

Stochastic quasi-geostrophic equation ^{*}

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Abstract

In this note we study the 2d stochastic quasi-geostrophic equation in \mathbb{T}^2 for general parameter $\alpha \in (0, 1)$ and multiplicative noise. We prove the existence of martingale solutions and pathwise uniqueness under some condition in the general case, i.e. for all $\alpha \in (0, 1)$. In the subcritical case $\alpha > 1/2$, we prove existence and uniqueness of (probabilistically) strong solutions and construct a Markov family of solutions. In particular, it is uniquely ergodic for $\alpha > \frac{2}{3}$ provided the noise is non-degenerate. In this case, the convergence to the (unique) invariant measure is exponentially fast. In the general case, we prove the existence of Markov selections.

1. Introduction and notation—Consider the following two dimensional (2D) stochastic quasi-geostrophic equation in the periodic domain $\mathbb{T}^2 = \mathbb{R}^2 / (2\pi\mathbb{Z})^2$:

$$\frac{\partial \theta(t, x)}{\partial t} = -u(t, x) \cdot \nabla \theta(t, x) - \kappa(-\Delta)^\alpha \theta(t, x) + G(\theta, \xi)(t, x), \quad (1.1)$$

with initial condition $\theta(0, x) = \theta_0(x)$, where $\theta(t, x)$ is a real-valued function of x and t , $0 < \alpha < 1$, $\kappa > 0$ are real numbers. u is determined by θ through a stream function ψ via the following relations:

$$u = (u_1, u_2) = (-R_2\theta, R_1\theta). \quad (1.2)$$

Here R_j is the j -th periodic Riesz transform and $\xi(t, x)$ is a Gaussian random field, white noise in time, subject to the restrictions imposed below. The variable θ represents the potential temperature, and u is the fluid velocity. The case $\alpha = \frac{1}{2}$ is called the critical case, the case $\alpha > \frac{1}{2}$ sub-critical and the case $\alpha < \frac{1}{2}$ super-critical. This equation is important model in geophysical fluid dynamics. The case $\alpha = 1/2$ exhibits similar features (singularities) as the 3D Navier-Stokes equations and can therefore serve as a model case for the latter. In the deterministic case this equation has been intensively investigated because of both its mathematical importance and its background in geophysical fluid dynamics. The existence of weak solutions in the deterministic case has been obtained in [7]. In the following, we will restrict ourselves to flows which have zero average on the torus, i.e. $\int_{\mathbb{T}^2} \theta dx = 0$. Set $H = L^2(\mathbb{T}^2)$ and let $|\cdot|$ and $\langle \cdot, \cdot \rangle$ denote the norm and inner product in H , respectively. We recall that on \mathbb{T}^2 , $\sin(k \cdot x)$, $\cos(k \cdot x)$ form an eigenbasis of $-\Delta$. Here $k \in \mathbb{Z}^2 \setminus \{0\}$, $x \in \mathbb{T}^2$ and the corresponding eigenvalues are $|k|^2$. Define $\|f\|_{H^s}^2 := \sum_k |k|^{2s} \langle f, e_k \rangle^2$ and let H^s denote the Sobolev space of all

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f for which $\|f\|_{H^s}$ is finite. Set $\Lambda = (-\Delta)^{1/2}$. Define the linear operator $A : D(A) \subset H \rightarrow H$ as $Au = \kappa(-\Delta)^\alpha u$. The operator A is positive definite and selfadjoint. Denote the eigenvalues of A by $0 < \lambda_1 \leq \lambda_2 \leq \dots$, and by e_1, e_2, \dots the corresponding complete orthonormal system in H of eigenvectors of A . We also denote $\|u\| = |A^{1/2}u|$, then $\|\theta\|^2 \geq \lambda_1|\theta|^2$.

2. Existence and uniqueness of solutions —By the above definitions Eqs (1.1)-(1.2) turn into the abstract stochastic evolution equation

$$\begin{cases} d\theta(t) + A\theta(t)dt + u(t) \cdot \nabla\theta(t)dt = G(\theta(t))dW(t), \\ \theta(0) = x, \end{cases} \quad (2.1)$$

where u satisfies (1.2) and $W(t)$ is a cylindrical Wiener process in a separable Hilbert space K defined on a probability space (Ω, \mathcal{F}, P) . Here G is a mapping from H^α to $L^2(K, H)$.

Definition 2.1 We say that there exists a martingale solution to (2.1) if there exists a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$, a cylindrical Wiener process W on the space K and a progressively measurable process $\theta : [0, T] \times \Omega \rightarrow H$, such that for P -a.e $\omega \in \Omega$

$$\theta(\cdot, \omega) \in L^\infty(0, T; H) \cap L^2(0, T; H^\alpha) \cap C([0, T]; H_w) \quad (2.2)$$

and P -a.s.

$$\langle \theta(t), \psi \rangle + \int_0^t \langle A^{1/2}\theta(s), A^{1/2}\psi \rangle ds - \int_0^t \langle u(s) \cdot \nabla\psi, \theta(s) \rangle ds = \langle x, \psi \rangle + \langle \int_0^t G(\theta(s))dW(s), \psi \rangle, \quad (2.3)$$

for all $t \in [0, T]$ and all $\psi \in C^1(\mathbb{T}^2)$. Here $C([0, T]; H_w)$ denotes the space of H -valued weakly continuous functions on $[0, T]$.

Remark 2.2 Note that for regular functions θ and v , we have $\langle u(s) \cdot \nabla(\theta(s) + \psi), \theta(s) + \psi \rangle = 0$, so $\langle u(s) \cdot \nabla\theta(s), \psi \rangle = -\langle u(s) \cdot \nabla\psi, \theta(s) \rangle$. Thus the integral equation (2.3) corresponds to equation (2.1).

Definition 2.3 We say that there exists a (probabilistically strong) solution to (2.1) over the time interval $[0, T]$ if for every probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$ with an \mathcal{F}_t -Wiener process W , there exists a progressively measurable process $\theta : [0, T] \times \Omega \rightarrow H$ such that (2.2) and (2.3) hold.

2.1. The general case—Consider the following condition:

(G.1) $G : H \rightarrow L_2(K, H)$ is continuous and $|G(\theta)|_{L_2(K, H)}^2 \leq \lambda_0|\theta|^2 + \rho, \theta \in H$ for some positive real numbers λ_0 and ρ .

By the compactness method based on fractional Sobolev spaces in [3], we obtain the existence of martingale solutions.

Theorem 2.1.1 Let $\alpha \in (0, 1)$. Under Assumption (G.1), there exists a martingale solution $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P, W, \theta)$ to (2.1).

Let $f_n, n \in \mathbb{N}$, be an ONB of K .

Theorem 2.1.2 Let $\alpha \in (0, 1)$. If $G \in L_2(K, H)$ satisfies (G.1) and also the following conditions: for all $\theta \in H^\alpha \cap L^p(\mathbb{T}^2)$,

$$\int (\sum_j |G(\theta)(f_j)|^2)^{p/2} d\xi \leq C(\int |\theta|^p d\xi + 1), \forall t > 0, \quad (2.4)$$

with $2 < p < \infty$ for some constant $C := C(p) > 0$ and for all $\theta_1, \theta_2 \in H^\alpha \cap L^p(\mathbb{T}^2)$,

$$\int (\sum_j |(G(\theta_1) - G(\theta_2))(f_j)|^2)^{p/2} d\xi \leq C \int |\theta_1 - \theta_2|^p d\xi, \quad (2.5)$$

then there exists a martingale solution $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P, W, \theta)$ to (2.1). Moreover, if $\theta_0 \in L^p(\mathbb{T}^2)$ with $p > 2$, then $E \sup_{t \in [0, T]} \|\theta(t)\|_{L^p}^p < \infty$.

Theorem 2.1.3 Let $\alpha \in (0, 1)$. If G satisfies the Lipschitz condition

$$\|G(u) - G(v)\|_{L_2(K, H)}^2 \leq \beta|u - v|^2 + \beta_1\|u - v\|_{H^\alpha}^2$$

for all $u, v \in H^\alpha$, for some $\beta \in \mathbb{R}$ independent of u, v , and $\beta_1 < 2\kappa$, then (2.1) admits at most one solution in the sense of Definition 2.3 such that $E \sup_{t \in [0, T]} |\theta(t)|^4 < \infty$ and

$$\int_0^T \|\Lambda^{1-\alpha+\varepsilon}\theta(t)\|_{L^p}^q dt < \infty, \quad \frac{1}{p} + \frac{\alpha}{q} = \frac{\alpha + \varepsilon}{2} \quad P - a.s.,$$

where $\varepsilon \in (0, \alpha]$ and $q < \infty$.

2.2. The subcritical case—In this section, we will consider the subcritical case.

Theorem 2.2.1 Assume $\alpha > \frac{1}{2}$. If G satisfies the following condition

$$\|\Lambda^{-1/2}(G(u) - G(v))\|_{L_2(K, H)}^2 \leq \beta|\Lambda^{-1/2}(u - v)|^2 + \beta_1|\Lambda^{\alpha-\frac{1}{2}}(u - v)|^2, \quad (2.6)$$

for all $u, v \in H^\alpha$, for some $\beta \in \mathbb{R}$ independent of u, v , and $\beta_1 < 2\kappa$, then (2.1) admits at most one probabilistically strong solution in the sense of Definition 2.3 such that

$$\sup_{t \in [0, T]} \|\theta(t)\|_{L^q} < \infty, \quad P - a.s.,$$

with $0 < 1/q < \alpha - \frac{1}{2}$, and $E \sup_{t \in [0, T]} |\Lambda^{-1/2}\theta(t)|^2 < \infty$.

Corollary 2.2.2 Assume $\alpha > \frac{1}{2}$. If there exists a probabilistically strong solution θ_2 in the sense of Definition 2.3 such that

$$\sup_{t \in [0, T]} \|\theta_2(t)\|_{L^q} < \infty, \quad P - a.s.$$

for some q with $0 < 1/q < \alpha - \frac{1}{2}$ and G satisfies (2.6), then θ_2 is the only solution to (2.1) such that $E \sup_{t \in [0, T]} |\Lambda^{-1/2}\theta_2(t)|^2 < \infty$.

Theorem 2.2.3 Assume $\alpha > \frac{1}{2}$ and that G satisfies (2.6), (G.1), (2.4) and (2.5) with $0 < 1/p < \alpha - \frac{1}{2}$. Then for each initial condition $\theta_0 \in L^p$, there exists a pathwise unique probabilistically strong solution θ of equation (2.1) over $[0, T]$ with initial condition $\theta(0) = \theta_0$ such that $E \sup_{t \in [0, T]} |\Lambda^{-1/2}\theta(t)|^2 < \infty$. Moreover, the solution satisfies $\sup_{t \in [0, T]} \|\theta(t)\|_{L^p} < \infty$, $P - a.s.$

Theorem 2.2.4 (Markov property) Assume $\alpha > \frac{1}{2}$ and that G satisfies (G.1), (2.6) and (2.4), (2.5) with $0 < 1/p < \alpha - \frac{1}{2}$. If $\theta_0 \in L^p$, then for every bounded, $\mathcal{B}(H)$ -measurable $F : H \rightarrow \mathbb{R}$, and all $s, t \in [0, T]$, $s \leq t$

$$E(F(\theta(t)) | \mathcal{F}_s)(\omega) = E(F(\theta(t, s, \theta(s)(\omega)))) \text{ for } P - a.s. \omega \in \Omega.$$

Here $\theta(t, s, \theta(s)(\omega))$ denotes the solution to (3.1) starting from $\theta(s)$ at time s satisfying

$$E \sup_{t \in [s, T]} |\Lambda^{-1/2}\theta(t)|^2 < \infty.$$

Set $p_t(x, dy) := P \circ (\theta(t, x))^{-1}(dy)$, $0 \leq t \leq T$, $x \in H$, and for $\mathcal{B}(H)$ -measurable $F : H \rightarrow \mathbb{R}$, and $t \in [0, T]$, $x \in H$, $P_t F(x) := \int F(y) p_t(x, dy)$, provided F is $p_t(x, dy)$ -integrable. Then by Theorem 2.2.4, we have for $F : H \rightarrow \mathbb{R}$, bounded and $\mathcal{B}(H)$ -measurable, $s, t \geq 0$, $P_s(P_t F)(x) = P_{s+t} F(x)$, $x \in L^p$ for some p with $0 \leq 1/p < \alpha - \frac{1}{2}$.

3. Ergodicity and Exponential convergence for $\alpha > \frac{2}{3}$

Assumption 3.1 There are an isomorphism Q_0 of H and a number $s \geq 1$ such that $G = A^{-\frac{s+\alpha}{2\alpha}} Q_0^{1/2}$, and furthermore, G satisfies (2.4) for some fixed $p \in ((\alpha - \frac{1}{2})^{-1}, \infty)$ and $f_j = e_j$, (which is e.g. always the case if $Q_0 = I$).

Set $\mathcal{W} = D(\Lambda^s)$ and $|x|_{\mathcal{W}} = |\Lambda^s x|$. Then by a similar method as in [4] and using the abstract results in [5] for exponential convergence, we obtain the following results.

Theorem 3.2 Assume $\alpha > \frac{2}{3}$ and Assumption 3.1. Then there exists a unique invariant measure ν on \mathcal{W} for the transition semigroup $(P_t)_{t \geq 0}$. Moreover:

- (i) The invariant measure ν is ergodic in the sense of [2].
- (ii) The transition semigroup $(P_t)_{t \geq 0}$ is \mathcal{W} -strong Feller, irreducible, and therefore strongly mixing in the sense of [2].
- (iii) Moreover, we additionally assume that $s > 3 - 2\alpha$. Then there is $C_{\text{exp}} > 0$ and $a > 0$ such that $\|P_t^* \delta_{x_0} - \mu\|_{TV} \leq \|P_t^* \delta_{x_0} - \mu\|_V \leq C_{\text{exp}}(1 + \|x_0\|_{L^q}^q) e^{-at}$, for all $t > 0$ and $x_0 \in \mathcal{W}$, where $\|\cdot\|_{TV}$ is the total variation distance for measures.

4. Markov Selections in the general case—By using the abstract results for Markov selections in [6], we obtain the following results.

Theorem 4.1 Assume G satisfies (G.1). Then there exists an almost sure Markov family $(P_{x_0})_{x_0 \in H}$ for Eq. (2.1).

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