Sub- and supercritical stochastic quasi-geostrophic equation *

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Abstract

In this paper we study the 2D stochastic quasi-geostrophic equation on $\mathbb{T}^2$ for general parameter $\alpha \in (0,1)$ and multiplicative noise. We prove the existence of weak solutions for additive noise, the existence of martingale solutions and Markov selections for multiplicative noise and under some condition pathwise uniqueness for all $\alpha \in (0,1)$. In the subcritical case $\alpha > 1/2$, we prove existence and uniqueness of (probabilistically) strong solutions. In particular, we prove ergodicity provided the noise is non-degenerate for $\alpha > \frac{2}{3}$. In this case, the convergence to the (unique) invariant measure is exponentially fast.

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1 Introduction

Consider the following two dimensional (2D) stochastic quasi-geostrophic equation in the periodic domain $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$:

$$\frac{\partial \theta(t,\xi)}{\partial t} = -u(t,\xi)\cdot\nabla\theta(t,\xi) - \kappa(-\triangle)^\alpha\theta(t,\xi) + (G(\theta)\eta)(t,\xi), \tag{1.1}$$

with initial condition

$$\theta(0,\xi) = \theta_0(\xi), \tag{1.2}$$

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where $\theta(t,\xi)$ is a real-valued function of $\xi \in \mathbb{T}^2$ and $t \geq 0$, $0 < \alpha < 1, \kappa > 0$ are real numbers. u is determined by θ through a stream function ψ via the following relations:

$$u = (u_1, u_2) = (-R_2\theta, R_1\theta) = R^\perp\theta. \tag{1.3}$$

Here R_j is the j-th periodic Riesz transform and $\eta(t,\xi)$ is a Gaussian random field, white noise in time, subject to the restrictions imposed below. The case $\alpha = \frac{1}{2}$ is called the critical case, the case $\alpha > \frac{1}{2}$ sub-critical and the case $\alpha < \frac{1}{2}$ super-critical.

This equation is an important model in geophysical fluid dynamics. The case $\alpha = 1/2$ exhibits similar features (singularities) as the 3D Navier-Stokes equations and can therefore serve as a model case for the latter. In the deterministic case this equation has been intensively investigated because of both its mathematical importance and its background in geophysical fluid dynamics (see for instance [CV06], [Re95], [CW99], [Ju03], [Ju04], [KNV07] and the references therein). In the deterministic case, the global existence of weak solutions has been obtained in [Re95] and one most remarkable result in [CV06] gives the existence of a classical solution for $\alpha = 1/2$. In [KNV07] another very important result is proved, namely that solutions for $\alpha = 1/2$ with periodic C^∞ data remain C^∞ for all times.

In this paper we study the 2D stochastic quasi-geostrophic equation on $\mathbb{T}^2$ for general parameter $\alpha \in (0,1)$ and for both additive as well as multiplicative noise.

For $\alpha \in (0,1)$: We prove the existence of weak solutions in the sense of Definition 3.1 (ii) with additive noise (Theorem 3.4). We also prove the existence of martingale solutions for multiplicative noise under two different assumptions on G (see (G.1) and (G.2) in Section 4): under (G.1) we use Galerkin approximations and the compactness method in [FG95] (Theorem 4.2) and under (G.2) we use Aldous's criterion (Theorem 4.5). In order to prove the existence of (probabilistically strong) solutions and ergodicity in subsequent sections, we need L^p norm estimates for solutions, which are obtained by using the L^p-Itô formula proved in [Kr10]. But these L^p-norm estimates we cannot prove by Galerkin approximation, instead we use another approximation (Theorem 4.3). Pathwise uniqueness is obtained under some extra condition on the solution (Theorem 5.6). But, in general, we cannot prove a solution satisfies this condition, except for very special cases (see Remark 5.7). Using an abstract result for obtaining Markov selections from [GRZ09], we prove the existence of an a.s. Markov family (Theorem 6.5).

For $\alpha > 1/2$: We obtain pathwise uniqueness (Theorem 5.1) and therefore get a (probabilistically strong) solution (Theorem 5.4) by the Yamada-Watanabe Theorem. In particular, it follows that the laws of the solutions form a Markov process. For this, we need to show decay of the solutions' L^p-norm for suitable p.

For $\alpha = 1/2$: Using a result from the deterministic case in [KN09] and [CV06], we also prove that there exists a unique solution of the 2D stochastic quasi-geostrophic equation in the critical case driven by real linear multiplicative noise (Remark 5.7).

Then we prove the ergodicity of the solution in the subcritical case, provided that the noise is non-degenerate and regular. The proof follows from employing the weak-strong uniqueness principle in [FR08] (Theorem 7.1.3) and as usual first establishing the strong Feller property (Theorem 7.1.2). Though one would expect to get ergodicity for $\alpha > \frac{1}{2}$, surprisingly it turns out that one needs $\alpha > \frac{2}{3}$. As the dynamics exists only in the martingale sense and standard tools of stochastic analysis are not available, the computations are made for an approximating cutoff dynamics, which is equal to the original dynamics on a small random time interval. As the noise is non-degenerate, we can use the Bismut-Elworthy-Li formula to prove the strong Feller property. Since in our case $\alpha < 1$, it is more difficult to use the H^α-norm to control the nonlinear term even though the equation is on $\mathbb{T}^2$. To prove the weak-strong uniqueness principle we need some regularity for the trajectories of the noise. Therefore, we need conditions on G so that it is enough regularizing. However, in order to apply the Bismut-Elworthy-Li formula, we also need G^{-1} to be regularizing enough. As a result, $\alpha > 2/3$ is required (see Remark 7.1.1 below for details). It seems difficult to use the Kolmogorov equation method as in [DD03], [DO06] or a coupling approach as in [O08] in our situation (see Remark 7.1.1 below).

In order to prove the exponential convergence (see Theorem 7.4.5), we need to show decay of the solutions' L^p-norm for suitable p. To prove this, we also need the improved positivity lemma (see Lemma 7.4.1 below).

This paper is organized as follows. In Section 2, we introduce some notations as preparation. In Section 3, for additive noise we prove the existence of weak solutions (in the sense of Definition 3.1 (ii) below). In Section 4, we prove the existence of martingale solutions for general parameter $\alpha \in (0,1)$ and multiplicative noise. In Section 5, we prove pathwise uniqueness (under some extra condition on the solutions) for all $\alpha \in (0,1)$. Furthermore, we get the existence and uniqueness of (probabilistically strong) solutions for multiplicative noise in the subcritical case. Moreover, we prove the Markov property for this unique solution. For the general case $\alpha \in (0,1)$ the existence of Markov selections is obtained in Section 6. In Section 7, for $\alpha > 2/3$, and provided the noise is non-degenerate, we prove the ergodicity of the solution and the exponential convergence to the (unique) invariant measure.

2 Notations and Preliminaries

We consider the usual abstract form of equations (1.1)-(1.3). In the following, we will restrict ourselves to flows which have zero average on the torus, i.e.

$$\int_{\mathbb{T}^2} \theta d\xi = 0.$$

Thus (1.3) can be restated as

$$u = \left(-\frac{\partial \psi}{\partial \xi_2}, \frac{\partial \psi}{\partial \xi_1}\right) \text{ and } (-\triangle)^{1/2}\psi = -\theta.$$

Set $H = \{f \in L^2(\mathbb{T}^2) : \int_{\mathbb{T}^2} f d\xi = 0\}$ and let $|\cdot|$ and $\langle .,. \rangle$ denote the norm and inner product in H respectively. On the periodic domain $\mathbb{T}^2$, $\{\sin(k\xi) | k \in \mathbb{Z}^2_+\} \cup \{\cos(k\xi) | k \in \mathbb{Z}^2_-\}$ form an eigenbasis of $-\triangle$. Here $\mathbb{Z}^2_+ = \{(k_1,k_2) \in \mathbb{Z}^2 | k_2 > 0\} \cup \{(k_1,0) \in \mathbb{Z}^2 | k_1 > 0\}, \mathbb{Z}^2_- = \{(k_1,k_2) \in \mathbb{Z}^2 | -k \in \mathbb{Z}^2_+\}, x \in \mathbb{T}^2$, and the corresponding eigenvalues are $|k|^2$. Define

$$\|f\|^2_{H^s} = \sum_k |k|^{2s} \langle f, e_k \rangle^2$$

and let H^s denote the Sobolev space of all f for which $\|f\|_{H^s}$ is finite. Set $\Lambda = (-\triangle)^{1/2}$. Then

$$\|f\|_{H^s} = |\Lambda^s f|.$$

By the singular integral theory of Calderón and Zygmund (cf [St70, Chapter 3]), for any $p \in (1, \infty)$, there is a constant $C = C(p)$, such that

$$\|u\|_{L^p} \leq C(p)\|\theta\|_{L^p}. \tag{2.1}$$

Fix $\alpha \in (0,1)$ and define the linear operator $A : D(A) = H^{2\alpha}(\mathbb{T}^2) \subset H \to H$ as $Au := \kappa(-\triangle)^\alpha u$. The operator A is positive definite and selfadjoint with the same eigenbasis as that of $-\triangle$ mentioned above. Denote the eigenvalues of A by $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, and renumber the above eigenbasis correspondingly as e_1, e_2,.... We also set $\|u\| := |A^{1/2}u|$, then $\|\theta\|^2 \geq \lambda_1|\theta|^2$.

First we recall the following important product estimates (cf. [Re95, Lemma A.4]):

Lemma 2.1 Suppose that $s > 0$ and $p \in (1, \infty)$. If $f, g \in \mathcal{S}$, the Schwartz class, then

$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}}\|g\|_{H^{s,p_2}} + \|g\|_{L^{p_3}}\|f\|_{H^{s,p_4}}), \tag{2.2}$$

with $p_i \in (1, \infty), i = 1, ..., 4$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

We shall use as well the following standard Sobolev inequality (cf. [St70, Chapter V]):

Lemma 2.2 Suppose that $q > 1, p \in [q, \infty)$ and

$$\frac{1}{p} + \frac{\sigma}{2} = \frac{1}{q}.$$

Suppose that $\Lambda^\sigma f \in L^q$, then $f \in L^p$ and there is a constant $C \geq 0$ such that

$$\|f\|_{L^p} \leq C\|\Lambda^\sigma f\|_{L^q}.$$

3 Existence of solutions for additive noise

In this section, we consider the abstract stochastic evolution equation in place of Eqs (1.1)-(1.3),

$$\begin{cases} d\theta(t) + A\theta(t)dt + u(t) \cdot \nabla\theta(t)dt = G(\theta(t))dW(t), \\ \theta(0) = \theta_0 \in H, \end{cases} \tag{3.1}$$

where u satisfies (1.3) and $W(t)$ is a cylindrical Wiener process in a separable Hilbert space K defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$. Here G is a measurable mapping from H^α to $L_2(K, H)$.

Definition 3.1 (i) We say that there exists a (probabilistically) strong solution to (3.1) over the time interval $[0, T]$ if for every probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ with an $\mathcal{F}_t$-Wiener process W, there exists an $\mathcal{F}_t$-adapted process $\theta : [0, T] \times \Omega \to H$ such that for $P - a.s.$ $\omega \in \Omega$

$$\theta(\cdot, \omega) \in L^\infty(0, T; H) \cap L^2(0, T; H^\alpha) \cap C([0, T]; H_w)$$

and P-a.s.

$$\langle\theta(t),\varphi\rangle+\int_0^t\langle A^{1/2}\theta(s),A^{1/2}\varphi\rangle ds-\int_0^t\langle u(s)\cdot\nabla\varphi,\theta(s)\rangle ds=\langle\theta_0,\varphi\rangle+\langle\int_0^t G(\theta(s))dW(s),\varphi\rangle,$$

for all $t\in[0,T]$ and all $\varphi\in C^1(\mathbb{T}^2)$, (assuming also that all integrals in the equation are defined). Here $C([0,T];H_w)$ denotes the space of H-valued weakly continuous functions on $[0,T]$.

(ii) If θ is not an $\mathcal{F}_t$-adapted process, then for additive noise the equation is still defined. In this case we call θ a (probabilistically) weak solution.

Remark 3.2 Note that, because $divu=0$ for regular functions θ and v, we have

$$\langle u(s)\cdot\nabla(\theta(s)+\psi),\theta(s)+\psi\rangle=0,$$

so

$$\langle u(s)\cdot\nabla\theta(s),\psi\rangle=-\langle u(s)\cdot\nabla\psi,\theta(s)\rangle.$$

Thus the integral equation in Definition 3.1 corresponds to equation (3.1).

Assumption 3.3 Assume that G does not depend on θ and $\mathrm{Tr}(\Lambda^{2(1+\sigma-\alpha)+\varepsilon}\mathrm{GG}^*)<\infty$ for some $\varepsilon>0$, where $\sigma:=(1-2\alpha)\vee 0$.

Consider the O-U equation

$$dz(t)+Az(t)dt=GdW(t).$$

It is known that the process

$$z(t)=\int_0^t e^{-(t-s)A}GdW(s)$$

is a solution with continuous trajectories.

Under Assumption 3.3 by standard methods we obtain that $\sup_{0\leq t\leq T}\|\nabla z(t)\|_{L^q}<\infty\ P-a.s.$ with $q=(\frac{1}{\alpha}+\varepsilon)\vee 2$ for some $\varepsilon>0$ (see e.g. the proof of [DZ92, Theorem 5.16]).

Theorem 3.4 Let $\alpha\in(0,1)$ and suppose that Assumption 3.3 holds. Then for each initial condition $\theta_0\in H$, there exists a weak solution θ of equation (3.1) over $[0,T]$ with initial condition $\theta(0)=\theta_0$.

Proof By the classical change of variable $v(t)=\theta(t)-z(t)$ we obtain the differential equation

$$\frac{dv(t)}{dt}+Av(t)+u(t)\cdot\nabla(v(t)+z(t))=0. \tag{3.2}$$

For almost all given paths of the process $z(t)$ we study this equation as a deterministic evolution equation.

Let P_n be the orthogonal projection in H onto the linear space spanned by $e_1,...e_n$. Consider the ordinary differential equation

$$\frac{dv_n(t)}{dt}+Av_n(t)+P_n(u_n(t)\cdot\nabla(v_n(t)+z(t)))=0,$$

with initial condition

$$v_n(0) = P_n v_0.$$

Here u_n satisfies (1.3) with θ replaced by $v_n + z$.

Its solution satisfies

$$\frac{1}{2}\frac{d}{dt}|v_n|^2 + \|v_n\|^2 = \langle -u_n(t)\cdot\nabla(v_n(t)+z(t)), v_n(t)\rangle.$$

Here $\omega \in \Omega$ is fixed. For simplicity, in the following estimate, we set $v = v_n$ and $u(t) = u_v(t) + u_z(t)$, u_v and u_z satisfying (1.3) with θ replaced by v and z, respectively. We have

$$\begin{aligned}|\langle -u(t)\cdot\nabla(v(t)+z(t)), v(t)\rangle| =& |\langle u_v(t)\cdot\nabla z(t), v(t)\rangle + \langle u_z(t)\cdot\nabla z(t), v(t)\rangle| \\ \leq& C\|\nabla z\|_{L^q}\|v\|_{L^p}^2 + C\|\nabla z\|_{L^q}\|z\|_{L^p}\|v\|_{L^p}.\end{aligned}$$

Here $\frac{1}{q}+\frac{2}{p}=1$. Since

$$\|v\|_{L^p}^2 \leq C\|v\|_{H^{\alpha-\varepsilon_1}}^2 \leq C\|v\|^{2\beta}|v|^{2(1-\beta)},$$

where $\beta = \frac{\alpha-\varepsilon_1}{\alpha}$, by Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}|v|^2 + \|v\|^2 \leq \varepsilon\|v\|^2 + C(\varepsilon)|v|^2 + C(\varepsilon)|v|^2\|\nabla z\|_{L^q}^{1/(1-\beta)} + C\|\nabla z\|_{L^q}^4.$$

Therefore, for all $t \in [0,T]$,

$$|v(t)|^2 \leq e^{\int_0^t C(1+\|\nabla z(s)\|_{L^q}^{1/(1-\beta)})ds}|v_0|^2 + C\int_0^t e^{\int_\tau^t C(1+\|\nabla z(s)\|_{L^q}^{1/(1-\beta)})ds}\|\nabla z(\tau)\|_{L^q}^4 d\tau, \tag{3.3}$$

and for $[r,t] \subset [0,T]$,

$$\int_r^t \|v\|^2 d\tau \leq |v(r)|^2 + C\int_r^t (|v|^2 + |v|^2\|\nabla z\|_{L^q}^{1/(1-\beta)} + \|\nabla z\|_{L^q}^4)d\tau. \tag{3.4}$$

Then by Assumption 3.3, all the terms in (3.3) and (3.4) containing z are uniformly bounded in t. Therefore, from (3.3) and (3.4) (which hold true for v_n) we obtain that the sequence v_n is bounded in $L^\infty(0,T;H)$ and in $L^2(0,T;H^\alpha)$. It is obvious that there exists an element $v \in L^\infty(0,T;H)\cap L^2(0,T;H^\alpha)$ and a sub-sequence v_m' such that

$$v_m' \to v \text{ in } L^2(0,T;H^\alpha) \text{ weakly, and in } L^\infty(0,T;H) \text{ weak-star, as } m\to\infty.$$

In order to prove the strong convergence in $L^2(0,T;H)$, we need to use [FG 95, Theorem 2.1]. So we just need to prove that $\|v_n\|_{W^{\gamma,2}(0,T,H^{-3})}$ is bounded for some $1/2 < \gamma < 1$. Then by compact embedding, we have $v_m' \to v$ in $L^2(0,T;H)\cap C([0,T];H^{-\beta})$ strongly for some $\beta > 3$. Note that v_n also satisfies

$$\langle v_n(t),\psi\rangle + \int_0^t \langle A^{1/2}v_n(s), A^{1/2}\psi\rangle ds - \int_0^t \langle u_n(s)\cdot\nabla\psi, v_n(s)+z(s)\rangle ds = \langle P_n v_0,\psi\rangle, \tag{3.5}$$

for all $t\in[0,T]$ and all $\psi \in C^1(\mathbb{T}^2)$. Then taking the limit in (3.5), we obtain the result.

Now decompose v_n as

$$v_n(t) = P_n v_0 - \int_0^t Av_n(s)ds - \int_0^t P_n(u_n(s)\cdot\nabla(v_n(s)+z(s)))ds.$$

By (3.4) we obtain

$$\|\int_0^\cdot Av_n(s)ds\|_{W^{1,2}(0,T,H^{-\alpha})} \leq C.$$

And by $H^2 \subset L^\infty$, we have for $\theta \in H^1, \psi \in H^3$,

$$|\langle u\cdot\nabla\theta,\psi\rangle| = |\langle u\cdot\nabla\psi,\theta\rangle| \leq |\theta|^2\|\nabla\psi\|_\infty \leq |\theta|^2\|\psi\|_{H^3}.$$

Then

$$\|P_n(u_n\cdot\nabla(v_n+z))\|_{L^2(0,T;H^{-3})} \leq T^{1/2}\sup_{0\leq s\leq T}|v_n(s)+z(s)|^2 \leq C,$$

whence

$$\|\int_0^\cdot P_n(u_n(s)\cdot\nabla(v_n(s)+z(s)))ds\|_{W^{1,2}(0,T,H^{-3})} \leq C.$$

Clearly for a Banach space B, $W^{1,2}(0,T;B) \subset W^{\gamma,2}(0,T;B)$. So we have proved

$$\|v_n\|_{W^{\gamma,2}(0,T,H^{-3})} \leq C.$$

Thus the assertion follows.

□

4 Martingale solutions in the general case

In this section, we consider multiplicative noise in the general case $\alpha \in (0,1)$. First we introduce the following definition of a martingale solution.

Definition 4.1 We say that there exists a martingale solution of the equation (3.1) if there exists a stochastic basis $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in[0,T]},P)$, a cylindrical Wiener process W on the space K and a progressively measurable process $\theta:[0,T]\times\Omega\to H$, such that for P-a.e. $\omega\in\Omega$,

$$\theta(\cdot,\omega) \in L^\infty(0,T;H)\cap L^2(0,T;H^\alpha)\cap C([0,T];H^{-\beta}),$$

where $\beta > 3$, and such that P-a.s.

$$\langle\theta(t),\phi\rangle + \int_0^t \langle A^{1/2}\theta(s),A^{1/2}\phi\rangle ds - \int_0^t \langle u(s)\cdot\nabla\phi,\theta(s)\rangle ds = \langle\theta_0,\phi\rangle + \langle\int_0^t G(\theta(s))dW(s),\phi\rangle, \tag{4.1}$$

for $t\in[0,T]$ and all $\phi\in C^1(\mathbb{T}^2)$.

Let $f_n, n\in\mathbb{N}$, be an ONB of K and consider the following two conditions:

(G.1)(i) $|G(\theta)|^2_{L_2(K,H)} \leq \lambda_0|\theta|^2+\rho, \theta\in H^\alpha$, for some positive real numbers λ_0 and ρ.

(ii) If $y, y_n \in H^\alpha$ such that $y_n\to y$ in H, then $\lim_{n\to\infty}\|G(y_n)^*(v)-G(y)^*(v)\|_K = 0$ for all $v\in C^\infty(\mathbb{T}^2)$.

(G.2)For $y \in K$

$$G(u)y = \sum_{k=1}^{\infty}(b_k \Lambda^{\alpha} u + c_k u)\langle y, f_k\rangle_K, u \in H^{\alpha},$$

where $b_k, c_k \in C^{\infty}(\mathbb{T}^2)$ satisfying $\sum_k b_k^2(\xi) < 2\kappa, \sum_k c_k^2(\xi) < M, \xi \in \mathbb{T}^2$.

Theorem 4.2 Let $\alpha \in (0,1)$. Under Assumption (G.1), there exists a martingale solution $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P, W, \theta)$ to (3.1).

Proof [Step 1] Let P_n be the orthogonal projection in H onto the space spanned by $e_1, ...e_n$. Consider the Faedo-Galerkin approximation.

$$\begin{cases} d\theta_n(t) + A\theta_n(t)dt + P_n(u_n(t) \cdot \nabla\theta_n(t))dt = P_n G(\theta_n(t))dW(t), \\ \theta_n(0) = P_n\theta_0, \end{cases} \tag{4.2}$$

where u_n satisfy (1.3) with θ replaced by θ_n. Since all the coefficients are smooth in $P_n H$, this equation has a martingale solution $\theta_n \in L^2(\Omega; C([0,T]; P_n H))$.

Since we have

$$\langle u_n(t) \cdot \nabla\theta_n(t), \theta_n\rangle = 0,$$

by Itô's formula, for all $p \geq 2$ we have

$$d|\theta_n(t)|^p + p|\theta_n(t)|^{p-2}\|\theta_n\|^2 dt \leq p|\theta_n(t)|^{p-2}\langle G(\theta_n)dW(t), \theta_n\rangle + \frac{1}{2}p(p-1)|\theta_n|^{p-2}|P_n G(\theta_n)|^2_{L_2(K,H)}dt.$$

By classical arguments, we easily show that there exist positive constants $C_1(p), C_2$, for each $p \geq 2$, such that (cf [FG95, Appendix 1])

$$E(\sup_{0\leq s\leq T}|\theta_n(s)|^p) \leq C_1(p), \tag{4.3}$$

and

$$E\int_0^T \|\theta_n(s)\|^2 ds \leq C_2. \tag{4.4}$$

[Step 2] Now decompose θ_n as

$$\theta_n(t) = P_n\theta_0 - \int_0^t A\theta_n(s)ds - \int_0^t P_n(u_n(s) \cdot \nabla\theta_n(s))ds + \int_0^t P_n G(\theta_n(s))dW(s).$$

By (4.4) we obtain

$$E\|\int_0^t A\theta_n(s)ds\|_{W^{1,2}(0,T,H^{-\alpha})} \leq C.$$

And by $H^2 \subset L^{\infty}$ we have for $\theta \in H^1, v \in H^3$

$$|\langle u \cdot \nabla\theta, v\rangle| = |\langle u \cdot \nabla v, \theta\rangle| \leq |\theta|^2\|\nabla v\|_{\infty} \leq |\theta|^2\|v\|_{H^3}.$$

Then

$$E\|P_n(u_n \cdot \nabla\theta_n)\|_{L^2(0,T;H^{-3})} \leq T^{1/2}E[\sup_{0\leq s\leq T}|\theta_n(s)|^2] \leq C,$$

whence

$$E\|\int_0^t P_n(u_n(s)\cdot\nabla\theta_n)ds\|_{W^{1,2}(0,T,H^{-3})}\le C.$$

By [FG95, Lemma 2.1], Assumption (G.1), and (4.3), (4.4), we have

$$E\|\int_0^t P_nG(\theta_n(s))dW(s)\|_{W^{\gamma,2}(0,T;H)}\le C.$$

Clearly, for a Banach space B, $W^{1,2}(0,T;B)\subset W^{\gamma,2}(0,T;B)$ for $0<\gamma<1$. So, we have proved

$$E\|\theta_n\|_{W^{\gamma,2}(0,T,H^{-3})}\le C.$$

Recalling (4.4), this implies that the laws $\mathcal{L}(\theta_n), n\in\mathbb{N}$ are bounded in probability in

$$L^2(0,T;H^\alpha)\cap W^{\gamma,2}(0,T,H^{-3})$$

and thus are tight in $L^2(0,T;H)$ by [FG95, Theorem 2.1].

Arguing similarly for the term $\int_0^t P_nG(\theta_n(s))dW(s)$, on the basis of the estimate (4.3), we apply [FG95,Theorem 2.2] and have that the family $\mathcal{L}(\theta_n), n\in\mathbb{N}$, is tight in $C([0,T];H^{-\beta})$, for all given $\beta>3$. Thus, we find a subsequence, still denoted by θ_n, such that $\mathcal{L}(\theta_n)$ converges weakly in

$$L^2(0,T;H)\cap C(0,T,H^{-\beta}).$$

By Skorohod's embedding theorem, there exist a stochastic basis $(\Omega^1,\mathcal{F}^1,\{\mathcal{F}_t^1\}_{t\in[0,T]},P^1)$ and, on this basis, $L^2(0,T;H)\cap C(0,T,H^{-\beta})$-valued random variables $\theta^1,\theta_n^1, n\ge 1$, such that θ_n^1 has the same law as θ_n on $L^2(0,T;H)\cap C(0,T,H^{-\beta})$, and $\theta_n^1\to\theta^1$ in $L^2(0,T;H)\cap C(0,T,H^{-\beta}), P^1$ -a.s. For θ_n^1 we also have (4.3) and (4.4). Hence it follows that

$$\theta^1(\cdot,\omega)\in L^2(0,T;H^\alpha)\cap L^\infty(0,T;H) \text{ for } P^1-a.e\ \omega\in\Omega.$$

For each θ_n^1 we have that u_n^1 satisfies (1.3) with θ replaced by θ_n^1.

For each $n\ge 1$, define the process

$$M_n^1(t):=\theta_n^1(t)-P_n\theta_0^1+\int_0^t A\theta_n^1(s)ds+\int_0^t P_n(u_n^1(s)\cdot\nabla\theta_n^1(s))ds.$$

In fact M_n^1 is a square integrable martingale with respect to the filtration

$$\{\mathcal{G}_n^1\}_t=\sigma\{\theta_n^1(s), s\le t\}.$$

Then by a standard method (cf [FG95], [DZ92]) we obtain the martingale solution.

$\square$

In order to get an estimate for the L^p norm, we need to use another approximation.

Theorem 4.3 Let $\alpha\in(0,1)$. If $G\in L_2(K,H)$ satisfies (G.1) and also the following conditions: for all $\theta\in H^\alpha\cap L^p(\mathbb{T}^2)$,

$$\int(\sum_j|G(\theta)(f_j)|^2)^{p/2}d\xi\le C(\int|\theta|^pd\xi+1),\forall t>0, \tag{4.5}$$

with $2 < p < \infty$ for some constant $C := C(p) > 0$ and for all $\theta_1, \theta_2 \in H^\alpha \cap L^p(\mathbb{T}^2)$,

$$\int (\sum_j |(G(\theta_1) - G(\theta_2))(f_j)|^2)^{p/2} d\xi \leq C \int |\theta_1 - \theta_2|^p d\xi, \tag{4.6}$$

then there exists a martingale solution $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P, W, \theta)$ to (3.1). Moreover, if $\theta_0 \in L^p(\mathbb{T}^2)$ with $p > 2$, then

$$E \sup_{t\in[0,T]} \|\theta(t)\|_{L^p} < \infty.$$

Remark 4.4 Typical examples for G satisfying (4.5) have the following form: for $\theta \in H^\alpha$

$$G(\theta)y = \sum_{k=1}^{\infty} b_k \langle y, f_k\rangle_K \theta, y \in K$$

where b_k are C^∞ functions on $\mathbb{T}^2$ satisfying $\sum_{k=1}^\infty b_k^2(\xi) \leq M$.

Proof [Step 1] We first establish the existence of L^p-bounded solutions of the linear equation:

$$d\theta(t) + A\theta(t)dt + w(t)\cdot\nabla\theta(t)dt = k_\delta * G(\theta)dW(t), \tag{4.7}$$

with a given coefficient function $w(t)$ which satisfies $divw(t) = 0$ and $\sup_{t\in[0,T]} \|w(t)\|_{C^3} \leq C$. Here $k_\delta * G(\theta)$ means for $y \in K$, $k_\delta * G(\theta)(y) = k_\delta * (G(\theta)(y))$, where k_δ is the periodic Poisson kernel in $\mathbb{T}^2$ given by $\widehat{k_\delta}(\zeta) = e^{-\delta|\zeta|}, \zeta \in \mathbb{Z}^2$. First, we consider G not depending on θ. Now take $z = \int_0^t e^{-(t-s)A} k_\delta * GdW(s), v = \theta - z$. We have

$$dv(t) + Av(t)dt + w(t)\cdot\nabla(v + z(t))dt = 0,$$

which is easily seen to have a solution $v \in C([0,T]; H) \cap L^2([0,T]; H^\alpha)$. We have for any $s > 0$,

$$\frac{d}{dt}|\Lambda^s v|^2 + 2|\Lambda^{s+\alpha}v|^2 \leq C(\|w\|_{C^3(\mathbb{T}^2)})|\Lambda^s v|^2 + |\Lambda^{s+\alpha}v|^2 + C(|\Lambda^{s+\alpha}z|).$$

By this estimate and a standard argument we prove that if $v(t_0) \in H^s$, then $v \in C([t_0,T], H^s) \cap L^2([t_0,T], H^{s+\alpha})$. Then we obtain $v \in C((0,T]; H^s)$ for any $3 > s > 0$. Thus we get the existence of L^p-bounded solutions for additive noise. Then consider the mapping $\Gamma : L^1(\Omega, L^\infty([0,T], L^p)) \to L^1(\Omega, L^\infty[0,T], L^p))$ defined by $\Gamma(\theta_1) = \theta$, where θ satisfies (4.7) with $G(\theta)$ replaced by $G(\theta_1)$. Thus, by considering the norm $[E\sup_{s\in[0,T]}(e^{-\beta s}\|\theta(s)\|_{L^p}^p)]^{1/p}$ for suitable $\beta \in (0,\infty)$ and a similar calculation as (4.9) below, we obtain Γ maps $L^1(\Omega, L^\infty[0,T], L^p))$ into itself and is a contraction. Thus, the equation $\theta_1 = \Gamma(\theta_1)$ has a unique solution. Hence (4.7) has a unique L^p bounded solution.

[Step 2] Now we construct an approximation of (3.1).

We pick a smooth $\phi \geq 0$, with supp$\phi \subset [1,2]$, $\int_0^\infty \phi = 1$, and for $\delta > 0$ let

$$U_\delta[\theta](t) := \int_0^\infty \phi(\tau)(k_\delta * R^\perp\theta)(t - \delta\tau)d\tau,$$

where k_δ is the periodic Poisson Kernel in $\mathbb{T}^2$ given by $\widehat{k_\delta}(\zeta) = e^{-\delta|\zeta|}, \zeta \in \mathbb{Z}^2$, and we set $\theta(t) = 0, t < 0$. We take a zero sequence δ_n and consider the equation:

$$d\theta_n(t) + A\theta_n(t)dt + u_n(t)\cdot\nabla\theta_n(t)dt = k_{\delta_n} * G(\theta)dW(t), \tag{4.8}$$

with initial data $\theta_n(0)=\theta_0$ and $u_n=U_{\delta_n}[\theta_n]$. For a fixed n, this is a linear equation in θ_n on each subinterval $[t_k,t_{k+1}]$ with $t_k=k\delta_n$, since u_n is determined by the values of θ_n on the two previous subintervals. By [Step1], we obtain the existence of a solution to (4.8).

[Step 3] It is sufficient to show that θ_n converge to the solution of (3.1). This follows by similar arguments as in the proof of Theorem 4.2. Just as in Theorem 4.2, we only need to prove

$$E\|\theta_n\|_{W^{\gamma,2}(0,T,H^{-3})}\leq C.$$

Here we can't bound $|u_n|$ by $|\theta_n|$, pointwise in time. Instead, we have

$$\sup_{[0,t]}|u_n|\leq C\sup_{[0,t]}|\theta_n|.$$

Thus by a small modification of the proof of Theorem 4.2, we get the martingale solution $(\Omega,\mathcal{F},\{\mathcal{F}_t\},P,W,\theta)$ to (3.1).

[Step 4] Now we prove the last statement. It is sufficient to prove that

$$E\sup_{t\in[0,T]}\|\theta_n(t)\|_{L^p}^p\leq C.$$

We write for simplicity $\theta(t)=\theta_n(t,\xi)$. By [Kr10, Lemma 5.1], we have

$$\begin{aligned}\|\theta(t)\|_{L^p}^p=&\|\theta_0\|_{L^p}^p+\int_0^t[-p\int_{\mathbb{T}^2}|\theta(s)|^{p-2}\theta(s)(\Lambda^{2\alpha}\theta(s)+u(s)\cdot\nabla\theta(s))d\xi\\&\quad+\frac{1}{2}p(p-1)\int_{\mathbb{T}^2}|\theta(s)|^{p-2}(\sum_j|k_{\delta_n}*G(\theta(s))(f_j)|^2)d\xi]ds\\&+p\int_0^t\int_{\mathbb{T}^2}|\theta(s)|^{p-2}\theta(s)k_{\delta_n}*G(\theta(s))d\xi dW(s)\\\leq&\|\theta_0\|_{L^p}^p+\int_0^t\frac{1}{2}p(p-1)\int_{\mathbb{T}^2}|\theta(s)|^{p-2}(\sum_j|k_{\delta_n}*G(\theta(s))(f_j)|^2)d\xi ds\\&+p\int_0^t\int_{\mathbb{T}^2}|\theta(s)|^{p-2}\theta(s)k_{\delta_n}*G(\theta(s))d\xi dW(s)\\\leq&\|\theta_0\|_{L^p}^p+\int_0^t(\varepsilon\int_{\mathbb{T}^2}|\theta(s)|^pd\xi+C(\varepsilon)\int(\sum_j|k_{\delta_n}*G(\theta(s))(f_j)|^2)^{p/2}d\xi)ds\\&+p\int_0^t\int_{\mathbb{T}^2}|\theta(s)|^{p-2}\theta(s)k_{\delta_n}*G(\theta(s))d\xi dW(s).\end{aligned}$$

Then by the Burkholder-Davis-Gundy inequality, Minkowski's inequality and the same estimate as in the proof of (6.4) in [Kr10] and (4.1) we have

$$
\begin{aligned}
E\sup_{s\in[0,t]}\|\theta(s)\|_{L^p}^p \leq & E\|\theta_0\|_{L^p}^p + E\int_0^t (\varepsilon\int_{\mathbb{T}^2}|\theta(s)|^p d\xi + C\int(\sum_j |k_{\delta_n} * G(\theta(s))(f_j)|^2)^{p/2} d\xi)ds \\
& + pE(\int_0^t (\int_{\mathbb{T}^2}|\theta(s)|^{p-1}(\sum_j |k_{\delta_n} * G(\theta(s))(f_j)|^2)^{1/2} d\xi)^2 ds)^{1/2} \\
\leq & E\|\theta_0\|_{L^p}^p + E\int_0^t (\varepsilon\int_{\mathbb{T}^2}|\theta(s)|^p d\xi + C\int(\sum_j |k_{\delta_n} * G(\theta(s))(f_j)|^2)^{p/2} d\xi)ds \\
& + pE\sup_{s\in[0,t]}\|\theta(s)\|_{L^p}^{p-1}(\int_0^t (\int_{\mathbb{T}^2}(\sum_j |k_{\delta_n} * G(\theta(s))(f_j)|^2)^{p/2} d\xi)^{2/p} ds)^{1/2} \\
\leq & E\|\theta_0\|_{L^p}^p + E\int_0^t (\varepsilon\int_{\mathbb{T}^2}|\theta(s)|^p d\xi + C\int(\sum_j |G(\theta(s))(f_j)|^2)^{p/2} d\xi)ds \\
& + C(T)E\sup_{s\in[0,t]}\|\theta(s)\|_{L^p}^{p-1}(\int_0^t (\int_{\mathbb{T}^2}(\sum_j |G(\theta(s))(f_j)|^2)^{p/2} d\xi)ds)^{1/p} \\
\leq & E\|\theta_0\|_{L^p}^p + \varepsilon E\sup_{s\in[0,t]}\|\theta(s)\|_{L^p}^p + C_1 E\int_0^t \|\theta(s)\|_{L^p}^p ds + C_2 \qquad (4.9) \\
\leq & E\|\theta_0\|_{L^p}^p + \varepsilon E\sup_{s\in[0,t]}\|\theta(s)\|_{L^p}^p + C_1\int_0^t E\sup_{s\in[0,\sigma]}\|\theta(s)\|_{L^p}^p d\sigma + C_2.
\end{aligned}
$$

By Gronwall's lemma, the assertion follows. $\square$

Theorem 4.5 Let $\alpha \in (0,1)$. Under Assumption (G.2), there exists a martingale solution $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P, W, \theta)$ to (3.1).

Proof The proof is similar to the one for Theorem 4.2. The only difference is the proof of $\theta(\cdot,\omega) \in C([0,T]; H^{-\beta})$. Here by Aldous' criterion it suffices to check that for all stopping times $\tau_n \leq T$ and $\delta_n \to 0$,

$$\lim_n E\|\theta_n(\tau_n+\delta_n) - \theta_n(\tau_n)\|_{H^{-\beta}} = 0.$$

This can however be checked easily. $\square$

5 Uniqueness of solutions

In this section, we will prove pathwise uniqueness for equation (3.1). First we prove uniqueness in the subcritical case.

Theorem 5.1 Assume $\alpha > \frac{1}{2}$. If G satisfies the following condition

$$\|\Lambda^{-1/2}(G(u)-G(v))\|_{L_2(K,H)}^2 \leq \beta|\Lambda^{-1/2}(u-v)|^2 + \beta_1|\Lambda^{\alpha-\frac{1}{2}}(u-v)|^2, \tag{5.1}$$

for all $u, v \in H^\alpha$, for some $\beta \in \mathbb{R}$ independent of u, v, and $\beta_1 < 2\kappa$, then (3.1) admits at most one probabilistically strong solution in the sense of Definition 3.1 such that

$$\sup_{t\in[0,T]}\|\theta(t)\|_{L^q} < \infty, \qquad P-a.s.,$$

with $0 < 1/q < \alpha - \frac{1}{2}$, and

$$E \sup_{t\in[0.T]} |\Lambda^{-1/2}\theta(t)|^2 < \infty.$$

Remark If in Remark 4.4 $b_k = \mu_k e_k$ for $\mu_k \in \mathbb{R}$, then (5.1) is satisfied.

Proof Let θ_1, θ_2 be two solutions of (3.1), and let $\{e_k\}_{k\in\mathbb{N}}$ be the eigenbasis of A from above. Then their difference $\theta = \theta_1 - \theta_2$ satisfies

$$\langle\psi, \theta(t)\rangle - \int_0^t \langle u\cdot\nabla\psi, \theta_1\rangle ds - \int_0^t \langle u_2\cdot\nabla\psi, \theta\rangle ds + \kappa \int_0^t \langle\theta, \Lambda^{2\alpha}\psi\rangle ds = \int_0^t \langle\psi, (G(\theta_1) - G(\theta_2))dW(s)\rangle. \tag{5.2}$$

Now set $\phi_k = \langle e_k, \theta(t)\rangle, \varphi_k = \langle\Lambda^{-1}e_k, \theta(t)\rangle$. Itô's formula and (5.2) yield

$$\begin{aligned}
\phi_k\varphi_k =& \int_0^t \phi_k d\varphi_k + \int_0^t \varphi_k d\phi_k + \langle\varphi_k, \phi_k\rangle(t)\\
=&2\int_0^t \langle u\cdot\nabla e_k, \theta_1\rangle\langle\Lambda^{-1}\theta, e_k\rangle + \langle u_2\cdot\nabla e_k, \theta\rangle\langle\Lambda^{-1}\theta, e_k\rangle - \kappa\langle\Lambda^{2\alpha}e_k, \theta\rangle\langle\Lambda^{-1}\theta, e_k\rangle ds\\
&+2\int_0^t \langle\Lambda^{-1}\theta, e_k\rangle\langle e_k, (G(\theta_1) - G(\theta_2))dW(s)\rangle + \int_0^t \langle(G(\theta_1) - G(\theta_2))^* e_k, (G(\theta_1) - G(\theta_2))^*\Lambda^{-1}e_k\rangle ds.
\end{aligned} \tag{5.3}$$

The dominated theorem implies:

$$\sum_{k\le N} \int_0^t \langle u\cdot\nabla e_k, \theta_1\rangle\langle\Lambda^{-1}\theta, e_k\rangle ds \to \int_0^t {}_{H^{-1}}\langle u\cdot\nabla\theta_1, \Lambda^{-1}\theta\rangle_{H^1} ds, N \to \infty,$$

$$\sum_{k\le N} \int_0^t \langle u_2\cdot\nabla e_k, \theta\rangle\langle\Lambda^{-1}\theta, e_k\rangle ds \to \int_0^t {}_{H^{-1}}\langle u_2\cdot\nabla\theta, \Lambda^{-1}\theta\rangle_{H^1} ds, N \to \infty,$$

and

$$\sum_{k\le N} \int_0^t \langle\Lambda^{2\alpha}e_k, \theta\rangle\langle\Lambda^{-1}\theta, e_k\rangle ds \to \int_0^t \langle\theta, \Lambda^{2\alpha-1}\theta\rangle ds, N \to \infty.$$

Furthermore, since

$$\int_0^t |\Lambda^{-1/2}\theta|^2 \|\Lambda^{-1/2}(G(\theta_1) - G(\theta_2))\|^2_{L_2(K,H)} ds \le C \sup_{s\le t} |\theta(s)|^2 \int_0^t \|\Lambda^{-1/2}(G(\theta_1) - G(\theta_2))\|^2_{L_2(K,H)} ds < \infty,$$

we obtain

$$\sum_{k\le N} \int_0^t \langle\Lambda^{-1}\theta, e_k\rangle\langle e_k, (G(\theta_1) - G(\theta_2))dW(s)\rangle \to M_t := \int_0^t \langle\Lambda^{-1/2}\theta, \Lambda^{-1/2}(G(\theta_1) - G(\theta_2))dW(s)\rangle, N \to \infty.$$

Finally, the following inequality holds:

$$\sum_{k\le N} \int_0^t \langle(G(\theta_1) - G(\theta_2))^* e_k, (G(\theta_1) - G(\theta_2))^*\Lambda^{-1}e_k\rangle ds \le \int_0^t \|\Lambda^{-1/2}(G(\theta_1) - G(\theta_2))\|^2_{L_2(K,H)} ds.$$

Thus, summing up over $k \le N$ in (5.3) and letting $N \to \infty$ we obtain

$$\begin{aligned}
&|\Lambda^{-1/2}\theta|^2 + 2\kappa \int_0^t |\Lambda^{\alpha-\frac{1}{2}}\theta|^2 ds \\
\le& 2M(t) + 2\int_0^t {}_{H_{-1}}\langle u\cdot\nabla\theta_1, \Lambda^{-1}\theta\rangle_{H_1} + {}_{H_{-1}}\langle u_2\cdot\nabla\theta, \Lambda^{-1}\theta\rangle_{H_1} ds \\
&+ \int_0^t \|\Lambda^{-1/2}(G(\theta_1)-G(\theta_2))\|^2_{L_2(K,H)} ds.
\end{aligned}$$

By [Re95] we have

$$ {}_{H^{-1}}\langle u\cdot\nabla\theta_1, \Lambda^{-1}\theta\rangle_{H^1} = 0,$$

and

$$\begin{aligned}
|{}_{H_{-1}}\langle u_2\cdot\nabla\theta, \Lambda^{-1}\theta\rangle_{H_1}| \le& \|u_2\|_{L^q}\|\theta\|_{L^p}\|\nabla\Lambda^{-1}\theta\|_{L^p} \le C\|u_2\|_{L^q}\|\theta\|_{H^{1/q}}\|\nabla\Lambda^{-1}\theta\|_{H^{1/q}} \\
\le& C\|\theta_2\|_{L^q}\|\Lambda^{-1}\theta\|^2_{H^{1+\frac{1}{q}}} \le C\|\theta_2\|_{L^q}\|\Lambda^{-1}\theta\|^{2/N}_{H^{1/2}}\|\Lambda^{-1}\theta\|^{2(1-\frac{1}{N})}_{H^{\frac{1}{2}+\alpha}} \\
\le& \varepsilon|\Lambda^{\alpha-\frac{1}{2}}\theta|^2 + C\|\theta_2\|^N_{L^q}|\Lambda^{-1/2}\theta|^2.
\end{aligned}$$

Here $\frac{1}{q}+\frac{2}{p}=1$ for $0<1/q<\alpha-1/2, N=\frac{\alpha}{\alpha-\frac{1}{2}-\frac{1}{q}}$ and we use $H^{1/q}\hookrightarrow L^p$ continuously.

Now by (5.1) we have

$$|\Lambda^{-1/2}\theta|^2 \le M(t) + \int_0^t C\|\theta_2\|^N_{L^q}|\Lambda^{-1/2}\theta|^2 ds + \beta\int_0^t |\Lambda^{-1/2}(\theta_1-\theta_2)|^2 ds.$$

Let

$$\tau_n^1 := \inf\{t>0, \|\theta_2(t)\|_{L^q} > n\}.$$

Then by the weak continuity of θ_2, τ_n^1 are stopping times with respect to $\mathcal{F}_{t+}$, $(\mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s)$and $\|\theta_2(t\wedge\tau_n^1)\|_{L^q} \le n$ for large n. Also let τ_n^2 be a localizing sequence of stopping times for M and $\tau_n := \tau_n^1 \wedge \tau_n^2$. Then, since $M(t\wedge\tau_n)$ is a martingale with respect to $\mathcal{F}_{t+}$, we get

$$\begin{aligned}
E|\Lambda^{-1/2}\theta(t\wedge\tau_n)|^2 \le& Cn^N E\int_0^{t\wedge\tau_n} |\Lambda^{-1/2}\theta|^2 ds + \beta E\int_0^{t\wedge\tau_n} |\Lambda^{-1/2}\theta|^2 ds \\
=& C(n)\int_0^t E|\Lambda^{-1/2}\theta(s\wedge\tau_n)|^2 ds + \beta\int_0^t E|\Lambda^{-1/2}\theta(s\wedge\tau_n)|^2 ds.
\end{aligned}$$

By Gronwall's inequality, we get $|\Lambda^{-1/2}\theta(t\wedge\tau_n)|^2 = 0$ $P-a.s.$, and recalling that $\tau_n \to T$ as $n\to\infty$, we obtain that $\theta(t)=0$ $P-a.s.$ for $t\le T$, thus completing the proof. □

From the proof we immediately obtain the following result.

Corollary 5.2 Assume $\alpha > \frac{1}{2}$. If there exists a probabilistically strong solution θ_2 in the sense of Definition 3.1 such that

$$\sup_{t\in[0,T]} \|\theta_2(t)\|_{L^q} < \infty, \qquad P-a.s.$$

for some q with $0 < 1/q < \alpha - \frac{1}{2}$ and G satisfies (5.1), then θ_2 is the only solution to (3.1) such that

$$E \sup_{t\in[0,T]} |\Lambda^{-1/2}\theta_2(t)|^2 < \infty.$$

Thus, combining Theorem 5.1, Theorem 4.3 and the Yamada-Watanabe Theorem in [Ku07], we get the following existence and uniqueness result.

Theorem 5.3 Assume $\alpha > \frac{1}{2}$ and that G satisfies (5.1), (G.1) (4.5) and (4.6) for some p with $0 < 1/p < \alpha - \frac{1}{2}$. Then for each initial condition $\theta_0 \in L^p$, there exists a pathwise unique probabilistically strong solution θ of equation (3.1) over $[0, T]$ with initial condition $\theta(0) = \theta_0$ such that

$$\sup_{t\in[0,T]} \|\theta(t)\|_{L^p} < \infty, \qquad P-a.s,$$

and

$$E \sup_{t\in[0,T]} |\Lambda^{-1/2}\theta(t)|^2 < \infty.$$

Combining Theorem 5.3 and Corollary 5.2, we obtain the following more general existence and uniqueness result.

Theorem 5.4 Assume $\alpha > \frac{1}{2}$ and that G satisfies (5.1), (G.1), (4.5) and (4.6) with $0 < 1/p < \alpha - \frac{1}{2}$. Then for each initial condition $\theta_0 \in L^p$, there exists a pathwise unique probabilistically strong solution θ of equation (3.1) over $[0, T]$ with initial condition $\theta(0) = \theta_0$ such that

$$E \sup_{t\in[0,T]} |\Lambda^{-1/2}\theta(t)|^2 < \infty.$$

Moreover, the solution satisfies

$$\sup_{t\in[0,T]} \|\theta(t)\|_{L^p} < \infty, \qquad P-a.s..$$

Theorem 5.5 (Markov property) Assume $\alpha > \frac{1}{2}$ and that G satisfies (G.1),(5.1) and (4.5),(4.6) with $0 < 1/p < \alpha - \frac{1}{2}$. If $\theta_0 \in L^p$, then for every bounded, $\mathcal{B}(H)$-measurable $F : H \to \mathbb{R}$, and all $s, t \in [0, T]$, $s \le t$

$$E(F(\theta(t))|\mathcal{F}_s)(\omega) = E(F(\theta(t, s, \theta(s)(\omega)))) \text{ for } P-a.s.\omega \in \Omega.$$

Here $\theta(t, s, \theta(s)(\omega))$ denotes the solution to (3.1) starting from $\theta(s)$ at time s satisfying

$$E \sup_{t\in[s,T]} |\Lambda^{-1/2}\theta(t)|^2 < \infty.$$

Proof By Theorem 5.4, we have $\theta(t) = \theta(t, s, \theta(s))$ P-a.s.. Then by the same arguments as in [PR07, Proposition 4.3.3] and the Yamada-Watanabe Theorem in [RSZ08], the assertion follows. □

Set

$$p_t(x, dy) := P \circ (\theta(t, x))^{-1}(dy), 0 \le t \le T, x \in H.$$

Here and in the following, we use $\theta(t,x)$ to denote a solution with initial value x. We set for $\mathcal{B}(H)$-measurable $F: H \to \mathbb{R}$, and $t \in [0,T], x \in H$

$$P_t F(x) := \int F(y) p_t(x, dy),$$

provided F is $p_t(x,dy)$-integrable. Then by Theorem 5.5, we have for $F: H \to \mathbb{R}$, bounded and $\mathcal{B}(H)$-measurable, $s,t \geq 0$,

$$P_s(P_t F)(x) = P_{s+t}F(x), x \in L^p \text{ with } 0 < 1/p < \alpha - \frac{1}{2}.$$

Theorem 5.6 Let $\alpha \in (0,1)$. If G satisfies the Lipschitz condition

$$\|G(u) - G(v)\|^2_{L_2(K,H)} \leq \beta |u - v|^2 + \beta_1 \|u - v\|^2_{H_\alpha} \tag{5.4}$$

for all $u, v \in H^\alpha$, for some $\beta \in \mathbb{R}$ independent of u, v, and $\beta_1 < 2\kappa$, then (3.1) admits at most one solution in the sense of Definition 3.1 such that

$$E \sup_{t\in[0,T]} |\theta(t)|^4 < \infty$$

and

$$\int_0^T \|\Lambda^{1-\alpha+\varepsilon}\theta(t)\|^q_{L^p} dt < \infty, \frac{1}{p} + \frac{\alpha}{q} = \frac{\alpha+\varepsilon}{2} \qquad P - a.s.,$$

where $\varepsilon \in (0,\alpha]$ and $q < \infty$.

Proof By the same argument as in the proof of Theorem 5.1, we get (5.2). Set $\phi_k := \langle e_k, \theta(t)\rangle$. Then Itô's formula and (5.2) yield

$$\begin{aligned}
\phi_k^2 =& 2\int_0^t \phi_k d\phi_k + [\phi_k](t) \\
=& 2\int_0^t \langle u \cdot \nabla e_k, \theta_1\rangle\langle\theta, e_k\rangle + \langle u_2 \cdot \nabla e_k, \theta\rangle\langle\theta, e_k\rangle - \kappa\langle\Lambda^{2\alpha}e_k, \theta\rangle\langle\theta, e_k\rangle ds \\
&+2\int_0^t \langle\theta, e_k\rangle\langle e_k, (G(\theta_1) - G(\theta_2))dW(s)\rangle + \int_0^t \langle(G(\theta_1) - G(\theta_2))^* e_k, (G(\theta_1) - G(\theta_2))^* e_k\rangle ds.
\end{aligned} \tag{5.5}$$

Since

$$\begin{aligned}
|\langle u_2 \cdot \nabla\theta, \varphi\rangle| \leq& \|\Lambda^{1-\alpha+\varepsilon}\varphi\|_{L^{p_1}} \|\Lambda^{\alpha-\varepsilon}(u_2\theta)\|_{L^{p_1'}} \leq C\|\Lambda^{1-\alpha+\varepsilon}\varphi\|_{L^{p_1}}(\|\theta_2\|_{L^{q_1}}\|\Lambda^{\alpha-\varepsilon}\theta\|_{L^{q_2}} + \|\theta\|_{L^{q_1}}\|\Lambda^{\alpha-\varepsilon}\theta_2\|_{L^{q_2}}) \\
\leq& C\|\Lambda^{1-\alpha+\varepsilon}\varphi\|_{L^{p_1}}(|\theta_2| + |\theta_1|)^{2-\beta-\gamma}(|\Lambda^\alpha\theta_1| + |\Lambda^\alpha\theta_2|)^{\beta+\gamma} \\
\leq& C\|\Lambda^{1-\alpha+\varepsilon}\varphi\|^{p_2'}_{L^{p_1}}(|\theta_2|^2 + |\theta_1|^2) + |\Lambda^\alpha\theta_2|^2 + |\Lambda^\alpha\theta_1|^2,
\end{aligned}$$

the term $u_2 \cdot \nabla\theta$ can be considered as an element in $(H^{1-\alpha+\varepsilon,p_1})'$. Here $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p_1'}$ and $\beta + \gamma = \frac{1}{\alpha}(\alpha - \varepsilon + \frac{2}{p_1}), p_2 = 2/(\beta+\gamma)$.

By a similar calculation for $\langle u\cdot\nabla\theta_1,\theta\rangle$, the dominated convergence theorem yields the following:

$$\sum_{k\leq N}\int_0^t\langle u\cdot\nabla e_k,\theta_1\rangle\langle\theta,e_k\rangle ds\to\int_0^t {}_{(H^{1-\alpha+\varepsilon,p_1})'}\langle u\cdot\nabla\theta_1,\theta\rangle_{H^{1-\alpha+\varepsilon,p_1}}ds, N\to\infty,$$

$$\sum_{k\leq N}\int_0^t\langle u_2\cdot\nabla e_k,\theta\rangle\langle\theta,e_k\rangle ds\to\int_0^t {}_{(H^{1-\alpha+\varepsilon,p_1})'}\langle u_2\cdot\nabla\theta,\theta\rangle_{H^{1-\alpha+\varepsilon,p_1}}ds, N\to\infty,$$

and

$$\sum_{k\leq N}\int_0^t\langle\Lambda^{2\alpha}e_k,\theta\rangle\langle\theta,e_k\rangle ds\to\int_0^t\langle\theta,\Lambda^{2\alpha}\theta\rangle ds, N\to\infty.$$

Furthermore, since

$$\int_0^t|\theta|^2\|G(u)-G(v)\|^2_{L_2(K,H)}ds\leq C\sup_{s\leq t}|\theta(s)|^2\int_0^t\|G(u)-G(v)\|^2_{L_2(K,H)}ds<\infty,$$

we obtain

$$\sum_{k\leq N}\int_0^t\langle\theta,e_k\rangle\langle e_k,(G(\theta_1)-G(\theta_2))dW(s)\rangle\to M_t:=\int_0^t\langle\theta,(G(\theta_1)-G(\theta_2))dW(s)\rangle, N\to\infty.$$

Finally, the following inequality holds:

$$\sum_{k\leq N}\int_0^t\langle(G(\theta_1)-G(\theta_2))^*e_k,(G(\theta_1)-G(\theta_2))e_k\rangle ds\leq\int_0^t\|G(\theta_1)-G(\theta_2)\|^2_{L_2(K,H)}ds.$$

Thus, summing up over $k\leq N$ in (5.5) and letting $N\to\infty$ we obtain

$$\begin{aligned}|\theta(t)|^2+2\kappa\int_0^t|\Lambda^\alpha\theta|^2ds\leq&2M(t)+2\int_0^t\langle u\cdot\nabla\theta_1,\theta\rangle+\langle u_2\cdot\nabla\theta,\theta\rangle ds\\&+\int_0^t\|(G(\theta_1)-G(\theta_2))\|^2_{L_2(K,H)}ds.\end{aligned}$$

We have

$$\langle u_2\cdot\nabla\theta,\theta\rangle=0,$$

and by a similar calculation as in the proof of [Ju05, Theorem 3.3], we have

$$\begin{aligned}|\langle u\cdot\nabla\theta_1,\theta\rangle|\leq&\|\Lambda^{1-\alpha+\varepsilon}\theta_1\|_{L^{p_1}}\|\Lambda^{\alpha-\varepsilon}(u\theta)\|_{L^{p_1'}}\leq C\|\Lambda^{1-\alpha+\varepsilon}\theta_1\|_{L^{p_1}}\|\theta\|_{L^{q_1}}\|\Lambda^{\alpha-\varepsilon}\theta\|_{L^{q_2}}\\\leq&C\|\Lambda^{1-\alpha+\varepsilon}\theta_1\|_{L^{p_1}}|\theta|^{2-\beta-\gamma}|\Lambda^\alpha\theta|^{\beta+\gamma}\\\leq&\varepsilon|\Lambda^\alpha\theta|^2+C\|\Lambda^{1-\alpha+\varepsilon}\theta_1\|^{p_2'}_{L^{p_1}}|\theta|^2.\end{aligned}$$

Here $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{p_1'}$ and $\beta+\gamma=\frac{1}{\alpha}(\alpha-\varepsilon+\frac{2}{p_1}), p_2=2/(\beta+\gamma)$.

Now by (5.4) we have

$$|\theta(t)|^2\leq M(t)+\int_0^t C\|\Lambda^{1-\alpha+\varepsilon}\theta_1\|^{p_2'}_{L^{p_1}}|\theta|^2ds+\beta\int_0^t|(\theta_1-\theta_2)|^2ds.$$

Define the stopping time

$$\tau_n := \inf\{t > 0, \int_0^t \|\Lambda^{1-\alpha+\varepsilon}\theta_1\|_{L^{p_1}}^{p_2'} ds > n\}.$$

Applying Gronwall's lemma, we have

$$|\theta(t \wedge \tau_n)|^2 \leq |M(t \wedge \tau_n)| e^{\int_0^{t\wedge\tau_n} C\|\Lambda^{1-\alpha+\varepsilon}\theta_1\|_{L^{p_1}}^{p_2'} ds + \beta t} \leq |M(t \wedge \tau_n)| e^{Cn+\beta t}.$$

Consequently,

$$\begin{aligned} E|\theta(t \wedge \tau_n)|^4 \leq & e^{2Cn+2\beta t} E \int_0^{t\wedge\tau_n} |\theta|^2 \|G(\theta_1) - G(\theta_2)\|_{L_2(K,H)}^2 ds \\ \leq & \beta^2 e^{2Cn+2\beta t} \int_0^t E|\theta(s \wedge \tau_n)|^4 ds. \end{aligned}$$

By Gronwall's lemma, we get $|\theta(t \wedge \tau_n)|^2 = 0$ $P - a.s.$, and recalling that $\tau_n \to T$ as $n \to \infty$, we obtain that $\theta(t) = 0$ $P - a.s.$ for $t \leq T$, thus completing the proof. $\square$

Remark 5.7 For $\alpha = 1/2$, consider

$$d\theta = [A\theta + u \cdot \nabla\theta)]dt + \sum_{j=1}^m b_j \theta \circ dw_j(t), \tag{5.6}$$

for $b_j \in \mathbb{R}$, and independent 1-dimensional Brownian motions w_j. Consider the process

$$\beta(t) = e^{-\sum_{j=1}^m b_j w_j(t)}.$$

Then, the process $v(t)$ defined by transformation

$$v(t) = \beta(t)\theta(t),$$

satisfies the equation (which depends on a random parameter)

$$\frac{dv}{dt} = Av + \beta^{-1} u_v \cdot \nabla v. \tag{5.7}$$

Then by the same argument as in the proof of [CC04, Theorem 3.1], we obtain the local existence and uniqueness of smooth solutions starting from H^1 periodic initial data. More precisely, for P-almost every $\omega \in \Omega$, there exists a time $t(\omega, |\Lambda\theta_0|)$, such that $v \in C((0,t), H^m)$ for any $m > 0$. On the other hand, by the same arguments as in [CV06, Section 2], we obtain for any $T > 0$, there exists $M(\omega, |\Lambda\theta_0|)$ such that

$$\|v(t,\cdot)\|_\infty \leq M \text{ for } t \in [0,T].$$

Then

$$\|\beta^{-1}u_v(t,\cdot)\|_{\mathrm{BMO}} \leq M_1(\omega, |\Lambda\theta_0|, T) \text{ for } t \in [0,T].$$

Hence by [KN09, Theorem 1.1], we obtain that there exists $\gamma(\omega, |\Lambda\theta_0|, T) > 0$, such that

$$\|v(\cdot,t)\|_{C^\gamma(\mathbb{T}^2)} \leq C(\omega, |\Lambda\theta_0|, T).$$

Then by the same arguments as in the proofs of [CW07, Theorem 3.1] and [CV06, Theorem 10], we obtain

$$\|v(\cdot,t)\|_{C^1(\mathbb{T}^2)} \leq C_1(\omega, |\Lambda\theta_0|, T) \text{ for } t \in [0,T].$$

By this a-priori bound and the local existence, we obtain a global regular solution v for P-almost every $\omega \in \Omega$. Define

$$\theta(t,\xi) := \beta(t)^{-1} v(t,\xi).$$

Then we obtain a solution θ such that

$$\sup_{t\in[0,T]} \|\Lambda^{1-\alpha+\varepsilon}\theta\|_{L^p} < \infty, \frac{1}{p} \leq \frac{\alpha+\varepsilon}{2} \qquad P-a.s..$$

So, for this special linear multiplicative noise, we obtain a solution satisfying the condition in Theorem 5.6. Unfortunately, we don't get this result for more general noise and $\alpha = \frac{1}{2}$ since the results and the method in the deterministic case (e.g. [CV06], [KNV07], [KN09]) cannot be applied directly

6 Markov selections in the general case

In this section, we will use [GRZ09, Theorem 4.7] to get an almost sure Markov family $(P_x)_{x\in L^2}$ for Eq. (3.1). Here we will use the same notation as in [GRZ09]. Below we choose

$$H = \mathbb{Y} = L^2(\mathbb{T}^2)$$

and

$$\mathbb{X} = (H^{2+2\alpha})^*, \qquad \mathbb{X}^* = H^{2+2\alpha}.$$

Then $\mathbb{X}$ is a Hilbert space and $\mathbb{X}^* \subset \mathbb{Y}$ compactly. Let $\mathcal{E} = \{e_i, i \in \mathbb{N}\}$ be the orthonormal basis of H introduced in Section 2. We define the operator $\mathcal{A}$ as follows: for $\theta \in C^\infty(\mathbb{T}^2)$

$$\mathcal{A}(\theta) := -\kappa(-\Delta)^\alpha\theta - u \cdot \nabla\theta,$$

where u satisfies (1.3). Then by Lemma 6.3 below, $\mathcal{A}$ can be extended to an operator $\mathcal{A} : H \to \mathbb{X}$. For θ not in H define $\mathcal{A}(\theta) := \infty$.

Set

$$\Omega := C([0,\infty); \mathbb{X}),$$

and let $\mathcal{B}$ denote the σ-field of Borel sets of Ω and let $\mathcal{P}(\Omega)$ denote the set of all probability measures on $(\Omega, \mathcal{B})$. Define the canonical process $\xi : \Omega \to \mathbb{X}$ as

$$\xi_t(\omega) = \omega(t).$$

For each t, $\mathcal{B}_t = \sigma(\xi_s : 0 \leq s \leq t)$. Given $P \in \mathcal{P}(\Omega)$ and $t > 0$, let $P(\cdot|\mathcal{B}_t)(\omega)$ denote a regular conditional probability distribution of P given $\mathcal{B}_t$. In particular, $P(\cdot|\mathcal{B}_t)(\omega) \in \mathcal{P}(\Omega)$ for every $\omega \in \Omega$ and for any bounded $\mathcal{B}$-measurable function f on Ω

$$E^P[f|\mathcal{B}_t] = \int_\Omega f(y)P(dy|\mathcal{B}_t), \quad P-a.s.,$$

and there exists a P-null set $N \in \mathcal{B}_t$ such that for every ω not in N

$$P(\cdot|\mathcal{B}_t)(\omega)_{|\mathcal{B}_t} = \delta_\omega (= \text{ Dirac measure at } \omega),$$

hence

$$P(\{y : y(s) = \omega(s), s \in [0,t]\}|\mathcal{B}_t)(\omega) = 1.$$

In particular, we can consider $P(\cdot|\mathcal{B}_t)(\omega)$ as a measure on $(\Omega^t, \mathcal{B}^t)$, i.e.,

$$P(\cdot|\mathcal{B}_t)(\omega) \in \mathcal{P}(\Omega^t),$$

where $\Omega^t := C([t,\infty); \mathbb{X})$ and $\mathcal{B}^t := \sigma(\xi_s : s \geq t)$.

We say $P \in \mathcal{P}(\Omega)$ is concentrated on the paths with values in H, if there exists $A \in \mathcal{B}$ with $P(A) = 1$ such that $A \subset \{\omega \in \Omega : \xi_t(\omega) \in H, \forall t \geq 0\}$. The set of such measures is denoted by $\mathcal{P}_H(\Omega)$. The shift operator $\Phi_t : \Omega \to \Omega^t$ is defined by

$$\Phi_t(\omega)(s) = \omega(s-t), \ s \geq t.$$

Following [GRZ09, Definitions 2.5], we introduce the following notions.

Definition 6.1 A family $(P_x)_{x \in H}$ of probability measures in $\mathcal{P}_H(\Omega)$, is called an *almost sure Markov family* if for any $A \in \mathcal{B}$, $x \mapsto P_x(A)$ is $\mathcal{B}(H)/\mathcal{B}([0,1])$-measurable, and for each $x \in H$ there exists a Lebesgue null set $T_{P_x} \subset (0,\infty)$ such that for all t not in T_{P_x} and P_x-almost all $\omega \in \Omega$

$$P_x(\cdot|\mathcal{B}_t)(\omega) = P_{\omega(t)} \circ \Phi_t^{-1}.$$

We now introduce the following notion of a martingale solution to Eq. (3.1) and write $\xi(t)$ instead of ξ_t.

Definition 6.2 Let $x_0 \in H$. A probability measure $P \in \mathcal{P}(\Omega)$ is called a martingale solution of Eq. (3.1) with initial value x_0, if:
(M1) $P(\xi(0) = x_0) = 1$ and for any $n \in \mathbb{N}$

$$P\{\xi \in \Omega : \int_0^n \|\mathcal{A}(\xi(s))\|_{\mathbb{X}} ds + \int_0^n \|G(\xi(s))\|^2_{L_2(K;H)} ds < +\infty\} = 1;$$

(M2) for every $l \in \mathcal{E}$, the process

$$M_l(t,\xi) :=_{\mathbb{X}} \langle \xi(t), l \rangle_{\mathbb{X}^*} - \int_0^t {}_{\mathbb{X}}\langle \mathcal{A}(\xi(s)), l \rangle_{\mathbb{X}^*} ds$$

is a continuous square-integrable $\mathcal{F}_t$-martingale under P, whose quadratic variation process is given by

$$\langle M_l \rangle(t,\xi) := \int_0^t \|G^*(\xi(s))(l)\|^2_K ds,$$

where the asterisk denotes the adjoint operator of $G(\xi(s))$;

(M3) for any $p \in \mathbb{N}$, there exist a continuous positive real function $t \mapsto C_{t,p}$ (only depending on p and $\mathcal{A}, G$), a lower semi-continuous positive real functional $\mathcal{N}_p : \mathbb{Y} \to [0, \infty]$, and a Lebesgue null set $T_P \subset (0, \infty)$ such that for all $0 \le s \in [0, \infty)\backslash T_P$ and for all $t \ge s$

$$E^P[\sup_{r\in[s,t]} |\xi(r)|^{2p} + \int_s^t \mathcal{N}_p(\xi(r))dr|\mathcal{B}_s] \le C_{t-s}(|\xi(s)|^{2p} + 1).$$

Remark This definition of martingale solution is different from Definition 4.1. By [On05, Theorem 2], it follows that it is more general than Definition 4.1. We will use Definition 6.2 below.

First, we prove the following lemma.

Lemma 6.3 For any $\theta_1, \theta_2 \in C^\infty(\mathbb{T}^2)$,

$$\|(-\Delta)^\alpha\theta_1 - (-\Delta)^\alpha\theta_2\|_{\mathbb{X}} \le C_1|\theta_1 - \theta_2|,$$

$$\|u_1 \cdot \nabla\theta_1 - u_2 \cdot \nabla\theta_2\|_{\mathbb{X}} \le C_2(|\theta_1| + |\theta_2|)|\theta_1 - \theta_2|,$$

for constants C_1, C_2. In particular, the operator $\mathcal{A} : C^\infty(\mathbb{T}^2) \to \mathbb{X}$ extends to an operator $\mathcal{A} : H \to \mathbb{X}$ by continuity.

Proof We only prove the second assertion, the first can be proved analogously. By the Sobolev embedding theorem we have

$$\begin{aligned}
&\|u_1 \cdot \nabla\theta_1 - u_2 \cdot \nabla\theta_2\|_{\mathbb{X}} \\
=&\sup_{w\in C^\infty(\mathbb{T}^2):\|w\|_{H^{2+2\alpha}}\le 1} |\langle u_1 \cdot \nabla\theta_1 - u_2 \cdot \nabla\theta_2, w\rangle| \\
=&\sup_{w\in C^\infty(\mathbb{T}^2):\|w\|_{H^{2+2\alpha}}\le 1} |\langle u_1 \cdot \nabla w, \theta_1\rangle - \langle u_2 \cdot \nabla w, \theta_2\rangle| \\
=&\sup_{w\in C^\infty(\mathbb{T}^2):\|w\|_{H^{2+2\alpha}}\le 1} |\langle (u_1 - u_2) \cdot \nabla w, \theta_1\rangle + \langle u_2 \cdot \nabla w, \theta_1 - \theta_2\rangle| \\
\le& C[\sup_{w\in C^\infty(\mathbb{T}^2):\|w\|_{H^{2+2\alpha}}\le 1} \|\nabla w\|_{C(\mathbb{T}^2)}](|u_1 - u_2| \cdot |\theta_1| + |\theta_1 - \theta_2| \cdot |u_2|) \\
\le& C(|\theta_1| + |\theta_2|)|\theta_1 - \theta_2|.
\end{aligned}$$

In the last inequality we use (2.1) and the constant C changes from line to line. $\square$

In order to use [GRZ09, Theorem 4.7], we define the functional $\mathcal{N}_1$ on $\mathbb{Y}$ as follows:

$$\mathcal{N}_1(\theta) := \begin{cases} |\Lambda^\alpha\theta|^2, & \text{if } \theta \in H^\alpha, \\ +\infty, & \text{otherwise }. \end{cases}$$

It is obvious that $\mathcal{N}_1 \in \mathfrak{U}^2$, defined in [GRZ09, Section 4]. We recall that a lower semicontinuous function $\mathcal{N} : \mathbb{Y} \to [0, \infty]$ belongs to $\mathfrak{U}^2$ if $\mathcal{N}(x) = 0$ implies $x = 0$, $\mathcal{N}(cy) \le c^2\mathcal{N}(y), \forall c \ge 0, y \in \mathbb{Y}$ and $\{y \in \mathbb{Y} : \mathcal{N}(y) \le 1\}$ is relatively compact in $\mathbb{Y}$.

Theorem 6.4 Let $\alpha \in (0, 1)$ and assume G satisfies (G.1). Then for each $x_0 \in H$, there exists a martingale solution $P \in \mathcal{P}(\Omega)$ starting from x_0 to Eq. (3.1) in the sense of Definition 6.2.

Proof We only need to check (C1)-(C3) in [GRZ09, Section 4] for the above $\mathcal{A}$ and G.

(C1) holds since Lemma 6.3 implies demi-continuity of $\mathcal{A}$ and G.

(C2) follows, because noting that for $\theta \in \mathbb{X}^*$

$$\langle u \cdot \nabla\theta, \theta\rangle = 0,$$

we have

$$\langle \mathcal{A}(\theta), \theta\rangle = -\mathcal{N}_1(\theta).$$

Also (C3) is clear since by Lemma 6.3

$$\|\mathcal{A}(\theta)\|_{\mathbb{X}} \le C|\theta|^2$$

and

$$\|G(\theta)\|_{L_2(K;H)} \le C(|\theta| + 1).$$

□

The set of all such martingale solutions with initial value x_0 is denoted by $\mathcal{C}(x_0)$. Using [GRZ09, Theorem 4.7], we now obtain the following:

Theorem 6.5 Let $\alpha \in (0,1)$. Assume G satisfies (G.1). Then there exists an almost sure Markov family $(P_{x_0})_{x0\in H}$ for Eq. (3.1) and $P_{x_0} \in \mathcal{C}(x_0)$ for each $x_0 \in H$.

7 Ergodicity for $\alpha > \frac{2}{3}$

In this section, we assume that $\alpha > \frac{2}{3}$, $K = H$, and that G satisfies:

Assumption 7.1 There are an isomophism Q_0 of H and a number $s \ge 1$ such that $G = A^{-\frac{s+\alpha}{2\alpha}} Q_0^{1/2}$, and furthermore, G satisfies (4.5) for some fixed $p \in ((\alpha - \frac{1}{2})^{-1}, \infty)$ and $f_j = e_j$, (which is e.g. always the case if $Q_0 = I$).

For $x := \theta_0 \in L^p$, let P_x denote the law of the corresponding solution θ to (3.1). Then by Theorems 5.4 and 5.5 the measures $P_x, x \in L^p$, form a Markov process. Let $(P_t)_{t\ge0}$ be the associated transition semi-group on $\mathcal{B}_b(H)$, defined as

$$P_t(\varphi)(x) := E_x[\varphi(\xi_t)], \qquad x \in L^p, \varphi \in \mathcal{B}_b(H), \tag{7.1}$$

where E_x denotes expectation under P_x.

7.1 The strong Feller property for $\alpha > \frac{2}{3}$

In this subsection we prove that its transition semigroup has the strong Feller property under appropriate conditions.

Remark 7.1.1 (i) Since in our case $\alpha < 1$, the linear part $(-\Delta)^\alpha$ in (1.1) is less regularizing. As $G = A^{-\frac{s+\alpha}{2\alpha}} Q_0^{1/2}$, we get the trajectories z of the associated O-U process to be in $C([0,\infty), H^{s+2\alpha-1-\varepsilon})$ for every $\varepsilon > 0$ (c.f. [DZ92, Theorem 5.16], [DO06, Proposition 3.1]). However, in order to prove the weak-strong uniqueness principle (see (7.2) and Theorem 7.1.3

below) and the strong Feller property of the semigroup associated with the solution of the cut-off equation (see Proposition 7.1.4 below), we need $z \in C([0,\infty), H^{s+1-\alpha+\sigma_1})$ for some $\sigma_1 > 0$. Therefore, we need $s + 2\alpha - 1 > s + 1 - \alpha$, i.e. $\alpha > \frac{2}{3}$. The situation of the 3D-Navier-Stokes equation is different. While in our case the needed regularity of z is higher than the regularity of our solution space $C((0,\infty), H^s)$ for the cutoff equation (7.2), for the 3-D Navier-Stokes equation the needed regularity of z is the same as for the solution of the cutoff equation.

(ii) Since $\alpha < 1$, we can't use the same type of estimate as in [FR08] (c.f. [FR08, Lemma D.2]) to obtain our results. We use Lemma 2.1 and choose suitable parameters (s, σ_1, σ_2) such that the approach in [FR08] can be modified to apply here (see (7.6)-(7.10), (7.13) and so on).

(iii) It seems difficult to use the Kolmogorov equation method as in [DD03], [DO06] or a coupling approach as in [O08] in our situation. In fact, to get a uniform H^s-norm estimate for the solutions of the Galerkin approximations of the equation (1.1) for some $s > 0$, the regularity, needed for the trajectories of the associated Ornstein-Uhlenbeck (O-U) process z is higher than H^s, which is entirely different from the situation of the 3-D Navier-Stokes equation. According to the method in [DD03], DO06] and [O08], we should use the solutions' $H^{s+\alpha}$-norm to control the $H^{s+\alpha}$-norm of the derivative of the solutions as required for the Bismut-Elworthy-Li formula. In particular, the associated O-U process z should be also in $H^{s+\alpha}$. However, under Assumption 7.1 for the noise, our O-U process z is only in $L^2([0,T], H^{s+2\alpha-1})$. As a result, for their method to apply here, we need even $\alpha \geq 1$.

Fix $s \geq 1$ as in Assumption 7.1 and set $\mathcal{W} := H^s$ and $|x|_{\mathcal{W}} := \|x\|_{H^s}$.

Now we state the main result of this section.

Theorem 7.1.2 Under Assumption 7.1, $(P_t)_{t\geq 0}$ is $\mathcal{W}$-strong Feller, i.e. for every $t > 0$ and $\psi \in \mathcal{B}_b(H)$, $P_t\psi \in C_b(\mathcal{W})$.

We shall use [FR08, Theorem 5.4], which is an abstract result to prove the strong Feller property. In order to use [FR08, Theorem 5.4], we follow the idea of [FR08, Theorem 5.11] to construct $P_x^{(R)}$. We introduce an equation which differs from the original one by a cut-off only, so that with large probability they have the same trajectories on a small random time interval (see (7.3) below). We consider the equation

$$d\theta(t) + A\theta(t)dt + \chi_R(|\theta|_{\mathcal{W}}^2)u(t) \cdot \nabla\theta(t)dt = GdW(t), \tag{7.2}$$

where $\chi_R : \mathbb{R} \to [0,1]$ is of class C^∞ such that $\chi_R(|\theta|) = 1$ if $|\theta| \leq R$, $\chi_R(|\theta|) = 0$ if $|\theta| > R+1$ and with its first derivative bounded by 1. Then, if we can prove the following Theorem 7.1.3 and Proposition 7.1.4, Theorem 7.1.2 follows.

Theorem 7.1.3 (Weak-strong uniqueness) Suppose Assumption 7.1 holds. Then for every $x \in \mathcal{W}$, Eq. (7.2) has a unique martingale solution $P_x^{(R)}$, with

$$P_x^{(R)}[C([0,\infty); \mathcal{W})] = 1.$$

Let $\tau_R : \Omega \to [0,\infty]$ be defined as

$$\tau_R(\omega) = \inf\{t \geq 0 : |\omega(t)|_{\mathcal{W}}^2 \geq R\},$$

and $\tau_R(\omega)=\infty$ if this set is empty. If $x\in\mathcal{W}$ and $|x|^2_{\mathcal{W}}<R$, then

$$\lim_{\varepsilon\to 0} P^{(R)}_{x+h}[\tau_R\geq\varepsilon]=1,\ \text{uniformly in } h\in\mathcal{W},|h|_{\mathcal{W}}<1. \tag{7.3}$$

Moreover,

$$E^{P^{(R)}_x}[\varphi(\xi_t)1_{[\tau_R\geq t]}]=E^{P_x}[\varphi(\xi_t)1_{[\tau_R\geq t]}], \tag{7.4}$$

for every $t\geq 0$ and $\varphi\in\mathcal{B}_b(H)$, where P_x is the martingale solution of (3.1).

Proof Let z denote the solution to

$$dz(t)+Az(t)dt=GdW(t),$$

with initial data $z(0)=0$ and let $v^{(R)}_x$ be the solution to the auxiliary problem

$$\frac{dv^{(R)}(t)}{dt}+Av^{(R)}(t)+u^{(R)}(t)\cdot\nabla(v^{(R)}(t)+z(t))\chi_R(|v^{(R)}+z|^2_{\mathcal{W}})=0, \tag{7.5}$$

with $v^{(R)}(0)=x$. Here $u^{(R)}(t)=u_{v^{(R)}}(t)+u_z(t)$, $u_{v^{(R)}}$ and u_z satisfy (1.3) with θ replaced by $v^{(R)}$ and z, respectively. Moreover, define $\theta^{(R)}:=v^{(R)}+z$, which is a martingale solution to equation (7.2). We denote its law on Ω by $P^{(R)}_x$. By Assumption 7.1 the trajectories of the noise belong to

$$\Omega^*:=\bigcap_{\beta\in(0,\frac12),\kappa\in[0,\frac{s+\alpha}{2\alpha}-\frac{1}{2\alpha})} C^\beta([0,\infty);D(A^\kappa)),$$

with probability one. Hence, the analyticity of the semigroup generated by A implies that for each $\omega\in\Omega^*$, $z(\omega)\in C([0,\infty),D(\Lambda^{s+2\alpha-1-\varepsilon}))$ for every $\varepsilon>0$.

Now, for $\omega\in\Omega^*$ we prove that Eq. (7.5) with $z(\omega)$ replacing z has a unique global weak solution in the space $C([0,\infty);\mathcal{W})$. First, we obtain the following a-priori estimate for suitable $\sigma_1,\sigma_2>0$ with $\sigma_2\leq s,\sigma_2+\sigma_1=1,s+\sigma_1-\alpha+1<s+2\alpha-1<s+\alpha$, where we used that $\alpha>\frac23$ since $0<\sigma_1<3\alpha-2$:

$$\begin{aligned}
\frac12\frac{d}{dt}|\Lambda^s v^{(R)}|^2+\kappa|\Lambda^{s+\alpha}v^{(R)}|^2 \leq& C\chi_R(|\theta^{(R)}|^2_{\mathcal{W}})|\Lambda^{s-\alpha+1}R(u^{(R)}\theta^{(R)})|\cdot|\Lambda^{s+\alpha}v^{(R)}|\\
\leq& C\chi_R(|\theta^{(R)}|^2_{\mathcal{W}})|\Lambda^{s-\alpha+1+\sigma_1}\theta^{(R)}||\Lambda^{\sigma_2}\theta^{(R)}|\cdot|\Lambda^{s+\alpha}v^{(R)}|\\
\leq& C\chi_R(|\theta^{(R)}|^2_{\mathcal{W}})(|\Lambda^{s-\alpha+1+\sigma_1}v^{(R)}|+|\Lambda^{s-\alpha+1+\sigma_1}z|)\cdot|\Lambda^{s+\alpha}v^{(R)}|\\
\leq& C\chi_R(|\theta^{(R)}|^2_{\mathcal{W}})(C|\Lambda^s v^{(R)}|^{1-r}|\Lambda^{s+\alpha}v^{(R)}|^r+|\Lambda^{s-\alpha+1+\sigma_1}z|)\cdot|\Lambda^{s+\alpha}v^{(R)}|\\
\leq& C\chi_R(|\theta^{(R)}|^2_{\mathcal{W}})(|\Lambda^s v^{(R)}|^2+|\Lambda^{s-\alpha+1+\sigma_1}z|^2)+\frac{\kappa}{2}|\Lambda^{s+\alpha}v^{(R)}|^2\\
\leq& C\chi_R(|\theta^{(R)}|^2_{\mathcal{W}})(C(R)+|\Lambda^{s-\alpha+1+\sigma_1}z|^2)+\frac{\kappa}{2}|\Lambda^{s+\alpha}v^{(R)}|^2,
\end{aligned} \tag{7.6}$$

where $r:=\frac{1-\alpha+\sigma_1}{\alpha}$. Here in the second inequality we used Lemmas 2.1 and 2.2, and in the fourth inequality we used the Gagliardo-Nirenberg inequality and in the fifth inequality we used Young's inequality. Then as in Theorem 3.4, we prove (7.5) has a weak solution in $L^\infty([0,T],\mathcal{W})$.

[Continuity] For each $\omega \in \Omega^*$, σ_1 and σ_2 as above, since $s-\alpha+1+\sigma_1 < s+2\alpha-1$, we have $z \in C([0,\infty); D(\Lambda^{s-\alpha+1+\sigma_1}))$. For $s > 3-3\alpha$, $s_0 = s-\alpha$, multiplying the equations (7.5) by $\frac{d}{dt}\Lambda^{2s_0}v^{(R)}$, we obtain

$$\begin{aligned}\frac{\kappa}{2}\frac{d}{dt}|\Lambda^{s_0+\alpha}v^{(R)}|^2+|\Lambda^{s_0}\dot{v}^{(R)}|^2 \leq & C\chi_R(|\theta^{(R)}|^2_{\mathcal{W}})|\Lambda^{s_0+1}R(u^{(R)}\theta^{(R)})|\cdot|\Lambda^{s_0}\dot{v}^{(R)}| \\ \leq & C\chi_R(|\theta^{(R)}|^2_{\mathcal{W}})|\Lambda^{s_0+1+\sigma_1}\theta^{(R)}||\Lambda^{\sigma_2}\theta^{(R)}|\cdot|\Lambda^{s_0}\dot{v}^{(R)}| \\ \leq & C\chi_R(|\theta^{(R)}|^2_{\mathcal{W}})(|\Lambda^{s+\alpha}v^{(R)}|^2+|\Lambda^{s_0+\alpha}v^{(R)}|^2+|\Lambda^{s_0+1+\sigma_1}z|^2) \\ & +\frac{1}{2}|\Lambda^{s_0}\dot{v}^{(R)}|^2.\end{aligned} \tag{7.7}$$

Here in the second inequality we used Lemmas 2.1 and 2.2, and in the third inequality we used the Gagliardo-Nirenberg inequality and Young's inequality.

As $\int_0^T|\Lambda^{s+\alpha}v^{(R)}(t_1)|^2dt_1$ can be dominated by the same arguments as (7.6), we get an a-priori estimate for the time derivative $\frac{d}{dt}v^{(R)}$ in $L^2(0,T;H^{s_0})$. Then by [Te84], we obtain $v^{(R)} \in C([0,T],\mathcal{W})$.

[Uniqueness] Let θ_1,θ_2 be two solutions of Eq. (7.5) in $C([0,\infty);\mathcal{W})$ and set $w := \theta_1-\theta_2$ and $u_w := u_1-u_2$. Then by a similar argument as in the proof of Theorem 5.1, we have for small $\varepsilon_0 > 0$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\Lambda^{s_0}w|^2+\kappa|\Lambda^{s_0+\alpha}w|^2 =& -(\chi_R(|\theta_1|^2_{\mathcal{W}})-\chi_R(|\theta_2|^2_{\mathcal{W}}))\langle\Lambda^{s_0+\varepsilon_0-\alpha}(u_1\cdot\nabla\theta_1),\Lambda^{s_0+\alpha-\varepsilon_0}w\rangle \\ & -\chi_R(|\theta_2|^2_{\mathcal{W}})\langle\Lambda^{s_0-\alpha}(u_1\cdot\nabla w+u_w\cdot\nabla\theta_2),\Lambda^{s_0+\alpha}w\rangle \\ =& I+II+III.\end{aligned}$$

As

$$|\chi_R(|\theta_1|^2_{\mathcal{W}})-\chi_R(|\theta_2|^2_{\mathcal{W}})| \leq C(R)|w|_{\mathcal{W}}[1_{[0,R+1]}(|\theta_1|^2_{\mathcal{W}})+1_{[0,R+1]}(|\theta_2|^2_{\mathcal{W}})],$$

we have for σ_1,σ_2 as above,

$$\begin{aligned}I \leq & C[1_{[0,R+1]}(|\theta_1|^2_{\mathcal{W}})+1_{[0,R+1]}(|\theta_2|^2_{\mathcal{W}})]|w|_{\mathcal{W}}\cdot|\Lambda^{s_0-\alpha+\varepsilon_0+1+\sigma_1}\theta_1||\Lambda^{\sigma_2}\theta_1|\cdot|\Lambda^{s_0+\alpha-\varepsilon_0}w| \\ \leq & C(R,|\theta_1|_{\mathcal{W}},|\theta_2|_{\mathcal{W}})|w|_{\mathcal{W}}|\Lambda^{s_0+\alpha-\varepsilon_0}w| \\ \leq & C(R,|\theta_1|_{\mathcal{W}},|\theta_2|_{\mathcal{W}})|\Lambda^{s_0}w|^2+\frac{\kappa}{4}|w|^2_{\mathcal{W}},\end{aligned} \tag{7.8}$$

where $s_0+\alpha = s$. Here in the first inequality we used Lemmas 2.1 and 2.2, and in the third inequality we used the Gagliardo-Nirenberg inequality and Young's inequality. In a similar way, we obtain

$$II \leq C(R,|\theta_1|_{\mathcal{W}})|\Lambda^{s_0}w|^2+\frac{\kappa}{4}|w|^2_{\mathcal{W}},$$

and

$$III \leq C(R,|\theta_2|_{\mathcal{W}})|\Lambda^{s_0}w|^2+\frac{\kappa}{4}|w|^2_{\mathcal{W}}.$$

Then we obtain

$$\frac{1}{2}\frac{d}{dt}|\Lambda^{s_0}w|^2+\kappa|\Lambda^{s_0+\alpha}w|^2 \leq C(R,\sup_{t\in[0,T]}|\theta_1(t)|_{\mathcal{W}},\sup_{t\in[0,T]}|\theta_2(t)|_{\mathcal{W}})|\Lambda^{s_0}w|^2+\frac{3\kappa}{4}|w|^2_{\mathcal{W}}.$$

By Gronwall's lemma we have $|\Lambda^{s_0}w| = 0$, which implies $w = 0$.

So Eq. (7.5) has a unique global weak solution in the space $C([0,\infty);\mathcal{W})$.

Next, we prove (7.3). In order to do so, it is sufficient to show that $P_x^{(R)}[\tau_R < \varepsilon] \le C(\varepsilon, R)$ with $C(\varepsilon,R) \downarrow 0$ as $\varepsilon \downarrow 0$, for all $x \in \mathcal{W}$, with $|x|^2_{\mathcal{W}} \le \frac{R}{8}$. So, fix $\varepsilon > 0$ small enough, let $\Theta_{\varepsilon,R} := \sup_{t\in[0,\varepsilon]} |\Lambda^{s-\alpha+1+\sigma_1} z(t)|$ and assume that $\Theta^2_{\varepsilon,R} \le \frac{R}{8}$. Setting $\varphi(t) := |v^{(R)}|^2_{\mathcal{W}} + \Theta^2_{\varepsilon,R}$, by (7.6) we get $\dot{\varphi} \le C(R)$. This implies, together with the bounds on x and $\Theta_{\varepsilon,R}$, that

$$|\theta^{(R)}(t)|^2_{\mathcal{W}} \le 2(|v^{(R)}(t)|^2_{\mathcal{W}} + |z(t)|^2_{\mathcal{W}}) \le R,$$

for ε small enough. In particular, since this holds for all $t \le \varepsilon$, it follows that $\tau_R \ge \varepsilon$. Hence

$$P_x^{(R)}[\tau_R < \varepsilon] \le P_x^{(R)}[\sup_{t\in[0,\varepsilon]} |\Lambda^{s+1+\sigma_1-\alpha} z(t)|^2 > \frac{R}{8}].$$

Letting $\varepsilon \downarrow 0$, we have $P_x^{(R)}[\tau_R < \varepsilon] \to 0$, and the claim is proved, since the probability above is independent of x.

Finally, the same arguments as in the proof of Theorem 5.1 imply that

$$\theta_x(t \wedge \tau_R(\theta_x^{(R)})) = \theta_x^{(R)}(t \wedge \tau_R(\theta_x^{(R)})) \quad \forall t, P_x - a.s..$$

Moreover, since θ is H-valued weakly continuous, we obtain $\tau_R(\theta_x^{(R)}) = \tau_R(\theta)$. □

In order to apply [FR08, Theorem 5.4], we now only need the following result.

Proposition 7.1.4 For every $R > 0$, the transition semi-group $(P_t^{(R)})_{t\ge0}$ associated to Eq. (7.2) is $\mathcal{W}$-strong Feller.

Proof We shall provide formal estimates, that can, however, be made rigorous through Galerkin approximations. Let $(\Sigma, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathbb{P})$ be a filtered probability space, $(W_t)_{t\ge0}$ a cylindrical Wiener process on H and, for every $x \in \mathcal{W}$, let $\theta_x^{(R)}$ be the solution to Eq. (7.2). By the Bismut, Elworthy and Li formula,

$$D_y(P_t^{(R)}\psi)(x) = \frac{1}{t} E^{\mathbb{P}}[\psi(\theta_x^{(R)}(t)) \int_0^t \langle G^{-1} D_y \theta_x^{(R)}(s), dW(s)\rangle],$$

where $D_y(P_t^{(R)}\psi)$ denotes $\langle D(P_t^{(R)}\psi), y\rangle$ for $y \in H$, and thus, for $\|\psi\|_\infty \le 1$, by the B-D-G inequality

$$|(P_t^{(R)}\psi)(x_0+h) - (P_t^{(R)}\psi)(x_0)| \le \frac{C}{t} \sup_{\eta\in[0,1]} E^{\mathbb{P}}[(\int_0^t |G^{-1} D_h \theta^{(R)}_{x_0+\eta h}(s)|^2 ds)^{1/2}].$$

The proposition is proved once we prove that the right-hand side of the above inequality converges to 0 as $|h|_{\mathcal{W}} \to 0$.

Fix $x \in \mathcal{W}$, $y \in H$ and write $\theta = \theta_x^{(R)}, D\theta = D_y\theta, Du = D_y u$. The term $D\theta$ solves the following equation

$$\frac{d}{dt} D\theta + \kappa\Lambda^{2\alpha}(D\theta) = -[\chi_R(|\theta|^2_{\mathcal{W}})[Du\cdot\nabla\theta + u\cdot\nabla D\theta] + 2\chi_R'(|\theta|^2_{\mathcal{W}})\langle\theta, D\theta\rangle_{\mathcal{W}} u\cdot\nabla\theta].$$

Multiplying the above equation with $\Lambda^{2s}D\theta$ and taking the inner product in L^2, we have

$$\frac{1}{2}\frac{d}{dt}|\Lambda^s D\theta|^2 + \kappa|\Lambda^{s+\alpha}(D\theta)|^2 = -\langle[\chi_R(|\theta|^2_{\mathcal{W}})[Du\cdot\nabla\theta + u\cdot\nabla D\theta] + 2\chi_R'(|\theta|^2_{\mathcal{W}})\langle\theta, D\theta\rangle_{\mathcal{W}} u\cdot\nabla\theta], \Lambda^{2s} D\theta\rangle.$$

For the first term on the left hand side, we have for $|\theta|^2_{\mathcal{W}} \le R$

$$
\begin{aligned}
|\langle Du\cdot\nabla\theta, \Lambda^{2s}D\theta\rangle| =&|\langle \Lambda^{s-\alpha}(Du\cdot\nabla\theta), \Lambda^{s+\alpha}D\theta\rangle|\\
\le& C|\Lambda^{s-\alpha+1+\sigma_1}\theta|\cdot|\Lambda^{\sigma_2}D\theta|\cdot|\Lambda^{s+\alpha}D\theta| + C|\Lambda^{s-\alpha+1+\sigma_1}D\theta|\cdot|\Lambda^{\sigma_2}\theta|\cdot|\Lambda^{s+\alpha}D\theta|\\
\le& \varepsilon|\Lambda^{s+\alpha}D\theta|^2 + C(C(R)+|\Lambda^{s+\alpha}v|^2+|\Lambda^{s-\alpha+1+\sigma_1}z|^2)|\Lambda^s D\theta|^2,
\end{aligned}
\tag{7.9}
$$

for σ_1,σ_2 as above, where we used Lemmas 2.1, 2.2 in the first inequality as well as the Gagliardo-Nirenberg inequality and Young's inequality in the second inequality.

The second term can be estimated similarly. For the third term, by Lemmas 2.1, 2.2 we have

$$
\begin{aligned}
|\langle u\cdot\nabla\theta, \Lambda^{2s}D\theta\rangle| =&|\langle \Lambda^{s-\alpha}(u\cdot\nabla\theta), \Lambda^{s+\alpha}D\theta\rangle|\\
\le& C|\Lambda^{s-\alpha+1+\sigma_1}\theta||\Lambda^{\sigma_2}\theta|\cdot|\Lambda^{s+\alpha}D\theta|\\
\le& C(|\Lambda^{s+\alpha}v|+|\Lambda^{s-\alpha+1+\sigma_1}z|)|\Lambda^s\theta||\Lambda^{s+\alpha}D\theta|.
\end{aligned}
\tag{7.10}
$$

Then we obtain

$$
\frac{1}{2}\frac{d}{dt}|\Lambda^s D\theta|^2 + \kappa|\Lambda^{s+\alpha}(D\theta)|^2 \le \frac{\kappa}{2}|\Lambda^{s+\alpha}(D\theta)|^2 + C(C(R)+|\Lambda^{s+\alpha}v|^2+|\Lambda^{s-\alpha+1+\sigma_1}z|^2)|\Lambda^s D\theta|^2.
$$

From Gronwall's inequality we finally obtain

$$
\int_0^t |\Lambda^{s+\alpha}(D\theta(l))|^2 dl \le \exp(C\int_0^t (C(R)+|\Lambda^{s+\alpha}v|^2+|\Lambda^{s-\alpha+1+\sigma_1}z|^2 dl))|\Lambda^s h|^2.
$$

By (7.6) we obtain

$$
E\int_0^t |\Lambda^{s+\alpha}(D\theta(l))|^2 dl \le \sum_{n=1}^{\infty}\exp(Ct(C(R)+cn^2))P(\sup_{(0,t)}|\Lambda^{s-\alpha+1+\sigma_1}z|>n)|\Lambda^s h|^2.
$$

Because of Assumption 7.1 and since z is a Gaussian process, one deduces that there exist $\eta, C>0$ such that

$$
P[\sup_{l\in[0,t]}|\Lambda^{s-\alpha+1+\sigma_1}z(l)|^2 > R_0] \le Ce^{-\eta\frac{R_0^2}{t}},
$$

(see e.g. [FR07, Proposition 15]). Then for $t_0^2 \le \frac{\eta}{cC}$, we obtain

$$
E\int_0^{t_0} |\Lambda^{s+\alpha}(D\theta(s))|^2 ds \le c(t_0,R)|\Lambda^s h|^2,
$$

which, as $G = Q_0^{-1/2}\Lambda^{s+\alpha}$, implies the assertion for t_0. For general t, by the semigroup property the assertion follows easily. $\square$

7.2 A support theorem for $\alpha > 2/3$

A Borel probability measure μ on H is fully supported on $\mathcal{W}$ if $\mu(U)>0$ for every non-empty open set $U\subset\mathcal{W}$. Set $\mathcal{W}_1 := D(\Lambda^{s-\alpha+1+\sigma_1})$, where σ_1 is the same as in the proof of Theorem 7.1.3 and we will use it below.

Lemma 7.2.1 (Approximate controllability) Let $R > 0$, $T > 0$. Let $x \in \mathcal{W}$ and $y \in \mathcal{W}$, with $Ay \in \mathcal{W}_1$, such that

$$|x|^2_{\mathcal{W}} \le \frac{R}{2} \qquad |y|^2_{\mathcal{W}} \le \frac{R}{2}.$$

Then there exist (a control function) $\omega \in \mathrm{Lip}([0,\mathrm{T}];\mathcal{W}_1)$ and

$$\theta \in C([0,T];\mathcal{W}) \cap L^2([0,T];D(\Lambda^{s+\alpha})),$$

such that θ solves the equation

$$\theta(t) - x + \int_0^t A\theta(r) + \chi_R(|\theta|^2_{\mathcal{W}})u(r)\cdot\nabla\theta(r)dr = \omega(t) \quad dt - a.e.t \in [0,T], \tag{7.11}$$

with $\theta(0) = x$ and $\theta(T) = y$, and

$$\sup_{t\in[0,T]} |\theta(t)|^2_{\mathcal{W}} \le R. \tag{7.12}$$

Proof First consider $\omega = 0$. Then by an inequality similar to (7.6), we get

$$\frac{d}{dt}|\theta|^2_{\mathcal{W}} + \kappa|\Lambda^\alpha\theta|^2_{\mathcal{W}} \le C(R)|\theta|^2_{\mathcal{W}}.$$

Hence by Gronwall's lemma $\theta(t) \in D(\Lambda^{s+\alpha})$ for almost every $t \in [0,T]$ and, by solving again the equation with one of these regular points as initial condition, by Lemma 2.1 we have

$$\frac{d}{dt}|\Lambda^{\alpha+s}\theta|^2 + \kappa|\Lambda^{2\alpha+s}\theta|^2_{\mathcal{W}} \le C|\Lambda^{2\alpha+s}\theta||\Lambda^{s+1+\sigma}\theta|\|\theta\|_{L^p} \le C(R)|\Lambda^{s+\alpha}\theta|^2 + \frac{\kappa}{2}|\Lambda^{s+2\alpha}\theta|^2,$$

where $\sigma = \frac{2}{p} < 2\alpha - 1$ and where we used the L_p-estimate in the same way as in the proof of [Re95, Theorem 3.3]. Then we find a small $T_* \in (0, \frac{T}{2})$ such that $|\theta(t)|^2_{\mathcal{W}} \le R$ and $A\theta(T_*) \in \mathcal{W}_1$ for all $t \le T_*$. Define θ to be the solution above for $t \in [0, T_*]$ and extended by linear interpolation between y and $\theta(T_*)$ in $[T_*, T]$. Then obviously (7.12) follows.

Next, if we set

$$\eta := \partial_t\theta + A\theta + \chi_R(|\theta|^2_{\mathcal{W}})u\cdot\nabla\theta, \quad T_* \le t \le T,$$

$\omega := 0$ for $t \le T_*$ and $\omega(t) = \int_{T_*}^t \eta_s ds$ for $t \in [T_*, T]$, we also have (7.11). It remains to prove that $\eta \in L^\infty(0,T;\mathcal{W}_1)$. For the first two terms of η this is obvious. For the non-linear term we have that

$$|u\cdot\nabla\theta|_{\mathcal{W}_1} \le C|\Lambda^{2\alpha}\theta|^2_{\mathcal{W}_1},$$

for any $\theta \in D(\Lambda^{s+\sigma_1+1+\alpha})$. □

Let $l \in (0, \frac{1}{2})$ and $p > 1$ such that $l - \frac{1}{p} > 0$. Under this assumption we see that for every $\alpha_1 < \frac{s+\alpha-1}{2\alpha}$ the map

$$\omega \mapsto z(\cdot,\omega) : W^{l,p}([0,T];D(A^{\alpha_1})) \to C([0,T];D(A^{\alpha_1+l-\frac{1}{p}-\varepsilon}))$$

is continuous, for all $\varepsilon > 0$, where z is the solution to the Stokes problem

$$z(t) + \int_0^t Az(s)ds = \omega(t).$$

In particular, it is possible to find $\alpha_1 \in (0, \frac{s+\alpha-1}{2\alpha})$, s and p such that the above map is continuous from $W^{l,p}([0,T]; D(A^{\alpha_1}))$ to $C([0,T]; D(\Lambda^{s-\alpha+1+\sigma_1}))$.

Lemma 7.2.2 (Continuity with respect to the control functions) Let l, p and α_1 be chosen as above, and let $\omega_n \to \omega$ in $W^{l,p}([0,T]; D(A^{\alpha_1}))$. Let θ be the solution to equation (7.11) corresponding to ω and some initial condition x, and let

$$\tau = \inf\{t \geq 0 : |\theta(t)|^2_{\mathcal{W}} \geq R\},$$

where as usual we set $\inf \emptyset = \infty$. For each $n \in \mathbb{N}$, define similarly θ_n and τ_n corresponding to ω_n with the same initial condition x. If $\tau > T$, then $\tau_n > T$ for n large enough and

$$\theta_n \to \theta \text{ in } C([0,T]; \mathcal{W}).$$

Proof Set $v_n := \theta_n - z_n$ for each $n \in \mathbb{N}$, and $v := \theta - z$, where z_n, z are the solutions to the Stokes problem corresponding to ω_n, ω respectively. Since $\omega_n \to \omega$ in $W^{l,p}([0,T]; D(A^{\alpha_1}))$, we can find a common lower bound for $(\tau_n)_{n\in\mathbb{N}}$ and τ. For every time smaller than this lower bound t_0, by (7.6) we have

$$\sup_{(0,t_0)} |\Lambda^s \theta_n|^2 \leq R, \qquad \sup_{(0,t_0)} |\Lambda^s \theta|^2 \leq R, \qquad \sup_{(0,t_0)} |\Lambda^{s-\alpha+1+\sigma_1} z_n| \leq C(R),$$

and

$$\sup_{(0,t_0)} |\Lambda^{s-\alpha+1+\sigma_1} z| \leq C(R), \qquad \int_0^{t_0} |\Lambda^{s+\alpha} v_n(l)|^2 dl \leq C(R), \qquad \int_0^{t_0} |\Lambda^{s+\alpha} v(l)|^2 dl \leq C(R),$$

where $C(R)$ is a constant depending only on R. Moreover, we obtain for $t \leq t_0$

$$\begin{aligned}
\frac{d}{dt}|v - v_n|^2_{\mathcal{W}} + 2\kappa|\Lambda^\alpha(v_n - v)|^2_{\mathcal{W}} =& \langle u_n \cdot \nabla\theta_n, \Lambda^{2s}(v - v_n)\rangle - \langle u \cdot \nabla\theta, \Lambda^{2s}(v - v_n)\rangle \\
=& [\langle (u_{v_n} - u_v) \cdot \nabla\theta_n, \Lambda^{2s}(v - v_n)\rangle + \langle u \cdot \nabla(v_n - v), \Lambda^{2s}(v - v_n)\rangle \\
&+ \langle (u_{z_n} - u_z) \cdot \nabla\theta_n, \Lambda^{2s}(v - v_n)\rangle + \langle u \cdot \nabla(z_n - z), \Lambda^{2s}(v - v_n)\rangle].
\end{aligned}$$

For the first term on the right hand side, by using Lemmas 2.1, 2.2 we have

$$\begin{aligned}
|\langle (v_n - v) \cdot \nabla\theta_n, \Lambda^{2s}(v - v_n)\rangle| \leq& C|\Lambda^{s+\alpha}(v - v_n)||\Lambda^{s-\alpha+1+\sigma_1}(v - v_n)||\Lambda^{\sigma_2}\theta_n| \\
&+ C|\Lambda^{s+\alpha}(v - v_n)||\Lambda^{s-\alpha+1+\sigma_1}\theta_n||\Lambda^{\sigma_2}(v - v_n)| \\
\leq& \frac{\kappa}{4}|\Lambda^{s+\alpha}(v - v_n)|^2 + (C(R) + |\Lambda^s v_n|^2 + |\Lambda^{s+\alpha} v_n|^2)|\Lambda^s(v - v_n)|^2 \\
&+ c|\Lambda^{s-\alpha+1+\sigma_1} z_n|^2 |\Lambda^s(v - v_n)|^2.
\end{aligned} \tag{7.13}$$

The other term can be estimated similarly. Then we obtain

$$\begin{aligned}
&\frac{d}{dt}|v - v_n|^2_{\mathcal{W}} + 2\kappa|\Lambda^\alpha(v_n - v)|^2_{\mathcal{W}} \\
\leq& \kappa|\Lambda^\alpha(v_n - v)|^2_{\mathcal{W}} + C(C(R) + |\Lambda^\alpha v_n|^2_{\mathcal{W}} + |\Lambda^\alpha v|^2_{\mathcal{W}})(|v - v_n|^2_{\mathcal{W}} + |\Lambda^{s-\alpha+1+\sigma_1}(z - z_n)|^2).
\end{aligned}$$

Here σ_1, σ_2 are as above. Then by Gronwall's lemma

$$|v - v_n|_{\mathcal{W}}^2 \le \Theta_n \exp(C \int_0^t (C(R) + |\Lambda^\alpha v_n|_{\mathcal{W}}^2 + |\Lambda^\alpha v|_{\mathcal{W}}^2) dl) \int_0^t (C(R) + |\Lambda^\alpha v_n|_{\mathcal{W}}^2 + |\Lambda^\alpha v|_{\mathcal{W}}^2) dl,$$

where $\Theta_n = \sup_{[0,T]} |\Lambda^{s-\alpha+1+\sigma_1}(z - z_n)|$. We conclude $\theta_n \to \theta$ in $C([0,T]; \mathcal{W})$. Now, since $\tau > T$, if $S = \sup_{t\in[0,T]} |\Lambda^s \theta(t)|^2$, then $S < R$ and we find $\delta > 0$ (depending only on R and S) and $n_0 \in \mathbb{N}$ such that $\Theta_n < \delta$ and $|v_n - v|_{\mathcal{W}}^2 < \delta$ for all $n \ge n_0$, and so

$$|\theta_n(t)|_{\mathcal{W}} \le |v_n(t) - v(t)|_{\mathcal{W}} + \Theta_n + |\theta(t)|_{\mathcal{W}} \le 2\sqrt{\delta} + \sqrt{S} \le \sqrt{R - \delta}.$$

Then $\tau_n > T$ for all $n \ge n_0$. $\square$

Theorem 7.2.3 Suppose Assumption 7.1 holds and for $x \in H$ let P_x be the distribution of the solution of (3.1) with initial value $\theta(0) = x$. Then for every $x \in \mathcal{W}$ and every $T > 0$, the image measure of P_x at time T is fully supported on $\mathcal{W}$.

Proof Fix $x \in \mathcal{W}$ and $T > 0$. We need to show that for every $y \in \mathcal{W}$ and $\varepsilon > 0$, $P_x[|\theta_T - y|_{\mathcal{W}} < \varepsilon] > 0$. Let $\bar{y} \in \mathcal{W} \cap D(A)$ such that $A\bar{y} \in \mathcal{W}_1$ and $|y - \bar{y}|_{\mathcal{W}} < \frac{\varepsilon}{2}$. Choose $R > 0$ such that $3|x|_{\mathcal{W}}^2 < R$ and $3|y|_{\mathcal{W}}^2 < R$. Then by Theorem 7.1.3,

$$\begin{aligned} P_x[|\theta_T - y|_{\mathcal{W}} < \varepsilon] \ge & P_x[|\theta_T - \bar{y}|_{\mathcal{W}} < \frac{\varepsilon}{2}] \ge P_x[|\theta_T - \bar{y}|_{\mathcal{W}} < \frac{\varepsilon}{2}, \tau_R > T] \\ = & P_x^{(R)}[|\theta_T - \bar{y}|_{\mathcal{W}} < \frac{\varepsilon}{2}, \tau_R > T]. \end{aligned}$$

By Lemma 7.2.1, there is a control $\bar{\omega} \in W^{l,p}([0,T]; D(A^{\alpha_1}))$, with l, p and α_1 chosen as in Lemma 7.2.2, such that the solution $\bar{\theta}$ to the control problem (7.11) corresponding to $\bar{\omega}$ satisfies $\bar{\theta}(0) = x, \bar{\theta}(T) = \bar{y}$ and $|\bar{\theta}(t)|_{\mathcal{W}}^2 \le \frac{2}{3}R$. By Lemma 7.2.2, there exists $\delta > 0$ such that for all $\omega \in W^{l,p}([0,T]; D(A^{\alpha_1}))$ with $|\omega - \bar{\omega}|_{W^{l,p}([0,T];D(A^{\alpha_1}))} < \delta$, we have

$$|\theta(T, \omega) - \bar{y}|_{\mathcal{W}} < \frac{\varepsilon}{2} \text{ and } \sup_{t\in[0,T]} |\theta(t, \omega)|_{\mathcal{W}}^2 < R,$$

where $\theta(\cdot, \omega)$ is the solution to the control problem (7.11) corresponding to ω and starting at x. Hence

$$P_x^{(R)}[|\theta_T - \bar{y}|_{\mathcal{W}} < \frac{\varepsilon}{2}, \tau_R > T] \ge P_x^{(R)}[|\eta - \bar{\omega}|_{W^{l,p}([0,T];D(A^{\alpha_1}))} < \delta],$$

where $\eta_t = \theta_t - x + \int_0^t (A\theta_s + \chi_R(|\theta_s|_{\mathcal{W}}^2) u \cdot \nabla\theta_s) ds$, hence $\theta_T = \theta(T, \eta)$, and the right hand side of the inequality above is strictly positive since by Assumption 7.1 η is a Brownian motion in $D(A^{\alpha_1})$.

7.3 Existence of invariant measures for $\alpha > \frac{2}{3}$

In this subsection, we prove the existence of invariant measures. Let θ_n denote the solution of the usual Galerkin approximation

$$\begin{cases} d\theta_n(t) + A\theta_n(t)dt + P_n(u_n(t) \cdot \nabla\theta_n(t))dt = P_n G(\theta_n(t))dW(t), \\ \theta_n(0) = P_n x. \end{cases}$$

Lemma 7.3.1 Let $\alpha > \frac{2}{3}$. If $x \in H^1, n \in \mathbb{N}, t > 0$, then there exist $\delta_1 > 0$ and $\gamma_0 > 0$ such that

$$E[\int_0^t |A^{\delta_1}\theta_n|_{\mathcal{W}}^{2\gamma_0} dr] \leq C(1+t)(|x|^2+1),$$

where C is independent of x and R.

Proof We apply Itô's formula to the function $(1+|\Lambda^\delta\theta|^2)^{-p}$ for $\delta > 2-2\alpha$ and get

$$\begin{aligned}
&\frac{1}{(1+|\Lambda^\delta\theta|^2)^p} - \frac{1}{(1+|\Lambda^\delta x|^2)^p}\\
=&2p\int_0^t \frac{|\Lambda^{\delta+\alpha}\theta|^2}{(1+|\Lambda^\delta\theta|^2)^{p+1}}dr + 2p\int_0^t \frac{\langle\Lambda^{\delta-\alpha}(u\cdot\nabla\theta), \Lambda^{\delta+\alpha}\theta\rangle}{(1+|\Lambda^\delta\theta|^2)^{p+1}}dr\\
&-2p\int_0^t \frac{\langle\Lambda^\delta\theta, \Lambda^\delta G dW_r\rangle}{(1+|\Lambda^\delta\theta|^2)^{p+1}} - p\int_0^t \frac{\mathrm{Tr}[\mathrm{GG}^*\Lambda^{2\delta}]}{(1+|\Lambda^\delta\theta|^2)^{p+1}}dr\\
&+2p(p+1)\int_0^t \frac{|\Lambda^\delta G\theta|^2}{(1+|\Lambda^\delta\theta|^2)^{p+1}}dr,
\end{aligned}$$

where for simplicity we write $\theta = \theta_n$. Choosing σ_1', σ_2' with $\sigma_2' \leq \delta, \sigma_2' + \sigma_1' = 1, \delta + \sigma_1' - \alpha + 1 < \delta + \alpha$ the non-linear part is estimated as follows:

$$\begin{aligned}
|\langle\Lambda^{\delta-\alpha}(u\cdot\nabla\theta), \Lambda^{\delta+\alpha}\theta\rangle| \leq& C|\Lambda^{\delta-\alpha+1+\sigma_1'}\theta|\cdot|\Lambda^{\sigma_2'}\theta||\Lambda^{\delta+\alpha}\theta|\\
\leq& C|\Lambda^\delta\theta|^m + |\Lambda^{\delta+\alpha}\theta|^2,
\end{aligned}$$

with $m = \frac{2(3\alpha-1-\sigma_1')}{2\alpha-1-\sigma_1'}$.

Then for p big enough we obtain

$$E\int_0^t \frac{|\Lambda^{\delta+\alpha}\theta_n|^2}{(1+|\Lambda^\delta\theta_n|^2)^{p+1}}dr \leq C(1+t).$$

Since by Young's inequality

$$|\Lambda^{\delta+\alpha}\theta_n|^{2\gamma_p} \leq c[\frac{|\Lambda^{\delta+\alpha}\theta_n|^2}{(1+|\Lambda^\delta\theta_n|^2)^{p+1}} + 1 + |\Lambda^\delta\theta_n|^2],$$

for $\delta \leq \alpha$ we obtain

$$E[\int_0^t |\Lambda^{\delta+\alpha}\theta_n|^{2\gamma_p} dr] \leq C(1+t)(|x|^2+1). \tag{7.14}$$

If $\delta > \alpha$, we already know that some power of $|\Lambda^\delta\theta_n|$ is integrable with respect to $dt \otimes P$. Then one proceeds as in the previous case to obtain (7.14). We choose $\delta + \alpha > s$ and obtain the assertions. □

Theorem 7.3.2 Let $\alpha > \frac{2}{3}$ and suppose Assumption 7.1 holds. Then there exists a unique invariant measure ν on $\mathcal{W}$ for the transition semigroup $(P_t)_{t\geq 0}$. Moreover:

(i) The invariant measure ν is ergodic.

(ii) The transition semigroup $(P_t)_{t\geq 0}$ is $\mathcal{W}$-strong Feller, irreducible, and therefore strongly mixing. Furthermore, $P_t(x, dy), t > 0, x \in \mathcal{W}$, are mutually equivalent.

(iii) There are $\delta_1 > 0$ and $\gamma_0 > 0$ such that

$$\int |A^{\delta_1} x|_{\mathcal{W}}^{2\gamma_0} d\nu < \infty.$$

Proof Choose $x_0 \in H^1$ and define

$$\mu_t = \frac{1}{t}\int_0^t P_r^* \delta_{x_0} dr.$$

Since

$$\int |A^{\delta_1} x|_{\mathcal{W}}^{2\gamma_0} \mu_t(dx) = \frac{1}{t} E_{x_0}[\int_0^t |A^{\delta_1}\theta|_{\mathcal{W}}^{2\gamma_0} dr],$$

by Lemma 7.3.1 we obtain

$$\int |A^{\delta_1} x|_{\mathcal{W}}^{2\gamma_0} \mu_t(dx) \le C.$$

This implies that μ_t is tight on $\mathcal{W}$. The strong Feller property of P_t follows from Theorem 7.1.2. Hence, a limit point of μ_t is an invariant measure for $(P_t)_{t\ge 0}$. Therefore, by Doob's theorem, the strong mixing property is a consequence of the irreducibility.

Remark 7.3.3 If we don't assume that G satisfies (4.5), the solution of equation (3.1) may be not unique. Then we can also prove the above results for each Markov selection $P_x, x \in \mathcal{W}$, corresponding to (3.1) and the respective semigroup $(P_t)_{t\ge 0}$ by similar arguments as [R08].

Remark 7.3.4 (i) (Mildly degenerate noise) We can also consider the ergodicity of the equation driven by a mildly degenerate noise as in [EH01]. For this we have to use an extension of the Bismut-Elworthy-Li formula. We have the same problem as explained in Remark 7.1.1. So, we can just get the result for $\alpha > 2/3$.

(ii) (Degenerate noise) There are many papers considering 2D Navier-Stokes equation driven by degenerate noise. Contrary to the 2D Navier-Stokes equation, no Foias-Prodi type estimate is available for the quasi-geostrophic equation. It seems impossible to use a coupling approach as in [KS02], [BKL02], [M02] to prove ergodicity in the case where equation (3.1) is driven by a degenerate noise. It also seems difficult to use the method in [HM06] to prove ergodicity.

7.4 Exponential convergence for $\alpha > \frac{2}{3}$

As will be seen below, we shall need uniform L^p-estimates, and a crucial ingredient to prove them is Krylov's L^p-Itô formula. In order to obtain a uniform estimate, the L^p-estimate known from the deterministic case (see e.g. [Re95]) is not strong enough for our purpose. Therefore, we need the following result, which is an improved version of the "positivity lemma" from [Re95, Lemma 3.2].

Lemma 7.4.1 (Improved positivity Lemma) For $\alpha \in (0,1)$, and $\theta \in L^p$ with $\Lambda^{2\alpha}\theta \in L^p$, for some $2 < p < \infty$,

$$\int |\theta|^{p-2}\theta(\Lambda^{2\alpha} - \frac{2\lambda_1}{p})\theta \ge 0.$$

Proof Denote the semigroup with respect to $-\Lambda^{2\alpha}+\frac{2\lambda_1}{p}$ and $-\Lambda^{2\alpha}$ in L^2 by P_t^0 and P_t^1, respectively. Then we have $P_t^0 f=e^{2t\lambda_1/p}P_t^1 f$. Since

$$\|P_t^1 f\|_{L^2}\leq e^{-\lambda_1 t}\|f\|_{L^2},$$

and

$$\|P_t^1 f\|_{L^\infty}\leq \|f\|_{L^\infty},$$

by the interpolation theorem, we have

$$\|P_t^1 f\|_{L^p}\leq e^{-2\lambda_1 t/p}\|f\|_{L^p}.$$

Thus,

$$\|P_t^0 f\|_{L^p}\leq \|f\|_{L^p}.$$

Then we have

$$\frac{d}{dt}\|P_t^0\theta\|_{L^p}^p=\int |P_t^0\theta|^{p-2}(P_t^0\theta)(P_t^0(-\Lambda^{2\alpha}+\frac{2\lambda_1}{p})\theta)dx\leq 0.$$

Letting $t\to 0$, we obtain our result. □

Proposition 7.4.2 Let $\alpha>\frac{1}{2}$. For $x\in L^p$, let θ denote the solution of equation (3.1). Then for $2<p<\infty$

$$E\|\theta(t)\|_{L^p}^p\leq \|x\|_{L^p}^p e^{-\lambda_1 t}+\frac{C}{\lambda_1}(1-e^{-\lambda_1 t}).$$

Proof Let θ_n be the approximation defined in the proof of Theorem 4.3. Using [Kr10, Lemma 5.1] for θ_n, we obtain

$$\begin{aligned}\|\theta(t)\|_{L^p}^p=&\|\theta(s)\|_{L^p}^p+\int_s^t[-p\int_{\mathbb{T}^2}|\theta(l)|^{p-2}\theta(l)(\Lambda^{2\alpha}\theta(l)+u(l)\cdot\nabla\theta(l))d\xi dl\\&+\frac{1}{2}p(p-1)\int_{\mathbb{T}^2}|\theta(l)|^{p-2}(\sum_j|k_{\delta_n}*G(e_j)|^2)d\xi]dl+p\int_s^t\int_{\mathbb{T}^2}|\theta(l)|^{p-2}\theta(l)k_{\delta_n}*Gd\xi dW(l)\\\leq&\|\theta(s)\|_{L^p}^p-2\lambda_1\int_s^t\int_{\mathbb{T}^2}|\theta(l)|^pd\xi dl+\int_s^t\frac{1}{2}p(p-1)\int_{\mathbb{T}^2}|\theta(l)|^{p-2}(\sum_j|k_{\delta_n}*G(e_j)|^2)d\xi dl\\&+p\int_s^t\int_{\mathbb{T}^2}|\theta(l)|^{p-2}\theta(l)k_{\delta_n}*Gd\xi dW(l)\\\leq&\|\theta(s)\|_{L^p}^p-2\lambda_1\int_s^t\int_{\mathbb{T}^2}|\theta(l)|^pd\xi dl+\int_s^t(\varepsilon\int_{\mathbb{T}^2}|\theta(l)|^pd\xi+C(\varepsilon)\int(\sum_j|k_{\delta_n}*G(e_j)|^2)^{p/2}d\xi)dl\\&+p\int_s^t\int_{\mathbb{T}^2}|\theta(l)|^{p-2}\theta(l)k_{\delta_n}*Gd\xi dW(l),\end{aligned}$$

where we used Lemma 7.4.1 to get the first inequality and for simplicity we write $\theta(t)=\theta_n(t,x)$. Taking expectation we obtain

$$E\|\theta_n(t)\|_{L^p}^p\leq E\|\theta_n(s)\|_{L^p}^p-E\lambda_1\int_s^t\int_{\mathbb{T}^2}|\theta_n(l)|^pd\xi dl+C(\varepsilon,p)(t-s).$$

Then by Gronwall's lemma we have

$$E\|\theta_n(t)\|_{L^p}^p \leq \|\theta_n(0)\|_{L^p}^p e^{-\lambda_1 t} + \frac{C}{\lambda_1}(1-e^{-\lambda_1 t}).$$

Then taking the limit $n \to \infty$ in the above inequality we deduce

$$E\|\theta(t)\|_{L^p}^p \leq \|x\|_{L^p}^p e^{-\lambda_1 t} + \frac{C}{\lambda_1}(1-e^{-\lambda_1 t}).$$

□

Lemma 7.4.3 Let $\alpha > 2/3$. Suppose Assumption 7.1 is satisfied with $s > 3-2\alpha$. Let θ denote the solution of (3.1) and take p as in Assumption 7.1. Then for every $R_0 \geq 1$, there exist values $T_1 = T_1(R_0)$ and $K_1 = K_1(R_0)$ such that if $|\theta_0| \leq R_0$, $\sup_{t\in[0,T_1]} \|\theta\|_{L^p}^p \leq R_0$, and $\sup_{t\in[0,T_1]} |\Lambda^{s-\alpha+1+\sigma_1+\delta} z(t)|^2 \leq R_0$ for some $0 < \delta < 3\alpha - 2 - \sigma_1$, then $|\Lambda^{s+\delta}\theta(T_1)|^2 \leq K_1$.

Proof By Itô's formula, we obtain that there exists $K_0 = K_0(R_0) > 0$ and for P-a.s. ω, $\exists$ $t_0(\omega) > 0$ such that

$$|\Lambda^\alpha \theta(t_0)|^2 \leq K_0.$$

For any $r > 0$, by Lemmas 2.1, 2.2 we have the following a-priori estimate for $N = \frac{\alpha}{\alpha - \frac{1}{2} - \frac{1}{p}}$ and $\sigma = \frac{2}{p}$,

$$\begin{aligned}\frac{d}{dt}|\Lambda^r v|^2 + |\Lambda^{r+\alpha} v|^2 \leq & |\langle u \cdot \nabla\theta, \Lambda^{2r} v\rangle| \\ \leq & C|\Lambda^{r+\alpha} v| \cdot |\Lambda^{r-\alpha+1+\sigma}\theta| \cdot \|\theta\|_{L^p} \\ \leq & \frac{1}{4}|\Lambda^{r+\alpha} v|^2 + C\|\theta\|_{L^p}^N |\Lambda^r v|^2 + C|\Lambda^{r-\alpha+1+\sigma} z|^2 \cdot \|\theta\|_{L^q}^2.\end{aligned} \tag{7.15}$$

We choose the approximation θ_n as in the proof of Theorem 4.3 with initial time $t = 0$ replaced by the initial time $t = t_0(\omega)$ and $\theta_n(t_0) = \theta(t_0)$. Set $z_n = \int_{t_0}^t e^{-(t-s)A} k_{\delta_n} * G dW(s)$. Then we have the following L^p-norm estimate of $v_n := \theta_n - z_n$,

$$\frac{d}{dt}\|v_n\|_{L^p}^p \leq Cp\|\nabla z_n\|_\infty (\|v_n\|_{L^p}^p + \|z_n\|_{L^p}\|v_n\|_{L^p}^{p-1}).$$

Thus we have

$$\frac{d}{dt}\|v_n\|_{L^p} \leq C\|\nabla z_n\|_\infty (\|v_n\|_{L^p} + \|z_n\|_{L^p}).$$

Then by Gronwall's lemma and since $s > 3 - 2\alpha$, we obtain the desired uniform L^p-norm estimates for θ_n. Moreover, by (7.15) and Gronwall's lemma we obtain the uniform H^r-norm estimates for v_n. By a similar argument as in the proof of Theorem 3.4 we have that v_n converges to some process $\tilde{v}$ in $L^2([t_0,T],H)$ such that $\tilde{v} + z$ is the solution of (3.1) in $[t_0,T]$. Then by the uniqueness proof in Theorem 5.1 we have $\tilde{v} = v$ in $[t_0,T]$, which implies that $v \in L^\infty([t_0,\infty),H^r) \cap L^2_{\mathrm{loc}}([t_0,\infty),H^{r+\alpha})$ P-a.s.. Therefore, (7.15) also holds for v with $t \in [t_0,\infty)$.

Then by (7.15) for $r = \alpha$, we obtain that there exist $K_1 = K_1(R_0) > 0$ and $t_1 = t_1(\omega) > t_0(\omega)$ such that $|\Lambda^{2\alpha} v(t_1)| \leq K_1$. Using (7.15) for $r = 2\alpha$ we obtain that there exists $T_0 = T_0(R_0)$ such that $|\Lambda^{2\alpha} v(T_0)| \leq K_1$. Then we proceed analogously and obtain that there exists $T_1 = T_1(R_0)$ such that $|\Lambda^{s+\delta} v(T_1)| \leq K_1$ for some $0 < \delta < 3\alpha - 2 - \sigma_1$. □

Lemma 7.4.4 Let $\alpha > 2/3$. Suppose Assumption 7.1 holds with $s > 3 - 2\alpha$. Then for each $R \geq 1$ there are $T_1 > 0$ and a compact subset $K \subset \mathcal{W}$ such that

$$\inf_{\|x\|_{L^p} \leq R} P_{T_1}(x, K) > 0,$$

for p as in Assumption 7.1.

Proof Define $K := \{x : |\Lambda^{s+\delta} x|^2 \leq K_1(R_0)\}$, where $K_1(R_0), \delta$ comes from the previous lemma. By Lemma 7.4.3, for $R \leq R_0$ we have

$$\begin{aligned} \inf_{\|x\|_{L^p} \leq R} P_{T_1}(x, K) \geq \inf_{\|x\|_{L^p} \leq R} (1 - P_x[\sup_{t \in [0,T_1]} |\Lambda^{s-\alpha+1+\sigma_1+\delta} z(t)|^2 > R_0] \\ - P_x[\sup_{t \in [0,T_1]} \|\theta\|^p_{L^p} > R_0]), \end{aligned}$$

where we used Lemma 7.4.3 in the last step. Under Assumption 7.1, since z is a Gaussian process, one deduces that there exist $\eta, C > 0$ such that

$$P_x[\sup_{t \in [0,T_1]} |\Lambda^{s-\alpha+1+\sigma_1+\delta} z(t)|^2 > R_0] \leq C e^{-\eta \frac{R_0^2}{T_1}},$$

(see e.g. [FR06, Proposition 15]). By Theorem 4.3, we obtain

$$\sup_{\|x\|_{L^p} \leq R} P_x[\sup_{t \in [0,T_1]} \|\theta\|^p_{L^p} > R_0] \leq \sup_{\|x\|_{L^p} \leq R} \frac{E_x[\sup_{t \in [0,T_1]} \|\theta\|^p_{L^p}]}{R_0} \leq \frac{C(R)}{R_0}.$$

Choosing R_0 big enough, we prove the assertion. □

The exponential convergence now follows from Lemma 7.4.4 and an abstract result of [GM05, Theorem 3.1]. For p as in Assumption 7.1, let $V : L^p \to \mathbb{R}$ be a measurable function and define $\|\phi\|_V := \sup_{x \in L^p} \frac{|\phi(x)|}{V(x)}$ and $\|\nu\|_V := \sup_{\|\phi\|_V \leq 1} \langle \nu, \phi \rangle$ for a signed measure ν.

Theorem 7.4.5 Let $\alpha > 2/3$. Suppose that Assumption 7.1 holds with $s > 3 - 2\alpha$ and let $V(x) := 1 + \|x\|^p_{L^p}$ for p as in Assumption 7.1. Then there exist $C_{\exp} > 0$ and $a > 0$ such that

$$\|P_t^* \delta_{x_0} - \mu\|_{TV} \leq \|P_t^* \delta_{x_0} - \mu\|_V \leq C_{\exp}(1 + \|x_0\|^p_{L^p}) e^{-at},$$

for all $t > 0$ and $x_0 \in Ł^p$, where $\| \cdot \|_{TV}$ is the total variation distance on measures.

Proof By a similar argument as the proof of Lemma 7.4.3 we obtain $P_t(x, \mathcal{W}) = 1$ for $x \in L^p$. By [GM05, Theorem 3.1], we need to verify the following four conditions,

1. the measures $(P_t(x, \cdot))_{t>0, x \in L^p}$ are equivalent,
2. $x \to P_t(x, \Gamma)$ is continuous in $\mathcal{W}$ for all $t > 0$ and all Borel sets $\Gamma \subset H$,
3. for each $R \geq 1$ there exist $T_1 > 0$ and a compact subset $K \subset \mathcal{W}$ such that

$$\inf_{\|x\|_{L^p} \leq R} P_{T_1}(x, K) > 0,$$

4. there exist $k, b, c > 0$ such that for all $t \geq 0$,

$$E^{P_x}[\|\theta(t)\|^p_{L^p}] \leq k \|x\|^p_{L^p} e^{-bt} + c.$$

Condition 1 can be verified by [GM05, Lemma 3.2] and $P_t(x, \mathcal{W}) = 1$ for $x \in L^p$. The other conditions can be verified by Theorem 7.3.2, Lemma 7.4.4 and Proposition 7.4.2. □

Remark 7.4.6 (i)For $\alpha > \frac{3}{4}$ we can get a better result following a similar argument as in [R08]. Namely, there exist $C_{\exp} > 0$ and $a > 0$ such that

$$\|P_t^*\delta_{x_0} - \mu\|_{TV} \leq \|P_t^*\delta_{x_0} - \mu\|_V \leq C_{\exp}(1 + |x_0|^2)e^{-at},$$

for all $t > 0$ and $x_0 \in H$. Here P_t could be every Markov selection associated to the solution of equation (3.1).

(ii)The reason why $\alpha > \frac{3}{4}$ is needed, is as follows: As in Theorem 7.1.2, we can prove P_t is H^s-strong Feller with $s > 3 - 3\alpha$. And for a solution θ of equation (3.1) starting from $x \in H$, we can prove that it will enter H^α only under our condition on the noise. If the process θ enters H^s, we can prove that it satisfies the above four conditions. Hence, to obtain exponential convergence for every $x \in H$, we need the process starting from $x \in H$ to enter H^s. Hence we need $3 - 3\alpha < s < \alpha$, i.e. $\alpha > \frac{3}{4}$.

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