Differential structure and Laplace operator associated with the gamma measure

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Abstract

Let $\mathbb{K}(\mathbb{R}^d)$ denote the cone of discrete Radon measures on $\mathbb{R}^d$. A fundamental example of a probability measure on $\mathbb{K}(\mathbb{R}^d)$ is given by the gamma measure, which has many distinguished properties. Our aim in this paper is to define and study a Laplace operator which is naturally associated with the gamma measure. To this end, we introduce differentiation on $\mathbb{M}(\mathbb{R}^d)$, the space of Radon measures on $\mathbb{R}^d$. For a 'differentiable' function $F : \mathbb{M}(\mathbb{R}^d) \to \mathbb{R}$, we define its gradient $(\nabla^{\mathbb{M}} F)(\eta)$ as a function of $\eta \in \mathbb{M}(\mathbb{R}^d)$ taking values in a tangent space $T_\eta(\mathbb{M})$ to $\mathbb{M}(\mathbb{R}^d)$ at point η. It is intuitively clear that we should have two types of such objects: one related to transformations of the support of a Radon measure, which we call intrinsic transformations, and one related to transformations of masses, which we call extrinsic transformations. We also combine the two types of tangent spaces/gradients into a full tangent space/gradient. We consider the corresponding Dirichlet forms

$$\mathcal{E}^{\mathbb{M}}(F,G) = \int_{\mathbb{K}(\mathbb{R}^d)} \langle (\nabla^{\mathbb{M}} F)(\eta), (\nabla^{\mathbb{M}} G)(\eta) \rangle_{T_\eta(\mathbb{M})}\, d\mathcal{G}(\eta),$$

where $\mathcal{G}$ denotes the gamma measure. Through integration by parts with respect to $\mathcal{G}$, we explicitly find the generators of these Dirichlet form. These generators can, in a certain sense, be thought of as Laplace operators on $\mathbb{K}(\mathbb{R}^d)$ associated with the gamma measure. Finally, we establish essential self-adjointness of these generators in $L^2(\mathcal{G})$ on a proper set of 'test' functions on $\mathbb{K}(\mathbb{R}^d)$.

1 Introduction

Handling and modeling complex systems have become an essential part of modern science. For a long time, complex systems have been treated in physics, where e.g. methods of probability theory are used to determine their macroscopic behavior by their microscopic properties. Nowadays, complex systems, including ecosystems, biological populations, societies, and financial markets, play an important role in various fields, like biology, chemistry, robotics, computer science, and social science.

A mathematical tool to study complex systems is infinite dimensional analysis. Such studies are often related to a probability measure μ defined on an infinite dimensional state space. The most 'traditional' example of a measure μ is Gaussian (white noise) measure, which is defined on the Schwartz space of tempered distributions, $\mathcal{S}'(\mathbb{R}^d)$, see e.g. [4,5,10]. Another example of measure μ is Poisson random measure on $\mathbb{R}^d$. This is a probability measure on the configuration space $\Gamma(\mathbb{R}^d)$ consisting of all locally finite subsets of $\mathbb{R}^d$. A configuration $\gamma = \{x_i\} \in \Gamma(\mathbb{R}^d)$ may be interpreted either as a collection of indistinguishable physical particles located at points x_i, or as a population of a species whose individuals occupy points x_i, or otherwise depending on the type of the problem. The Poisson measure corresponds to a system without interaction between its entities. In order to describe an interaction, one introduces Gibbs perturbations of the Poisson measure, i.e., Gibbs measures on $\Gamma(\mathbb{R}^d)$.

In papers [1,2], some elements of analysis and geometry on the configuration space $\Gamma(\mathbb{R}^d)$ were introduced. In particular, for each $\gamma = \{x_i\} \in \Gamma(\mathbb{R}^d)$, a tangent space to $\Gamma(\mathbb{R}^d)$ at point γ was defined as

$$T_\gamma(\Gamma) := L^2(\mathbb{R}^d \to \mathbb{R}^d, \gamma),$$

where we identified γ with the Radon measure $\sum_i \delta_{x_i}$. A gradient of a 'differentiable' function $F : \Gamma(\mathbb{R}^d) \to \mathbb{R}$ was explicitly identified as a function

$$\Gamma(\mathbb{R}^d) \ni \gamma \mapsto (\nabla^\Gamma F)(\gamma) \in T_\gamma(\Gamma).$$

This, in turn, led to a Dirichlet form

$$\mathcal{E}^\Gamma(F,G) = \int_{\Gamma(\mathbb{R}^d)} \langle (\nabla^\Gamma F)(\gamma), (\nabla^\Gamma G)(\gamma) \rangle_{T_\gamma(\Gamma)} \, d\mu(\gamma),$$

where μ is either Poisson measure or a Gibbs measure. Denote by $-L^\Gamma$ the generator of the Dirichlet form $\mathcal{E}^\Gamma$. Then, in the case where μ is Poisson measure, the operator L^Γ can be understood as a Laplace operator on the configuration space $\Gamma(\mathbb{R}^d)$.

Assume that the dimension d of the underlying space $\mathbb{R}^d$ is ≥ 2. By using the theory of Dirichlet forms, it was shown that there exists a diffusion process on $\Gamma(\mathbb{R}^d)$ which has generator L^Γ, see [1,2,19,20,23,34]. In particular, this diffusion process has

μ as an invariant measure. (For $d = 1$, in order to construct an associated diffusion process an extension of $\Gamma(\mathbb{R}^d)$ is required.)

A further fundamental example of a probability measure on an infinite dimensional space is given by the gamma measure [6,30,32,33]. This measure, denoted in this paper by $\mathcal{G}$, was initially defined through its Fourier transform as a probability measure on the Schwartz space of tempered distributions, $\mathcal{S}'(\mathbb{R}^d)$. A more delicate analysis shows that the gamma measure is concentrated on the smaller space $\mathbb{M}(\mathbb{R}^d)$ of all Radon measures on $\mathbb{R}^d$. More precisely, $\mathcal{G}$ is concentrated on the cone of discrete Radon measures on $\mathbb{R}^d$, denoted by $\mathbb{K}(\mathbb{R}^d)$. By definition, $\mathbb{K}(\mathbb{R}^d)$ consists of all Radon measures of the form $\eta = \sum_i s_i \delta_{x_i}$. It should be stressed that, with $\mathcal{G}$-probability one, the countable set of positions, $\{x_i\}$, is dense in $\mathbb{R}^d$. As for the weights s_i, with $\mathcal{G}$-probability one, we have $\eta(\mathbb{R}^d) = \sum_i s_i = \infty$, but for each compact set $A \subset \mathbb{R}^d$, $\eta(A) = \sum_{i:\, x_i \in A} s_i < \infty$. Elements $\eta \in \mathbb{K}(\mathbb{R}^d)$ may model, for example, biological systems, so that the points x_i represent location of some organisms, and the values s_i are a certain attribute attached to these organisms, like their weight or height.

We note that a study of countable dense random subsets $\{x_i\}$ of $\mathbb{R}^d$ leads to "situations in which probabilistic statements about such sets can be uninformative" [13], see also [3]. It is the presence of the weights s_i in random discrete measures that makes a real difference.

Among all measures on $\mathbb{K}(\mathbb{R}^d)$, the gamma measure has many distinguished properties. In particular, the gamma measure is quasi-invariant with respect to a natural group of transformations of the weights s_i [30], see also [16]. An infinite dimensional analog of the Lebesgue measure is absolutely continuous with respect to the gamma measure [30,31]. Furthermore, the gamma measure belongs to the class of five Meixner-type Lévy measures (this class also includes Gaussian and Poisson measures). Each measure μ from this Meixner-type class admits a 'nice' orthogonal decomposition of $L^2(\mu)$ in orthogonal polynomials of infinitely many variables, see [14, 15, 17, 18]. In particular, in the case of the gamma measure $\mathcal{G}$, these orthogonal polynomials are an infinite dimensional counterpart of the Laguerre polynomials on the real line.

The aims of the present paper are:

(i) Introduce differentiation on $\mathbb{M}(\mathbb{R}^d)$: for a 'differentiable' function $F : \mathbb{M}(\mathbb{R}^d) \to \mathbb{R}$, we would like to define its gradient $(\nabla^{\mathbb{M}} F)(\eta)$ as a function of $\eta \in \mathbb{M}(\mathbb{R}^d)$ taking value at η in a tangent space $T_\eta(\mathbb{M})$ to $\mathbb{M}(\mathbb{R}^d)$ at point η.

(ii) Discuss an integration by parts formula for the gamma measure.

(iii) Introduce and study the corresponding Dirichlet form

$$\mathcal{E}^{\mathbb{M}}(F, G) = \int_{\mathbb{K}(\mathbb{R}^d)} \langle (\nabla^{\mathbb{M}} F)(\eta), (\nabla^{\mathbb{M}} G)(\eta) \rangle_{T_\eta(\mathbb{M})} \, d\mathcal{G}(\eta).$$

In particular, find an explicit form of the generator $-L^{\mathbb{M}}$ of this Dirichlet form on a proper set of 'test' functions on $\mathbb{K}(\mathbb{R}^d)$. Note that the operator $L^{\mathbb{M}}$ can, in a certain sense, be thought of as a Laplace operator on $\mathbb{K}(\mathbb{R}^d)$, associated with the gamma measure.

(iv) Prove that the generator $-L^{\mathbb{M}}$, defined on the set of 'test' functions on $\mathbb{K}(\mathbb{R}^d)$, is essentially self-adjoint in $L^2(\mathbb{K}(\mathbb{R}^d), \mathcal{G})$.

Let us briefly discuss the structure of the paper. In Section 2, we introduce basic notions related to differentiation on $\mathbb{M}(\mathbb{R}^d)$, like a tangent space and a gradient of a function on $\mathbb{M}(X)$. It is intuitively clear that we should have two types of such objects: one related to transformations of the support of a Radon measure, which we call intrinsic transformations, and one related to transformations of masses, which we call extrinsic transformations. We also combine the two types of tangent spaces/gradients into a full tangent space/gradient.

In Section 3, we explicitly construct the gamma measure $\mathcal{G}$ on $\mathbb{K}(\mathbb{R}^d)$. In Section 4, we discuss an integration by parts formula for the gamma measure. Using that the gamma measure is quasi-invariant with respect to the extrinsic transformations [30], we derive an integration by parts formula for the extrinsic directional derivative. We also present an alternative proof of this formula, which uses, in particular, the Mecke identity for the Poisson measure. We note that this alternative proof admits a straightforward generalization to a wide class of Lévy measures on $\mathbb{K}(\mathbb{R}^d)$. It should be noted that such measures are not necessarily quasi-invariant with respect to the extrinsic transformations of $\mathbb{K}(\mathbb{R}^d)$. On the other hand, we show that, even in the case of the gamma measure, an integration by parts formula is absent for the intrinsic directional derivative. Note that the gamma measure is not quasi-invariant with respect to the intrinsic transformations of $\mathbb{K}(\mathbb{R}^d)$.

In Section 5, we construct and study the respective Dirichlet forms on the space $L^2(\mathbb{K}(\mathbb{R}^d), \mathcal{G})$. These Dirichlet forms are related to the intrinsic, extrinsic, and full gradients. We carry out integration by parts with respect to the measure $\mathcal{G}$ and derive generators of these bilinear forms. It should be stressed that, in the case of the intrinsic gradient, despite the absence of integration by parts formula for the directional derivative, it is indeed possible to give an explicit form of the generator of the corresponding Dirichlet form on a dense domain.

Finally, in Section 6, we prove the essential self-adjointness in $L^2(\mathbb{K}(\mathbb{R}^d), \mathcal{G})$ of the generators of the Dirichlet forms on a proper set of 'test' functions on $\mathbb{K}(\mathbb{R}^d)$. To this end, we construct a unitary isomorphism between $L^2(\mathbb{K}(\mathbb{R}^d), \mathcal{G})$ and the symmetric Fock space $\mathcal{F}(\mathcal{H})$ over the space

$$\mathcal{H} = L^2(\mathbb{R}^d \times (0, \infty), dx\, s^{-1} e^{-s}\, ds)).$$

We show that the semigroup $(\mathbf{T}_t)_{t\geq 0}$ in $L^2(\mathbb{K}(\mathbb{R}^d), \mathcal{G})$ which corresponds to the Dirichlet form is unitary isomorphic to the second quantization of a respective semigroup $(T_t)_{t\geq 0}$ in $\mathcal{H}$. It can be shown that this semigroup $(T_t)_{t\geq 0}$ generates a diffusion on $\mathbb{R}^d \times (0, \infty)$.

In particular, in the extrinsic case, the respective diffusion on $\mathbb{R}^d \times (0, \infty)$ is related to a simple space-time transformation of the square of the 0-dimensional Bessel process on $[0, \infty)$.

In the forthcoming paper [8], by using the theory of Dirichlet forms, we will prove the existence of a diffusion on $\mathbb{K}(\mathbb{R}^d)$ with generator $L^{\mathbb{M}}$. We will also explicitly construct the Markov semigroup of kernels on $\mathbb{K}(\mathbb{R}^d)$ which corresponds to this diffusion. Note that many results of the present paper admit a (rather straightforward) generalization to a wide class of Lévy measures on $\mathbb{K}(\mathbb{R}^d)$. Furthermore, we plan to study equilibrium dynamics on $\mathbb{K}(\mathbb{R}^d)$ for which a Gibbs perturbation of the gamma measure (see [9]) is a symmetrizing (and hence invariant) measure.

2 Differentiation on the space of Radon measures

Let X denote the Euclidean space $\mathbb{R}^d$, $d \in \mathbb{N}$, and let $\mathcal{B}(X)$ denote the Borel σ-algebra on X. Let $\mathbb{M}(X)$ denote the space of all (nonnegative) Radon measures on $(X, \mathcal{B}(X))$. The space $\mathbb{M}(X)$ is equipped with the vague topology, i.e., the coarsest topology making all mappings

$$\mathbb{M}(X) \ni \eta \mapsto \langle \varphi, \eta \rangle := \int_X \varphi \, d\eta, \quad \varphi \in C_0(X),$$

continuous. Here $C_0(X)$ is the space all continuous functions on X with compact support. It is well known (see e.g. [12, 15.7.7]) that $\mathbb{M}(X)$ is a Polish space. Let $\mathcal{B}(\mathbb{M}(X))$ denote the Borel σ-algebra on $\mathbb{M}(X)$.

We would like to introduce an appropriate notion of a gradient $\nabla^{\mathbb{M}}$ of a 'differentiable' function $F : \mathbb{M}(X) \to \mathbb{R}$. For each $\eta \in \mathbb{M}(X)$, the gradient $(\nabla^{\mathbb{M}} F)(\eta)$ should take its value in a tangent space $T_\eta(\mathbb{M})$ to $\mathbb{M}(X)$ at point η. To this end, we need appropriate one-parameter groups of transformations of $\mathbb{M}(X)$. It is intuitively clear that we should have two types of groups of transformations: one for the support of a Radon measure η and one for its masses. We will call transformations of the support intrinsic transformations, and transformations of the masses extrinsic transformations.

So we start with a transformation of the support. By analogy with the case of the configuration space $\Gamma(X)$ (see [1]), we proceed as follows. We fix any $v \in C_0^\infty(X \to X)$, a smooth, compactly supported vector field over X. Let $(\phi_t^v)_{t \in \mathbb{R}}$ be the corresponding one-parameter group of diffeomorphisms of X which are equal to the identity outside a compact set in X. More precisely, $(\phi_t^v)_{t \in \mathbb{R}}$ is the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{d}{dt} \phi_t^v(x) = v(\phi_t^v(x)), \\ \phi_0^v(x) = x. \end{cases} \tag{1}$$

We naturally lift the action of this group to the space $\mathbb{M}(X)$. For each $\eta \in \mathbb{M}(X)$, we define $\phi_t^v(\eta) \in \mathbb{M}(X)$ as the pushforward of η under the mapping ϕ_t^v. Thus,

$$\phi_t^v(\eta)(\Delta) := \eta((\phi_t^v)^{-1}\Delta) = \eta(\phi_{-t}^v\,\Delta), \quad \Delta \in \mathcal{B}(X).$$

Hence, for each $f \in L^1(X, \eta)$,

$$\langle f, \phi_t^v(\eta)\rangle = \langle f \circ \phi_t^v, \eta\rangle. \tag{2}$$

For a function $F : \mathbb{M}(X) \to \mathbb{R}$, we define the intrinsic derivative of F in direction v by

$$(\nabla_v^{\mathrm{int}} F)(\eta) := \frac{d}{dt}\Big|_{t=0} F(\phi_t^v(\eta)), \quad \eta \in \mathbb{M}(X), \tag{3}$$

provided the derivative on the right hand side of formula (3) exists. As an intrinsic tangent space to $\mathbb{M}(X)$ at point $\eta \in \mathbb{M}(X)$ we choose the space

$$T_\eta^{\mathrm{int}}(\mathbb{M}) := L^2(X \to X\,\eta),$$

i.e., the space of X-valued functions on X which are square integrable with respect to the measure η. The intrinsic gradient of F at point η is, by definition, the element $(\nabla^{\mathrm{int}} F)(\eta)$ of $T_\eta^{\mathrm{int}}(\mathbb{M})$ satisfying

$$\begin{aligned}(\nabla_v^{\mathrm{int}} F)(\eta) &= ((\nabla^{\mathrm{int}} F)(\eta), v)_{T_\eta^{\mathrm{int}}(\mathbb{M})} \\ &= \int_X \langle (\nabla^{\mathrm{int}} F)(\eta, x), v(x)\rangle_X \, d\eta(x), \quad v \in C_0^\infty(X \to X).\end{aligned} \tag{4}$$

(In the above formula, $\langle\cdot,\cdot\rangle_X$ denotes the usual scalar product in X.) Note that, since η is a Radon measure on X, we indeed have $v \in T_\eta^{\mathrm{int}}(\mathbb{M})$.

We will now introduce an extrinsic gradient. We fix any $h \in C_0(X)$. We consider the one-parameter group of transformations of $\mathbb{M}(X)$ given through multiplication of each measure $\eta \in \mathbb{M}(X)$ by the function $e^{th(x)}$, $t \in \mathbb{R}$. Thus, for each $\eta \in \mathbb{M}(X)$, we define $M_{th}(\eta) \in \mathbb{M}(X)$ by

$$dM_{th}(\eta)(x) := e^{th(x)}\, d\eta(x). \tag{5}$$

The extrinsic derivative of a function $F : \mathbb{M}(X) \to \mathbb{R}$ in direction h is defined by

$$(\nabla_h^{\mathrm{ext}} F)(\eta) := \frac{d}{dt}\Big|_{t=0} F(M_{th}(\eta)), \quad \eta \in \mathbb{M}(X), \tag{6}$$

provided the derivative on the right hand side of (6) exists. As an extrinsic tangent space to $\mathbb{M}(X)$ at point $\eta \in \mathbb{M}(X)$ we choose

$$T_\eta^{\mathrm{ext}}(\mathbb{M}) := L^2(X, \eta).$$

The extrinsic gradient of F at point η is defined to be the element $(\nabla^{\mathrm{ext}}F)(\eta)$ of $T_\eta^{\mathrm{ext}}(\mathbb{M})$ satisfying

$$(\nabla_h^{\mathrm{ext}}F)(\eta) = ((\nabla^{\mathrm{ext}}F)(\eta), h)_{T_\eta^{\mathrm{ext}}(\mathbb{M})} = \int_X (\nabla^{\mathrm{ext}}F)(\eta, x)h(x)\, d\eta(x), \quad h \in C_0(X). \tag{7}$$

We finally combine the intrinsic and extrinsic differentiation. For any $\eta \in \mathbb{M}(X)$, the full tangent space to $\mathbb{M}(X)$ at point η is defined by

$$T_\eta(\mathbb{M}) := T_\eta^{\mathrm{int}}(\mathbb{M}) \oplus T_\eta^{\mathrm{ext}}(\mathbb{M}).$$

We define the full gradient $\nabla^{\mathbb{M}} := (\nabla^{\mathrm{int}}, \nabla^{\mathrm{ext}})$.

Let us now explicitly calculate the gradient $\nabla^{\mathbb{M}}F$ for F from a certain class of 'smooth' cylinder functions on $\mathbb{M}(X)$. We denote by $\mathcal{FC}_b^\infty(\mathcal{D}(X), \mathbb{M}(X))$ the set of all functions $F : \mathbb{M}(X) \to \mathbb{R}$ of the form

$$F(\eta) = g(\langle f_1, \eta\rangle, \dots, \langle f_N, \eta\rangle), \tag{8}$$

where $g \in C_b^\infty(\mathbb{R}^N)$ (an infinitely differentiable function on $\mathbb{R}^N$ which, together with all its derivatives, is bounded), $f_1 \dots, f_N \in \mathcal{D}(X)$, and $N \in \mathbb{N}$. Here $\mathcal{D}(X) := C_0^\infty(X)$ is the space of all smooth, compactly supported functions on X.

For F of the form (8) and any $v \in C_0^\infty(X \to X)$, we have, by virtue of (1)–(3),

$$\begin{aligned}
(\nabla_v^{\mathrm{int}}F)(\eta) &= \frac{d}{dt}\Big|_{t=0} F(\phi_t^v(\eta)) = \frac{d}{dt}\Big|_{t=0} g(\langle f_1, \phi_t^v\eta\rangle, \dots, \langle f_N, \phi_t^v\eta\rangle) \\
&= \frac{d}{dt}\Big|_{t=0} g(\langle f_1 \circ \phi_t^v, \eta\rangle, \dots, \langle f_N \circ \phi_t^v, \eta\rangle) \\
&= \sum_{i=1}^N (\partial_i g)(\langle f_1, \eta\rangle, \dots, \langle f_N, \eta\rangle) \int_X \langle \nabla f_i(x), v(x)\rangle_X \, d\eta(x).
\end{aligned} \tag{9}$$

Here $\partial_i g$ denotes the partial derivative of g in the i-th variable. By (4) and (9), we get

$$(\nabla^{\mathrm{int}}F)(\eta, x) = \sum_{i=1}^N (\partial_i g)(\langle f_1, \eta\rangle, \dots, \langle f_N, \eta\rangle)\nabla f_i(x). \tag{10}$$

For each $h \in C_0(X)$, using (5) and (6), we have

$$\begin{aligned}
(\nabla_h^{\mathrm{ext}}F)(\eta) &= \frac{d}{dt}\Big|_{t=0} F(M_{th}(\eta)) = \frac{d}{dt}\Big|_{t=0} g(\langle f_1 e^{th}, \eta\rangle, \dots, \langle f_N e^{th}, \eta\rangle) \\
&= \sum_{i=1}^N (\partial_i g)(\langle f_1, \eta\rangle, \dots, \langle f_N, \eta\rangle) \int_X f_i(x)h(x)\, d\eta(x).
\end{aligned} \tag{11}$$

Hence, by (7),

$$(\nabla^{\mathrm{ext}} F)(\eta, x) = \sum_{i=1}^{N} (\partial_i g)(\langle f_1, \eta\rangle, \dots, \langle f_N, \eta\rangle) f_i(x). \tag{12}$$

By (10) and (12), we get

$$(\nabla^{\mathbb{M}} F)(\eta, x) = \sum_{i=1}^{N} (\partial_i g)(\langle f_1, \eta\rangle, \dots, \langle f_N, \eta\rangle)(\nabla f_i, f_i).$$

3 Gamma measure

Assume that μ is a probability measure on $(\mathbb{M}(X), \mathcal{B}(\mathbb{M}(X)))$. Is it possible to have an integration by parts formula with respect to μ? In Section 4 below, we will show that the answer to this question is positive if μ is the gamma measure and we deal with the extrinsic derivative only. By similar methods, an integration by parts formula for the extrinsic derivative can be derived for a wide class of Lévy measures, as well as Gibbs perturbations of the gamma measure [9].

So, in this section, we will discuss a construction of the gamma measure. We equip $\mathcal{D}(X)$ with the standard nuclear space topology, see e.g. [4, Chap. 1, subsec. 1.2] for details. Consider the standard triple $\mathcal{D}(X) \subset L^2(X, dx) \subset \mathcal{D}'(X)$, where $\mathcal{D}'(X)$ is the dual space of $\mathcal{D}(X)$ with respect to zero space $L^2(X, dx)$. Let $\mathscr{C}(\mathcal{D}'(X))$ denote the cylinder σ-algebra on $\mathcal{D}'(X)$, i.e., the minimal σ-algebra on $\mathcal{D}'(X)$ with respect to which each mapping

$$\mathcal{D}'(X) \ni \omega \mapsto \langle \varphi, \omega\rangle, \quad \varphi \in \mathcal{D}(X),$$

is measurable. Here $\langle \varphi, \omega\rangle$ denotes the dual pairing between ω and φ. The gamma measure is defined as the unique probability measure $\mathcal{G}$ on $(\mathcal{D}'(X), \mathscr{C}(\mathcal{D}'(X)))$ which has Laplace transform

$$\int_{\mathcal{D}'(X)} e^{\langle \varphi, \omega\rangle}\, d\mathcal{G}(\omega) = \exp\left[-\int_X \log(1 - \varphi(x))\, dx\right], \quad \varphi \in \mathcal{D}(X),\ \varphi < 1.$$

To prove that the gamma measure is concentrated on $\mathbb{M}(X)$, we will now present an explicit construction of the gamma measure.

Recall that we denote by $\mathbb{K}(X)$ the cone of discrete Radon measures on X:

$$\mathbb{K}(X) := \left\{ \eta = \sum_i s_i \delta_{x_i} \in \mathbb{M}(X) \mid s_i > 0,\ x_i \in X \right\}.$$

Here, δ_{x_i} is the Dirac measure with mass at x_i, the atoms x_i are assumed to be distinct and their total number is at most countable. By convention, the cone $\mathbb{K}(X)$ contains the null mass $\eta = 0$, which is represented by the sum over the empty set of indices i. We denote $\tau(\eta) := \{x_i\}$, i.e., the set on which the measure η is concentrated. For $\eta \in \mathbb{K}(X)$ and $x \in \tau(\eta)$, we denote by $s(\eta, x)$ the mass of η at point x, i.e., $s(\eta, x) := \eta(\{x\})$. Thus, each $\eta \in \mathbb{K}(X)$ can be written in the form $\eta = \sum_{x\in\tau(\eta)} s(\eta, x)\delta_x$.

Note that the closure of $\mathbb{K}(X)$ in the vague topology coincides with $\mathbb{M}(X)$. As shown in [9], $\mathbb{K}(X) \in \mathcal{B}(\mathbb{M}(X))$. We denote by $\mathcal{B}(\mathbb{K}(X))$ the trace σ-algebra of $\mathcal{B}(\mathbb{M}(X))$ on $\mathbb{K}(X)$. Since $\mathbb{M}(X)$ is a Polish space, $(\mathbb{K}(X), \mathcal{B}(\mathbb{K}(X)))$ is a standard Borel space, see e.g. [24, Theorem 5.2.2].

Remark 1. It is an open problem whether one can introduce a metric on $\mathbb{K}(X)$ making it a Polish space and being compatible with the vague topology inherited from $\mathbb{M}(X)$. We refer to [9, Remark 2.1] for a discussion of this problem.

Proposition 2. *There exists a unique probability measure $\mathcal{G}$ on $(\mathbb{K}(X), \mathcal{B}(\mathbb{K}(X)))$ which has Laplace transform*

$$\int_{\mathbb{K}(X)} e^{\langle\varphi,\eta\rangle}\, d\mathcal{G}(\eta) = \exp\left[-\int_X \log(1-\varphi(x))\, dx\right], \quad \varphi \in C_0(X),\ \varphi < 1. \tag{13}$$

Proof. Denote $\mathbb{R}_+^* := (0, \infty)$ and define a metric on $\mathbb{R}_+^*$ by

$$d_{\mathbb{R}_+^*}(s_1, s_2) := |\log(s_1) - \log(s_2)|, \quad s_1, s_2 \in \mathbb{R}_+^*.$$

Then $\mathbb{R}_+^*$ becomes a locally compact Polish space, and any set of the form $[a, b]$, with $0 < a < b < \infty$, is compact. We denote $\hat{X} := X \times \mathbb{R}_+^*$ and define the configuration space over $\hat{X}$ by

$$\Gamma(\hat{X}) := \left\{\gamma \subset \hat{X} \mid |\gamma \cap \Lambda| < \infty \text{ for each compact } \Lambda \subset \hat{X}\right\}.$$

Here $|\gamma \cap \Lambda|$ denotes the number of points in the set $\gamma \cap \Lambda$. One can identify a configuration $\gamma \in \Gamma(\hat{X})$ with Radon measure $\sum_{(x,s)\in\gamma} \delta_{(x,s)}$ from $\mathbb{M}(\hat{X})$. The space $\Gamma(\hat{X})$ is endowed with the vague topology, i.e., the weakest topology on $\Gamma(\hat{X})$ with respect to which all maps

$$\Gamma(\hat{X}) \mapsto \langle f, \gamma\rangle := \int_{\hat{X}} f(x, s)\, d\gamma(x, s) = \sum_{(x,s)\in\gamma} f(x, s), \quad f \in C_0(\hat{X}),$$

are continuous. Let $\mathcal{B}(\Gamma(\hat{X}))$ denote the Borel σ-algebra on $\Gamma(\hat{X})$. We denote by π the Poisson measure on $(\Gamma(\hat{X}), \mathcal{B}(\Gamma(\hat{X})))$ with intensity measure

$$d\varkappa(x, s) := dx\, d\lambda(s), \tag{14}$$

where

$$d\lambda(s) := \frac{1}{s}\, e^{-s}\, ds. \tag{15}$$

The measure π can be characterized as the unique probability measure on $\Gamma(\hat{X})$ which satisfies the Mecke identity: for each measurable function $F : \Gamma(\hat{X}) \times \hat{X} \to [0, \infty]$, we have

$$\int_{\Gamma(\hat{X})} d\pi(\gamma) \int_{\hat{X}} d\gamma(x,s)\, F(\gamma, x, s) = \int_{\Gamma(\hat{X})} d\pi(\gamma) \int_{\hat{X}} d\varkappa(x,s)\, F(\gamma \cup \{(x,s)\}, x, s). \quad (16)$$

Denote by $\Gamma_p(\hat{X})$ the set of so-called pinpointing configurations in $\hat{X}$. By definition, $\Gamma_p(\hat{X})$ consists of all configurations $\gamma \in \Gamma(\hat{X})$ such that if $(x_1, s_1), (x_2, s_2) \in \gamma$ and $(x_1, s_1) \neq (x_2, s_2)$, then $x_1 \neq x_2$. Thus, a configuration $\gamma \in \Gamma_p(\hat{X})$ can not contain two points (x, s_1) and (x, s_2) with $s_1 \neq s_2$. As easily seen, $\Gamma_p(\hat{X}) \in \mathcal{B}(\Gamma(\hat{X}))$. Since the Lebesgue measure dx is non-atomic, the set

$$\big\{(x_1, s_1, x_2, s_2) \in \hat{X}^2 \mid x_1 = x_2\big\}$$

is of zero $\varkappa^{\otimes 2}$-measure. Denote by $\mathcal{B}_c(\hat{X})$ the set of all Borel measurable sets in $\hat{X}$ which have compact closure. Fix any $\Lambda \in \mathcal{B}_0(\hat{X})$. Using the distribution of the configuration $\gamma \cap \Lambda$ under π (see e.g. [12]), we conclude that

$$\pi\big(\gamma \in \Gamma(\hat{X}) \mid \exists (x_1, s_1), (x_2, s_2) \in \gamma \cap \Lambda :\ x_1 = x_2,\ s_1 \neq s_2\big) = 0.$$

Hence, $\pi(\Gamma_p(\hat{X})) = 1$.

For each $\gamma \in \Gamma_p(\hat{X})$ and $A \in \mathcal{B}_c(X)$, we define a local mass by

$$\mathfrak{M}_A(\gamma) := \int_{\hat{X}} \chi_A(x) s\, d\gamma(x,s) = \sum_{(x,s) \in \gamma} \chi_A(x) s \in [0, \infty]. \quad (17)$$

Here χ_A denotes the indicator function of the set A. The set of pinpointing configurations with finite local mass is defined by

$$\Gamma_{pf}(\hat{X}) := \big\{\gamma \in \Gamma_p(\hat{X}) \mid \mathfrak{M}_A(\gamma) < \infty \text{ for each } A \in \mathcal{B}_c(X)\big\}.$$

As easily seen, $\Gamma_{pf}(\hat{X}) \in \mathcal{B}(\Gamma(\hat{X}))$ and we denote by $\mathcal{B}(\Gamma_{pf}(\hat{X}))$ the trace σ-algebra of $\mathcal{B}(\Gamma(\hat{X}))$ on $\Gamma_{pf}(\hat{X})$. For each $A \in \mathcal{B}_c(X)$, using the Mecke identity (16), we get

$$\int_{\Gamma_p(\hat{X})} \mathfrak{M}_A(\gamma)\, d\pi(\gamma) = \int_{\Gamma_p(\hat{X})} d\pi(\gamma) \int_A d\varkappa(x,s)\, s = \int_A dx < \infty.$$

Therefore, $\pi(\Gamma_{pf}(\hat{X})) = 1$ and we can consider π as a probability measure on $(\Gamma_{pf}(\hat{X}), \mathcal{B}(\Gamma_{pf}(\hat{X})))$.

We construct a bijective mapping $\mathscr{R} : \Gamma_{pf}(\hat{X}) \to \mathbb{K}(X)$ by setting, for each $\gamma = \{(x_i, s_i)\} \in \Gamma_{pf}(\hat{X})$, $\mathscr{R}\gamma := \sum_i s_i \delta_{x_i} \in \mathbb{K}(X)$. By [9, Theorem 6.2], we have

$$\mathcal{B}(\mathbb{K}(X)) = \big\{\mathscr{R}A \mid A \in \mathcal{B}(\Gamma_{pf}(\hat{X}))\big\}.$$

Hence, both $\mathscr{R}$ and its inverse $\mathscr{R}^{-1}$ are measurable mappings. We define $\mathcal{G}$ to be the pushforward of the measure π under $\mathscr{R}$. One can easily check that $\mathcal{G}$ has Laplace transform (13) and this Laplace transform indeed uniquely characterizes this measure. □

Corollary 3. *For each measurable function $F : \mathbb{K}(X) \times X \to [0, \infty]$, we have*

$$\int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_X d\eta(x)\, F(\eta, x) = \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_{\hat{X}} dx\ ds\, e^{-s} F(\eta + s\delta_x, x). \tag{18}$$

Proof. By the proof of Proposition 2 (in particular, using the Mecke identity), we see that the left hand side of (18) is equal to

$$\begin{aligned}&\int_{\Gamma_{pf}(\hat{X})} d\pi(\gamma) \int_{\hat{X}} d\gamma(x,s)\, sF(\mathscr{R}\gamma, x)\\ &\quad = \int_{\Gamma_{pf}(\hat{X})} d\pi(\gamma) \int_{\hat{X}} d\varkappa(x,s)\, sF(\mathscr{R}(\gamma \cup \{(x,s)\}), x),\end{aligned}$$

which is equal to the right hand side of (18). □

Remark 4. In fact, identity (18) uniquely characterizes the gamma measure $\mathcal{G}$, i.e., if a probability measure μ on $\mathbb{K}(X)$ satisfies identity (18) with $\mathcal{G}$ being replaced by μ, then $\mu = \mathcal{G}_\theta$. See [9, Theorem 6.3] for a proof of this statement.

Remark 5. By using either the Laplace transform of the gamma measure (formula (13)) or formula (18), one can easily show that the gamma measure has all moments finite, that is, for each $A \in \mathcal{B}_c(X)$ and $n \in \mathbb{N}$, we have

$$\int_{\mathbb{K}(X)} \langle \chi_A, \eta \rangle^n \, d\mathcal{G}(\eta) = \int_{\mathbb{K}(X)} \eta(A)^n \, d\mathcal{G}(\eta) < \infty. \tag{19}$$

4 Extrinsic integration by parts formula for the gamma measure

In this section, we will derive an integration by parts formula for the extrinsic directional derivative with respect to the gamma measure $\mathcal{G}$. We will present two proofs of this formula. The first proof will use the quasi-invariance of $\mathcal{G}$ under the transformations $M_h : \mathbb{M}(X) \to \mathbb{M}(X)$ for $h \in C_0(X)$. The second proof will use the Mecke-type formula (18). It is the second proof that allows for a generalization of the extrinsic integration by parts formula to a wide class of Lévy measures on $\mathbb{K}(X)$ which are not necessarily quasi-invariant under the M_h transformations.

It is important to note that a similar integration by parts formula for the intrinsic differentiation fails, see Remark 12 below. Note that, by Skorohod's theorem on absolute continuity of Poisson measures (see [28]), the pushforward of $\mathcal{G}$ under ϕ_t^v is not absolutely continuous with respect to $\mathcal{G}$ (unless, of course, ϕ_t^v is the identity mapping). Despite this, analogously to the second proof of the extrinsic integration by parts formula, in Section 5 below, we will be able to carry out an integration by parts in the Dirichlet form generated by the intrinsic gradient.

So we start with citing Theorem 3.1 in [30].

Theorem 6 ([30])**.** *Let $h \in C_0(X)$ and let $M_h\mathcal{G}$ denote the pushforward of the gamma measure $\mathcal{G}$ under M_h. Then $M_h\mathcal{G}$ is absolutely continuous with respect to $\mathcal{G}$ and the corresponding Radon–Nikodym derivative is given by*

$$\frac{dM_h\mathcal{G}}{d\mathcal{G}}(\eta) = \exp\left(-\int_X h(x)\,dx + \int_X (1-e^{-h(x)})\,d\eta(x)\right). \tag{20}$$

Definition 7. Let $h \in C_0(X)$. We define the *extrinsic logarithmic derivative of the gamma measure $\mathcal{G}$ along h* as the following function on $\mathbb{K}(X)$:

$$B_h^{\mathrm{ext}}(\eta) := \langle h,\eta\rangle - \int_X h(x)\,dx, \quad \eta \in \mathbb{K}(X).$$

Below we denote by $\mathcal{FC}_b^\infty(\mathcal{D}(X),\mathbb{K}(X))$ the set of the functions on $\mathbb{K}(X)$ which are restrictions of functions from $\mathcal{FC}_b^\infty(\mathcal{D}(X),\mathbb{M}(X))$ to $\mathbb{K}(X)$, i.e., they have the form (8) with $\eta \in \mathbb{K}(X)$. We now formulate the main result of this section.

Theorem 8. *Let $F, G \in \mathcal{FC}_b^\infty(\mathcal{D}(X),\mathbb{K}(X))$ and $h \in C_0(X)$. Then*

$$\begin{aligned}\int_{\mathbb{K}(X)} &(\nabla_h^{\mathrm{ext}}F)(\eta)G(\eta)\,d\mathcal{G}(\eta)\\ &= -\int_{\mathbb{K}(X)} F(\eta)(\nabla_h^{\mathrm{ext}}G)(\eta)\,d\mathcal{G}(\eta) + \int_{\mathbb{K}(X)} F(\eta)G(\eta)B_h^{\mathrm{ext}}(\eta)\,d\mathcal{G}(\eta).\end{aligned} \tag{21}$$

The first proof. By (6), we have

$$\int_{\mathbb{K}(X)} (\nabla_h^{\mathrm{ext}}F)(\eta)G(\eta)\,d\mathcal{G}(\eta) = \int_{\mathbb{K}(X)} \frac{d}{dt}\Big|_{t=0} F(M_{th}(\eta))\,G(\eta)\,d\mathcal{G}(\eta). \tag{22}$$

Assume that F is of the form (8). Analogously to (11), we have for each $t \in [-1,1]$,

$$\begin{aligned}&\left|\frac{d}{dt}F(M_{th}(\eta))\right|\\ &\quad \le \sum_{i=1}^N \big|(\partial_i g)(\langle f_1,\eta\rangle,\dots,\langle f_N,\eta\rangle)\big| \int_X |f_i(x)h(x)|\,e^{|h(x)|}\,d\eta(x) \in L^1(\mathbb{K}(X),\mathcal{G}).\end{aligned} \tag{23}$$

By (22), (23), the dominated convergence theorem, and Theorem 6, we have

$$
\begin{aligned}
&\int_{\mathbb{K}(X)} (\nabla_h^{\mathrm{ext}} F)(\eta) G(\eta)\, d\mathcal{G}(\eta) = \frac{d}{dt}\Big|_{t=0} \int_{\mathbb{K}(X)} F(M_{th}(\eta))\, G(\eta)\, d\mathcal{G}(\eta)\\
&\quad = \frac{d}{dt}\Big|_{t=0} \int_{\mathbb{K}(X)} F(\eta)\, G(M_{-th}(\eta))\, d(M_{th}\mathcal{G})(\eta)\\
&\quad = \frac{d}{dt}\Big|_{t=0} \int_{\mathbb{K}(X)} F(\eta)\, G(M_{-th}(\eta)) \exp\left(-\int_X th(x)\, dx + \int_X (1-e^{-th(x)})\, d\eta(x)\right)\, d\mathcal{G}(\eta).
\end{aligned} \tag{24}
$$

Note that

$$
\sup_{t\in[-1,1]} (1-e^{-th(x)}) \le 1 - e^{-|h(x)|}. \tag{25}
$$

Furthermore, we state that, for each $\varphi \in C_0(X)$, $\varphi \ge 0$, we have

$$
\int_{\mathbb{K}(X)} \langle \varphi, \eta\rangle \exp\left(\int_X (1-e^{-|h(x)|})\, d\eta(x)\right)\, d\mathcal{G}(\eta) < \infty. \tag{26}
$$

Indeed, by Theorem 6,

$$
\begin{aligned}
&\int_{\mathbb{K}(X)} \langle \varphi, \eta\rangle \exp\left(-\int_X |h(x)|\, dx + \int_X (1-e^{-|h(x)|})\, d\eta(x)\right)\, d\mathcal{G}(\eta)\\
&\quad = \int_{\mathbb{K}(X)} \langle \varphi, \eta\rangle\, d(M_{|h|}\mathcal{G})(\eta) = \int_{\mathbb{K}(X)} \langle \varphi e^{|h|}, \eta\rangle\, d\mathcal{G}(\eta) < \infty.
\end{aligned}
$$

Hence, analogously to (23) and (24), we conclude from (24)–(26) and the dominated convergence theorem that

$$
\begin{aligned}
&\int_{\mathbb{K}(X)} (\nabla_h^{\mathrm{ext}} F)(\eta) G(\eta)\, d\mathcal{G}(\eta)\\
&= \int_{\mathbb{K}(X)} F(\eta)\, \frac{d}{dt}\Big|_{t=0} \left[G(M_{-th}(\eta)) \exp\left(-\int_X th(x)\, dx + \int_X (1-e^{-th(x)})\, d\eta(x)\right)\right]\, d\mathcal{G}(\eta).
\end{aligned}
$$

From here, formula (21) easily follows. □

To give the second proof of formula (21), we first need some preparation. For our future use, we will now treat both the intrinsic and extrinsic gradients.

Let $F : \mathbb{K}(X) \to \mathbb{R}$, $\eta \in \mathbb{K}(X)$, and $x \in \tau(\eta)$. We define

$$
(\nabla_x^X F)(\eta) := \nabla_y\big|_{y=x} F(\eta - s(\eta,x)\delta_x + s(\eta,x)\delta_y), \tag{27}
$$

$$
(\nabla_x^{\mathbb{R}_+^*})(\eta) := \frac{d}{du}\Big|_{u=s(\eta,x)} F(\eta - s(\eta,x)\delta_x + u\delta_x), \tag{28}
$$

provided the derivatives on the right hand side of (27) and (28) exist. Here the variable y is from X, ∇_y is the usual gradient on X in the y variable, and the variable u is from $\mathbb{R}_+^*$.

Proposition 9. *For each $F \in \mathcal{FC}_b^\infty(\mathcal{D}(X), \mathbb{K}(X))$, $\eta \in \mathbb{K}(X)$, and $x \in \tau(\eta)$, we have*

$$(\nabla^{\text{int}} F)(\eta, x) = \frac{1}{s(\eta, x)} (\nabla_x^X F)(\eta), \tag{29}$$

$$(\nabla^{\text{ext}} F)(\eta, x) = (\nabla_x^{\mathbb{R}_+^*} F)(\eta). \tag{30}$$

Proof. For simplicity of notation, we consider the case of $F(\eta) = g(\langle f, \eta \rangle)$, where $g \in C_b^\infty(\mathbb{R})$ and $f \in \mathcal{D}(X)$. We have

$$F(\eta - s(\eta, x)\delta_x + s(\eta, x)\delta_y) = g\big(\langle f, \eta \rangle - s(\eta, x)f(x) + s(\eta, x)f(y)\big).$$

Hence

$$\nabla_y F(\eta - s(\eta, x)\delta_x + s(\eta, x)\delta_y) = g'\big(\langle f, \eta \rangle - s(\eta, x)f(x) + s(\eta, x)f(y)\big)s(\eta, x)\nabla f(y).$$

Setting $y = x$, we get

$$\nabla_y\big|_{y=x} F(\eta - s(\eta, x)\delta_x + s(\eta, x)\delta_y) = g'(\langle f, \eta \rangle)s(\eta, x)\nabla f(x). \tag{31}$$

By virtue of (10) and (27), formula (31) yields (29). Analogously

$$F(\eta - s(\eta, x)\delta_x + u\delta_x) = g\big(\langle f, \eta \rangle - s(\eta, x)f(x) + uf(x)\big).$$

Hence

$$\frac{d}{du} F(\eta - s(\eta, x)\delta_x + u\delta_x) = g'\big(\langle f, \eta \rangle - s(\eta, x)f(x) + uf(x)\big)f(x). \tag{32}$$

Formulas (12), (28), and (32) imply (30). □

Remark 10. It is possible to give a clear informal interpretation of formulas (29) and (30). Indeed, taking $\eta = \sum_i s_i \delta_{x_i}$, we have

$$(\nabla^{\text{int}} F)\left(\sum_i s_i \delta_{x_i}, x_j\right) = \frac{1}{s_j} \nabla_{x_j} F\left(\sum_i s_i \delta_{x_i}\right),$$

$$(\nabla^{\text{ext}} F)\left(\sum_i s_i \delta_{x_i}, x_j\right) = \frac{\partial}{\partial s_j} F\left(\sum_i s_i \delta_{x_i}\right).$$

Here ∇_{x_j} is the gradient in the x_j variable, and $\frac{\partial}{\partial s_j}$ is the partial derivative in the s_j variable.

The second proof of Theorem 8. Using Corollary 3 and formulas (7), (28), and (30), we get

$$\begin{aligned}
&\int_{\mathbb{K}(X)} (\nabla_h^{\mathrm{ext}} F)(\eta) G(\eta)\, d\mathcal{G}(\eta) = \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_X d\eta(x)\, (\nabla^{\mathrm{ext}} F)(\eta, x) h(x) G(\eta) \\
&\quad = \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_X d\eta(x)\, (\nabla_x^{\mathbb{R}_+^*} F)(\eta) h(x) G(\eta) \\
&\quad = \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_X dx \int_{\mathbb{R}_+^*} ds\, e^{-s} \left(\frac{d}{ds} F(\eta + s\delta_x) \right) h(x) G(\eta + s\delta_x) \\
&\quad = - \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_X dx \int_{\mathbb{R}_+^*} ds\, e^{-s} F(\eta + s\delta_x) h(x) \left(\frac{d}{ds} G(\eta + s\delta_x) \right) \\
&\qquad + \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_X dx \int_{\mathbb{R}_+^*} ds\, e^{-s} F(\eta + s\delta_x) h(x) G(\eta + s\delta_x) \\
&\qquad - \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_X dx \left(e^{-s} F(\eta + s\delta_x) G(\eta + s\delta_x) \right)\Big|_{s=0} h(x) \\
&\quad = - \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_X d\eta(x) F(\eta) h(x) (\nabla_x^{\mathbb{R}_+^*} G)(\eta) \\
&\qquad + \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_X d\eta(x) F(\eta) h(x) G(\eta) - \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) F(\eta) G(\eta) \int_X dx\, h(x).
\end{aligned}$$

From here, formula (21) follows. □

Remark 11. It is evident that the second proof of Theorem 8 admits a straightforward generalization to a wide class of Lévy measures on $\mathbb{K}(X)$. For example, let $\rho : [0, \infty) \to [0, \infty)$ be a smooth bounded function such that $\rho(0) > 0$ and $\rho(s) = 0$ for all $s \geq R$ for some $R > 0$. Consider the Poisson measure on $\hat{X}$ with intensity $dx\, ds\, \frac{1}{s}\, \rho(s)$. This Poisson measure is concentrated on $\Gamma_{pf}(\hat{X})$. Let μ be the pushforward of this Poisson measure under the mapping $\mathscr{R} : \Gamma_{pf}(\hat{X}) \to \mathbb{K}(X)$. Thus, μ is a Lévy measure on $\mathbb{K}(X)$. According to [16], this measure is not quasi-invariant under the action of the M_h transformations. However, we can easily derive an extrinsic integration by parts formula with respect to the measure μ.

Remark 12. Let us explain why a counterpart of Theorem 8 does not hold for the intrinsic differentiation. Analogously to the second proof of Theorem 8 and using Corollary 3 and formulas (4), (27), (29), we get, for $v \in C_0^\infty(X \to X)$ and $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}(X), \mathbb{K}(X))$,

$$\begin{aligned}
&\int_{\mathbb{K}(X)} (\nabla_v^{\mathrm{int}} F)(\eta) G(\eta)\, d\mathcal{G}(\eta) = \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_X d\eta(x)\, \langle (\nabla_x^X F)(\eta), v(x) \rangle_X\, G(\eta) \\
&\quad = \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_X dx \int_{\mathbb{R}_+^*} ds\, \frac{e^{-s}}{s}\, \langle \nabla_x^X F(\eta + s\delta_x), v(x) \rangle_X\, G(\eta + s\delta_x)
\end{aligned}$$

$$= \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_{\mathbb{R}_+^*} ds\, \frac{e^{-s}}{s} \int_X dx\, \langle \nabla_x^X F(\eta + s\delta_x), v(x)\rangle_X\, G(\eta + s\delta_x).$$

For a fixed $\eta \in \mathbb{K}(X)$ and $s \in \mathbb{R}_+^*$, we get

$$\int_X dx\, \langle \nabla_x^X F(\eta + s\delta_x), v(x)\rangle_X\, G(\eta + s\delta_x)$$
$$= -\int_X dx\, F(\eta + s\delta_x)\langle \nabla_x^X G(\eta + s\delta_x), v(x)\rangle_X + \int_X dx\, F(\eta + s\delta_x) G(\eta + s\delta_x) \operatorname{div} v(x),$$

where $\operatorname{div} v(x)$ denotes the divergence of $v(x)$. Clearly

$$-\int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_{\mathbb{R}_+^*} ds\, \frac{e^{-s}}{s} \int_X dx\, F(\eta + s\delta_x)\langle \nabla_x^X G(\eta + s\delta_x), v(x)\rangle_X$$
$$= -\int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_X d\eta(x)\, F(\eta)\langle (\nabla_x^X G)(\eta), v(x)\rangle_X.$$

However, by Corollary 3,

$$\int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_{\mathbb{R}_+^*} ds\, \frac{e^{-s}}{s} \int_X dx\, |F(\eta + s\delta_x) G(\eta + s\delta_x) \operatorname{div} v(x)|$$
$$= \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) |F(\eta) G(\eta)| \int_X d\eta(x)\, \frac{|\operatorname{div} v(x)|}{s(\eta, x)}$$
$$= \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) |F(\eta) G(\eta)| \sum_{x \in \tau(\eta)} |\operatorname{div} v(x)|. \tag{33}$$

Recall that, for any open ball B in X, with $\mathcal{G}$ probability 1, $\tau(\eta) \cap B$ is an infinite (and even dense) subset of B. Therefore, if $\operatorname{div} v \neq 0$, $\sum_{x\in\tau(\eta)} |\operatorname{div} v(x)| = \infty$ $\mathcal{G}$-a.s. Hence, in this case, the integrals in (33) are infinite unless $|FG| = 0$ $\mathcal{G}$-a.s.

5 Dirichlet forms

Having arrived at notions of both a gradient and a tangent space to $\mathbb{M}(X)$, we would like to construct a corresponding Dirichlet form on the space $L^2(\mathbb{K}(X), \mathcal{G})$. This, in turn, should lead us, in future, to a diffusion process on $\mathbb{K}(X)$. In fact, we will consider different types of Dirichlet forms, corresponding to the intrinsic gradient ∇^{int}, extrinsic gradient ∇^{ext}, and the full gradient $\nabla^{\mathbb{M}}$. Furthermore, in the case of the intrinsic gradient (full gradient, respectively), we will use a coefficient in the Dirichlet form which depends on masses only. The sense of this coefficient will become clear below.

We first note that $\mathcal{FC}_b^\infty(\mathcal{D}(X), \mathbb{K}(X))$ is a dense subset of $L^2(\mathbb{K}(X), \mu)$ for any probability measure μ on $\mathbb{K}(X)$ (see [7, Corollary 6.2.8] for a proof of this rather obvious statement). In particular, $\mathcal{FC}_b^\infty(\mathcal{D}(X), \mathbb{K}(X))$ is dense in $L^2(\mathbb{K}(X), \mathcal{G})$.

We fix a measurable function $c : \mathbb{R}_+^* \to [0, \infty)$ which satisfies

$$\int_{\mathbb{R}_+^*} c(s) e^{-s}\, ds < \infty. \tag{34}$$

We define symmetric bilinear forms on $L^2(\mathbb{K}(X), \mathcal{G})$ by

$$\begin{aligned}
\mathcal{E}^{\mathrm{int}}(F, G) &:= \int_{\mathbb{K}(X)} \langle (\nabla^{\mathrm{int}} F)(\eta), c(s(\eta, \cdot))(\nabla^{\mathrm{int}} G)(\eta)\rangle_{T_\eta^{\mathrm{int}}(\mathbb{M})}\, d\mathcal{G}(\eta), \\
&= \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_X d\eta(x) \left\langle (\nabla^{\mathrm{int}} F)(\eta, x),\, c(s(\eta, x))(\nabla^{\mathrm{int}} G)(\eta, x)\right\rangle_X , \qquad (35)\\
\mathcal{E}^{\mathrm{ext}}(F, G) &:= \int_{\mathbb{K}(X)} \langle (\nabla^{\mathrm{ext}} F)(\eta), (\nabla^{\mathrm{ext}} G)(\eta)\rangle_{T_\eta^{\mathrm{ext}}(\mathbb{M})}\, d\mathcal{G}(\eta), \qquad (36)\\
\mathcal{E}^{\mathbb{M}}(F, G) &:= \mathcal{E}^{\mathrm{int}}(F, G) + \mathcal{E}^{\mathrm{ext}}(F, G), \quad F, G \in \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(X), \mathbb{K}(X)). \qquad (37)
\end{aligned}$$

It follows from formulas (10) and (12) that, for each $F \in \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(X), \mathbb{K}(X))$, there exist a constant $C_1 > 0$ and a set $\Lambda \in \mathcal{B}_c(X)$ such that

$$\max\{\|\nabla^{\mathrm{int}} F(\eta, x)\|_X, |\nabla^{\mathrm{ext}} F(\eta, x)|\} \le C_1 \chi_\Lambda(x), \quad \eta \in \mathbb{K}(X),\ x \in \tau(\eta).$$

(Here $\|\cdot\|_X$ is the norm on X determined by the scalar product $\langle \cdot, \cdot \rangle_X$.) Therefore, by Corollary 3 and (34), the integrals in (35) and (36) indeed make sense and are finite.

We may also give an equivalent representation of the bilinear forms $\mathcal{E}^{\mathrm{int}}$, $\mathcal{E}^{\mathrm{ext}}$.

Lemma 13. *For any* $F, G \in \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(X), \mathbb{K}(X))$,

$$\mathcal{E}^{\mathrm{int}}(F, G) = \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_{\hat{X}} dx\, ds\, e^{-s}\, \frac{c(s)}{s^2} \big\langle \nabla_x F(\eta + s\delta_x), \nabla_x G(\eta + s\delta_x)\big\rangle_X , \tag{38}$$

$$\mathcal{E}^{\mathrm{ext}}(F, G) = \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_{\hat{X}} dx\, ds\, e^{-s} \left(\frac{d}{ds}\, F(\eta + s\delta_x)\right) \left(\frac{d}{ds} G(\eta + s\delta_x)\right). \tag{39}$$

Proof. Formulas (38), (39) directly follow from Corollary 3, Proposition 9, and formulas (35), (36). □

The lemma below shows that the introduced symmetric bilinear forms are well defined on $L^2(\mathbb{K}(X), \mathcal{G})$.

Lemma 14. *Let* $F, G \in \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(X), \mathbb{K}(X))$ *and let* $F = 0$ $\mathcal{G}$*-a.e. Then* $\mathcal{E}^\sharp(F, G) = 0$, $\sharp = \mathrm{int}, \mathrm{ext}, \mathbb{M}$.

Proof. For each $A \in \mathcal{B}_c(X)$, making use of Corollary 3, we get

$$\int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_{\hat{X}} dx\, ds\, e^{-s} |F(\eta + s\delta_x)| \chi_A(x) = \int_{\mathbb{K}(X)} d\mathcal{G}(\eta)\, |F(\eta)|\, \eta(A) = 0.$$

Hence $F(\eta + s\delta_x) = 0$ $d\mathcal{G}(\eta)\, dx\, ds$-a.e. on $\mathbb{K}(X) \times \hat{X}$. From here and Lemma 13, the statement easily follows. □

Lemma 15. *For $\sharp = \mathrm{int}, \mathrm{ext}, \mathbb{M}$, the bilinear form $(\mathcal{E}^\sharp, \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(X), \mathbb{K}(X)))$ is a pre-Dirichlet form on $L^2(\mathbb{K}(X), \mathcal{G})$ (i.e., if it is closable, then its closure is a Dirichlet form).*

Proof. The assertion follows, by standard methods, directly from [19, Chap. I, Proposition 4.10] (see also [19, Chap. II, Exercise 2.7]). □

Analogously to (27), (28), we define, for a function $F : \mathbb{K}(X) \to \mathbb{R}$, $\eta \in \mathbb{K}(X)$, and $x \in \tau(\eta)$,

$$(\Delta_x^X F)(\eta) := \Delta_y\big|_{y=x} F(\eta - s(\eta, x)\delta_x + s(\eta, x)\delta_y), \tag{40}$$

$$(\Delta_x^{\mathbb{R}_+^*} F)(\eta) := \left(\frac{d^2}{du^2} - \frac{d}{du}\right)\Big|_{u=s(\eta,x)} F(\eta - s(\eta, x)\delta_x + u\delta_x). \tag{41}$$

Here and below, Δ denotes the usual Laplacian on X (Δ_y denoting the Laplacian in the y variable). Explicitly, for a function $F \in \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(X), \mathbb{K}(X))$ of the form (8), we get

$$\begin{aligned}(\Delta_x^X F)(\eta, x) &= s^2 \sum_{i,j=1}^N (\partial_i\partial_j g)(\langle f_1, \eta\rangle, \dots, \langle f_N, \eta\rangle)\langle \nabla f_i, \nabla f_j\rangle_X \\ &\quad + s \sum_{i=1}^N (\partial_i g)(\langle f_1, \eta\rangle, \dots, \langle f_N, \eta\rangle)\Delta f_i(x), \end{aligned}\tag{42}$$

$$\begin{aligned}(\Delta_x^{\mathbb{R}_+^*} F)(\eta, x) &= \sum_{i,j=1}^N (\partial_i\partial_j g)(\langle f_1, \eta\rangle, \dots, \langle f_N, \eta\rangle) f_i(x) f_j(x) \\ &\quad - \sum_{i=1}^N (\partial_i g)(\langle f_1, \eta\rangle, \dots, \langle f_N, \eta\rangle) f_i(x).\end{aligned}$$

Proposition 16. *Assume that, for some $C_2 > 0$ and $n \in \mathbb{N}$,*

$$c(s) \le C_2\, s, \quad s \in (0, 1] \tag{43}$$

and

$$\int_1^\infty e^{-s} c(s)^2\, ds < \infty. \tag{44}$$

For each $F \in \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(X), \mathbb{K}(X))$, we define

$$(L^{\mathrm{int}} F)(\eta) := \int_X d\eta(x)\, \frac{c(s(\eta, x))}{s(\eta, x)^2}\, (\Delta_x^X F)(\eta), \tag{45}$$

$$(L^{\mathrm{ext}} F)(\eta) := \int_X d\eta(x)\, (\Delta_x^{\mathbb{R}_+^*} F)(\eta) + \int_X \frac{d}{ds}\Big|_{s=0} F(\eta + s\delta_x)\, dx, \quad \eta \in \mathbb{K}(X), \tag{46}$$

$$L^{\mathbb{M}}F := L^{\mathrm{int}}F + L^{\mathrm{ext}}F. \tag{47}$$

Then, for $\sharp = \mathrm{int}, \mathrm{ext}, \mathbb{M}$*,* $(L^\sharp, \mathcal{FC}_b^\infty(\mathcal{D}(X), \mathbb{K}(X))$ *is a symmetric operator in* $L^2(\mathbb{K}(X), \mathcal{G})$ *which satisfies*

$$\mathcal{E}^\sharp(F, G) = (-L^\sharp F, G)_{L^2(\mathbb{K}(X),\mathcal{G})}\,, \quad F, G \in \mathcal{FC}_b^\infty(\mathcal{D}(X), \mathbb{K}(X)). \tag{48}$$

The bilinear form $(\mathcal{E}^\sharp, \mathcal{FC}_b^\infty(\mathcal{D}(X), \mathbb{K}(X)))$ *is closable on* $L^2(\mathbb{K}(X), \mathcal{G})$ *and its closure, denoted by* $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$*, is a Dirichlet form. The operator* $(-L^\sharp, \mathcal{FC}_b^\infty(\mathcal{D}(X), \mathbb{K}(X)))$ *has Friedrichs' extension, which we denote by* $(-L^\sharp, D(L^\sharp))$*.*

Proof. Let $\sharp = \mathrm{int}$. Let $F, G \in \mathcal{FC}_b^\infty(\mathcal{D}(X), \mathbb{K}(X))$. Note that there exists $A \in \mathcal{B}_c(X)$ such that, for any $\eta \in \mathbb{K}(X)$ and $s \in \mathbb{R}_+^*$, the function $X \ni x \mapsto \nabla_x G(\eta + s\delta_x)$ vanishes outside of A. Hence, integration by parts in the x variable in formula (38) yields

$$\mathcal{E}^{\mathrm{int}}(F, G) = \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_{\hat{X}} dx\, ds\, e^{-s}\, \frac{c(s)}{s^2}\big(-\Delta_x F(\eta + s\delta_x)\big) G(\eta + s\delta_x). \tag{49}$$

Note that, for F of the form (8), we have

$$\begin{aligned}(\Delta_x F)(\eta + s\delta_x) &= s^2 \sum_{i,j=1}^N (\partial_i \partial_j g)\big(\langle f_1, \eta\rangle + s f_1(x), \dots, \langle f_N, \eta\rangle + s f_N(x)\big)\langle \nabla f_i, \nabla f_j\rangle_X \\ &\quad + s \sum_{i=1}^N (\partial_i g)\big(\langle f_1, \eta\rangle + s f_1(x), \dots, \langle f_N, \eta\rangle + s f_N(x)\big)\Delta f_i(x). \end{aligned} \tag{50}$$

Hence, by (43), (44), and (50), the function under the sign of integral on the right hand side of (49) is integrable. By Corollary 3, (45), and (49), we get

$$\begin{aligned}\mathcal{E}^{\mathrm{int}}(F, G) &= \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_{\hat{X}} d\eta(x)\, \frac{c(s(\eta, x))}{s(\eta, x)^2}\, (-\Delta_x^X F)(\eta, x) G(\eta) \\ &= \int_{\mathbb{K}(X)} (-L^{\mathrm{int}}F)(\eta) G(\eta)\, d\mathcal{G}(\eta). \end{aligned} \tag{51}$$

We have to prove that $L^{\mathrm{int}}F \in L^2(\mathbb{K}(X), \mathcal{G})$. By (42), there exist $C_3 > 0$ and $\Lambda \in \mathcal{B}_0(X)$ such that

$$|(\Delta_x^X F)(\eta, x)| \le C_3\, \chi_\Lambda(x)(s(\eta, x) + s(\eta, x)^2), \quad \eta \in \mathbb{K}(X),\ x \in \tau(\eta). \tag{52}$$

Hence, by (45),

$$|(L^{\mathrm{int}}F)(\eta)| \le C_3 \int_\Lambda d\eta(x)\, r(s(\eta, x)), \quad \eta \in \mathbb{K}(X), \tag{53}$$

where

$$r(s) := c(s)(1+s^{-1}), \quad s \in [0,\infty). \tag{54}$$

By Corollary 3, (43), (44), (53), and (53),

$$\begin{aligned}
\int_{\mathbb{K}(X)} (L^{\mathrm{int}}F)(\eta)^2\, d\mathcal{G}(\eta) &\le C_3^2 \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \left(\int_\Lambda d\eta(x)\, r(s(\eta,x)) \right)^2 \\
&= C_3^2 \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_\Lambda d\eta(x)\, r(s(\eta,x))^2 \\
&\quad + C_3^2 \int_{\mathbb{K}(X)} \mathcal{G}(\eta) \int_\Lambda d\eta(x) r(s(\eta,x)) \int_\Lambda d(\eta - s(\eta,x)\delta_x)(y) r(s(\eta,y)) \\
&= C_3^2 \int_\Lambda dx \int_{\mathbb{R}_+^*} ds\, e^{-s} r(s)^2 + C_3^2 \left(\int_\Lambda dx \int_{\mathbb{R}_+^*} ds\, e^{-s} r(s) \right)^2 < \infty.
\end{aligned}$$

Thus, the bilinear form $(\mathcal{E}^{\mathrm{int}}, \mathcal{F}C_b^\infty(\mathcal{D}(X), \mathbb{K}(X)))$ has L^2-generator. Hence, it is closable, and by Lemma 15 its closure is a Dirichlet form. The last statement of the proposition about Friedrich's extension is a standard fact of functional analysis.

Next, we consider the case $\sharp = \mathrm{ext}$. In principal, formula (48) with the L^{ext} given by (46) can be easily derived from Theorem 8 and formulas (7), (12). However, we will now give a direct proof of (48), which is similar to the second proof of Theorem 8. By Corollary 3, Lemma 13, and formulas (41), (46),

$$\begin{aligned}
\mathcal{E}^{\mathrm{ext}}(F,G) &= \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_X dx \int_{\mathbb{R}_+^*} ds\, e^{-s} \left(\frac{d}{ds} F(\eta + s\delta_x) \right) \left(\frac{d}{ds} G(\eta + s\delta_x) \right) \\
&= - \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_X dx \int_{\mathbb{R}_+^*} ds\, e^{-s} \left(\left(\frac{d^2}{ds^2} - \frac{d}{ds} \right) F(\eta + s\delta_x) \right) G(\eta + s\delta_x) \\
&\quad - \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_X dx \left(e^{-s} \left(\frac{d}{ds} F(\eta + s\delta_x) \right) G(\eta + s\delta_x) \Big|_{s=0} \right) \\
&= - \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) \int_X d\eta(x) (\Delta_x^{\mathbb{R}_+^*} F)(\eta) G(\eta) - \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) G(\eta) \int_X dx \frac{d}{ds}\Big|_{s=0} F(\eta + s\delta_x) \\
&= \int_{\mathbb{K}(X)} d\mathcal{G}(\eta) (-L^{\mathrm{ext}}F)(\eta) G(\eta).
\end{aligned} \tag{55}$$

It can be easily verified that, for each $F \in \mathcal{F}C_b^\infty(\mathcal{D}(X), \mathbb{K}(X))$, the function on the right hand side of formula (46) belongs to $L^2(\mathbb{K}(X), \mathcal{G})$. So the statement of the proposition regarding the extrinsic case (and hence also regarding the case $\sharp = \mathbb{M}$) indeed holds. □

The condition (43) of Proposition 16 excludes the important case $c(s) = 1$. For this choice of $c(s)$, the informal intrinsic generator, denoted by L^{int}, has the form

$$(L^{\mathrm{int}}F)(\eta) = \int_X \eta(dx)\, \frac{1}{s(\eta,x)^2}\, (\Delta_x^X F)(\eta, x).$$

Using the estimate (52) and Corollary 3, it is easy to prove that, for each $F \in \mathcal{FC}_b^\infty(\mathcal{D}(X), \mathbb{K}(X))$, $L^{\mathrm{int}}F \in L^1(\mathbb{K}(X), \mathcal{G})$. Hence, analogously to the proof of Proposition 16, we get

$$\mathcal{E}^{\mathrm{int}}(F, G) = \int_{\mathbb{K}(X)} (-L^{\mathrm{int}}F)(\eta) G(\eta)\, d\mathcal{G}(\eta), \quad F, G \in \mathcal{FC}_b^\infty(\mathcal{D}(X), \mathbb{K}(X)). \tag{56}$$

We note that formula (56) holds despite the absence of the integration by parts formula for the intrinsic directional derivative (Remark 12). However, generally speaking, for $F \in \mathcal{FC}_b^\infty(\mathcal{D}(X), \mathbb{K}(X))$, $L^{\mathrm{int}}F \notin L^2(\mathbb{K}(X), \mathcal{G})$. Still, even in this case, one can prove the closability of the corresponding bilinear form.

Proposition 17. *Assume* (34) *holds. Then the bilinear forms* $(\mathcal{E}^{\mathrm{int}}, \mathcal{FC}_b^\infty(\mathcal{D}(X), \mathbb{K}(X)))$ *and* $(\mathcal{E}^{\mathbb{M}}, \mathcal{FC}_b^\infty(\mathcal{D}(X), \mathbb{K}(X)))$ *are closable on* $L^2(\mathbb{K}(X), \mathcal{G})$. *Their closures,* $(\mathcal{E}^{\mathrm{int}}, D(\mathcal{E}^{\mathrm{int}}))$ *and* $(\mathcal{E}^{\mathbb{M}}, D(\mathcal{E}^{\mathbb{M}}))$*, are Dirichlet forms. We denote by* $(-L^{\mathrm{int}}, D(L^{\mathrm{int}}))$ *and* $(-L^{\mathbb{M}}, D(L^{\mathbb{M}}))$ *the respective generators of these forms.*

Proof. Similar to the proof of [15, Theorem 6.3] (see also [19, subsec. 2.3]). □

According to Proposition 17, $(-L^{\mathrm{int}}, D(L^{\mathrm{int}}))$ and $(-L^{\mathbb{M}}, D(L^{\mathbb{M}}))$ are densely defined linear operators in $L^2(\mathbb{K}(X), \mathcal{G})$. However, the only elements of $D(L^{\mathrm{int}})$ and $D(L^{\mathbb{M}})$ which we know are constant functions. We will now show how our construction of the Dirichlet forms can be modified in order to gain a much better understanding of the corresponding generators L^{int} and $L^{\mathbb{M}}$. To this end, instead of the set $\mathcal{FC}_b^\infty(\mathcal{D}(X), \mathbb{K}(X))$, we will use another set of 'test' functions on $\mathbb{K}(X)$.

Denote by $\mathcal{D}(\hat{X})$ the space of all infinitely differentiable functions on $\hat{X}$ which have compact support in $\hat{X}$. In particular, the support of each $\varphi \in \mathcal{D}(\hat{X})$ is a subset of some set $A \times [a, b]$, where $A \in \mathcal{B}_c(X)$ and $0 < a < b < \infty$. We denote by $\mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}), \Gamma(\hat{X}))$ the set of all cylinder functions $F : \Gamma(\hat{X}) \to \mathbb{R}$ of the form

$$F(\gamma) = g(\langle \varphi_1, \gamma \rangle, \dots, \langle \varphi_N, \gamma \rangle), \quad \gamma \in \Gamma(\hat{X}), \tag{57}$$

where $g \in C_b^\infty(\mathbb{R}^N)$, $\varphi_1 \dots, \varphi_N \in \mathcal{D}(\hat{X})$, and $N \in \mathbb{N}$. Next, we define

$$\begin{aligned}&\mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X))\\ &\qquad := \big\{F : \mathbb{K}(X) \to \mathbb{R} \mid F(\eta) = G(\mathscr{R}^{-1}\eta) \text{ for some } G \in \mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}), \Gamma(\hat{X}))\big\}.\end{aligned}$$

For $\varphi \in \mathcal{D}(\hat{X})$ and $\eta \in \mathbb{K}(X)$, we denote

$$\langle\!\langle \varphi, \eta \rangle\!\rangle := \langle \varphi, \mathscr{R}^{-1}\eta \rangle = \sum_{x \in \tau(\eta)} \varphi(x, s(\eta, x))) = \int_X \frac{\varphi(x, s(\eta, x))}{s(\eta, x)}\, d\eta(x).$$

Then, each function $F \in \mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X))$ has the form

$$F(\eta) = g\big(\langle\!\langle \varphi_1, \eta \rangle\!\rangle, \dots, \langle\!\langle \varphi_N, \eta \rangle\!\rangle\big), \quad \eta \in \mathbb{K}(X), \tag{58}$$

with $g, \varphi_1 \ldots, \varphi_N$ and N as in (57).

Analogously to (9) and (11), for a function F of the form (58), $v \in C_0^\infty(X \to X)$, and $h \in C_0(X)$, we easily calculate:

$$(\nabla_v^{\rm int} F)(\eta) = \sum_{i=1}^N (\partial_i g)\big(\langle\!\langle \varphi_1, \eta \rangle\!\rangle, \ldots, \langle\!\langle \varphi_N, \eta \rangle\!\rangle\big) \sum_{x \in \tau(\eta)} \langle \nabla_y\big|_{y=x} \varphi_i(y, s(\eta, x)), v(x)\rangle_X$$

$$= \sum_{i=1}^N (\partial_i g)\big(\langle\!\langle \varphi_1, \eta \rangle\!\rangle, \ldots, \langle\!\langle \varphi_N, \eta \rangle\!\rangle\big) \int_X \frac{1}{s(\eta, x)} \langle \nabla_y\big|_{y=x} \varphi_i(y, s(\eta, x)), v(x)\rangle_X \, d\eta(x),$$

$$(\nabla_h^{\rm ext} F)(\eta, x) = \sum_{i=1}^N (\partial_i g)\big(\langle\!\langle \varphi_1, \eta \rangle\!\rangle, \ldots, \langle\!\langle \varphi_N, \eta \rangle\!\rangle\big) \sum_{x \in \tau(\eta)} \frac{\partial}{\partial u}\Big|_{u=s(\eta,x)} \varphi(x, u) s(\eta, x) h(x)$$

$$= \sum_{i=1}^N (\partial_i g)\big(\langle\!\langle \varphi_1, \eta \rangle\!\rangle, \ldots, \langle\!\langle \varphi_N, \eta \rangle\!\rangle\big) \int_X \frac{\partial}{\partial u}\Big|_{u=s(\eta,x)} \varphi(x, u) h(x) \, d\eta(x).$$

Hence,

$$(\nabla^{\rm int} F)(\eta, x) = \sum_{i=1}^N (\partial_i g)\big(\langle\!\langle \varphi_1, \eta \rangle\!\rangle, \ldots, \langle\!\langle \varphi_N, \eta \rangle\!\rangle\big) \frac{1}{s(\eta, x)} \nabla_y\big|_{y=x} \varphi_i(y, s(\eta, x)), \tag{59}$$

$$(\nabla^{\rm ext} F)(\eta, x) = \sum_{i=1}^N (\partial_i g)\big(\langle\!\langle \varphi_1, \eta \rangle\!\rangle, \ldots, \langle\!\langle \varphi_N, \eta \rangle\!\rangle\big) \frac{\partial}{\partial u}\Big|_{u=s(\eta,x)} \varphi(x, u). \tag{60}$$

Furthermore, formulas (29), (30) clearly remain true for each $F \in \mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X))$, $\eta \in \mathbb{K}(X)$, and $x \in \tau(\eta)$.

We note that $\mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}), \Gamma(\hat{X}))$ is a dense subset of $L^2(\Gamma(\hat{X}), \zeta)$ for any probability measure ζ on $\Gamma(\hat{X})$. Hence, $\mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X))$ is a dense subset of $L^2(\mathbb{K}(X), \mu)$ for any probability measure μ on $\mathbb{K}(X)$, in particular, $\mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X))$ is dense in $L^2(\mathbb{K}(X), \mathcal{G})$. About the coefficient function $c : \mathbb{R}_+^* \to [0, \infty)$ we will now only assume that it is measurable and locally bounded (compare with (34)). We define symmetric bilinear forms $\mathcal{E}_1^{\rm int}$, $\mathcal{E}_1^{\rm ext}$, $\mathcal{E}_1^{\mathbb{M}}$ just as the bilinear forms $\mathcal{E}^{\rm int}$, $\mathcal{E}^{\rm ext}$, $\mathcal{E}^{\mathbb{M}}$ but on the domain $\mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X))$ rather than $\mathcal{FC}_b^\infty(\mathcal{D}(X), \mathbb{K}(X))$ (recall formulas (35)–(37)).

It follows from formulas (59) and (60) that, for each $F \in \mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X))$, there exist a constant $C_4 > 0$, a set $A \in \mathcal{B}_c(X)$ and an interval $[a, b]$ with $0 < a < b < \infty$ such that

$$\max\{\|\nabla^{\rm int} F(\eta, x)\|_X, |\nabla^{\rm ext} F(\eta, x)|\} \le C_4 \, \chi_A(x) \, \chi_{[a,b]}(s(\eta, x)), \quad \eta \in \mathbb{K}(X), \ x \in \tau(\eta). \tag{61}$$

Since the function c is locally bounded, there exists a constant $C_5 > 0$ such that

$$c(s(\eta, x)) \chi_{[a,b]}(s(\eta, x)) \le C_5, \quad \eta \in \mathbb{K}(X), \ x \in \tau(\eta). \tag{62}$$

Therefore, by (19), (61), and (62), the integrals in (35) and (36) indeed make sense and are finite for any $F, G \in \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X))$. Next, Lemmas 13, 14, and 15 admit a straightforward generalization to the new domain. Using (40) and (41), we get, for a function a function $F \in \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X))$ of the form (58), we get

$$\begin{aligned}(\Delta_x^X F)(\eta) = &\sum_{i,j=1}^N (\partial_i \partial_j g)(\langle\!\langle \varphi_1, \eta \rangle\!\rangle, \dots, \langle\!\langle \varphi_N, \eta \rangle\!\rangle) \\ &\times \langle \nabla_y \big|_{y=x} \varphi_i(y, s(\eta, x)), \nabla_y \big|_{y=x} \varphi_j(y, s(\eta, x)) \rangle_X \\ &+ \sum_{i=1}^N (\partial_i g)(\langle\!\langle \varphi_1, \eta \rangle\!\rangle, \dots, \langle\!\langle \varphi_N, \eta \rangle\!\rangle) \Delta_y \big|_{y=x} \varphi_i(y, s(\eta, x)), \end{aligned} \tag{63}$$

and similarly, we calculate $(\Delta_x^{\mathbb{R}_+^*} F)(\eta)$.

A counterpart of Proposition 16 holds.

Proposition 18. *Assume that the function $c : \mathbb{R}_+^* \to [0, \infty)$ is measurable and locally bounded. For each $F \in \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X))$, we define*

$$(L_1^{\text{int}} F)(\eta) := \int_X d\eta(x) \, \frac{c(s(\eta, x))}{s(\eta, x)^2} \, (\Delta_x^X F)(\eta), \tag{64}$$

$$(L_1^{\text{ext}} F)(\eta) := \int_X d\eta(x) \, (\Delta_x^{\mathbb{R}_+^*} F)(\eta), \quad \eta \in \mathbb{K}(X), \tag{65}$$

$$L_1^{\mathbb{M}} F := L_1^{\text{int}} F + L_1^{\text{ext}} F. \tag{66}$$

Then, for $\sharp = \text{int}, \text{ext}, \mathbb{M}$, $(L_1^\sharp, \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X)))$ is a symmetric operator in $L^2(\mathbb{K}(X), \mathcal{G})$ which satisfies

$$\mathcal{E}_1^\sharp(F, G) = (-L_1^\sharp F, G)_{L^2(\mathbb{K}(X), \mathcal{G})}, \quad F, G \in \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X)).$$

The bilinear form $(\mathcal{E}_1^\sharp, \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X)))$ is closable on $L^2(\mathbb{K}(X), \mathcal{G})$ and its closure, denoted by $(\mathcal{E}_1^\sharp, D(\mathcal{E}_1^\sharp))$, is a Dirichlet form. The operator $(-L_1^\sharp, \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X)))$ has Friedrichs' extension, which we denote by $(-L_1^\sharp, D(L_1^\sharp))$.

Proof. We first note that, for a fixed $F \in \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X))$, there exist $A \in \mathcal{B}_c(X)$ and an interval $[a, b]$ with $0 < a < b < \infty$ such that the functions

$$\hat{X} \ni (x, s) \mapsto \nabla_x F(\eta + s\delta_x), \quad \hat{X} \ni (x, s) \mapsto \frac{d}{ds} F(\eta + s\delta_x)$$

vanish outside the set $A \times [a, b]$. Let $\sharp = \text{int}$ and let $F, G \in \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X))$. Using the counterpart of Lemma 13 and integrating by parts in the x variable, we see that the formula (49) holds, with $\mathcal{E}^{\mathrm{int}}$ being replaced by $\mathcal{E}_1^{\mathrm{int}}$. Note that, for F of the form (58), we have

$$\begin{aligned}(\Delta_x F)(\eta+s\delta_x)\\ &=\sum_{i,j=1}^N(\partial_i\partial_j g)\big(\langle\!\langle\varphi_1,\eta\rangle\!\rangle+\varphi_1(x,s),\dots,\langle\!\langle\varphi_N,\eta\rangle\!\rangle+\varphi_N(x,s)\big)\langle\nabla_x\varphi_i(x,s),\nabla_x\varphi_j(x,s)\rangle_X\\ &\quad+\sum_{i=1}^N(\partial_i g)\big(\langle\!\langle\varphi_1,\eta\rangle\!\rangle+\varphi_1(x,s),\dots,\langle\!\langle\varphi_N,\eta\rangle\!\rangle+\varphi_N(x,s)\big)\Delta_x\varphi_i(x,s).\end{aligned}\tag{67}$$

Hence, the function under the sign of integral on the right hand side of (49) is integrable. By Corollary 3, (49), (63), (64), and (67), the formula (51) holds, with $\mathcal{E}^{\mathrm{int}}$ and L^{int} being respectively replaced by $\mathcal{E}_1^{\mathrm{int}}$ and L_1^{int}. By (63) and the local boundedness of the function c, there exist $C_6>0$ and $A\in\mathcal{B}_0(X)$ such that

$$\frac{c(s(\eta,x))}{s(\eta,x)^2}\,|(\Delta_x F)(\eta)|\le C_6\chi_A(x),\quad \eta\in\mathbb{K}(X),\ x\in\tau(\eta).$$

Hence, by (19) and (64), we get $L_1^{\mathrm{int}}F\in L^2(\mathbb{K}(X),\mathcal{G})$. Thus, the bilinear form $(\mathcal{E}_1^{\mathrm{int}},\mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}),\mathbb{K}(X)))$ has L^2-generator. Hence, the statement of the proposition regarding $\sharp=\mathrm{int}$ holds.

The proof for $\sharp=\mathrm{ext}$ (and so also for $\sharp=\mathbb{M}$) is similar. We only note that, for each function $F\in\mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}),\mathbb{K}(X))$, the boundary term $\frac{d}{ds}\big|_{s=0}F(\eta+s\delta_x)$ is always equal to 0. □

Remark 19. Let us quickly note some natural choices of the coefficient function $c(s)$. Choosing $c(s)=1$, the intrinsic Dirichlet form becomes the closure of the bilinear form

$$\mathcal{E}_1^{\mathrm{int}}(F,G):=\int_{\mathbb{K}(X)}\langle(\nabla^{\mathrm{int}}F)(\eta),(\nabla^{\mathrm{int}}G)(\eta)\rangle_{T_\eta^{\mathrm{int}}(\mathbb{K})}\,d\mathcal{G}(\eta),\quad F,G\in\mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}),\mathbb{K}(X)).$$

The choice of $c(s)=s$ yields, in fact, the Dirichlet form which is associated with a diffusion process on $\mathbb{K}(X)$ of the type $\eta(t)=\sum_{i=1}^\infty s_i\delta_{x_i(t)}$, where $(x_i(t))_{i=1}^\infty$ are independent Brownian motions on X, see [8]. When we choose $c(s)=s^2$, the generator of the intrinsic Dirichlet form becomes (see (64))

$$(L_1^{\mathrm{int}}F)(\eta)=\int_X d\eta(x)\,(\Delta_x^X F)(\eta).$$

We finish this section with a proposition which shows a connection between the Dirichlet forms $(\mathcal{E}^\sharp,D(\mathcal{E}^\sharp))$ and $(\mathcal{E}_1^\sharp,D(\mathcal{E}_1^\sharp))$.

Proposition 20. *Assume that the function c satisfies* (34). *For $\sharp=\mathrm{int},\mathrm{ext},\mathbb{M}$, the Dirichlet form $(\mathcal{E}_1^\sharp,D(\mathcal{E}_1^\sharp))$ is an extension of $(\mathcal{E}^\sharp,D((\mathcal{E}^\sharp))$, i.e., $D(\mathcal{E}^\sharp)\subset D(\mathcal{E}_1^\sharp)$ and, for any $F,G\in D(\mathcal{E}^\sharp)$, $\mathcal{E}^\sharp(F,G)=\mathcal{E}_1^\sharp(F,G)$.*

Proof. It suffices to prove that

$$\mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(X),\mathbb{K}(X))\subset D(\mathcal{E}_1^\sharp)$$

and for any $F\in\mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(X),\mathbb{K}(X))$, $\mathcal{E}^\sharp(F)=\mathcal{E}_1^\sharp(F)$. Here $\mathcal{E}^\sharp(F):=\mathcal{E}^\sharp(F,F)$, and similarly for $\mathcal{E}_1$. So, on $D(\mathcal{E}_1^\sharp)$ we consider the norm

$$\|F\|_{D(\mathcal{E}_1^\sharp)}:=\mathcal{E}_1^\sharp(F)^{1/2}+\|F\|_{L^2(\mathbb{K}(X),\mathcal{G})}\,. \tag{68}$$

For simplicity of notation, assume that $F\in\mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(X),\mathbb{K}(X))$ is of the form $F(\eta)=g(\langle f,\eta\rangle)$, where $g\in C_b^\infty(\mathbb{R})$ and $f\in\mathcal{D}(X)$. For each $n\in\mathbb{N}$, we fix any function $u_n\in C^\infty(\mathbb{R})$ such that

$$\chi_{[1/n,\,\infty)}\le u_n\le\chi_{[1/(2n),\,\infty)} \tag{69}$$

and

$$|u_n'(t)|\le 4n\,\chi_{[1/(2n),\,1/n]}(t),\quad t\in\mathbb{R}. \tag{70}$$

For $n\in\mathbb{N}$, let $v_n\in C^\infty(\mathbb{R})$ be such that

$$\chi_{(-\infty,n+1]}\le v_n\le\chi_{(-\infty,n+2]} \tag{71}$$

and

$$|v_n'(t)|\le 2\,\chi_{[n+1,\,n+2]}(t),\quad t\in\mathbb{R}. \tag{72}$$

We define

$$h_n(s):=su_n(s)v_n(s),\quad s\in\mathbb{R}_+^*,\ n\in\mathbb{N}, \tag{73}$$

and

$$\varphi_n(x,s):=f(x)h_n(s),\quad (x,s)\in\hat{X},\ n\in\mathbb{N}. \tag{74}$$

Note that $h_n\in C_0^\infty(\mathbb{R}_+^*)$ and $\varphi_n\in\mathcal{D}(\hat{X})$. Let

$$F_n(\eta):=g(\langle\!\langle\varphi_n,\eta\rangle\!\rangle),\quad \eta\in\mathbb{K}(X),\ n\in\mathbb{N}, \tag{75}$$

each F_n being an element of $\mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}(\hat{X}),\mathbb{K}(X))$. For each $\eta\in\mathbb{K}(X)$,

$$\langle\!\langle\varphi_n,\eta\rangle\!\rangle=\sum_{x\in\tau(\eta)}f(x)s(\eta,x)u_n(s(\eta,x))v_n(s(\eta,x))\to\langle f,\eta\rangle\quad\text{as } n\to\infty. \tag{76}$$

Hence, by the dominated convergence theorem, $F_n\to F$ in $L^2(\mathbb{K}(X),\mathcal{G})$. Note that

$$F_n(\eta+s\delta_x)=g(\langle\!\langle\varphi_n,\eta\rangle\!\rangle+\varphi_n(x,s)),\quad \eta\in\mathbb{K}(X),\ (x,s)\in\hat{X}. \tag{77}$$

Using the counterpart of Lemma 13 for $\mathcal{E}_1^\sharp$ and formulas (69)–(77), one can easily show that

$$\mathcal{E}_1^\sharp(F_n-F_m)\to 0\quad\text{as } n,m\to\infty. \tag{78}$$

Since $(\mathcal{E}_1^\sharp,D(\mathcal{E}_1^\sharp))$ is a closed bilinear form on $L^2(\mathbb{K}(X),\mathcal{G})$, we therefore have $F\in D(\mathcal{E}_1^\sharp)$. Furthermore, analogously to the proof of (78), we get $\mathcal{E}_1^\sharp(F_n)\to\mathcal{E}^\sharp(F)$ as $n\to\infty$. Hence $\mathcal{E}^\sharp(F)=\mathcal{E}_1^\sharp(F)$. □

Remark 21. Let $\sharp = \mathrm{int}, \mathrm{ext}, \mathbb{M}$. For $\sharp = \mathrm{int}, \mathbb{M}$, assume that condition (34) is satisfied and the dimension d of the underlying space X is ≥ 2. In the forthcoming paper [8], for $\sharp = \mathrm{int}, \mathrm{ext}, \mathbb{M}$, we will prove the existence of a conservative diffusion process on $\mathbb{K}(X)$ (i.e., a conservative strong Markov process with continuous sample paths in $\mathbb{K}(X)$) which is properly associated with the Dirichlet form $(\mathcal{E}_1^\sharp, D(\mathcal{E}_1^\sharp))$, see [19] for details on diffusion processes properly associated with a Dirichlet form. In particular, this diffusion process is $\mathcal{G}$-symmetric and has $\mathcal{G}$ as an invariant measure.

Remark 22. In view of Proposition 20 there is a natural question whether the Dirichlet forms $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$ and $(\mathcal{E}_1^\sharp, D(\mathcal{E}_1^\sharp))$ coincide. We do not expect a positive answer to this question. Furthermore, we do not expect the existence of a conservative diffusion process on $\mathbb{K}(X)$ which is properly associated with the Dirichlet form $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$.

6 Essential self-adjointness of the generators

In this section, for $\sharp = \mathrm{int}, \mathrm{ext}, \mathbb{M}$, we will discuss the essential self-adjointness of the operator $(L_1^\sharp, D(L_1^\sharp))$ on the domain $\mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X))$. Since we will not anymore use the Dirichlet forms $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$ and their generators $(L^\sharp, D(L^\sharp))$, with some abuse of notation, we will drop the lower index 1 from the notations $(\mathcal{E}_1^\sharp, D(\mathcal{E}_1^\sharp))$ and $(L_1^\sharp, D(L_1^\sharp))$, thus denoting these Dirichlet forms and their generators by $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$ and $(L^\sharp, D(L^\sharp))$, respectively.

Theorem 23. *Let $\sharp = \mathrm{int}, \mathrm{ext}, \mathbb{M}$. Let the function $c : \mathbb{R}_+^* \to [0, \infty)$ be measurable and locally bounded. For $\sharp = \mathbb{M}$, assume additionally that*

$$c(s) = a_1 s + a_2 s^2 + a_3 s^3 \tag{79}$$

for some $a_i \geq 0$, $i = 1, 2, 3$, $\max\{a_1, a_2, a_3\} > 0$. Then the operator $(L^\sharp, \mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X)))$ is essentially self-adjoint on $L^2(\mathbb{K}(X), \mathcal{G})$. In particular, the operator $(L^\sharp, D(L^\sharp))$ is the closure of $(L^\sharp, \mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}), \mathbb{K}(X)))$ on $L^2(\mathbb{K}(X), \mathcal{G})$.

Proof. Fix any $F \in \mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}), \Gamma(\hat{X}))$ and $\gamma \in \Gamma(\hat{X})$. Consider the function

$$\hat{X} \setminus \gamma \ni (x, s) \mapsto F(\gamma + \delta_{(x,s)}).$$

It is evident that this function admits a unique extension by continuity to the whole space $\hat{X}$. We denote the resulting function by $F(\gamma + \delta_{(x,s)})$, although $\gamma + \delta_{(x,s)}$ is not necessarily an element of $\Gamma(\hat{X})$. Note that $F(\gamma + \delta_{(x,s)})$ is a smooth functions of $(x, s) \in \hat{X}$.

We preserve the notation $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$ for the realization of the respective Dirichlet form on $\Gamma_{pf}(\hat{X})$. Thus, $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$ is the closure of the bilinear form

$$(\mathcal{E}^\sharp, \mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}), \Gamma(\hat{X})))$$

on $L^2(\Gamma(\hat{X}),\pi)$. Furthermore, by the counterpart of Lemma 13 for the domain $\mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}),\mathbb{K}(X))$, we get, for any $F,G\in\mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}),\Gamma(\hat{X}))$,

$$\begin{aligned}
\mathcal{E}^{\rm int}(F,G) &= \int_{\Gamma_{pf}(\hat{X})} d\pi(\gamma)\int_{\hat{X}} dx\,ds\,e^{-s}\,\frac{c(s)}{s^2}\big\langle \nabla_x F(\gamma+\delta_{(x,s)}),\nabla_x G(\gamma+\delta_{(x,s)})\big\rangle_X\,,\\
\mathcal{E}^{\rm ext}(F,G) &= \int_{\Gamma_{pf}(\hat{X})} d\pi(\gamma)\int_{\hat{X}} dx\,ds\,e^{-s}\left(\frac{d}{ds}\,F(\gamma+\delta_{(x,s)})\right)\left(\frac{d}{ds}G(\gamma+\delta_{(x,s)})\right),\\
\mathcal{E}^{\mathbb{K}}(F,G) &= \mathcal{E}^{\rm int}(F,G)+\mathcal{E}^{\rm ext}(F,G).
\end{aligned}\tag{80}$$

We keep the notation $(L^\sharp,D(L^\sharp))$ for the generator of the closed bilinear form $(\mathcal{E}^\sharp,D(\mathcal{E}^\sharp))$ on $L^2(\Gamma_{pf},\pi)$. We easily conclude from Proposition 18 that

$$\mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}),\Gamma(\hat{X}))\subset D(L^\sharp)$$

and for each $F\in\mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}),\Gamma(\hat{X}))$ and $\gamma\in\Gamma(\hat{X})$

$$(L^{\rm int}F)(\gamma)=\int_{\hat{X}} d\gamma(x,s)\,\frac{c(s)}{s}\,(\Delta_x^X F)(\gamma),\tag{81}$$

$$(L^{\rm ext}F)(\gamma)=\int_{\hat{X}} d\gamma(x,s)\,s\,(\Delta_x^{\mathbb{R}_+^*}F)(\gamma),\tag{82}$$

$$(L^{\mathbb{K}}F)(\gamma)=(L^{\rm int}F)(\gamma)+(L^{\rm ext}F)(\gamma),\tag{83}$$

with

$$\begin{aligned}
(\Delta_x^X F)(\gamma) :=&\,\Delta_y\big|_{y=x}F(\gamma-\delta_{(x,s)}+\delta_{(y,s)}),\\
(\Delta_x^{\mathbb{R}_+^*}F)(\gamma) :=&\left(\frac{d^2}{du^2}-\frac{d}{du}\right)\Big|_{u=s}F(\gamma-\delta_{(x,s)}+\delta_{(x,u)}).
\end{aligned}$$

We equivalently have to prove that the symmetric operator $(L^\sharp,\mathcal{FC}_b^\infty(\mathcal{D}(\hat{X}),\Gamma(\hat{X})))$ is essentially self-adjoint on $L^2(\Gamma(\hat{X}),\pi)$. Denote by $(H^\sharp,D(H^\sharp))$ the closure of this symmetric operator on $L^2(\Gamma(\hat{X}),\pi)$. So we have to prove that the operator $(H^\sharp,D(H^\sharp))$ is self-adjoint.

It is not hard to check by approximation that, for each $\varphi\in\mathcal{D}(\hat{X})$ and $n\in\mathbb{N}$, $F=\langle\varphi,\cdot\rangle^n\in D(H^\sharp)$ and $(H^\sharp F)(\gamma)$ is given by the right hand sides of formulas (81)–(83), respectively. Hence, by the polarization identity (e.g. [4, Chap. 2, formula (2.17)]), we have

$$\langle\varphi_1,\cdot\rangle\cdots\langle\varphi_n,\cdot\rangle\in D(H^\sharp),\quad \varphi_1,\dots,\varphi_n\in\mathcal{D}(\hat{X}),\ n\in\mathbb{N},\tag{84}$$

and again the action of $H^\sharp$ onto a function F as in (84) is given by the right hand side of formulas (81)–(83), respectively. Let $\mathscr{P}$ denote the set of all functions on $\Gamma(\hat{X})$ which are finite sums of functions as in (84) and constants. Thus, $\mathscr{P}$ is a set of polynomials on $\Gamma(\hat{X})$, and $\mathscr{P}\subset D(H^\sharp)$. Furthermore,

$$(-H^\sharp F,G)_{L^2(\Gamma(\hat{X}),\pi)}=\mathcal{E}^\sharp(F,G),\quad F,G\in\mathscr{P},\ \sharp=\text{int},\text{ext},\mathbb{M}.\tag{85}$$

In formula (85), $\mathcal{E}^\sharp(F,G)$ is given by formulas (80).

For a real separable Hilbert space $\mathcal{H}$, we denote by $\mathcal{F}(\mathcal{H})$ the symmetric Fock space over $\mathcal{H}$. Thus, $\mathcal{F}(\mathcal{H})$ is the real Hilbert space

$$\mathcal{F}(\mathcal{H})=\bigoplus_{n=0}^{\infty}\mathcal{F}^{(n)}(\mathcal{H}),$$

where $\mathcal{F}^{(0)}(\mathcal{H}):=\mathbb{R}$, and for $n\in\mathbb{N}$, $\mathcal{F}^{(n)}(\mathcal{H})$ coincides with $\mathcal{H}^{\odot n}$ as a set, and for any $f^{(n)},g^{(n)}\in\mathcal{F}^{(n)}(\mathcal{H})$

$$(f^{(n)},g^{(n)})_{\mathcal{F}^{(n)}(\mathcal{H})}:=(f^{(n)},g^{(n)})_{\mathcal{H}^{\odot n}}\,n!\,.$$

Here $\odot$ stands for symmetric tensor product.

Recall the measure $\varkappa$ on $\hat{X}$ defined by formulas (14), (15). Let

$$I:L^2(\Gamma(\hat{X}),\pi)\to\mathcal{F}(L^2(\hat{X},\varkappa)) \tag{86}$$

denote the unitary isomorphism which is derived through multiple stochastic integrals with respect to the centered Poisson random measure on $\hat{X}$ with intensity measure $\varkappa$, see e.g. [29]. Denote by $\tilde{\mathscr{P}}$ the subset of $\mathcal{F}(L^2(\hat{X},\varkappa))$ which is the linear span of vectors of the form

$$\varphi_1\odot\varphi_2\odot\dots\odot\varphi_n,\quad \varphi_1,\dots,\varphi_n\in\mathcal{D}(\hat{X}),\ n\in\mathbb{N}$$

and the vacuum vector $\Psi=(1,0,0,\dots)$. For any $\varphi\in\mathcal{D}(\hat{X})$, denote by M_φ the operator of multiplication by the function $\langle\varphi,\cdot\rangle$ in $L^2(\Gamma(\hat{X}),\pi)$. Using the representation of the operator $IM_\varphi I^{-1}$ as a sum of creation, neutral, and annihilation operators in the Fock space (see e.g. [29]), we easily conclude that $I\mathscr{P}=\tilde{\mathscr{P}}$.

We define a bilinear form $(\tilde{\mathcal{E}}^\sharp,\tilde{\mathscr{P}})$ by

$$\tilde{\mathcal{E}}^\sharp(f,g):=\mathcal{E}^\sharp(I^{-1}f,I^{-1}g),\quad f,g\in\tilde{\mathscr{P}}$$

on $\mathcal{F}(L^2(\hat{X},\varkappa))$.

For each $(x,s)\in\hat{X}$, we define an annihilation operator at (x,s) as follows:

$$\partial_{(x,s)}:\tilde{\mathscr{P}}\to\tilde{\mathscr{P}}$$

is the linear map given by

$$\partial_{(x,s)}\Psi:=0,\quad \partial_{(x,s)}\varphi_1\odot\varphi_2\odot\dots\odot\varphi_n:=\sum_{i=1}^{n}\varphi_i(x,s)\varphi_1\odot\varphi_2\odot\dots\odot\check{\varphi}_i\odot\dots\odot\varphi_n, \tag{87}$$

where $\check{\varphi}_i$ denotes the absence of φ_i. We will preserve the notation $\partial_{(x,s)}$ for the operator $I\partial_{(x,s)}I^{-1}:\mathscr{P}\to\mathscr{P}$. This operator admits the following explicit representation:

$$\partial_{(x,s)}F(\gamma)=F(\gamma+\delta_{(x,s)})-F(\gamma)$$

for π-a.a. $\gamma\in\Gamma(\hat X)$, see e.g. [11,21]. Note that

$$\nabla_x F(\gamma+\delta_{(x,s)})=\nabla_x\big(F(\gamma+\delta_{(x,s)})-F(\gamma)\big),\quad \frac{d}{ds}\,F(\gamma+\delta_{(x,s)})=\frac{d}{ds}\big(F(\gamma+\delta_{(x,s)})-F(\gamma)\big).$$

Hence, by (80), for any $F,G\in\mathscr{P}$,

$$\begin{aligned}
\mathcal{E}^{\rm int}(F,G)&=\int_{\Gamma(\hat X)}d\pi(\gamma)\int_{\hat X}dx\,ds\,e^{-s}\,\frac{c(s)}{s^2}\big\langle\nabla_x\,\partial_{(x,s)}F(\gamma),\nabla_x\,\partial_{(x,s)}G(\gamma)\big\rangle_X\,,\\
\mathcal{E}^{\rm ext}(F,G)&=\int_{\Gamma(\hat X)}d\pi(\gamma)\int_{\hat X}dx\,ds\,e^{-s}\left(\frac{\partial}{\partial s}\,\partial_{(x,s)}F(\gamma)\right)\left(\frac{\partial}{\partial s}\partial_{(x,s)}G(\gamma)\right),\\
\mathcal{E}^{\mathbb{M}}(F,G)&=\mathcal{E}^{\rm int}(F,G)+\mathcal{E}^{\rm ext}(F,G).
\end{aligned}$$

Hence, for any $f,g\in\tilde{\mathscr{P}}$,

$$\begin{aligned}
\tilde{\mathcal{E}}^{\rm int}(f,g)&=\int_{\hat X}d\varkappa(x,s)\,\frac{c(s)}{s}\sum_{i=1}^d\left(\frac{\partial}{\partial x^i}\,\partial_{(x,s)}f,\frac{\partial}{\partial x^i}\,\partial_{(x,s)}g\right)_{\mathcal{F}(L^2(\hat X,\varkappa))},\\
\tilde{\mathcal{E}}^{\rm ext}(f,g)&=\int_{\hat X}d\varkappa(x,s)\,s\left(\frac{\partial}{\partial s}\,\partial_{(x,s)}f,\frac{\partial}{\partial s}\,\partial_{(x,s)}g\right)_{\mathcal{F}(L^2(\hat X,\varkappa))},\\
\tilde{\mathcal{E}}^{\mathbb{M}}(f,g)&=\tilde{\mathcal{E}}^{\rm int}(f,g)+\tilde{\mathcal{E}}^{\rm ext}(f,g).
\end{aligned}\tag{88}$$

Consider the bilinear forms

$$\begin{aligned}
\mathfrak{E}^{\rm int}(\varphi,\psi)&:=\int_{\hat X}d\varkappa(x,s)\,\frac{c(s)}{s}\big\langle\nabla_x\varphi(x,s),\nabla_x\psi(x,s)\big\rangle_X,\\
\mathfrak{E}^{\rm ext}(\varphi,\psi)&:=\int_{\hat X}d\varkappa(x,s)\,s\left(\frac{\partial}{\partial s}\varphi(x,s)\right)\left(\frac{\partial}{\partial s}\psi(x,s)\right),\\
\mathfrak{E}^{\mathbb{M}}(\varphi,\psi)&:=\mathfrak{E}^{\rm int}(\varphi,\psi)+\mathfrak{E}^{\rm ext}(\varphi,\psi),\quad \varphi,\psi\in\mathcal{D}(\hat X),
\end{aligned}\tag{89}$$

on $L^2(\hat X,\varkappa)$. We easily calculate the L^2-generators of these bilinear forms:

$$\mathfrak{E}^\sharp(\varphi,\psi)=(-\mathfrak{L}^\sharp\varphi,\psi)_{L^2(\hat X,\varkappa)},\quad \varphi,\psi\in\mathcal{D}(\hat X),\tag{90}$$

where for $\varphi\in\mathcal{D}(\hat X)$

$$\begin{aligned}
(\mathfrak{L}^{\rm int}\varphi)(x,s)&=\frac{c(s)}{s}\,\Delta_x\varphi(x,s),\\
(\mathfrak{L}^{\rm ext}\varphi)(x,s)&=s\left(\frac{\partial^2}{\partial s^2}-\frac{\partial}{\partial s}\right)\varphi(x,s),\\
\mathfrak{L}^{\mathbb{M}}\varphi=\mathfrak{L}^{\rm int}\varphi+\mathfrak{L}^{\rm ext}\varphi&=\frac{c(s)}{s}\,\Delta_x\varphi(x,s)+s\left(\frac{\partial^2}{\partial s^2}-\frac{\partial}{\partial s}\right)\varphi(x,s).
\end{aligned}\tag{91}$$

Let us now recall the notion of a differential second quantization. Let $(\mathscr{A}, \mathscr{D})$ be a densely defined symmetric operator in a real, separable Hilbert space $\mathcal{H}$. We denote by $\mathcal{F}_{\mathrm{alg}}(\mathscr{D})$ the subset of the Fock space $\mathcal{F}(\mathcal{H})$ which is the linear span of the vacuum vector Ψ and vectors of the form $\varphi_1 \odot \varphi_2 \odot \cdots \odot \varphi_n$, where $\varphi_1, \dots, \varphi_n \in \mathscr{D}$ and $n \in \mathbb{N}$. The differential second quantization $d\,\mathrm{Exp}(\mathscr{A})$ is defined as the symmetric operator in $\mathcal{F}(\mathcal{H})$ with domain $\mathcal{F}_{\mathrm{alg}}(\mathscr{D})$ which acts as follows:

$$\begin{aligned} d\,\mathrm{Exp}(\mathscr{A})\Psi &:= 0, \\ d\,\mathrm{Exp}(\mathscr{A})\varphi_1 \odot \varphi_2 \odot \cdots \odot \varphi_n &:= \sum_{i=1}^{n} \varphi_1 \odot \varphi_2 \odot \cdots \odot (\mathscr{A}\varphi_i) \odot \cdots \odot \varphi_n. \end{aligned} \tag{92}$$

By e.g. [4, Chap. 6, subsec. 1.1], if the operator $(\mathscr{A}, \mathscr{D})$ is essentially self-adjoint on $\mathcal{H}$, then the differential second quantization $(d\,\mathrm{Exp}(\mathscr{A}), \mathcal{F}_{\mathrm{alg}}(\mathscr{D}))$ is essentially self-adjoint on $\mathcal{F}(\mathcal{H})$.

Now, we note that $\tilde{\mathscr{P}} = \mathcal{F}_{\mathrm{alg}}(\mathcal{D}(\hat{X}))$. By (87)–(92) (see also [4, Chap. 6, Sect. 1]), an easy calculation shows that

$$\tilde{\mathscr{E}}^\sharp(f,g) = (d\,\mathrm{Exp}(-\mathfrak{L}^\sharp)f, g)_{\mathcal{F}(L^2(\hat{X},\varkappa))}, \quad f, g \in \tilde{\mathscr{P}},\ \sharp = \mathrm{int}, \mathrm{ext}, \mathbb{M}.$$

Hence, by (85),

$$\tilde{H}^\sharp f = d\,\mathrm{Exp}(\mathfrak{L}^\sharp) f, \quad f \in \tilde{\mathscr{P}},\ \sharp = \mathrm{int}, \mathrm{ext}, \mathbb{M}. \tag{93}$$

Here $\tilde{H}^\sharp := I H^\sharp I^{-1}$. To prove the theorem, it suffices to show that the operator $(H^\sharp, \mathscr{P})$ is essentially self-adjoint on $L^2(\mathbb{K}(X), \mathcal{G})$, or equivalently the operator $(\tilde{H}^\sharp, \tilde{\mathscr{P}})$ is essentially self-adjoint on $\mathcal{F}(L^2(\hat{X}, \varkappa))$. By (93), the theorem will follow from the lemma below. □

Lemma 24. *Under the assumptions of Theorem 23, the operator $(\mathfrak{L}^\sharp, \mathcal{D}(\hat{X}))$ is essentially self-adjoint on $L^2(\hat{X}, \varkappa)$, $\sharp = \mathrm{int}, \mathrm{ext}, \mathbb{M}$.*

Proof. We will only discuss the hardest case $\sharp = \mathbb{M}$. We denote by $(\mathfrak{L}^{\mathbb{M}}, D(\mathfrak{L}^{\mathbb{M}}))$ the closure of the symmetric operator $(\mathfrak{L}^{\mathbb{M}}, \mathcal{D}(\hat{X}))$ on $L^2(\hat{X}, \varkappa)$. We denote by $\mathcal{S}(X)$ the Schwartz space of real-valued, rapidly decreasing functions on X (see e.g. [25, Sect. V.3]).

Claim. For each $f \in \mathcal{S}(X)$ and $k \in \mathbb{N}$, the function $\varphi(x,s) = f(x)s^k$ belongs to $D(\mathfrak{L}^{\mathbb{M}})$, and $\mathfrak{L}^{\mathbb{M}}\varphi$ is given by the right hand side of (91).

Indeed, for any functions $f \in \mathcal{D}(X)$ and $g \in C_0^\infty(\mathbb{R}_+^*)$, we have $f(x)g(s) \in \mathcal{D}(\hat{X}) \subset D(\mathfrak{L}^{\mathbb{K}})$. Hence, by approximation, we can easily conclude that, for any functions $f \in \mathcal{S}(X)$ and $g \in C_0^\infty(\mathbb{R}_+^*)$, we have $f(x)g(s) \in D(\mathfrak{L}^{\mathbb{K}})$.

Fix any function $u \in C^\infty(\mathbb{R})$ such that $\chi_{[1,\infty)} \le u \le \chi_{[1/2,\infty)}$. Let

$$C_7 := \max_{t \in [1/2,\,1]} \max\{\, |u'(t)|, |u''(t)| \,\} < \infty.$$

For $n \in \mathbb{N}$, let $u_n(t) := u(nt)$, $t \in \mathbb{R}$. Then

$$\chi_{[1/n,\,\infty)} \le u_n \le \chi_{[1/(2n),\,\infty)} \tag{94}$$

and

$$|u_n'(t)| \le C_7\, n\chi_{[1/(2n),\,1/n]}(t), \quad |u_n''(t)| \le C_7\, n^2\chi_{[1/(2n),\,1/n]}(t), \quad t \in \mathbb{R},\ n \in \mathbb{N}. \tag{95}$$

We also fix any function $v \in C^\infty(\mathbb{R})$ such that $\chi_{(-\infty,\,1]} \le v \le \chi_{(-\infty,\,2]}$. For $n \in \mathbb{N}$, set $v_n(t) := v(t-n)$, $t \in \mathbb{R}$. Hence

$$\chi_{(-\infty,\,n+1]} \le v_n \le \chi_{(-\infty,\,n+2]}, \tag{96}$$

and for some $C_8 > 0$

$$\max\{\,|v_n'(t)|, |v_n''(t)|\,\} \le C_8\, \chi_{[n+1,\,n+2]}, \quad t \in \mathbb{R},\ n \in \mathbb{N}. \tag{97}$$

We fix any $k \in \mathbb{N}$ and set

$$g_n(s) := s^k u_n(s) v_n(s), \quad s \in \mathbb{R}_+^*,\ n \in \mathbb{N}. \tag{98}$$

Clearly, $g_n \in C_0^\infty(\mathbb{R}_+^*)$. We fix $f \in \mathcal{S}(X)$ and set

$$\varphi_n(x,s) := f(x) g_n(s), \quad (x,s) \in \hat{X},\ n \in \mathbb{N}. \tag{99}$$

Thus, $\varphi_n \in D(\mathfrak{L}^{\mathbb{K}})$. By the dominated convergence theorem,

$$\varphi_n(x,s) \to \varphi(x,s) := f(x)s^k \quad \text{in } L^2(\hat{X},\varkappa) \text{ as } n \to \infty. \tag{100}$$

We fix any $\psi \in \mathcal{D}(\hat{X})$. Then

$$(-\mathfrak{L}^{\mathbb{M}}\varphi_n, \psi)_{L^2(\hat{X},\varkappa)} = \mathfrak{E}^{\mathbb{M}}(\varphi_n, \psi), \quad n \in \mathbb{N}. \tag{101}$$

It is easy to see that

$$\lim_{n\to\infty} \mathfrak{E}^{\mathbb{M}}(\varphi_n, \psi) = \mathfrak{E}^{\mathbb{M}}(\varphi, \psi). \tag{102}$$

In (101) and (102) , $\mathfrak{E}^{\mathbb{M}}(\cdot,\cdot)$ is given by the formulas in (89). Hence

$$\lim_{n\to\infty} (\mathfrak{L}^{\mathbb{M}}\varphi_n, \psi)_{L^2(\hat{X},\varkappa)} = (\mathfrak{L}^{\mathbb{M}}\varphi, \psi)_{L^2(\hat{X},\varkappa)}. \tag{103}$$

We stress that, in (103), the function $\mathfrak{L}^{\mathbb{M}}\varphi \in L^2(\hat{X},\varkappa)$ is given by formulas in (91), however we do not yet state that $\varphi \in D(\mathfrak{L}^{\mathbb{M}})$.

By using (94)–(99), it can be easily shown that

$$\sup_{n\in\mathbb{N}} \|\mathfrak{L}^{\mathbb{M}}\varphi_n\|_{L^2(\hat{X},\varkappa)} < \infty.$$

Hence, by the Banach–Alaoglu and Banach–Saks theorems (see e.g. [19, Appendix, Sect. 2]), there exists a subsequence $(\varphi_{n_j})_{j=1}^\infty$ of $(\varphi_n)_{n=1}^\infty$ such that the sequence $(\mathfrak{L}^{\mathbb{M}}\xi_i)_{i=1}^\infty$ converges in $L^2(\hat{X},\varkappa)$. Here

$$\xi_i := \frac{1}{i}\sum_{j=1}^{i}\varphi_{n_j}, \quad i\in\mathbb{N}.$$

We note that, for each $i\in\mathbb{N}$, $\xi_i\in\mathcal{D}(\hat{X})$, and by (100)

$$\xi_i\to\varphi \quad \text{in } L^2(\hat{X},\varkappa) \text{ as } i\to\infty. \tag{104}$$

Furthermore, by (103),

$$\lim_{i\to\infty}(\mathfrak{L}^{\mathbb{M}}\xi_i,\psi)_{L^2(\hat{X},\varkappa)} = (\mathfrak{L}^{\mathbb{M}}\varphi,\psi)_{L^2(\hat{X},\varkappa)}, \quad \psi\in\mathcal{D}(\hat{X}).$$

Hence

$$\mathfrak{L}^{\mathbb{M}}\xi_i\to\mathfrak{L}^{\mathbb{M}}\varphi \quad \text{in } L^2(\hat{X},\varkappa) \text{ as } i\to\infty. \tag{105}$$

By (104) and (105), we conclude that $\xi_i\to\varphi$ in the graph norm of the operator $(\mathfrak{L}^{\mathbb{M}}, D(\mathfrak{L}^{\mathbb{M}}))$. Thus, the claim is proven.

We next note that

$$L^2(\hat{X},\varkappa) = L^2(X,dx)\otimes L^2(\mathbb{R}_+^*,\lambda) \tag{106}$$

(recall (14)). Evidently, $\mathcal{S}(X)$ is a dense subset of $L^2(X,dx)$. Furthermore, the functions $\{s^k\}_{k=1}^\infty$ form a total set in $L^2(\mathbb{R}_+^*,\lambda)$ (i.e., the linear span of this set is dense in $L^2(\mathbb{R}_+^*,\lambda)$). Indeed, consider the unitary operator

$$L^2(\mathbb{R}_+^*,\lambda)\ni g(s)\mapsto \frac{g(s)}{s}\in L^2(\mathbb{R}_+^*, se^{-s}\,ds).$$

Under this unitary operator, the set $\{s^k\}_{k=1}^\infty$ goes over into the set $\{s^k\}_{k=0}^\infty$. But the measure $\chi_{\mathbb{R}_+^*}(s)se^{-s}\,ds$ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ has Laplace transform which is analytic in a neighborhood of zero, hence the set of polynomials is dense in $L^2(\mathbb{R}_+^*, se^{-s}\,ds)$. Therefore, the set

$$\Upsilon := \mathrm{l.\,s.}\{f(x)s^k \mid f\in\mathcal{S}(X),\ k\in\mathbb{N}\}$$

is dense in $L^2(\hat{X},\varkappa)$. Here l. s. denotes the linear span. By the Claim, the set Υ is a subset of $D(\mathfrak{L}^{\mathbb{M}})$. Note also that the operator $\mathfrak{L}^{\mathbb{M}}$ maps the set Υ into itself.

Since the symmetric operator $(\mathfrak{L}^{\mathbb{M}}, D(\mathfrak{L}^{\mathbb{M}}))$ is an extension of the operator $(\mathfrak{L}^{\mathbb{M}},\Upsilon)$, to prove that $(\mathfrak{L}^{\mathbb{M}}, D(\mathfrak{L}^{\mathbb{M}}))$ is a self-adjoint operator, it suffices to prove that the operator $(\mathfrak{L}^{\mathbb{M}},\Upsilon)$ is essentially self-adjoint.

We denote by $L^2_{\mathbb{C}}(\hat{X},\varkappa)$ the complex Hilbert space of all complex-valued $\varkappa$-square-integrable functions on $\hat{X}$. Let $\Upsilon_{\mathbb{C}}$ denote the complexification of Υ, i.e., the set of all functions of the form $\varphi_1 + i\varphi_2$, where $\varphi_1, \varphi_2 \in \Upsilon$. Analogously, we define $L^2_{\mathbb{C}}(X, dx)$ and $\mathcal{S}_{\mathbb{C}}(X)$, the Schwartz space of complex-valued, rapidly decreasing functions on X. We extend the operator $\mathfrak{L}^{\mathbb{M}}$ by linearity to $\Upsilon_{\mathbb{C}}$.

Recall that the Fourier transform determines a unitary operator

$$\mathfrak{F} : L^2_{\mathbb{C}}(X, dx) \to L^2_{\mathbb{C}}(X, dx).$$

This operator leaves the Schwartz space $\mathcal{S}_{\mathbb{C}}(X)$ invariant, and furthermore

$$\mathfrak{F} : \mathcal{S}_{\mathbb{C}}(X) \to \mathcal{S}_{\mathbb{C}}(X)$$

is a bijective mapping. Under $\mathfrak{F}$, the Laplace operator Δ goes over into the operator of multiplication by $-\|x\|^2_X$, see e.g. [26, Sect. IX.1]. Using (106), we obtain the unitary operator

$$\mathfrak{F} \otimes \mathbf{1} : L^2_{\mathbb{C}}(\hat{X}, \varkappa) \to L^2_{\mathbb{C}}(\hat{X}, \varkappa).$$

Here $\mathbf{1}$ denotes the identity operator. Clearly $\mathfrak{F} \otimes \mathbf{1} : \Upsilon_{\mathbb{C}} \to \Upsilon_{\mathbb{C}}$ is a bijective mapping. We define an operator $\mathfrak{R}^{\mathbb{M}} : \Upsilon_{\mathbb{C}} \to \Upsilon_{\mathbb{C}}$ by

$$\mathfrak{R}^{\mathbb{M}} := (\mathfrak{F} \otimes \mathbf{1}) \mathfrak{L}^{\mathbb{M}} (\mathfrak{F} \otimes \mathbf{1})^{-1}.$$

Explicitly, for each $\varphi \in \Upsilon_{\mathbb{C}}$,

$$(\mathfrak{R}^{\mathbb{M}}\varphi)(x, s) = -\frac{c(s)}{s} \|x\|^2_X \, \varphi(x, s) + s \left(\frac{\partial^2}{\partial s^2} - \frac{\partial}{\partial s} \right) \varphi(x, s). \tag{107}$$

It suffices to prove that the operator $(\mathfrak{R}^{\mathbb{M}}, \Upsilon_{\mathbb{C}})$ is essentially self-adjoint on $L^2_{\mathbb{C}}(\hat{X}, \varkappa)$.

Since the operator $(\mathfrak{R}^{\mathbb{M}}, \Upsilon_{\mathbb{C}})$ is non-positive, by the Nussbaum theorem [22], it suffices to prove that, for each function

$$\varphi(x, s) = f(x) s^k \tag{108}$$

with $f \in \mathcal{D}(X)$ and $k \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} \|(\mathfrak{R}^{\mathbb{M}})^n \varphi\|^{-1/2n}_{L^2_{\mathbb{C}}(\hat{X}, \varkappa)} = \infty. \tag{109}$$

For a function $\varphi(x, s)$ of the form (108), by virtue of (79) and (107), we get

$$\begin{aligned}(\mathfrak{R}^{\mathbb{M}}\varphi)(x, s) &= -(a_1 s^k + a_2 s^{k+1} + a_3 s^{k+2}) \|x\|^2_X \, f(x) + (k(k-1) s^{k-1} - k s^k) f(x) \\ &= (\mathfrak{R}^{\mathbb{M}}_{-1}\varphi)(x, s) + (\mathfrak{R}^{\mathbb{M}}_0\varphi)(x, s) + (\mathfrak{R}^{\mathbb{M}}_1\varphi)(x, s) + (\mathfrak{R}^{\mathbb{M}}_2\varphi)(x, s).\end{aligned} \tag{110}$$

Here

$$(\mathfrak{R}^{\mathbb{M}}_{-1}\varphi)(x, s) = k(k-1) s^{k-1} f(x),$$

$$
\begin{aligned}
(\mathfrak{R}_0^{\mathbb{M}}\varphi)(x,s) &= (-a_1\|x\|_X^2 - k)s^k f(x),\\
(\mathfrak{R}_1^{\mathbb{M}}\varphi)(x,s) &= -a_2\|x\|_X^2 s^{k+1} f(x),\\
(\mathfrak{R}_2^{\mathbb{M}}\varphi)(x,s) &= -a_3\|x\|_X^2 s^{k+2} f(x).
\end{aligned}
\tag{111}
$$

For $l \in \mathbb{N}$, denote

$$
m_l := \int_{\mathbb{R}_+^*} s^l \, d\lambda(s) = \int_0^\infty s^{l-1} e^{-s} \, ds. \tag{112}
$$

Since the Laplace transform of the measure $\chi_{\mathbb{R}_+^*}(s)e^{-s}\,ds$ on $\mathbb{R}$ is analytic in a neighborhood of zero, there exists a constant $C_9 \geq 1$ such that

$$
m_l \leq C_9^l \, l!\,, \quad l \in \mathbb{N}. \tag{113}
$$

Consider a product $\mathfrak{R}_{i_1}^{\mathbb{M}} \cdots \mathfrak{R}_{i_n}^{\mathbb{M}}\varphi$, where $i_1, \dots, i_n \in \{-1, 0, 1, 2\}$. Denote by l_j the number of the $\mathfrak{R}_j^{\mathbb{M}}$ operators among the operators $\mathfrak{R}_{i_1}^{\mathbb{M}}, \dots, \mathfrak{R}_{i_n}^{\mathbb{M}}$. Thus, $l_{-1}+l_0+l_1+l_2 = n$. Note that the function $f(x)$ has a compact support in X, hence the function $\|x\|_X^2$ is bounded on $\operatorname{supp}(f)$. Recall also the estimate

$$
(2j)! \leq 4^j \, (j!)^2, \quad j \in \mathbb{N}. \tag{114}
$$

Hence, by (110)–(114), we get:

$$
\|\mathfrak{R}_{i_1}^{\mathbb{M}} \cdots \mathfrak{R}_{i_n}^{\mathbb{M}}\varphi\|_{L^2(\hat{X},\varkappa)} \leq C_{10}^n (k - l_{-1} + l_1 + 2l_2)! \, (k - l_{-1} + l_1 + 2l_2)^{2l_{-1}+l_0} \tag{115}
$$

for some constant $C_{10} > 0$ which is independent of l_{-1}, l_0, l_1, l_2, n. Since $j! \leq j^j$, we get from (115)

$$
\begin{aligned}
\|\mathfrak{R}_{i_1}^{\mathbb{M}} \cdots \mathfrak{R}_{i_n}^{\mathbb{M}}\varphi\|_{L^2(\hat{X},\varkappa)} &\leq C_{10}^n (k - l_{-1} + l_1 + 2l_2)^{k-l_{-1}+l_1+2l_2+2l_{-1}+l_0}\\
&= C_{10}^n (k - l_{-1} + l_1 + 2l_2)^{k+l_{-1}+l_0+l_1+2l_2}\\
&\leq C_{10}^n (k+2n)^{k+2n}.
\end{aligned}
$$

Therefore,

$$
\|(\mathfrak{R}^{\mathbb{M}})^n\varphi\|_{L^2_{\mathbb{C}}(\hat{X},\varkappa)} \leq (4C_{10})^n (k+2n)^{k+2n}.
$$

From here (109) follows. □

Let us recall the notion of a second quantization in a symmetric Fock space. Let $\mathcal{H}$ be a real separable Hilbert space, and let $\mathcal{F}(\mathcal{H})$ be the symmetric Fock space over $\mathcal{H}$. Let B be a bounded linear operator in $\mathcal{H}$, and assume that the operator norm of B is ≤ 1. We define the second quantization of B as a bounded linear operator $\operatorname{Exp}(B)$ in $\mathcal{H}$ which satisfies $\operatorname{Exp}(B)\Psi := \Psi$ (Ψ being the vacuum vector in $\mathcal{F}(\mathcal{H})$) and for each $n \in \mathbb{N}$, the restriction of $\operatorname{Exp}(B)$ to $\mathcal{F}^{(n)}(\mathcal{H})$ coincides with $B^{\otimes n}$.

Let the conditions of Theorem 23 be satisfied. For $\sharp = \mathrm{int}, \mathrm{ext}, \mathbb{M}$, recall the non-positive self-adjoint operator $(\mathfrak{L}^\sharp, D(\mathfrak{L}^\sharp))$ in $L^2(\hat{X},\varkappa)$. By Lemma 24, this operator is essentially self-adjoint on $\mathcal{D}(\hat{X})$ and, for each $\varphi \in \mathcal{D}(\hat{X})$, $\mathfrak{L}^\sharp\varphi$ is given by (91). Recall the unitary operator I in formula (86). In view of the bijective mapping $\mathscr{R}: \Gamma_{pf}(\hat{X}) \to \mathbb{K}(X)$, we can equivalently treat the operator I as a unitary operator

$$I: L^2(\mathbb{K}(X), \mathcal{G}) \to \mathcal{F}(L^2(\hat{X}, \varkappa)) \tag{116}$$

(recall that the Poisson measure π is concentrated on $\Gamma_{pf}(\hat{X})$).

Corollary 25. *Let the conditions of Theorem 23 be satisfied. Then, for* $\sharp = \mathrm{int}, \mathrm{ext}, \mathbb{M}$, *we have*

$$Ie^{tL^\sharp}I^{-1} = \mathrm{Exp}(e^{t\mathfrak{L}^\sharp}), \quad t \ge 0,$$

i.e., under the unitary isomorphism (116), *the semigroup* $(e^{tL^\sharp})_{t\ge0}$ *with generator* $(L^\sharp, D(L^\sharp))$ *goes over into the semigroup* $(\mathrm{Exp}(e^{t\mathfrak{L}^\sharp}))_{t\ge0}$ *— the second quantization of the semigroup* $(e^{t\mathfrak{L}^\sharp})_{t\ge0}$ *with generator* $(\mathfrak{L}^\sharp, D(\mathfrak{L}^\sharp))$.

Proof. It follows from the proof of Theorem 23 that

$$IL^\sharp I^{-1}f = d\,\mathrm{Exp}(\mathfrak{L}^\sharp)f, \quad f \in \mathcal{F}_{\mathrm{alg}}(\mathcal{D}(\hat{X})),$$

and the operator $d\,\mathrm{Exp}(\mathfrak{L}^\sharp)$ is essentially self-adjoint on $\mathcal{F}_{\mathrm{alg}}(\mathcal{D}(\hat{X}))$. From here the result immediately follows (cf. e.g. [4, Chap. 6, subsec. 1.1]). □

Remark 26. Consider the operator $(\mathfrak{L}^{\mathrm{ext}}, D(\mathfrak{L}^{\mathrm{ext}}))$. We define the linear operator

$$\mathfrak{L}^{\mathrm{ext}}_{\mathbb{R}_+^*}u(s) := s\left(\frac{\partial^2}{\partial s^2} - \frac{\partial}{\partial s}\right)u(s), \quad u \in C_0^\infty(\mathbb{R}_+^*).$$

It follows from the proof of Lemma 24 that this operator is essentially self-adjoint on $L^2(\mathbb{R}_+^*, \lambda)$, and we denote by $(\mathfrak{L}^{\mathrm{ext}}_{\mathbb{R}_+^*}, D(\mathfrak{L}^{\mathrm{ext}}_{\mathbb{R}_+^*}))$ the closure of this operator. Recall that $L^2(\hat{X}, \varkappa) = L^2(X, dx) \otimes L^2(\mathbb{R}_+^*, \lambda)$. Using (91), it is easy to show that

$$\mathfrak{L}^{\mathrm{ext}} = \mathbf{1} \otimes \mathfrak{L}^{\mathrm{ext}}_{\mathbb{R}_+^*}.$$

Using e.g. [27, Chap. XI], we easily conclude that $(\mathfrak{L}^{\mathrm{ext}}_{\mathbb{R}_+^*}, D(\mathfrak{L}^{\mathrm{ext}}_{\mathbb{R}_+^*}))$ is the generator of the Markov process $Y(t)$ on $\mathbb{R}_+ = [0, \infty)$ given by the following space-time transformation of the square of the 0-dimensional Bessel process $Q(t)$:

$$Y(t) = e^{-2t}Q((e^{2t} - 1)/2).$$

Note that, for each starting point $s > 0$, the process $Y(t)$ is at 0 (so that it has exited $\mathbb{R}_+^*$) with probability $\exp(-s/(1 - e^{-t}))$, and once $Y(t)$ reaches zero it stays there forever (i.e., does not return to $\mathbb{R}_+^*$).

Acknowledgments

The authors acknowledge the financial support of the SFB 701 "Spectral structures and topological methods in mathematics" (Bielefeld University) and the Research Group "Stochastic Dynamics: Mathematical Theory and Applications" (Center for Interdisciplinary Research, Bielefeld University).

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