div} b_0 = 0$. Finally,

$$
\text{div} (A \nabla \psi) = \langle b_0, \nabla \psi \rangle = L \psi.
$$

Let $R_0 < N < R_1 < R$. Let us take $\psi = \zeta_N(V) - R_1$, where $\zeta_N \in C^2(\mathbb{R})$, $\zeta_N(t) = t$ if $t \leq N$, $\zeta_N(t) = (R_1 + N)/2$ if $t \geq R_1$, and $0 \leq \zeta_N \leq 1$, $\zeta_N'' \leq 0$. Note that the function $\psi$ belongs to the class $C^2(\mathbb{R}^d)$ and vanishes if $V \geq R_1$. Taking into account that $\psi \leq 0$, $\nabla \psi = \zeta_N'(V) \nabla V$, $L \psi = LV$ on $\Omega_0$ and that outside of

$$
\Omega_0 = \{ V < R_0 \}
$$

we have

$$
L \psi = \zeta_N'(V) LV + \zeta_N''(V) \langle A \nabla V, \nabla V \rangle \leq \zeta_N'(V) LV \leq \zeta_N'(V) \Psi - \zeta_N'(V) W,
$$

we conclude that (2.16) yields the estimate

$$
\int_{\Omega_0} v^2 \zeta_N'(V) W \varrho \, dx \leq \int_{\Omega_0} v^2 |L V| \varrho \, dx + 2 \int_{\Omega} |A^{1/2} \nabla v|^2 \psi |\varrho| \varrho \, dx
$$

$$
+ \int_{\Omega_0} v^2 \Psi \varrho \, dx + 2 \int_{\Omega} \langle F, \nabla \psi \rangle v \varrho \, dx + 2 \int_{\Omega} \langle F, \nabla v \rangle \psi \varrho \, dx.
$$

Since

$$
2 \int_{\Omega} \langle F, \nabla \psi \rangle v \varrho \, dx \leq \int_{\Omega} \zeta_N(V) \left[ \frac{1}{2} v^2 W + 2W^{-1} \langle F, \nabla V \rangle^2 \right] \varrho \, dx,
$$

$$
2 \int_{\Omega} \langle F, \nabla v \rangle \psi \varrho \, dx \leq \int_{\Omega} \left[ |A^{1/2} \nabla v|^2 |\psi| + |A^{-1/2} F|^2 |\psi| \right] \varrho \, dx,
$$

we have

$$
\int_{\Omega_0} v^2 \zeta_N'(V) W \varrho \, dx \leq \int_{\Omega} v^2 |L V| \varrho \, dx + 2 \int_{\Omega} |A^{1/2} \nabla v|^2 |\psi| |\varrho| \varrho \, dx
$$

$$
+ \int_{\Omega} v^2 \Psi \varrho \, dx + 2 \int_{\Omega} \langle F, \nabla \psi \rangle v \varrho \, dx + 2 \int_{\Omega} \langle F, \nabla v \rangle \psi \varrho \, dx.
$$
we arrive at the estimate
\[
\int_{\Omega \setminus \Omega_0} v^2 \zeta_N(V) W \varrho \, dx \leq 2 \int_{\Omega_0} v^2 |LV| \varrho \, dx + 6 \int_{\Omega} |A^{1/2} \nabla v| |\psi| \varrho \, dx
+ 2 \int_{\Omega \setminus \Omega_0} v^2 \Psi \varrho \, dx + 4 \int_{\Omega} |A^{-1/2} F|^2 |A^{1/2} \nabla V|^2 W^{-1} \varrho \, dx
+ 2 \int_{\Omega} |A^{-1/2} F|^2 |\psi| \varrho \, dx,
\]
which completes the proof by letting \( N \to R \), since \(|\psi| \leq R\).

\( \square \)

**Corollary 2.5.** Suppose that \( v \) is a solution to (2.13) on the domain \( \Omega = \{ V < R \} \), \( R \geq 1 \), where \( V \in C^2(\Omega) \) is a nonnegative function and there exist a measurable function \( W \geq 1 \) and a number \( R_0 \in (0, R) \) such that
\[
 LV(x) \leq -W(x) \quad \text{if} \ V(x) \geq R_0.
\]
If \( |A^{1/2} \nabla V|^2 \leq C_V + C_V W \) with some number \( C_V \geq 1 \), then for any \( k \geq 0 \) we have
\[
\int_{R_0 < V < R} v^{2k} W \varrho \, dx
\leq M(R_0) C_V R^{k+1} \left( \sup_{V < R_0} (\varrho v^2) + \int_{V < R} \left[ |A^{1/2} \nabla v|^2 + |A^{-1/2} F|^2 \right] \varrho \, dx \right), \tag{2.17}
\]
where the number \( M(R_0) \) is independent of \( v \) and depends only on \( R_0 \) and the bounds on the coefficients on \( \{ V < R_0 \} \).

In the formulation of the next proposition two numbers \( P(\Omega_0) \) and \( H_0 = H_0(\Omega_0) \) are employed. The first one depends only on the domain \( \Omega_0 = \{ V < R_0 \} \), where \( R_0 > 0 \) will be picked later. This is the number in the Poincaré inequality
\[
\int_{\Omega_0} \varphi^2 \, dx \leq P(\Omega_0) \int_{\Omega_0} |\nabla \varphi|^2 \, dx
\]
valid for every function \( \varphi \in W^{2,1}(\Omega_0) \) with zero integral over \( \Omega_0 \). There is also a refined version of this inequality: if \( S_0 \) is a fixed ball containing the closure of \( \Omega_0 \) (for the later use we assume also that \( \operatorname{dist}(\Omega_0, \partial S_0) = 1 \)), then
\[
\int_{S_0} \varphi^2 \, dx \leq P(S_0, \Omega_0) \int_{S_0} |\nabla \varphi|^2 \, dx
\]
for every function \( \varphi \in W^{2,1}(S_0) \) with zero integral over \( \Omega_0 \) (see [24, Theorem 4.4.2]).

The second number \( H_0 = H_0(\Omega_0, S_0) \) is Harnack’s constant for the operator \( L \) on the same ball \( S_0 \). With this constant one has (2.4) for every positive solution \( u \) of the equation \( L^* u = 0 \) on \( S_0 \), namely,
\[
\sup_{x \in U} u(x) \leq H_0 \inf_{x \in U} u(x) \quad \text{(2.18)}
\]
for each ball \( U \subset \Omega_0 \). This number depends only on \( S_0, \Omega_0 \), and the coefficients of \( L \) through the \( W^{p,1}(S_0) \)-norms of \( a^{ij} \), the \( L^p(S_0) \)-norms of \( b^i \), and \( \inf_{S_0} \det A \).

**Proposition 2.6.** Suppose that there exist a function \( V \in C^2(\mathbb{R}^d) \), a locally integrable function \( W \geq 1 \) and a number \( C_V \geq 1 \) such that
\[
\lim_{|x| \to +\infty} V(x) = +\infty, \quad \langle A \nabla V, \nabla V \rangle \leq C_V + C_V W
\]
and for some $R_0 > 0$ we have

$$LV(x) \leq -W(x) \quad \text{if} \quad x \not\in \Omega_0 := \{V < R_0\}.$$  

Let $q$ be the unique probability solution of the equation $L^*q = 0$. Assume also that

$$\int_{\mathbb{R}^d} |A^{-1/2}F|^2 q \, dx \leq M_F < \infty \quad (2.19)$$

and for some numbers $m \geq 1$ and $t > 1$

$$\int_{\mathbb{R}^d} V^{2m+1+t} W^t q \, dx \leq M_0 < \infty. \quad (2.20)$$

Then, there exists a solution $w$ of equation (2.11) with the following property:

$$\int_{\mathbb{R}^d} WV^m |w| \, dx \leq (M_*M_F)^{t/2} \frac{1}{t-1} + (M_*M_F)^{t/2} 2^{2m+1+t} M_0, \quad (2.21)$$

where $M_*$ is a number that depends only on the constants in (1.3) and (1.4) for a fixed ball $S_0$ containing the closure of $\Omega_0$, say, a fixed ball $S_0$ such that $\text{dist}(S_0, \Omega_0) = 1$, and also on the integral of $|W|$ over $\{V \leq 1\}$.

**Proof.** We seek for a solution $w$ of the form

$$w = vq,$$

where $v$ satisfies equation (2.13). Let $U_n = \{V < n\}, n > R_0$. Let $v_n$ be the solution to the Dirichlet problem

$$\text{div}(qA\nabla v_n) + \langle b_0, \nabla v_n \rangle = \text{div}(Fq), \quad v_n |_{\partial U_n} = 0.$$  

This solution exists due to our assumptions about the coefficients, see [21]. Multiplying the equation by $v_n$, integrating over $U_n$ and using the integration by parts formula we obtain the equality

$$-\int_{U_n} |A^{1/2} \nabla v_n|^2 q \, dx + \int_{U_n} v_n \langle b_0, \nabla v_n \rangle \, dx = -\int_{U_n} \langle v_n, F \rangle q \, dx,$$

where the second term on the left vanishes, since $\text{div} b_0 = 0$ and $v_n \nabla v_n = \nabla (v_n^2)/2$. The integrand on the right is estimated by $|A^{1/2} \nabla v_n|^2/2 + |A^{-1/2} F|^2/2$, which yields the estimate

$$\int_{U_n} |A^{1/2} \nabla v_n|^2 q \, dx \leq \int_{U_n} |A^{-1/2} F|^2 q \, dx.$$  

Therefore,

$$\int_{U_n} |A^{1/2} \nabla v_n|^2 q \, dx \leq M_F. \quad (2.22)$$

We now change the function $v_n$ (keeping the same notation) by subtracting its integral over the domain $\Omega_0 = \{V < R_0\}$, which yields a function satisfying the same equation (but not the boundary condition, of course) and having the zero integral over $\Omega_0$. Obviously, these new functions $v_n$ satisfy (2.22). The Poincaré inequality (see above) yields the bound

$$\int_{\Omega_0} v_n^2 q \, dx \leq \sup_{\Omega_0} q \int_{\Omega_0} v_n^2 \, dx \leq P(\Omega_0) \sup_{\Omega_0} q \int_{\Omega_0} |\nabla v_n|^2 \, dx \leq c(\Omega_0) P(\Omega_0) \sup_{\Omega_0} q (\inf_{\Omega_0} q)^{-1} \int_{\Omega_0} |A^{1/2} \nabla v_n|^2 q \, dx \leq c(\Omega_0) P(\Omega_0) H_0 M_F. \quad (2.23)$$
However, we need more: we need a bound on the integral of $|v_n|^2 (|LV| + 1)$ over $\Omega_0$. Since $|LV| + 1$ is integrable on $\Omega_0$, it suffices to have a uniform bound on $\sup_{\Omega_0} |v_n|$. The desired bound is ensured by (2.3), where we take $U = S_0$ (a ball whose interior contains the closure of $\Omega_0$) and $\Omega = S_1$ is the ball with the same center and the radius increased by 1. Again by the Poincaré inequality we obtain

$$
\|v_n\|_{L^2(S_1)}^2 \leq \|v_n\|_{L^2(S_1)}^2 |S_1| \leq C(S_1) P(S_1, \Omega_0) H_0 M F.
$$

So we have

$$
\sup_{\Omega_0} |v_n|^2 \leq M_1 M F, \quad M_1 = C'(p, K, S_0, S_1)^2 c(S_1) P(S_1, \Omega_0)^2 |S_1|,
$$

where the number $K$ is determined by (1.3) and (1.4) according to (2.3).

By the previous proposition (see (2.17)) we arrive at the following estimate for all $k \geq n$:

$$
\int_{U_n \setminus \Omega_0} v_k^2 W q \, dx \leq 2 \int_{\Omega_0} v_n^2 |LV| q \, dx + 6n M F + C_V (6n + 1) M F
$$

$$
\leq 2 M_1 M F \int_{\Omega_0} |LV| q \, dx + 6n M F + C_V (6n + 1) M F \leq M_2 M F n,
$$

where $M_2$ is a number determined by the regarded norms of the coefficients on the ball $S_1$, $\sup_{S_1} q$, $\inf_{S_1} \det A$, and also some universal constants (entering through the Poincaré, Sobolev, and Harnack inequalities). Increasing $M_2$ we can assume that

$$
\int_{U_n} v_k^2 W q \, dx \leq M_3 n \quad \forall n, k \geq n.
$$

(2.24)

It follows by (2.22), (2.23) and the Poincaré inequality that on every fixed ball $U$ the sequence of functions $v_n$ with $n \geq n(U)$ is bounded in the Sobolev norm of $W^{p,1}(U)$. Since these functions satisfy the elliptic equation whose coefficients satisfy the above mentioned conditions, we conclude by (2.2) that this sequence is bounded also in the Sobolev space $W^{p,1}(U)$, where $p = p(U) > d$, hence is uniformly bounded and contains a subsequence convergent uniformly on $U$ to some function $v$. Using the diagonal procedure we pick a subsequence convergent locally uniformly to a common function $v$ such that $v \in W^{p,1}(U)$ for every ball $U$ with the respective $p = p(U) > d$. It is also possible to ensure that on each ball the restrictions of $v_n$ converge to the restriction of $v$ weakly in the respective $W^{p,1}(U)$. Obviously, $v$ satisfies the desired equation on the whole space. By Fatou’s theorem and (2.24) we have

$$
\int_{U_n} v^2 W q \, dx \leq M_3 n \quad \forall n.
$$

(2.25)

We now show that $W^{m}v q$ is integrable on the whole space. For any $n > 1$, by the Cauchy inequality and (2.25) we have

$$
\int_{n-1 \leq V \leq n} W^{m} |v| q \, dx \leq n^m \int_{n-1 \leq V \leq n} W |v| q \, dx
$$

$$
\leq M_3^{1/2} n^{m+1/2} \left( \int_{n-1 \leq V \leq n} W q \, dx \right)^{1/2}
$$

$$
\leq M_3^{1/2} n^{-t} + M_3^{1/2} n^{2m+1+t} \int_{n-1 \leq V \leq n} W q \, dx
$$

$$
\leq M_3^{1/2} n^{-t} + M_3^{1/2} n^{2m+1+t} \int_{n-1 \leq V \leq n} W q \, dx.
$$
The integral of $W V^m |v| \varrho$ over $\{V \leq 1\}$ is dominated by the square root of $M_3$ multiplied by the integral of $W \varrho$ over $\{V \leq 1\}$. Therefore, increasing $M_3$, we arrive at the estimate

$$\int_{\mathbb{R}^d} W V^m |v| \varrho \, dx \leq M_3^{1/2} \frac{1}{t-1} + M_3^{1/2} 2^{m+1+t} \int_{\mathbb{R}^d} V^{2m+1+t} W \varrho \, dx,$$

which is the desired bound.

\[ \square \]

**Proposition 2.7.** If $F = 0$, then any solution $w$ of equation (2.11) satisfying the condition

$$\int_{\mathbb{R}^d} (|LV| + \langle A\nabla V, \nabla V \rangle) |w| \, dx < \infty$$

has the form $w = \lambda \varrho$, where $\lambda$ is a constant. Therefore, a solution to (2.11) in the class of functions satisfying the above condition is unique up to adding functions of the form $\lambda \varrho$.

**Proof.** It suffices to show that any solution $w$ of the homogeneous equation is proportional to $\varrho$, because $\varrho$ is a solution. Let $v = w/\varrho$. Then $v$ satisfies equation (2.13). Let $f$ be a smooth function on $[0, +\infty)$ and let $\psi \in C_0^2(\mathbb{R}^d)$. We have (see [3, Lemma 1])

$$\int_{\mathbb{R}^d} |\sqrt{A} \nabla v^+|^2 f''(v^+) \psi \varrho \, dx = \int_{\mathbb{R}^d} f(v^+) L \varrho \, dx - f'(0) \int_{\mathbb{R}^d} v^{-1} L \varrho \, dx,$$

where $v^+ = \max\{v, 0\}$ and $v^- = -\min\{v, 0\}$. Set $f(t) = (1 + t)^{-1}$ and $\psi_N = \varphi(V/N)$, where $\varphi \in C_0^\infty(\mathbb{R})$, $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ if $|x| \leq 1$, $|\varphi'| \leq 1$, $|\varphi''| \leq 1$. Then

$$L \psi_N = N^{-1} \varphi''(V/N) L V + N^{-2} \varphi''(V/N) \langle A\nabla V, \nabla V \rangle.$$

Hence

$$2 \int_{\mathbb{R}^d} |\sqrt{A} \nabla v^+|^2 (1 + v^+)^{-3} \psi_N \varrho \, dx$$

$$\leq N^{-1} \int_{\mathbb{R}^d} (|L V| + \langle A\nabla V, \nabla V \rangle) \varrho \, dx + N^{-1} \int_{\mathbb{R}^d} (|L V| + \langle A\nabla V, \nabla V \rangle) |w| \, dx.$$

The right-hand side tends to zero as $N \to \infty$. In addition, $\psi_N \to 1$. Hence $\nabla v^+ = 0$ a.e., so $v^+ = \text{const}$. Replacing $w$ by $-w$, we conclude that $v^- = \text{const}$. Thus, $v = \text{const}$. \[ \square \]

Note that the integrability condition required in this proposition is fulfilled if we have the estimate $|L V| + \langle A\nabla V, \nabla V \rangle \leq C_V + C_V V^m W$, assumed in the main theorem and $V^m W$ is integrable.

It should be also observed that the uniform bound (1.4) was never used in this section in its full strength: it would suffice to require this lower bound on each ball $U$ with a constant $c(U)$ depending on $U$.

### 3. Proofs

We first prove the continuity result, which is very simple.

**Proof of Proposition 1.1.** Let $\alpha_n \to \alpha$ in $[0, 1]$. As explained in the previous section, it follows from our assumptions that, for every ball $U$, the restrictions of the densities $g_\alpha$ to $U$ are uniformly bounded in the Sobolev norm of $W^{p,1}(U)$, hence are uniformly bounded and uniformly Hölder continuous. Therefore, there is a subsequence $\{\alpha_{n_j}\}$ such that the functions $g_\alpha$ with the respective indices converge uniformly on balls to some continuous function $\varrho$. Since the measures $\mu_\alpha$ are uniformly tight by assumption, we conclude that $\nu = \varrho \, dx$ is a probability measure. By convergence of densities we obtain convergence in variation, i.e. convergence of densities in $L^1(\mathbb{R}^d)$. It is clear that $\nu$ satisfies the equation
\( L^*_\nu = 0 \) (here the local \( L^1 \)-continuity of the coefficients in \( \alpha \) is used to take limits under the integral sign), whence by the assumed uniqueness we have \( \nu = \mu_\alpha \). Since this is true for any subsequence in the original sequence, our assertion is proven. \( \Box \)

**Remark 3.1.** A similar (and even simpler) proof yields an alternative continuity result under incomparable conditions: if the functions \((x, \alpha) \mapsto a^i_\alpha(x)\) and \((x, \alpha) \mapsto b^i_\alpha(x)\) are continuous, the matrix \( A(x) \) is nonnegative-definite (possibly degenerate), the family \( \{\mu_\alpha\} \) is uniformly tight, and for each \( \alpha \) the measure \( \mu_\alpha \) is a unique probability solution to the equation \( L^*_\alpha \mu = 0 \), then the mapping \( \alpha \mapsto \mu_\alpha \) is continuous with values in the space of measures with the weak topology.

**Remark 3.2.** In the situation of the theorem one can deal with parameters \( \alpha \) belonging to a compact interval \([\tau, \tau]\) in \((0, 1)\). Since by our assumption the mapping \( \alpha \mapsto b_\alpha|_U \) is \( L^p \)-differentiable for every ball \( U \), it is possible to choose versions of the functions \( b_\alpha \) that are absolutely continuous in \( \alpha \), namely, one can use the version given by

\[
   b_\alpha(x) = b_\tau(x) + \int_\tau^0 \partial_s b_\alpha(x) \, ds.
\]

For this version we have

\[
   |b_\alpha(x)| \leq |b_\tau(x)| + \int_\tau^\tau |\partial_s b_\alpha(x)| \, ds,
\]

hence \( \sup_\alpha |b_\alpha(x)| \) is locally integrable (this function is Lebesgue measurable, since the coefficients are jointly Borel measurable, see [2, Corollary 2.12.8]), moreover, it is locally in \( L^p \). Then the function \( \sup_\alpha |L_\alpha V(x)| \) is locally integrable, because the terms with second derivatives of \( V \) are locally uniformly bounded in \( x \) and \( \alpha \). Hence in (1.7) without loss of generality we can assume that \( \sup_\alpha L_\alpha V(x) \leq -W(x) \) on the whole space. However, it will be more convenient to redefine \( W \) by 1 on a suitable ball.

**Proof of Theorem 1.2.** By assumption, \( L_\alpha V(x) \leq -W \leq -1 \) outside of some ball. We can assume that \( V \geq 1 \). Since \( V \) is continuous and \( \lim_{|x| \to \infty} V(x) = +\infty \), we can assume that this holds outside of the set \( \Omega_0 = \{V \geq R_0\} \) for some \( R_0 \geq 1 \). Let us set \( W(x) = 1 \) if \( x \in \Omega_0 \). Then \( W \geq 1 \).

Let us prove the differentiability with respect to \( \alpha \) in the special case where \( A_\alpha(x) \) and \( b_\alpha(x) \) do not depend on \( \alpha \) for all \( x \) outside of some common ball \( U \), i.e.,

\[
   A_{\alpha+h}(x) = A_\alpha(x), \quad b_{\alpha+h}(x) = b_\alpha(x) \quad \text{for all } h \text{ and all } x \notin U.
\]

Suppose that \( \alpha \) is fixed and a sequence of nonzero numbers \( h_k \) tends to zero. Set

\[
   \delta_k \varrho_k = h_k^{-1} (\varrho_\alpha - \varrho_{\alpha-h_k}), \quad \delta_k b = h_k^{-1} (b_\alpha - b_{\alpha-h_k}),
   \delta_k a^{ij} = h_k^{-1} (a^{ij}_\alpha - a^{ij}_{\alpha-h_k}), \quad \delta_k A^{i,j} = (\delta_k a^{ij})_{i,j \leq d}.
\]

Of course, these functions depend also on \( \alpha \), which is suppressed in our notation, since \( \alpha \) is a fixed point where the differentiability is verified. Observe that \( \delta_k a^{ij} = 0 \) and \( \delta_k b = 0 \) outside of \( U \) for all \( k \). Each function \( \delta_k \varrho \) satisfies the equation

\[
   L^*_\alpha \delta_k \varrho = \text{div}(F_k \varrho_\alpha),
\]

where

\[
   F_k \varrho_\alpha = -\delta_k A \nabla \varrho_{\alpha-h_k} - \delta_k \partial_{x^i} A^i_{\alpha-h_k} \varrho_{\alpha-h_k} + \delta_k b \varrho_{\alpha-h_k},
   \delta_k \partial_{x^i} A^i = h_k^{-1} (\partial_{x^i} a^{i1}_\alpha - \partial_{x^i} a^{i1}_{\alpha-h_k}, \ldots, \partial_{x^i} a^{id}_\alpha - \partial_{x^i} a^{id}_{\alpha-h_k}).
\]

The vector field \( F_k \) vanishes outside of \( U \) and its \( L^r(U) \)-norm is bounded by a number \( C(U) \) independent of \( k \) for some \( r = r(U) > d \). Indeed, by (2.2) the functions \( \varrho_{\alpha-h_k} \) are
uniformly bounded on $U$. By Proposition 1.1 they converge to $\varrho_\alpha$ in $L^1(U)$ and pointwise, hence also in $L^q(U)$ for each $q < \infty$. According to (1.5), for some $p_0 = p_0(U) > d$, the mappings $\delta_k \partial_{x_i} A^i$ and $\delta_k b$ converge in $L^{p_0}(U)$ to $\partial_{x_i} A^i_\alpha$ and $\partial_{x} b$, respectively. Therefore,

$$-\delta_k \partial_{x_i} A^i g_{\alpha-h_k} + \delta_k b g_{\alpha-h_k} \rightarrow -\partial_{x_i} A^i_\alpha g_{\alpha} + \partial_{x} b g_{\alpha}$$

in $L^{p_0}(U)$. In addition, again by (2.2) the mappings $\nabla g_{\alpha-h_k}$ are uniformly bounded in $L^{p}(U)$, where $p = p(U) > d$, and by convergence of $g_{\alpha-h_k}$ to $g_{\alpha}$ they weakly converge in $L^{p}(U)$ to $\nabla g_{\alpha}$. Next, according to (1.5) and the comment made below (1.6), the mappings $\delta_k A$ converge to $\partial_{x} A_\alpha$ in $L^{p_1}(U)$ for some $p_1 = p_1(U) > dp/(p-d)$. By Hölder’s inequality the mappings $\delta_k A \nabla g_{\alpha-h_k}$ are uniformly bounded in $L^s(U)$ with $s = p_1p/(p_1 + p)$. Note that $s > d$. Recall also that by Lemma 2.1 the functions $g_{\alpha}$ are locally uniformly separated from zero. Therefore, we have the following weak convergence in $L^s(U)$ with $s > d$:

$$F_k \rightarrow F := -\partial_{x_i} A_\alpha \nabla g_{\alpha}/g_{\alpha} - \partial_{x_i} A^i_\alpha + \partial_{x} b.$$

Note also that $\delta_k \varrho$ satisfies the following conditions:

$$\int_{\mathbb{R}^d} \delta_k \varrho \, dx = 0, \quad \int_{\mathbb{R}^d} (|L_\alpha V| + \langle A_\alpha \nabla V, V \rangle) |\delta_k \varrho| \, dx < \infty.$$

Indeed, by Lemma 2.2 for every $\alpha$ the function $V^m W \varrho_\alpha$ is integrable, where $m$ is the number from the hypotheses of the theorem, moreover,

$$\sup_{\alpha} \int_{\mathbb{R}^d} V^m W \varrho_\alpha \, dx \leq M_1 < \infty. \quad (3.2)$$

According to the results of the previous section (see Proposition 2.6 and Proposition 2.7), for each $k$, there are a solution $u_k$ to the equation $L^*_\alpha u_k = \text{div}(F_k \varrho_\alpha)$ and a constant $\lambda_k$ such that

$$\delta_k \varrho = u_k - \lambda_k \varrho_\alpha$$

and

$$\int_{\mathbb{R}^d} V^m W |u_k| \, dx \leq M,$$

where $M$ does not depend on $k$. The latter follows by (2.21), since we have (2.20) and (2.19) holds for $F = F_k$ with a constant independent of $k$ due to the fact that the fields $F_k$ have supports in $U$ and are uniformly bounded in $L^{p}(U)$. It is clear that

$$\lambda_k = \int_{\mathbb{R}^d} u_k \, dx.$$

Hence $|\lambda_k| \leq \|u_k\|_{L^1(\mathbb{R}^d)} \leq M$. Therefore, by (3.2) we have

$$\int_{\mathbb{R}^d} V^m W |\delta_k \varrho| \, dx \leq M + M \|V^m W\|_{L^1(\mu_\alpha)} \leq M + MM_1.$$

Passing to a subsequence and using our local estimates (2.2), we can assume that the functions $\delta_k \varrho$ converge locally uniformly to some function $w_\alpha$. By Fatou’s theorem

$$\int_{\mathbb{R}^d} V^m W |w_\alpha| \, dx \leq M + M \|V^m W\|_{L^1(\mu_\alpha)} \leq M + MM_1.$$

In addition, the functions $\delta_k \varrho$ converge to $w_\alpha$ in $L^1(\mathbb{R}^d)$ (since $\lim_{|x| \rightarrow \infty} W(x) = +\infty$) and

$$\int_{\mathbb{R}^d} w_\alpha \, dx = 0.$$
Moreover, the function $w_\alpha$ satisfies the equation

$$L_\alpha^* w_\alpha = \text{div}(\varrho_\alpha F).$$

It remains to observe that $w_\alpha$ satisfies the hypotheses of Proposition 2.7 about uniqueness. Thus, for each sequence $h_k \to 0$ the continuous functions $\delta_k \varrho$ converge locally uniformly to one and the same limit $w_\alpha$. Therefore, we have $w_\alpha = \partial_\alpha \varrho_\alpha$.

We now proceed to the general case. We can assume that our parameters take values in an interval of length less than $c_0(2\lambda_0 + 1)^{-1}$, where $\lambda_0$ and $c_0$ are constants from (1.4) and (1.6), which by (1.6) yields the estimate

$$\|A_\alpha(x) - A_{\alpha_0}(x)\| \leq \frac{1}{2c_0 + 1}$$

for all $\alpha, \alpha_0$ and $x$. It follows that for any number $\theta \in [0, 1]$ we have

$$\|((\theta A_\alpha(x) + (1 - \theta)A_{\alpha_0}(x))^{-1/2} A_\alpha(x)^{1/2}\| \leq 2. \quad (3.3)$$

Indeed, let us observe that for any nonnegative operator $T$ and symmetric operator $D$ such that $T \geq c_0 I$ and $\|D\| \leq \varepsilon I$, where $\varepsilon < c_0/2$, we have $T - \varepsilon I \leq T + D \leq T + \varepsilon I$ in the sense of quadratic forms, hence (see [19, Chapter VIII, Problem 50])

$$(T - \varepsilon I)^{1/2} - T^{1/2} \leq (T + D)^{1/2} - T^{1/2} \leq (T + \varepsilon I)^{1/2} - T^{1/2},$$

so that

$$-\frac{\varepsilon}{2c_0^{1/2}} I \leq (T + D)^{1/2} - T^{1/2} \leq \frac{\varepsilon}{(2c_0)^{1/2}} I,$$

which yields that $\|(T + D)^{1/2} - T^{1/2}\| \leq \varepsilon(2c_0)^{-1/2}$. Therefore,

$$\|(T + D)^{-1/2} T^{1/2}\| = \|(T + D)^{-1/2}((T + D)^{1/2} + (T^{1/2} - (T + D)^{1/2}))\|
\leq 1 + \|(T + D)^{-1/2}\| \|T^{1/2} + (T + D)^{1/2})\|
\leq 1 + \|(T + D)^{-1/2}\| \| \frac{\varepsilon}{(2c_0)^{1/2}} I \leq 1 + c_0^{1/2} \|(T + D)^{-1/2}\|.$$

This yields (3.3) if we take $T = A_\alpha(x)$ and $D = (1 - \theta)(A_{\alpha_0}(x) - A_\alpha(x))$, that is, $T + D = \theta A_\alpha(x) + (1 - \theta)A_{\alpha_0}(x)$, because $\theta A_\alpha(x) + (1 - \theta)A_{\alpha_0}(x) \geq c_0 I$, hence

$$\|((\theta A_\alpha(x) + (1 - \theta)A_{\alpha_0}(x))^{-1/2}\| \leq c_0^{-1/2}.$$

Let $\psi_N(x) = \psi(|x| - N + 1)$, where $\psi \in C^\infty(\mathbb{R})$, $0 \leq \psi \leq 1$, $-2 \leq \psi' \leq 0$, $\psi(s) = 1$ if $s \leq 1$ and $\psi(s) = 0$ if $s \geq 2$. In addition, we take $\psi$ such that $|\psi'(s)|^2 \leq C_0 \psi(s)$ with some $C_0 > 0$. Then $0 \leq \psi_N \leq 1$, $\psi_N(x) = 1$ if $|x| < N$ and $\psi_N(x) = 0$ if $|x| > N + 1$, $|\nabla \psi_N| \leq 2$, $|\nabla \psi_N|^2 \leq C_0 \psi_N$. Fix some $\alpha_0$ (say, the middle of the interval) and set

$$L_{\alpha,N} = \psi_N L_{\alpha} + (1 - \psi_N) L_{\alpha_0},$$

$$A_{\alpha,N} = \psi_N A_{\alpha} + (1 - \psi_N) A_{\alpha_0}, \quad b_{\alpha,N} = \psi_N b_{\alpha} + (1 - \psi_N) b_{\alpha_0}.$$

We observe that the corresponding coefficients $a^{ij}_{\alpha,N}(x)$ and $b^{i}_{\alpha,N}(x)$ do not depend on $\alpha$ if $|x| > N + 1$, once $N$ is fixed. Moreover,

$$L_{\alpha,N} V = \psi_N L_{\alpha} V + (1 - \psi_N) L_{\alpha_0} V \leq -W$$

and

$$\langle A_{\alpha,N} \nabla V, \nabla V \rangle \leq \varepsilon V W.$$
outside of the same ball as in the case of $L_\alpha$. We also have
\[
|L_{\alpha,N} V| + |A_{\alpha,N}^{-1/2} (b_{\alpha,N} - \text{div} A_{\alpha,N})|^2 + |A_{\alpha,N}^{-1/2} (\partial_\alpha b_{\alpha,N} - \partial_\alpha \text{div} A_{\alpha,N})|^2 \leq \tilde{C}_V + \tilde{C}_V V^m \quad (3.4)
\]
with $\tilde{C}_V = 18 C_V + 16 \lambda_0^0 c_0^{-1}$, since
\[
\text{div} A_{\alpha,N} = \psi_N \text{div} A_\alpha + (1 - \psi_N) \text{div} A_{\alpha_0} + (A_\alpha - A_{\alpha_0}) \nabla \psi_N,
\]
\[
\partial_\alpha \text{div} A_{\alpha,N} = \psi_N \text{div} \partial_\alpha A_\alpha + \partial_\alpha A_\alpha \nabla \psi_N, \quad \partial_\alpha b_{\alpha,N} = \psi_N \partial_\alpha b_\alpha,
\]
where $\|A_\alpha(x) - A_{\alpha_0}(x)\| \leq \lambda_0$ by (1.6). Indeed,
\[
A_{\alpha,N}^{-1/2} (b_{\alpha,N} - \text{div} A_{\alpha,N}) = \psi_N A_{\alpha,N}^{-1/2} A_\alpha^{-1/2} A_{\alpha_0}^{-1/2} (b_\alpha - \text{div} A_\alpha)
+ (1 - \psi_N) A_{\alpha,N}^{-1/2} A_\alpha^{-1/2} A_{\alpha_0}^{-1/2} (b_{\alpha_0} - \text{div} A_{\alpha_0}) + A_{\alpha,N}^{-1/2} (A_\alpha - A_{\alpha_0}) \nabla \psi_N,
\]
\[
A_{\alpha,N}^{-1/2} (\partial_\alpha b_{\alpha,N} - \partial_\alpha \text{div} A_{\alpha,N}) = \psi_N A_{\alpha,N}^{-1/2} A_\alpha^{-1/2} A_{\alpha_0}^{-1/2} (\partial_\alpha b_\alpha - \partial_\alpha \text{div} A_\alpha)
+ A_{\alpha,N}^{-1/2} \partial_\alpha A_\alpha \nabla \psi_N.
\]
Hence, by (3.3), the norm of the first of these two vectors is dominated by the sum of the norms of $A_\alpha^{-1/2} (b_\alpha - \text{div} A_\alpha)$ and $A_{\alpha_0}^{-1/2} (b_{\alpha_0} - \text{div} A_{\alpha_0})$ and $c_0^{-1/2} \lambda_0$, and similarly for the second vector.

In addition, for every ball $U$ we have
\[
\sup_{\alpha} \|a_{ij}^{\alpha,N}\|_{W^{p,1}(U)} \leq 2 \sup_{\alpha} \|a_{ij}^{\alpha_0}\|_{W^{p,1}(U)}, \quad \sup_{\alpha} \|b_{\alpha,N}\|_{L^p(U)} \leq \sup_{\alpha} \|b_{\alpha_0}\|_{L^p(U)}
\]
for the corresponding $p = p(U) > d$, and also
\[
\psi_N(x) A_\alpha(x) + (1 - \psi_N(x)) A_{\alpha_0}(x) \geq c_0 \cdot 1 \quad \forall x.
\]
Defining $S_{\alpha,N}$ for the mapping $A_{\alpha,N}$ by the same formula as $S_\alpha$ for $A_\alpha$, due to (1.6), we have
\[
\sup_{\alpha,N} \|S_{\alpha,N}\| \leq \lambda_0 < \infty. \quad (3.5)
\]

For each $N$ there exist probability solutions $\varrho_{\alpha,N}$ of the equations
\[
L_\alpha^* \varrho_{\alpha,N} = 0.
\]
As shown above, there exist the derivatives $w_{\alpha,N} = \partial_\alpha \varrho_{\alpha,N}$ satisfying the equations
\[
L_\alpha^* w_{\alpha,N} = \text{div} (B_{\alpha,N} \varrho_{\alpha,N} - R_{\alpha,N} \varrho_{\alpha,N} - S_{\alpha,N} \nabla \varrho_{\alpha,N}),
\]
where $B_{\alpha,N} = \partial_\alpha b_{\alpha,N}$, $R_{\alpha,N} = \text{div} S_{\alpha,N}$, as in the case of the original operators $L_\alpha$. Moreover, we have
\[
\int_{\mathbb{R}^d} V^m W |w_{\alpha,N}| \, dx \leq M, \quad (3.6)
\]
where $M$ does not depend on $N$ and $\alpha$. Indeed, as above, we can construct a solution $\tilde{w}_{\alpha,N}$ for which this estimate holds. To this end we observe that by Lemma 2.3 and (3.4) we have
\[
\int_{\mathbb{R}^d} \left| A_{\alpha,N}^{-1/2} \nabla \varrho_{\alpha,N} \right|^2 \varrho_{\alpha,N} \, dx \leq \int_{\mathbb{R}^d} |A_{\alpha,N}^{-1/2} (b_{\alpha,N} - \text{div} A_{\alpha,N})|^2 \varrho_{\alpha,N} \, dx
\leq (1 + c_0) \tilde{C}_V + (1 + c_0) \tilde{C}_V \int_{\mathbb{R}^d} V^m W \varrho_{\alpha,N} \, dx,
\]
which is uniformly bounded in $\alpha$ and $N$, since the integrals of $V^m W \varrho_{\alpha,N}$ are uniformly bounded by Lemma 2.2. Therefore, by the equality

$$A_{\alpha,N}^{-1/2} S_{\alpha,N} \nabla \varrho_{\alpha,N} = A_{\alpha,N}^{-1/2} S_{\alpha,N} A_{\alpha,N}^{-1/2} A_{\alpha,N}^{1/2} \nabla \varrho_{\alpha,N}$$

combined with (1.4) and (3.5) we have

$$\sup_{\alpha,N} \int_{\mathbb{R}^d} \left| A_{\alpha,N}^{-1/2} S_{\alpha,N} \nabla \varrho_{\alpha,N} \right|^2 \varrho_{\alpha,N} \, dx \leq c_0^{-1} \lambda_0^2 \sup_{\alpha,N} \int_{\mathbb{R}^d} \left| A_{\alpha,N}^{1/2} \nabla \varrho_{\alpha,N} \right|^2 \varrho_{\alpha,N} \, dx < \infty.$$ 

We have also a uniform bound for the integrals of $|A_{\alpha,N}^{-1/2} (B_{\alpha,N} - R_{\alpha,N})|^2 \varrho_{\alpha,N}$, because

$$B_{\alpha,N} - R_{\alpha,N} = \psi_N(B_{\alpha} - R_{\alpha}) - S_{\alpha} \nabla \psi_N,$$

so that

$$|A_{\alpha,N}^{-1/2} (B_{\alpha,N} - R_{\alpha,N})| \leq C_2 + C_2 V^m W$$

with some constant $C_2$. Hence we ensure condition (2.19), so that estimate (2.21) in Proposition 2.6 yields (3.6) for $\tilde{w}_{\alpha,N}.$ By the uniqueness result (Proposition 2.7) we have the equality

$$w_{\alpha,N}(x) = \tilde{w}_{\alpha,N}(x) - \varrho_{\alpha,N}(x) \int_{\mathbb{R}^d} \tilde{w}_{\alpha,N} \, dx,$$

which yields the desired estimate (3.6) for $w_{\alpha,N},$ because the integrals of $V^m W \varrho_{\alpha,N}$ are uniformly bounded by Lemma 2.2.

Passing to a subsequence and using (2.2), we conclude that for each $\alpha$ the sequence of functions $\varrho_{\alpha,N}$ converges uniformly in $x$ to the unique solution $\varrho_{\alpha}$ of the equation $L_{\alpha}^* \mu = 0$ and the sequence of functions $w_{\alpha,N}$ converges to a solution $w_{\alpha}$ of the equation

$$L_{\alpha}^* w = \text{div} (B_{\alpha} \varrho_{\alpha} - R_{\alpha} \varrho_{\alpha} - S_{\alpha} \nabla \varrho_{\alpha})$$

(3.7)

that satisfies the same bound as in (3.6). It follows that

$$\int_{\mathbb{R}^d} w_{\alpha} \, dx = 0,$$

hence $w_{\alpha}$ is a unique solution to (3.7) with zero integral such that $V^m W w_{\alpha}$ is integrable.

We now observe that the solutions $w_{\alpha}$ (as well as $w_{\alpha,N}$) satisfying the conditions

$$\int_{\mathbb{R}^d} w_{\alpha} \, dx = 0, \quad \sup_{\alpha} \int_{\mathbb{R}^d} V^m W |w_{\alpha}| \, dx < \infty$$

(3.8)

are continuous in $\alpha$ locally uniformly in $x$. Indeed, if $\alpha_k \to \alpha$, then the sequence $\{w_{\alpha_k}\}$ contains a subsequence convergent locally uniformly in $x$ to some function $w$ (this follows by (2.2)). Due to our assumption that the mappings $\alpha \mapsto \partial_\alpha a_{ij}^0$, $\alpha \mapsto \partial_\alpha b_i^0$, $\alpha \mapsto \partial_\alpha \partial_\beta a_{ij}$ are continuous with values in $L^1(U)$ for each ball $U$ and $\varrho_{\alpha}(x)$ is jointly continuous by Proposition 1.1, we see that $w$ is a solution to the equation

$$L_{\alpha}^* w = \text{div} (B_{\alpha} \varrho_{\alpha} - R_{\alpha} \varrho_{\alpha} - S_{\alpha} \nabla \varrho_{\alpha})$$

and this solution satisfies the above estimate by Fatou’s theorem. Hence $w$ coincides with our unique solution $w_{\alpha}$. Since this is true for each sequence $\{\alpha_k\}$, we obtain that $\{w_{\alpha_k}\}$ converges to $w_{\alpha}$. The continuity is proven.

It remains to show that $w_{\alpha} = \partial_\alpha \varrho_{\alpha}$. Let us fix $x$. By the Newton–Leibniz formula

$$\varrho_{\alpha,N}(x) = \varrho_{\alpha_0,N}(x) + \int_{\alpha_0}^{\alpha} w_{s,N}(x) \, ds.$$
All limiting functions $g_\alpha(x)$ and $w_\alpha(x)$ are continuous in $\alpha$, as shown above. In addition, letting $U(x,r)$ be the ball of radius $r$ centered at $x$, we have by (2.2)

$$|w_{\alpha,N}(x)| \leq \|w_{\alpha,N}\|_{W^{1,1}(U(x,1))} \leq C\left(\|w_{\alpha,N}\|_{L^1(U(x,2))} + \|B_{\alpha,N}g_{\alpha,N} - R_{\alpha,N}g_{\alpha,N} - S_{\alpha,N}\nabla g_{\alpha,N}\|_{L^p(U(x,2))}\right).$$

The right-hand side is uniformly bounded in $\alpha$ and $N$ (once $x$ is fixed). Therefore,

$$\sup_{\alpha,N} |w_{\alpha,N}(x)| < \infty.$$

Passing to the limit as $N \to \infty$, we obtain the equality

$$g_\alpha(x) = g_{\alpha_0}(x) + \int_{\alpha_0}^\alpha w_s(x) \, ds,$$

whence we conclude that $\partial_\alpha g_\alpha(x) = w_\alpha(x)$. The assertion about the $L^1$-differentiability of $\alpha \mapsto g_\alpha$ follows from (3.8), which allows to show the $L^1$-convergence to zero of the ratio

$$\frac{g_{\alpha+h} - g_{\alpha}}{h} - w_\alpha = h^{-1} \int_{\alpha}^{\alpha+h} (w_s - w_\alpha) \, ds,$$

reducing it to the $L^1$-convergence on balls. The general case of the theorem is proven.

In the special case where the diffusion matrix does not depend on $\alpha$ we have $R_\alpha = 0$ and $S_\alpha = 0$, so in the right-hand side of (3.7) we have only one term $\text{div}(B_\alpha g_\alpha)$ with $B_\alpha = \partial_\alpha b_\alpha$. Hence we can obtain (2.19) immediately from the given bound on $B_\alpha$. \qed

**Remark 3.3.** (i) It follows from the proof (or from the $L^1$-differentiability) that

$$\int_{\mathbb{R}^d} \partial_\alpha g_\alpha(x) \, dx = 0.$$

(ii) The main theorem can be combined with the results of [5] on Sobolev regularity of non-homogeneous equations in order to ensure the differentiability of $\alpha \mapsto g_\alpha$ with values in $W^{r,1}(\mathbb{R}^d)$. For example, suppose that in the main theorem $A_\alpha$, $A^{-1}_\alpha$, $\nabla a_{ij}$, and $\partial_\alpha A_{ij}$ are uniformly bounded and

$$|b_\alpha|^p + |\partial_\alpha b_\alpha|^p + |\partial_\alpha \text{div } A_\alpha|^p \leq C_V + C_V V^m W$$

with some $p > d$. Then we have the differentiability in $W^{r,1}(\mathbb{R}^d)$ for any $r < p$.

(iii) Condition (1.6) has been essential in estimating the integral of the expression $|A^{1/2}_\alpha S_\alpha \nabla g_\alpha|^2 / g_\alpha$ and a similar integral for $g_{\alpha,N}$, since we have had an a priori bound just for the integral of $|\nabla g_\alpha|^2 / g_\alpha$, so that a growing $\|S_\alpha\|$ could destroy this estimate. However, under assumptions similar to those used in Corollary 1.5 it is proved in [8] that there is a bound for the integral of $|\nabla g_\alpha|^p / g_\alpha$ with a sufficiently large $p > d$. This enables us to replace (1.6) by an exponential bound and make our condition on $\partial_\alpha A_\alpha$ closer to that of the condition on $\partial_\alpha b_\alpha$.

(iv) The main theorem and its corollaries extend to the case where the parameter $\alpha$ takes values in $\mathbb{R}^n$; this case can be also deduced from the scalar case.

(v) Analogous results can be obtained by the same method for equations on manifolds; some ingredients of the proofs are already developed in [13].

(vi) Finally, let us observe that a similar method enables one to obtain higher differentiability of $g_\alpha$ with respect to $\alpha$ (considered in [18] for coefficients of class $C^k_b$), which will be the subject of another paper (in order not to overload this paper with additional technicalities).
Remark 3.4. It would be tempting to prove the theorem along the following lines: it is known that under our assumptions the solutions $\varrho_\alpha$ can be obtained as limits of the normalized positive solutions to the equations $L^*_\alpha \varrho_{\alpha,n} = 0$ on increasing domains $U_n = \{ V < n \}$; e.g., one can use solutions to boundary value problems with constant boundary conditions. Such solutions are differentiable with respect to $\alpha$ and the derivative in $\alpha$ satisfies the required equation in $U_n$. Then the problem is to obtain convergence of these derivatives. Proposition 2.4 seems to be a suitable tool, moreover, we apply it in a similar situation, however, in that situation we deal with zero boundary condition, which is very essential.

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