Localization of solutions to stochastic porous media equations: finite speed of propagation

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Abstract

It is proved that the solutions to the low diffusion stochastic porous media equation

$$dX - \Delta(|X|^{m-1}X)dt = \sigma(X)dW_t,\ 1 < m \leq 5,$$

in $\mathcal{O} \subset \mathbb{R}^d,\ d = 1, 2, 3$, have the property of finite speed of propagation of disturbances for $\mathbb{P}$-a.s. $\omega \in \Omega$ on a sufficiently small time interval $(0, t(\omega))$.

Key words and phrases: Wiener process, porous media equation, energy method, stochastic flow.
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1 Introduction

Let $\mathcal{O}$ be a bounded and open domain of $\mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$.

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Consider the stochastic porous media equation

$$
\begin{array}{lll}
(1.1) \qquad & dX - \Delta(|X|^{m-1}X)dt = \sigma(X)dW_t, & t \geq 0, \\
& X = 0 & \text{on } \partial\mathcal{O}, \\
& X(0) = x & \text{in } \mathcal{O},
\end{array}
$$

where $m \geq 1$, W_t is a Wiener process in $L^2(\mathcal{O})$ of the form

$$
W_t = \sum_{k=1}^{N} \beta_k(t) e_k. \tag{1.2}
$$

$\{\beta_k\}_{k=1}^N$ is a sequence of mutually independent Brownian motions on a filtered probability space $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$ while $\{e_k\}_{k\in N}$ is an orthonormal system in $L^2(\mathcal{O})$ and

$$
\sigma(X)W_t = \sum_{k=1}^{N} \mu_k X e_k \beta_k(t), \tag{1.3}
$$

where $\{\mu_k\}$ is a sequence of nonnegative numbers.

We assume that $e_k \in C^2(\overline{\mathcal{O}})$ and

$$
\sum_{k=1}^{N} \mu_k^2 e_k^2(x) \geq \rho > 0, \quad \forall x \in \overline{\mathcal{O}}. \tag{1.4}
$$

Let $H_0^1(\mathcal{O})$, $H^{-1}(\mathcal{O})$ denote standard Sobolev spaces on $\mathcal{O}$ with the norms $\|\cdot\|_1$ and $|\cdot|_{-1}$, respectively. The norm of $L^p(\mathcal{O})$, $1 \leq p \leq \infty$, is denoted by $|\cdot|_p$ and the scalar product by $(\cdot,\cdot)$. The scalar product in $H^{-1}(\mathcal{O})$ is denoted by $(\cdot,\cdot)_{-1}$. The space $H^{-1}(\mathcal{O})$ will be denoted by H and set $A = -\Delta$, $D(A) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$.

An $H^{-1}(\mathcal{O})$-valued continuous $\mathcal{F}_t$-adapted process $X = X(t,\xi)$ is called a *strong solution to* (1.1) on $(0,T) \times \mathcal{O}$ if

$$
X \in L^2(\Omega, C([0,T]; H)) \cap L^\infty(0,T; L^2(\Omega \times \mathcal{O})), \; t \in [0,T], \tag{1.5}
$$

$$
|X|^{m-1}X \in L^2(0,T; L^2(\Omega, H_0^1(\mathcal{O}))), \tag{1.6}
$$

$$
X(t) = x + \int_0^t \Delta(|X(s)|^{m-1}X(s))ds + \int_0^t \sigma(X(s))dW_s. \tag{1.7}
$$

Here we use the standard notation $L^p(E;B)$, $p \in [0,\infty]$, for a measure space $(E,\mathcal{E},\mu)$ and a Banach space B, i.e., $L^p(E;B)$ denotes the space of all B-valued measurable maps $f : E \to B$ such that $|f|_B^p$ is μ-integrable.

The main result of this work, Theorem 2.3 below, amounts to saying that if $1 < m \leq 5$, which is the case of slow diffusion under stochastic perturbation, then the process $X = X(t,\cdot)$ has the property of finite speed propagation of disturbances in the following sense (see [4]): if $x = 0$ in $B_r(\xi_0) = \{\xi \in \mathcal{O};\ |\xi - \xi_0| < r\}$, then there is a function $r = r(t,\omega)$, decreasing in t, such that $X(t,\xi,\omega) = 0$ in $B_{r(t,\omega)}(\xi_0)$ for $0 \leq t \leq t(\omega)$, for $\mathbb{P}$-a.e. $\omega \in \Omega$. In this sense, we speak about finite speed of propagation of $X(t)$. This localization property for stochastic porous media equations has resisted its proof for quite some time, because the stochastic perturbation is a serious obstacle to adapt localization proofs and techniques from the known deterministic case. This lock was broken by the results in [6], and, particularly [9], which allow to transform the problem to a deterministic partial differential equation (PDE) with random coefficients. This latter PDE, however, is not of porous media or any other known type, so that the necessary estimates become much more complicated, but eventually lead to success.

We mention that in the case $0 < m < 1$ (fast diffusion) the solution $X = X(t,x)$ has a finite extinction property with positive probability (see [7]) which also can be seen as a localization property of stochastic flows associated with equation (1.1).

The main result, Theorem 2.3, is formulated in Section 2 and proved in Section 3 via some arguments inspired by the local energy method of S.N. Antontsev [1] (see also [2], [3], [4], [11], [12], [18] for some recent results on the localization of solutions to deterministic porous media equations). However, the overlap is not large. In a few words, the idea of the proof is to reduce equation (2.5) to a random partial differential equation on $(0,T)\times\mathcal{O}$ and combine the energy method from [1]–[3], with some sharp L^∞ estimates obtained in the authors' work [9].

Here, the discussion is confined to stochastic porous media equations with Dirichlet homogeneous boundary conditions because the previous existence theory we invoke and use here was developed so far in this case only. However, one might expect that everything extends *mutatis mutandis* to the Neumann reflection conditions on boundary. As regards the case $\mathcal{O} = \mathbb{R}^d$, this still remains open. We shall use standard notations and results for spaces of infinite dimensional adapted stochastic processes (see [10], [15]).

2 The main result

Proposition 2.1 *Assume that* $x \in L^{m+1}(\mathcal{O})$*. Then equation* (1.1) *has a unique strong solution* X*. If* $x \geq 0$ *a.e. in* $\mathcal{O}$*, then* $X \geq 0$ *a.e. in* $\Omega \times (0,T) \times \mathcal{O}$ *and*

$$
\begin{aligned}
E\int_0^T ds \int_{\mathcal{O}} \left|\nabla(|X|^{m-1}X)\right|^2 d\xi + \sup_{t\in[0,T]} E\int_{\mathcal{O}} |X(t,\xi)|^{m+1} d\xi \\
\leq C\int_{\mathcal{O}} |x|^{m+1} d\xi.
\end{aligned} \tag{2.1}
$$

Remark 2.2 Existence and uniqueness, as well as nonnegativity of solutions to equation (1.1) has been discussed in several papers (see [5], [6], [17]). But the notion of solution was different. More precisely, solutions were not required to satisfy (1.6), but only that

$$
t \mapsto \int_0^t |X(s)|^{m-1} X(s) ds
$$

is a continuous process in $H_0^1(\mathcal{O})$, and that (1.7) holds with the Laplacian in front of the ds-integral. We refer to [16] for a detailed discussion. In the present paper, we need the stronger notion of solution as in (1.5)–(1.7). For very recent results on existence of such "strong" solutions for general SPDE of gradient type, including our situation as a special case, we refer to [13].

Proof. We only give a sketch of the proof since the techniques are very close to [5], [6]. Let $\beta(r)$ denote by the function $|r|^{m-1}r$. We proceed as in [5], [6] and consider the approximating equation

$$
\begin{aligned}
& dX_\lambda - \Delta\beta_\lambda(X_\lambda)dt = \sigma(X_\lambda)dW_t \ \text{ in } (0,T)\times\mathcal{O},\ \lambda > 0, \\
& X_\lambda(0) = x \ \text{ in } \mathcal{O},
\end{aligned} \tag{2.2}
$$

where $\beta_\lambda = \beta(1+\lambda\beta)^{-1}$, $\lambda > 0$, and 1 denotes the identity map. Equation (2.2) has a unique solution X_λ in the sense of (1.5) to (1.7), which satisfies also (see [6], Lemma 3)

$$
\sup_{t\in[0,T]} E|X_\lambda(t)|_2^2 \leq C|x|_2^2,
$$

where C is a positive constant independent of λ.

Now, let $\varphi(x)=\int_\Omega j_\lambda(x(\xi))d\xi$, where $j_\lambda(r)=\int_0^r\beta_\lambda(s)ds$ ($\in C^\infty(\mathbb{R})$. By a suitable regularization, we apply Itô's formula for φ in (2.5) and obtain that

$$\begin{aligned}
&E\int_{\mathcal{O}} j_\lambda(X_\lambda(t,\xi))d\xi+E\int_0^t\int_{\mathcal{O}}|\nabla\beta_\lambda(X_\lambda(s,\xi))|^2d\xi\,ds\\
&\qquad=\int_{\mathcal{O}} j_\lambda(x(\xi))d\xi+E\int_0^t\sum_{k=1}^N\int_{\mathcal{O}}\beta'_\lambda(X_\lambda(s,\xi))|X_\lambda(s,\xi)e_k|^2d\xi\,ds\\
&\qquad\le\int_{\mathcal{O}}|x|^{m+1}(\xi)d\xi+CE\int_0^t\int_{\mathcal{O}}|X_\lambda(s,\xi)|^{m+1}d\xi\,ds,\\
&\qquad\qquad\qquad\forall\lambda>0,\ t\in[0,T].
\end{aligned}\tag{2.3}$$

Taking into account that

$$j_\lambda(X_\lambda)=\frac{1}{m+1}\ |(1+\lambda\beta)^{-1}X_\lambda|^{m+1}+\frac{1}{2\lambda}\ |X_\lambda-(1+\lambda\beta)^{-1}X_\lambda|^2,$$

by (2.3) we obtain that

$$\begin{aligned}
&\frac{1}{m+1}\ E\int_{\mathcal{O}}|(1+\lambda\beta)^{-1}X_\lambda(t,\xi)|^{m+1}d\xi+E\int_0^t\int_{\mathcal{O}}|\nabla\beta_\lambda(X_\lambda(s,\xi))|^2d\xi\,dx\\
&\qquad\le C\int_{\mathcal{O}}|x(\xi)|^{m+1}d\xi,\ \forall\lambda>0,\ t\in[0,T].
\end{aligned}\tag{2.4}$$

Moreover, arguing as in [5], [6], that is, by applying the Itô formula to the function $\varphi(X)=|X^-|_{m+1}^{m+1}$, it follows that $X_\lambda\ge0$ a.e. in $\Omega\times(0,T)\times\mathcal{O}$.

This yields via the Burkholder-Davis-Gundy inequality (see the proof of Theorem 2.2 in [6]) that

$$E\sup_{0\le t\le T}|X_\lambda(t)-X_\mu(t)|_{-1}^2e^{-\alpha t}\le C\max\{\lambda,\mu\},\ \lambda,\mu>0,$$

for some $\alpha>0$ and, therefore,

$$\begin{aligned}
X_\lambda\ \longrightarrow\ X\ \ &\text{strongly in } L^2(\Omega,C([0,T];H))\\
&\text{and weakly star in } L^\infty(0,T;L^2(\Omega\times\mathcal{O})),
\end{aligned}$$

Moreover, by (2.4) it follows that

$$\beta_\lambda(X_\lambda)\ \longrightarrow\ \eta\ \ \text{weakly in } L^2(\Omega\times(0,T);H_0^1(\mathcal{O})),$$

where $\eta \in \beta(X)$ a.e. in $\Omega \times (0,T) \times \mathcal{O}$.

Then, letting λ tend to zero in (2.2), we obtain (1.5)–(1.7) and $X \geq 0$, as claimed. Also, (2.4) implies (2.1) by lower semicontinuity. $\square$

Everywhere in the sequel $B_r(\xi_0)$ shall denote the open ball $\{\xi; |\xi - \xi_0| < r\}$, and $\Sigma_r(\xi_0) = \{\xi \in R^d;\ |\xi - \xi_0| = r\}$ its boundary, and $B_r^c(\xi_0) = \mathcal{O} \setminus B_r(\xi_0)$, $\xi_0 \in \mathcal{O}$. As mentioned in the introduction, $\mathcal{O}$ is an open and bounded domain of R^d with smooth boundary $\partial\mathcal{O}$, $d = 1, 2, 3$. Everywhere below, X is the solution to equation (1.1) in the sense of definition (1.5)–(1.7) with initial data x.

Below, we are only concerned with small $T > 0$, so we may assume that $T \leq 1$. Furthermore, for a function $g : [0,1] \to \mathbb{R}$, we define its α-Hölder norm, $\alpha \in (0,1)$, by

$$|g|_\alpha := \sup_{\substack{s,t \in [0,1] \\ s \neq t}} \frac{|g(t) - g(s)|}{|t - s|^\alpha}\,.$$

Let for $\alpha \in (0, \frac{1}{2})$

$$\Omega_{H,R}^\alpha = \{\omega \in \Omega \mid |\beta_k(\omega)|_\alpha \leq R,\ 1 \leq k \leq N\}\,.$$

Then, $\Omega_{H,R}^\alpha \nearrow \Omega$ as $R \to \infty$ $\mathbb{P}$-a.s.

Now, we are ready to formulate the main result.

Theorem 2.3 *Assume that $d = 1, 2, 3$ and $1 < m \leq 5$, and that $x \in L^\infty(\mathcal{O})$, $x \geq 0$, is such that*

$$\text{support}\{x\} \subset B_{r_0}^c(\xi_0), \tag{2.5}$$

where $r_0 > 0$ and $\xi_0 \in \mathcal{O}$. Fix $\alpha \in (0, \frac{1}{2})$ and let for $R > 0$

$$\delta(R) := \left(\frac{1}{m+1} \left(\frac{\rho}{2}\right)^{1/2} c_1^{-1} \left(\sum_{k=1}^N |\nabla e_k|_\infty \mu_k \right)^{-1} \times \exp\left[\frac{1}{2}\,(1-m) \left(\frac{1}{2}\, c_2 + \sum_{k=1}^N |e_k|_\infty \mu_k \right) \right] \right) \wedge 1,$$

where c_1, c_2 (depending on R) are as in Lemma 3.1 below and ρ as in (1.4). Define for $T \in (0,1]$

$$\Omega_T^{\delta(R)} := \left\{ \sup_{t \in [0,T]} |\beta_k(t)| \leq \delta(R) \textit{ for all } 1 \leq k \leq N \right\}.$$

Then, for $\omega \in \Omega_T^{\delta(R)} \cap \Omega_{H,R}^\alpha$*, there is a decreasing function* $r(\cdot, \omega) : [0, T] \to (0, r_0]$*, and* $t(\omega) \in (0, T]$ *such that for all* $0 \le t \le t(\omega)$*,*

$$\begin{aligned} &X(t,\omega) = 0 \ \text{ on } B_{r(t,\omega)}(\xi_0) \supset B_{r(t(\omega),\omega)}(\xi_0), \quad \text{and} \\ &X(t,\omega) \not\equiv 0 \ \text{ on } B^c_{r(t,\omega)} \subset B^c_{r(t(\omega),\omega)}(\xi_0). \end{aligned} \tag{2.6}$$

Since $\Omega_T^{\delta(R)} \nearrow \Omega$ *as* $T \to 0$ *up to a* $\mathbb{P}$*-zero set, and hence*

$$\mathbb{P}\left(\bigcup_{M\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \Omega_{1/N}^{\delta(M)} \cap \Omega_{H,M}^\alpha \right) = 1,$$

it follows that we have finite speed of propagation of disturbances ("localization") for $(X_t)_{t\ge0}$ $\mathbb{P}$*-a.s..*

As explicitly follows from the proof, the function $t \to r(t)$ is a process adapted to the filtration $\{\mathcal{F}_t\}$.

Roughly speaking, Theorem 2.3 amounts to saying that, for $\omega \in \Omega_T^{\delta(R)} \cap \Omega_{H,R}^\alpha$ and for a time interval $[0, t(\omega)]$ sufficiently small, the stochastic flow $X = X(t, \xi, \omega)$ propagates with finite speed.

If we set $r^T(\omega) = \lim\limits_{t\to T} r(t, \omega)$, we see by (2.6) that $X(t, \omega) = 0$ on $B_{r^T(\omega)}$, $\forall t \in (0, t(\omega))$ and $X(t) \not\equiv 0$ on $B^c_{r^T(\omega)}$. It is not clear whether $r^T(\omega) = 0$ for some $T > 0$, that is, whether the "hole filling" property holds in this case (see [18]).

It should be mentioned also that the assumption $x \ge 0$ in $\mathcal{O}$ was made only to give a physical meaning to the propagation process.

The conditions $m \le 5$ and $x \in L^\infty(\mathcal{O})$ might seem unnatural, but they are technical assumptions required by the work [9] on which the present proof essentially relies.

3 Proof of Theorem 2.3

For the proof we shall take $\xi_0 = 0 \in \mathcal{O}$ and set $B_r = B_r(0)$. The method of the proof relies on some sharp integral energy type estimates of $X = X(t)$ on arbitrary balls $B_r \subset \mathcal{O}$.

It is convenient to rewrite equation (1.1) as a deterministic equation with random coefficients. To this aim we consider the transformation

$$y(t) = e^{\mu(t)} X(t), \ t \ge 0, \tag{3.1}$$

where $\mu(t) = -\sum_{k=1}^{N} \mu_k e_k \beta_k(t)$.

Then we have (see, [6], Lemma 4.1)

$$
\begin{aligned}
&\frac{dy}{dt} - e^{\mu}\Delta(y^m e^{-m\mu}) + \frac{1}{2}\widetilde{\mu} y = 0, \ t > 0, \ \mathbb{P}\text{-a.s.}, \\
&y(0) = x, \\
&y^m \in H_0^1(\mathcal{O}), \ \forall t > 0, \ \mathbb{P}\text{-a.s.},
\end{aligned}
\tag{3.2}
$$

where

$$
\widetilde{\mu} = \sum_{k=1}^{N} \mu_k^2 e_k^2. \tag{3.3}
$$

By Proposition 2.1, we have $\mathbb{P}$-a.s.

$$
y \geq 0, \ y^m(t)e^{\mu(t)} \in H_0^1(\mathcal{O}) \cap L^{\frac{m+1}{m}}(\mathcal{O}), \ \text{a.e. } t \geq 0. \tag{3.4}
$$

As a matter of fact, in [9], a sharper result on equation (3.2) was proved. Namely, one has

Lemma 3.1 *Assume that* $1 \leq d \leq 3$ *and* $m \in [1, 5]$. *Then, if* $x \in L^\infty(\mathcal{O})$, *the solution* y *to* (3.2) *satisfies* $\mathbb{P}$*-a.s. for every* $T > 0$

$$
y \in L^\infty((0,T) \times \mathcal{O}) \cap C([0,T]; H), \tag{3.5}
$$

$$
y^m \in L^2(0,T; H_0^1(\mathcal{O})), \ \frac{dy}{dt} \in L^2(0,T; H). \tag{3.6}
$$

Moreover, for every $T \in (0,1]$, $\alpha \in (0, \frac{1}{2})$, $R > 0$, *there exist constants* $c_1, c_2 > 0$ *depending on* α, R, $\mathcal{O}$, $|x|_\infty$, $\max\limits_{1 \leq k \leq N}(|e_k|_\infty, |\nabla e_k|_\infty, |\Delta e_k|_\infty)$, *but not on* T *such that* $\mathbb{P}$*-a.s. on* $\Omega_{H,R}^\alpha$,

$$
\|y\|_{L^\infty((0,T)\times\mathcal{O})} \leq c_1 \exp\left[c_2 \max_{1 \leq k \leq N} \sup_{t \in [0,T]} |\beta_k(t)|\right]. \tag{3.7}
$$

The first part of Lemma 3.1 is just Theorem 2.1 in [9], while (3.2) follows by the proof of Lemma 3.1 in [9] (see (3.25)–3.28)).

Before we introduce our crucial energy functional ϕ in (3.14) below and explaining the idea of the proof subsequently, we need some preparations by a few estimates on the solution y to (3.2). Everywhere in the following we fix $\alpha \in (0, \frac{1}{2})$, $\alpha > 0$ and assume that $x \geq 0$ so that (3.4) holds and fix $T \in (0, 1]$.

By Green's formula, it follows from (3.2) that

$$\begin{aligned} &\frac{1}{m+1}\int_{\mathcal{O}} y^{m+1}(t,\xi)\psi(\xi)d\xi + \int_0^t ds \int_{\mathcal{O}} \nabla(y^m e^{-m\mu})\cdot\nabla(e^\mu y^m \psi)d\xi \\ &+\frac{1}{2}\int_0^t ds\int_{\mathcal{O}} \widetilde{\mu} y^{m+1}\psi\, d\xi = \frac{1}{m+1}\int_{\mathcal{O}} x^{m+1}(\xi)\psi(\xi)d\xi,\ t\in(0,T), \end{aligned} \tag{3.8}$$

for all $\psi \in C_0^\infty(\mathcal{O})$.

Fix $r > 0$ and let $\rho_\varepsilon \in C^\infty(R^+)$ be a cut-off function such that $\rho_\varepsilon(s) = 1$ for $0 \leq s \leq r + \varepsilon$, $\rho_\varepsilon(s) = 0$ for $s \geq r + 2\varepsilon$ and

$$\lim_{\varepsilon\to 0}\left|\rho_\varepsilon'(s) + \frac{1}{\varepsilon}\right|\chi_\varepsilon(s) = 0, \tag{3.9}$$

uniformly in $s \in [0, \infty)$. Roughly speaking, this means that ρ_ε is a smooth approximation of the function $\gamma_\varepsilon(s) = 1$ on $[0, r+\varepsilon]$, $\gamma_\varepsilon(s) = 0$ on $[r+2\varepsilon, \infty)$, $\gamma_\varepsilon(s) = -\frac{1}{\varepsilon}(s - r - \varepsilon) + 1$ on $[r + \varepsilon, r + 2\varepsilon]$.

If in (3.8) we take $\psi = \rho_\varepsilon(|\xi|)$ (for ε small enough), setting $\psi_\varepsilon(\xi) = \rho_\varepsilon(|\xi|)$, $\xi \in \mathcal{O}$, we obtain that

$$\begin{aligned} &\frac{1}{m+1}\int_{\mathcal{O}}(y(t,\xi))^{m+1}\rho_\varepsilon(|\xi|)d\xi + \int_0^t ds\int_{\mathcal{O}}\nabla(ye^{-\mu})^m\cdot\nabla(e^\mu y^m\psi_\varepsilon)d\xi \\ &\qquad+\frac{1}{2}\int_0^t ds\int_{\mathcal{O}}\widetilde{\mu} y^{m+1}\psi_\varepsilon d\xi = \frac{1}{m+1}\int_{\mathcal{O}} x^{m+1}\psi_\varepsilon d\xi. \end{aligned} \tag{3.10}$$

On the other hand, we have

$$\begin{aligned} \int_{\mathcal{O}}\nabla(ye^{-\mu})^m\cdot\nabla(e^\mu y^m\psi_\varepsilon)d\xi &= \int_{\mathcal{O}}|\nabla(ye^{-\mu})^m|^2\psi_\varepsilon e^{(m+1)\mu}d\xi \\ &+(m+1)\frac{1}{2}\int_{\mathcal{O}}(\nabla(ye^{-\mu})^m\cdot\nabla\mu)e^\mu y^m\psi_\varepsilon d\xi \\ &+\int_{\mathcal{O}}(\nabla(ye^{-\mu})^m\cdot\nu)(s,\xi)\rho_\varepsilon'(|\xi|)(e^\mu y^m)(s,\xi d\xi, \end{aligned} \tag{3.11}$$

where $\nu(\xi) = \dfrac{\xi}{|\xi|} \cdot$ (Since $\mu \in C^2(\overline{\mathcal{O}})$, the above calculation is justified.)

Everywhere in the following, the estimates are taken on the set $\Omega^{\alpha}_{H,R} \cap \Omega^{\delta(R)}_{T}$.

We set $B^{\varepsilon}_{r} = B_{r+2\varepsilon} \setminus B_{r+\varepsilon}$. Then, by (3.10), (3.11), we see that

$$
\begin{aligned}
&\frac{1}{m+1}\int_{B_{r+\varepsilon}} y^{m+1}(t,\xi)d\xi + \int_0^t ds \int_{B_{r+2\varepsilon}} \psi_\varepsilon e^{(m+1)\mu}|\nabla(ye^{-\mu})^m|^2 d\xi\, ds \\
&\quad + \frac{1}{2}\int_0^t ds \int_{B_{r+2\varepsilon}} \psi_\varepsilon \widetilde{\mu} y^{m+1} d\xi\, ds \\
&\quad = \frac{1}{m+1}\int_{B_{r+2\varepsilon}} \psi_\varepsilon x^{m+1} d\xi \\
&\quad -(m+1)\int_0^t \int_{B_{r+2\varepsilon}} (\nabla(ye^{-\mu})^m \cdot \nabla\mu)\psi_\varepsilon e^{\mu} y^m d\xi\, ds \\
&\quad - \int_0^t \int_{B^{\varepsilon}_{r}} (\nabla(ye^{-\mu})^m \cdot \nu)(s,\xi)(e^{\mu}y^m)(s,\xi)\rho'_\varepsilon(|\xi|)d\xi\, ds.
\end{aligned}
\tag{3.12}
$$

On the other hand, we have

$$
\begin{aligned}
&\int_0^t \int_{B^{\varepsilon}_{r}} |(\nabla(ye^{-\mu})^m \cdot \nu)e^{\mu}y^m \rho'_\varepsilon(|\cdot|)| d\xi\, ds \\
&\le \left(\int_0^t\int_{B^{\varepsilon}_{r}} |\rho'_\varepsilon(|\cdot|)|\,|\nabla(ye^{-\mu})^m|^2 e^{(m+1)\mu} d\xi\, ds\right)^{\frac{1}{2}} \\
&\qquad \times \left(\int_0^t\int_{B^{\varepsilon}_{r}} e^{(1-m)\mu} y^{2m} |\rho'_\varepsilon(|\cdot|)| d\xi\, ds\right)^{\frac{1}{2}}.
\end{aligned}
\tag{3.13}
$$

We introduce the energy function

$$
\phi(t,r) = \int_0^t \int_{B_r} |\nabla(ye^{-\mu})^m|^2 e^{(m+1)\mu} d\xi\, ds,\ \ t \in [0,T],\ \ r \ge 0. \tag{3.14}
$$

In order to prove (2.6), our aim in the following is to show that ϕ satisfies a differential inequality of the form

$$
\frac{\partial \phi}{\partial r}\,(t,r) \ge Ct^{\theta-1}(\phi(t,r))^{\delta} \text{ on } \Omega^{\alpha}_{H,R} \cap \Omega^{\delta(R)}_{T} \text{ for } t \in [0,T],\ \ r \in [0,r_0],
$$

where $0 < \theta < 1$ and $0 < \delta < 1$ and from which (2.6) will follow.

Taking into account that function ϕ is absolutely continuous in r, we have by (3.9), a.e. on $(0, r_0)$,

$$\lim_{\varepsilon \to 0} \int_0^t \int_{B_r^\varepsilon} |\rho_\varepsilon'(|\cdot|)||\nabla(ye^{-\mu})^m|^2 e^{(m+1)\mu} d\xi\, ds = \frac{\partial \phi}{\partial r}(t,r).$$

Then, letting $\varepsilon \to 0$ in (3.12), (3.13), we obtain that

$$\begin{aligned}
&\frac{1}{m+1}\int_{B_r} y^{m+1}(t,\xi)d\xi + \phi(t,r) + \frac{1}{2}\int_0^t \int_{B_r} \widetilde{\mu} y^{m+1} d\xi\, ds \\
&\leq \frac{1}{m+1}\int_{B_r} x^{m+1} d\xi - (m+1)\int_0^t \int_{B_r} (\nabla(ye^{-\mu})^m \cdot \nabla\mu) e^{\mu} y^m d\xi\, ds \\
&+ \left(\frac{\partial \phi}{\partial r}(t,r)\right)^{\frac{1}{2}} \left(\int_0^t ds \int_{\Sigma_r} y^{2m} e^{(1-m)\mu} d\xi\right)^{\frac{1}{2}}, \\
&\qquad \text{on } \Omega_{H,R}^{\alpha} \cap \Omega_T^{\delta(R)},\ t \in [0,T],\ r \in [0, r_0].
\end{aligned} \tag{3.15}$$

In order to estimate the right-hand side of (3.15), we introduce the following notations

$$K(t,r) = \frac{1}{2}\int_0^t \int_{B_r} \widetilde{\mu} y^{m+1} ds\, d\xi \tag{3.16}$$

$$H(t,r) = \sup\left\{\frac{1}{m+1}\int_{B_r} y^{m+1}(s,\xi)d\xi,\ 0 \leq s \leq t\right\}, \tag{3.17}$$

and note that by assumption (1.4) we have

$$K(t,r) \geq \frac{1}{2}\,\rho \int_0^t \int_{B_r} y^{m+1} d\xi\, ds,\ \forall t \in [0,T],\ r \in [0, r_0]. \tag{3.18}$$

Then (3.15) yields, for $r \in (0, r_0]$,

$$\begin{aligned}
&H(t,r) + \phi(t,r) + K(t,r) \\
&\quad \leq (m+1)\int_0^t \int_{B_r} |(\nabla(ye^{-\mu})^m \cdot \nabla\mu) e^{\mu} y^m| d\xi\, ds \\
&\quad + \left(\frac{\partial \phi}{\partial r}(t,r)\right)^{\frac{1}{2}} \left(\int_0^t \int_{\Sigma_r} y^{2m} e^{(1-m)\mu} d\xi\, ds\right)^{\frac{1}{2}}
\end{aligned} \tag{3.19}$$

because $x \equiv 0$ on B_r. We note that, by the trace theorem, the surface integral arising in the right-hand side of formula (3.19) is well defined because $\nabla(ye^{-\mu})^m \in L^2([0,T] \times \mathcal{O})$ and, by Lemma 3.1, $y \in L^\infty((0,T) \times \mathcal{O})$ $\mathbb{P}$-a.s.

Now, we are going to estimate the right-hand side of (3.19).

By Cauchy–Schwarz and (3.18), we have

$$
\begin{aligned}
&\int_0^t \int_{B_r} |(\nabla(ye^{-\mu})^m \cdot \nabla\mu)e^{\mu}y^m| d\xi\, ds \\
&\le \|y^{m-1}e^{(1-m)\mu}|\nabla\mu|^2\|^{1/2}_{L^\infty((0,T)\times\mathcal{O})} \\
&\times \left(\int_0^t ds \int_{B_r} |\nabla(y^m e^{-m\mu})|^2 e^{(m+1)\mu} d\xi\right)^{\frac{1}{2}} \left(\int_0^t ds \int_{B_r} y^{m+1} d\xi\right)^{\frac{1}{2}} \\
&\le (2\rho^{-1})^{1/2}\|y^{m-1}e^{(1-m)\mu}|\nabla\mu|^2\|^{1/2}_{L^\infty((0,T)\times\mathcal{O})}(\phi(t,r))^{\frac{1}{2}}(K(t,r))^{\frac{1}{2}} \\
&\le \frac{1}{2(m+1)}\,(\phi(t,r) + K(t,r)), \\
&\qquad\qquad \forall t \in (0,T],\ r \in (0,r_0], \quad \text{on } \Omega^\alpha_{H,R} \cap \Omega^{\delta(R)}_T,
\end{aligned}
\tag{3.20}
$$

by the definition of $\delta(R)$.

By (3.19), it follows that

$$
\begin{aligned}
&H(t,r) + \phi(t,r) + K(t,r) \\
&\le \left(\frac{\partial\phi}{\partial r}\,(t,r)\right)^{\frac{1}{2}} \left(\int_0^t ds \int_{\Sigma_r} y^{2m}e^{(1-m)\mu} d\xi\right)^{\frac{1}{2}} \\
&\qquad \forall t \in [0,T],\ r \in [0,r_0], \quad \text{on } \Omega^\alpha_{H,R} \cap \Omega^{\delta(R)}_T.
\end{aligned}
\tag{3.21}
$$

In order to estimate the surface integral from the right-hand side of (3.21), we invoke the following interpolation-trace inequality (see, e.g., Lemma 2.2 in [12])

$$
|z|_{L^2(\Sigma_r)} \le C(|\nabla z|_{L^2(B_r)} + |z|_{L^{\sigma+1}(B_r)})^\theta |z|^{1-\theta}_{L^{\sigma+1}(B_r)},
\tag{3.22}
$$

for all $\sigma \in [0,1]$ and $\theta = (d(1-\sigma) + \sigma + 1)/(d(1-\sigma) + 2(\sigma+1))$. Clearly, $\theta \in \left[\frac{1}{2}, 1\right)$.

We shall apply this inequality for $z = (y^m e^{-\mu})^m$ and $\sigma = \frac{1}{m}$. We obtain, by (3.17) that

$$
\begin{aligned}
&\left(\int_{\Sigma_r} y^{2m} e^{(1-m)\mu} d\xi\right)^{\frac{1}{2}} \le \|e^{(1+m)\mu}\|^{1/2}_{L^\infty((0,T)\times\mathcal{O})} \left(\int_{\Sigma_r} (ye^{-\mu})^{2m} d\xi\right)^{\frac{1}{2}}\\
&\le C\|e^{(1+m)\mu}\|^{1/2}_{L^\infty((0,T)\times\mathcal{O})}(|\nabla(ye^{-\mu})^m|_{L^2(B_r)}+|y^m e^{-m\mu}|_{L^{\frac{m+1}{m}}(B_r)})^\theta |y^m e^{-m\mu}|^{1-\theta}_{L^{\frac{m+1}{m}}(B_r)}\\
&\le \widetilde{C}\left(\left(\int_{B_r} |\nabla(y^m e^{-m\mu})|^2 e^{(m+1)\mu} d\xi\right)^{\frac{1}{2}} + H^{\frac{m}{m+1}}(t,r)\right)^\theta (H^{\frac{m}{m+1}}(t,r))^{1-\theta},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \Omega^\alpha_{H,R}\cap\Omega^{\delta(R)}_T,
\end{aligned}
$$

where, as will be the case below, $\widetilde{C}$ is a positive function of $\omega\in\Omega^\alpha_{H,R}\cap\Omega^{\delta(R)}_T$, independent of t and r, which may change below from line to line.

Integrating over $(0,t)$ and applying first Minkowski's (since $\theta\ge\frac{1}{2}$) and then Hölder's inequality yields

$$
\begin{aligned}
&\left(\int_0^t ds \int_{\Sigma_r} y^{2m} e^{(1-m)\mu} d\xi\right)^{\frac{1}{2}}\\
&\le \widetilde{C}\left(\int_0^t ds \left(\int_{B_r} |\nabla(y^m e^{-m\mu})|^2 e^{(m+1)\mu} d\xi + H^{\frac{2m}{m+1}}(s,r)\right)^\theta H^{\frac{2m(1-\theta)}{m+1}}(s,r)\right)^{\frac{1}{2}}\\
&\le \widetilde{C} H^{\frac{m(1-\theta)}{m+1}}(t,r) t^{\frac{1-\theta}{2}}((\phi(t,r))^{\frac{1}{2}} + H^{\frac{m}{m+1}}(t,r))^\theta, \quad \text{on } \Omega^\alpha_{H,R}\cap\Omega^{\delta(R)}_T.
\end{aligned}
$$

Substituting the latter into (3.21), we obtain that

$$
\begin{aligned}
\phi + H &\le \widetilde{C} t^{\frac{1-\theta}{2}} \left(\frac{\partial\phi}{\partial r}\right)^{\frac{1}{2}} (\phi^{\frac{1}{2}} + H^{\frac{m}{m+1}})^\theta H^{\frac{m(1-\theta)}{m+1}}\\
&\le \widetilde{C} t^{\frac{1-\theta}{2}} \left(\frac{\partial\phi}{\partial r}\right)^{\frac{1}{2}} \left(\phi^{\frac{1}{2}} H^{\frac{m(1-\theta)}{(m+1)\theta}} + H^{\frac{m}{(m+1)\theta}}\right)^\theta,\\
&\qquad \forall t\in[0,T],\ r\in[0,r_0], \quad \text{on } \Omega^\alpha_{H,R}\cap\Omega^{\delta(R)}_T.
\end{aligned}
\tag{3.23}
$$

On the other hand, for $H_0 = H(T,r_0)$, we have the estimate

$$
\begin{aligned}
\phi^{\frac{1}{2}} H^{\frac{m(1-\theta)}{(m+1)\theta}} + H^{\frac{m}{(m+1)\theta}} &\le \phi^{\frac{1}{2}} H^{\frac{m(1-\theta)}{(m+1)\theta}} + H_0^{\frac{m}{m+1}-\frac{1}{2}} H^{\frac{m(1-\theta)}{(m+1)\theta}+\frac{1}{2}}\\
&\le \widetilde{C}(\phi+H)^{\frac{1}{2}+\frac{m(1-\theta)}{(m+1)\theta}},
\end{aligned}
$$

where $\widetilde{C} := 2\max(1, H_0^{\frac{m-1}{2(m+1)}})$ and where we used that by Young's inequality, for all $p, q \in (0, \infty)$,

$$\phi^p H^q \le (\phi + H)^{p+q}.$$

Substituting the latter into (3.23) yields

$$\phi + H \le \widetilde{C} t^{\frac{1-\theta}{2}} \left(\frac{\partial \phi}{\partial r}\right)^{\frac{1}{2}} (\phi + H)^{\frac{\theta}{2} + \frac{m(1-\theta)}{m+1}} \text{ on } (0,T) \times (0, r_0) \times \Omega^{\alpha}_{H,R} \cap \Omega^{\delta(R)}_T,$$

and therefore

$$\left(\frac{\partial \phi}{\partial r}(t,r)\right)^{\frac{1}{2}} \ge \widetilde{C} t^{\frac{\theta-1}{2}} (\phi(t,r))^{\frac{2-\theta}{2} - \frac{m(1-\theta)}{m+1}} \tag{3.24}$$

$$\text{on } (0,T) \times (0, r_0) \times \Omega^{\alpha}_{H,R} \cap \Omega^{\delta(R)}_T.$$

Equivalently,

$$\frac{\partial \varphi}{\partial r}(t,r) \ge \widetilde{C} t^{\theta - 1}, \text{ on } (0,T) \times (r(t), r_0) \times \Omega^{\alpha}_{H,R} \cap \Omega^{\delta(R)}_T, \tag{3.25}$$

where

$$\varphi(t,r) = (\phi(t,r))^{\theta + \frac{2m(1-\theta)}{m+1} - 1}, \tag{3.26}$$

and

$$r(t) := \inf\{r \ge 0 \mid \phi(t,r) > 0\} \wedge r_0.$$

We note that, by continuity,

$$\phi(t, r(t)) = 0$$

and that, since $t \mapsto \phi(t,r)$ is increasing, we have $\phi(t,r) > 0$, if $r > r(t)$, and that $t \mapsto r(t)$ is decreasing in t. Furthermore, the same is true for φ defined in (3.26), since $\theta + \frac{2m(1-\theta)}{m+1} - 1 > 0$, because $0 < \theta < 1$ and $m > 1$.

Moreover, by (3.17) and (3.23) we see that

$$X(t,\xi) = 0 \qquad \text{for} \quad \xi \in B_{r(t)}.$$

We recall that $r(t) = r(t,\omega)$ depends on $\omega \in \Omega$. Now, fix $\omega \in \Omega^{\alpha}_{H,R} \times \Omega^{\delta(R)}_T$. Our aim is to show that

$$\exists\, t(\omega) \in (0, T] \text{ such that } r(t,\omega) > 0,\ \forall t \in [0, t(\omega)]. \tag{3.27}$$

Since we already noted that $\phi(t,r) > 0$, if $r > r(t)$, by (3.21), (3.26) and (3.1), we deduce the property in (2.6) from (3.27). To show (3.27), we first note that by (3.25) for all $t \in (0,T)$

$$\varphi(t,r_0)(\omega) \geq \widetilde{C} t^{\theta-1}(r_0 - r(t,\omega)),$$

hence

$$r(t,\omega) \geq r_0 - (\widetilde{C}(\omega))^{-1} t^{1-\theta} \varphi(t,r_0)(\omega).$$

So, because $0 < \theta < 1$, we can find $t = t(\omega) \in (0,T)$, small enough, so that the right-hand side is strictly positive. Now, (3.27) follows, since, as noted earlier, $t \mapsto r(t,\omega)$ is decreasing in t, which completes the proof of (2.6). By elementary considerations for $\delta > 0$, we have

$$\mathbb{P}(\Omega_T^\delta) \geq 2^N \left(1 - \sqrt{\frac{T}{2\pi\delta^2}}\, e^{-\delta^2/(2T)}\right)^N.$$

Hence $\Omega_T^\delta \nearrow \Omega$ as $T \to 0$ up to a $\mathbb{P}$-zero set and the last part of of the assertion also follows. □

Remark 3.2 In the deterministic case, for $\mathcal{O} = \mathbb{R}^d$ the finite speed propagation property: support $\{x\} \subset B_{r_0}(\xi_0) \Longrightarrow$ support $\{X(t)\} \subset B_{r(t)}(\widetilde{\xi_0})$ for some $\widetilde{\xi_0} \in \mathbb{R}^d$ and $r = r(t)$, follows by the comparison principle $X(t,\xi) \leq U(t+\tau,\, \xi - \widetilde{\xi_0})$, where $U = U(t,\xi)$ is the Barenblatt source solution

$$U(t,\xi) = t^{-\frac{d}{(m-1)d+2}} \left[C - \frac{m-1}{2m((m-1)d+2)}\, \frac{|\xi|^2}{t^{\frac{2}{(m-1)d+2}}}\right]_+^{\frac{1}{m-1}} \tag{3.28}$$

(see [18]) and which has the support in $\{(t,\xi);\ |\xi|^2 \leq C_1 t^{\frac{2}{(m-1)d+2}}\}$.

At least in the simpler case, where the noise is not function valued, i.e. independent of ξ, this is similar in the stochastic case. More precisely, for $d = 1$, $\mathcal{O} = \mathbb{R}^1$ and $W(t) = \beta(t) =$ standard, real-valued Brownian motion, the function

$$Z(t,\xi) = U\left(\int_0^t k(s)ds, \xi\right) k(t),\ \ k(t) = e^{\beta(t) - \frac{1}{2}t}$$

is a solution to (1.1) and support $Z \subset \left\{(t,\xi);\ |\xi|^2 \leq C_1 \left(\int_0^t k(s)ds\right)^{\frac{2}{3}}\right\}$ (see [14] for details). (We are indebted to the referee for pointing this out to us.) However, on bounded domains, it is not clear, whether this is applicable.

Remark 3.3 The finite dimensional structure of the Wiener process $W(t)$ was essential for the present approach, which is based on sharp estimates on solutions to equation (3.2). A direct application of the above energy method in $L^2(\Omega; L^2(0,T; H^{-1}(\mathcal{O})))$ failed for general cylindrical Wiener processes $W(t)$.

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