GIBBS STATES OF AMORPHOUS MEDIA

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Abstract. We study a class of Gibbs measures of classical particle spin systems with unbounded pair interactions on a graph given by a random configuration of points distributed according to a random point process in $\mathbb{R}^d$. We prove the existence conditions and study support of these measures. Moreover we show their measurability with respect to the random parameter and derive averaged moment estimates.

1. Introduction

The aim of this paper is to study a class of Gibbs random fields describing equilibrium states of the following model of interacting particle system. A countable collection (configuration) γ of point particles is chaotically distributed over a Euclidean space $X = \mathbb{R}^n$, with their positions realized by a given random point process in X. Each of the particles possesses internal structure characterized by a marks (or spin) $\sigma(x) \in S$, where $S = \mathbb{R}^m$ is another Euclidean space. The spin-spin interaction is supposed to be pair-wise, with the intensity depending on the location of particles. Such settings corresponds to the physical model called an *amorphous crystal*, where the spins are displacements of the anharmonic oscillators from their random equilibrium positions in X.

To describe possible configurations of spins attached to γ, we consider the product space $S^\gamma := \prod_{x\in\gamma} S_x,\ S_x = S$, so that any element of S^γ has the form $\sigma = (\sigma(x))_{x\in\gamma},\ \sigma(x) \in S$. The space S^γ is endowed with the product topology and the associated Borel σ-algebra $\mathcal{B}(S^\gamma)$. Our aim is to construct Gibbs measures ν_γ on $(S^\gamma, \mathcal{B}(S^\gamma))$ (for almost all configurations γ) which correspond to the (heuristic) energy function

$$E(\sigma) := \sum_{\{x,y\}\subset\gamma} W_{x,y}(\sigma(x), \sigma(y)) + \sum_{x\in\gamma} V_x(\sigma(x)), \tag{1.1}$$

where $W_{x,y} : S \times S \to \mathbb{R}$ and $V_x : S \to \mathbb{R}$ are given interaction potentials satisfying certain stability assumptions (for a rigorous definition see Section 3). The first sum in (1.1) runs over all (unordered) pairs of distinct points x, y from γ. Actually, the expression (1.1) makes sense only for finite configurations γ (having however the probability zero). Taking a standard root in equilibrium statistical mechanics,

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Gibbs measures are defined via their conditional finite volume distributions (*specifications*) and the *Dobrushin-Lanford-Ruelle* (DLR for short) equation, see Section 3. The fundamental question arising here is whether the set $\mathcal{G}(S^\gamma)$ of all Gibbs measures associated with E is non-empty. The answer certainly depends on the structure of the underlying configuration γ and properties of the functions W and V. To make the situation simpler, we assume that the pair interaction has *finite range*, i.e.,

$$W_{x,y} \equiv 0 \quad \text{if} \quad |x-y| > R \tag{1.2}$$

for some fixed $R > 0$. In this case, a crucial characteristic of the configuration γ is the behavior of the number $n_{\gamma,R}(x)$ of its elements $y \neq x$ in the ball of radius R centred at x.

Observe that the configuration γ can be endowed with a natural *graph* structure $(\mathcal{V}, \mathcal{E})$ by defining the set of vertices $\mathcal{V} :=\gamma$ and the set of (unordered) edges

$$\mathcal{E} := \{\{x,y\} \subset \gamma : |x-y| \leq R\}\,. \tag{1.3}$$

Then $n_{\gamma,R}(x)$ represents the degree the vertex $x \in \gamma$, that is, the number of its nearest neighbors. In this setting, the energy function E describes an infinite system of classical spins $(\sigma(x))_{x\in\gamma}$ coupled via the nearest neighbor interaction. There is an extensive literature on such models in the situation of bounded degree graphs (that is, when the function $n_{\gamma,R}$ is globally bounded), in particular, in the case where γ is a regular integer (or group) lattice (see, e.g., the seminal papers [21, 31]). In the situation of unbounded degree graphs and unbounded spins, the question of existence of Gibbs measures was first studied in [13], where certain growth conditions on the function $n_{\gamma,R}$ and stability conditions on the potentials V_x, $W_{x,y}$ were posed. The next core extension, including the system of our interest, is to consider Gibbs measures on random graphs. By now, even the initial question about existence of such measures has been remaining open (except the case of a compact single spin space S where the answer is always positive, see e.g. Proposition 5.3 in [28]). So far, there exist only few publications on the mathematically rigorous theory of systems living on amorphous substances, which are mostly dealing with systems of bounded spins in the so-called *annealed approach* (see [10] and references therein).

In the present work, we consider a system of interacting anharmonic oscillators on a random set γ represented by a typical element of the space $\Gamma(X)$ of locally finite configurations in X, equipped with a probability measure μ (e.g. a Poisson or, more generally, Gibbs measure). More specifically, we suppose that the pair potential $W_{x,y}$ (not necessary attractive) is a bilinear form on S and the one-particle potential V_x has a super-quadratic growth. In general, the random variable $n_{\gamma,R}$ appears to be unbounded and non-stationary for μ-a.a. configurations γ. In Section 2, we study its behavior and derive certain bounds on its growth for a typical γ (under mild condition of boundedness of correlation functions of the measure μ). In Section 3, we prove that the set $\mathcal{G}(S^\gamma)$ is non-empty; its elements can be interpreted as *quenched states* in the terminology commonly accepted in the theory of disordered systems (see e.g. [4]). Moreover, we describe the support of the corresponding Gibbs measures and prove uniform estimates of their exponential moments. As is conventional for systems of unbounded spins, we have to confine ourselves to a proper subset (latter on denoted by $\mathcal{G}_{\alpha,p}(S^\gamma)$) of tempered Gibbs measures with controlled growth. In Section 4 we comment on the results obtained and outline their possible extensions.

To prove the results mentioned above, we develop an analytic method based on exponential moment bounds for local Gibbs specification and their weak dependence on boundary condition. Such technique is suited well to work with spatially irregular systems, for its implementations to particular models see [13, 14, 17]. In view of specific properties of the graph $(\mathcal{V},\mathcal{E})$ associated with a typical configuration $\gamma \in \Gamma(X)$, such as the unboundedness of the degree function $n_{\gamma,R}(x)$, $x \in \gamma$, and the lack of spatial transitivity of $\mathcal{V} = \gamma$ as a subset of X, the two fundamental approaches to the study of Gibbs random fields in statistical mechanics – Ruelle's (super-) stability technique [30, 31] and general Dobrushin's existence and uniqueness criteria [8] – are not directly applicable to our model. Moreover, the uniqueness problem in the considered model remains open except for a special case of the convex attractive interaction discussed in Section 2.4 of [6]. As was shown [6], for ferromagnetic models with the interaction (1.1) there might exist multiple Gibbs measures, which means that the map $\Gamma(X) \ni \gamma \to \{\nu_\gamma \in \mathcal{G}(S^\gamma)\}$ is set-valued.

In Section 5 we prove that there exist *measurable selections* $\Gamma(X) \ni \gamma \mapsto \nu_\gamma \in \mathcal{G}(S^\gamma)$. Such measurability is a key property if one speaks about *averages with respect to disorder*, that is about expectations $\int_{\Gamma(X)} \Phi(\mathbb{E}_{\nu_\gamma} F)\mu(d\gamma)$ for appropriate functions $\Phi : \mathbb{R} \to \mathbb{R}$ and $F : S^\gamma \to \mathbb{R}$. The measurable maps $\gamma \mapsto \nu_\gamma$ are then called *random Gibbs measures*, see e.g. Section 6.2 in [4]. The novelty of our situation is that the measures $\nu_\gamma \in \mathcal{G}_{\alpha,p}(S^\gamma)$ live (for different γ) on different spaces, and it is not clear in what sense this measurability can be understood. In this paper, we will identify the spaces S^γ that support measures ν_γ with the fibres of a natural bundle over $\Gamma(X)$ and extend ν_γ to the total space $\mathfrak{X}$ of this bundle by setting $\nu_\gamma(\mathfrak{X} \setminus S^\gamma) = 0$. It turns out that the space $\mathfrak{X}$ can be identified with the *marked configuration space* $\Gamma(X, S)$. For the definitions and description of main structures on marked configuration spaces we refer to [1], [5], [19] . In addition, necessary information on topological structure of $\Gamma(X, S)$ is given in Appendix; the latter material may be of independent interest in the theory of marked point processes and its applications. The mentioned embedding in the extended space $\Gamma(X, S)$ enables us to give a constructive procedure of obtaining measurable selections $\gamma \mapsto \nu_\gamma$ by means of *Komlós' theorem*. This theorem is a renowned tool in the probability theory providing a.s.-convergence of Cesàro means for integrable functions. On the physical level we see here a certain analogy with the *Newman–Stein approach* which uses space averaging to control the chaotic size dependence (see [25, 26]). Finally, we obtain *à-priori* bounds on the quenched moments of the random Gibbs measures, with the constants explicitly computable in terms of the model parameters.

2. Estimates for a typical configuration.

Let us consider the space $\Gamma(X)$ of locally finite configurations (subsets) of X, that is,

$$\Gamma(X) := \{\gamma \subset X : \; N(\gamma \cap \Lambda) < \infty \text{ for any compact } \Lambda \subset X\},$$

where $N(A)$ denotes cardinality of the set A. We equip $\Gamma(X)$ with the vague topology, that is, the weakest topology that makes continuous all mappings

$$\Gamma(X) \ni \gamma \mapsto \langle f, \gamma \rangle := \sum_{x\in\gamma} f(x),$$

$f \in C_0(X)$ (=: the set of continuous functions on X with compact support). It is known that this topology is completely metrizable, which makes $\Gamma(X)$ a Polish space (see. e.g. 15.7.7 in [11] or Proposition 3.17 in [29]). An explicit construction of the appropriate metric can be found in [16]. By $\mathcal{P}(\Gamma(X))$ we denote the space of all probability measures on the Borel σ-algebra $\mathcal{B}(\Gamma(X))$ of $\Gamma(X)$.

Let us fix some $\mu \in \mathcal{P}(\Gamma(X))$. A measurable symmetric (w.r.t. permutations of its coordinate) function

$$0 \leq k_m : X^m \to \mathbb{R}, \quad m \in \mathbb{N},$$

is called the m-th order *correlation function* of μ if for any non-negative measurable symmetric function $f : X^m \to \mathbb{R}$ the following equality holds

$$\begin{aligned} &\int_{\Gamma(X)} \sum_{\{x_1,\dots,x_m\}\subset\gamma} f(x_1,...,x_m)\mu(d\gamma) \\ = \quad &\frac{1}{m!}\int_{X^m} f(x_1,...,x_m)k_m(x_1,...,x_m)dx_1...dx_m. \end{aligned} \tag{2.1}$$

From now on we assume that all correlation functions of μ up to some order $M \in \mathbb{N}$ *exist* and are *bounded*, i.e.,

$$||k_m||_\infty := \operatorname*{ess\,sup}_{X^m} k_m(x_1,...,x_m) < \infty, \quad 1 \leq m \leq M. \tag{2.2}$$

Remark 2.1. The condition (2.2) holds for a wide class of measures on $\Gamma(X)$. In statistics of point processes, correlation functions k_m appear as densities (w.r.t. $dx_1...dx_m$) of the so-called m-th factorial moment measures for μ (see e.g. Section 5.4 of [7]). According to (2.1), $k_m(x_1,...,x_m)dx_1\,...dx_m$ can be interpreted as the $\mu(d\gamma)$-expectation for finding particles from $\gamma \in \Gamma(X)$ in each of the infinitesimal volumes $dx_1\,,...,\,dx_m$ (see e.g. § 4.1.1 in [30] and Section 3 in [22]). For a standard Poisson point process $\mu := \pi_z$ with the activity parameter $z > 0$ and Lebesgue intensity measure zdx, the correlation functions $k_m(x_1,...,x_m)$ are just constants z^m, $m \in \mathbb{N}$. If there exists $\zeta > 0$ such that $||k_m||_\infty \leq \zeta^m$ for all $m \in \mathbb{N}$, we say that the correlations functions k_m are sub-Poissonian or satisfy Ruelle's bound. Such measures typically arise in classical statistical mechanics as Gibbs modifications of the Poisson measure π_z by means of stable interactions, see [30, 31]. Note that any probability measure μ on $\Gamma(X)$ obeying the Ruelle bound is uniquely determined by its correlation functions $(k_m)_{m\in\mathbb{N}}$ and has all local moments finite, i.e.,

$$\int\limits_{\Gamma(X)} |\langle f,\gamma\rangle|^N \mu(d\gamma) < \infty, \qquad f \in C_0(X), \quad N \in \mathbb{N}.$$

General criteria allowing for reconstruction of the state $\mu \in \mathcal{P}(\Gamma(X))$ from a given system of correlation functions $(k_m)_{m\in\mathbb{N}}$ are established in [1, 15, 22].

Recall that for each $x \in \gamma$, being considered as a vertex of the associated graph $(\mathcal{V}, \mathcal{E})$ (cf. (1.3)), we have defined its degree as

$$n_{\gamma,R}(x) := N\left(\{y \in \gamma :\ y \sim x\}\right) \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}, \tag{2.3}$$

where $y \sim x$ means that $\{x, y\} \in \mathcal{E}$, i.e., $0 < |y - x| \leq R$. For $\alpha, r > 0$ to be specified below, we introduce the weights

$$w_\alpha(x) := e^{-\alpha|x|}, \quad x \in X,$$

and consider the following functions on $\Gamma(X)$

$$a_{\alpha,r}(\gamma) = \sum_{\substack{\{x,y\}\subset\gamma \\ |y-x|\le R}} w_\alpha(x)\left[n_{\gamma,R}(x)n_{\gamma,R}(y)\right]^r, \tag{2.4}$$

$$b_\alpha(\gamma) = \sum_{x\in\gamma} w_\alpha(x) = \langle w_\alpha, \gamma\rangle . \tag{2.5}$$

One can prove that $a_{\alpha,r}$ and b_α are $\mathcal{B}(\Gamma(X))$-measurable by using expansion of γ over its n-particle subsets ($n \in \mathbb{Z}_+$) in finite volumes $\Lambda \subset X$.

Proposition 2.2. *Let μ satisfy condition (2.2) with some integer $M \ge 2$. Then for any $\alpha > 0$ and $0 \le r \le M/2 - 1$, we have inclusions $a_{\alpha,r}$, $b_\alpha \in L^1(\Gamma(X), \mu)$.*

Proof. 1) Let us first check that $b_\alpha \in L^1(\Gamma(X), \mu)$. Applying (2.1) to the function $w_\alpha \in L^1(X)$ we obtain

$$\begin{aligned}\int_{\Gamma(X)} b_\alpha(\gamma)\ \mu(d\gamma) &= \int_{\Gamma(X)} \sum_{x\in\gamma} w_\alpha(x)\ \mu(d\gamma) \\ &= \int_X w_\alpha(x)k_1(x)dx \le ||k_1||_\infty \int_X e^{-\alpha|x|}dx < \infty. \end{aligned}\tag{2.6}$$

2) Now we proceed to proving the inclusion $a_{\alpha,r} \in L^1(\Gamma(X), \mu)$. Observe that, because $n_{\gamma,R}(x)n_{\gamma,R}(y)$ is either 0 or ≥ 1, we have $a_{\alpha,r}(\gamma) \le a_{\alpha,r'}(\gamma)$ whenever $r \le r'$. Thus it is sufficient to prove the inclusion $a_{\alpha,r} \in L^1(\Gamma(X), \mu)$ just for $r = M/2 - 1$.

Let us fix some $x \in \gamma$. Clearly, for any $y \in \gamma$ such that $|x - y| \le R$ we have

$$n_{\gamma,R}(y) \le n_{\gamma,2R}(x),$$

which yields

$$\begin{aligned}\sum_{y\in\gamma\setminus\{x\}} \left[n_{\gamma,R}(x)n_{\gamma,R}(y)\right]^r &\le n_{\gamma,R}(x)\left[n_{\gamma,R}(x)n_{\gamma,2R}(x)\right]^r \\ &\le n_{\gamma,2R}(x)^{2r+1} = n_{\gamma,2R}(x)^{M-1}.\end{aligned}$$

Observe that

$$n_{\gamma,2R}(x) = N\left(\{y \in \gamma : 0 < |x-y| \le 2R\}\right) = \sum_{\substack{y\in\gamma \\ y\ne x}} \mathbf{1}_{B_{2R}}(y-x),$$

where B_{2R} is the closed ball of radius $2R$ centred at 0 and $\mathbf{1}_{B_{2R}}$ is the corresponding indicator function. Thus, we have the multinomial expansion

$$\begin{aligned}n_{\gamma,2R}(x)^{M-1} &= \left(\sum_{y\in\gamma\setminus\{x\}} \mathbf{1}_{B_{2R}}(y-x)\right)^{M-1} \\ &= \sum_{y_1,\dots,y_{M-1}\in\gamma\setminus\{x\}} \prod_{k=1}^{M-1} \mathbf{1}_{B_{2R}}(y_k - x) \\ &= \sum_{j=1}^{M-1} c_j \sum_{\{y_1,\dots,y_j\}\in\gamma\setminus\{x\}} \prod_{k=1}^{j} \mathbf{1}_{B_{2R}}(y_k - x)\end{aligned}\tag{2.7}$$

with the coefficients

$$c_j := \sum_{\substack{i_1,\dots,i_j\in\mathbb{N}\\ i_1+\dots+i_j=M-1}} \frac{(M-1)!}{i_1!\dots i_j!}, \quad 1\le j\le M-1. \tag{2.8}$$

Let us introduce notations $\bar{y}_j := (y_0, y_1, ..., y_j) \in \gamma^{j+1}$ and $\{\bar{y}_j\} := \{y_0, y_1, ..., y_j\} \subset \gamma$ for the vector and configuration with components y_0, $y_1, ..., y_j \in \gamma$, respectively, and consider functions

$$f_j(\bar{y}_j) = w_\alpha(y_0)\prod_{k=1}^{j}\mathbf{1}_{B_{2R}}(y_k - y_0)$$

and

$$\hat{f}_j(\bar{y}_j) = \sum_{s\in S_{j+1}} f_j(s(\bar{y}_j)),$$

where S_i is the permutation group of order i. By construction $\hat{f}_j$ is a symmetric function that dominates f_j and hence it satisfies Eq. (2.1), that is,

$$\int_{\Gamma(X)}\sum_{\{\bar{y}_j\}\subset\gamma}\hat{f}_j(\bar{y}_j)\mu(d\gamma) = \frac{1}{(j+1)!}\int_{X^{j+1}}\hat{f}_j(\bar{y}_j)k_{j+1}(\bar{y}_j)d\bar{y}_j.$$

Thus we have the following estimates

$$a_{\alpha,r}(\gamma) \le \sum_{x\in\gamma} w_\alpha(x)n_{\gamma,2R}(x)^{M-1} = \sum_{j=1}^{M-1} c_j \sum_{\{\bar{y}_j\}\subset\gamma}\hat{f}_j(\bar{y}_j)$$

and

$$\int_{\Gamma(X)} a_{\alpha,r}(\gamma)\mu(d\gamma) \le \sum_{j=1}^{M-1}\frac{c_j}{(j+1)!}\int_{X^{j+1}}\hat{f}_j(\bar{y}_j)k_{j+1}(\bar{y}_j)d\bar{y}_j.$$

Finally, because of the symmetricity of the correlation functions, we obtain the estimate

$$\begin{aligned}\int_{\Gamma(X)} a_{\alpha,r}(\gamma)\mu(d\gamma) &\le \sum_{j=1}^{M-1}\frac{c_j\,|S_{j+1}|}{(j+1)!}\int_{X^{j+1}} w_\alpha(y_0)\prod_{k=1}^{j}\mathbf{1}_{B_{2R}}(y_k-y_0)k_{j+1}(\bar{y}_j)d\bar{y}_j \\ &\le \|k\|_\infty \sum_{j=1}^{M-1} c_j \mathrm{Vol}(B_{2R})^j \int_X e^{-\alpha|x|}dx < \infty, \end{aligned} \tag{2.9}$$

where $\mathrm{Vol}(B_{2R})$ is the volume of the ball B_{2R} and $||k||_\infty := \max_{1\le m\le M}||k_m||_\infty$, and the proof is complete. ■

Corollary 2.3. *Under the conditions of Proposition 2.2 we have $a_{\alpha,r}(\gamma)$, $b_\alpha(\gamma) < \infty$ for μ-a.a. γ.*

Remark 2.4. For the Poisson measure $\mu := \pi_z$ on $\Gamma(X)$ there is an alternative way of proving Proposition 2.2 based on the Mecke identity (see e.g. Proposition 13.1.VII in [7]). The later states that for any measurable function $F : X \times \Gamma(X) \to \mathbb{R}_+$

$$\int_{\Gamma(X)}\sum_{x\in\gamma}F(x,\gamma)\pi_z(d\gamma) = \int_{\Gamma(X)}\int_X F(x,\gamma\cup\{x\})zdx\pi_z(d\gamma).$$

By the translation invariance of π_z this immediately yields

$$\int_{\Gamma(X)} b_\alpha(\gamma)\ \pi_z(d\gamma) = z \int_X e^{-\alpha|x|} dx$$

and

$$\begin{aligned}\int_{\Gamma(X)} a_{\alpha,r}(\gamma)\pi_z(d\gamma) &\leq \int_{\Gamma(X)} \sum_{x\in\gamma} w_\alpha(x) \left[n_{\gamma,2R}(x)\right]^{2r+1} \pi_z(d\gamma) \\ &= z \int_{\Gamma(X)} \left[n_{\gamma,2R}(0)\right]^{2r+1} \pi_z(d\gamma) \int_X e^{-\alpha|x|} dx,\end{aligned}$$

which are finite for any $\alpha, r > 0$. Similar reasonings work also for Gibbs measures μ of Ruelle's type by applying the Georgii-Nguyen-Zessin identity for them.

3. Construction of Gibbs measures

The aim of this section is to construct a class of quenched Gibbs measures on the product space S^γ. Following the standard *Dobrushin-Lanford-Ruelle approach* in statistical mechanics, see e.g. the monographs [9, 28], the Gibbs random fields are described through a system of their local conditional distributions constituting the so-called Gibbsian specification. In the practical realization of this approach for our model, the main technical problem is to control the spatial irregularity of the configuration γ and unboundedness of the interaction. To illustrate the key ideas, we focus our attention on the case of pair interactions having *finite radius* and *quadratic growth*. Possible generalizations are discussed in Section 4.

In what follows, we will write $|\cdot|$ for the corresponding Euclidean norms in both X and S. Let

$$J : X \to S \otimes S$$

be a bounded continuous matrix-valued mapping with

$$\operatorname{supp} J \subset B_R = \{x \in X \ : \ |x| \leq R\}. \tag{3.1}$$

Set

$$||J||_\infty := \sup_{x\in X} \|J(x)\|_{S\otimes S}$$

and assume that it is finite. For any $x, y \in X$ define a pair potential

$$W_{x,y} : S \times S \to \mathbb{R}$$

by the formula

$$W_{x,y}(u, v) = J(x - y)u \cdot v, \quad u, v \in S,$$

where $\cdot$ denotes the Euclidean inner product in S. Let $V : S \to \mathbb{R}$ be a continuous function satisfying the *super-quadratic* growth estimate

$$V(u) \geq a_V\, |u|^q - b_V, \quad u \in S, \tag{3.2}$$

for some constants $a_V, b_V > 0$ and $q > 2$. The latter condition is aimed to compensate the destabilizing effects caused by the unbounded pair interactions $W_{x,y}$ by means of the strong enough growth of the one-particle potentials $V_x := V$. Note that the case $q = 2$ cannot be covered by our scheme because of the absence of the uniform bound on vertex degrees in the underlying graph $(\mathcal{V}, \mathcal{E})$.

Let $\mathcal{F}(\gamma)$ be the family of all finite subsets of γ. For any $\eta \in \mathcal{F}(\gamma)$, $\sigma_\eta = (\sigma(x))_{x\in\eta} \in S^\eta$ and $\xi = (\xi(y))_{y\in\gamma} \in S^\gamma$ define the relative local interaction energy

$$E_\eta(\sigma_\eta \,|\xi\,) = \sum_{\{x,y\}\subset\eta} W_{x,y}(\sigma(x),\sigma(y)) + \sum_{\substack{x\in\eta\\ y\in\gamma\setminus\eta}} W_{x,y}(\sigma(x),\xi(y)) + \sum_{x\in\eta} V_x(\sigma(x)). \quad (3.3)$$

The corresponding specification kernel $\Pi_\eta(d\sigma\,|\xi\,) \in \mathcal{P}(S^\gamma)$ is given by the formula

$$\int_{S^\gamma} f(\sigma)\Pi_\eta(d\sigma|\xi) = Z(\xi)^{-1}\int_{S^\eta} f(\sigma_\eta \times \xi_{\gamma\setminus\eta})\exp\ [-E_\eta(\sigma_\eta\,|\xi\,)]\, d\sigma_\eta, \quad (3.4)$$

where $f \in L^\infty(S^\gamma)$ (=: the set of bounded Borel function on S^γ) and

$$Z(\xi) = \int_{S^\eta} \exp\ [-E_\eta(\sigma_\eta\,|\xi\,)]\, d\sigma_\eta$$

is a normalizing factor. Observe that the integral in the right-hand side of (3.4) is well-defined because of estimate (3.2). For each fixed $\xi \in S^\gamma$, $\Pi_\eta(d\sigma|\xi)$ is a probability measure on S^γ and, for each fixed $B \in \mathcal{B}(S^\gamma)$, the map $S^\gamma \ni \xi \to \Pi_\eta(B|\xi) \in [0,1]$ is measurable. The family $\Pi_\gamma := \{\Pi_\eta(d\sigma|\xi)\}_{\eta\in\mathcal{F}(\gamma),\xi\in S^\gamma}$ constitutes a Gibbsian specification (see e.g. [9, 28]). In particular, it satisfies the consistency property

$$\int_{S^\gamma} \Pi_{\eta_1}(B\,|\sigma\,)\Pi_{\eta_2}(d\sigma\,|\xi\,) = \Pi_{\eta_2}(B\,|\xi\,), \quad (3.5)$$

which holds for any $B \in \mathcal{B}(S^\gamma)$, $\xi \in S^\gamma$ and $\eta_1, \eta_2 \in \mathcal{F}(\gamma)$ such that $\eta_1 \subset \eta_2$.

Let ν be a probability measure on S^γ. We say that ν is a *quenched Gibbs measure* associated with the (heuristic) energy functional (1.1) if it satisfies the DLR equation

$$\nu(B) = \int_{S^\gamma} \Pi_\eta(B|\xi)\nu(d\xi) \quad (3.6)$$

for all $B \in \mathcal{B}(S^\gamma)$ and $\eta \in \mathcal{F}(\gamma)$. For a given $\gamma \in \Gamma(X)$, we denote by $\mathcal{G}(S^\gamma)$ the set of all such measures.

Observe that $S^\gamma = S^{\gamma_1} \times S^{\gamma_2}$ for any disjoint decomposition $\gamma = \gamma_1 \sqcup \gamma_2$. If, moreover, the distance between configurations γ_1 and γ_2 is greater than R, we have the following simple result.

Lemma 3.1. *Let $\gamma = \gamma_1 \cup \gamma_2$, $\gamma_1 \cap \gamma_2 = \emptyset$, and $\inf_{\substack{x_1\in\gamma_1\\ x_2\in\gamma_2}} |x_1 - x_2| > R$. Consider Gibbs measures $\nu^{(1)} \in \mathcal{G}(S^{\gamma_1})$ and $\nu^{(2)} \in \mathcal{G}(S^{\gamma_2})$. Then*

$$\nu := \nu^{(1)} \otimes \nu^{(2)} \in \mathcal{G}(S^\gamma).$$

Proof. It follows directly from definition (3.4) that the specification Π splits into the product of the corresponding specifications $\Pi^{(1)}$ and $\Pi^{(2)}$on the spaces S^{γ_1} and S^{γ_2} respectively, that is,

$$\Pi_\eta(d\sigma|\xi) = \Pi^{(1)}_{\eta_1}(d\sigma_1|\xi_1) \otimes \Pi^{(2)}_{\eta_2}(d\sigma_2|\xi_2),$$

where $\eta_k := \eta \cap \gamma_k$, $\sigma_k := (\sigma(x))_{x\in\eta_k}$, $\xi_k := (\xi(y))_{y\in\eta_k} \in S^{\gamma_k}$, $k = 1,2$. A direct calculation shows that the measure $\nu := \nu^{(1)} \otimes \nu^{(2)}$ satisfies the DLR equation (3.6) if and only if $(\nu^{(1)}, \Pi^{(1)})$ and $(\nu^{(2)}, \Pi^{(2)})$ satisfy the corresponding DLR equations on S^{γ_1} and S^{γ_2} respectively. ∎

Our next goal is to prove the existence of Gibbs measures supported on certain spaces of tempered sequences from S^γ for μ-a.a. $\gamma \in \Gamma(X)$. Let us assume that the measure μ satisfies condition (2.2) with an integer M (cf. Proposition 2.2) such that

$$M > \frac{2q}{q-2} > 2, \tag{3.7}$$

where q is the same as in (3.2). Fix a parameter

$$p \in \left[\frac{2M}{M-2}\ ,\ q\right]. \tag{3.8}$$

Setting

$$p' := 2\,(p-2)^{-1}\,, \tag{3.9}$$

we then have

$$\frac{2}{q-2} < p' \leq M/2 - 1.$$

Thus, according to Proposition 2.2, $a_{\alpha,p'}, b_\alpha \in L^1(\Gamma(X),\mu)$ for any $\alpha > 0$, and therefore

$$a_{\alpha,p'}(\gamma),\ b_\alpha(\gamma) < \infty \tag{3.10}$$

for μ-a.a. $\gamma \in \Gamma(X)$.

For $\sigma \in S^\gamma$ define the norm

$$\|\sigma\|_{\alpha,p} := \left(\sum_{x\in\gamma} |\sigma(x)|^p\, w_\alpha(x)\right)^{1/p} \tag{3.11}$$

and consider the Banach space

$$l^p_\alpha(\gamma, S) := \left\{\sigma \in S^\gamma : \|\sigma\|_{\alpha,p} < \infty\right\}.$$

We will denote by $\mathcal{G}_{\alpha,p}(S^\gamma) \subset \mathcal{G}(S^\gamma)$ the set of all Gibbs measures associated with E, which are supported on $l^p_\alpha(\gamma, S)$. These measures are called *tempered.*

Theorem 3.2. *Assume that conditions (3.7) and (3.8) are satisfied. Then the following statements hold for μ-a.a. $\gamma \in \Gamma(X)$:*
1) the set $\mathcal{G}_{\alpha,p}(S^\gamma)$ is not empty and weakly compact (in the topology inherited from $\mathcal{P}(S^\gamma)$);
2) for any $\lambda \in \mathbb{R}_+$ the exists a constant $\Xi_\gamma(\lambda) > 0$ such that every $\nu \in \mathcal{G}_{\alpha,p}(S^\gamma)$ satisfies the estimate

$$\int_{S^\gamma} \exp\ \left\{\lambda\, \|\sigma\|^p_{\alpha,p}\right\}\ \nu(d\sigma) \leq \exp\,\Xi_\gamma(\lambda). \tag{3.12}$$

The proof requires some technical preparations. We say that a configuration $\gamma \in \Gamma(X)$ is *connected* if $n_{\gamma,R}(x) \neq 0$ for any $x \in \gamma$. Obviously, for any $\gamma \in \Gamma(X)$ there exists a unique (finite of countable) decomposition

$$\gamma = \bigcup_k \gamma_k,\ \text{ where } \gamma_k \text{ are maximal (disjoint) connected components of } \gamma. \tag{3.13}$$

To prove the existence of $\nu \in \mathcal{G}_{\alpha,p}(S^\gamma)$ a *key issue* is to check that the family of specification kernels $(\Pi_\eta(d\sigma|\xi))_{\eta\in\mathcal{F}(\gamma)}$, with a fixed boundary condition $\xi \in l^p_\alpha(\gamma, S)$, is tight (cf. Proposition 3.7 below). As all its accumulation points certainly will be supported by $l^p_\alpha(\gamma, S)$ and solve the DLR equation, this immediately yields that $\mathcal{G}_{\alpha,p}(S^\gamma) \neq \emptyset$. To that end, in Proposition 3.3 we first establish the exponential moment bound for $\Pi_\eta(d\sigma|\xi)$, which is uniform with respect to all finite configurations $\eta \in \mathcal{F}(\gamma)$. We fix a non-empty configuration $\gamma \in \Gamma(X)$ such that (3.10) holds.

Proposition 3.3. *For any $\xi \in l^p_\alpha(\gamma, S)$ and $\lambda \in \mathbb{R}_+$ there exists a constant $\Theta_\gamma(\lambda,\xi) < \infty$ (depending also on $\alpha, p > 0$) such that*

$$\sup_{\eta\in\mathcal{F}(\gamma)} \int_{S^\gamma} e^{\lambda\|\sigma\|^p_{\alpha,p}} \Pi_\eta(d\sigma\,|\xi\,) \le \exp\,\Theta_\gamma(\lambda,\xi). \tag{3.14}$$

Proof. The proof is quite technical and we split it into 3 steps. First, we assume that γ is connected and prove estimate (3.14) for a one-point configuration $\eta = \{x\}$. Then we extend it to a general $\eta \in \mathcal{F}(\gamma)$ using the consistency property of specification Π, and finally, pass to a general (not necessarily connected) configuration γ.

Step 1. One-point estimate. Let us assume that $\gamma \in \Gamma(X)$ is connected (i.e., $n_{\gamma,R}(y) \ge 1$ for any $y \in \gamma$) and fix an $x \in \gamma$. For an arbitrary constant $\varkappa > 0$, any $y \in \gamma$ and $u, v \in S$ we have by Young's inequality

$$|W_{x,y}(u,v)| \le ||J||_\infty\,|u|\ \ |v| \le \varkappa\,(|u|^p + |v|^p) + C_{p,J}\varkappa^{-p'}, \tag{3.15}$$

where $p' = 2(p-2)^{-1}$ and

$$C_{p,J} := \left(\frac{||J||_\infty}{p}\right)^{\frac{p}{p-2}} (p-2). \tag{3.16}$$

Let us fix $\sigma, \zeta \in S^\gamma$ and set $u := \sigma(x)$, $\varkappa := \beta\,[n_{\gamma,R}(x)n_{\gamma,R}(y)]^{-1}$, where $\beta \in (0, e^{\alpha R}\lambda/2)$ and $\lambda > 0$ are arbitrary. Inequalities (3.2) and (3.15) imply the following estimate

$$\begin{aligned}
&-\beta\sum_{y\sim x}\frac{|u|^p + |\zeta(y)|^p}{n_{\gamma,R}(x)n_{\gamma,R}(y)} - C_{p,J}\beta^{-p'}\sum_{y\sim x}[n_{\gamma,R}(x)n_{\gamma,R}(y)]^{p'} - V(u) \qquad (3.17)\\
\le\quad & -\sum_{y\sim x} W_{x,y}(u,\zeta(y)) - V(u)\\
\le\quad & \beta\sum_{y\sim x}\frac{|u|^p + |\zeta(y)|^p}{n_{\gamma,R}(x)n_{\gamma,R}(y)} + C_{p,J}\beta^{-p'}\sum_{y\sim x}[n_{\gamma,R}(x)n_{\gamma,R}(y)]^{p'} - a_V\,|u|^q + b_V.
\end{aligned}$$

By the definition of $\Pi_x(d\sigma\,|\xi\,) := \Pi_{\{x\}}(d\sigma\,|\xi\,)$, cf. (3.3) and (3.4), we have

$$\begin{aligned}
\int_{S^\gamma} e^{\lambda|\sigma(x)|^p}\Pi_x(d\sigma\,|\zeta\,) &= Z_x(\xi)^{-1}\int_S \exp\left\{\lambda\,|u|^p - \sum_{y\sim x} W_{x,y}(u,\zeta(y)) - V(u)\right\}du,\\
Z_x(\xi) &= \int_S \exp\left\{-\sum_{y\sim x} W_{x,y}(u,\zeta(y)) - V(u)\right\}du.
\end{aligned}$$

Elementary calculations based on (3.17) show that

$$\begin{aligned}
Z_x(\zeta) &\ge \int_S \exp\left\{\beta\sum_{y\sim x}\frac{|u|^p + |\zeta(y)|^p}{n_{\gamma,R}(x)n_{\gamma,R}(y)} - C_{p,J}\beta^{-p'}\sum_{y\sim x}[n_{\gamma,R}(x)n_{\gamma,R}(y)]^{p'} - V(u)\right\}du\\
&= \exp\left\{-\beta\sum_{y\sim x}\frac{|\zeta(y)|^p}{n_{\gamma,R}(x)n_{\gamma,R}(y)} - C_{p,J}\beta^{-p'}\sum_{y\sim x}[n_{\gamma,R}(x)n_{\gamma,R}(y)]^{p'}\right\}\\
&\quad\times\int_S \exp\{-\beta\,|u|^p - V(u)\}\,du
\end{aligned}$$

and

$$\begin{aligned}
&\int_S \exp\left\{\lambda\,|u|^p - \sum_{y\sim x} W_{x,y}(u,\zeta(y)) - V(u)\right\} du \\
\le\ &\exp\left\{b_V + \beta\sum_{y\sim x}\frac{|\zeta(y)|^p}{n_{\gamma,R}(x)n_{\gamma,R}(y)} + C_{p,J}\beta^{-p'}\sum_{y\sim x}[n_{\gamma,R}(x)n_{\gamma,R}(y)]^{p'}\right\} \\
&\times \int_S \exp\left\{(\lambda+\beta)\,|u|^p - a_V\,|u|^q\right\} du.
\end{aligned}$$

This yields the following one-point estimate

$$\begin{aligned}
\int_{S^\gamma} e^{\lambda|\sigma(x)|^p}\Pi_x(d\sigma\,|\zeta) \quad \le \quad &\exp\left\{C_\lambda + 2\beta\sum_{y\sim x}\frac{|\zeta(y)|^p}{n_{\gamma,R}(x)n_{\gamma,R}(y)}\right. \\
&\left.+2C_{p,J}\beta^{-p'}\sum_{y\sim x}[n_{\gamma,R}(x)n_{\gamma,R}(y)]^{p'}\right\},
\end{aligned} \tag{3.18}$$

where

$$C_\lambda := b_V + \log\frac{\int_S \exp\left\{(\lambda+\beta)\,|u|^p - a_V\,|u|^q\right\} du}{\int_S \exp\left\{-\beta\,|u|^p - V(u)\right\} du} < \infty \tag{3.19}$$

because of the condition $q > p$. Since $\beta > 0$ can be taken arbitrary small, (3.19) states a weak dependence of the one-point exponential moments on boundary conditions.

Step 2. We still suppose that $\gamma \in \Gamma(X)$ is connected and extend the estimate (3.18) to an arbitrary $\eta \in \mathcal{F}(\gamma)$. Let us fix a tempered boundary condition $\xi \in l^p_\alpha(\gamma, S)$ and define $\xi_{\gamma\setminus\eta} := (\xi(y))_{y\in\gamma\setminus\eta}$. Integrating both sides of (3.18) with respect to $\Pi_\eta(\cdot\,|\xi)$ and taking into account the identity

$$\int_{S^\gamma} \Pi_x(\cdot\,|\zeta)\Pi_\eta(d\zeta\,|\xi) = \Pi_\eta(\cdot\,|\xi)$$

that holds for every $x \in \eta$ (cf. (3.5)), we get

$$\begin{aligned}
&\int_{S^\gamma} e^{\lambda|\sigma(x)|^p}\Pi_\eta(d\sigma\,|\xi) \\
\le\ &\exp\left\{C_\lambda + 2\beta\sum_{\substack{y\sim x\\ y\in\gamma\setminus\eta}}\frac{|\xi(y)|^p}{n_{\gamma,R}(x)n_{\gamma,R}(y)} + 2C_{p,J}\beta^{-p'}\sum_{y\sim x}[n_{\gamma,R}(x)n_{\gamma,R}(y)]^{p'}\right\}\times \\
&\times\int_{S^\gamma}\exp\left\{2\beta\sum_{\substack{y\sim x\\ y\in\eta}}\frac{|\zeta(y)|^p}{n_{\gamma,R}(x)n_{\gamma,R}(y)}\right\}\Pi_\eta(d\zeta\,|\xi).
\end{aligned} \tag{3.20}$$

The integral in the right-hand side can be estimated with the aid of the multiple Hölder inequality

$$\begin{aligned}
&\int_{S^\gamma} \exp\left\{2\beta \sum_{\substack{y\sim x\\ y\in\eta}} \frac{|\zeta(y)|^p}{n_{\gamma,R}(x)n_{\gamma,R}(y)}\right\} \Pi_\eta(d\zeta\,|\xi) \\
=\ & \int_{S^\gamma} \prod_{\substack{y\sim x\\ y\in\eta}} \left\{\exp\ [\lambda\,|\zeta(y)|^p]\right\}^{\frac{2\beta}{\lambda n_{\gamma,R}(x)n_{\gamma,R}(y)}} \Pi_\eta(d\zeta\,|\xi) \\
\leq\ & \prod_{\substack{y\sim x\\ y\in\eta}} \left\{\int_{S^\gamma} \exp\ [\lambda\,|\zeta(y)|^p]\ \Pi_\eta(d\zeta\,|\xi)\right\}^{\frac{2\beta}{\lambda n_{\gamma,R}(x)n_{\gamma,R}(y)}}.
\end{aligned}$$

Introducing the notation

$$M_{\lambda,x}(\eta;\xi) := \log\ \left\{\int_{S^\gamma} e^{\lambda|\zeta(x)|^p}\Pi_\eta(d\zeta\,|\xi)\right\}, \tag{3.21}$$

we obtain

$$\begin{aligned}
0\ \leq\ & M_{\lambda,x}(\eta;\xi) \leq C_\lambda + 2\beta \sum_{\substack{y\sim x\\ y\in\gamma\setminus\eta}} \frac{|\xi(y)|^p}{n_{\gamma,R}(x)n_{\gamma,R}(y)} \\
& + 2C_{p,J}\beta^{-p'} \sum_{y\sim x} [n_{\gamma,R}(x)n_{\gamma,R}(y)]^{p'} + \sum_{\substack{y\sim x\\ y\in\eta}} \frac{2\beta}{\lambda n_{\gamma,R}(x)n_{\gamma,R}(y)} M_{\lambda,y}(\eta;\xi).
\end{aligned} \tag{3.22}$$

We will estimate the weighted sum

$$\|M_\lambda(\eta;\xi)\|_\alpha := \sum_{x\in\eta} M_{\lambda,x}(\eta;\xi)w_\alpha(x).$$

Formula (3.22) implies that

$$\begin{aligned}
\|M_\lambda(\eta;\xi)\|_\alpha\ \leq\ & C_\lambda \sum_{x\in\eta} w_\alpha(x) + 2C_{p,J}\beta^{-p'} \sum_{x\in\eta} w_\alpha(x) \sum_{y\sim x} [n_{\gamma,R}(x)n_{\gamma,R}(y)]^{p'} \\
& + 2\beta \sum_{x\in\eta} w_\alpha(x) \sum_{\substack{y\in\gamma\setminus\eta\\ y\sim x}} \frac{|\xi(y)|^p}{n_{\gamma,R}(x)n_{\gamma,R}(y)} \\
& + 2\beta \sum_{x\in\eta} w_\alpha(x) \sum_{\substack{y\in\eta\\ y\sim x}} \frac{M_{\lambda,y}(\eta;\xi)}{\lambda n_{\gamma,R}(x)n_{\gamma,R}(y)}.
\end{aligned}$$

Recall that the functions $a_{\alpha,p'}$ and b_α are defined by formulae (2.4) and (2.5). Then

$$\begin{aligned}
\|M_\lambda(\eta;\xi)\|_\alpha\ \leq\ & C_\lambda b_\alpha(\gamma) + 2C_{p,J}\beta^{-p'} a_{\alpha,p'}(\gamma) + 2\beta \sum_{x\in\eta} w_\alpha(x) \sum_{\substack{y\in\gamma\setminus\eta\\ y\sim x}} \frac{|\xi(y)|^p}{n_{\gamma,R}(x)n_{\gamma,R}(y)} \\
& + 2\beta \sum_{x\in\eta} w_\alpha(x) \sum_{\substack{y\in\eta\\ y\sim x}} \frac{M_{\lambda,y}(\eta;\xi)}{\lambda n_{\gamma,R}(x)n_{\gamma,R}(y)}.
\end{aligned} \tag{3.23}$$

The triangle inequality $|y| \leq |x| + |y-x|$ implies that

$$w_\alpha(x) = e^{-\alpha|x|} \leq e^{\alpha|y-x|}e^{-\alpha|y|} = e^{\alpha|y-x|}w_\alpha(y) \leq e^{\alpha R}w_\alpha(y)$$

provided $|y-x| \leq R$. Changing the order of summation in the last two terms of (3.23), we obtain

$$\begin{aligned}\|M_\lambda(\eta;\xi)\|_\alpha &\leq C_\lambda b_\alpha(\gamma) + 2C_{p,J}\beta^{-p'}a_{\alpha,p'}(\gamma)\\ &\quad + e^{\alpha R}2\beta\sum_{y\in\gamma\setminus\eta}\frac{w_\alpha(y)\,|\xi(y)|^p}{n_{\gamma,R}(y)}\sum_{\substack{x\in\eta\\ x\sim y}}\frac{1}{n_{\gamma,R}(x)}\\ &\quad + e^{\alpha R}\frac{2\beta}{\lambda}\sum_{y\in\eta}\frac{w_\alpha(y)M_{\lambda,y}(\eta;\xi)}{n_{\gamma,R}(y)}\sum_{\substack{x\in\eta\\ x\sim y}}\frac{1}{n_{\gamma,R}(x)}.\end{aligned}$$

Taking into account that $\sum_{\substack{x\in\eta\\ x\sim y}}\frac{1}{n_{\gamma,R}(x)} \leq n_{\gamma,R}(y)$ for any $y \in \gamma$, we see that the following inequality holds

$$\begin{aligned}\|M_\lambda(\eta;\xi)\|_\alpha &\leq C_\lambda b_\alpha(\gamma) + 2C_{p,J}\beta^{-p'}a_{p'}(\gamma)\\ &\quad + e^{\alpha R}2\beta\sum_{y\in\gamma\setminus\eta}w_\alpha(y)\,|\xi(y)|^p + e^{\alpha R}\frac{2\beta}{\lambda}\sum_{y\in\eta}w_\alpha(y)M_{\lambda,y}(\eta;\xi)\\ &\leq C_\lambda b_\alpha(\gamma) + 2C_{p,J}\beta^{-p'}a_{\alpha,p'}(\gamma) + e^{\alpha R}2\beta\left\|\xi_{\gamma\setminus\eta}\right\|_{a,p}^p\\ &\quad + e^{\alpha R}\frac{2\beta}{\lambda}\|M_\lambda(\eta;\xi)\|_\alpha,\end{aligned} \tag{3.24}$$

which yields

$$\begin{aligned}\|M_\lambda(\eta;\xi)\|_\alpha &\leq \left(1-e^{\alpha R}2\beta\lambda^{-1}\right)^{-1}\\ \times&\left[C_\lambda b_\alpha(\gamma) + 2C_{p,J}\beta^{-p'}a_{\alpha,p'}(\gamma) + e^{\alpha R}2\beta\left\|\xi_{\gamma\setminus\eta}\right\|_{a,p}^p\right] < \infty.\end{aligned} \tag{3.25}$$

Observe that $1-e^{\alpha R}2\beta\lambda^{-1} > 0$ because of the condition $\beta \in (0, e^{\alpha R}\lambda/2)$.

To finish the proof we use Hölder's inequality and obtain the estimate

$$\begin{aligned}\int_{S^\gamma}\exp\left\{\lambda\|\sigma\|_{\alpha,p}^p\right\}\Pi_\eta(d\sigma\,|\xi) &= \exp\left\{\lambda\sum_{y\in\gamma\setminus\eta}|\xi(y)|^p\,w_\alpha(y)\right\}\\ &\quad\times\int_{S^\gamma}\prod_{x\in\eta}\left\{\exp\ [\delta\,|\sigma(x)|^p]\right\}^{\lambda w_\alpha(x)/\delta}\Pi_\eta(d\sigma\,|\xi)\\ &\leq \exp\left\{\lambda\left\|\xi_{\gamma\setminus\eta}\right\|_{\alpha,p}^p + \frac{\lambda}{\delta}\|M_\delta(\eta;\xi)\|_\alpha\right\},\end{aligned} \tag{3.26}$$

where $\delta := \lambda b_a(\gamma)$. By virtue of (3.25) applied to $\|M_\delta(\eta;\xi)\|_\alpha$ we conclude that for any $\eta \in \mathcal{F}(\gamma)$ and $\beta \in (0, e^{\alpha R}\lambda b_a(\gamma)/2)$ the following estimate holds

$$\int_{S^\gamma}\exp\left\{\lambda\|\sigma\|_{\alpha,p}^p\right\}\Pi_\eta(d\sigma\,|\xi) \leq \exp\Theta_\gamma(\lambda,\xi,\eta) < \infty,$$

where

$$\Theta_\gamma(\lambda,\xi,\eta) = \Xi_\gamma(\lambda) + \Psi_\gamma(\lambda)\left\|\xi_{\gamma\setminus\eta}\right\|_{\alpha,p}^p \tag{3.27}$$

with

$$\Xi_\gamma(\lambda) = \frac{C_\delta \lambda b_a(\gamma) + 2C_{p,J}\beta^{-p'}\lambda a_{\alpha,p'}(\gamma)}{\lambda b_a(\gamma) - 2e^{\alpha R}\beta}, \tag{3.28}$$

$$\Psi_\gamma(\lambda) = \frac{\lambda^2 b_a(\gamma)}{\lambda b_a(\gamma) - 2e^{\alpha R}\beta}, \tag{3.29}$$

and C_δ is defined by (3.19). It is clear now that estimate (3.14) holds with

$$\Theta_\gamma(\lambda,\xi) := \sup_{\eta\in\mathcal{F}(\gamma)} \Theta_\gamma(\lambda,\xi,\eta) = \Xi_\gamma(\lambda) + \Psi_\gamma(\lambda)\,\|\xi\|^p_{\alpha,p}\,. \tag{3.30}$$

Step 3. For a general (disconnected) configuration γ, consider its decomposition (3.13). The result follows from Steps 1, 2 applied to all connected components of γ and Lemma 3.1. ■

Remark 3.4. The application of Jensen's inequality to the right-hand side of formula (3.21) shows that $\int_{S^\gamma} |\sigma(x)|^p\,\Pi_\eta(d\sigma\,|\xi) \le \frac{1}{\lambda}M_{\lambda,x}(\eta;\xi)$ for any $x\in\gamma$, which together with (3.25) implies in turn the bound

$$\begin{aligned}\int_{S^\gamma} \|\sigma\|^p_{\alpha,p}\,\Pi_\eta(d\sigma\,|\xi) &\le \frac{1}{\lambda}\,\|M_\lambda(\eta;\xi)\|_\alpha \\ &\le \mathcal{A}_\gamma(\lambda,\xi) := \mathcal{B}_\gamma(\lambda) + \mathcal{C}(\lambda)\left\|\xi_{\gamma\setminus\eta}\right\|^p_{\alpha,p}\end{aligned} \tag{3.31}$$

with

$$\mathcal{B}_\gamma(\lambda) = \frac{C_\lambda b_\alpha(\gamma) + 2C_{p,J}\beta^{-p'}a_{\alpha,p'}(\gamma)}{\lambda - e^{\alpha R}2\beta}, \tag{3.32}$$

$$\mathcal{C}(\lambda) = \frac{e^{\alpha R}2\beta}{\lambda - e^{\alpha R}2\beta}, \tag{3.33}$$

which holds for any $\lambda>0$, $\beta\in(0,e^{\alpha R}\lambda/2)$ and $p'=2(p-2)^{-1}$. Here the constants $C_{p,J}$, C_λ are defined by formulae (3.16), (3.19) respectively. For further applications (see the proof of Proposition 5.5 below) it is important that $\mathcal{B}_\gamma(\lambda)$ depends *linearly* on $b_\alpha(\gamma)$ and $a_{\alpha,p'}(\gamma)$, which by Proposition 2.2 are *integrable* with respect to the underlying measure μ on $\Gamma(X)$.

Remark 3.5. Observe that $\mathcal{A}_\gamma(\lambda,\xi)$ and $\Theta_\gamma(\lambda,\xi)$ depend on the parameters β and λ. We do not discuss the question of their optimal choice here.

The constants in the right-hand sides of (3.14) and (3.31) depend on the boundary condition ξ. However, we can get rid of this dependence in the "thermodynamic" limit by choosing a particular sequence of configurations η exhausting γ. We have the following statement.

Proposition 3.6. *For any $\lambda\in\mathbb{R}$ and the constant $\Xi_\gamma(\lambda)=\Xi_\gamma(\lambda,\alpha,p)<\infty$ defined by formula (3.28) we have the estimate*

$$\begin{aligned}\limsup_{n\to\infty}\int_{S^\gamma} e^{\lambda\|\sigma\|^p_{\alpha,p}}\Pi_{\eta_n}(d\sigma\,|\xi) &\le \exp\,\Xi_\gamma(\lambda), \\ \limsup_{n\to\infty}\int_{S^\gamma} \|\sigma\|^p_{\alpha,p}\,\Pi_{\eta_n}(d\sigma\,|\xi) &\le \exp\,\mathcal{B}_\gamma(\lambda),\end{aligned} \tag{3.34}$$

which holds for all boundary conditions $\xi\in l^p_\alpha(\gamma,S)$ and any increasing sequence $(\eta_n)_{n\in\mathbb{N}}\subset\mathcal{F}(\gamma)$ that exhausts γ, that is, $\bigcup_{n\in\mathbb{N}}\eta_n=\gamma$.

Proof. Observe that $\lim_{n\to\infty} \left\| \xi_{\gamma\setminus\eta_n} \right\|^p_{\alpha,p} = 0$ for each $\xi \in l^p_\alpha(\gamma, S)$. Thus, the second term in the right-hand side of (3.27) and (3.31) vanishes as $n \to \infty$ and the claim follows. ■

Recall that $C_{\mathrm{b}}(S^\gamma)$ stands for the set of *bounded continuous* functions $f : S^\gamma \to \mathbb{R}$, whereas S^γ is endowed with the Polish topology of coordinate convergence. The *weak topology* on the set $\mathcal{P}(S^\gamma)$ of Borel probability measures on S^γ is defined as the coarsest topology making each of the following mappings continuous

$$\mathcal{P}(S^\gamma) \ni \mu \mapsto \mu(f) := \int_{S^\gamma} f d\mu, \quad f \in C_{\mathrm{b}}(S^\gamma).$$

Proposition 3.7. *For any $\xi \in l^p_\alpha(\gamma, S)$, the family $\{\Pi_\eta(d\sigma\,|\xi\,)\}_{\eta\in\mathcal{F}(\gamma)} \subset \mathcal{P}(S^\gamma)$ is relatively weakly compact.*

Proof. By Prokhorov's criterion (see e.g. Theorem 5.1 in [3] or 15.4.4 in [11]) the claim is equivalent to showing that for any $\xi \in l^p_\alpha(\gamma, S)$ the family of measures $\{\Pi_\eta(d\sigma\,|\xi\,)\}_{\eta\in\mathcal{F}(\gamma)}$ is tight. Let $B_\gamma(r)$ be the ball of radius r in $l^p_\alpha(\gamma, S)$, centered at 0. It is a compact subset of S^γ and denote by $(B_\gamma(r))^c$ its complement. By Chebyshev's inequality and (3.31) we have that

$$\Pi_\eta[(B_\gamma(r))^c\,|\xi] \le \frac{1}{r^p}\int_{S^\gamma} \|\sigma\|^p_{\alpha,p}\,\Pi_{\eta_n}(d\sigma\,|\xi\,) \le \frac{\mathcal{A}_\gamma(\lambda,\xi)}{r^p}$$

for any $r > 0$. Given $\varepsilon > 0$ we can set

$$r = \left[\frac{\mathcal{A}_\gamma(\lambda,\xi)}{\varepsilon}\right]^{1/p}, \tag{3.35}$$

and obtain the estimate

$$\sup_{\eta\in\mathcal{F}(\gamma)} \Pi_\eta[(B_\gamma(r))^c\,|\xi] \le \varepsilon, \tag{3.36}$$

which implies the tightness in question, and the result follows. ■

Proof of Theorem 3.2. Proposition 3.7 says that for any $\xi \in l^p_\alpha(\gamma, S)$ one finds an exhausting sequence $\boldsymbol{\eta} = (\eta_n)_{n\in\mathbb{N}} \subset \mathcal{F}(\gamma)$ such that $(\Pi_{\eta_n}(\cdot\,|\xi\,))_{n\in\mathbb{N}}$ is weakly converging to a certain probability measure $\nu^\xi_{\boldsymbol{\eta}}$ on S^γ. To check that $\nu^\xi_{\boldsymbol{\eta}}$ satisfies the DLR equation (3.6), we will use the *Feller property* of the specification $\Pi_\gamma := \{\Pi_\eta(d\sigma|\xi)\}_{\eta\in\mathcal{F}(\gamma),\xi\in S^\gamma}$. The later means that, for any $\eta \in \mathcal{F}(\gamma)$ and $f \in C_{\mathrm{b}}(S^\gamma)$, the function $\Pi_\eta f$ defined by

$$S^\gamma \ni \xi \mapsto \Pi_\eta f(\xi) := \int_{S^\gamma} f(\sigma)\Pi_\eta(d\sigma\,|\xi\,) \tag{3.37}$$

belongs to $C_{\mathrm{b}}(S^\gamma)$ as well. The proof of this property is quite standard and crucially exploits the continuity of the local interaction energy $E_\eta(\sigma_\eta\,|\xi\,)$ subject to boundary conditions $\xi \in S^\gamma$. For more details we refer to the proof of Lemma 2.10 in [18].

Note that a measure $\nu \in \mathcal{P}(S^\gamma)$ solves (3.6) if and only if for all $\eta \in \mathcal{F}(\gamma)$ and $f \in C_{\mathrm{b}}(S^\gamma)$

$$\int_{S^\gamma} \Pi_\eta f(\sigma)\nu(d\sigma) = \int_{S^\gamma} f(\sigma)\nu(dx). \tag{3.38}$$

For any $\eta \in \mathcal{F}(\gamma)$ one finds $n_\eta \in \mathbb{N}$ such that $\eta \subset \eta_n$ for all $n \ge n_\eta$. For such n, by the consistency property (3.5) we have

$$\int_{S^\gamma} \Pi_\eta f(\sigma)\Pi_{\eta_n}(d\sigma|\xi) = \int_{S^\gamma} f(\sigma)\Pi_{\eta_n}(d\sigma|\xi).$$

Due to the inclusion $\Pi_\eta f \in C_{\mathrm{b}}(S^\gamma)$, we can pass to the limit $n \to \infty$ and thus prove (3.38) for $\nu := \nu^\xi_{\boldsymbol{\eta}}$.

Next, we will show that $\nu^\xi_{\boldsymbol{\eta}}$ obeys the exponential moment estimate (3.12) for any $\lambda > 0$, which immediately implies that this measure is supported on $l^p_\alpha(\gamma, S)$. Note that the function

$$S^\gamma \ni \sigma \mapsto F(\sigma) := \exp\left\{\lambda \left\|\sigma\right\|^p_{\alpha,p}\right\} \in \mathbb{R}_+ \cup \{+\infty\} \tag{3.39}$$

is *lower semi-continuous* (i.e, its lower level sets $\{\sigma \in S^\gamma \,|F(\sigma) \le c\}$ are closed for all $c \in \mathbb{R}_+$). By (3.34) and the weak convergence $\Pi_{\eta_n}(\cdot\,|\xi\,) \to \nu^\xi_{\boldsymbol{\eta}}$, $n \to \infty$, we have

$$\int_{S^\gamma} e^{\lambda\|\sigma\|^p_{\alpha,p}} \nu^\xi_{\boldsymbol{\eta}}(d\sigma) \le \liminf_{n\to\infty} \int_{S^\gamma} e^{\lambda\|\sigma\|^p_{\alpha,p}} \Pi_{\eta_n}(d\sigma\,|\xi\,) \le \exp\, \Xi_\gamma(\lambda), \tag{3.40}$$

which implies that $\nu^\xi_{\boldsymbol{\eta}} \in \mathcal{G}_{\alpha,p}(S^\gamma)$.

Let us prove that the *à-priori* estimate (3.12) holds for every $\nu \in \mathcal{G}_{\alpha,p}(\mathcal{S}^\gamma)$, with the constant $\Xi_\gamma(\lambda)$ being the same as in (3.34). Let us consider a family of cut-off functions

$$F_N(\sigma) := \min\{F(\sigma); N\} = \min\left\{e^{\lambda\|\sigma\|^p_{\alpha,p}}\,;\, N\right\}, \quad N \in \mathbb{N}.$$

Then there is a bound

$$\int_{S^\gamma} F_N(\sigma)\Pi_\eta(d\sigma|\xi) \le N$$

for all $\eta \in \mathcal{F}(\gamma)$ and $\xi \in S^\gamma$. By Fatou's lemma and estimate (3.34) we have

$$\begin{aligned}
&\limsup_{n\to\infty} \int_{S^\gamma} \left[\int_{S^\gamma} F_N(\sigma)\Pi_{\eta_n}(d\sigma|\xi)\right] \nu(d\xi) \\
\le\ & \int_{S^\gamma} \left[\limsup_{n\to\infty} \int_{S^\gamma} F_N(\sigma)\Pi_{\eta_n}(d\sigma|\xi)\right] \nu(d\xi) \\
\le\ & \int_{S^\gamma} \left[\limsup_{n\to\infty} \int_{S^\gamma} e^{\lambda\|\sigma\|^p_{\alpha,p}}\Pi_{\eta_n}(d\sigma|\xi)\right] \nu(d\xi) \\
\le\ & \int_{S^\gamma} [\exp\, \Xi_\gamma(\lambda)]\ \nu(d\xi) = \exp\, \Xi_\gamma(\lambda)
\end{aligned}$$

(because $\Xi_\gamma(\lambda)$ is independent of ξ). It follows from the DLR equation (3.6) that

$$\int_{S^\gamma}\int_{S^\gamma} F_N(\sigma)\Pi_{\eta_n}(d\sigma|\xi)\nu(d\xi) = \int_{S^\gamma} F_N(\sigma)\nu(d\sigma), \quad n \in \mathbb{N},$$

which implies that

$$\int_{S^\gamma} F_N(\sigma)\nu(d\sigma) \le \exp\, \Xi_\gamma(\lambda), \quad N \in \mathbb{N}.$$

As ν is supported on $l^p_\alpha(\gamma, S)$ and $F_N(\sigma) \nearrow F(\sigma)$ as $N \to \infty$ for each $\sigma \in l^p_\alpha(\gamma, S)$, we can apply B. Levi's monotone convergence theorem yielding that

$$\int_{S^\gamma} \exp\ \left\{\lambda \left\|\sigma\right\|^p_{\alpha,p}\right\}\ \nu(d\sigma) = \lim_{N\to\infty} \int_{l^p_\alpha(\gamma,S)} F_N(\sigma)\nu(d\sigma) \le \exp\, \Xi_\gamma(\lambda).$$

A similar reasoning can be used to prove the compactness of $\mathcal{G}_{\alpha,p}(S^\gamma)$. Indeed, the uniform moment bound (3.12) implies that the set $\mathcal{G}_{\alpha,p}(S^\gamma)$ is tight and thus (by Prokhorov's theorem) relatively compact, whereas by the Feller property all its accumulation points solve the DLR equation and hence are Gibbs measures. ■

Remark 3.8. We have proved that $\mathcal{G}_{\alpha,p}(S^\gamma)$ is nonempty at least for all $\gamma \in \Gamma(X)$, for which both $b_\alpha(\gamma)$ and $a_{\alpha,p'}(\gamma)$ are finite. Using inequality (3.31) and applying the arguments similar to those used in the proof of estimate (3.12), we can verify that for any $\nu \in \mathcal{G}_{\alpha,p}(S^\gamma)$ the following moment estimate holds:

$$\int_{S^\gamma} \|\sigma\|_{\alpha,p}^p \ \nu(d\sigma) \le \mathcal{B}_\gamma := \inf_{\lambda,\beta} \frac{C_\lambda b_\alpha(\gamma) + 2C_{p,J}\beta^{-p'} a_{\alpha,p'}(\gamma)}{\lambda - e^{\alpha R} 2\beta}. \tag{3.41}$$

Here the constants $C_{p,J}$, C_λ are defined by formulae (3.16), (3.19) respectively, $p' = 2(p-2)^{-1}$, and the infimum is taken over all $\lambda > 0$, $\beta \in (0, e^{\alpha R}\lambda/2)$. Note (cf. Proposition 3.4) that B_γ depends *linearly* on $b_\alpha(\gamma)$ and $a_{\alpha,p'}(\gamma)$, which are integrable with respect to the underlying measure μ on $\Gamma(X)$.

4. Some remarks and extensions

1. Condition (3.7) establishes certain relation between the growth rate q of the self-interaction potential V and the number of bounded correlation functions k_m, $1 \le m \le M$, of μ. In the case where μ has bounded correlation functions of arbitrary order, condition (3.7) holds for any $q > 2$ and thus p can be any number from the interval $(2, q]$. In turn, higher values of q guarantee the existence of Gibbs measures with the smaller support set $l_\alpha^p(\gamma, S)$ (that corresponds to higher values of p). However, by our method we cannot control the case of $q = 2$, even when the particles $x \in \gamma$ are distributed according to the homogeneous Poisson random field $\mu := \pi_z$ on $\Gamma(X)$. In particular, the existence problem is still open for the important class of ferromagnetic harmonic systems on S^γ that are described by the energy functional (3.3) with $J(x-y) \le 0$ and $V(u) = a_V |u|^2$, $a_V > 0$.

2. The above assumptions on the potentials W and V can be weaken in the following way. First observe that the results of the previous section are in fact based only on estimate (3.15) and condition (3.1). Therefore Theorem 3.2 holds for any continuous $W_{x,y} : S \times S \to \mathbb{R}$ satisfying the at most polynomial growth estimate

$$|W_{x,y}(u,v)| \le C_1 (|u|^p + |v|^p) + C_2, \quad u, v \in S, \tag{4.1}$$

and the finite range condition $W_{x,y} \equiv 0$ if $|x-y| \le R$, for all $x, y \in X$ and some constants $C_1, C_2, R > 0$. Furthermore, we can consider the one-particle potentials V_x varying with x, provided all of them satisfy the same lower bound 3.2.

Second, we can drop the assumption of the *continuity* of the potentials $V_x : S \to \mathbb{R}$ and $W_{x,y} : S \times S \to \mathbb{R}$. Note that in this case the specification will be no longer Feller. Consequently, the topology of weak convergence could not help us to construct Gibbs measures $\nu \in \mathcal{G}_{\alpha,p}(S^\gamma)$ as accumulation points of the family $\{\Pi_\eta(d\sigma\,|\,\xi)\}_{\eta\in\mathcal{F}(\gamma)}$ with a fixed $\xi \in l_\alpha^p(\gamma, S)$. Instead, we can use the topology of *set-wise convergence* on the algebra $\mathcal{B}_0(S^\gamma) = \cup_{\eta\in\mathcal{F}(\gamma)} \mathcal{B}_0(S^\eta)$ of local subsets of S^γ. A key observation is that the moment bound (3.14) implies the *local equicontinuity* of the family $\{\Pi_\eta(d\sigma\,|\,\xi)\}_{\eta\in\mathcal{F}(\gamma)}$ (cf. Definition 4.6 in [9]) and hence its relative compactness in the last-named topology. This again ensures the existence of accumulation points $\nu_\eta^\xi \in \mathcal{P}(S^\gamma)$, each of them ought to be Gibbs due to the finite range condition (1.2) imposed on the interaction. For systems of unbounded spins on general graphs, a similar approach was realized in [13].

3. As already mentioned in the Introduction, in this paper we do not touch the question of uniqueness of $\nu \in \mathcal{G}(S^\gamma)$. This is a highly non-trivial problem and general conditions that guarantee that $N(\mathcal{G}(S^\gamma)) = 1$ due to the small interaction

strength $||J||_\infty \ll 1$ are not known. On the other hand, in [6] we studied a class of models with ferromagnetic pair interaction living on Poisson random graphs and showed the existence multiple Gibbs states, that is, that $N(\mathcal{G}(S^\gamma)) > 1$ (and therefore $= \infty$) for a.a. $\gamma \in \Gamma(X)$.

5. Measurable dependence on γ.

We have shown that the set $\mathcal{G}_{\alpha,p}(S^\gamma)$ is not empty for μ-a.a. $\gamma \in \Gamma(X)$. In the proof of Theorem 3.2, a measure $\nu_\gamma^\xi := \nu_{\gamma,\boldsymbol{\eta}}^\xi \in \mathcal{G}_{\alpha,p}(S^\gamma)$ has been constructed for each tempered $\xi \in \Gamma(X)$ as a limit of certain sequence of "finite volume" measures $\Pi_{\eta_n}(d\sigma|\xi)$, $n \in \mathbb{N}$. The sequence $\boldsymbol{\eta} = (\eta_n)_{n\in\mathbb{N}} \subset \mathcal{F}(\gamma)$ however depends on the "randomness" γ in some uncontrollable way (the so-called *chaotic size dependence*, see the discussion in [25, 26]). This does not answer the question whether there exist measurable selections $\Gamma(X) \ni \gamma \mapsto \nu_\gamma \in \mathcal{G}_{\alpha,p}(S^\gamma)$. In fact, it is known that for general models with random interaction the dependence of such limiting measures on the random parameter can fail to be measurable. An additional difficulty in our case is that the measures $\nu_\gamma \in \mathcal{G}_{\alpha,p}(S^\gamma)$ live (for different γ) on different spaces, and it is not clear in what sense this measurability can be understood.

In this paper, we will identify the spaces S^γ that support measures $\nu_\gamma \in \mathcal{G}_{\alpha,p}(S^\gamma)$ with the fibres of a natural bundle over $\Gamma(X)$ and extend ν_γ to the total space $\mathfrak{X}$ of this bundle by setting $\nu_\gamma(\mathfrak{X} \setminus S^\gamma) = 0$. It turns out that $\mathfrak{X}$ can be identified with the *marked configuration space* $\Gamma(X, S)$ (defined by expression (5.2) below). For basic definitions and description of main structures on marked configuration spaces we refer to e.g. [1], [5], [19] and Appendix, where we give necessary information on the topological structure of $\Gamma(X, S)$. We fix a topology on $\Gamma(X, S)$ (defined below, see (5.4)), which makes $\Gamma(X, S)$ a Polish space (see [5] for the explicit construction of the corresponding metric). In what follows, it will be equipped with the corresponding Borel σ-algebra $\mathcal{B}(\Gamma(X, S))$.

Let $\mathcal{P}(\Gamma(X, S))$ stand for the Polish space of all Borel probability measures on $\Gamma(X, S)$, which is equipped with the topology of weak convergence. By this definition, the measurability of the map $\Gamma(X) \ni \gamma \mapsto \nu_\gamma \in \mathcal{P}(\Gamma(X, S))$ is equivalent to the measurability of $\Gamma(X) \ni \gamma \mapsto \langle f, \nu_\gamma\rangle := \int_{\Gamma(X,S)} f(\hat{\eta})\nu_\gamma(d\hat{\eta})$ for each bounded continuous function $f : \Gamma(X, S) \to R$, which in turn is equivalent to the measurability of $\Gamma(X) \ni \gamma \mapsto \nu_\gamma(A)$ for each $A \in \mathcal{B}(\Gamma(X, S))$.

The main result of this section is the following

Theorem 5.1. *There exists a $\mathcal{B}(\Gamma(X))/\mathcal{B}(P(\Gamma(X, S)))$-measurable mapping*

$$\Gamma(X) \ni \gamma \mapsto \nu_\gamma \in \mathcal{P}(\Gamma(X, S)) \tag{5.1}$$

such that $\nu_\gamma \in G_{\alpha,p}(S^\gamma)$ for μ-a.a. $\gamma \in \Gamma(X)$.

The proof will go along the following lines. First, using the moment bound (5.10) and Prokhorov's theorem, we will construct an auxiliary measure $\hat{\nu}$ on $\Gamma(X, S)$ and define its conditional distribution (i.e., disintegration) $(\nu_\gamma)_{\gamma\in\Gamma(X)} \subset \mathcal{P}(S^\gamma)$ with respect to μ, so that the measurability required in (5.1) holds. Then, with the help of Komlós' theorem we will prove the inclusion $\nu_\gamma \in G_{\alpha,p}(S^\gamma)$.

In order to proceed, we need to introduce the necessary framework. Let us consider the product space $X \times S$. The canonical projection $p_X : X \times S \to X$ can be naturally extended to the corresponding configuration space $\Gamma(X \times S)$. Observe that the image $p_X(\hat{\gamma})$, $\hat{\gamma} \in \Gamma(X \times S)$, is a configuration in X that in general admits accumulation and multiple points. The marked configuration space $\Gamma(X,S)$ is defined in the following way

$$\Gamma(X,S) := \{\hat{\gamma} \in \Gamma(X \times S) : p_X(\hat{\gamma}) \in \Gamma(X)\} . \tag{5.2}$$

We will also use the larger space

$$\ddot{\Gamma}(X,S) := \left\{\hat{\gamma} \in \ddot{\Gamma}(X \times S) : p_X(\hat{\gamma}) \in \ddot{\Gamma}(X)\right\}, \tag{5.3}$$

where $\ddot{\Gamma}(X)$ and $\ddot{\Gamma}(X \times S)$ are the spaces of configurations of points with (finite) multiplicities in X and $X \times S$, respectively. Observe that $\ddot{\Gamma}(X)$ and $\ddot{\Gamma}(X \times S)$ can be equipped with the vague topology defined similarly to the vague topology on $\Gamma(X)$ and $\Gamma(X \times S)$ (see e.g. Section 2.4 in [12]).

The spaces $\Gamma(X,S)$ and $\ddot{\Gamma}(X,S)$ will be endowed with a (completely metrizable) topology which is defined as the weakest topology that makes the map

$$\Gamma(X,S) \ni \hat{\gamma} \mapsto \langle f, \hat{\gamma}\rangle \tag{5.4}$$

continuous for any $f \in C_{\mathrm{b}}(X \times S)$ with supp $f \subset S_\Lambda := \Lambda \times S$, where Λ is a compact subset of X. This topology has been used in e.g. [1], [5] and [19]; for a short account of its properties see Appendix. We equip $\Gamma(X,S)$ and $\ddot{\Gamma}(X,S)$ with the corresponding Borel σ-algebras. Note that $\Gamma(X,S)$ is a dense Borel set in $\ddot{\Gamma}(X,S)$.

Both spaces $\Gamma(X,S)$ and $\ddot{\Gamma}(X,S)$ have the structure of a fibre bundle over $\Gamma(X)$ and $\ddot{\Gamma}(X)$ respectively, with fibres $p_X^{-1}(\gamma)$, which can be identified with the product spaces

$$S^\gamma = \prod_{x\in\gamma} S_x, \quad S_x = S.$$

Thus each $\hat{\gamma} \in \Gamma(X,S)$ can be represented by the pair

$$\hat{\gamma} = (\gamma,\ \sigma_\gamma), \text{ where } \gamma = p_X(\hat{\gamma}) \in \Gamma(X),\ \sigma_\gamma = (\sigma_x)_{x\in\gamma} \in S^\gamma.$$

It follows directly from the definition of the corresponding topologies that the map $p_X : \Gamma(X,S) \to \Gamma(X)$ (resp. $\ddot{\Gamma}(X,S) \to \ddot{\Gamma}(X)$) is continuous. Thus for any configuration γ the spaces S^γ are Borel subsets of $\Gamma(X,S)$ (resp. $\ddot{\Gamma}(X,S)$).

Let us fix $u \in C_b(X \to S)$ with $||u||_\infty := \sup_{x\in X} |u(x)| < \infty$ and define the map

$$\Gamma(X) \ni \gamma \mapsto \xi_\gamma = (u(x))_{x\in\gamma} \in S^\gamma \tag{5.5}$$

(in particular, one may take $u \equiv const$). Obviously, we have the inclusion $\xi_\gamma \in l^p_\alpha(\gamma, S)$ for all $\gamma \in \Gamma(X)$ and any $\alpha, p > 0$.

Proposition 5.2. *Let ξ_γ be given by formula (5.5). Then the map*

$$\Gamma(X) \ni \gamma \mapsto \hat{\gamma} = (\gamma, \xi_\gamma) \in \Gamma(X,S)$$

is continuous.

Proof. According to the definition of the topology on $\Gamma(X,S)$, the claim is equivalent to the continuity of the map

$$\Gamma(x) \ni \gamma \mapsto F(\gamma) := \langle f, \hat{\gamma}\rangle, \quad \hat{\gamma} = (\gamma, \xi_\gamma),$$

for any $f \in C_b(X \times S)$ with supp $f \in S_\Lambda := \Lambda \times S$, where Λ is a compact subset of X. It is clear that $F(\gamma) = \langle g, \gamma\rangle$, where $g(x) = f(x, u(x))$, so that $g \in C_0(X)$ and the assertion follows from the definition of the topology on $\Gamma(X)$. ■

Let $\mathcal{B}_c(X)$ denote the collection of all convex compact subsets of X. Let us fix a set $\Delta \in \mathcal{B}_c(X)$ and define $\gamma_\Delta := \Delta \cap \gamma$, $\gamma \in \Gamma(X)$. Obviously $\gamma_\Delta \in \mathcal{F}(\gamma)$. Consider the measure $\hat{\nu}^\xi_\Delta(d\hat{\gamma})$ on $\Gamma(X,S)$ defined by the formula

$$\hat{\nu}^\xi_\Delta(d\hat{\gamma}) := \Pi_{\gamma_\Delta}(d\sigma_\gamma \left| \xi_\gamma \right.)\mu(d\gamma), \quad \gamma = p_X(\hat{\gamma}), \tag{5.6}$$

or, equivalently,

$$\int_{\Gamma(X,S)} F(\hat{\gamma})\hat{\nu}^\xi_\Delta(d\hat{\gamma}) = \int_{\Gamma(X)} \Phi_F(\gamma)\mu(d\gamma)$$

for any $F \in C_b(\Gamma(X,S))$, where

$$\Phi_F(\gamma) := \int_{S^\gamma} F(\gamma, \sigma_\gamma)\ \Pi_{\gamma_\Delta}(d\sigma_\gamma \left| \xi_\gamma \right.). \tag{5.7}$$

The measure $\hat{\nu}^\xi_\Delta$ is well-defined because of the next result that is also of its own interest, stating the following "almost Feller" property for the specification kernels $\Pi_{\gamma_\Delta}(d\sigma_\gamma \left| \xi_\gamma \right.)$. As will be clear from the proof below, the reason why this property only holds "almost" everywhere is that the projection map $\gamma \to \gamma_\Lambda$ is vague continuous at some $\gamma \in \Gamma(X)$ iff γ has no points on the bondary of Λ.

Proposition 5.3. *Let $\Delta \in \mathcal{B}_c(X)$ (in particular, Δ can be a closed ball or cube). Then there exists a set $M = M(\Delta) \subset \Gamma(X)$ such that $\mu(M) = 1$ and the function $\Phi_F : \Gamma(X) \to \mathbb{R}$ defined by the formula (5.7) is continuous at any $\gamma \in M$, for all $F \in C_b(\Gamma(X,S))$.*

Proof. For a fixed $\Delta \in \mathcal{B}_c(X)$ consider its R-neighborhood $\Delta^R := \cup_{x\in\Delta} B_R(x)$, where $B_R(x)$ is the closed ball of radius R centered at x. Obviously, Δ^R is convex and bounded. Denote by $\partial\Delta$ and $\partial\Delta^R$ the topological bondary or the sets Δ and Δ^R respectively, then $\partial\Delta$ and $\partial\Delta^R$ both have zero Lebesgue mass (see e.g. [20]). Define

$$M = M(\Delta) := \left\{ \gamma \in \Gamma(X) :\ \gamma \cap \left(\partial\Delta \cup \partial\Delta^R\right) = \emptyset \right\} \in \mathcal{B}(\Gamma(X)).$$

It follows from the boundedness of the first correlation function of the measure μ that $\mu(M) = 1$.

Fix $\gamma \in M$. Then

$$\begin{aligned}\Phi_F(\gamma) \quad &= \quad Z^{-1} \int\limits_{S^{\gamma_\Delta}} F(\gamma, \sigma_{\gamma_\Delta} \times \xi_{\gamma_{\Delta^c}})\ \times \\ &\qquad \exp\left[-\sum_{\{x,y\}\subset\gamma_\Delta} J(x-y)\sigma_x\sigma_y - \sum_{\substack{x\in\gamma_\Delta \\ z\in\gamma_{\Delta^c}}} J(x-z)\sigma_x\xi_z \right] \bigotimes_{x\in\gamma_\Delta} e^{-U(\sigma_x)} d\sigma_x,\end{aligned}$$

where

$$Z = \int\limits_{S^{\gamma_\Delta}} \exp\left[-\sum_{\{x,y\}\subset\gamma_\Delta} J(x-y)\sigma_x\sigma_y - \sum_{\substack{x\in\gamma_\Delta \\ z\in\gamma_{\Delta^c}}} J(x-z)\sigma_x\xi_z \right] \bigotimes_{x\in\gamma_\Delta} e^{-U(\sigma_x)} d\sigma_x.$$

The condition $\gamma \cap \partial\Delta = \emptyset$ implies that there exists a closed set $\Delta' \subset \Delta^R \diagdown \Delta$ such that

$$\gamma \cap (\Delta^R \diagdown \Delta) \subset \Delta'$$

and

$$\gamma \cap \partial \Delta' = \emptyset.$$

Let

$$\gamma \cap \Delta = \left\{ x^{(1)}, ..., x^{(m)} \right\}, \quad \gamma \cap \Delta' = \left\{ x^{(m+1)}, ..., x^{(N)} \right\}.$$

Then, in the particle representation we have

$$\begin{aligned} Z &= Z(x^{(1)}, ..., x^{(N)};\ \xi^{(m+1)}, .., \xi^{(N)}) \\ &= \int\limits_{S^{\gamma_\Delta}} \exp\left[-E(x^{(1)}, ..., x^{(N)};\ \xi^{(m+1)}, .., \xi^{(N)}) \right] \bigotimes_{k=1}^{m} e^{-U(\sigma_k)} d\sigma_k \ , \end{aligned}$$

$$\begin{aligned} \Phi_F(\gamma) &= Z^{-1} \int\limits_{S^{\gamma_\Delta}} F\left(\left\{ x^{(1)}, ..., x^{(N)} \right\} \cup \gamma_{\Delta^c},\ (\sigma_1, \ldots, \sigma_m) \times (\xi^{(m+1)}, ..., \xi^{(N)}) \times \xi(\gamma \diagdown \Delta^R) \right) \\ &\quad \times \exp\left[-E(x^{(1)}, ..., x^{(N)};\ \xi^{(m+1)}, ..., \xi^{(N)}) \right] \bigotimes_{k=1}^{m} e^{-U(\sigma_k)} d\sigma_k \end{aligned}$$

with $\xi^{(m+1)} := \xi(x^{(m+1)}), ..., \xi^{(N)} := \xi(x^{(N)})$ and

$$\begin{aligned} & E(x^{(1)}, ..., x^{(N)};\ \xi^{(m+1)}, ..., \xi^{(N)}) : \\ = & \sum_{\substack{k,j=1 \\ k \neq j}}^{m} J(x^{(k)} - x^{(j)}) \sigma_k \sigma_j - \sum_{k=1}^{m} \sum_{j=m+1}^{N} J(x^{(k)} - x^{(j)}) \sigma_k \xi^{(j)}. \end{aligned}$$

The required result follows now from the continuity of $F,\ J$ and ξ and Proposition 6.2. ■

Next we will show that for any ξ of the form (5.5) the family of measures $\left\{ \hat{\nu}_\Delta^\xi,\ \Delta \in \mathcal{B}_c(X) \right\}$ is tight. To this end we need some technical preparations. Let us introduce a Lyapunov-type function $F : \Gamma(X, S) \to \mathbb{R}_+$ given by the expression

$$F(\hat{\gamma}) = b_\alpha(\gamma) + \|\sigma_\gamma\|_{\alpha,p}^p, \quad \hat{\gamma} = (\gamma, \sigma_\gamma).$$

Lemma 5.4. *For any $R > 0$ the set*

$$M_R := \{ \hat{\gamma} :\ F(\hat{\gamma}) \leq R \} \tag{5.8}$$

is relatively compact in $\ddot{\Gamma}(X, S)$.

Proof. Let $K = \{ \gamma \in \Gamma(X) :\ b_\alpha(\gamma) < \infty \}$. A direct application of Lemma 6.4 in Appendix shows that the set K is relatively compact in $\ddot{\Gamma}(X)$. By Proposition 6.7, the corresponding set $K(S)_R$ defined by (6.5) is relatively compact in $\ddot{\Gamma}(X, S)$. Observe now that $M_R \subset K(S)_R$, and the result follows. ■

Proposition 5.5. *For any ξ of the form (5.5) the family of measures $\hat{\nu}_\Delta^\xi,\ \Delta \in \mathcal{B}_c(X)$, is relatively weakly compact in $\mathcal{P}(\ddot{\Gamma}(X, S))$*

Proof. It follows from the definition (5.6) of the measure $\hat{\nu}_\Delta(d\hat{\gamma}\,|\xi)$, inequality (2.6) and moment estimates (3.31)–(3.33) that

$$\begin{aligned} & \int_{\Gamma(X,S)} F(\hat{\gamma})\hat{\nu}_\Delta(d\hat{\gamma}\,|\xi) \qquad (5.9)\\ = & \int_{\Gamma(X)}\int_{S^\gamma}\left[b_\alpha(\gamma)+\|\sigma_\gamma\|^p_{\alpha,p}\right]\ \Pi_{\gamma_\Delta}(d\sigma_\gamma\,|\xi_\gamma)\mu(d\gamma)\\ \le & \int_{\Gamma(X)} b_\alpha(\gamma)\mu(d\gamma)+\left(\lambda-e^{\alpha R}2\beta\right)^{-1}e^{\alpha R}2\beta\int_{\Gamma(X)}\left\|\xi_{\gamma_{\Delta^c}}\right\|^p_{\alpha,p}\mu(d\gamma)\\ & +\left(\lambda-e^{\alpha R}2\beta\right)^{-1}\int_{\Gamma(X)}\left[C_\lambda b_\alpha(\gamma)+2C_{p,J}\beta^{-p'}a_{\alpha,p'}(\gamma)\right]\mu(d\gamma). \end{aligned}$$

Note that by (5.5)

$$\int_{\Gamma(X)}\left\|\xi_{\gamma_{\Delta^c}}\right\|^p_{\alpha,p}\mu(d\gamma)\le ||u||^p_\infty\int_{\Gamma(X)} b_\alpha(\gamma_{_{\Delta^c}})\mu(d\gamma),$$

and hence by Proposition 2.2

$$\sup_{\Delta\in\mathcal{B}_c(X)}\int_{\Gamma(X,S)} F(\hat{\gamma})\hat{\nu}_\Delta(d\hat{\gamma}\,|\xi)<\infty. \qquad (5.10)$$

The application of Chebyshev's inequality (similar to the proof of Proposition 3.7) shows that for any $\varepsilon>0$ there exists $R>0$ such that

$$\hat{\nu}_\Delta(M_R\,|\xi)\ge 1-\varepsilon$$

uniformly for all $\Delta\in\mathcal{B}_c(X)$. By Lemma 5.4 M_R is relatively compact in $\ddot{\Gamma}(X,S)$, which implies that the family of measures $\left\{\hat{\nu}^\xi_\Delta,\Delta\in\mathcal{B}_c(X)\right\}\subset\mathcal{P}(\ddot{\Gamma}(X,S))$ is tight. Then Prokhorov's theorem yields the result. ■

Corollary 5.6. *The family of measures* $\left\{\hat{\nu}^\xi_\Delta,\Delta\in\mathcal{B}_c(X)\right\}$ *contains a sequence* $\hat{\nu}^\xi_{\Delta_n}$, $n\in\mathbb{N}$, *which converges weakly to a probability measure* $\hat{\nu}^\xi$ *on* $\ddot{\Gamma}(X,S)$. *Without loss of generality we will assume that the sequence of sets* Δ_n *is increasing.*

Observe that the function $\hat{F}:=F\circ p_X$, where $F\in C_b(\ddot{\Gamma}(X))$, is continuous. The limit transition as $n\to\infty$ in the formula

$$\int_{\ddot{\Gamma}(X,S)}\hat{F}(\hat{\gamma})\hat{\nu}^\xi_{\Delta_n}(d\hat{\gamma})=\int_{\ddot{\Gamma}(X)}F(\gamma)\left(\int_{S^\gamma}\Pi_{\gamma_{\Delta_n}}(d\sigma_\gamma\,|\xi_\gamma)\right)\mu(d\gamma)=\int_{\ddot{\Gamma}(X)}F(\gamma)\mu(d\gamma)$$

shows that

$$\int_{\ddot{\Gamma}(X,S)}\hat{F}(\hat{\gamma})\hat{\nu}^\xi(d\hat{\gamma})=\int_{\ddot{\Gamma}(X)}F(\gamma)\mu(d\gamma),$$

or equivalently

$$\mu=p^*_X\hat{\nu}^\xi.$$

The application of Theorem 8.1 of [27] to the measurable map

$$p_X:\ddot{\Gamma}(X,S)\to\ddot{\Gamma}(X)$$

yields that there exists the corresponding regular conditional probability distribution ν^ξ_γ, $\gamma\in\ddot{\Gamma}(X)$, that is, a family of probability measures ν^ξ_γ on $\ddot{\Gamma}(X,S)$ such that for any measurable set $A \subset \ddot{\Gamma}(X,S)$ we have

$$\hat{\nu}^{\xi}(A) = \int_{\ddot{\Gamma}(X)} \nu_{\gamma}^{\xi}(A)\mu(d\gamma),$$

and the map

$$\ddot{\Gamma}(X) \ni \gamma \mapsto \nu_{\gamma}^{\xi}(A) \tag{5.11}$$

is measurable. Moreover, $\nu_{\gamma}^{\xi}(\ddot{\Gamma}(X,S) \diagdown p_X(\gamma)) = 0$ for μ-a.a. $\gamma \in \ddot{\Gamma}(X)$.

Thus ν_{γ}^{ξ} generates (for μ-a.a. $\gamma \in \ddot{\Gamma}(X)$) a probability measure on $p_X(\gamma) = S^{\gamma}$ (for which we preserve the notation ν_{γ}^{ξ}) such that the map

$$\ddot{\Gamma}(X) : \gamma \mapsto \int_{S^{\gamma}} F(\gamma, \sigma_{\gamma})\nu_{\gamma}^{\xi}(d\sigma_{\gamma})$$

is measurable for any $F \in C_b(\ddot{\Gamma}(X,S))$ and

$$\int_{\ddot{\Gamma}(X,S)} F(\gamma,\sigma_{\gamma})\hat{\nu}^{\xi}(d\hat{\gamma}) = \int_{\ddot{\Gamma}(X)} \left(\int_{S^{\gamma}} F(\gamma,\sigma_{\gamma})\nu_{\gamma}^{\xi}(d\sigma_{\gamma}) \right) \mu(d\gamma).$$

Recall that μ is concentrated on $\Gamma(X)$, so that we can replace the spaces $\ddot{\Gamma}(X,S)$, $\ddot{\Gamma}(X)$ by $\Gamma(X,S)$, $\Gamma(X)$ respectively and obtain the equality

$$\int_{\Gamma(X,S)} F(\gamma,\sigma_{\gamma})\hat{\nu}^{\xi}(d\hat{\gamma}) = \int_{\Gamma(X)} \left(\int_{S^{\gamma}} F(\gamma,\sigma_{\gamma})\nu_{\gamma}^{\xi}(d\sigma_{\gamma}) \right) \mu(d\gamma) \tag{5.12}$$

for any $F \in C_b(\Gamma(X,S))$.

Proposition 5.7. *For μ-a.a. $\gamma \in \Gamma(X)$ and any ξ of the form (5.5) we have the inclusion*

$$\nu_{\gamma}^{\xi} \in \mathcal{G}_{\alpha,p}(S^{\gamma}).$$

Proof. Let us first note that the weak convergence of the measures $\hat{\nu}_{\Delta_n}^{\xi}$, $n \in \mathbb{N}$, to $\hat{\nu}^{\xi}$ does not in general imply convergence of their conditional distributions $\Pi_{\gamma_{\Delta_n}}(d\sigma_{\gamma} \,|\xi_{\gamma})$ to $\nu_{\gamma}^{\xi}(d\sigma_{\gamma})$ for μ-a.a. γ. However, we can make use of Komlós' theorem (see e.g. [2]) and prove convergence (for μ-a.a. $\gamma \in \Gamma(X)$) of the Cesàro means $\frac{1}{N}\sum_{j=1}^{N} \Pi_{\gamma_j}(d\sigma_{\gamma} \,|\xi_{\gamma})$ for some subsequence $\gamma_j := \gamma_{\Delta_{n_j}}$, $j \in \mathbb{N}$, which will allow us show that $\hat{\nu}^{\xi}$ satisfies the DLR equation.

Let us fix an indexation $\mathfrak{i} = \{\mathfrak{i}_{\gamma}, \gamma \in \Gamma_X\}$ in Γ_X, where $\mathfrak{i}_{\gamma} \colon \gamma \to \mathbb{N}$ is a bijection for each $\gamma \in \Gamma_X$. The indexation $\mathfrak{i}$ defines a natural bijection

$$\Gamma_{X,1} \ni (\gamma, x) \mapsto (\gamma, \mathfrak{i}_{\gamma}(x)) \in \Gamma_X \times \mathbb{N}, \tag{5.13}$$

where

$$\Gamma_{X,1} := \{(\gamma,x) \in \Gamma_X \times X \colon\ x \in \gamma\}.$$

Moreover, the indexation $\mathfrak{i}$ can be constructed so that bijection (5.13) is measurable (see e.g. Lemma 2.3 in [11] or 1.1.4 in [12]). It generates a natural homeomorphism $\mathcal{I}_{\gamma} : S^{\gamma} \to S^{\mathbb{N}}$, $\gamma \in \Gamma_X$.

Observe that the space of $\mathcal{P}(S^{\mathbb{N}})$ of probability measures on the Polish space $S^{\mathbb{N}}$ is completely metrizable, and there exists a sequence of functions $f_m \in C_b(S^{\mathbb{N}})$, $m \in \mathbb{N}$, which form a separating class for $\mathcal{P}(S^{\mathbb{N}})$. Define functions

$$\hat{f}_{m,\gamma} := f_m \circ \mathcal{I}_{\gamma} \in C_b(S^{\gamma}), \quad m \in \mathbb{N}.$$

It is clear that the family $\{\hat{f}_{m,\gamma}\}_{m\in\mathbb{N}}$ is separating for $\mathcal{P}(S^{\gamma})$ (μ-a.s. in γ).

Consider a family of functions

$$g_{\Delta_n}^{(m)}(\gamma) = \int_{S^\gamma} \hat{f}_{m,\gamma}(\sigma)\Pi_{\gamma_{\Delta_n}}(d\sigma \,|\xi_\gamma), \quad n \in \mathbb{N}.$$

The measurability of the indexation $\mathfrak{i}$ implies that $g_{\Delta_n}^{(m)} \in L^1(\Gamma_X, \mu)$. Applying Komlós' theorem (similarly to the proof Theorem 3.6 in [14]) and taking into account formula (5.12), one can show that there exists a subsequence $\{n_j\}_{j\in\mathbb{N}}$ such that

$$\frac{1}{N}\sum_{j=1}^{N} g_{\Delta_{n_j}}^{(m)}(\gamma) \to \int_{S^\gamma} \hat{f}_m(\gamma,\sigma)\nu_\gamma^\xi(d\sigma), \quad N \to \infty,$$

for any $m \in \mathbb{N}$ and μ-a.a. $\gamma \in \Gamma_X$. This implies the weak convergence of measures

$$\frac{1}{N}\sum_{j=1}^{N} \Pi_{\gamma_j}(\cdot \,|\xi_\gamma) \to \nu_\gamma^\xi, \quad N \to \infty, \ \gamma_j := \gamma_{\Delta_{n_j}}, \tag{5.14}$$

on S^γ, for μ-a.a. $\gamma \in \Gamma_X$.

Now we are in a position to prove that the measure ν_γ^ξ satisfies the DLR equation

$$\nu_\gamma^\xi(d\sigma) = \int_{S^\gamma} \Pi_\eta(d\sigma, \xi)\nu_\gamma^\xi(d\xi), \quad \eta \in \mathcal{F}(\gamma), \tag{5.15}$$

for μ-a.a. $\gamma \in \Gamma_X$. Indeed, for a fixed γ and $\eta \in \mathcal{F}(\gamma)$, the relation (5.14) implies the convergence

$$\begin{aligned} &\int_{S^\gamma}\left(\int_{S^\gamma} f(\sigma)\Pi_\eta(d\sigma,\xi)\right)\frac{1}{N}\sum_{j=1}^{N}\Pi_{\gamma_j}(d\xi \,|\xi_\gamma) \\ \to \quad &\int_{S^\gamma}\left(\int_{S^\gamma} f(\sigma)\Pi_\eta(d\sigma,\xi)\right)\nu_\gamma^\xi(d\xi), \quad N \to \infty, \end{aligned} \tag{5.16}$$

where $f \in C_b(S^\gamma)$. Choose $j_0 \in \mathbb{N}$ such that $\eta \subset \gamma_{j_0} := \gamma \cap \Delta_{n_{j_0}}$. Then we have

$$\begin{aligned} &\lim_{N\to\infty}\int_{S^\gamma}\left(\int_{S^\gamma} f(\sigma)\Pi_\eta(d\sigma,\xi)\right)\frac{1}{N}\sum_{j=1}^{N}\Pi_{\gamma_j}(d\xi \,|\xi_\gamma) \\ = \quad &\lim_{N\to\infty}\int_{S^\gamma}\left(\int_{S^\gamma} f(\sigma)\Pi_\eta(d\sigma,\xi)\right)\frac{1}{N}\sum_{j=j_0}^{N}\Pi_{\gamma_j}(d\xi \,|\xi_\gamma) \\ = \quad &\lim_{N\to\infty}\int_{S^\gamma} f(\sigma)\frac{1}{N}\sum_{j=j_0}^{N}\Pi_{\gamma_j}(d\sigma \,|\xi_\gamma) \\ = \quad &\int_{S^\gamma} f(\sigma)\nu_\gamma^\xi(d\sigma), \end{aligned}$$

where we used the the consistency property (3.5) of specification Π and relation (5.14). Thus (5.15) holds.

By the arguments similar the proof of Theorem 3.2 one can show that the measure ν_γ^ξ satisfies estimate (3.12) and is therefore supported on $l_\alpha^p(\gamma, S)$. This completes the proof. ∎

Proof of Theorem 5.1. The result follows directly from formula (5.11) and Proposition 5.7. ∎

Remark 5.8. Let $\nu_\gamma \in \mathcal{G}_{\alpha,p}(S^\gamma)$, $\gamma \in \Gamma(X)$, be a family of Gibbs measures satisfying the measurability condition (5.1). For μ-a.a. $\gamma \in \Gamma(X)$ the measure ν_γ obeys the moment estimate (3.41). Integrating both sides of this inequality we obtain

$$\int_{\Gamma(X)} \int_{S^\gamma} \|\sigma\|^p_{\alpha,p} \;\nu_\gamma(d\sigma)\mu(d\gamma) \leq \left(\lambda - e^{\alpha R}2\beta\right)^{-1} C_\lambda \int_{\Gamma(X)} b_\alpha(\gamma)\mu(d\gamma) + 2C_{p,J}\beta^{-p'} \int_{\Gamma(X)} a_{\alpha,p'}(\gamma)\,\mu(d\gamma).$$

Estimates (2.6) and (2.9) imply now that

$$\int_{\Gamma(X)} \int_{S^\gamma} \|\sigma\|^p_{\alpha,p} \;\nu_\gamma(d\sigma)\mu(d\gamma) \leq \left(\lambda - e^{\alpha R}2\beta\right)^{-1} \int_X e^{-\alpha|x|}dx \times \left(C_\lambda ||k_1||_\infty + 2C_{p,J}\beta^{-p'}||k||_\infty \sum_{j=1}^{M-1} c_j \mathrm{Vol}(B_{2R})^j \right),$$

where $\mathrm{Vol}(B_{2R})$ is the volume of the ball B_{2R}, $||k||_\infty := \max_{1\leq m\leq M} ||k_m||_\infty$ and the constants c_j are given by formula (2.8).

6. Appendix: Some facts on topological structure of $\Gamma(X,S)$.

In this Appendix, we discuss topological properties of the marked configuration spaces $\Gamma(X,S)$ and $\ddot{\Gamma}(X,S)$ defined by expressions (5.2) and (5.3) respectively. Although most of these properties naturally extend those known for the configuration spaces $\Gamma(X)$ and $\ddot{\Gamma}(X)$, we could not find any adequate presentation in the literature. Here we work out the necessary topological issues, especially characterization of convergence and compactness criteria, which also may be of independent interest in the theory of marked point processes and its applications. As compared to the spaces $\Gamma(X)$ and $\ddot{\Gamma}(X)$, now we have not only to control the concentration of points x in bounded domains, but also to exclude the possibility that their marks s_x go to infinity. We suppose that the configuration spaces $\Gamma(X \times S)$, $\Gamma(X)$ and $\ddot{\Gamma}(X)$ are equipped with the corresponding vague topology.

1. **Topology**. There are two natural topologies on $\Gamma(X,S)$ and $\ddot{\Gamma}(X,S)$.

(i) Our "main" topology has been defined in Section 5. This is the weakest topology that makes the map

$$\Gamma(X,S) \ni \hat{\gamma} \mapsto \langle f, \hat{\gamma}\rangle \tag{6.1}$$

continuous for any $f \in C_b(X\times S)$ with supp $f \in S_\Lambda := \Lambda \times S$, where Λ is a compact subset of X. The space $(\Gamma(X,S), \tau)$ is Polish (see Section 2 in [5]). In the rest of this paper, we will call it the τ-topology.

(ii) We will also use the topology induced from $\Gamma(X \times S)$. This is the weakest topology that makes the map (6.1) continuous for any $f \in C_0(X \times S)$. We will call it the w-topology. Observe that $\Gamma(X,S)$ is not a closed subset of $\Gamma(X\times S)$. The w-topology. is weaker than the τ-topology. Observe that the map $p_X : \Gamma(X,S) \to \Gamma(X)$ is not w-continuous.

2. **Convergence.** The following statement is known **(**see e.g. Proposition 3.13 in [29]**):**

Lemma 6.1. *A sequence $\{\gamma_n,\ n\in\mathbb{N}\}\subset\Gamma(X)$ (resp. $\ddot{\Gamma}(X)$) converges to $\gamma\in\Gamma(x)$ (resp. $\ddot{\Gamma}(X)$) iff for any compact $\Lambda\subset X$ such that $\gamma\cap\partial\Lambda=\emptyset$, the number of elements of $\gamma_n\cap\Lambda$ stabilizes and $\gamma_n\cap\Lambda\to\gamma\cap\Lambda$ pointwise.*

Let $\hat{\gamma}_n,\ n\in\mathbb{N}$, be a sequence of elements of $\Gamma(X,S)$ and $\hat{\gamma}\in\Gamma(X,S)$.

Proposition 6.2. *The following statements are equivalent:*
1) $\hat{\gamma}_n\overset{\tau}{\to}\hat{\gamma}$;
2) for any compact $\Lambda\subset X$ such that $p_X(\hat{\gamma})\cap\partial\Lambda=\emptyset$, the number of elements of $\hat{\gamma}_n\cap S_\Lambda$ stabilizes and $\hat{\gamma}_n\cap S_\Lambda\to\hat{\gamma}\cap S_\Lambda$ pointwise;.
3) $\hat{\gamma}_n\overset{w}{\to}\hat{\gamma}$ and $p_X(\hat{\gamma}_n)\to p_X(\hat{\gamma})$ in $\Gamma(X)$.

Proof. 1) $\Longleftrightarrow$ 2) Similar to the proof of Lemma 6.1.
2) $\Longrightarrow$ 3) Direct calculation.
3) $\Longrightarrow$ 2) Fix a compact $\Lambda\subset X$ such that $p_X(\hat{\gamma})\cap\partial\Lambda=\emptyset$. The convergence $\gamma_n:=p_X(\hat{\gamma}_n)\to p_X(\hat{\gamma})=:\gamma$ implies that $\exists\ n_0>0$ such that $\forall\ n\geq n_0$ we have

$$N\left(\gamma_n\cap\Lambda\right)=N\left(\gamma\cap\Lambda\right)\tag{6.2}$$

and

$$(\gamma_n\cap\Lambda)\to(\gamma\cap\Lambda),\ n\to\infty,$$

pointwise. Observe that for any $\hat{\xi}\in\Gamma(X,S)$ we have

$$N\left(\hat{\xi}\cap S_\Lambda\right)=N\left(p_X(\xi)\cap\Lambda\right)=N\left(\hat{\xi}\cap(\Lambda\times B)\right)$$

for a compact $B\subset S$. Equality (6.2) implies that

$$N\left(\hat{\gamma}_n\cap S_\Lambda\right)=N\left(\hat{\gamma}\cap S_\Lambda\right)\text{ and }N\left(\hat{\gamma}_n\cap(\Lambda\times B)\right)=N\left(\hat{\gamma}\cap(\Lambda\times B)\right)$$

for a compact $B\subset S$. This together with the convergence $\hat{\gamma}_n\overset{w}{\to}\hat{\gamma}$ implies that $\hat{\gamma}_n\cap S_\Lambda\to\hat{\gamma}\cap S_\Lambda$ pointwise. ■

Remark 6.3. It is easy to see that Proposition 6.2 holds for a sequence $\hat{\gamma}_n\in\ddot{\Gamma}(X,S)$ converging to $\hat{\gamma}\in\ddot{\Gamma}(X,S)$, with $\ddot{\Gamma}(X)$ instead of $\Gamma(X)$ in part 3).

3. **Compactness.** The following criterion of compactness of a set K in $\Gamma(X)$ and $\ddot{\Gamma}(X)$ is well-known. Consider the following two conditions: (i) $\forall\Lambda\in\mathcal{B}_0(X)$, $\sup_{\gamma\in K}N(\gamma_\Lambda)<\infty$ and (ii) $\forall\Lambda\in\mathcal{B}_0(X)$, $\inf_{\gamma\in K}\min_{\{x_1,x_2\}\subset\gamma_\Lambda}|x_1-x_2|>0$.

Lemma 6.4. *(see e.g. Propositions 3.2.5 and 3.2.6 in* [12] *or* [16]*). The set K is relatively compact in $\ddot{\Gamma}(X)$ (resp. in $\Gamma(X)$) if and only if (i) (resp. (i) and (ii)) hold.*

Let now K be a relatively compact subset of $\ddot{\Gamma}(X)$, and consider the corresponding subset $K(S):=p_X^{-1}(K)$ of the marked configuration space $\ddot{\Gamma}(X,S)$:

$$K(S)=\left\{\hat{\gamma}\in\Gamma(X,S):p_X(\hat{\gamma})\in K\right\}.\tag{6.3}$$

Lemma 6.5. *$K(S)$ is relatively compact in $\ddot{\Gamma}(X\times S)$.*

Proof. The proof can be obtained by a direct application Lemma 6.4. Fix a compact $\hat{\Lambda}\subset X\times S$. It is sufficient to show that $\sup_{\hat{\gamma}\in K(S)}N\left(\hat{\gamma}_{\hat{\Lambda}}\right)<\infty$. Observe that $\Lambda:=p_X(\hat{\Lambda})$ is a compact subset of X. The relative compactness of $K\subset\ddot{\Gamma}(X)$ implies that $\sup_{\gamma\in K}N\left(\gamma_\Lambda\right)<\infty$. Moreover, we have $N\left(\hat{\gamma}_{\hat{\Lambda}}\right)\leq N\left(\hat{\gamma}_{S_\Lambda}\right)=N\left(\gamma_\Lambda\right)$, and the assertion follows. ■

Remark 6.6. It can be shown by similar arguments that $K(S)$ is relatively compact in $\Gamma(X \times S)$ provided K is relatively compact in $\Gamma(X)$.

Let us introduce the spaces

$$\Gamma(X,S)_r := \left\{(\gamma,\sigma_\gamma) \in \Gamma(X,S):\ \|\sigma_\gamma\|_{\alpha,p} \le r\right\} \tag{6.4}$$

and

$$\ddot{\Gamma}(X,S)_r := \left\{(\gamma,\sigma_\gamma) \in \ddot{\Gamma}(X,S):\ \|\sigma_\gamma\|_{\alpha,p} \le r\right\}.$$

Set

$$K(S)_r := K(S) \cap \Gamma(X,S)_r, \tag{6.5}$$

where the set $K(S)$ is defined by (6.3). We have the following statement.

Proposition 6.7. *1) Topologies τ and w coincide on $\ddot{\Gamma}(X,S)_r$*
2) $\ddot{\Gamma}(X,S)_r$ is a closed subset of $\ddot{\Gamma}(X,S)$ and $\ddot{\Gamma}(X \times S)$.
3) $\overline{\Gamma(X,S)}_r \subset \ddot{\Gamma}(X,S)_r$, where $\overline{\Gamma(X,S)}_r$ is the closure of $\Gamma(X,S)_r$ in $\ddot{\Gamma}(X,S)$ (or, equivalently, in $\ddot{\Gamma}(X \times S)$).
4) $\overline{K(S)}_r$ is a compact subset of $\ddot{\Gamma}(X,S)$.

Proof. 1) It is sufficient to prove that the map

$$\ddot{\Gamma}(X,S)_r \ni \hat{\gamma} \mapsto \langle f,\hat{\gamma}\rangle$$

is w-continuous for any $f \in C_b(X \times S)$ with supp $f \in S_\Lambda := \Lambda \times S$, where Λ is a compact subset of X. For this, fix an arbitrary function $\chi \in C_0(S)$ such that $\chi(\sigma) = 1$ if $\sigma \in B_{r_0}$ (the ball in S of radius r_0 centered at 0), where $r_0 > r\ \max_{x\in\Lambda} w_a(x)$, and define $g := \chi f$. Then (i) $\langle f,\hat{\gamma}\rangle = \langle g,\hat{\gamma}\rangle$ for $\hat{\gamma} \in \ddot{\Gamma}(X,S)_r$, and (ii) $g \in C_0(X \times S)$ so that the map $\hat{\gamma} \mapsto 2\,\langle g,\hat{\gamma}\rangle$ is w-continuous, and the result follows.

2) Let $\hat{\gamma}_n,\ n = 1,2,...,$ be a sequence of elements of $\ddot{\Gamma}(X,S)_r$ that converges to $\hat{\gamma} \in \Gamma(X \times S)$. Fix a compact $\Lambda \subset X$ such that $p_X(\gamma_n) \cap \partial\Lambda = \emptyset$ for all n and consider the compact $\hat{\Lambda} = \Lambda \times B_{r_0}$ in $X \times S$. Then $\hat{\gamma}_n \cap \partial\hat{\Lambda} = \emptyset$ for all n. Proposition 6.1 applied to the configuration space $\ddot{\Gamma}(X \times S)$ shows that $N\left(\hat{\gamma}_n \cap \hat{\Lambda}\right)$ stabilizes and $\hat{\gamma}_n \cap \hat{\Lambda} \to \hat{\gamma} \cap \hat{\Lambda}$ pointwise. Observe that $\hat{\gamma}_n \cap \hat{\Lambda} = \hat{\gamma}_n \cap S_\Lambda$, which implies that the number of elements of the sequence $\hat{\gamma}_n \cap S_\Lambda$ stabilizes. Thus, by claim 2) of Proposition 6.2, $\hat{\gamma}_n \xrightarrow{\tau} \hat{\gamma}$ and $N\left(p_X(\hat{\gamma}) \cap \Lambda\right) < \infty$, so that $\hat{\gamma} \in \ddot{\Gamma}(X,S)$. The bound $\|\sigma_\gamma\|_{\alpha,p} \equiv \langle \Phi,\hat{\gamma}\rangle \le r,\ \hat{\gamma} = (\gamma,\sigma_\gamma)$, where $\Phi(x,\sigma) = |\sigma|^p\, e^{-\alpha|x|}$, can be proved by the limit transition along any monotonically increasing sequence of functions $\Phi_n \in C_0(X \times S)$ that approximates Φ.

3) The claim follows directly from 2).

4) We have $\overline{K(S)}_r = \overline{K(S)} \cap \overline{\Gamma(X,S)}_r$, which is w-compact by Lemma 6.5. On the other hand, $\overline{K(S)}_r \subset \overline{\Gamma(X,S)}_r \subset \ddot{\Gamma}(X,S)_r$ so that w and τ topologies on $\overline{K(S)}_r$ coincide, and the result follows. ■

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References

[1] S. Albeverio, Yu. Kondratiev, E. Lytvynov, G. Us, Analysis and geometry on marked configuration spaces. Infinite dimensional harmonic analysis (Kyoto, 1999), 1–39, Gräbner, Altendorf, 2000.

[2] E. J. Balder, Infinite-dimensional extensions of a theorem of Komlós, *Prob. Th. Rel. Fields* **81** (1989) 185–188.

[3] P. Billingsley, *Convergence of Probability Measures.* 2d edition, New York, NY: John Wiley & Sons, 1999.

[4] A. Bovier, *Statistical Mechanics of Disordered Systems. A Mathematical Perspective.* Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2006.

[5] F. Conrad, M. Grothaus, N/V-limit for Langevin dynamics in continuum, *Reviews in Math. Physics* **23** (2011).

[6] A. Daletskii, Yu. Kondratiev, Yu. Kozitsky, T. Pasurek, Phase Transitions in a quenched amorphous ferromagnet, Preprint 12142, SFB 701, Universität Bielefeld (2012).

[7] D.J. Daley, D. Vere-Jones, *An Introduction to the Theory of Point Processes Volume I: Elementary Theory and Methods,* 2nd edition (Springer, New York, 2003).

[8] R. L. Dobrushin, Prescribing a system of random variables by conditional distributions, *Theory Probab. Appl.* **15** (1970), 101–118.

[9] H.-O. Georgii, *Gibbs Measures and Phase Transitions*, De Gruyter Studies in Mathematics Vol. 9. Berlin: de Gruyter 1988.

[10] H.-O. Georgii, O. Häggström, C. Maes, The random geometry of equilibrium phases. In: C. Domb and J.L. Lebowitz (eds.) *Phase Transitions and Critical Phenomena* Vol. **18**, Academic Press, London 2000, pp. 1-142.

[11] O. Kallenberg, *Random Measures,* 3rd edition, Berlin, Akademie-Verlag, 1983.

[12] J. Kerstan, K. Matthes, J. Mecke, *Infinitely Divisible Point Processes,* Wiley & Sons, 1978.

[13] Yu. Kondratiev, Yu. Kozitsky, T. Pasurek, Gibbs random fields with unbounded spins on unbounded degree graphs, *J. Appl. Probab.* **47** (2010), 856-875.

[14] Yu. Kondratiev, Yu. Kozitsky, T. Pasurek, Gibbs measures of disordered lattice systems with unbounded spins, *Markov Processes Relat. Fields* **18** (2012), 553–582.

[15] Yu. G. Kondratiev, T. Kuna, Harmonic analysis on configuration space I. General theory, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **5** (2002), 201–233.

[16] Yu. Kondratiev, O. Kutovyi, On the metrical properties of the configuration space, *Math. Nachr.* **279**, No. 7, 774–783 (2006).

[17] Yu. Kondratiev, T. Pasurek, M. Röckner, Gibbs measures of continuous systems: an analytic approach, *Reviews in Math. Physics* **10** (2012),

[18] Yu. Kozitsky, T. Pasurek, Euclidean Gibbs measures of interacting quantum anharmonic oscillators, *J. Stat. Phys.* **127** (2007), 985–1047.

[19] T. Kuna, Studies in Configuration Space Analysis and Applications, Ph.D. dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, 1999, in: *Bonner Math. Schrift.* **324**, Universität Bonn, Math. Inst., Bonn, 1999, 187 pp.

[20] R. Lang, A note on measurability of convex sets, *Arch. Math.* **47** (1986), 90–92.

[21] J. L. Lebowitz, E. Presutti, Statistical mechanics of systems of unbounded spins, *Commun. Math. Phys.* **50** (1976), 195–218.

[22] A. Lenard, Correlation functions and the uniqueness of the state in classical statistical mechanics. *Commun. Math. Phys.* **30** (1973), 35–44.

[23] A. Lenard, States of classical statistical mechanical systems of infinitely many particles. I. *Arch. Rational Mech. Anal.* **59** (1975), 219–239.

[24] A. Lenard, States of classical statistical mechanical systems of infinitely many particles. II. Characterization of correlation measures. *Arch. Rational Mech. Anal.* **59** (1975) 241–256.

[25] C. M. Newman, *Topics in disordered systems.* Lectures in Mathematics ETH Zürich, Birkhäser Verlag, Basel, 1997.

[26] C. M. Newman, D. L. Stein, Thermodynamic chaos and the structure of the short-range spin glasses. In *Mathematical aspects of spin glasses and neural networks,* eds. A. Bovier and P. Picco, 243-287, Progr. Probab.,41, Birkhäser Boston, Boston MA, 1998.

[27] K.R. Parthasarathy, *Probability Measures on Metric Spaces*, Probab. Math. Statist., Academic Press, New York, 1967.

[28] Ch. Preston, *Random Fields*, Lect. Notes Math. **534** (Springer, Berlin, 1976).
[29] S. Resnick, *Extreme Values, Regular Variation, and Point Processes*, Applied Probability, Springer, New York, 1987.
[30] D. Ruelle, *Statistical Mechanics. Rigorous Results* (Benjamins, New York, 1969).
[31] D. Ruelle, Superstable interactions in classical statistical mechanics, *Commun. Math. Phys.* **18** (1970), 127–159.

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