ASYMPTOTIC BEHAVIOR OF LARGE EIGENVALUES OF A MODIFIED JAYNES–CUMMINGS MODEL

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Abstract. We consider a class of unbounded self-adjoint operators with discrete spectrum obtained as a modification of the Hamiltonian of the Jaynes–Cummings model without rotating-wave approximation (RWA). The corresponding operators are defined by infinite Jacobi matrices and the purpose of this paper is to investigate the asymptotic behavior of large eigenvalues.

1. Introduction

1.1. Main result. We consider an infinite real Jacobi matrix

$$\begin{pmatrix} d(1) & a(1) & 0 & 0 & 0 & \dots \\ a(1) & d(2) & a(2) & 0 & 0 & \dots \\ 0 & a(2) & d(3) & a(3) & 0 & \dots \\ 0 & 0 & a(3) & d(4) & a(4) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.1}$$

where the form of entries $\{d(k)\}_{k=1}^{\infty}$, $\{a(k)\}_{k=1}^{\infty}$ is motivated by the structure of the Hamiltonian of the Jaynes–Cummings model without rotating-wave approximation (RWA). Following È. A. Tur [8] this model can be represented by the Jacobi matrix (1.1) with

$$\begin{cases} d(k) = k + c_0(-1)^k, \\ a(k) = c_1 k^{1/2} \end{cases} \tag{1.2}$$

where $c_0 \in \mathbb{R}$, $c_1 > 0$ are some constants.

Here we consider a "modified Jaynes–Cummings model", which means a Jacobi matrix (1.1) with

$$\begin{cases} d(k) = k^{\alpha} + v(k), \\ a(k) = c_1 k^{\gamma} \end{cases} \tag{1.3}$$

where $\alpha > \gamma > 0$, $c_1 > 0$ are some constants and v is real-valued, periodic of period $N \geq 1$, i.e., for any $k \in \mathbb{N}^* = \{1, 2, \dots\}$,

$$v(k+N) = v(k). \tag{1.4}$$

Let $l^2 = l^2(\mathbb{N}^*)$ denote the Hilbert space of square-summable complex sequences $x\colon \mathbb{N}^* \to \mathbb{C}$ equipped with the scalar product $\langle x, y\rangle := \sum_{k=1}^{\infty} \overline{x(k)}y(k)$ and with the

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norm $\|x\|_{l^2} := \big(\sum_{k=1}^{\infty} |x(k)|^2\big)^{1/2} < \infty$. We denote

$$\mathcal{D} := \Big\{x \in l^2 : \sum_{k=1}^{\infty} d(k)^2 |x(k)|^2 < \infty\Big\} \tag{1.5}$$

and define $J\colon \mathcal{D} \to l^2$ by the formula

$$(Jx)(k) = d(k)x(k) + a(k)x(k+1) + a(k-1)x(k-1) \tag{1.6}$$

where, by convention, $x(0) = 0$ and $a(0) = 0$. Then J is self-adjoint with compact resolvent and there exists an orthonormal basis $\{v_n\}_{n=1}^{\infty}$ such that $Jv_n = \lambda_n(J)v_n$ where $\{\lambda_n(J)\}_{n=1}^{\infty}$ is the non-decreasing sequence of real eigenvalues, i.e.,

$$\lambda_1(J) \le \cdots \le \lambda_n(J) \le \lambda_{n+1}(J) \le \dots .$$

In this paper we consider the above "modified Jaynes–Cummings model" with $\alpha = 1$ and prove the following

Theorem 1.1. *Let J be the self-adjoint operator defined in $l^2(\mathbb{N}^*)$ by (1.6) where*

$$\begin{cases} d(k) = k + v(k), \\ a(k) = c_1 k^{\gamma}. \end{cases} \tag{1.7}$$

Assume v is real-valued, periodic of period $N \ge 1$, $c_1 > 0$, $0 < \gamma < 1$ and denote

$$\langle v\rangle := \frac{1}{N} \sum_{1 \le k \le N} v(k), \tag{1.8}$$

$$\rho_N := \max_{1 \le k \le N} |v(k) - \langle v\rangle|. \tag{1.9}$$

If $\lambda_n(J)$ denotes the n-th eigenvalue of J, we have the large n asymptotic formula

$$\lambda_n(J) = n + \langle v\rangle + \mathrm{O}\big(n^{-\gamma/2} \ln n + n^{2\gamma-1}\big), \tag{1.10}$$

provided ρ_N is small enough.

In Section 1.2 we discuss the place of Theorem 1.1 among other known results and we precise the assumption concerning ρ_N. In Section 1.3 we state Theorem 1.2 which is a slight generalization of Theorem 1.1. In Section 2 we outline the main steps of the proof.

1.2. **Discussion.**

1.2.1. $\alpha - \gamma > 1$ *or not.* Concerning the asymptotic behavior of $\lambda_n(J)$ for the modified Jaynes–Cummings model, i.e., for J given by (1.6), (1.7) we observe that the analysis strongly depends on whether $\alpha - \gamma > 1$ or not. In fact except [2] all results known up to now concern the easy case $\alpha - \gamma > 1$ when it is possible to apply approximation methods based on an idea of successive diagonalizations described in [1]— see also [6].

The main purpose of this paper is to exhibit a radical change of the asymptotic behavior of $\lambda_n(J)$ in the case when $\alpha = 1$ and $0 < \gamma < \frac{1}{2}$. The new phenomenon consists in the absence of a periodic modulation of large eigenvalues. The case of the Jaynes–Cummings model, i.e., $\alpha = 1$ and $\gamma = \frac{1}{2}$ is more complicated to analyze, but a similar phenomenon holds.

1.2.2. *Known estimates.* Let us discuss the nature of known asymptotic estimates for eigenvalues of the modified Jaynes–Cummings model. First of all we cite the paper of A. Boutet de Monvel, S. Naboko, L.O. Silva [1] treating the case $\alpha = 2$ and $\gamma = \frac{1}{2}$. This work ensures the large n asymptotic estimate

$$\lambda_n(J) = n^2 + v(n) + \mathrm{O}(n^{-1}). \tag{1.11}$$

Then the works of M. Malejki [7] and A. Boutet de Monvel, L. Zielinski [3] ensure the large n asymptotic estimate

$$\lambda_n(J) = n^\alpha + v(n) + \mathrm{O}(n^{\gamma-2\kappa} + n^{2\gamma-\alpha}) \tag{1.12}$$

where $\kappa \coloneqq \alpha - 1 - \gamma > 0$. We observe that under the additional conditions $\alpha \le 2$ and $\gamma < \frac{2}{3}(\alpha - 1)$ we have $\alpha - 2\gamma > 0$ and $2\kappa - \gamma = 2(\alpha - 1) - 3\gamma > 0$, hence we obtain the large n asymptotic behavior of the difference

$$\lambda_n(J) - n^\alpha = v(n) + \mathrm{o}(1), \tag{1.13}$$

reflecting the oscillations determined by the periodic nature of v. Consequently in the case when v is not constant and $\alpha = 1$ the asymptotic behavior of $\lambda_n(J) - n$ given by (1.13) is quite different from the assertion of Theorem 1.1 ensuring $\lambda_n(J) - n \to \langle v \rangle$ as $n \to \infty$.

1.2.3. *Case* $0 < \gamma < \alpha = 1$. This is the case considered in Theorem 1.1. First of all we observe that for $v(k) = \text{const}$ the result of [2] ensures the large n behavior

$$\lambda_n(J) = n + \langle v \rangle + \mathrm{O}(n^{2\gamma-1}) \tag{1.14}$$

and $\mathrm{O}(n^{2\gamma-1})$ cannot be replaced by $\mathrm{o}(n^{2\gamma-1})$. Moreover in the case $\frac{1}{2} \le \gamma < 1$, Theorem 1.1 follows easily from [2]. Indeed, if a bounded sequence $\{v(k)\}_{k=1}^\infty$ is replaced by 0 then the error term is of order $\mathrm{O}(1)$ due to the min-max principle and it can be included in the remainder $\mathrm{O}(n^{2\gamma-1})$ since $\frac{1}{2} \le \gamma < 1$. Thus Theorem 1.1 is new only when $\gamma < \frac{1}{2}$.

1.2.4. *Assumption on* γ. From now one we always assume

$$\gamma \le \frac{1}{2}.$$

This assumption is sufficient to prove some partial results (Propositions 2.1, 2.2 and 2.5), but other (Propositions 2.4 and 2.6) are proved under the stronger assumption $\gamma < \frac{1}{2}$. The case $\gamma = \frac{1}{2}$ should require a more involved analysis.

1.2.5. *Assumption on* ρ_N. In Theorem 1.1 we made a purely qualitative assumption on ρ_N claiming that (1.10) holds provided ρ_N is "small enough". Now we give quantitative assumptions. For $N = 2$ it suffices to take

$$\rho_2 < \frac{1}{2}.$$

For $N \ge 3$ we consider the $N \times N$ Vandermonde matrix $M \coloneqq \left(\mathrm{e}^{2\pi \mathrm{i} jk/N}\right)_{j,k=0}^{N-1}$ which is invertible since $\det M \ne 0$ and we denote

$$\|M^{-1}\| \coloneqq \sup_{\substack{w \in \mathbb{C}^N \\ |w|=1}} |M^{-1}w|, \tag{1.15}$$

where $|w| = (|w_1|^2 + \cdots + |w_N|^2)^{1/2}$ for $w = (w_1, \ldots, w_N) \in \mathbb{C}^N$. We prove that (1.10) holds if

$$\rho_N < \min\Big\{\frac{1}{2}, \frac{1}{\pi N \|M^{-1}\|}\Big\}.$$

1.3. **A generalization.** We assume that the entries $\{d(k)\}_{k=1}^{\infty}$, $\{a(k)\}_{k=1}^{\infty}$ are real,

$$d(k) = k + v(k), \tag{1.16}$$

where v is periodic of period $N \geq 1$ and

$$a(k) \xrightarrow[k\to\infty]{} \infty. \tag{1.17}$$

Theorem 1.1 is a simple application of the following more general result.

Theorem 1.2. *Let J be the self-adjoint operator defined in $l^2(\mathbb{N}^*)$ by* (1.6) *with real entries $\{d(k)\}_{k\geq 1}$, $\{a(k)\}_{k\geq 1}$ such that*

(i) *$a(k) \to \infty$ as $k \to \infty$,*
(ii) *there exist real constants $0 < \gamma < \frac{1}{2}$, $\gamma_1 < 1$, $C_0 > 0$ such that for any $k \geq 1$*

$$0 < a(k) \leq C_0 k^{\gamma}, \tag{1.18}$$

$$|a(k+1) - a(k)| \leq C_0 k^{-\gamma_1}, \tag{1.19}$$

(iii) *$d(k) = k + v(k)$ with v real-valued of period $N \geq 1$.*

Let ρ_N be given by (1.9)*. We also assume*

(iv) *$\rho_N < \frac{1}{2}$,*
(v) *$\rho_N < 1/(\pi N \|M^{-1}\|)$ if $N \geq 3$.*

We have then the large n estimate

$$\lambda_n(J) = n + \langle v \rangle + \mathrm{O}\big(a(n)^{-1/2} \ln n \, + n^{\gamma - \gamma_1}\big). \tag{1.20}$$

Proof scheme. It is based on Propositions all stated in Section 2, as follows:

$$\left.\begin{matrix}\text{Proposition 2.5}\\ \text{Proposition 2.6}\end{matrix}\right\} \Longrightarrow \text{Theorem 1.2.}$$

See end of Section 2.5. Proposition 2.6 uses [4] to compare eigenvalues of two Jacobi matrices. Proposition 2.5 derives from trace formulas:

$$\left.\begin{matrix}\text{Proposition 2.1}\\ \text{Lemma 2.3}\end{matrix}\right\} \Longrightarrow \left.\begin{matrix}\text{Proposition 2.4}\\ \text{Proposition 2.2}\end{matrix}\right\} \Longrightarrow \text{Proposition 2.5.}$$

Proof of Theorem 1.1. Theorem 1.2 applies with $a(k) = c_1 k^{\gamma}$, $\gamma < \frac{1}{2}$ and $\gamma_1 = 1 - \gamma$. For these data the asymptotic formula (1.20) takes the form given in (1.10). □

2. Outline

2.1. **Contents.** Our approach uses special properties of auxiliary operators J_n and J_n' acting in $\mathcal{H} := l^2(\mathbb{Z})$. They are presented in Sections 2.3 and 2.4, respectively.

In Sections 3 and 4 we investigate the operators J_n with frozen off-diagonal entries. The simple structure of J_n allows us to establish a trace formula (Proposition 2.1).

In Section 5 we investigate operators J_n' which differ from J_n by an additional cut-off in the configuration space but a trace formula remains valid (Proposition 2.2).

In Section 6 we deduce spectral asymptotics for J_n' from the trace formula (Proposition 2.5). In Section 7 we compare the n-th eigenvalue of J by that of J_n' giving a large

n estimate of the difference (Proposition 2.6). We thus obtain the large n asymptotics of the n-th eigenvalue of J as claimed in Theorem 1.2.

In Section 8 we prove auxiliary results mainly used in Section 5.

2.2. **Notations.** Let $\mathcal{H} := l^2(\mathbb{Z})$ denote the Hilbert space of square-summable complex sequences $x\colon \mathbb{Z} \to \mathbb{C}$ whose scalar product is $\langle x, y\rangle = \sum_{k\in\mathbb{Z}} \overline{x(k)}y(k)$, with norm

$$\|x\|_{\mathcal{H}} := \Big(\sum_{k\in\mathbb{Z}} |x(k)|^2\Big)^{1/2} < \infty. \tag{2.1}$$

We denote by $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$ equipped with the operator norm $\|\cdot\|_{\mathcal{B}(\mathcal{H})}$. Let $\{\mathrm{e}_k\}_{k\in\mathbb{Z}}$ be the canonical basis of $\mathcal{H}$, i.e., $\mathrm{e}_k(k) = 1$ and $\mathrm{e}_k(j) = 0$ if $j \neq k$. We define the shift $S \in \mathcal{B}(\mathcal{H})$ by

$$S\mathrm{e}_k = \mathrm{e}_{k+1} \tag{2.2}$$

for any $k \in \mathbb{Z}$. We also consider the closed linear operator $\Lambda\colon \mathcal{H}_1 \to \mathcal{H}$ satisfying

$$\Lambda\mathrm{e}_k = k\mathrm{e}_k \tag{2.3}$$

for any $k \in \mathbb{Z}$, whose domain is

$$\mathcal{H}_1 = \Big\{x \in \mathcal{H} : \sum_{k\in\mathbb{Z}} k^2|x(k)|^2 < \infty\Big\}. \tag{2.4}$$

If $b\colon \mathbb{Z} \to \mathbb{C}$ then $b(\Lambda)$ denotes the closed linear operator satisfying $b(\Lambda)\mathrm{e}_k = b(k)\mathrm{e}_k$ for any $k \in \mathbb{Z}$. Further on we assume that $v\colon \mathbb{Z} \to \mathbb{R}$ is periodic of period $N \geq 1$, hence

$$\|v(\Lambda)\|_{\mathcal{B}(\mathcal{H})} = \sup_{j\in\mathbb{Z}}|v(j)| = \max_{1\leq j\leq N}|v(j)|.$$

Assumption on v. Since $\lambda_n(J + \mu) = \lambda_n(J) + \mu$ holds for any $\mu \in \mathbb{R}$, we may assume further on, without loss of generality, that

$$\langle v\rangle = 0. \tag{2.5}$$

Let ρ_N be as in (1.9). Under assumption (2.5) we find

$$\rho_N = \|v(\Lambda)\|_{\mathcal{B}(\mathcal{H})}. \tag{2.6}$$

2.3. **Operators J_n.**

2.3.1. *Spectrum of J_n.* For each $n \geq 1$ we define an operator $J_n\colon \mathcal{H}_1 \to \mathcal{H}$ by

$$J_n := J_n^0 + v(\Lambda) \tag{2.7}$$

where $J_n^0\colon \mathcal{H}_1 \to \mathcal{H}$ is given by

$$J_n^0 := \Lambda + a(n)(S + S^{-1}). \tag{2.8}$$

We first show (Lemma 3.1) that J_n^0 is unitary equivalent with Λ. Therefore its spectrum is $\sigma(J_n^0) = \sigma(\Lambda) = \mathbb{Z}$. Then, by an elementary perturbation argument using (2.6),

$$\sigma(J_n) \subset \bigcup_{k\in\mathbb{Z}} [k - \rho_N,\, k + \rho_N]. \tag{2.9}$$

Since all eigenvalues of Λ and J_n^0 are simple, the additional assumption $\rho_N < \frac{1}{2}$ ensures that all eigenvalues of J_n are simple as well.

2.3.2. *Trace formula for J_n.* Our key result is the following trace formula.

Proposition 2.1 (trace formula for J_n)**.** *Let J_n be as above, acting in $l^2(\mathbb{Z})$. Assume*
(i) $v\colon \mathbb{Z} \to \mathbb{R}$ *is periodic of period* $N \geq 1$,
(ii) $\langle v \rangle = 0$,
(iii) $\rho_N < \frac{1}{2}$,
(iv) $a(k) = \mathrm{O}(k^{1/2})$ *as* $k \to \infty$,
(v) $a(k) \to \infty$ *as* $k \to \infty$.

Let moreover $\chi\colon \mathbb{R} \to \mathbb{C}$ be such that

$$\chi(\lambda) = \int_{-\infty}^{\infty} \hat{\chi}(t) \mathrm{e}^{\mathrm{i}t\lambda} \mathrm{d}t \tag{2.10}$$

with $\hat{\chi} \in C_0^\infty(\mathbb{R})$ and denote, for $n \geq 1$,

$$\mathcal{G}_n^0 := \sum_{k \in \mathbb{Z}} \big(\chi\big(\lambda_k(J_n) - n\big) - \chi(k-n)\big). \tag{2.11}$$

We have then the large n estimate

$$\mathcal{G}_n^0 = \mathrm{O}\big(a(n)^{-1/2} \ln n\big). \tag{2.12}$$

Remark. The assumption $\hat{\chi} \in C_0^\infty(\mathbb{R})$ implies $\chi \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz class of rapidly decreasing functions on $\mathbb{R}$. Then $\hat{\chi}$ is the Fourier transform of χ:

$$\hat{\chi}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}t\lambda} \chi(\lambda) \, \frac{\mathrm{d}\lambda}{2\pi}\,. \tag{2.13}$$

Proof scheme. After initial steps described in Section 3 the proof is completed in Section 4, according to the following scheme:

$$\text{Lemma 4.1} \implies \text{Proposition 3.3} \implies \left.\begin{array}{r}\text{Proposition 3.2}\\ \text{Lemma 4.2}\end{array}\right\} \implies \text{Proposition 2.1.} \qquad \square$$

2.4. **Operators J_n'.**

2.4.1. *Spectrum of J_n'.* For each $n \geq 1$ we introduce an operator $J_n'\colon \mathcal{H}_1 \to \mathcal{H}$ which is intermediary between J_n and J, and defined by

$$J_n' = \Lambda + v(\Lambda) + a(n)\left(S\,\theta^+(\tfrac{\Lambda}{n}) + \theta^+(\tfrac{\Lambda}{n})\,S^{-1}\right) \tag{2.14}$$

where $\theta^+ \in C^\infty(\mathbb{R})$ is a cut-off such that

$$\theta^+(t) = \begin{cases} 1 & \text{if } t \geq \frac{1}{2}\,, \\ 0 & \text{if } t \leq \frac{1}{3}\,. \end{cases}$$

The reason of introducing these operators is that J_n' commutes with the projector Π_+ on the closed subspace $l^2(\mathbb{N}^*)$ generated by $\{\mathrm{e}_k\}_{k=1}^\infty$. Moreover we have

$$J_n' \mathrm{e}_k = \begin{cases} J_n^+ \mathrm{e}_k & \text{if } k \geq 1, \\ (k + v(k))\mathrm{e}_k & \text{if } k \leq 0, \end{cases} \tag{2.15}$$

where J_n^+ is a Jacobi operator from the class of operators investigated by P. A. Cojuhari, J. Janas [5], i.e., a self-adjoint bounded from below operator on $l^2(\mathbb{N}^*)$ with compact resolvent. Thus J_n^+ can be diagonalized in an orthonormal basis $\{v_{n,k}\}_{k=1}^\infty$, i.e., for any $k \geq 1$,

$$J_n^+ v_{n,k} = \lambda_k(J_n^+) v_{n,k}$$

and we find the spectrum $\sigma(J_n') = \{\lambda_k(J_n')\}_{k\in\mathbb{Z}}$ with

$$\lambda_k(J_n') = \begin{cases} \lambda_k(J_n^+) & \text{if } k \geq 1, \\ k + v(k) & \text{if } k \leq 0. \end{cases} \tag{2.16}$$

2.4.2. *Trace formula for J_n'.* We show that a trace formula still holds for J_n'.

Proposition 2.2 (trace formula for J_n')**.** *Let J_n' be as above, acting in $l^2(\mathbb{Z})$. Under assumptions and with notations of Proposition 2.1, we consider*

$$\mathcal{G}_n^+ := \sum_{k\in\mathbb{Z}} \big(\chi\big(\lambda_k(J_n') - n\big) - \chi(k-n)\big). \tag{2.17}$$

We have the large n behavior

$$\mathcal{G}_n^+ = \mathrm{O}\big(a(n)^{-1/2} \ln n\big). \tag{2.18}$$

To prove this behavior we use estimate (2.19) from

Lemma 2.3. *Let J_n, J_n' be as above and $\chi \in \mathcal{S}(\mathbb{R})$. Then*

$$\|\chi(J_n' - n) - \chi(J_n - n)\|_{\mathcal{B}_1(\mathcal{H})} = \mathrm{O}(n^{\gamma-1}). \tag{2.19}$$

Proof. See Section 8.3. □

Proof of Proposition 2.2. Consequence of Proposition 2.1 and Lemma 2.3. By (2.12),

$$\mathcal{G}_n^0 = \mathrm{O}\big(a(n)^{-1/2} \ln n\big).$$

By Lemma 2.3,

$$\mathcal{G}_n^+ - \mathcal{G}_n^0 = \mathrm{O}(n^{\gamma-1}). \tag{2.20}$$

Here $\gamma = \frac{1}{2}$ by our assumption $a(n) = \mathrm{O}(n^{1/2})$. Hence $\mathcal{G}_n^+ - \mathcal{G}_n^0 = \mathrm{O}(n^{-1/2})$. Moreover, $n^{-1/2}a(n)^{1/2} = \mathrm{O}(n^{-1/4})$, hence

$$n^{-1/2} = \mathrm{O}\big(a(n)^{-1/2}\big) = \mathrm{O}\big(a(n)^{-1/2} \ln n\big). \tag{2.21}$$

Thus, estimate (2.18) follows from (2.12) and (2.20). □

2.5. **Eigenvalue asymptotics.**

2.5.1. *Estimates of eigenvalues of J_n'.* Here is a better description of $\sigma(J_n')$ for large n.

Proposition 2.4 (spectrum of J_n')**.** *Let J_n' be as above, acting in $l^2(\mathbb{Z})$. Assume*

(i) *$v\colon \mathbb{Z} \to \mathbb{R}$ is periodic of period $N \geq 1$,*
(ii) *$\langle v \rangle = 0$,*
(iii) *$\rho_N < \frac{1}{2}$,*
(iv) *$a(k) = \mathrm{O}(k^\gamma)$ with $\gamma < \frac{1}{2}$.*

Let ε be such that $0 < \varepsilon < \frac{1}{2} - \rho_N$.

Then there is $n_\varepsilon \geq 1$ such that for any $n \geq n_\varepsilon$ the spectrum of J_n' is discrete, all eigenvalues of J_n' are simple and there is exactly one eigenvalue of J_n' in each interval $(k - \frac{1}{2}, k + \frac{1}{2}]$, $k \in \mathbb{Z}$, i.e.

$$\sigma(J_n') = \{\lambda_k(J_n')\}_{k\in\mathbb{Z}}$$

with, for each $k \in \mathbb{Z}$,

$$\sigma(J_n') \cap \big(k - \tfrac{1}{2}, k + \tfrac{1}{2}\big] = \{\lambda_k(J_n')\}. \tag{2.22}$$

Moreover for any $n \geq n_\varepsilon$ we have the estimates

$$\sup_{k\in\mathbb{Z}} |\lambda_k(J'_n) - k| \leq \rho_N + \varepsilon, \tag{2.23}$$

$$\sup_{k\in\mathbb{Z}} |\lambda_{k+N}(J'_n) - \lambda_k(J'_n) - N| = \mathrm{O}(n^{\gamma-1}). \tag{2.24}$$

Sketch of proof. We use the method of "approximative diagonalization" (see [2, 3]) in the case $v = 0$. The general case follows similarly as before, i.e., the control of the perturbed spectrum is ensured by the condition $\rho_N + \varepsilon < \frac{1}{2}$ for $n \geq n_\varepsilon$. The details are given in Sections 5.2 and 5.3. □

2.5.2. *Estimate of $\lambda_n(J'_n)$.* We show the following estimate of Tauberian nature.

Proposition 2.5. *Let J'_n be as above, acting in $l^2(\mathbb{Z})$. We assume:*

(i) *Estimate* (2.12) *holds if χ is given by* (2.10) *with $\hat{\chi} \in C_0^\infty(\mathbb{R})$.*
(ii) $\rho_N < \frac{1}{2}$.
(iii) $\rho_N < 1/(\pi N\|M^{-1}\|)$ *if* $N \geq 3$.
(iv) *For any $\varepsilon > 0$ one can find $n_\varepsilon \geq 1$ such that estimates* (2.23) *and* (2.24) *hold for any $n \geq n_\varepsilon$.*

Then we have the large n estimate

$$\lambda_n(J'_n) = n + \mathrm{O}\big(a(n)^{-1/2}\ln n\big). \tag{2.25}$$

Proof scheme. The proof is given in Section 6 according to the scheme:

$$\left.\begin{matrix}\text{Proposition 2.4}\\ \text{Proposition 2.2}\end{matrix}\right\} \implies \text{Proposition 2.5.}$$ □

2.5.3. *Estimate of $\lambda_n(J'_n) - \lambda_n(J)$.* In the last step we prove the following relation between eigenvalues of J'_n and J:

Proposition 2.6. *Let assume $a(k)$ satisfies* (1.18), (1.19) *with $0 < \gamma < \gamma_1 < 1$. Then we have the large n estimate*

$$\lambda_n(J) = \lambda_n(J'_n) + \mathrm{O}(n^{\gamma-\gamma_1}). \tag{2.26}$$

Sketch of proof. The proof is by comparison of eigenvalues of two Jacobi matrices using [4]. Details are given in Section 7. □

Proof of Theorem 1.2. Clearly follows from Propositions 2.6, 2.5 with 2.4, and 2.1:

$$\begin{aligned}\lambda_n(J) - n &= \lambda_n(J'_n) + \mathrm{O}(n^{\gamma-\gamma_1}) && \text{by (2.26)}\\ &= n + \mathrm{O}\big(a(n)^{-1/2}\ln n\big) + \mathrm{O}(n^{\gamma-\gamma_1}) && \text{by (2.25)}\end{aligned}$$

which is (1.20) when $\langle v\rangle = 0$. □

3. First considerations

3.1. **The first step.** The starting point of our analysis is the following simple result.

Lemma 3.1. *For every $t \in \mathbb{R}$ one has*

$$\mathrm{e}^{t(S-S^{-1})}(\Lambda + t(S+S^{-1}))\mathrm{e}^{-t(S-S^{-1})} = \Lambda. \tag{3.1}$$

Proof. Since (3.1) holds when $t = 0$, it suffices to check that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{e}^{t(S-S^{-1})}\,(\Lambda + t(S+S^{-1}))\mathrm{e}^{-t(S-S^{-1})}\Big) = 0 \tag{3.2}$$

holds for all $t \in \mathbb{R}$. However the left hand side of (3.2) has the form

$$\mathrm{e}^{t(S-S^{-1})}\,([S-S^{-1},\Lambda] + S + S^{-1})\mathrm{e}^{-t(S-S^{-1})}$$

and the direct computation of the commutator

$$[S^k,\Lambda]\mathrm{e}_n = S^k\Lambda\mathrm{e}_n - \Lambda S^k\mathrm{e}_n = S^k n\mathrm{e}_n - (n+k)\mathrm{e}_{n+k} = -kS^k\mathrm{e}_n$$

for $k \in \mathbb{Z}$ gives $[S-S^{-1},\Lambda] = -S - S^{-1}$, completing the proof of (3.1). ☐

3.2. **Reformulations.** We denote by Q^* the adjoint of $Q \in \mathcal{B}(\mathcal{H})$ and we write

$$\operatorname{Im} Q \coloneqq \frac{1}{2\mathrm{i}}(Q - Q^*).$$

If J_n^0 is given by (2.8), then Lemma 3.1 with $t = a(n)$ gives

$$\Lambda = \mathrm{e}^{2\mathrm{i}a(n)\operatorname{Im} S}\,J_n^0\,\mathrm{e}^{-2\mathrm{i}a(n)\operatorname{Im} S}. \tag{3.3}$$

Thus J_n^0 is unitary equivalent to Λ as claimed at the beginning of Section 2.3. We will use this fact in the following way. Instead of investigating directly the operators $J_n = J_n^0 + v(\Lambda)$ we will work with the operators

$$L_n \coloneqq \mathrm{e}^{2\mathrm{i}a(n)\operatorname{Im} S}\,J_n\,\mathrm{e}^{-2\mathrm{i}a(n)\operatorname{Im} S}. \tag{3.4}$$

We denote by $\mathcal{B}_1(\mathcal{H})$ the ideal of trace class operators on $\mathcal{H}$ with the norm

$$\|Q\|_{\mathcal{B}_1(\mathcal{H})} \coloneqq \operatorname{tr}(Q^*Q)^{1/2}.$$

Further on we assume that χ is given by (2.10) with $\hat{\chi} \in C_0^\infty((0,4\pi))$. Then $\chi \in \mathcal{S}(\mathbb{R})$ where $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space of rapidly decreasing functions on $\mathbb{R}$ and $\chi(L_n - n)$, $\chi(\Lambda - n) \in \mathcal{B}_1(\mathcal{H})$ (see Lemma 8.1 in Section 8). Since the eigenvalues of J_n coincide with eigenvalues of L_n, the quantity (2.11) can be expressed in the form

$$\mathcal{G}_n^0 = \operatorname{tr}\chi(L_n - n) - \operatorname{tr}\chi(\Lambda - n). \tag{3.5}$$

Before proving Proposition 2.1 we consider its modification

Proposition 3.2. *Let L_n be the operator defined by (3.4), acting in $l^2(\mathbb{Z})$. We assume*

(i) $\rho_N < \frac{1}{2}$,
(ii) $a(k) \to \infty$ *as* $k \to \infty$,
(iii) $a(k) = \mathrm{O}(k^{1/2})$.

Assume moreover that

(iv) χ *is given by* (2.10) *with* $\hat{\chi} \in C_0^\infty(\mathbb{R})$,
(v) $\theta \in C_0^\infty((\frac{1}{2}, 2))$ *is such that* $\theta(t) = 1$ *if* $\frac{3}{4} \le t \le \frac{3}{2}$.

If we denote

$$\mathcal{G}_n \coloneqq \operatorname{tr}\big(\theta(\Lambda/n)\chi(L_n - n)\big) - \operatorname{tr}\big(\theta(\Lambda/n)\chi(\Lambda - n)\big), \tag{3.6}$$

then we have the large n behavior

$$\mathcal{G}_n = \mathrm{O}\left(a(n)^{-1/2}\ln n\right). \tag{3.7}$$

3.3. **Proof of Proposition 3.2.**

Proof. (a) *First step.* Taking $L_n - n$ and $\Lambda - n$ instead of λ in (2.10) we find

$$\chi(L_n - n) - \chi(\Lambda - n) = \int_{-\infty}^{\infty} \hat{\chi}(t)\, \mathrm{e}^{-\mathrm{i}tn} \left(\mathrm{e}^{\mathrm{i}tL_n} - \mathrm{e}^{\mathrm{i}t\Lambda}\right) \mathrm{d}t. \tag{3.8}$$

Introducing

$$U_n(t) \coloneqq \mathrm{e}^{-\mathrm{i}t\Lambda}\mathrm{e}^{\mathrm{i}tL_n} \tag{3.9}$$

we can express the right-hand side of (3.6) in the form

$$\mathcal{G}_n = \int_{-\infty}^{\infty} \hat{\chi}(t)\ \mathrm{tr}\Big(\theta(\Lambda/n)\mathrm{e}^{\mathrm{i}t(\Lambda-n)}(U_n(t) - I)\Big)\, \mathrm{d}t. \tag{3.10}$$

Since $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} U_n(t) = \mathrm{e}^{-\mathrm{i}t\Lambda}(L_n - \Lambda)\mathrm{e}^{\mathrm{i}tL_n}$ we have

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} U_n(t) = H_n(t)U_n(t), \quad U_n(0) = I \tag{3.11}$$

with

$$H_n(t) \coloneqq \mathrm{e}^{-\mathrm{i}t\Lambda}(L_n - \Lambda)\mathrm{e}^{\mathrm{i}t\Lambda}. \tag{3.12}$$

Then we introduce the operators

$$h_n \coloneqq -\theta(\Lambda/n)\big(\Lambda - n - \tfrac{1}{2}\big)^{-1} \tag{3.13}$$

which allow us to write

$$h_n\, \mathrm{i}\, \frac{\mathrm{d}}{\mathrm{d}t}\, \mathrm{e}^{\mathrm{i}t(\Lambda-n-1/2)} = \theta(\Lambda/n)\, \mathrm{e}^{\mathrm{i}t(\Lambda-n-1/2)} \tag{3.14}$$

and using (3.14) in (3.10) we find

$$\mathcal{G}_n = \int_{-\infty}^{\infty} \hat{\chi}(t)\mathrm{e}^{\mathrm{i}t/2}\ \mathrm{tr}\bigg(h_n\Big(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\, \mathrm{e}^{\mathrm{i}t(\Lambda-n-1/2)}\Big)(U_n(t) - I)\bigg)\, \mathrm{d}t. \tag{3.15}$$

Then by integration by parts

$$\mathcal{G}_n = \mathcal{G}_n' - \mathrm{i}\mathcal{G}_n'' \tag{3.16}$$

with

$$\mathcal{G}_n' = \int_{-\infty}^{\infty} \tilde{\chi}(t)\ \mathrm{tr}\Big(\mathrm{e}^{\mathrm{i}t(\Lambda-n)}h_n(U_n(t) - I)\Big)\mathrm{d}t, \tag{3.17}$$

where $\tilde{\chi}(t) \coloneqq \frac{1}{2}\hat{\chi}(t) - \mathrm{i}\frac{\mathrm{d}\hat{\chi}}{\mathrm{d}t}(t)$ and

$$\mathcal{G}_n'' = \mathrm{i}\int_{-\infty}^{\infty} \hat{\chi}(t)\ \mathrm{tr}\Big(\mathrm{e}^{\mathrm{i}t(\Lambda-n)}h_nH_n(t)U_n(t)\Big)\mathrm{d}t. \tag{3.18}$$

(b) *Next step.* We use the analytic expansion formula

$$U_n(t) = I + \mathrm{i}\int_0^t \mathrm{d}t_1\, H_n(t_1)\ + \sum_{\nu=2}^{\infty} \mathrm{i}^\nu \int_0^t \mathrm{d}t_1 \ldots \int_0^{t_{\nu-1}} \mathrm{d}t_\nu\, H_n(t_1)\ldots H_n(t_\nu).$$

For this purpose we introduce more notations. For $t_1, t \in \mathbb{R}$ we denote

$$g_{n,1}(t;t_1) = \mathrm{i}\ \mathrm{tr}\Big(\mathrm{e}^{\mathrm{i}t(\Lambda-n)}h_nH_n(t_1)\Big) \tag{3.19}$$

and more generally for $\nu \in \mathbb{N}^*$, $(t_1, \ldots, t_\nu) \in \mathbb{R}^\nu$ we introduce

$$g_{n,\nu}(t;t_1,\ldots,t_\nu) = \mathrm{i}^\nu\, \mathrm{tr}\Big(\mathrm{e}^{\mathrm{i}t(\Lambda-n)}h_nH_n(t_1)\ldots H_n(t_\nu)\Big). \tag{3.20}$$

Then the analytic expansion of $U_n(t)$ allows us to express

$$\mathcal{G}'_n = \sum_{\nu=1}^{\infty} \mathcal{G}'_{n,\nu}, \tag{3.21}$$

where

$$\mathcal{G}'_{n,1} = \int_{-\infty}^{\infty} \mathrm{d}t \, \tilde{\chi}(t) \int_0^t \mathrm{d}t_1 \, g_{n,1}(t; t_1)$$

and

$$\mathcal{G}'_{n,\nu} = \int_{-\infty}^{\infty} \mathrm{d}t \, \tilde{\chi}(t) \int_0^t \mathrm{d}t_1 \dots \int_0^{t_{\nu-1}} \mathrm{d}t_\nu \, g_{n,\nu}(t; t_1, \dots, t_\nu)$$

for $\nu \geq 2$. Similarly

$$\mathcal{G}''_n = \sum_{\nu=1}^{\infty} \mathcal{G}''_{n,\nu}, \tag{3.22}$$

holds with

$$\mathcal{G}''_{n,1} = \int_{-\infty}^{\infty} \mathrm{d}t \, \hat{\chi}(t) \, g_{n,1}(t; t),$$

$$\mathcal{G}''_{n,2}(n) = \int_{-\infty}^{\infty} \mathrm{d}t \, \hat{\chi}(t) \int_0^t \mathrm{d}t_2 \, g_{n,2}(t; t, t_2),$$

and, for $\nu \geq 3$,

$$\mathcal{G}''_{n,\nu} = \int_{-\infty}^{\infty} \mathrm{d}t \, \hat{\chi}(t) \int_0^t \mathrm{d}t_2 \dots \int_0^{t_{\nu-1}} \mathrm{d}t_\nu \, g_{n,\nu}(t; t, t_2, \dots, t_\nu).$$

To complete the proof we need estimates (3.23) and (3.24) from the next proposition.

Proposition 3.3. *Let* $\tau_0 > 0$ *be such that* $\operatorname{supp} \hat{\chi} \subset [-\tau_0, \, \tau_0]$. *There exists a constant* $C > 0$ *such that*

$$|g_{n,1}(t; t_1)| \leq C a(n)^{-1/2} \ln n \tag{3.23}$$

holds for $t, t_1 \in [-\tau_0, \, \tau_0]$ *and*

$$\int_0^{4\pi} |g_{n,\nu}(t; t_1, \dots, t_\nu)| \, \mathrm{d}t_\nu \leq C^\nu a(n)^{-1/2} \ln n \tag{3.24}$$

holds for any $\nu \geq 2$ *and* $t, t_1, \dots, t_{\nu-1} \in [-\tau_0, \, \tau_0]$.

Proof. Estimates (3.23) and (3.24) are proven in Sections 4.2 and 4.4 respectively. □

End of proof of Proposition 3.2. (c) *Last step.* Estimates (3.23) and (3.24) ensure existence of a constant $C_0 > 0$ such that

$$|\mathcal{G}'_{n,\nu}| + |\mathcal{G}''_{n,\nu}| \leq \frac{C_0^\nu}{\nu!} \, a(n)^{-1/2} \ln n. \tag{3.25}$$

It is clear that (3.25) allows us to estimate $\mathcal{G}_n$ by $\mathrm{O}(a(n)^{-1/2} \ln n)$. □

4. Proof of Proposition 2.1

The proof scheme is as follows and it remains essentially to prove Proposition 3.3:

$$\left.\begin{array}{r}\text{Lemma 4.1} \implies \text{Proposition 3.3} \implies \text{Proposition 3.2} \\ \text{Lemma 4.2}\end{array}\right\} \implies \text{Proposition 2.1.}$$

4.1. **Notations.** Since v is of period N we can express

$$v(k) = \sum_{\omega\in\Omega} c_\omega \mathrm{e}^{\mathrm{i}\omega k} \tag{4.1}$$

where $c_\omega \in \mathbb{C}$ are constants and $\Omega = \{2\pi k/N\}_{k=0,1,\dots,N-1}$. Moreover our assumption $\langle v\rangle = 0$ ensures $c_0 = 0$ and we can express

$$v(\Lambda) = \sum_{w\in\Omega^*} c_\omega \mathrm{e}^{\mathrm{i}\omega\Lambda} \tag{4.2}$$

with $\Omega^* = \Omega\setminus\{0\}$. Due to (3.3) and (3.4) we find

$$L_n = \Lambda + \tilde{V}_n \tag{4.3}$$

with

$$\tilde{V}_n = \mathrm{e}^{2\mathrm{i}a(n)\operatorname{Im} S} v(\Lambda)\mathrm{e}^{-2\mathrm{i}a(n)\operatorname{Im} S} \tag{4.4}$$

and we consider the decomposition

$$\tilde{V}_n = \sum_{\omega\in\Omega^*} \tilde{V}_n^\omega \tag{4.5}$$

with

$$\tilde{V}_n^\omega := c_\omega \mathrm{e}^{2\mathrm{i}a(n)\operatorname{Im} S}\mathrm{e}^{\mathrm{i}\omega\Lambda}\mathrm{e}^{-2\mathrm{i}a(n)\operatorname{Im} S}. \tag{4.6}$$

Moreover we use the notation

$$H_n(t) = \mathrm{e}^{-\mathrm{i}t\Lambda}\tilde{V}_n\mathrm{e}^{\mathrm{i}t\Lambda} = \sum_{\omega\in\Omega^*} H_n^\omega(t) \tag{4.7}$$

with

$$H_n^\omega(t) := \mathrm{e}^{-\mathrm{i}t\Lambda}\tilde{V}_n^\omega\mathrm{e}^{\mathrm{i}t\Lambda} \tag{4.8}$$

and for $\nu\in\mathbb{N}^*$, $\underline{t} = (t_1,\dots,t_\nu)\in\mathbb{R}^\nu$, $\underline{\omega} = (\omega_1,\dots,\omega_\nu)\in(\Omega^*)^\nu$, we write

$$H_n^{\underline{\omega}}(\underline{t}) := H_n^{\omega_1}(t_1)\dots H_n^{\omega_\nu}(t_\nu). \tag{4.9}$$

This notation allows us to decompose

$$g_{n,\nu}(t;\underline{t}) = \sum_{\underline{\omega}\in(\Omega^*)^\nu} g_n^{\underline{\omega}}(t;\underline{t}) \tag{4.10}$$

with

$$g_n^{\underline{\omega}}(t;\underline{t}) = \mathrm{i}^\nu \operatorname{tr}\Big(\mathrm{e}^{\mathrm{i}t(\Lambda-n)}h_n H_n^{\underline{\omega}}(\underline{t})\Big). \tag{4.11}$$

4.2. **Proof of Proposition 3.3 – estimate (3.23).**

Proof. Let $b\colon\mathbb{Z}\to\mathbb{C}$ be bounded and $q\colon\mathbb{C}\to\mathbb{C}$ be continuous. Then the operator $b(\Lambda)q(S)\in\mathcal{B}(\mathcal{H})$ has the kernel

$$\langle \mathrm{e}_j, b(\Lambda)q(S)\mathrm{e}_k\rangle = b(j)\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(j-k)\xi}q(\mathrm{e}^{\mathrm{i}\xi})\,\frac{\mathrm{d}\xi}{2\pi}. \tag{4.12}$$

If $\operatorname{supp} b$ is bounded then $b(\Lambda)q(S)\in\mathcal{B}_1(\mathcal{H})$ and its trace is given by

$$\operatorname{tr}\Big(b(\Lambda)q(S)\Big) = \sum_{k\in\mathbb{Z}}\langle \mathrm{e}_k, b(\Lambda)q(S)\mathrm{e}_k\rangle = \sum_{k\in\mathbb{Z}} b(k)\int_0^{2\pi} q(\mathrm{e}^{\mathrm{i}\xi})\frac{\mathrm{d}\xi}{2\pi}. \tag{4.13}$$

Since $\mathrm{e}^{-\mathrm{i}\omega\Lambda}S\mathrm{e}^{\mathrm{i}\omega\Lambda} = \mathrm{e}^{-\mathrm{i}\omega}S$ we have

$$\mathrm{e}^{-\mathrm{i}\omega\Lambda}\mathrm{e}^{2\mathrm{i}a(n)\operatorname{Im} S}\mathrm{e}^{\mathrm{i}\omega\Lambda} = \mathrm{e}^{2\mathrm{i}a(n)\operatorname{Im}(\mathrm{e}^{-\mathrm{i}\omega}S)} \tag{4.14}$$

and find the expression

$$\tilde{V}_n^{\omega} = c_\omega \mathrm{e}^{\mathrm{i}\omega\Lambda} \mathrm{e}^{\mathrm{i}\psi_n^{\omega}(S)} \tag{4.15}$$

with

$$\psi_n^{\omega}(S) = 2a(n) \operatorname{Im}\Big((\mathrm{e}^{-\mathrm{i}\omega} - 1)S\Big). \tag{4.16}$$

Therefore

$$H_n^{\omega}(t) = c_\omega \mathrm{e}^{\mathrm{i}\omega\Lambda} \mathrm{e}^{\mathrm{i}\psi_n^{\omega}(t,S)} \tag{4.17}$$

holds with

$$\psi_n^{\omega}(t,S) = \psi_n^{\omega}(\mathrm{e}^{-\mathrm{i}t} S). \tag{4.18}$$

Applying (4.13) we find

$$g_n^{\omega}(t;t_1) = \mathrm{i}\, c_\omega \sum_{k\in\mathbb{Z}} \frac{\theta(k/n)}{k-n-\frac{1}{2}}\, \mathrm{e}^{\mathrm{i}t(k-n)+\mathrm{i}\omega k}\, h_n^{\omega}(t_1) \tag{4.19}$$

with

$$h_n^{\omega}(t_1) := \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\psi_n^{\omega}(t_1,\mathrm{e}^{\mathrm{i}\xi})} \,\frac{\mathrm{d}\xi}{2\pi}. \tag{4.20}$$

Since $\operatorname{supp}\theta \subset [\frac{1}{2},\, 2]$,

$$\sum_{k\in\mathbb{Z}} \frac{|\theta(k/n)|}{|k-n-\frac{1}{2}|} \le \sum_{c_1 n\le k\le c_2 n} \frac{C}{|k-n-\frac{1}{2}|} \le C' \ln n, \tag{4.21}$$

hence the estimate $|g_n^{\omega}(t;t_1)| \le C'|c_\omega| \times |h_n^{\omega}(t_1)| \times \ln n$. Recall we want to prove that $g_{n,1}(t;t_1) = \mathrm{O}\big(a(n)^{-1/2}\ln n\big)$. Since

$$g_{n,1}(t;t_1) = \sum_{\omega\in\Omega^*} g_n^{\omega}(t;t_1)$$

where $\#\Omega^* < \infty$, it only remains to show that

$$h_n^{\omega}(t_1) = \mathrm{O}\big(a(n)^{-1/2}\big). \tag{4.22}$$

It suffices to observe that

$$\psi_n^{\omega}(t_1,\mathrm{e}^{\mathrm{i}\xi}) = 2a(n)\operatorname{Im}\Big((\mathrm{e}^{-\mathrm{i}\omega}-1)\mathrm{e}^{\mathrm{i}(\xi-t_1)}\Big) = -4a(n)\sin\frac{\omega}{2}\cos\Big(\xi - t_1 - \frac{\omega}{2}\Big)$$

has non-degenerate critical points at $\xi = t_1 + \frac{\omega}{2}$ and $\xi = t_1 + \frac{\omega}{2} + \pi$. Then the stationary phase method gives (4.22). □

4.3. **Auxiliary results.** For $\underline{\omega} = (\omega_1,\dots,\omega_\nu) \in (\Omega^*)^\nu$ we write $|\underline{\omega}|_1 = \omega_1 + \cdots + \omega_\nu$ and using induction with respect to ν we prove

$$H_n^{\underline{\omega}}(\underline{t}) = c_{\underline{\omega}}\, \mathrm{e}^{\mathrm{i}|\omega|_1\Lambda} \mathrm{e}^{\mathrm{i}\psi_n^{\underline{\omega}}(\underline{t},S)} \tag{4.23}$$

holds with some real phase functions $\psi_n^{\underline{\omega}}$ and $c_{\underline{\omega}} := c_{\omega_1}\dots c_{\omega_\nu}$ To begin we observe that due to (4.16)-(4.18) in the case $\nu = 1$ the formula (4.23) holds with $\psi_n^{\omega} = \operatorname{Im}\Psi_n^{\omega}$ where

$$\Psi_n^{\omega}(t,\mathrm{e}^{\mathrm{i}\xi}) = 2a(n)\big(\mathrm{e}^{-\mathrm{i}\omega}-1\big)\mathrm{e}^{\mathrm{i}(\xi-t)}. \tag{4.24}$$

Next we write $\underline{\omega} = (\underline{\omega}',\omega) \in (\Omega^*)^{\nu-1}\times\Omega^*$, $\underline{t} = (\underline{t}',t) \in \mathbb{R}^{\nu-1}\times\mathbb{R}$ and assume that

$$H_n^{\underline{\omega}'}(\underline{t}') = c_{\underline{\omega}'}\, \mathrm{e}^{\mathrm{i}|\underline{\omega}'|_1\Lambda} \mathrm{e}^{\mathrm{i}\psi_n^{\underline{\omega}'}(\underline{t}',S)} \tag{4.25}$$

holds with $\psi_n^{\underline{\omega}'} = \operatorname{Im}\Psi_n^{\underline{\omega}'}$ and

$$\Psi_n^{\underline{\omega}'}(\underline{t}',\mathrm{e}^{\mathrm{i}\xi}) = \Psi_n^{\underline{\omega}'}(\underline{t}',1)\mathrm{e}^{\mathrm{i}\xi}. \tag{4.26}$$

Then writing

$$H_n^{\underline{\omega}'}(\underline{t}')H_n^{\omega}(t) = c_{\underline{\omega}'}c_\omega\, \mathrm{e}^{\mathrm{i}|\underline{\omega}'|_1\Lambda}\mathrm{e}^{\mathrm{i}\psi_n^{\underline{\omega}'}(\underline{t}',S)}\mathrm{e}^{\mathrm{i}\omega\Lambda}\mathrm{e}^{\mathrm{i}\psi_n^{\omega}(t,S)} \tag{4.27}$$

and using

$$\mathrm{e}^{-\mathrm{i}\omega\Lambda}\mathrm{e}^{\mathrm{i}\psi_n^{\underline{\omega}'}(\underline{t}',S)}\mathrm{e}^{\mathrm{i}\omega\Lambda} = \mathrm{e}^{\mathrm{i}\psi_n^{\underline{\omega}'}(\underline{t}',\,\mathrm{e}^{-\mathrm{i}\omega}S)} \tag{4.28}$$

we obtain (4.23) taking $\psi_n^{\underline{\omega}} = \operatorname{Im}\Psi_n^{\underline{\omega}}$ where

$$\Psi_n^{\underline{\omega}}(\underline{t}',t,\mathrm{e}^{\mathrm{i}\xi}) = \Psi_n^{\underline{\omega}'}(\underline{t}',\mathrm{e}^{\mathrm{i}(\xi-\omega)}) + \Psi_n^{\omega}(t,\mathrm{e}^{\mathrm{i}\xi}). \tag{4.29}$$

Moreover (4.26), (4.24) and (4.29) ensure

$$\Psi_n^{\underline{\omega}}(\underline{t},\mathrm{e}^{\mathrm{i}\xi}) = \Psi_n^{\underline{\omega}}(\underline{t},1)\mathrm{e}^{\mathrm{i}\xi}. \tag{4.30}$$

Next we show

Lemma 4.1. *Let $\Psi_n^{\underline{\omega}}$ be defined as above. Then there exist $c_0 > 0$ and a measurable function $\eta_{\,n}^{\,\underline{\omega}}\colon \mathbb{R}^{\nu-1} \to [0,2\pi)$ such that one has*

$$|\Psi_n^{\underline{\omega}}(\underline{t},\mathrm{e}^{\mathrm{i}\xi})| = |\Psi_n^{\underline{\omega}}(\underline{t},1)| \geq c_0 a(n)|t-\eta_{\,n}^{\,\underline{\omega}}(\underline{t}')|_{\mathrm{mod}\,\pi}, \tag{4.31}$$

where $\underline{t} = (\underline{t}',t) \in \mathbb{R}^{\nu-1}\times\mathbb{R}$ and $|s|_{\mathrm{mod}\,\pi} := \operatorname{dist}(s+\pi\mathbb{Z},\,\pi\mathbb{Z})$.

Proof. Due to (4.24) we have

$$\frac{\Psi_n^{\underline{\omega}'}(\underline{t}',\mathrm{e}^{\mathrm{i}\omega})}{\Psi_n^{\omega}(t,1)} = \Phi_n^{\underline{\omega}}(\underline{t}')\mathrm{e}^{\mathrm{i}t} \tag{4.32}$$

and using (4.29) we find

$$\frac{\Psi_n^{\underline{\omega}}(\underline{t},1)}{\Psi_n^{\omega}(t,1)} = 1 + \Phi_n^{\underline{\omega}}(\underline{t}')\mathrm{e}^{\mathrm{i}t} \tag{4.33}$$

and it is clear that

$$|\Phi_n^{\underline{\omega}}(\underline{t}')| \leq \frac{1}{2} \implies \left|\frac{\Psi_n^{\underline{\omega}}(\underline{t},1)}{\Psi_n^{\omega}(t,1)}\right| \geq \frac{1}{2}\,. \tag{4.34}$$

Let $-\eta_n^{\underline{\omega}}(\underline{t}')$ be the argument of $\Phi_n^{\underline{\omega}}(\underline{t}')$, i.e.

$$\frac{\Psi_n^{\underline{\omega}}(\underline{t},1)}{\Psi_n^{\omega_\nu}(t,1)} = 1 + |\Phi_n^{\underline{\omega}}(\underline{t}')|\mathrm{e}^{\mathrm{i}(t-\eta_n^{\underline{\omega}}(\underline{t}'))}, \tag{4.35}$$

hence

$$\operatorname{Im}\left(\frac{\Psi_n^{\underline{\omega}}(\underline{t},1)}{\Psi_n^{\omega}(t,1)}\right) = |\Phi_n^{\underline{\omega}}(\underline{t}')|\sin(t-\eta_n^{\underline{\omega}}(\underline{t}')) \tag{4.36}$$

and

$$|\Phi_n^{\underline{\omega}}(\underline{t}')| > \frac{1}{2} \implies \left|\frac{\Psi_n^{\underline{\omega}}(\underline{t},1)}{\Psi_n^{\omega}(t,1)}\right| > \frac{1}{2}|\sin(t-\eta_n^{\underline{\omega}}(\underline{t}'))|. \tag{4.37}$$

Thus combining (4.34) and (4.37) we observe that we can always estimate

$$|\Psi_n^{\underline{\omega}}(\underline{t},1)| \geq \frac{|\Psi_n^{\omega}(t,1)|}{2\pi}|t-\eta_n^{\underline{\omega}}(\underline{t}')|_{\mathrm{mod}\,\pi} \tag{4.38}$$

and $|\Psi_n^{\omega}(t,1)| = 4a(n)\sin(\omega/2) \geq 4a(n)\sin(\pi/N)$ completes the proof. ∎

4.4. **Proof of Proposition 3.3 – estimate (3.24).**

Proof. To complete the proof of Proposition 3.3 it remains to show estimate (3.24). Using (4.23) we have

$$g_n^{\omega}(t,\underline{t}) = \mathrm{i}^{\nu} c_{\underline{\omega}}\, \mathrm{tr}\Big(h_n \mathrm{e}^{\mathrm{i}t(\Lambda-n)+\mathrm{i}|\underline{\omega}|_1\Lambda} \mathrm{e}^{\mathrm{i}\psi_n^{\omega}(\underline{t},S)}\Big). \tag{4.39}$$

As in Section 4.2 we obtain

$$g_n^{\omega}(t,\underline{t}) = \mathrm{i}^{\nu} c_{\underline{\omega}} \sum_{k\in\mathbb{Z}} \frac{\theta(k/n)}{k-n-\frac{1}{2}}\, \mathrm{e}^{\mathrm{i}t(k-n)+\mathrm{i}k|\underline{\omega}|_1} h_n^{\omega}(\underline{t}) \tag{4.40}$$

with

$$h_n^{\omega}(\underline{t}) = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\psi_n^{\omega}(\underline{t},\mathrm{e}^{\mathrm{i}\xi})} \frac{\mathrm{d}\xi}{2\pi}. \tag{4.41}$$

Let $\tilde{\eta}_n^{\omega}(\underline{t}) \in [0,\, 2\pi)$ denote the argument of $\Psi_n^{\omega}(\underline{t},1)$. Then

$$\psi_n^{\omega}(\underline{t},\mathrm{e}^{\mathrm{i}\xi}) = \mathrm{Im}\Big(\Psi_n^{\omega}(\underline{t},1)\mathrm{e}^{\mathrm{i}\xi}\Big) = |\Psi_n^{\omega}(\underline{t},1)|\, \sin(\tilde{\eta}_n^{\omega}(\underline{t})+\xi) \tag{4.42}$$

and the stationary phase formula allows us to estimate

$$|h_n^{\omega}(\underline{t})| \le C_0 |\Psi_n^{\omega}(\underline{t},1)|^{-1/2}. \tag{4.43}$$

Then similarly as in Section 4.2 we have

$$|g_n^{\omega}(t,\underline{t})| \le C_0 \ln n\, |\Psi_n^{\omega}(\underline{t},1)|^{-1/2} \tag{4.44}$$

and due to Lemma 4.1 the left hand side of (3.24) can be estimated by

$$C_1 \ln n \int_0^{4\pi} |a(n)|^{-1/2} |t_{\nu} - \eta_n^{\omega}(\underline{t}')|_{\mathrm{mod}\,\pi}^{-1/2} \mathrm{d}t_{\nu}. \tag{4.45}$$

Since $t \to |t|^{-1/2}$ is locally integrable on $\mathbb{R}$ it is clear that the quantity (4.45) can be estimated by $C \ln n\, |a(n)|^{-1/2}$, which completes the proof. □

4.5. **End of proof of Proposition 2.1.** We use estimate (4.46) from the next lemma.

Lemma 4.2. *Let L_n, θ be as in Proposition 3.2 and $\chi \in \mathcal{S}(\mathbb{R})$. Then we have the large n estimate*

$$\|\big(I-\theta(\Lambda/n)\big)\chi(L_n-n)\|_{\mathcal{B}_1(\mathcal{H})} = \mathrm{O}(n^{\gamma-1}). \tag{4.46}$$

Proof. See Section 8.2. □

End of proof of Proposition 2.1. It is obvious that Lemma 4.2 still holds with L_n replaced by Λ, hence the large n estimate

$$\mathcal{G}_n^0 - \mathcal{G}_n = \mathrm{O}(n^{\gamma-1}). \tag{4.47}$$

Since $n^{\gamma-1} = \mathrm{O}(a(n)^{-1/2}\ln n)$ (see (2.21)) it is clear that Proposition 3.2 and Lemma 4.2 imply Proposition 2.1. □

5. Proof of Proposition 2.4

5.1. **Operators J_n^+.** Let $\{e_n^+\}_{n=1}^\infty$ be the canonical basis of $l^2 = l^2(\mathbb{N}^*)$, i.e., $e_n^+(k) = 0$ when $k \neq n$ and $e_n^+(n) = 1$. If T is a self-adjoint operator which is bounded from below and has compact resolvent, let $(\lambda_k(T))_{k=1}^\infty$ denote the sequence of its eigenvalues enumerated in non-decreasing order with repetitions according to their multiplicities.

Let $S^+ \in \mathcal{B}(l^2)$ be the shift operator defined by

$$S^+ e_n^+ = e_{n+1}^+ \tag{5.1}$$

and let $\Lambda^+ \colon \mathcal{D} \to l^2$ be the closed linear operator defined by

$$\Lambda^+ e_n^+ = n e_n^+. \tag{5.2}$$

For every $b \colon \mathbb{N}^* \to \mathbb{C}$ we denote by $b(\Lambda^+)$ the closed linear operator satisfying

$$b(\Lambda^+) e_n^+ = b(n) e_n^+$$

for any $n \geq 1$. With these notations the operator J defined by (1.6) can be written in the form

$$J = \Lambda^+ + v(\Lambda^+) + 2\,\mathrm{Re}\left(S^+ a(\Lambda^+)\right), \tag{5.3}$$

where $\mathrm{Re}\, Q := \frac{1}{2}(Q + Q^*)$. We identify $l^2 = l^2(\mathbb{N}^*)$ with the closed subspace of $\mathcal{H} = l^2(\mathbb{Z})$ generated by $\{e_n\}_{n=1}^\infty$. Since $\mathcal{D} = \mathcal{H}_1 \cap l^2$ is invariant by J'_n we can define the restriction

$$J_n^+ := J'_n|_{\mathcal{D}}.$$

Then we can express

$$J_n^+ = \Lambda^+ + v(\Lambda^+) + A_n^+ \tag{5.4}$$

with

$$A_n^+ := 2\,\mathrm{Re}\left(S^+ a_n^+(\Lambda^+)\right), \tag{5.5}$$

where $a_n^+(k) := a(n)\theta^+(k/n)$ for any $k \geq 1$.

5.2. **Proof of Proposition 2.4 – estimate (2.23).**

Proof. We denote $\mathrm{Im}\, Q := \frac{1}{2\mathrm{i}}(Q - Q^*)$ and for $t \in \mathbb{R}$ we introduce

$$G_n(t) := e^{\mathrm{i}tB_n^+}(\Lambda^+ + tA_n^+)e^{-\mathrm{i}tB_n^+} \tag{5.6}$$

where

$$B_n^+ := 2\,\mathrm{Im}(S^+ a_n^+(\Lambda^+)). \tag{5.7}$$

We observe that $\lambda_k(J_n^+) = \lambda_k(L_n^+)$ holds with

$$L_n^+ := e^{\mathrm{i}B_n^+}\, J_n^+\, e^{-\mathrm{i}B_n^+} \tag{5.8}$$

and $L_n^+ - G_n(1) = e^{\mathrm{i}B_n^+} v(\Lambda^+) e^{-\mathrm{i}B_n^+}$ ensures $||L_n^+ - G_n(1)||_{\mathcal{B}(l^2)} = ||v(\Lambda^+)||_{\mathcal{B}(l^2)}$, hence the min-max principle allows us to estimate

$$|\lambda_k(G_n(1)) - \lambda_k(L_n^+)| \leq ||v(\Lambda_+)||_{\mathcal{B}(l^2)} \leq \rho_N. \tag{5.9}$$

Next we observe that the derivative of $t \to G_n(t)$ is

$$G'_n(t) = e^{\mathrm{i}tB_n^+}([\mathrm{i}B_n^+,\, \Lambda^+ + tA_n^+] + A_n^+)e^{-\mathrm{i}tB_n^+} \tag{5.10}$$

and similarly as in Section 2, $[S^+,\, \Lambda^+] = -S^+$ allows us to compute

$$[\mathrm{i}B_n^+, \Lambda^+] = 2\,\mathrm{Re}[S^+ a_n^+(\Lambda^+), \Lambda^+] = 2\,\mathrm{Re}[S^+, \Lambda^+] a_n^+(\Lambda^+) = -A_n^+, \tag{5.11}$$

hence

$$G'_n(t) = e^{\mathrm{i}tB_n^+}[\mathrm{i}B_n^+,\, tA_n^+]e^{-\mathrm{i}tB_n^+}. \tag{5.12}$$

However due to the min-max principle we have

$$|\lambda_k(G_n(1)) - \lambda_k(G_n(0))| \le \|G_n(1) - G_n(0)\|_{\mathcal{B}(l^2)} \tag{5.13}$$

and $G_n(1) - G_n(0) = \int_0^1 G_n'(s)\,\mathrm{d}s$ allows us to estimate the right-hand side of (5.13) by

$$\sup_{0\le s\le 1} \|G_n'(s)\|_{\mathcal{B}(l^2)} \le \|[\mathrm{i}B_n^+,\, A_n^+]\|_{\mathcal{B}(l^2)}. \tag{5.14}$$

In order to estimate the norm of $[\mathrm{i}B_n^+,\, A_n^+] = 2\,\mathrm{Re}[S^+a_n^+(\Lambda^+), A_n^+]$ we observe that

$$\begin{aligned}[S^+a_n^+(\Lambda^+), A_n^+] &= [S^+a_n^+(\Lambda^+),\, S^+a_n^+(\Lambda^+) + a_n^+(\Lambda^+)(S^+)^*] \\ &= S^+a_n^+(\Lambda^+)^2(S^+)^* - a_n^+(\Lambda^+)(S^+)^*S^+a_n^+(\Lambda^+) \\ &= a_n^+(\Lambda^+ - I)^2 - a_n^+(\Lambda^+)^2.\end{aligned}$$

However

$$|a_n^+(k-1)^2 - a_n^+(k)^2| = a(n)^2|\theta^+((k-1)/n)^2 - \theta^+(k/n)^2| \le Cn^{2\gamma-1}$$

allows us to estimate the norm of the right hand side of (5.14) by $\mathrm{O}(n^{2\gamma-1})$, hence

$$|\lambda_k(G_n(1)) - \lambda_k(G_n(0))| \le Cn^{2\gamma-1}. \tag{5.15}$$

Due to $\lambda_k(J_n^+) = \lambda_k(L_n^+)$ and $\lambda_k(G_n(0)) = \lambda_k(\Lambda^+) = k$, we can estimate

$$\begin{aligned}|\lambda_k(J_n^+) - k| &= |\lambda_k(L_n^+) - \lambda_k(G_n(0))| \\ &\le |\lambda_k(L_n^+) - \lambda_k(G_n(1))| + |\lambda_k(G_n(1)) - \lambda_k(G_n(0))| \\ &\le \rho_N + Cn^{2\gamma-1}. \end{aligned}\tag{5.16}$$

Let $\varepsilon > 0$. Since $\gamma < \frac{1}{2}$ ensures $n^{2\gamma-1} \to 0$ as $n \to \infty$, we can find n_ε large enough to ensure $\rho_N + Cn_0^{2\gamma-1} \le \rho_N + \varepsilon$. If $\rho_N + \varepsilon < \frac{1}{2}$ and $n \ge n_\varepsilon$ then all eigenvalues of J_n^+ are simple and the interval $(k - \frac{1}{2},\, k + \frac{1}{2}]$ contains exactly one eigenvalue of J_n' for any $k \in \mathbb{Z}$, i.e., $\sigma(J_n^+) = \{\lambda_k(J_n^+)\}_{k\in\mathbb{Z}}$ holds with

$$\sigma(J_n^+) \cap \big(k - \tfrac{1}{2},\, k + \tfrac{1}{2}\big] = \{\lambda_k(J_n^+)\}. \tag{5.17}$$

We complete the proof due to (2.16). □

5.3. **Proof of Proposition 2.4 – estimate (2.24).**

Proof. We first note that

$$J_n' = \Lambda + v(\Lambda) + 2\,\mathrm{Re}\big(Sa_n^+(\Lambda)\big), \tag{5.18}$$

with

$$a_n^+(k) := a(n)\theta^+(k/n).$$

Next we observe that

$$\begin{aligned}S^{-N}v(\Lambda)S^N &= v(\Lambda + N) = v(\Lambda), \\ S^{-N}\theta^+(\Lambda/n)S^N &= \theta^+((\Lambda + N)/n),\end{aligned}$$

and due to $|\theta^+((\lambda + N)/n) - \theta^+(\lambda/n)| \le C/n$ we have

$$S^{-N}J_n'S^N = J_n' + N + R_n \tag{5.19}$$

with $\|R_n\|_{\mathcal{B}(\mathcal{H})} = \mathrm{O}(n^{\gamma-1})$. Moreover

$$\sigma(J_n') = \sigma(S^{-N}J_n'S^N) = \sigma(J_n' + N + R_n) \subset \bigcup_{j\in\mathbb{Z}} \Delta_{j,n} \tag{5.20}$$

holds with

$$\Delta_{j,n} \coloneqq \big[\lambda_j(J'_n) + N - \|R_n\|_{\mathcal{B}(\mathcal{H})},\, \lambda_j(J'_n) + N + \|R_n\|_{\mathcal{B}(\mathcal{H})}\big].$$

However by definition of $\Delta_{j,n}$ we have

$$\lambda \in \Delta_{j,n} \implies |\lambda - \lambda_j(J'_n) - N| \le \|R_n\|_{\mathcal{B}(\mathcal{H})}$$

and using assumption (2.23) we find

$$\lambda \in \Delta_{j,n} \implies |\lambda - j - N| \le \rho_N + Cn^{2\gamma-1} + \|R_n\|_{\mathcal{B}(\mathcal{H})}.$$

However $\gamma < \frac{1}{2}$ ensures $n^{2\gamma-1} \to 0$ as $n \to \infty$ and we can find $n_0 \in \mathbb{N}^*$ such that

$$n \ge n_0 \implies \rho_N + Cn^{2\gamma-1} + \|R_n\|_{\mathcal{B}(\mathcal{H})} < \frac{1}{2}.$$

Therefore denoting $\Delta_r \coloneqq (r - \frac{1}{2},\, r + \frac{1}{2})$ we find

$$n \ge n_0 \implies \Delta_{j,n} \subset \Delta_{j+N}$$

and we conclude

$$n \ge n_0 \implies \lambda_{k+N}(J'_n) \in \Delta_{k+N} \cap \bigcup_{j\in\mathbb{Z}} \Delta_{j,n} = \Delta_{k,n},$$

which implies $|\lambda_{k+N}(J'_n) - \lambda_k(J'_n) - N| \le \|R_n\|_{\mathcal{B}(l^2)} = \mathrm{O}(n^{\gamma-1})$. □

6. Proof of Proposition 2.5

Proof. At the beginning we write

$$\operatorname{tr}\chi(\Lambda - n) = \sum_{l\in\mathbb{Z}} \chi(l) = \sum_{k=0}^{N-1} \sum_{m\in\mathbb{Z}} \chi(Nm + k) \tag{6.1}$$

and introduce $\chi_k(\lambda) = \chi(N\lambda + k)$ $(k = 0, \dots, N-1)$. Then the Poisson summation formula allows us to express (6.1) in the form

$$\sum_{k=0}^{N-1} \sum_{m\in\mathbb{Z}} \chi_k(m) = \sum_{k=0}^{N-1} \sum_{m\in\mathbb{Z}} \hat{\chi}_k(2m\pi) \tag{6.2}$$

with

$$\hat{\chi}_k(t) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}t\lambda}\, \chi(N\lambda + k)\, \frac{\mathrm{d}\lambda}{2\pi} = \mathrm{e}^{\mathrm{i}kt/N}\, \frac{\hat{\chi}(t/N)}{N} \tag{6.3}$$

(see (2.13)). We denote $\rho'_N \coloneqq \rho_N + \varepsilon_0$ with $\varepsilon_0 > 0$ fixed small enough to ensure $\rho'_N < \frac{1}{2}$ and $\rho'_N < 1/(\pi N ||M^{-1}||)$ if $N \ge 3$. Then the assumption (iv) ensures

$$n \ge n_0 \implies |\lambda_k(J'_n) - k| \le \rho'_N. \tag{6.4}$$

Further on we always assume $n \ge n_0$ and consider

$$r_n(k) \coloneqq \lambda_{n+k}(J'_n) - n - k \in [-\rho'_N,\, \rho'_N]. \tag{6.5}$$

However

$$r_n(N + k) - r_n(k) = \lambda_{n+k+N}(J'_n) - N - \lambda_{n+k}(J'_n)$$

and using assumption (2.24) we can estimate

$$|r_n(mN + k) - r_n(k)| \le Cmn^{\gamma-1}. \tag{6.6}$$

Then we write

$$\operatorname{tr} \chi(J_n' - n) = \sum_{k=0}^{N-1} \sum_{m \in \mathbb{Z}} \chi(\lambda_{n+mN+k}(J_n') - n) \tag{6.7}$$

and for every fixed $\varepsilon > 0$ we have

$$\operatorname{tr} \chi(J_n' - n) - \sum_{k=0}^{N-1} \sum_{|m| \le n^\varepsilon} \chi(\lambda_{n+mN+k}(J_n') - n) = \mathrm{O}\,(n^{-\infty}). \tag{6.8}$$

Moreover $|m| \le n^\varepsilon$ ensures

$$\begin{aligned} \lambda_{n+mN+k}(J_n') - n &= mN + k + r_n(mN+k) \\ &= mN + k + r_n(k) + \mathrm{O}(n^{\varepsilon+\gamma-1}) \end{aligned} \tag{6.9}$$

and we obtain

$$\operatorname{tr} \chi(J_n' - n) \; - \sum_{k=0}^{N-1} \sum_{|m| \le n^\varepsilon} \chi(mN + k + r_n(k)) = \mathrm{O}\,(n^{2\varepsilon+\gamma-1}). \tag{6.10}$$

Then denoting $\chi_{n,k}(\lambda) = \chi(\lambda N + k + r_n(k))$ we can write

$$\operatorname{tr} \chi(J_n' - n) \; - \sum_{k=0}^{N-1} \sum_{m \in \mathbb{Z}} \chi_{n,k}(m) = \mathrm{O}\,(n^{2\varepsilon+\gamma-1}) \tag{6.11}$$

and Poisson summation formula allows us to express (6.11) in the form

$$\operatorname{tr} \chi(J_n' - n) \; - \sum_{k=0}^{N-1} \sum_{m \in \mathbb{Z}} \hat{\chi}_{n,k}(2\pi m) = \mathrm{O}(n^{2\varepsilon+\gamma-1}) \tag{6.12}$$

with

$$\hat{\chi}_{n,k}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}t\lambda}\, \chi_{n,k}(\lambda)\, \frac{\mathrm{d}\lambda}{2\pi} = \mathrm{e}^{\mathrm{i}(k+r_n(k))t/N}\, \frac{\hat{\chi}(t/N)}{N} \tag{6.13}$$

(see (2.13)). Let us fix $j = 1, \dots, N$ and take $\hat{\chi} \in C_0^\infty(\mathbb{R})$ such that $\hat{\chi}(2\pi m/N) = N\delta_{m,j}$ for $m \in \mathbb{Z}$. Then

$$\begin{aligned} \sum_{k=0}^{N-1} \sum_{m \in \mathbb{Z}} (\hat{\chi}_{n,k}(2\pi m) - \hat{\chi}_k(2\pi m)) &= \sum_{k=0}^{N-1} (\hat{\chi}_{n,k}(2\pi j) - \hat{\chi}_k(2\pi j)) \\ &= \sum_{k=0}^{N-1} (z_{k+1}(n)^j - w_{k+1}^j), \end{aligned}$$

where $z_{k+1}(n) := \mathrm{e}^{2\pi\mathrm{i}(k+r_n(k))/N}$ and $w_{k+1} := \mathrm{e}^{2\pi\mathrm{i}k/N}$ for $k = 0, \dots, N-1$. Introducing $F_j \colon \mathbb{C}^N \to \mathbb{C}$ by the formula

$$F_j(z_1, \dots, z_N) = \sum_{k=0}^{N-1} \frac{z_{k+1}^j}{j}$$

and $z(n) = (z_1(n), \dots, z_N(n))$, $w = (w_1, \dots, w_N)$ we find

$$j\big(F_j(z(n)) - F_j(w)\big) = \operatorname{tr} \chi(J_n' - n) - \operatorname{tr} \chi(\Lambda - n) + \mathrm{O}(n^{2\varepsilon+\gamma-1}). \tag{6.14}$$

Let $\varepsilon \le \frac{1}{8}$. Then $2\varepsilon + \gamma - 1 \le -\frac{1}{4} \le -\frac{\gamma}{2}$ and due to (2.21) it is clear that (6.14) implies

$$j\big(F_j(z(n)) - F_j(w)\big) = \mathcal{G}_n^+ + \mathrm{O}\left(a(n)^{-1/2} \ln n\right), \tag{6.15}$$

where $\mathcal{G}_n^+$ is as in (2.18). Thus Proposition 2.2 ensures

$$|F_j(z(n)) - F_j(w)| \leq C\mathcal{G}_n^+. \tag{6.16}$$

We notice that the estimate (6.16) holds for every $j = 1, \dots, N$ and further on we consider $F(z) = (F_1(z), \dots, F_N(z)) \in \mathbb{C}^N$. Then $F'(z) = (z_l^{j-1})_{j,l=1}^N$ and $F'(w) = M$. Introducing

$$G(z) = \int_0^1 (F'(w + t(z - w)) - M)\,\mathrm{d}t$$

we can express

$$F(z) - F(w) - M(z - w) = G(z)(z - w)$$

and

$$z(n) - w = M^{-1}(F(z(n)) - F(w)) - M^{-1}G(z(n))(z(n) - w).$$

We denote $z(n,t) = w + t(z(n) - w)$ and we want to estimate

$$F'(z(n,t)) - M = (z_l(n,t)^{j-1} - w_l^{j-1})_{j,l=1}^N \quad (0 \leq t \leq 1)$$

Case $N \geq 3$. Then estimating

$$\begin{aligned}
|z_l(n,t)^{j-1} - w_l^{j-1}| &\leq (j-1)|z_l(n,t) - w_l| \\
&\leq Nt|z_l(n) - w_l| \\
&= Nt|\mathrm{e}^{2\mathrm{i}\pi r_n(l-1)/N} - 1| \\
&= 2Nt\,|\sin(\pi r_n(l-1)/N)| \\
&\leq 2\pi\rho'_N t
\end{aligned}$$

we deduce easily $||F'(z(n,t)) - M|| \leq 2\pi N\rho'_N t$ and $\|G(z)\| \leq \pi N\rho'_N$, hence

$$|z(n) - w| \leq ||M^{-1}(F(z(n)) - F(w))|| + \mu_N|z(n) - w|$$

holds with $\mu_N := \pi N\rho'_N||M^{-1}||$. Therefore we can estimate

$$(1 - \mu_N)|z(n) - w| \leq ||M^{-1}(F(z(n)) - F(w))|| \leq C|F(z(n)) - F(w)| \tag{6.17}$$

and our choice of ρ'_N ensures $\mu_N < 1$, hence it is clear that (6.17) implies

$$r_n(k) = \mathrm{O}(|F(z(n)) - F(w)|)$$

for $k = 0, \dots, N-1$ and due to (6.16) the proof of Proposition 2.5 is done for $N \geq 3$.
Case $N = 2$. We have $(w_1, w_2) = (1, -1)$,

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad G(z) = \begin{pmatrix} 0 & 0 \\ (z_1 - 1)/2 & (z_2 + 1)/2 \end{pmatrix}$$

and

$$\rho_2 < \frac{1}{2} \implies (|z_1 - 1|^2 + |z_1 - 1|^2)^{1/2} \leq 2\mu$$

with a certain $\mu < 1$. Then $\|M^{-1}G(z(n))\| \leq \mu$. For $N = 2$ the estimate (6.17) still holds with $\mu_2 < 1$, hence the assertion of Proposition 2.5 also still holds. □

7. Proof of Proposition 2.6

Proof. Let $\tilde{J}_n$ be the operator acting in $\ell^2 = \ell^2(\mathbb{N}^*)$ and defined by

$$\tilde{J}_n := \Lambda^+ + 2\,\mathrm{Re}\left(S^+ \tilde{a}_n(\Lambda^+)\right) + v(\Lambda^+) \tag{7.1}$$

with

$$\tilde{a}_n(k) := \begin{cases} a(k) & \text{if } n - C_0(n+1)^\gamma \le k \le n + C_0(n+1)^\gamma, \\ a_n^+(k) & \text{otherwise,} \end{cases} \tag{7.2}$$

with C_0 large enough. Let us fix $n_0 \ge (2C_0)^{1/(1-\gamma)}$ and assume $n \ge n_0$. Then

$$k \ge n - C_0 n^{1-\gamma} \ge n/2 \implies a_n^+(k) = a(n)$$

and due to $\Delta a(k) := a(k+1) - a(k) = \mathrm{O}(k^{-\gamma_1})$ we have the estimate

$$\sup_{k\in\mathbb{Z}} |\tilde{a}_n(k) - a_n^+(k)| \le \sup_{|i|\le C_0 n^\gamma} |a_n(n+i) - a(n)| \le C_1 n^{\gamma-\gamma_1}, \tag{7.3}$$

hence $||\tilde{J}_n - J_n^+||_{\mathcal{B}(l^2)} = \mathrm{O}(n^{\gamma-\gamma_1})$ and

$$|\lambda_n(\tilde{J}_n) - \lambda_n(J_n^+)| \le ||\tilde{J}_n - J_n^+||_{\mathcal{B}(l^2)} \tag{7.4}$$

follows from the min-max principle. To complete the proof we show that the estimates

$$\lambda_n(J) = \lambda_n(\tilde{J}_n) + \mathrm{O}(n^{-\nu}) \tag{7.5}$$

hold for any $\nu > 0$ under the assumption that C_0 is chosen large enough in (7.2).

For this purpose we will use a property of Jacobi matrices proved in [4]. We fix C_0 large enough and for $\lambda, \lambda' > 0$ we define

$$\kappa(\lambda) := \lambda + C_0\lambda^\gamma \text{ and } \kappa(\lambda, \lambda') := \lambda' - C_0\lambda^\gamma. \tag{7.6}$$

We denote $\lambda_n := \lambda_n(J)$ and $\lambda_n := \lambda_n - \lambda_n^{-\nu}$ where $\nu \ge 1$ is fixed. Since $|\lambda_n(J) - n| \le \frac{1}{2}$ for $n \ge n_0$ we deduce

$$\kappa(\lambda_n, \lambda_n') \le k \le \kappa(\lambda_n) \implies J\mathrm{e}_k^+ = \tilde{J}_n \mathrm{e}_k^+ \tag{7.7}$$

for $n \ge n_0$ due to (7.2). We notice that (7.6) defines $\kappa(\lambda)$, $\kappa(\lambda, \lambda')$ considered in [4, Theorem 2.3] applied to J, i.e., to the case of the diagonal entries $d_k = k + v(k)$ and off-diagonal entries $b_k = a(k)$ (corresponding to the values $c = 1$, $\alpha = 1$, $\beta = \gamma$ in [4, Theorem 2.3]. The condition (7.7) allows us to use [4, Theorem 2.3] with $\lambda = \lambda_n$, $\lambda' = \lambda_n'$ and $J_{\lambda,\lambda'} = \tilde{J}_n$ for $n \ge n_0$, which ensures

$$\mathrm{card}\left(\sigma(J) \cap (\lambda_n - \lambda_n^{-\nu}, \lambda_n]\right) \le \mathrm{card}\left(\sigma(\tilde{J}_n) \cap \left(\lambda_n - 2\lambda_n^{-\nu},\, \lambda_n + \lambda_n^{-\nu}\right]\right). \tag{7.8}$$

However

$$\sigma(J) \cap \left(\lambda_n - \lambda_n^{-\nu}, \lambda_n\right] = \{\lambda_n\} \tag{7.9}$$

due to $\lambda_n = \lambda_n(J)$ and

$$\sigma(\tilde{J}_n) \cap \left(\lambda_n - 2\lambda_n^{-\nu}, \lambda_n + \lambda_n^{-\nu}\right] \subset \{\lambda_n(\tilde{J}_n)\}, \tag{7.10}$$

hence (7.8) ensures that "$\subset$" can be replaced by "$=$" in (7.10). Thus

$$\lambda_n(\tilde{J}_n) \in \left(\lambda_n(J) - 2\lambda_n(J)^{-\nu}, \lambda_n(J) + \lambda_n(J)^{-\nu}\right]$$

holds for $n \ge n_0$ and (7.5) follows, completing the proof of Proposition 2.6. □

8. Appendix

8.1. **Auxiliary lemmas.**

Lemma 8.1. *Let* $\chi \in C^2(\mathbb{R})$ *and let* $C > 0$ *be such that*

$$|\chi(\lambda)| + |\chi'(\lambda)| + |\chi''(\lambda)| \le C(1+\lambda^2)^{-1} \quad (\lambda \in \mathbb{R}) \tag{8.1}$$

Assume that $L, L' \colon \mathcal{D} \to \mathcal{H}$ *are self-adjoint and their common domain* $\mathcal{D}$ *is invariant with respect to* $h \in \mathcal{B}(\mathcal{H})$. *If* $Lh - hL' \in \mathcal{B}(\mathcal{H})$ *then*

$$||\chi(L)h - h\chi(L')||_{\mathcal{B}(\mathcal{H})} \le C_\chi ||Lh - hL'||_{\mathcal{B}(\mathcal{H})} \tag{8.2}$$

Proof. For x from the domain of L we can write

$$\big(\mathrm{e}^{\mathrm{i}tL}h - h\mathrm{e}^{\mathrm{i}tL'}\big)x = \mathrm{i}t \int_0^1 \mathrm{e}^{\mathrm{i}stL}(Lh - hL')\mathrm{e}^{\mathrm{i}(1-s)tL'} x \,\mathrm{d}s. \tag{8.3}$$

Since (8.1) ensures that $t \to \hat{\chi}(t)$ and $t \to t\hat{\chi}(t)$ are integrable on $\mathbb{R}$, we can express

$$\big(\chi(L)h - h\chi(L')\big)x = \int_{-\infty}^{\infty} \hat{\chi}(t)\,\big(\mathrm{e}^{\mathrm{i}tL}h - h\mathrm{e}^{\mathrm{i}tL'}\big)x \,\mathrm{d}t, \tag{8.4}$$

and using (8.4) we obtain (8.2) with $C_\chi = \int_{-\infty}^{\infty} |t\hat{\chi}(t)|\,\mathrm{d}t$. □

Lemma 8.2. *Let* $\theta \in C_0^\infty\big((\frac{1}{2}, 2)\big)$ *be such that* $\theta(t) = 1$ *if* $\frac{3}{4} \le t \le \frac{3}{2}$ *and let* χ *be as in Lemma 8.1. Then one has*

$$||(I - \theta(\Lambda/n))\chi(J_n - n)||_{\mathcal{B}(\mathcal{H})} = \mathrm{O}(n^{\gamma-1}). \tag{8.5}$$

Proof. Since $\sup_{\lambda\in\mathbb{R}} |(1-\theta(\lambda/n))\chi(\lambda - n)| = \mathrm{O}(n^{-2})$ we have

$$||(1 - \theta(J_n/n))\chi(J_n - n)||_{\mathcal{B}(\mathcal{H})} = \mathrm{O}(n^{-2}). \tag{8.6}$$

We deduce (8.5) combining (8.6) with the estimate

$$||\theta(J_n/n) - \theta(\Lambda/n)||_{\mathcal{B}(\mathcal{H})} \le C||(J_n - \Lambda)/n||_{\mathcal{B}(\mathcal{H})} = \mathrm{O}(n^{\gamma-1}), \tag{8.7}$$

which follows from Lemma 8.1 with $L = J_n/n$, $L' = \Lambda/n$ and $h = I$. □

Lemma 8.3. *Let* χ *be as in Lemma 8.1. Then*

$$||\chi(J_n' - n) - \chi(J_n - n)||_{\mathcal{B}(\mathcal{H})} = \mathrm{O}(n^{\gamma-1}) \tag{8.8}$$

Proof. Let θ be as in Lemma 8.2. Then using (8.5) and a similar estimate

$$||\chi(J_n' - n)(I - \theta(\Lambda/n))||_{\mathcal{B}(\mathcal{H})} = \mathrm{O}(n^{\gamma-1}) \tag{8.9}$$

we find that in order to prove (8.8) it suffices to check

$$||\chi(J_n' - n)\theta(\Lambda/n) - \theta(\Lambda/n)\chi(J_n - n)||_{\mathcal{B}(\mathcal{H})} = \mathrm{O}(n^{\gamma-1}). \tag{8.10}$$

We observe that $(J_n - J_n')\theta(\Lambda/n) = 0$ by definition of J_n, J_n' and θ, hence using Lemma 8.1 with $L = J_n$, $L' = J_n'$, $h = \theta(\Lambda/n)$ we can estimate the left-hand side of (8.10) by

$$C_\chi ||J_n'\theta(\Lambda/n) - \theta(\Lambda/n)J_n||_{\mathcal{B}(\mathcal{H})} = C_\chi ||\,[J_n,\, \theta(\Lambda/n)]\,||_{\mathcal{B}(\mathcal{H})} \tag{8.11}$$

and to complete the proof we observe that the right-hand side of (8.11) is $\mathrm{O}(n^{\gamma-1})$. □

Lemma 8.4. *Let* χ *be as in Lemma 8.1. Then one has the estimate*

$$\sup_n ||\chi(J_n - n)||_{\mathcal{B}_1(\mathcal{H})} < \infty. \tag{8.12}$$

Proof. We write $\chi = \chi_1\chi_0$ with $\chi_0(\lambda) = (1+\lambda^2)^{-1}$. Since $|\chi_1(\lambda)| = |\chi(\lambda)|(1+\lambda^2) \le C$,

$$||\chi(J_n - n)||_{\mathcal{B}_1(\mathcal{H})} \le C||\chi_0(J_n - n)||_{\mathcal{B}_1(\mathcal{H})} = C||\chi_0(L_n - n)||_{\mathcal{B}_1(\mathcal{H})},$$

where $L_n := \mathrm{e}^{\mathrm{i}A'_n} J_n \mathrm{e}^{-\mathrm{i}A'_n}$ and it suffices to show that

$$\sup_n ||((L_n - n)^2 + I)^{-1}||_{\mathcal{B}_1(\mathcal{H})} = \sup_n ||(L_n - n - \mathrm{i})^{-1}||^2_{\mathcal{B}_2(\mathcal{H})} < \infty, \tag{8.13}$$

where $||Q||_{\mathcal{B}_2(\mathcal{H})} := \big(\mathrm{tr}(Q^*Q)\big)^{1/2}$. We observe that

$$||L_n - \Lambda|| = ||\tilde{V}_n||_{\mathcal{B}(\mathcal{H})} = ||v(\Lambda)||_{\mathcal{B}(\mathcal{H})} < \frac{1}{2}$$

and

$$(L_n - n - \mathrm{i})^{-1} = (\Lambda - n - \mathrm{i})^{-1} + (L_n - n - \mathrm{i})^{-1}\tilde{V}_n(\Lambda - n - \mathrm{i})^{-1},$$

hence the estimate

$$||(\Lambda - n - \mathrm{i})^{-1}||^2_{\mathcal{B}_2(\mathcal{H})} = ||(\Lambda - \mathrm{i})^{-1}||^2_{\mathcal{B}_2(\mathcal{H})} = \sum_{k\in\mathbb{Z}}(1 + k^2)^{-1}$$

allows us to complete the proof. □

8.2. **Proof of Lemma 4.2.**

Proof. Assume that $\chi \in C^2(\mathbb{R})$ is such that

$$|\chi(\lambda)| + |\chi'(\lambda)| + |\chi''(\lambda)| \le C(1 + \lambda^2)^{-2} \quad (\lambda \in \mathbb{R}) \tag{8.14}$$

holds with a certain constant $C > 0$. Then we can write $\chi = \chi_1\chi_0$ with $\chi_0(\lambda) = (1 + \lambda^2)^{-1}$. Since $\chi_1(\lambda) = \chi(\lambda)(1 + \lambda^2)$ satisfies the hypothesis of Lemma 8.1, we complete the proof estimating the left-hand side of (4.46) by

$$||(I - \theta(\Lambda/n))\chi_1(J_n - n)||_{\mathcal{B}(\mathcal{H})}||\chi_0(J_n - n)||_{\mathcal{B}_1(\mathcal{H})}.$$ □

8.3. **Proof of Lemma 2.3.**

Proof. We write $\chi = \chi_1\chi_2$ with $\chi_2(\lambda) = (1 + \lambda^2)^{-1}$. Then $\chi_1 \in \mathcal{S}(\mathbb{R})$ and we can express $\chi(J'_n - n) - \chi(J_n - n)$ in the form

$$(\chi_1(J'_n - n) - \chi_1(J_n - n))\chi_2(J'_n - n) + \chi_1(J_n - n)(\chi_2(J'_n - n) - \chi_2(J_n - n)).$$

Thus the left-hand side of (2.19) can be estimated by the sum of

$$||\chi_1(J'_n - n) - \chi_1(J_n - n)||_{\mathcal{B}(\mathcal{H})}||\chi_2(J'_n - n)||_{\mathcal{B}_1(\mathcal{H})} \tag{8.15}$$
$$||\chi_1(J_n - n)||_{\mathcal{B}_1(\mathcal{H})}||\chi_2(J'_n - n) - \chi_2(J_n - n)||_{\mathcal{B}(\mathcal{H})}. \tag{8.16}$$

To complete the proof we observe that the assertion of Lemma 4.2 holds with J'_n instead of J_n, hence using Lemma 8.3 and Lemma 8.4 with χ_1, χ_2 instead of χ we can estimate (8.15) and (8.16) by $\mathrm{O}(n^{\gamma-1})$. □

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