On surface measures generated by differentiable measures

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Abstract We study surface measures on level sets of functions on general probability spaces with measures differentiable along vector fields and suggest a new simple construction. Our construction applies also to level sets of mappings with values in finite-dimensional spaces. The standard surface measures arising for Gaussian measures in the Malliavin calculus can be obtained on this way.

Keywords: Malliavin calculus, surface measure, differential measure, Gaussian measure

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Introduction

Surface measures on general spaces have become a popular subject of study in the recent years due to development of the Malliavin calculus, geometric measure theory and metric measure spaces, and infinite-dimensional stochastic analysis, see [2], [3], [12], [13], [14], [15], [16], [18], [21], [23], [24], [33] and [34], where one can find discussions of diverse problems explicitly or implicitly connected with surface measures in infinite dimensions. A rich theory of surface measures on infinite-dimensional spaces equipped with differentiable measures was worked out by A.V. Uglanov in the 70–80s and presented in his book [40] (see also [38], [39], and [41]). In the same years, an approach to surface measures for Gaussian volume measures was developed in the Malliavin calculus which provided efficient tools for the study of induced measures. For this approach, see [1], [26], [5], [6], and [7]; far reaching generalizations to the case of differentiable measures were obtained in [27], [28], [29], [30], and [31]. A close construction for configuration spaces was presented in [19]. Hausdorff measures associated with Gaussian measures were studied in [20]; more references for the Gaussian case can be found in [7] and [9].

The goal of our paper is to introduce a construction of surface measures that follows Malliavin's idea, but applies to nonlinear spaces and requires less regularity of the function generating the surface. In the Gaussian case this construction applies to one-fold Malliavin differentiable functions with gradients having divergences and also contains some novelties when applied to surfaces of higher codimension. In the nondegenerate case, our surface measures are equivalent to the standard ones. However, our approach even in the known cases leads to much shorter and simpler proofs, in particular, the existence of surface measures is proved in few lines. In addition, we answer a question posed by M. Röckner on continuous dependence of surface measures on the parameter y determining the level set $F^{-1}(y)$.

Let μ be a bounded nonnegative Radon measure on a completely regular topological space X defined on the Borel σ-field $\mathcal{B}$ (see [8] for definitions). We shall assume that μ is concentrated on a countable union of metrizable compacts, which is always the case if X is Souslin or metrizable or if μ is Gaussian. Given a measurable function $F\colon X\to\mathbb{R}$ or a measurable mapping $F\colon X\to\mathbb{R}^d$, we can take the image-measure $\mu\circ F^{-1}$ defined by the formula

$$\mu\circ F^{-1}(B) := \mu(F^{-1}(B))$$

on the Borel σ-field in $\mathbb{R}$ or $\mathbb{R}^d$, respectively, and find the so-called conditional measures μ^y on X such that the function $B\mapsto\mu^y(B)$ is μ-measurable for each $B\in\mathcal{B}$, μ^y is concentrated on $F^{-1}(y)$ for every y (or $\mu\circ F^{-1}$-a.e. y) and μ is the integral of μ^y against $\mu\circ F^{-1}$, which is written as

$$\mu = \mu^y\cdot\mu\circ F^{-1}(dy),$$

in the sense that

$$\int_X f(x)\,\mu(dx) = \int\int_X f(x)\mu^y(dx)\mu\circ F^{-1}(dy)$$

for every bounded Borel function f on X, where the integral exists due to the assumption of measurability for μ^y, see [8, Chapter 10] or [9, Chapter 1] for details. This classical construction competes, however, with another natural concept, that of a surface measure. The latter is usually defined in a more special situation, where one can consider suitable neighborhoods of the "surfaces" $\{F = y\}$ and obtain a reasonable limit after appropriate scaling. For example, the usual surface measure in $\mathbb{R}^d$ arises as a limit of the ratio of the volume of the ε-neighborhood of the surface and ε, as $\varepsilon \to 0$. The proposed construction of a surface measure σ^y on the level set $F^{-1}(y)$ is this: we introduce a certain weight function θ_F and set

$$\int f(x)\sigma^y(dx) = \lim_{r\to 0} \frac{1}{r} \int_{\{y<F<y+r\}} f(x)\theta_F(x)\mu(dx)$$

for a suitable class of functions f (say, bounded Lipschitzian). Unlike the case of conditional measures, such constructions require certain constraints on measures and functions in question. In the case of a Gaussian measure μ on a locally convex space X this construction applies to F in the first Sobolev class $W^{p,1}(\mu)$ and we take $\theta_F = |D_H F|^2$, where $D_H F$ is the Sobolev gradient of F along the Cameron–Martin space H of the measure μ. The weight function θ_F can be later dismissed provided it is sufficiently nondegenerate; its purpose is to allow degenerate F and lower the required order of differentiability of F. The approach suggested in this paper can be also of interest for the study of surface measures on metric measure spaces (see [4], [17], [22], [25], and [37]).

Why is not it enough to deal with conditional measures that exist in much greater generality? The reason is essentially the same as in the finite-dimensional case: the Gauss–Ostrogradskii–Stokes formula and integrations by parts. This explains at once why certain smoothness restrictions on the volume measure and the function generating level sets are needed. Another reason is that conditional measures μ^y depend not only on the level sets $F^{-1}(y)$, but also on the image-measure $\mu \circ F^{-1}$ (though, for induced measures with positive densities this dependence reduces to a constant factor for each fixed y). Our construction shares this property, but allows a modification that does not.

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1. Non-normalized surface measures

The measure μ mentioned above is fixed throughout. Let $\mathcal{F}$ be a class of bounded $\mathcal{B}$-measurable functions. We assume throughout that $\mathcal{F}$ satisfies the following conditions:

(F1) $\mathcal{F}$ is a linear space and $\varphi(f) \in \mathcal{F}$ for all $f \in \mathcal{F}$ and all Lipschitzian functions φ on $\mathbb{R}$.

(F2) the space $\mathcal{P}(X)$ of Radon probability measures on $\mathcal{B}$ is sequentially complete in the topology on the space of bounded measures generated by duality with $\mathcal{F}$ and the corresponding convergence yields weak convergence, which means that, whenever $\{\mu_n\}$ is a sequence in $\mathcal{P}(X)$ such that the integrals of every function in $\mathcal{F}$ against the measures μ_n have a finite limit, there is a measure $\mu \in \mathcal{P}(X)$ such that for each bounded continuous function f we have

$$\int_X f(x)\mu(dx) = \lim_{n\to\infty} \int_X f(x)\mu_n(dx).$$

For example, if X is a complete separable metric space, then the class of all bounded Lipschitzian functions on X satisfies both conditions (F1) and (F2) (see [8, Corollary 8.6.3]). We recall that weak convergence of a sequence of Radon probability measures to a Radon probability measure μ is equivalent to the relation $\mu(W) \le \liminf_{n\to\infty} \mu_n(W)$ for every open set W (see [8, Section 8.2]). In case of metric spaces it suffices to have convergence of the integrals of bounded Lipschitzian functions.

For many applications, it is possible to take for $\mathcal{F}$ exactly the class of all bounded Lipschitzian functions.

It follows from (F1) that $1 \in \mathcal{F}$ and that $fg \in \mathcal{F}$ for all $f, g \in \mathcal{F}$. Indeed, $f^2 \in \mathcal{F}$ for all $f \in \mathcal{F}$, because we can take for φ a function in $C_b^\infty(\mathbb{R})$ that coincides with x^2 on the bounded range of f, so it remains to use the equality $2fg = (f+g)^2 - f^2 - g^2$.

It follows from (F2) that $\mathcal{F}$ separates Radon measures on X, i.e., two measures coincide provided that they assign equal integrals to each $f \in \mathcal{F}$.

Let v be a vector field on X understood as a linear mapping

$$v\colon \mathcal{F}\to L^1(\mu), \quad f\mapsto \partial_v f,$$

such that

$$\partial_v(\varphi\circ f)=\varphi'(f)\partial_v f \quad \mu\text{-a.e.} \tag{1.1}$$

for all $f\in\mathcal{F}$ and all Lipschitzian functions φ on the real line. Similarly we can define more general vector fields for which functions $\partial_v f$ belong to the space $L^0(\mu)$ of μ-measurable functions.

Applying this relation to φ such that $\varphi(t)=t$ on a sufficiently large interval we obtain the Leibniz rule

$$\partial_v(fg)=f\partial_v g+g\partial_v f \quad \forall\, f,g\in\mathcal{F}. \tag{1.2}$$

It is worth noting that $\partial_v 1=0$, because we can take $\varphi=1$ in (1.1) or, alternatively, we can take $f=g=1$ in (1.2).

Suppose that μ is Skorohod differentiable along v in the following sense: there is a bounded measure $d_v\mu$ on $\mathcal{B}$, called the Skorohod derivative of μ along v, such that

$$\int_X \partial_v f(x)\mu(dx)=-\int_X f(x)d_v\mu(dx) \quad \forall\, f\in\mathcal{F}. \tag{1.3}$$

We say that μ is Fomin differentiable along v if $d_v\ll\mu$; in that case the Radon–Nikodym density of $d_v\mu$ with respect to μ is denoted by β_v and is called the logarithmic derivative of μ along v or divergence of v with respect to μ.

For example, if μ is a measure on $\mathbb{R}^d$ with a smooth density ϱ and v is a nonzero constant vector, then $d_v\mu$ is given by density $\partial_v\varrho$ and $\beta_v=(\partial_v\varrho)/\varrho$, which explains the terminology. For a survey of the theory of differentiable measures, see [9].

The original definition of Fomin dealt with constant vector fields on linear spaces. Differentiability of measures along non-constant vector fields was considered already in the 80–90s (sometimes implicitly) in the Malliavin calculus and its modifications (see [36], [35], [5], and [19]).

Observe that $d_v\mu(X)=0$, which follows by (1.3) applied to $f=1$, so $d_v\mu$ is necessarily a signed measure.

We need an extension of ∂_v to functions outside of $\mathcal{F}$.

Definition 1.1. *We say that a function $\Psi\in L^1(\mu)$ belongs to $\mathfrak{D}_v$ if $\Psi\in L^1(d_v\mu)$ and there is a sequence of functions $f_n\in\mathcal{F}$ converging to Ψ in $L^1(\mu)$ and in $L^1(d_v\mu)$ such that the functions $\partial_v f_n$ converge in $L^1(\mu)$ to some function w and the functions $f_n\partial_v g$ are uniformly integrable for each $g\in\mathcal{F}$ (the latter holds if $\{f_n\}$ converges to Ψ in $L^p(\mu)$ for some $p>1$ and all functions $\partial_v g$ for $g\in\mathcal{F}$ belong to $L^q(\mu)$, $q=p/(p-1)$). Then we set $\partial_v\Psi:=w$.*

The function w (if exists) is uniquely defined. Indeed, for each $g\in\mathcal{F}$ we have

$$\begin{aligned}\int_X g(x)w(x)\mu(dx)&=\lim_{n\to\infty}\int_X g(x)\partial_v f_n(x)\mu(dx)\\ &=\lim_{n\to\infty}\int_X[\partial_v(gf_n)(x)-f_n\partial_v g]\mu(dx)=-\lim_{n\to\infty}\int_X(gf_n)(x)d_v\mu(dx)-\int_X\Psi\partial_v g\mu(dx)\\ &=-\int_X(g\Psi)(x)d_v\mu(dx)-\int_X\Psi\partial_v g\mu(dx).\end{aligned}$$

We shall assume that $F\colon X\to\mathbb{R}$ is a $\mathcal{B}$-measurable function such that

(F3) $\psi(F)\in\mathfrak{D}_v$ for each Lipschitzian function ψ on $\mathbb{R}$ with compact support.

Then $\partial_v F$ can be defined as follows: this a measurable function such that $\partial_v(\psi\circ F)=\psi'(F)\partial_v F$ a.e. for each Lipschitzian function with compact support.

Let us assume that

(F4) $\partial_v F\in L^1(\mu)$.

Set

$$\nu=:(\partial_v F)\cdot\mu, \quad \eta:=d_v\mu\circ F^{-1}.$$

The conditional measures on the level sets $F^{-1}(y)$ generated by the measure ν will be denoted by ν^y.

Let us assume that $d_v\mu$ vanishes on each set $F^{-1}(y)$. This is automatically true in the case of Fomin's differentiability if $\mu\circ F^{-1}$ has no atoms.

Another concept coming along with surface measures is capacity. Suppose that $\mathcal{F}$ is equipped with a norm $\|\cdot\|_{\mathcal{F}}$ such that convergence in this norm yields convergence in $L^1(\mu)$. In practical situations, this will be often the norm of a suitable Sobolev space $W^{p,1}(\mu)$, but so far no Sobolev spaces are needed. This norm generates a capacity (about capacities, see [9]). Namely, for every open set $U\subset X$ we define its capacity associated with $\mathcal{F}$ by the formula

$$C_{\mathcal{F}}(U)=\inf\{\|f\|_{\mathcal{F}}\colon\ f\in\mathcal{F},\ f\ge 0,\ f\ge 1\ \ \mu\text{-a.e. on } U\}.$$

For any set $B\subset X$ let

$$C_{\mathcal{F}}(B)=\inf\{C_{\mathcal{F}}(U)\colon\ U\supset B \text{ is open}\}.$$

Typically, capacities of the sort we consider are tight (see [32], [30], [31]), i.e., for each $\varepsilon>0$ there is a compact set K_ε such that $C_{\mathcal{F}}(X\backslash K_\varepsilon)<\varepsilon$. However, we do not assume this property.

Recall that a function f is called $C_{\mathcal{F}}$-quasi-continuous if, for each n, there is a closed set A_n such that $C_{\mathcal{F}}(X\backslash A_n)<1/n$ and $f|_{A_n}$ is continuous.

It is known that each function $f\in\mathcal{F}$ has a $C_{\mathcal{F}}$-quasi-continuous version (see [9, Section 8.13]), provided that the norm $\|\cdot\|_{\mathcal{F}}$ is strictly convex, as is the case of the L^p-norm with $p\in(1,+\infty)$, and, more generally, with any norm of the form $\|f\|_{\mathcal{F}}=\|T^{-1}f\|_{L^p(m)}$, where m is a probability measure and T is a bounded linear operator from $L^p(m)$ to $L^1(\mu)$; in particular, the latter case covers most of Sobolev classes. However, in place of such assumptions we simply assume that

(F5) F *has a quasi-continuous version.*

We now fix a quasi-continuous version of F; the results below refer to this version!

By using (F1)–(F4), for any $\psi\in C_0^\infty(\mathbb{R})$ we have

$$\begin{aligned}\int\psi'(t)\nu\circ F^{-1}(dt)&=\int_X\psi'(F(x))\partial_v F(x)\,\mu(dx)=\int_X\partial_v(\psi\circ F)(x)\mu(dx)\\&=-\int_X\psi(F(x))d_v\mu(dx)=-\int\psi(t)\eta(dt).\end{aligned}$$

Therefore, $(\nu\circ F^{-1})'=\eta$ in the sense of distributions. It follows that the measure $\nu\circ F^{-1}$ on the real line has a density ϱ_1 of bounded variation and

$$\varrho_1(t)=\eta((-\infty,t))=d_v\mu(x\colon\ F(x)<t).$$

By our assumption that $d_v\mu$ vanishes on each set $\{F=t\}$ this density is continuous. In particular, this holds if μ is Fomin differentiable along v.

The same holds if we replace ν by the measure $f\cdot\nu$, where $f\in\mathcal{F}$. Indeed, in that case

$$\begin{aligned}\int\psi'(t)(f\cdot\nu)\circ F^{-1}(dt)&=\int_X\partial_v(\psi\circ F)(x)f(x)\mu(dx)\\&=-\int_X\psi(F(x))f(x)d_v\mu(dx)-\int_X\psi(F(x))\partial_v f(x)\mu(dx),\end{aligned}$$

so that

$$((f\cdot\nu)\circ F^{-1})'=(f\cdot d_v\mu)\circ F^{-1}+(\partial_v f\cdot\mu)\circ F^{-1}$$

and

$$\|((f\cdot\nu)\circ F^{-1})'\|\le\|f\cdot d_v\mu+(\partial_v f\cdot\mu)\|.$$

As above, the measure $(f\cdot\nu)\circ F^{-1}$ on the real line has a continuous density ϱ_f of bounded variation. Therefore,

$$|\varrho_f(y)|\le\|f\cdot d_v\mu+(\partial_v f\cdot\mu)\|\le\|d_v\mu\|\cdot\|f\|_\infty+\|\mu\|\cdot\|\partial_v f\|_{L^1(\mu)}. \tag{1.4}$$

We need also a similar estimate with an L^p-norm of f, which holds if $d_v f=\beta_v\cdot\mu$, where $\beta_v\in L^q(\mu)$, $q=p/(p-1)$. Then

$$|\varrho_f(y)|\le\|\beta_v\|_{L^q(\mu)}\|f\|_{L^p(\mu)}+\|\mu\|\cdot\|\partial_v f\|_{L^1(\mu)}. \tag{1.5}$$

We now introduce non-normalized surface measures σ^y. The definition employs only the differentiability of the distribution functions

$$\Phi_f(t) = \int_{\{F<t\}} f(x)\nu(dx)$$

at a given point. However, for deriving further properties of our surface measures we shall need some additional assumptions.

Definition 1.2. *Given $y \in \mathbb{R}$, suppose that Φ_f is differentiable at y for each $f \in \mathcal{F}$. The measure σ^y on X is defined as follows*:

$$\int_X f(x)\sigma^y(dx) = \varrho_f(y), \quad f \in \mathcal{F}. \tag{1.6}$$

The hypothesis of differentiability of Φ_f is fulfilled if μ is Fomin differentiable along v and F satisfies the aforementioned assumptions (F1)–(F4).

This construction is close to the one described in [5], [6], [7] and later developed in [29] in the case of measures on locally convex spaces differentiable along constant vectors, but it requires only one-fold differentiability of F; in [29] the membership of F in the second Sobolev class is required and in [18], in the Gaussian case, also the second derivative is used (the function F is in the first Sobolev class, but its normalized Malliavin gradient must be also in the first Sobolev class).

By our construction,

$$\varrho_f(y) = \lim_{n\to\infty} n \int_{y<F<y+1/n} f(x)\nu(dx) = \lim_{n\to\infty} n \int_y^{y+1/n} \varrho_f(s)ds. \tag{1.7}$$

It is worth noting that we have another version of (1.4), namely, for each n we have the estimate

$$\left| n \int_{y<F<y+1/n} f(x)\nu(dx) \right| \le \|d_v\mu\| \cdot \|f\|_\infty + \|\mu\| \cdot \|\partial_v f\|_{L^1(\mu))}. \tag{1.8}$$

Obviously,

$$f \mapsto \varrho_f(y)$$

is a linear functional, so we have to show that this functional is represented by a bounded measure. As we have seen, this follows immediately by our assumption about $\mathcal{F}$ provided that μ is Fomin differentiable along v and F satisfies the stated assumptions.

It should be noted that the "non-normalized" surface measures introduced above are still not true "surface measures", since they depend not only of the level sets $F^{-1}(y)$, but also on the whole function F. For example, it we replace F by $2F$, the set $F^{-1}(0)$ does not change, but our measure σ^0 obviously does: it will be multiplied by 4, because the sets $\{0 < 2F < r\}$ are the old sets $\{0 < F < r/2\}$, so when evaluating the derivative of the distribution function at zero we get the factor 2; another factor 2 comes from $\partial_v(2F)$. Obviously, the whole thing depends also on our choice of the vector field v. This is a certain disadvantage of our definition, which will be partially overcome below (by passing to surface measures normalized by weights), but one should bear in mind that even dealing with very nice functions F on infinite-dimensional spaces, the known constructions do not really define surface measures on individual level sets $F^{-1}(y)$, as it happens with usual nice surfaces in $\mathbb{R}^d$, it is still necessary that each fixed surface be included in a special family of level sets. On the other hand, by using weight functions one can obtain "geometric surface measures" on the basis of our surface measures in case of a reasonable individual surface.

Lemma 1.3. *Suppose that there is $p > 1$ such that*

$$\|f\|_{L^p(\mu)} + \|\partial_v f\|_{L^1(\mu)} \le \|f\|_{\mathcal{F}}, \quad f \in \mathcal{F}. \tag{1.9}$$

Assume also that $\beta_v \in L^{p/(p-1)}(\mu)$. Then, for every open set $W \subset X$ and any $r > 0$, we have

$$\nu(W \cap \{y < F < y + r\}) \le rC(\mu)C_{\mathcal{F}}(W), \quad C(\mu) = \|\mu\| + \|\beta_v\|_{L^q(\mu)}. \tag{1.10}$$

Proof. Let $f \in \mathcal{F}$, $f \geq 0$ and $f \geq 1$ μ-a.e. on W. Then $f \geq 1$ ν-a.e. on W, hence on account of (1.5) and (1.9) we obtain

$$\nu(W \cap \{y < F < y + r\}) \leq \int_{W\cap\{y<F<y+r\}} f(x)\nu(dx)$$

$$\leq \int_{y<F<y+r} f(x)\nu(dx) \leq rC(\mu)(\|f\|_{L^p(\mu)} + \|\partial_v f\|_{L^1(\mu)}) \leq rC(\mu)\|f\|_{\mathcal{F}},$$

which yields the announced estimate by taking inf in f. □

Theorem 1.4. *Suppose that the assumptions of the previous lemma hold along with* (F1)–(F5). *For each $y \in \mathbb{R}$, the measure σ^y on $\mathcal{B}$ exists, is concentrated on the set $F^{-1}(y)$ and vanishes on all sets of $C_{\mathcal{F}}$-capacity zero. In addition, for $\nu \circ F^{-1}$-a.e. y, we have the equality*

$$\sigma^y = \varrho_1(y)\nu^y.$$

Proof. We can assume that $y = 0$. We know that for every $f \in \mathcal{F}$ the distribution function of the measure $(f \cdot \nu) \circ F^{-1}$ is differentiable at zero and its derivative is $\varrho_f(0)$, which must be the integral of f with respect to the desired surface measure. Clearly, this value is the limit of the integrals of f over the sets $B_n = \{0 < F < n^{-1}\}$ with respect to the measure $n \cdot \nu$. Such an integral can be written as the integral of f against the measure $\nu_n := nI_{B_n} \cdot \nu$. The nonnegative measures ν_n are uniformly bounded, since their values on the whole space X converge to $\varrho_1(0)$. It follows from our condition on $\mathcal{F}$ that there is a bounded nonnegative measure σ^0 on X such that the aforementioned integrals converge to the integral of f against σ^0. Indeed, this is true in the class of probability measures, to which everything reduces if $\varrho_1(0) > 0$, but in case of $\varrho_1(0) = 0$ we have convergence to zero in variation.

It follows from the definition of $\varrho_f(y)$ that

$$\int_{-\infty}^{+\infty}\int_X f(x)\sigma^y(dx)dy = \int_X f(x)\nu(dx) = \int_{\mathbb{R}}\int_X f(x)\nu^y(dx)\nu \circ F^{-1}(dy).$$

The integral on the left can be written as

$$\int_{-\infty}^{+\infty}\int_X f(x)\frac{1}{\varrho_1(y)}\sigma^y(dx)\varrho_1(y)dy = \int_{\mathbb{R}}\int_X f(x)\frac{1}{\varrho_1(y)}\sigma^y(dx)\nu \circ F^{-1}(dy),$$

hence the measure $\sigma^y/\varrho_1(y)$ coincides with the conditional measure ν^y for $\nu \circ F^{-1}$-a.e. y due to our assumption that $\mathcal{F}$ separates measures on $\mathcal{B}$ and the essential uniqueness of conditional measures.

Let us show that $\sigma^y(X\backslash F^{-1}(y)) = 0$. We can assume again that $y = 0$. It suffices to show that σ^0 vanishes on each set $U := \{|F| > \delta\}$, where $\delta > 0$. By assumption, for each n, there is a closed set A_n such that $C_{\mathcal{F}}(X\backslash A_n) < 1/n$ and $F|_{A_n}$ is continuous. The sets

$$U_n = U \cap (X\backslash A_n)$$

are open, because $\{|F| \leq \delta\} \cap A_n$ is closed by the continuity of the restriction $F|_{A_n}$. We have $U \subset \bigcap_{n=1}^{\infty} U_n$. Let $k > 1/\delta$. Then $\nu_k(U) = 0$, so by the lemma we have

$$\nu_k(U_n) = \nu_k(X\backslash A_n) \leq C(\mu)n^{-1},$$

hence $\sigma^0(U_n) \leq C(\mu)n^{-1}$, which yields that $\sigma^0(U) = 0$. Note that we could not derive this directly from the equality $\nu_k(U) = 0$, because U need not be open.

We now prove that $\sigma^y(B) = 0$ for every set $B \in \mathcal{B}$ of zero $C_{\mathcal{F}}$-capacity. Again it suffices to consider the case $y = 0$. Let $\varepsilon > 0$. By definition, there is an open set U containing B such that $C_{\mathcal{F}}(U) < \varepsilon$. Therefore, there is a function $f \in \mathcal{F} \geq 0$ such that $f \geq 1$ on μ-a.e. U and $\|f\|_{\mathcal{F}} < \varepsilon$. It follows by the lemma that $|\nu_n(U)| \leq \varepsilon C(\mu)$, which yields that $|\sigma^0(U)| \leq \varepsilon C(\mu)$. Letting $\varepsilon \to 0$ we arrive at the desired conclusion. □

In the considered situation we have the following version of the Gauss–Ostrogradskii–Stokes formula with our non-normalized surface measure. Set

$$V_r = F^{-1}(-\infty, r), \quad S_r = F^{-1}(r).$$

Proposition 1.5. *Let u be another vector field along which μ is differentiable, satisfying the same hypotheses as v. Then*

$$\int_{V_r} \beta_u(x)\mu(dx) = -\int_{S_r} \frac{\partial_u F(x)}{\partial_v F(x)}\, \sigma^r(dx).$$

Proof. Let $\psi_h(s) = 1$ if $s \le r$, $\psi_h(s) = 0$ if $s \ge r+h$, $\psi_h(s) = C - s/h$ if $r < s < r+h$, $C = 1 + r/h$. Then $\psi_h'(s) = -1/h$ in the interval $(r, r+h)$ and $\psi_h' = 0$ outside the closure of this interval. We have $\partial_u(\psi_h \circ F) = -h^{-1}\partial_u F$ on the set $\{r < F < r+h\}$ and

$$\int_X \psi_h(F(x))\delta_\mu u(x)\mu(dx) = -\int_X \partial_u \psi_h(F(x))\mu(dx) = h^{-1}\int_{r<F<r+h} \partial_u F(x)\mu(dx).$$

As $h \to 0$, the left-hand side of this identity tends to the integral of β_u over V_r and the right-hand side tends to the surface integral of the function $\partial_u F/\partial_v F$ against the surface measure σ^r. □

Remark 1.6. It follows from our construction that the mapping $y \mapsto \sigma^y$ is continuous provided that the space of probability measures is equipped with the weak topology. Indeed, according to (1.6), whenever $y_j \to y$, for each $f \in \mathcal{F}$, the integral of f against σ^{y_j} converges to the integral of f against σ^y, which by our assumption yields weak convergence.

In the framework described above there is no natural way of normalizing our non-normalized surface measures. One way of making the construction more invariant is this: assuming that $\partial_v F$ is quasi-continuous and positive quasi-everywhere and that $1/\partial_v F$ is μ-integrable, so that the measure $|\partial_v F|^{-2}\cdot\nu$ is finite, one can take new measures

$$\sigma_0^y := |\partial_v F|^{-2}\cdot \nu^y$$

on the same level sets $F^{-1}(y)$. These measures are finite for $\nu\circ F^{-1}$-a.e. y, hence also for $\mu\circ F^{-1}$-a.e. y. At the level of conditional measures the difference between our surface measures and the standard ones is that the former are proportional to conditional measures for ν and the latter are proportional to conditional measures for μ itself, so in case $\partial_v F \ne 0$ our surface measures are equivalent to the standard ones.

Let us now recall some concepts related to Gaussian measures (see [7], [9], and [10]). Let X be a locally convex space and let X^* be its topological dual. A Radon probability measure μ on X is called centered Gaussian if, for every functional $l \in X^*$, the induced measure $\mu\circ l^{-1}$ on the real line is centered Gaussian, i.e., is Dirac's measure at the origin or has a symmetric Gaussian density with respect to Lebesgue measure. The Cameron–Martin space H of μ consists of all vectors h with finite norm

$$|h|_H := \sup\{l(h)\colon\ l \in X^*,\ \|l\|_{L^2(\mu)} \le 1\}.$$

It is known that H with this norm is a separable Hilbert space compactly embedded into X; the corresponding inner product is denoted by $(\cdot,\cdot)_H$. A typical example: μ is the countable power of the standard Gaussian measure on the real line, X is the space $\mathbb{R}^\infty$ of all real sequences (the countable power of the real line), and $H = l^2$.

For every $h \in H$, there is a measurable liner functional $\widehat{h}$, belonging to the closure of X^* in $L^2(\mu)$, such that

$$l(h) = \int_X l(x)\widehat{h}(x)\mu(dx) \quad \forall\, l \in X^*.$$

The measure μ is Fomin differentiable along the constant vector field h and $\beta_h = -\widehat{h}$.

The Sobolev class $W^{p,1}(\mu)$, $p \in [1,+\infty)$, is defined as the completion of the class $\mathcal{FC}$ of functions of the form

$$f(x) = f_0(l_1(x),\dots,l_n(x)), \quad f_0 \in C_b^\infty(\mathbb{R}^n),\ l_i \in X^*,$$

with respect to the Sobolev norm

$$\|f\|_{p,1} = \|f\|_{L^p(\mu)} + \|D_H f\|_{L^p(\mu)} = \|f\|_{L^p(\mu)} + \left(\int_X |D_H f(x)|_H^p \mu(dx)\right)^{1/p},$$

where the gradient $D_H f(x) \in H$ is defined by

$$(D_H f(x), h)_H = \partial_h f(x) = \lim_{t\to 0} t^{-1}(f(x+th) - f(x)).$$

If $\{e_n\}$ is an orthogonal basis in H, then the vector $D_H f(x)$ has coordinates $\partial_{e_n} f(x)$. In the aforementioned case of the standard Gaussian product-measure functions of class $\mathcal{FC}$ are just smooth functions with bounded derivatives in finitely many variables, and $D_H f(x) = \nabla f(x)$. Similarly one defines Sobolev classes $W^{p,1}(\mu, E)$ of mappings with values in a separable Hilbert space E; in this case $D_H f(x)$ is an operator between H and E and the Hilbert–Schmidt norm of such operators is used to define the Sobolev norm. This means that in place of $|D_H f(x)|_H$ in the previous formula we use $\Big(\sum_{n=1}^\infty |\partial_{e_n} f(x)|_E^2\Big)^{1/2}$.

As a result of completing, every Sobolev function $f \in W^{p,1}(\mu)$ obtains a gradient $D_H f$, which is an L^p-mapping with values in H. It satisfies the integration by parts formula

$$\int_X \psi(x)(D_H f(x), h)_H \mu(dx) = -\int_X f(x)[\partial_h\psi(x) - \psi(x)\widehat{h}(x)]\mu(dx)$$

for all $\psi \in \mathcal{FC}$. Actually, this equality extends to $\psi \in W^{q,1}(\mu)$, $q = p/(p-1)$. By using this directional integration by parts formula, one can show that μ is differentiable along vector fields $v \in W^{p,1}(\mu, H)$. In this case for $v(x) = \sum_{n=1}^\infty v_n(x)e_n$ we have

$$\beta_v(x) = \sum_{n=1}^{\infty} (\partial_{e_n} v(x) - v_n(x)\widehat{e}_n(x)),$$

where the series converges in $L^p(\mu)$.

Inductively one defines higher Sobolev classes $W^{p,k}(\mu, E)$ with derivatives up to order k. For example, the class $W^{p,2}(\mu)$ consists of functions $f \in W^{p,1}(\mu)$ such that $D_H f \in W^{p,1}(\mu, H)$. Therefore, the measure μ is differentiable along the gradient field $v = D_H F$ once $F \in W^{p,2}(\mu)$.

In the case of a centered Gaussian measure μ with the Cameron–Martin space H and F belonging to the Sobolev class $W^{p,2}(\mu)$ (or belonging to $W^{p,1}(\mu)$ and having the gradient with divergence) our construction leads to usual surface measures considered in [1], [26], [7], [9], and [18] by taking $v = D_H F$ and $\partial_v F = |D_H F|_H^2$. If X is a Banach space, then we take for $\mathcal{F}$ the class of all bounded Lipschitzian functions. Conditions (F1)–(F5) are known to hold in this case. The case of a general locally convex space with a Radon Gaussian measure reduces to this one by the Tsirelson linear isomorphism theorem (see [7]).

We emphasize that for better surfaces (existing individually such as level sets of continuously Fréchet differentiable functions with nondegenerate derivatives) there is no need to involve variable vector fields D_H: it becomes much simpler to define surface measures locally by using only constant vector fields of differentiability of μ. In that case no second derivatives of F appear at all and in this way we recover the existence results of [40] (even under weaker assumptions).

In place of a Gaussian measure μ, it is possible to consider a Radon probability measure μ on a locally convex space X that is Fomin differentiable along a continuously embedded dense Hilbert space H; then one can also define Sobolev classes. This situation studied in [29], [30], and [31] has been the most general considered so far in the linear case. In the recent paper [18], similar results have been reproved in the Gaussian case (note that the proof of the fact that the constructed surface measures are concentrated on the corresponding level sets, given in [18], is incorrect if the defining function is not continuous: there might be no nonzero continuous functions with support in the set $\{|F| > r\}$). The approach suggested here leads to much shorter and simpler proofs than in the cited papers.

Remark 1.7. (i) It is worth noting that the case of a Fréchet space reduces to that of a separable reflexive Banach space, since every Radon measure on a Fréchet space is concentrated on a compactly embedded separable reflexive space (see [8, Theorem 7.12.4]). For many measures on Banach spaces (Gaussian, differentiable), the class of bounded Lipschitzian functions is a suitable candidate for $\mathcal{F}$, since such functions are almost everywhere differentiable with respect to such measures.

(ii) The class $\mathcal{FC}$ of bounded Lipschitzian cylindrical functions satisfies condition (F1), but not always (F2). However, for μ with compact support it suffices to ensure (F1) only on this compact support, so that $\mathcal{FC}$ works well. It should be also noted that it is possible to define surface measures locally in a suitable sense (for example, on compact sets) by replacing μ by $\zeta \cdot \mu$, where $\zeta \geq 0$ is a bump function whose support gives the desired localization. For example, in the Gaussian case or in the case of a differentiable measure on a Banach space, it is always possible to choose ζ in a such a way that its support will be compact and contain a given compact set and the measure $\zeta \cdot \mu$ will remain Fomin differentiable along the same directions as μ. This can give local surface measures in more general situations where there are no global surface measures. A possible way of gluing these local surface measures is based on establishing their uniform tightness.

If X is equipped with a suitable tangent structure enabling us to consider v not as a differentiation, but as a true vector field possessing the corresponding norm $|v(x)|$, one might try to use fields of unit length, but again the question of their choice arises. The choice $v = D_H F$ in the Gaussian case mentioned above is connected with another natural object: H-neighborhoods of sets. Given a Borel set B, we take the set $B^r = B + rU_H$, where U_H is the unit ball in the Cameron–Martin space H. The set B^r in general is much smaller than the usual metric r-neighborhood of B. Then, for certain "surfaces" B, the surface measure of B can be obtained as a limit of $\mu(B^r)/r$ as $r \to 0$. However, a precise definition on this way is more involved.

Among various restrictions on μ and F imposed above, certainly, the most stringent one is the existence of vector fields of differentiability for μ. For example, in many cases, given a measure μ on a metric space, one can take for $\mathcal{F}$ the space of bounded Lipschitzian functions; in many cases, such functions possess appropriate gradients μ-almost everywhere, so if F is locally Lipschitzian, then the only problem is to find suitable differentiability fields for the measure. It is not always possible to build such fields from constant vector fields (this happens already for distributions of diffusion processes with non-constant diffusion coefficients, see [9, Chapter 4]). It would be interesting to study vector fields of differentiability of measures in the framework of metric measure spaces.

2. Surface measures on surfaces of higher codimension

The construction developed in the previous section works also in the case of surfaces of higher codimension, but requires a bit more regularity of the mapping

$$F = (F_1, \dots, F_d)\colon\ X \to \mathbb{R}^d$$

on the level sets of which we wish to define surface measures. We recall that conditional measures are not sensitive at all to this change, they exist even for mappings with values in quite general infinite-dimensional spaces.

Now we need d vector fields $v_1, \dots, v_d$ along which the measure μ is differentiable. However, in the multidimensional case it is reasonable to modify our conditions on $\mathcal{F}$ as follows:

$$\varphi(f_1, \dots, f_n) \in \mathcal{F} \quad \forall\, f_1, \dots, f_n \in \mathcal{F}$$

for all Lipschitzian functions φ on $\mathbb{R}^n$ and

$$\partial_{v_i}(\varphi(f_1, \dots, f_n)) = \sum_{j=1}^{n} \partial_{x_j}\varphi(f_1, \dots, f_n)\partial_{v_i} f_j.$$

Suppose that $\psi(F) \in \mathfrak{D}_{v_i}$ for all Lipschitzian functions ψ on $\mathbb{R}^d$ with compact support and each $i = 1, \dots, d$. This enables us to define functions $\partial_{v_i} F_j$ as we have done in the one-dimensional case.

In place of $\partial_v F$ we now take the determinant Δ_F of the so-called Malliavin matrix

$$(\sigma_{ij})_{i,j \leq d}, \quad \sigma_{ij} := \partial_{v_i} F_j.$$

The minor in the Malliavin matrix corresponding to the element $\sigma_{ij}(x)$ is denoted by $M^{ij}(x)$. If the matrix $(\sigma_{ij}(x))_{i,j\leq d}$ is invertible, the inverse matrix will be denoted by $(\gamma^{ij}(x))_{i,j\leq d}$.

Suppose that $\Delta_F \in L^1(\mu)$. Set
$$\nu = \Delta_F \cdot \mu.$$
Let $U_r = \{x \in \mathbb{R}^d\colon\ |x| < r\}$ and $W_r = \{|F| < r\}$.

Proposition 2.1. *Suppose that $fM^{ij} \in \mathfrak{D}_{v_j}$ for all $i, j \le d$ and all $f \in \mathcal{F}$ vanishing outside of W_r. Then the measure $\nu \circ F^{-1}$ is absolutely continuous on U_r and has a density ϱ of class $BV(U_r)$. In particular, $\varrho \in L^{d/(d-1)}(U_r)$.*

If $\Delta_F(x) \neq 0$ μ-a.e. on W_r, then this density belongs to $W^{1,1}(U_r)$.

Finally, if
$$u_i := \frac{I_{W_r}}{\Delta_F} \sum_{j\le d} [\partial_{v_j} M^{ij} + M^{ij}\beta_{v_j}] \in L^s(\nu), \quad \text{where } s > d, \tag{2.1}$$
then this density belongs to $W^{p,1}(U_r)$ with some $p > d$ and has a continuous version.

Proof. Note that
$$\sum_{k\le d} M^{ij}(x)\sigma_{jk}(x) = \Delta_F(x)\delta_{ik},$$
where δ_{ik} is Kronecker's symbol. Indeed, this is true for invertible matrices, but remains valid for any matrix by approximation by invertible matrices. Let $\psi \in C_0^\infty(U_r)$. We have
$$\int_{U_r} \partial_{y_i}\psi(y)\nu\circ F^{-1}(dy) = \int_{W_r} \partial_{y_i}\psi(F(x))\Delta_F(x)\mu(dx)$$
$$= \int_X \sum_{j,k\le d} M^{ij}(x)\sigma_{jk}(x)[\partial_{y_k}\psi(F(x))]\mu(dx) = \int_X \sum_{j\le d} \partial_{v_j}(\psi\circ F)(x)M^{ij}(x)\mu(dx)$$
$$= -\sum_{j\le d}\int_X (\psi\circ F)(x)M^{ij}(x)d_{v_j}\mu(dx) - \sum_{j\le d}\int_X (\psi\circ F)(x)\partial_{v_j}M^{ij}(x)\mu(dx)$$
$$= -\sum_{j\le d}\int_X (\psi\circ F)(x)[\partial_{v_j}M^{ij}(x) + M^{ij}(x)\beta_{v_j}(x)]\mu(dx).$$
The right-hand side can be written as the integral of ψ with respect to a bounded measure on U_r, hence the measure $\nu\circ F^{-1}$ on U_r has a density ϱ of class $BV(U_r)$. By the Sobolev embedding theorem $\varrho \in L^{d/(d-1)}(U_r)$.

In case the measure ν is equivalent to μ the right-hand side can be written as the integral of $\psi g_i \varrho$, where g_i is the conditional expectation of the ν-integrable function $-u_i$ with respect to the measure ν and the σ-field generated by F. Therefore, $\varrho \in W^{1,1}(U_r)$.

Note that
$$\partial_{y_i}\varrho = g_i\varrho.$$
By Jensen's inequality for conditional expectations the inclusion $|u_i|^s \in L^1(\nu)$ yields the inclusion $|g_i|^s\varrho \in L^1(U_r)$.

We now show that $\partial_{y_i}\varrho$ is better integrable under the assumptions of the last assertion. Suppose that $\varrho \in L^p(U_r)$ for some $p \ge 1$. By Hölder's inequality we have $g_i\varrho \in L^{sp/(p+s)}(U_r)$. Therefore, $\varrho \in W^{p_1,1}(U_r)$ with $p_1 = sp/(p+s)$, which in case $p_1 < d$ by the Sobolev embedding yields that $\varrho \in L^{p_2}(U_r)$ with
$$p_2 = \frac{dp_1}{d-p_1} = p\frac{ds}{ds - p(s-d)} \ge p\frac{ds}{ds-s+d} = \lambda p, \quad \lambda = \frac{ds}{ds-s+d} > 1.$$
If $p_1 = d$, then $\varrho \in L^q(U_r)$ for any $q < \infty$, hence $\partial_{y_i}\varrho \in W^{s-\varepsilon,1}(U_r)$ for any $\varepsilon > 0$. Therefore, in finitely many steps we arrive at the situation where $\partial_{y_i}\varrho \in W^{p,1}(U_r)$ with some $p > d$. So the Sobolev embedding ensures a continuous density. □

We now give a constructive sufficient condition for the continuity of densities of multidimensional distributions related to μ rather than ν. This requires, however, second derivatives of F. In the next proposition we assume that $\partial_{v_k}\partial_{v_j}F_i$ can be defined in the same sense as $\partial_{v_j}F_i$ above by using that $\psi(\partial_{v_j}F) \in \mathfrak{D}_{v_k}$ for Lipschitzian functions on $\mathbb{R}^d$ with compact support.

Proposition 2.2. (i) *Suppose that for every $r \in \mathbb{N}$ there is $\varepsilon_r > 0$ such that the functions*

$$\exp\Bigl(\frac{\varepsilon_r}{\Delta_F^2}\Bigr),\ \exp\bigl(\varepsilon_r|\sigma_{ij}\beta_{v_k}|\bigr),\ \exp\bigl(\varepsilon_r|\partial_{v_k}\partial_{v_j}F_i|\bigr)$$

are μ-integrable on the set $\{|F| < r\}$. Then the measure $\mu\circ F^{-1}$ has a continuous density without zeros.

(ii) *Suppose that for every $r \in \mathbb{N}$ there is $p_r > d$ such that the functions*

$$|\gamma^{ij}\beta_{v_k}|^{p_r},\ |\partial_{v_k}\gamma^{ki}|^{p_r}$$

are μ-integrable on the set $\{|F| < r\}$. Then the measure $\mu \circ F^{-1}$ has a continuous density.

Proof. (i) We shall use the following result (see [11] or [9, Proposition 6.4.1]): if a nonnegative function ϱ on a ball $U \subset \mathbb{R}^d$ belongs to the Sobolev class $W^{1,1}(U)$ and there is $\varepsilon > 0$ such that $\varrho\exp(\varepsilon|\nabla\varrho|/\varrho) \in L^1(U)$, where we set $\nabla\varrho/\varrho = 0$ on the set $\{\varrho = 0\}$, then ϱ has a continuous version that is either identically zero or positive.

Let us fix $r \in \mathbb{N}$ and let U be the open ball of radius r in $\mathbb{R}^d$ centered at the origin. Let $\varphi \in C_0^\infty(U)$. We have

$$\begin{aligned}\int_U \partial_{y_i}\varphi(y)\mu\circ F^{-1}(dy) &= \int_X \partial_{y_i}\varphi(F(x))\mu(dx) = \int_X \sum_{k,j\le d}\gamma^{ik}\sigma_{kj}\partial_{y_j}\varphi(F(x))\mu(dx)\\ &= \int_X \sum_{k\le d}\gamma^{ik}\partial_{v_k}(\varphi\circ F)(x)\mu(dx) = -\int_{|F|<r}\sum_{k\le d}\varphi(F(x))[\partial_{v_k}\gamma^{ik} + \gamma^{ik}\beta_{v_k}]\mu(dx)\\ &= -\int_U \varphi(y)\eta_i(y)\mu\circ F^{-1}(dy),\end{aligned}$$

where η is the conditional expectation of the function $\sum_k[\partial_{v_k}\gamma^{ik} + \gamma^{ik}\beta_{v_k}]I_{\{|F|<r\}}$ with respect to the measure μ and the σ-field generated by F. It follows that the generalized derivative of the measure $\mu\circ F^{-1}$ on U in the variable y_i is the measure $\eta_i\cdot(\mu\circ F^{-1}) \ll \mu\circ F^{-1}$. Therefore, μ on U has a density $\varrho \in W^{1,1}(U)$ and $\partial_{y_i}\varrho/\varrho = \eta_i$. By our assumption and Jensen's inequality for conditional expectations, we arrive at the condition mentioned above.

(ii) If we are given that μ has a locally bounded density ϱ, then the previous relation can be written as

$$\int_U \partial_{y_i}\varphi(y)\varrho(y)dy = -\int_U \varphi(y)\eta_i(y)\varrho(y)dy,$$

which means that $\partial_{y_i}\varrho = \eta_i\varrho$ on U in the sense of distributions. We obtain again that $\varrho \in W^{1,1}(U)$, but now we conclude that $\partial_{y_i}\varrho \in L^{p_r}(U)$ by the same iteration of the Sobolev embedding theorem as above. Therefore, by the Sobolev embedding theorem ϱ has a continuous density (now it is not asserted that it is positive). ☐

Definition 2.3. *The surface measure σ^y on $F^{-1}(y)$ is defined by the formula*

$$\int_X f(x)\sigma^y(dx) = \varrho_f(y), \quad f \in \mathcal{F}.$$

This definition means that

$$\int_X f(x)\sigma^y(dx) = \lim_{r\to 0}\frac{1}{|U_r|}\int_{\{|F-y|<r\}} f(x)\nu(dx),$$

where $|U_r|$ is the usual volume of the ball U_r. The existence of the limit in the right-hand side is the only condition required by the definition and this condition is fulfilled in the situation of Proposition 2.1.

As in the previous section, we have to show that this relation defines a bounded measure.

Theorem 2.4. *Under the assumptions of Proposition* 2.1*, the assertions of Theorem* 1.4 *hold.*

The proof is essentially the same, however, we should note that the assumptions are now stronger.

Remark 2.5. Since $\sigma^y = \varrho_1(y)\nu^y$, every ν-integrable $\mathcal{B}$-measurable function g is σ^y-integrable for $\nu\circ F^{-1}$-almost every y. This enables us to define surface measures for $g\cdot\nu$.

References

[1] Airault H., Malliavin P. *Intégration géometrique sur l'espaces de Wiener*. Bull. Sci. Math. (2). 1988. V. 112, N 1. P. 3–52.

[2] Ambrosio L., Di Marino S. *Equivalent definitions of BV space and of total variation on metric measure spaces*. J. Funct. Anal. 2014. V. 266. P. 4150–4188.

[3] Ambrosio L., Miranda M. (Jr.), Maniglia S., Pallara D. *BV functions in abstract Wiener spaces*. J. Funct. Anal. 2010. V. 258, N 3. P. 785–813.

[4] Ambrosio L., Tilli P. Topics on analysis in metric spaces. Oxford University Press, Oxford, 2004.

[5] Bogachev V.I. *Smooth measures, the Malliavin calculus and approximation in infinite dimensional spaces*. Acta Univ. Carolinae, Math. et Phys. 1990. V. 31, N 2. P. 9–23.

[6] Bogachev V.I. *Differentiable measures and the Malliavin calculus*. J. Math. Sci. (New York). 1997. V. 87, N 4. P. 3577–3731.

[7] Bogachev V.I. Gaussian measures. Amer. Math. Soc., Providence, Rhode Island, 1998.

[8] Bogachev V.I. Measure theory. V. 1, 2. Springer, Berlin, 2007.

[9] Bogachev V.I. Differentiable measures and the Malliavin calculus. Amer. Math. Soc., Providence, Rhode Island, 2010.

[10] Bogachev V.I. Gaussian measures on infinite-dimensional spaces. In: Real and Stochastic Analysis. Current Trends (M.M. Rao ed.), pp. 1–83. World Sci., Singapore, 2014.

[11] Bogachev V.I., Mayer-Wolf E. *Absolutely continuous flows generated by Sobolev class vector fields in finite and infinite dimensions*. J. Funct. Anal. 1999. V. 167, N 1. P. 1–68.

[12] Bogachev V.I., Pilipenko A.Yu., Rebrova E.A. Classes of functions of bounded variation on infinite-dimensional domains. Dokl. Akad. Nauk. 2013. V. 451, N 2. P. 127–131 (in Russian); English transl.: Dokl. Math. 2013. V. 88, N 1. P. 391–395.

[13] Bogachev V.I., Pilipenko A.Yu., Shaposhnikov A.V. *Sobolev functions on infinite-dimensional domains*. J. Math. Anal. Appl. 2014. V. 419. V. 1023–1044.

[14] Caselles V., Lunardi A., Miranda M. (jun.), Novaga M. *Perimeter of sublevel sets in infinite dimensional spaces*. Adv. Calc. Var. 2012. V. 5, N 1. P. 59–76.

[15] Celada P., Lunardi A. *Traces of Sobolev functions on regular surfaces in infinite dimensions*. J. Funct. Anal. 2014. V. 266. P. 1948–1987.

[16] Chaari S., Cipriano F., Kuo H.-H., Ouerdiane H. Surface measures on the dual space of the Schwartz space. Comm. Stoch. Anal. 2010. V. 4. P. 467–480.

[17] Cheeger J. *Differentiability of Lipschitz functions on metric measure spaces*. Geom. Funct. Anal. 1999. V. 9. P. 428–517.

[18] Da Prato G., Lunardi A., Tubaro L. *Surface measures in infinite dimension*. Rendic. Lincei. 2014. V. 25, N 3. P. 309–330.

[19] Davydov Yu.A., Lifshits M.A., Smorodina N.V. Local properties of distributions of stochastic functionals. Translated from the Russian. Amer. Math. Soc., Providence, Rhode Island, 1998 (Russian ed.: Moscow, 1995).

[20] Feyel D., La Pradelle A. de. *Hausdorff measures on the Wiener space*. Potential. Anal. 1992. V. 1, N 2. P. 177–189.

[21] Fukushima M., Hino M. *On the space of BV functions and a related stochastic calculus in infinite dimensions*. J. Funct. Anal. 2001. V. 183, N 1. P. 245–268.

[22] Heinonen J. *Nonsmooth calculus*. Bull. Amer. Math. Soc. 2007. V. 44, N 2. P. 163–232.

[23] Hino M. *Sets of finite perimeter and the Hausdorff–Gauss measure on the Wiener space*. J. Funct. Anal. 2010. V. 258, N 5. P. 1656–1681.

[24] Hino M. *Dirichlet spaces on H-convex sets in Wiener space*. Bull. Sci. Math. 2011. V. 135. P. 667–683. Erratum: ibid., 2013. V. 137. P. 688–689.

[25] Keith S. *A differentiable structure for metric measure spaces*. Adv. Math. 2004. V. 183, N 2. P. 271–315.

[26] Malliavin P. Stochastic analysis. Springer-Verlag, Berlin, 1997.

[27] Pugachev O.V. *Surface measures in infinite-dimensional spaces*. Mat. Zametki. 1998. V. 63, N 1. P. 106–114 (in Russian); English transl.: Math. Notes. 1998. V. 63, N 1. P. 94–101.

[28] Pugachev O.V. *The Gauss–Ostrogradskii formula in infinite-dimensional space*. Mat. Sb. 1998. V. 189, N 5. P. 115–128 (in Russian); English transl.: Sb. Math. 1998. V. 189, N 5. P. 757–770.

[29] Pugachev O.V. *Construction of non-Gaussian surface measures by the Malliavin method*. Mat. Zametki. 1999. V. 65, N 3. P. 377–388 (in Russian); English transl.: Math. Notes. 1999. V. 65, N 3-4. P. 315–325.

[30] Pugachev O.V. *Tightness of Sobolev capacities in infinite dimensional spaces.* Infin. Dim. Anal. Quantum Probab. Relat. Top. 1999. V. 2, N 3. P. 427–440.

[31] Pugachev O.V. *Capacities and surface measures in locally convex spaces.* Teor. Verojatn. i Primen. 2008. V. 53, N 1. P. 178–188 (in Russian); English transl.: Theory Probab. Appl. 2009. V. 53, N 1. P. 179–190.

[32] Röckner M., Schmuland B. *Tightness of general $C_{1,p}$ capacities on Banach spaces.* J. Funct. Anal. 1992. V. 108, N 1. P. 1–12.

[33] Röckner M., Zhu R.-Ch., Zhu X.-Ch. *The stochastic reflection problem on an infinite dimensional convex set and BV functions in a Gelfand triple.* Ann. Probab. 2012. V. 40, N 4. P. 1759–1794.

[34] Röckner M., Zhu R.-Ch., Zhu X.-Ch. *BV functions in a Gelfand triple for differentiable measures and its applications.* Forum Math. 2015.

[35] Smorodina N.V. *Differential calculus in configuration space and stable measures. I.* Teor. Verojatn. i Primen. 1988. V. 33, N 3. P. 522–534 (in Russian); English transl.: Theory Probab. Appl. 1988. V. 33, N 3. P. 488–499.

[36] Stroock D. *The Malliavin calculus, a functional analytic approach.* J. Funct. Anal. 1981. V. 44, N 2. P. 212–257.

[37] Sturm K.-T. *On the geometry of metric measure spaces. I, II.* Acta Math. 2006. V. 196, N 1. P. 65–131, 133–177.

[38] Uglanov A.V. *Surface integrals in a Banach space.* Mat. Sb. 1979. V. 110, N 2. P. 189–217 (in Russian); English transl.: Math. USSR Sb. 1981. V. 38. P. 175–199.

[39] Uglanov A.V. *Surface integrals in Fréchet spaces.* Mat. Sb. 1998. V. 189, N 11. P. 139–157 (in Russian); English transl.: Sb. Math. 1998. V. 189, N 11. P. 1719–1737.

[40] Uglanov A.V. Integration on infinite-dimensional surfaces and its applications. Kluwer Acad. Publ., Dordrecht, 2000.

[41] Uglanov A.V. *Surface integrals in locally convex spaces.* Tr. Mosk. Mat. Ob-va. 2001. V. 62. P. 262–285 (in Russian); English transl.: Trans. Mosc. Math. Soc. 2001. P. 249–270.

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