From resolvents to càdlàg processes through compact excessive functions and applications to singular SDE on Hilbert spaces

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Abstract. We present an approach for constructing càdlàg strong Markov processes given a resolvent of kernels. The conditions imposed on the resolvent are checkable in applications and allow the control of the set of admissible starting points of the process. The main application is to singular SDEs on Hilbert spaces.

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1 Introduction

A fundamental problem of potential theory is the question which conditions on a given resolvent $(U_\alpha)_{\alpha>0}$ on a general topological space imply that it is the resolvent of a (strong) Markov process with càdlàg sample paths (see [BeBo 04] and the references therein). More concretely, it is an open question what are the necessary and sufficient conditions to be imposed on a strongly Feller resolvent (i.e., each U_α maps bounded measurable functions to bounded continuous functions) on a Polish space, so that such a Markov process exists. (This latter question was posed to the authors by Y. LeJan several years ago).

From the point of view of applications there was some criticism whether so much effort should be invested in the above questions, because in concrete cases usually one does not know much about the resolvent under consideration and should rather concentrate on the associated stochastic differential equation (SDE) or martingale problem and construct the Markov process through its solution. There are, however, many interesting cases, where the coefficients are too singular so that the SDEs or martingale problem have resisted all attempts to be solved at all or at least not for all starting points or the admissible starting points could not be identified explicitly. On the other hand, techniques to analyze the underlying generators, nowadays called Kolmogorov operators, have significantly advanced, so that a lot of information for the corresponding resolvents can be proved analytically in many applications with singular coefficients (cf. e.g. [DaRö 02], [DaRö 09], [Da 04], and [ASZ 09]).

The purpose of this paper is to develop a general approach for constructing càdlàg strong Markov processes given a resolvent $(U_\alpha)_{\alpha>0}$ (of kernels), imposing conditions on the resolvents (see $(H1)-(H3)$ in Subsection 2.1 below for the fundamental ones), which are checkable with modern techniques in applications, and which give rise to a control of the set of admissible starting points. In our main application to singular SDE on Hilbert spaces (see Section 6 below), we, however, even prove that we can start at every point in the state space. Further applications are under investigation and will be contained in a future paper.

The organization of this paper is as follows. In Section 2 we present some preliminaries on resolvents of kernels and recall some basic notions of Potential Theory (as e.g. excessive functions and measures, fine topology, polar set, energy functionals, etc). Section 3 deals with modifications of resolvents on inessential sets. Section 4 is devoted to compact excessive functions ("Lyapunov functions") and capacities. Section 5 contains our main abstract results (see in particular, Theorem 5.2 and Corollary 5.3) on the existence of a càdlàg strong Markov processes associated with a given resolvent. In particular, we study the stability of the results under subordination (cf. Corollary 5.4 and the subsequent Example). These results are complemented by analyzing, witch additional conditions ensure that the Markov process is even standard (see the definition in front of Theorem 5.5 below), in Subsection 5.3. These results have already been used in [BeCoRö 10] to prove for the first time that a large class of Lévy processes on infinite dimensional state spaces are in fact standard. Finally, we apply our results to singular SDE on Hilbert spaces in Section 6. These results improve previously known results (e.g., those in [RöSo 06], if $\psi \equiv 0$, i.e., if there is no "Burgers-part" in the SDE) to non-continuous drifts.

2 Preliminaries on resolvents of kernels

2.1 Excessive functions

Let $(E, \mathcal{B})$ be a Lusin measurable space (i.e., it is measurable isomorphic to a Borel subset of a metrizable compact space endowed with the Borel σ-algebra).

We consider a sub-Markovian resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ and we denote by $\mathcal{E}(\mathcal{U})$ the set of all $\mathcal{B}$-measurable $\mathcal{U}$*-excessive functions:* $u \in \mathcal{E}(\mathcal{U})$ if and only if u is a non-negative numerical $\mathcal{B}$-measurable function, $\alpha U_\alpha u \le u$ for all $\alpha > 0$ and $\lim_{\alpha\to\infty} \alpha U_\alpha u(x) = u(x)$ for all $x \in E$. If $\beta > 0$ we denote by $\mathcal{U}_\beta$ the sub-Markovian resolvent of kernels $(U_{\beta+\alpha})_{\alpha>0}$. Furthermore, for a set of functions $\mathcal{F}$ we denote the subset of all its bounded and non-negative elements by $b\mathcal{F}$ and $\mathcal{F}_+$ respectively.

We assume that:

$(H1)$ $\mathcal{C}$ is a vector lattice of bounded, $\mathcal{B}$-measurable, real-valued functions on E, $1 \in \mathcal{C}$, and there exists a countable subset of $\mathcal{C}_+$ separating the points of E.

The following two properties of $\mathcal{U} = (U_\alpha)_{\alpha>0}$ will be considered in the sequel:

$(H2)$ $U_\alpha(\mathcal{C}) \subset \mathcal{C}$ for all $\alpha > 0$;

$(H3)$ $\lim\limits_{\alpha\to\infty} \alpha U_\alpha f(x) = f(x)$ for all $f \in \mathcal{C}$ and $x \in E$.

If w is a $\mathcal{U}_\beta$*-supermedian function* (i.e., $\alpha U_{\beta+\alpha} w \le w$ for all $\alpha > 0$), its $\mathcal{U}_\beta$-excessive regularization $\widehat{w}$ is given by $\widehat{w}(x) = \sup_\alpha \alpha U_{\beta+\alpha} w(x)$.

Notation. For a family $\mathcal{G}$ of real valued functions on E we denote by $\sigma(\mathcal{G})$ the σ-algebra generated by $\mathcal{G}$ and by $\mathcal{T}(\mathcal{G})$ the topology generated by $\mathcal{G}$.

Proposition 2.1. *If conditions* $(H1)$, $(H2)$, *and* $(H3)$ *are verified and* $\beta > 0$ *then*

$(H4)$ $\sigma(\mathcal{E}(\mathcal{U}_\beta)) = \mathcal{B}$ and all the points of E are non-branch points with respect to $\mathcal{U}_\beta$,

that is $1 \in \mathcal{E}(\mathcal{U}_\beta)$ *and if* $u, v \in \mathcal{E}(\mathcal{U}_\beta)$ *then for all* $x \in E$ *we have* $\inf(u, v)(x) = \widehat{\inf(u, v)}(x)$.

Proof. Let $\mathcal{F}_0$ be a a countable subset of $\mathcal{C}_+$ separating the points of E. By $(H3)$ $\sigma(\mathcal{F}_0) \subset \sigma(\mathcal{E}(\mathcal{U}_\beta))$ and thus $\sigma(\mathcal{F}_0) = \mathcal{B}$ by Lusin's Theorem. If $f, g \in \mathcal{C}_+$ then the function $v := \inf(U_\beta f, U_\beta g)$ is $\mathcal{U}_\beta$-supermedian and by $(H2)$ it belongs to $\mathcal{C}$, hence again by $(H3)$ we see that $\widehat{v} = v$. By Lemma 1.2.10 in [BeBo 04] and since $\sigma(\mathcal{C}_+) = \mathcal{B}$ we conclude that the set of all non-branch points (with respect to $\mathcal{U}_\beta$) equals E. □

Recall now some facts on Ray cones and Ray topologies; for more details see [BeBo 04] and also [BeBoRö 06a] for the non-transient case.

If $\beta > 0$ then a *Ray cone* associated with $\mathcal{U}_\beta$ is a cone $\mathcal{R}$ of bounded $\mathcal{U}_\beta$-excessive functions such that: $U_\alpha(\mathcal{R}) \subset \mathcal{R}$ for all $\alpha > 0$, $U_\beta\big((\mathcal{R} - \mathcal{R})_+\big) \subset \mathcal{R}$, $\sigma(\mathcal{R}) = \mathcal{B}$, it is min-stable, separable in the supremum norm and $1 \in \mathcal{R}$. Below if we say Ray cone it is always meant to be associated with one fixed resolvent $\mathcal{U}_\beta$. A *Ray topology* on E is a topology generated by a Ray cone.

In the sequel we also consider the following condition stronger than $(H3)$:

$(H3u)$ $\quad \lim_{\alpha \to \infty} ||\alpha U_\alpha f - f||_\infty = 0$ for all $f \in \mathcal{C}$.

Remark. *An example where conditions $(H1)$, $(H2)$, and $(H3)$ hold but not condition $(H3u)$ is given by the resolvent constructed in Proposition 5.2 from [DaRö 02].*

Proposition 2.2. *The following assertions hold.*

(i) If condition $(H4)$ is verified then there exists a Ray cone $\mathcal{R}$ associated with $\mathcal{U}_\beta$ and conditions $(H1)$, $(H2)$, and $(H3)$ hold taking $\mathcal{C} = \mathcal{R} - \mathcal{R}$.

(ii) Conversely, assume that conditions $(H1)$, $(H2)$, and $(H3)$ hold. Then there exists a Ray cone $\mathcal{R}$ such that the Ray topology generated by $\mathcal{R}$ is smaller than $\mathcal{T}(\mathcal{C})$ (the topology on E generated by $\mathcal{C}$). If condition $(H3u)$ is verified and $\mathcal{T}(\mathcal{C})$ is generated by a countable subfamily of $\mathcal{C}$, then $\mathcal{T}(\mathcal{C})$ is a Ray topology.

Proof. (i) For the existence of a Ray cone see, e.g., Proposition 1.5.1 from [BeBo 04] and [BeBoRö 06b]. Conditions $(H2)$ and $(H3)$ are direct consequences of the properties of $\mathcal{R}$.

(ii) Note that by Proposition 2.1 it follows that all the points of E are non-branch points with respect to $\mathcal{U}_\beta$. Let $\mathcal{F}_0$ be a a countable subset of $\mathcal{C}_+$ separating the points of E and $\mathcal{R}_0 := U_\beta(\mathcal{F}_0) \cup \mathbb{Q}_+$. Recall that a Ray cone $\mathcal{R}$ is given by the closure in the sup norm of $\bigcup_n \mathcal{R}_n$, where $\mathcal{R}_n$ is defined inductively as follows:
$\mathcal{R}_{n+1} := \mathbb{Q}_+ \cdot \mathcal{R}_n \cup (\sum_f \mathcal{R}_n) \cup (\bigwedge_f \mathcal{R}_n) \cup (\cup_{\alpha \in \mathbb{Q}_+^*} U_\alpha(\mathcal{R}_n)) \cup U_\beta((\mathcal{R}_n - \mathcal{R}_n)_+)$. Since $\mathcal{C}$ is a vector lattice, we deduce by $(H2)$ that $\mathcal{R}_n \subset \mathcal{C}$ for all n and thus $\mathcal{T}(\mathcal{R}) \subset \mathcal{T}(\mathcal{C})$.

If condition $(H3u)$ holds and $\mathcal{T}(\mathcal{C})$ is generated by a countable family of $\mathcal{C}$, we may assume that $\mathcal{F}_0$ generates $\mathcal{T}(\mathcal{C})$ and because $U_\alpha(\mathcal{F}_0) \subset \mathcal{R}$ for all $\alpha > 0$ it follows that every $f \in \mathcal{F}_0$ is $\mathcal{T}(\mathcal{R})$-continuous and therefore we have also $\mathcal{T}(\mathcal{C}) \subset \mathcal{T}(\mathcal{R})$. □

Corollary 2.3. *The following assertions are equivalent for a sub-Markovian resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on a Lusin measurable space $(E, \mathcal{B})$.*

(i) Condition $(H4)$ is verified.

(ii) There exists a vector lattice $\mathcal{C}$ such that conditions $(H1)$, $(H2)$, and $(H3)$ hold.

(iii) For some $\beta > 0$ there exists a Ray cone associated with $\mathcal{U}_\beta$.

In the sequel we assume that conditions $(H1)$, $(H2)$, and $(H3)$ hold.

If $M \in \mathcal{B}$ and $u \in \mathcal{E}(\mathcal{U}_\beta)$, then the *reduced function* (with respect to $\mathcal{U}_\beta$) *of* u *on* M is the function $R^M_\beta u$ defined by

$$R^M_\beta u := \inf\{v \in \mathcal{E}(\mathcal{U}_\beta) : v \geq u \text{ on } M\} .$$

The reduced function $R^M_\beta u$ is universally $\mathcal{B}$-measurable.

2.2 Fine topology, right processes in different topologies

In this subsection we collect the basic notions and results on the analytic and probabilisitic potential theory associated with a sub-Markovian resolvent of kernels; for further details see, e.g., Ch. 1 from [BeBo 04].

A metrizable Lusin topology on E is called *natural* if its Borel σ-algebra is precisely $\mathcal{B}$ and it is smaller than the *fine topology* on E (i.e., the topology generated by $\mathcal{E}(\mathcal{U}_\beta)$).

Recall that if $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is the resolvent associated with a right process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ with state space E (endowed with a natural topology), i.e.,

$$U_\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f \circ X_t \,\mathrm{d}t$$

for all $\alpha > 0$, $x \in E$ and $Uf := \sup_{\beta>0} U_\beta f$, $f \in p\mathcal{B}$ (:= the set of all positive $\mathcal{B}$-measurable functions on E), then by a theorem of Hunt we have:

$$R^M_\beta u(x) = E^x(e^{-\alpha D_M} u \circ X_{D_M} \,;\, D_M < \infty) , \quad \widehat{R^M_\beta} u(x) = E^x(e^{-\alpha T_M} u \circ X_{T_M} \,;\, T_M < \infty)$$

where $D_M(\omega) := \inf\{t \geq 0 : X_t(\omega) \in M\}$, $T_M(\omega) := \inf\{t > 0 : X_t(\omega) \in M\}$, $\omega \in \Omega$.

Remark. *The Markov processes occurring in this subsection will be (Borel) right processes with respect to a metrizable topology and recall that a right process is strong Markov; see, e.g., [Sh 88] for details. However, motivated by the relevant applications presented in Subsection 6.2 below, in the main results from Subsection 5.2 (Theorem 5.2 and Corollary 5.3) we also consider strong Markov processes which have right continuous paths or are càdlàg with respect to topologies which are not necessary metrizable; see, e.g., [MaRö 92] for more details on this type of processes.*

A set $M \in \mathcal{B}$ is called *polar* (resp. *μ-polar*; where μ is a σ-finite measure on $(E, \mathcal{B})$) if $\widehat{R^M_\beta} 1 = 0$ (resp. $\widehat{R^M_\beta} 1 = 0$ μ-a.e.).

Let $\mathsf{Exc}(\mathcal{U})$ be the set of all $\mathcal{U}$-excessive measures on E: $\xi \in \mathsf{Exc}(\mathcal{U})$ if and only if it is a σ-finite measure on $(E, \mathcal{B})$ such that $\xi \circ \alpha U_\alpha \leq \xi$ for all $\alpha > 0$. Recall that if $\xi \in \mathsf{Exc}(\mathcal{U})$ then actually $\xi \circ \alpha U_\alpha \nearrow \xi$ as $\alpha \to \infty$. We denote by $\mathsf{Pot}(\mathcal{U})$ the set of all *potential* $\mathcal{U}$-excessive measures: if $\xi \in \mathsf{Exc}(\mathcal{U})$ then $\xi \in \mathsf{Pot}(\mathcal{U})$ if $\xi = \mu \circ U$, where μ is a σ-finite on $(E, \mathcal{B})$. If $\beta > 0$ then the *energy functional* $L_\beta : \mathsf{Exc}(\mathcal{U}_\beta) \times \mathcal{E}(\mathcal{U}_\beta) \longrightarrow \overline{\mathbb{R}}_+$ is defined by

$$L_\beta(\xi, u) := \sup\{\nu(u) : \mathsf{Pot}(\mathcal{U}_\beta) \ni \nu \circ U_\beta \leq \xi\}.$$

We recall that by Theorem 1.4.5 from [BeBo 04] for all $\xi \in \mathsf{Exc}(\mathcal{U}_\beta)$ and $u \in \mathcal{E}(\mathcal{U}_\beta)$ we have

$$L_\beta(\xi, u) := \sup\{\xi(f) : f \in p\mathcal{B}, U_\beta f \leq u\}.$$

Consequently, if $u = U_\beta f$, $f \in p\mathcal{B}$, then

(2.1a) $\qquad L_\beta(\xi, u) = \xi(f)$

and if $\xi = \mu \circ U_\beta \in \mathsf{Pot}(\mathcal{U}_\beta)$, then

(2.1b) $\quad L_\beta(\xi, u) = \int u\, d\mu.$

(2.2) Let E_1 be the set of all extreme points of the set $\{\xi \in \mathsf{Exc}(\mathcal{U}_\beta) : L_\beta(\xi, 1) = 1\}$, endowed with the σ-algebra $\mathcal{B}_1$ generated by the functionals $\widetilde{u}$, $\widetilde{u}(\xi) := L_\beta(\xi, u)$ for all $\xi \in \mathsf{Exc}(\mathcal{U}_\beta)$ and $u \in \mathcal{E}(\mathcal{U}_\beta)$. Then the following assertions hold.
(2.2a) $(E_1, \mathcal{B}_1)$ is a Lusin measurable space, the map $x \longmapsto \varepsilon_x \circ U_\beta$ identifies E with a subset of E_1, $E \in \mathcal{B}_1$, $\mathcal{B} = \mathcal{B}_1|_E$ and there exists a Markovian resolvent of kernels $\mathcal{U}^1 = (U^1_\alpha)_{\alpha>0}$ on $(E_1, \mathcal{B}_1)$ such that $\sigma(\mathcal{E}(\mathcal{U}^1_\beta)) = \mathcal{B}_1$, every point of E_1 is a non-branch point with respect to $\mathcal{U}^1_\beta$, $U^1_\beta(1_{E_1\setminus E}) = 0$, and $\mathcal{U}$ is the restriction of $\mathcal{U}^1$ to E. The set E_1 is called the *saturation* of E with respect to $\mathcal{U}$. E is a dense subset of E_1 with respect to the fine topology on E_1 generated by $\mathcal{E}(\mathcal{U}^1_\beta)$.
(2.2b) For every $u \in \mathcal{E}(\mathcal{U}_\beta)$ we consider the function $\widetilde{u} : E_1 \longrightarrow \overline{\mathbb{R}}_+$ defined above,

$$\widetilde{u}(\xi) := L_\beta(\xi, u), \quad \xi \in E_1.$$

Then by (2.1b) we have $\widetilde{u}(\varepsilon_x \circ U_\beta) = u(x)$ for all $x \in E$ and therefore, by the embedding of E in E_1 described in (2.2a),

$$\widetilde{u}|_E = u.$$

In addition, $\widetilde{u}$ is $\mathcal{U}^1_\beta$-excessive and it is the (unique) extension by fine continuity of u from E to E_1.
(2.3) By Sections 1.7 and 1.8 in [BeBo 04] and Theorem 1.3 in [BeBoRö 06a] we get that the following assertions are equivalent:
(2.3a) Every $\mathcal{U}_\beta$-excessive measure dominated by a potential is also a potential.
(2.3b) The set $E_1 \setminus E$ is a polar subset of E_1 (with respect to $\mathcal{U}^1$).
(2.3c) If E is endowed with a natural topology then there exists a right process with state space E having $\mathcal{U}$ as associated resolvent.

Note that:

– any Ray topology is natural;

– the resolvent $\mathcal{U}^1$ is always the resolvent of a right process with state space E_1 endowed with any natural topology.

Recall that a σ-finite measure μ on $(E, \mathcal{B})$ is called *reference measure* for the resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ if all the measures $U_\alpha(x, \cdot)$, $x \in E$, re absolutely continuous with respect to μ. (One says that a right process satisfies *hypothesis (L) of P.A. Meyer* provided that its resolvent family has a reference measure.)
(2.4) If μ is a reference measure for $\mathcal{U}$ then μ has fine full support, i.e., if $G \in \mathcal{B}$ is non-empty and finely open then $\mu(G) > 0$.
Indeed, this happens because if G is finely open then $U_\alpha(1_G) > 0$ on G. In particular,
(2.5) if a function v is $\mathcal{U}_\beta$-excessive and $\mu(v) = 0$ then $v(x) = 0$ for all $x \in E$.

We close this subsection with an example showing that conditions $(H1)$, $(H2)$, and $(H3)$ do not imply that the resolvent is associated with a right process.

Example. Assume that $\mathcal{U} = (U_\alpha)_{\alpha>0}$ satisfies conditions $(H1)$, $(H2)$, and $(H3u)$ and it is associated with a right process with state space E. Let $M \in \mathcal{B}$ be a subset of E which is not polar and such that $U_\alpha(1_M) = 0$, $\alpha > 0$. Let $F := E \setminus M$ and $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ be the restriction of $\mathcal{U}$ to F. We claim that $\mathcal{U}'$ satisfies conditions $(H1)$, $(H2)$, and $(H3u)$ (hence also $(H3)$) but there is no right process with state space F having $\mathcal{U}'$ as associated resolvent. Indeed, condition $(H3u)$ holds on F for $\mathcal{C}|_F$ and for every $x \in M$ the $\mathcal{U}'_\beta$-excessive measure $\varepsilon_x \circ U_\beta$ is an extreme point of $\{\xi \in \mathsf{Exc}(\mathcal{U}'_\beta) : L_\beta(\xi, 1) = 1\}$, hence the set $F_1 \setminus F$ is not polar. The assertion follows now by (2.3).

3 Modifications of a resolvent

3.1 Restriction

We denote by $\mathcal{A}(\mathcal{U})$ the family of all sets $A \in \mathcal{B}$ such that

$$R_\beta^{E\setminus A} 1 = 0 \text{ on } A.$$

It is easy to see that $\mathcal{A}(\mathcal{U})$ does not depend on β. The following properties hold.

- Every set $A \in \mathcal{A}(\mathcal{U})$ is finely open;

- If v is a $\mathcal{U}_\beta$-excessive function then the sets $[v < \infty]$ and $[v = 0]$ belong to $\mathcal{A}(\mathcal{U})$. Indeed, since $1 \le \frac{1}{n} v$ on the set $[v = \infty]$ for every $n \ge 1$, it follows that $R_\beta^{[v=\infty]} 1 = 0$ on $[v < \infty]$. We have clearly $[v = 0] = [\sup_n nv < \infty] \in \mathcal{A}(\mathcal{U})$.

- If $A \in \mathcal{A}(\mathcal{U})$ then $U_\beta(1_{E\setminus A}) = 0$ on A and therefore we may consider the *restriction* $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ *of* $\mathcal{U}$ *to* A, i.e., the sub-Markovian resolvent of kernels on $(A, \mathcal{B}|_A)$ defined as:

$$U'_\alpha f := U_\alpha \bar{f}|_A, \text{ for all } f \in p\mathcal{B}|_A,$$

where $\bar{f} \in p\mathcal{B}$ is such that $\bar{f}|_A = f$.

- If $A \in \mathcal{A}(\mathcal{U})$ then a function $u \in p\mathcal{B}|_A$ is $\mathcal{U}'_\beta$-excessive if and only if there exists a function $\bar{u} \in \mathcal{E}(\mathcal{U}_\beta)$ such that $u = \bar{u}|_A$. In particular, the resolvent $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ satisfies conditions $(H1)$, $(H2)$, and $(H3)$ for $\mathcal{C}|_A$ on the measurable space $(A, \mathcal{B}|_A)$.

- If $\mathcal{R}$ is a Ray cone associated with $\mathcal{U}_\beta$ then $\mathcal{R}|_A$ is a Ray cone associated with $\mathcal{U}'_\beta$.

Lemma 3.1. *If $A \in \mathcal{A}(\mathcal{U})$ then the following assertions hold.*

a) If ξ' is a measure on $(A, \mathcal{B}|_A)$ then $\xi' \in \mathsf{Exc}(\mathcal{U}'_\beta)$ if and only ξ' is the restriction to A of a $\mathcal{U}_\beta$-excessive measure on E. If $\xi' \in \mathsf{Exc}(\mathcal{U}'_\beta)$ then the measure ξ_o on E obtained by extending ξ' with zero on $E \setminus A$ belongs to $\mathsf{Exc}(\mathcal{U}_\beta)$ and for all $u \in \mathcal{E}(\mathcal{U}_\beta)$ we have

$$L_{\mathcal{U}'_\beta}(\xi', u|_A) = L_\beta(\xi_o, u),$$

where $L_{\mathcal{U}'_\beta}$ denotes the energy functional with respect to $\mathcal{U}'_\beta$.

b) Let μ be a finite measure on E carried by A. Then the following assertions are equivalent:

(b.i) If $\xi \in \mathsf{Exc}(\mathcal{U}_\beta)$ and $\xi \le \mu \circ U_\beta$ then ξ is a potential.

(b.ii) If $\xi' \in \mathsf{Exc}(\mathcal{U}'_\beta)$ and $\xi' \le \mu \circ U'_\beta$ then ξ' is a potential.

Proof. The proof *a*) is straightforward.

b) $(b.i) \Longrightarrow (b.ii)$. Let $\xi' \in \mathsf{Exc}(\mathcal{U}'_\beta)$, $\xi' \le \mu \circ U'_\beta$. Then by *a*) we have $\xi_o \in \mathsf{Exc}(\mathcal{U}_\beta)$ and $\xi_o \le \mu \circ U_\beta$, hence by hypothesis there exists a measure ν on $(E, \mathcal{B})$ such that $\xi_o = \nu \circ U_\beta$. Since $\nu \circ U_\beta \le \mu \circ U_\beta$, it follows that $\nu(R_\beta^{E\setminus A} 1) \le \mu(R_\beta^{E\setminus A} 1) = 0$. Hence the measure ν is also carried by A and thus $\xi' = \xi_o|_A = \nu \circ U'_\beta$.

$(b.ii) \Longrightarrow (b.i)$. Let $\xi \in \mathsf{Exc}(\mathcal{U}_\beta)$, $\xi \le \mu \circ U_\beta$. From $U_\beta(1_{E\setminus A}) = 0$ on A we deduce that ξ is carried by A and therefore $\xi \in \mathsf{Exc}(\mathcal{U}'_\beta)$. By $\xi \le \mu \circ U'_\beta$ and the hypothesis it follows that there exists a measure ν' on A such that $\xi = \nu' \circ U'_\beta$ and then clearly $\xi = \nu' \circ U_\beta$. $\square$

Corollary 3.2. *Assume that $\mathcal{U}$ is associated with a right process with state space E. If $A \in \mathcal{A}(\mathcal{U})$ then the restriction of $\mathcal{U}$ to A is the resolvent of a right process with state space A.*

Proof. The assertion follows from Lemma 3.1 b) and the equivalence between (2.3a) and (2.3c). □

(3.1) The process with state space A claimed to exist by Corollary 3.2 is called the *restriction of X to A* and we denote it by $\widetilde{X}$: $\widetilde{\Omega} := \{\omega \in \Omega : X_t(\omega) \in A$, for all $0 \leq t < \zeta(\omega)\}$, $\widetilde{P}^x := P^x|_{\widetilde{\Omega}}$ for all $x \in A$, and $\widetilde{X}_t(\omega) := X_t(\omega)$ if $\omega \in \widetilde{\Omega}$ (see, e.g., [Sh 88]).

3.2 Trivial modification

In the sequel λ will be a fixed finite measure on $(E, \mathcal{B})$.

Recall that a set $M \in \mathcal{B}$ is called λ-*inessential* (with respect to $\mathcal{U}$) provided that it is λ-negligible and $E \setminus M \in \mathcal{A}(\mathcal{U})$.

(3.2) Every λ-polar set is the subset of a $\mathcal{B}$-measurable λ-inessential set; cf. the first part of the proof of Theorem 1.7.29 in [BeBo 04].

(3.3) Suppose that $\mathcal{U}$ is associated with a right process with state space E. If $M \in \mathcal{B}$ is a λ-inessential set then we may consider the restriction $\widetilde{X}$ from (3.1) of the process X to $F := E \setminus M \in \mathcal{A}(\mathcal{U})$. Note that $U_\beta(1_M) = 0$ on F, the resolvent associated with the restriction of X to F is precisely the restriction $\mathcal{U}|_F$ of $\mathcal{U}$ to F, and because $\mathcal{E}(\mathcal{U}_\beta|_F) = \mathcal{E}(\mathcal{U}_\beta)|_F$, we deduce that the fine topology on F with respect to $\mathcal{U}|_F$ is the trace on F of the fine topology on E with respect to $\mathcal{U}$.

(3.4) We consider the *trivial modification of $\mathcal{U}$ on M* (see e.g. [BeBo 04] and [BeBoRö 06a]), namely the sub-Markovian resolvent $\mathcal{U}^o = (U^o_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ obtained from the resolvent $\mathcal{U}|_F$ as follows:

$$U^o_\alpha f = 1_F U_\alpha(f 1_F) + \frac{1}{\alpha} f 1_M, \quad \alpha > 0,\ f \in p\mathcal{B}.$$

Then $\mathcal{U}^o$ also satisfies conditions $(H1)$, $(H2)$, and $(H3)$ with $\mathcal{C}$ replaced by $1_F\mathcal{C} + 1_M\mathcal{C}$, F and $E \setminus F$ belong to $\mathcal{A}(\mathcal{U}^o)$, and we have: $\mathcal{U}^o$ is the resolvent of a right process with state space E if and only if $\mathcal{U}|_F$ is the resolvent of a right process with state space F. For the precise definition of the extended process from F to E we refer to [MaRö 92], page 118. A function $u \in p\mathcal{B}$ belongs to $\mathcal{E}(\mathcal{U}^o_\beta)$ if and only if $u|_F \in \mathcal{E}(\mathcal{U}_\beta|_F)$. Consequently we have: a subset Γ of E is finely open with respect to $\mathcal{U}^o$ if and only if there exists Γ_o, a finely open set with respect to $\mathcal{U}$, such that $\Gamma \cap F = \Gamma_o \cap F$. In particular, every topology on E which is natural with respect to $\mathcal{U}$ is also natural with respect to $\mathcal{U}^o$. A σ-finite measure ξ on E belongs to $\mathsf{Exc}(\mathcal{U}^o_\beta)$ if and only if $\xi|_F \in \mathsf{Exc}(\mathcal{U}_\beta|_F)$. If $\xi \in \mathsf{Exc}(\mathcal{U}^o_\beta)$ and $u \in \mathcal{E}(\mathcal{U}^o_\beta)$ then

(3.5) $$L_{\mathcal{U}^o_\beta}(\xi, u) = L_{\mathcal{U}_\beta|_F}(\xi|_F, u|_F) + \beta\xi(u 1_M).$$

4 Compact excessive functions and tightness of capacities

In this section let λ be a fixed finite (non-negative) measure on $(E, \mathcal{B})$.

Recall that an increasing sequence $(F_n)_n \subset \mathcal{B}^n$ is called λ-*nest* provided that $\inf_n R_\beta^{E\setminus F_n} u = 0$ λ-a.e. for some bounded strictly positive $\mathcal{U}_\beta$-excessive function u.

(4.1) If $(F_n)_n$ is a λ-nest then $(\overset{\circ f}{F_n})_n$ is also a λ-nest.

The assertion follows since $R^M u = R^{\overline{M}^f} u$, where $\overset{\circ f}{M}$ (resp. $\overline{M}^f$) denotes the fine interior (resp. the fine closure) of a set M.

Let $\mathcal{T}$ be a Hausdorff topology on E such that $\mathcal{B}(\mathcal{T}) = \mathcal{B}$. Set $u_o := U_\beta f_o$, where f_o is a bounded, strictly positive $\mathcal{B}$-measurable function, and consider the functional $M \longmapsto c_\lambda(M)$, $M \subset E$, defined as

$$c_\lambda(M) := \inf\{\lambda(R_\beta^G u_o) : G \in \mathcal{T}, M \subset G\}.$$

Recall that c_λ is a Choquet capacity on E; see, e.g., [BeBo 04].

Proposition 4.1. *Let $\mathcal{T}$ be a Hausdorff topology on E such that $\mathcal{B}(\mathcal{T}) = \mathcal{B}$. Then the following assertions are equivalent.*

(i) The capacity c_λ is tight, i.e., there exists a λ-nest of $\mathcal{T}$-compact sets.

(ii) For one strictly positive number $\beta > 0$ there exist a $\mathcal{U}_\beta$-excessive function v which is finite λ-a.e. and a bounded strictly positive $\mathcal{U}_\beta$-excessive function u such that the set $[\frac{v}{u} \leq \alpha]$ is relatively $\mathcal{T}$-compact for all $\alpha > 0$. Such a function $\frac{v}{u}$ is called compact Lyapunov function.

(iii) If $\beta > 0$ and, then there exists a $\mathcal{U}_\beta$-excessive function v which is finite λ-a.e. such that $\frac{v}{u_o}$ is a compact Lyapunov function.

Proof. $(i) \Longrightarrow (iii)$. By hypothesis there exists an increasing sequence of compact sets $(K_n)_{n\geq 1}$ such that $\inf_n \lambda(R_\beta^{E\setminus K_n} u_o) \leq \frac{1}{2^n}$ for all $n \geq 1$. We set

$$v := \sum_{n\geq 1} R_\beta^{E\setminus K_n} u_o.$$

Then clearly the function v is λ-integrable and since $R_\beta^{E\setminus K_n} u_o = u_o$ on $E \setminus K_n$ we conclude that $[v \leq n u_o] \subset K_{n+1}$ for all n.

$(ii) \Longrightarrow (i)$. We proceed as in Remark 3.3 from [BeBoRö 06a]. Let $v, u \in \mathcal{E}(\mathcal{U}_\beta)$ be such that $\frac{v}{u}$ is a compact Lyapunov function and for each n set $K_n := \overline{[v \leq nu]}$. Since $u \leq \frac{v}{n}$ on $E \setminus K_n$, we deduce that $\inf_n R_\beta^{E\setminus K_n} u = 0$ on the set $[v < \infty]$, hence λ-a.e. We conclude that $(K_n)_n$ is a λ-nest of $\mathcal{T}$-compact sets. □

Remark. *(i) If there exists a strictly positive constant k such that $k \leq U_\beta f_o$, where f_o is a bounded, strictly positive $\mathcal{B}$-measurable function (in particular, this happens if the resolvent $\mathcal{U}$ is Markovian), then in the above assertion (ii) one can take $u = 1$.*

(ii) Proposition 4.1 (the implication $(i) \Longrightarrow (iii)$) will be used to deduce the right continuity of the paths of a Markov process (in the proof Theorem 5.2 below); for further applications see [Be 11] and the survey article [BeRö 11].

5 Resolvents and càdlàg processes

5.1 Càdlàg modification of a right process

The next proposition completes the results from [Do 05] on the càdlàg modification of a right process.

Proposition 5.1. *Assume that E is a metrizable Lusin topological space and X is a right process with state space E. Then the following assertions hold.*

(i) If X possesses left limits in E P^λ-a.e., then there exists a càdlàg right process with state space E which is λ-equivalent with X (their resolvents coincide outside a λ-inessential set).

(ii) If X possesses left limits in E P^λ-a.e. and λ is a reference measure for its resolvent, then X is a càdlàg right process.

Proof. (i) Let

$$\Lambda := \{\omega \in \Omega : \ t \longmapsto X_t(\omega) \ \text{ is not càdlàg}\}.$$

Then $\Lambda \in \mathcal{F}^u$ (cf., e.g., Theorem 34 in [DeMe 78]), $P^\lambda(\Lambda) = 0$, and it is easy to see that if $s < t$ then $\theta_t^{-1}(\Lambda) \subset \theta_s^{-1}(\Lambda) \subset \Lambda$. In addition, we have $\bigcup_n \theta_{t_n}^{-1}(\Lambda) = \Lambda$ provided that $t_n \searrow 0$. We define the function v by

$$v(x) := P^x(\Lambda),\, x \in E.$$

From the above mentioned properties of Λ we deduce that $\lambda(v) = 0$ and v is $\mathcal{U}$-excessive. Indeed,

$$P_t v(x) = E^x[v(X_t)] = E^x[P^{X_t}(\Lambda)] = P^x(\theta_t^{-1}(\Lambda)),\ \sup_{t>0} P_t v(x) = P^x(\bigcup_{t>0} \theta_t^{-1}(\Lambda)) = v(x).$$

We consider the set

$$A = [v_o = 0],$$

where $v_o \in \mathcal{E}(\mathcal{U}_\beta)$ is such that $v_o \geq v$ and $\lambda(v_o) = \lambda(v) = 0$; the existence of the function v_o follows from [BeBo 04], Proposition 1.2.9. Clearly, the restriction $\widetilde{X}$ of X to A, see (3.1), is a process with càdlàg trajectories. The trivial extension of $\widetilde{X}$ to E is the claimed càdlàg process with state space E, because every point of $E \setminus A$ is a trap for the extension; see (3.48) in [MaRö 92].

(ii) If in addition λ is a reference measure for $\mathcal{U}$ then by (2.4) and since $\lambda(v_o) = 0$ we see that $v_o(x) = 0$ for all $x \in E$. Hence $A = E$ and we conclude that X is a càdlàg right process. □

5.2 Existence results for càdlàg processes

In order to state the first main result of the paper, it is convenient to extend the domain of definition of the energy functional as follows. For $u \in \mathcal{E}(\mathcal{U}_\beta)$ we consider the subset $E_{\mathcal{U}_\beta}(u)$ of $\mathsf{Exc}(\mathcal{U}_\beta)$ defined as

$$E_{\mathcal{U}_\beta}(u) := \{\xi \in \mathsf{Exc}(\mathcal{U}_\beta) : \ L_\beta(\xi, u) < \infty\}.$$

If $g \in bp\mathcal{B}$ and $u, v \in \mathcal{E}(\mathcal{U}_\beta)$, are such that $g = u - v$ on $[u + v < \infty]$, and $\xi \in E_{\mathcal{U}_\beta}(u + v)$, then $L_\beta(\xi, g)$ is defined naturally as $L_\beta(\xi, g) := L_\beta(\xi, u) - L_\beta(\xi, v)$.

Theorem 5.2. *Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a sub-Markovian resolvent of kernels and λ a positive finite measure on $(E, \mathcal{B})$. Assume that (H4) and the following three conditions (H5), (H6), and (H7) hold.*

(H5) E is endowed with a Hausdorff topology $\mathcal{T}$ with $\mathcal{B}(\mathcal{T}) = \mathcal{B}$ and there exists a λ-nest $(K_n)_n$ of metrizable $\mathcal{T}$-compact sets.

(H6) There exists a countable family $\mathcal{F}$ of bounded $\mathcal{B}$-measurable functions such that for each $\varphi \in \mathcal{F}$ there exist two $\mathcal{U}_\beta$-excessive functions u_φ, v_φ (where $\beta > 0$ is fixed), such that $u_\varphi + v_\varphi < \infty$ λ-a.e., $\varphi = u_\varphi - v_\varphi$ on $[u_\varphi + v_\varphi < \infty]$, and $\mathcal{T}|_{K_n}$ is generated by $\mathcal{F}|_{K_n}$ for all n.

(H7) Let $\mathcal{U}^o = (U^o_\alpha)_{\alpha>0}$ be the trivial modification of $\mathcal{U}$ on the λ-inessential set $M := \bigcup_{\varphi\in\mathcal{F}}[u_\varphi + v_\varphi = \infty]$. Suppose that there exists a function $g \in p\mathcal{B}$ with $U_\beta g < \infty$ on E, such that if $\xi, \eta \in \bigcap_{\varphi\in\mathcal{F}} E_{\mathcal{U}^o_\beta}(u_\varphi + v_\varphi + 1)$, $\int_{E\setminus M} g d(\xi + \eta) < \infty$, and $L_{\mathcal{U}^o_\beta}(\xi, \varphi) = L_{\mathcal{U}^o_\beta}(\eta, \varphi)$ for all $\varphi \in \mathcal{F}$, then $\xi = \eta$.

Then there exists a strong Markov process X with state space E (equipped with $\mathcal{T}$) such that:

(i) The trivial modification of $\mathcal{U}$ on a λ-inessential set is the resolvent of X.

(ii) X possesses P^λ-a.e. left limits in E and the process $t \longmapsto X_t$ is P^x-a.s. right continuous on $[0, \infty)$ for all $x \in E$.

(iii) X may be taken such that it is a right process having càdlàg trajectories with respect to a given metrizable Lusin topology on E which is smaller than $\mathcal{T}$.

Proof. For each $\varphi \in \mathcal{F}$ let u^o_φ (resp. v^o_φ) be the modification of u_φ (resp. v_φ) equal to φ (resp. zero) on M. Then by (3.4) u^o_φ and v^o_φ are real-valued $\mathcal{U}^o_\beta$-excessive functions and $\varphi = u^o_\varphi - v^o_\varphi$ on E for each $\varphi \in \mathcal{F}$.

Let further $(E^o_1, \mathcal{B}^o_1)$ be the saturation of E with respect to $\mathcal{U}^o$. Recall that $E \subset E^o_1$ (by the embedding $x \longmapsto \varepsilon \circ U^o_\beta$; see (2.2)) and let $\mathcal{U}^{o1} = (U^{o1}_\beta)_{\beta>0}$ be the extension of $\mathcal{U}^o$ from $(E, \mathcal{B})$ to $(E^o_1, \mathcal{B}^0_1)$, given by (2.2*a*). From (2.2*b*) applied to the resolvent $\mathcal{U}^o$, each $u \in \mathcal{E}(\mathcal{U}^o_\beta)$ has a unique $\overline{\mathbb{R}}_+$-valued extension (by fine continuity) $\widetilde{u} \in \mathcal{E}(\mathcal{U}^{o1}_\beta)$ from E to E^o_1,

$$\widetilde{u}(\xi) := L_{\mathcal{U}^o_\beta}(\xi, u), \quad \xi \in E^o_1.$$

We take the subset F of E^o_1 defined as

$$F := \bigcap_{\varphi\in\mathcal{F}} [\widetilde{u}^o_\varphi + \widetilde{v}^o_\varphi + \widetilde{U_\beta g} < \infty].$$

Then $E \subset F$ and the set $E^0_1 \setminus F$ is polar. In particular, $F \in \mathcal{A}(\mathcal{U}^{o1})$. Consequently, if $\varphi \in \mathcal{F}$, $\varphi = u^o_\varphi - v^o_\varphi$, then $\widetilde{u}^o_\varphi|_F$ and $\widetilde{v}^o_\varphi|_F$ are real-valued functions and $\varphi = \widetilde{u}^o_\varphi|_E - \widetilde{v}^o_\varphi|_E$. Therefore for each $\varphi \in \mathcal{F}$ we may consider the real-valued extension $\widetilde{\varphi}$ of the function φ from E to F, defined as $\widetilde{\varphi} := \widetilde{u}^o_\varphi|_F - \widetilde{v}^o_\varphi|_F$,

$$(5.1) \qquad \widetilde{\varphi}(\xi) := \widetilde{u}^o_\varphi(\xi) - \widetilde{v}^o_\varphi(\xi) = L_{\mathcal{U}^o_\beta}(\xi, u^o_\varphi) - L_{\mathcal{U}^o_\beta}(\xi, v^o_\varphi), \quad \xi \in F.$$

Because $\widetilde{\varphi}$ is the extension by fine continuity of the bounded function φ on E, and E is a finely dense subset of F, it follows that $\widetilde{\varphi}$ is a bounded function on F for all $\varphi \in \mathcal{F}$.

We claim that the family $\widetilde{\mathcal{F}} := \{\widetilde{\varphi} : \varphi \in \mathcal{F}\}$ separates the points of F. Indeed, let $\xi, \eta \in F$ be such that $\widetilde{\varphi}(\xi) = \widetilde{\varphi}(\eta)$ for all $\varphi \in \mathcal{F}$. By (5.1) we see that the last equality becomes $L_{\mathcal{U}^o_\beta}(\xi, \varphi) = L_{\mathcal{U}^o_\beta}(\eta, \varphi)$. Since ξ, η belong to F, it follows that $\xi, \eta \in \bigcap_{\varphi\in\mathcal{F}} E_{\mathcal{U}^o_\beta}(u_\varphi + v_\varphi + 1)$ and $\int_{E\setminus M} g d(\xi + \eta) < \infty$ because by (3.5) and Lemma 3.1 *a*)

$$\infty > \widetilde{U_\beta g}(\xi) = L_{\mathcal{U}^o_\beta}(\xi, U_\beta g) = L_{\mathcal{U}_\beta|_{E\setminus M}}(\xi|_{E\setminus M}, U_\beta g|_{E\setminus M}) + \beta\xi(1_M U_\beta g) \geq$$

$$L_\beta(1_{E\setminus M}\cdot\xi, U_\beta g) = \int_{E\setminus M} g d\xi.$$

Note that ξ and η are finite measures on E since using $(2.1a)$ $\xi(1) = L_{\mathcal{U}^o_\beta}(\xi, U^o_\beta 1) \le \frac{1}{\beta} L_{\mathcal{U}^o_\beta}(\xi, 1) = \frac{1}{\beta}$. The fact that $\widetilde{\mathcal{F}}$ separates the points of F follows now by $(H7)$.

From (2.3) we know that $\mathcal{U}^{o1}$ is the resolvent of a right process with state space E^o_1, endowed with the Ray topology. By Corollary 3.2 the restriction $\mathcal{U}^{o1}|_F$ of $\mathcal{U}^{o1}$ to F is the resolvent of a right process with state space F.

Let $\mathcal{T}_o$ be the topology on F generated by $\widetilde{\mathcal{F}}$. Since $\widetilde{\varphi}$ is finely continuous on F for all $\varphi \in \mathcal{F}$, we deduce that $\mathcal{T}_o$ is a natural (metrizable) topology on F and $(H6)$ implies that $\mathcal{T}_o|_{K_n} = \mathcal{T}|_{K_n}$ for all n. Again by (2.3) we get that $\mathcal{U}^{o1}|_F$ is the resolvent of a right process $\overline{X}$ with state space F, endowed with the topology $\mathcal{T}_o$.

Let K be the compactification of F with respect to $\mathcal{F}$. Since for every real-valued function $u \in \mathcal{E}(\mathcal{U}^o_\beta)$ the real-valued process $(e^{-\beta t}\widetilde{u} \circ \overline{X}_t)_{t\ge 0}$ is a right continuous ($\overline{P}^x$-integrable) supermartingale under $\overline{P}^x$ for $x \in F$, it follows that this process has left limits $\overline{P}^\lambda$-a.e. and we conclude that $\overline{X}$ has left limits in K $\overline{P}^\lambda$-a.e.

The next step is standard (cf., e.g., the proof of Theorem 3.7.7 from [BeBo 04]) but for the reader's convenience we repete here the arguments. Because $F \setminus K_n$ is a finely open subset of F (with respect to $\mathcal{U}^{o1}$) for all n, by $(*1)$ from [BeBoRö 06a], and using $(H5)$

(5.2) $\qquad \lambda(R^{F\setminus E}_\beta U^{o1}_\beta 1) \le \inf_n \lambda(R^{F\setminus K_n}_\beta U^{o1}_\beta 1) = \inf_n \lambda(R^{E\setminus K_n}_\beta U^o_\beta 1) = 0.$

Note that by $(H5)$ and since $\lambda(M) = 0$ we see that $(K_n)_n$ remains a λ-nest with respect to $\mathcal{U}^o$. We get

$$\lim_n \overline{E}^\lambda \int_{T_{F\setminus K_n}}^\infty e^{-\beta t} 1_{[t<\zeta]}\, dt = 0$$

and thus

$$\sup_n T_{F\setminus K_n} \ge \zeta \quad \overline{P}^\lambda - \text{a.e.}$$

Hence for every $\omega \in \Omega$ with $T_{F\setminus K_n}(\omega) < \zeta(\omega)$ we have $\overline{X}_t(\omega) \in K_n$, provided that $t < T_{F\setminus K_n}(\omega)$ and so $\overline{X}_{t-}(\omega) \in K_n$. By (4.1) the sequence $(\overset{\circ f}{K_n})_n$ is a λ-nest. Therefore the set $M_1 := \bigcap_n (F\setminus \overset{\circ f}{K_n})$ is λ-polar, $F \setminus E \subset M_1$.

Using $(H5)$ and Proposition 4.1 there exists a compact Lyapunov function of the form $\frac{v}{u_o}$, where $v \in \mathcal{E}(\mathcal{U}_\beta)$ is finite λ-a.e. and $u_o = U_\beta f_o$, with $f_o \in bp\mathcal{B}$, $f_o > 0$. We may suppose in addition that

(5.3) $\qquad [v \le n u_o] \subset K_{n+1}$ for all n

(see the proof of $(i) \Longrightarrow (iii)$ of Proposition 4.1). Consider the λ-polar set $M_2 := [v = \infty]$. By (3.2) there exists a λ-inessential set $M_o \in \mathcal{B}^o_1$, such that $M_1 \cup M_2 \subset M_o$.

Corollary 3.2, (3.3) and (3.1) imply that $\mathcal{U}^{o1}|_{F\setminus M_o} = \mathcal{U}^o|_{E\setminus M_o}$ is the resolvent of a right process with state space $E \setminus M_o$, namely the restriction $\widetilde{X}$ of $\overline{X}$ to the set $F \setminus M_o \in \mathcal{A}(\mathcal{U}^{o1})$.

Because $E \setminus M_o \subset \bigcup_n \overset{\circ f}{K_n}$, $\mathcal{T}_o|_{\overset{\circ f}{K_n}} = \mathcal{T}|_{\overset{\circ f}{K_n}}$ for all n, and by (3.3) we have that $\mathcal{T}_o|_{E\setminus M_o}$ is a natural topology for $\mathcal{U}^o|_{E\setminus M_o}$, we conclude that $\mathcal{T}|_{E\setminus M_o}$ is smaller than the fine topology on $E \setminus M_o$.

From (3.4) we deduce that the trivial modification of $\mathcal{U}^o|_{E\setminus M_o}$ on $E\cap M_o$ is the resolvent of a right process X with state space E endowed with a metrizable Lusin topology smaller than $\mathcal{T}$. Therefore assertion (i) holds. The above considerations imply that

$$\text{there exists } \ X_{t-}(\omega) = \widetilde{X}_{t-}(\omega) \in K_n \subset E \ \text{ for all } \omega \in \widetilde{\Omega} \text{ with } t < T_{F\setminus K_n}(\omega),$$

where the left limits are considered with respect to the topology $\mathcal{T}$. We conclude that the process X possesses P^λ-a.e. left limits in E.

We show that the paths $t \longmapsto X_t$ are P^x-a.s. right continuous on $[0,\infty)$ for all $x \in E$. If $x \in M_o \cap E$ then, the resolvent of X being obtained by a trivial modification on $E \cap M_o$, we have $P^x(X_t = x$ for all $t \geq 0) = 1$, so, the claimed right continuity is clear. Let now $x \in E \setminus M_o$. Since $M_2 \subset M_o$ we have $v(x) < \infty$. Hence $(e^{-\beta t} v \circ X_t)_{t\geq 0}$ and $(e^{-\beta t} u_o \circ X_t)_{t\geq 0}$ are two right continuous (P^x-integrable) supermartingales under P^x. Let $\Omega_o \subset \Omega$ be such that $P^x(\Omega_o) = 1$ and for all $\omega \in \Omega_o$ the path $t \longmapsto X_t(\omega)$ is right continuous when E is endowed with the metrizable topology $\mathcal{T}_o$ and $t \longmapsto v\circ X_t(\omega)$, $t \longmapsto u_o \circ X_t(\omega)$ are two real-valued right continuous functions on $[0,\infty)$. We claim that if $\omega \in \Omega_o$, then $t \longmapsto X_t(\omega)$ is right continuous on $[0,\infty)$ when E is endowed with the topology $\mathcal{T}$ too. Indeed, if $t_o \geq 0$ then $v(X_{t_o}(\omega)) < \infty$ and let n_o be a natural number (depending on ω and t_o) such that $v(X_{t_o}(\omega)) < n_o u_o(X_{t_o}(\omega))$. By the right continuity at t_o of $v \circ X_t(\omega)$ and $u_o \circ X_t(\omega)$ we see that there exists $\delta > 0$ such that $v(X_t(\omega)) < n_o u_o(X_t(\omega))$ for all $t \in [t_o, t_o + \delta)$. Therefore from (5.3) $X_t(\omega) \in K_{n_o+1}$ for all $t \in [t_o, t_o+\delta)$. But we already observed that $\mathcal{T}_o|_{K_{n_o+1}} = \mathcal{T}|_{K_{n_o+1}}$. Consequently, the right continuity of $t \longmapsto X_t(\omega)$ at t_o in the topology $\mathcal{T}_o$ implies the same continuity in the topology $\mathcal{T}$. The proof of (ii) is now complete.

Assertion (iii) follows by modifying the process X according with Proposition 5.1. □

Remark. (i) *It was proved in [LyRö 92] that condition* $(H5)$ *is necessary in order to get càdlàg trajectories for the obtained process; see also [BeBo 05].*

(ii) *Conditions* $(H6)$ *and* $(H7)$ *look technical but it is essentially what is minimally needed and it can be checked in many applications; see Section 5.3 below.*

(iii) *Under the assumptions of Theorem 5.2, let* $D \in \mathcal{B}$ *be a finely open set such that* $\mathcal{T}|_D$ *is metrizable Lusin and consider the kernels* V^D_α, $\alpha > 0$, *defined as*

$$V^D_\alpha f = E^{\cdot}\int_0^{T_{E\setminus D}} e^{-\alpha t} f(X_t)\, dt, \quad f \in p\mathcal{B}.$$

If we assume that V^D_α *are kernels on* $(E,\mathcal{B})$, *then the family* $\mathcal{V}^D$ *given by the restrictions to* D *of* V^D_α, $\alpha > 0$, *is the resolvent of a right process with state space* D. *The assertion follows from Theorem 3.6.9 ii) in [BeBo 04] and by (2.3a), since clearly* $\mathcal{T}|_D$ *becomes a natural topology for the resolvent* $\mathcal{V}^D$ *on* $(D, \mathcal{B}|_D)$.

Corollary 5.3. *Assume that* $(H4)$ *holds,* E *is endowed with a Hausdorff topology* $\mathcal{T}$ *with* $\mathcal{B}(\mathcal{T}) = \mathcal{B}$. *Then the following assertions hold.*

(i) *Suppose that* λ *is a reference measure for* $\mathcal{U}$, $(H5)$ *and* $(H6)$ *are satisfied,* $u_\varphi|_{K_n}$, $v_\varphi|_{K_n}$ *are real-valued functions for all* n *and* $\varphi \in \mathcal{F}$, *and*

$(H7')$ *there exists a function* $g \in p\mathcal{B}$ *with* $U_\beta g < \infty$ *on each* K_n, *such that if* $\xi, \eta \in \bigcap_{\varphi\in\mathcal{F}} E_{\mathcal{U}_\beta}(u_\varphi + v_\varphi + 1)$, $\int_E g d(\xi+\eta) < \infty$, *and* $L_{\mathcal{U}_\beta}(\xi,\varphi) = L_{\mathcal{U}_\beta}(\eta,\varphi)$ *for all* $\varphi \in \mathcal{F}$, *then* $\xi = \eta$.

Then the trivial modification of $\mathcal{U}$ on the polar set $N := E \setminus \bigcup_n K_n$ is the resolvent of a strong Markov process with state space E, such that assertions (ii) and (iii) from Theorem 5.2 hold.

(ii) *Assume that conditions $(H5)$, $(H6)$, and $(H7)$ are verified for every finite measure λ on E with u_φ and v_φ real-valued for all $\varphi \in \mathcal{F}$, and $U_\beta g < \infty$ on E. Then there exists a right process with state space E, having $\mathcal{U}$ as associated resolvent, provided that E is endowed with any natural topology. The process has càdlàg trajectories if the natural topology is smaller than $\mathcal{T}$.*

Proof. (i) Because λ is a reference measure, the set N is polar and therefore we may consider the restriction $\mathcal{U}|_{E\setminus N}$ of $\mathcal{U}$ to $E \setminus N$. One can see that $\mathcal{U}|_{E\setminus N}$ satisfies $(H5)$, $(H6)$, and $(H7)$ with the family $\mathcal{F}|_{E\setminus N}$ and $U_\beta g|_{E\setminus N}$ $(= U_\beta|_{E\setminus N}(g|_{E\setminus N}))$. Note that $(H7)$ for $\mathcal{U}|_{E\setminus N}$ follows from $(H7')$, using Lemma 3.1. Hence we can apply Theorem 5.2 for $\mathcal{U}|_{E\setminus N}$. Analyzing its proof, since $u_\varphi|_{E\setminus N}$, $v_\varphi|_{E\setminus N}$ are real-valued functions, one can see that $\mathcal{U}|_{E\setminus N}$ is the resolvent of the restriction $\widetilde{X}$ to $E \setminus N$ of the process $\overline{X}$ from F; observe that by (5.2) and because λ is a reference measure it follows that the set $F \setminus (E \setminus N)$ is polar. Note also that, using Proposition 5.1(ii), we see that the process $\widetilde{X}$ has càdlàg trajectories with respect to the topology on $E \setminus N$ generated by $\mathcal{F}|_{E\setminus N}$. In addition, assertions (ii) and (iii) from Theorem 5.2 hold for $\widetilde{X}$. Let X be the extension of $\widetilde{X}$ from $E \setminus N$ to E; see [MaRö 92], page 118. By (3.4) the trivial modification of $\mathcal{U}$ on N is the resolvent of X which verifies (ii) and (iii) too.

(ii) The additional hypothesis on u_φ and v_φ implies that the exceptional set $M = \bigcup_{\varphi\in\mathcal{F}}[u_\varphi + v_\varphi = \infty]$ is empty. We can apply Theorem 5.2 for all λ . By (5.2) verified for every λ we see that the set $E_1 \setminus E$ is polar. The existence of the right process follows now using (2.3). Let $\mathcal{T}_1$ be the given natural topology on E and assume that $\mathcal{T}_1 \subset \mathcal{T}$. The last assertion holds by Theorem 5.2, because $\mathcal{T}_1|_{K_n} = \mathcal{T}_o|_{K_n} = \mathcal{T}|_{K_n}$ for all $n \geq 1$. □

Corollary 5.4. *Assume that $\mathcal{U} = (U_\alpha)_{\alpha>0}$ satisfies $(H4)$, $(H5)$, $(H6)$ and $(H7)$ with u_φ and v_φ real-valued for all $\varphi \in \mathcal{F}$ and $g = 0$. Let $\mathcal{W} = (W_\alpha)_{\alpha>0}$ be a second sub-Markovian resolvent on $(E, \mathcal{B})$ such that $(H4)$ is satisfied by $\mathcal{W}$ too. Suppose in addition that*

$(H8)$ $\quad \mathcal{E}(\mathcal{U}_\beta) \subset \mathcal{E}(\mathcal{W}_{\beta'})$ *and* $W_{\beta'}(b\mathcal{E}(\mathcal{U}_\beta)) \subset [b\mathcal{E}(\mathcal{U}_\beta)]$ *for some* β *and* $\beta' > 0$,

where $[b\mathcal{E}(\mathcal{U}_\beta)]$ denotes the vector space spanned by $b\mathcal{E}(\mathcal{U}_\beta)$. Then $(H6)$ and $(H7)$ are satisfied by $\mathcal{W}$ too. In particular, the conclusion of Theorem 5.2 holds for $\mathcal{W}$.

If $\mathcal{U}$ is proper (i.e., the kernel $U_0 := \sup_{\alpha>0} U_\alpha$ is proper), then in the hypothesis $(H8)$ one can take $\beta = 0$ (with the notation $\mathcal{U}_0 = \mathcal{U}$).

Proof. We check that $(H5)$, $(H6)$ and $(H7)$ are satisfied by $\mathcal{W}$. Indeed, $(H8)$ implies

$$R^M_{\mathcal{W}_{\beta'}} u \leq R^M_\beta u \quad \text{for every } M \in \mathcal{B} \text{ and } u \in \mathcal{E}(\mathcal{U}_\beta),$$

where $R^M_{\mathcal{W}_{\beta'}} u$ denotes the reduced function of u on M w.r.t. $\mathcal{W}_{\beta'}$. Consequently, every λ-nest w.r.t. $\mathcal{U}$ is a λ-nest w.r.t. $\mathcal{W}$ and therefore $(H5)$ is satisfied by $\mathcal{W}$.

Let now $\xi, \eta \in \mathsf{Exc}(\mathcal{W}_{\beta'})$ be such that $L_{\mathcal{W}_{\beta'}}(\xi + \eta, u_\varphi + v_\varphi + 1) < \infty$ and $L_{\mathcal{W}_{\beta'}}(\xi, \varphi) = L_{\mathcal{W}_{\beta'}}(\eta, \varphi)$ for all $\varphi \in \mathcal{F}$. Let $(\mu_n)_n$, $(\nu_n)_n$ be two sequences of finite measures on $(E, \mathcal{B})$ such that $\mu_n \circ W_{\beta'} \nearrow \xi$ and $\nu_n \circ W_{\beta'} \nearrow \eta$. By $(H8)$ the sequences of measures $(\mu_n \circ U_\beta)_n$ and $(\nu_n \circ U_\beta)_n$ are increasing. Let $\xi' := \sup_n \mu_n \circ U_\beta$ and $\eta' := \sup_n \nu_n \circ U_\beta$. Using $(2.1b)$ we have $\xi'(1) = \sup_n \mu_n(U_\beta 1) = \sup_n L_{\mathcal{W}_{\beta'}}(\mu_n \circ W_{\beta'}, U_\beta 1) = L_{\mathcal{W}_{\beta'}}(\xi, U_\beta 1) \leq \frac{1}{\beta} L_{\mathcal{W}_{\beta'}}(\xi, 1) < \infty$. As a consequence, ξ' is a finite $\mathcal{U}_\beta$-excessive measure and analogously we see that $\eta' \in \mathsf{Exc}(\mathcal{U}_\beta)$. In addition, again by (2.1b) we have for every $v \in \mathcal{E}(\mathcal{U}_\beta)$

$$L_\beta(\xi', v) = \sup_n L_\beta(\mu_n \circ U_\beta, v) = \sup_n \mu_n(v) = L_{\mathcal{W}_{\beta'}}(\xi, v)$$

and similarly $L_\beta(\eta', v) = L_{\mathcal{W}_{\beta'}}(\eta, v)$. It follows that $L_\beta(\xi', \varphi) = L_\beta(\eta', \varphi)$ for all $\varphi \in \mathcal{F}$ and by $(H7)$ for $\mathcal{U}$ we obtain that $\xi' = \eta'$. Hence $L_{\mathcal{W}_{\beta'}}(\xi, v) = L_{\mathcal{W}_{\beta'}}(\eta, v)$ for all $v \in \mathcal{E}(\mathcal{U}_\beta)$. By the second inclusion of $(H8)$ we can take $v = W_{\beta'} u$ with $u \in b\mathcal{E}(\mathcal{U}_\beta)$. We conclude that $\xi(u) = \eta(u)$ for all $u \in b\mathcal{E}(\mathcal{U}_\beta)$ and so $\xi = \eta$. The assertion follows now by Theorem 5.2 applied to the resolvent $\mathcal{W}$.

Suppose that $\mathcal{U}$ is proper and let $f_0 \in bp\mathcal{B}$, $f_0 > 0$, be such that $Uf_0 \leq 1$. We take $\xi, \eta \in \mathsf{Exc}(\mathcal{W}_{\beta'})$ as before and we consider the measures $\xi' := \sup_n \mu_n \circ U$ and $\eta' := \sup_n \nu_n \circ U$. Then $\xi'(f_0) = \sup_n \mu_n(Uf_0) = \sup_n L_{\mathcal{W}_{\beta'}}(\mu_n \circ W_{\beta'}, Uf_0) = L_{\mathcal{W}_{\beta'}}(\xi, Uf_0) \leq L_{\mathcal{W}_{\beta'}}(\xi, 1) < \infty$. It follows that ξ' and η' are σ-finite measures and therefore $\xi', \eta' \in \mathsf{Exc}(\mathcal{U})$. As before we obtain for all $v \in \mathcal{E}(\mathcal{U})$

$$L_{\mathcal{W}_{\beta'}}(\xi, v) = L(\xi', v), \quad L_{\mathcal{W}_{\beta'}}(\eta, v) = L(\eta', v).$$

Let $\xi'' := \xi' - \xi' \circ \beta U_\beta$, $\eta'' := \eta' - \eta' \circ \beta U_\beta$. Then $\xi'', \eta'' \in \mathsf{Exc}(\mathcal{U}_\beta)$ and if $f \in p\mathcal{B}$, $f \leq f_0$, then

$$L_\beta(\xi'', Uf) = L_\beta(\xi'', U_\beta(f + \beta Uf)) = \xi''(f + \beta Uf) = \xi'(f) = L(\xi', Uf).$$

It follows that for all $v \in \mathcal{E}(\mathcal{U})$ we have $L_\beta(\xi'', v) = L(\xi', v)$ and analogously $L_\beta(\eta'', v) = L(\eta', v)$. We obtain $L_\beta(\xi'', \varphi) = L_\beta(\eta'', \varphi)$ for all $\varphi \in \mathcal{F}$ and by $(H7)$ $\xi'' = \eta''$. Hence $L_{\mathcal{W}_{\beta'}}(\xi, v) = L_{\mathcal{W}_{\beta'}}(\eta, v)$ for all $v \in \mathcal{E}(\mathcal{U})$ and again as before we conclude that $\xi = \eta$. □

The following example shows that Corollary 5.4 may be applied to prove that the càdlàg property of the trajectories of a Markov process is preserved by certain perturbations.

Example (Subordination by convolution semigroup). Assume that $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is associated with a right Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ with state space E endowed with a metrizable Lusin topology which is smaller than $\mathcal{T}$, such that $(H5)$, $(H6)$, and $(H7)$ hold for every finite measure λ. Let $(P_t)_{t\geq 0}$ be its transition function: $P_t f(x) = E^x(f \circ X_t)$, $f \in p\mathcal{B}$, $x \in E$.

Let $(\mu_t)_{t\geq 0}$ be a (vaguely continuous) convolution semigroup on $\mathbb{R}_+^*$ and for each $t \geq 0$ define the kernel P_t^μ on $(E, \mathcal{B})$ by

$$P_t^\mu f := \int_0^\infty P_s f \mu_t(ds) = \mu_t(P.f).$$

Then the family $(P_t^\mu)_{t\geq 0}$ is a sub-Markovian semigroup of kernels on $(E, \mathcal{B})$ called *subordinate to* $(P_t)_{t\geq 0}$ *by means* of $(\mu_t)_{t\geq 0}$. Let $\mathcal{U}^\mu = (U_\alpha^\mu)_{\alpha>0}$ be the resolvent of kernels induced by $(P_t^\mu)_{t\geq 0}$.

Assume that the process X is transient (i.e., the resolvent $\mathcal{U}$ is proper). Then $(P_t^\mu)_{t\geq 0}$ is the transition function of a right process with state space E, having càdlàg trajectories.

Indeed, by Corollary 5.4 and Corollary 5.3(ii) it is sufficient to prove that $(H8)$ is satisfied for $\beta = 0$ (since $\mathcal{U}$ is proper). One can check that $\mathcal{E}(\mathcal{U}) \subset \mathcal{E}(\mathcal{U}^\mu)$ and $P_t P_s^\mu = P_s^\mu P_t$ for all $s, t > 0$. Consequently, we see that $U_\alpha U_\beta^\mu = U_\beta^\mu U_\alpha$ for all $\alpha, \beta \geq 0$. Therefore $U_{\beta'}^\mu(Uf) \in \mathcal{E}(\mathcal{U})$ for all $f \in p\mathcal{B}$ and we conclude that $(H8)$ holds.

Remark. *In [BeTr 11] are given examples of Markov processes for which (under conditions closed to* $(H8)$*) the standardness property (in the Ray topology) is preserved by perturbation.*

5.3 Standard processes

In this subsection we investigate further properties of the Markov processes from Theorem 5.2 and Corollary 5.3, namely the quasi-left continuity (recall that a càdlàg process which satisfies it is called standard; see, e.g., [Sh 88] and the definition below). The main motivation is given by two applications: an approach using the Lyapunov functions to the Brownian motion on an abstract Wiener space and the proof of the standardness property for infinite dimensional Lévy processes on Hilbert spaces; cf. [BeCoRö 10]. Note that for the standard processes more intimate connections between the analytic and probabilistic potential theory may be established, in particular, the polar sets are precisely the capacity zero sets; see assertion (*iii*) of the next theorem and [BeTr 11] for a more detailed discussion.

We say that the right process X is *standard* if for every finite measure μ on E it possesses left limits in E P^μ-a.e. on $[0,\zeta)$ and for every increasing sequence $(T_n)_n$ of stopping times with $T_n \nearrow T$ we have $X_{T_n} \longrightarrow X_T$ P^μ-a.e. on $[T<\zeta]$, ζ being the life time of X.

Theorem 5.5. *Assume that conditions* $(H1)$, $(H2)$, $(H3)$, *and* $(H5)$ *are verified. Then the following assertions hold.*

(*i*) *If in addition*

$(H9)$ $\quad \mathcal{T}|_{K_n} = \mathcal{T}(\mathcal{C})|_{K_n}$ *for all* n*, where* $\mathcal{C}$ *is given by* $(H1)$,

then $(H6)$ *and* $(H7)$ *are also verified with* u_φ *real-valued,* $v_\varphi = 0$ *for every* $\varphi \in \mathcal{F}$ *(consequently,* $M=\emptyset$*) and* $g=0$*. In particular, the conclusion of Theorem 5.2 holds.*

(*ii*) *If* $\mathcal{T} = \mathcal{T}(\mathcal{C})$ *is metrizable Lusin, then there exists a standard process with state space* E *such that its resolvent equals* $\mathcal{U}$ λ*-quasi everywhere.*

(*iii*) *Suppose that* $(H9)$ *holds and let* μ *be a finite measure on* E *such that* $\mu\circ U_\beta \leq \lambda\circ U_\beta$*. Then for each* $N\in\mathcal{B}$ *we have*

$$c_\mu(N) = \mu(R^N_\beta u_o) = \sup\{\nu(u_o 1_N) : \nu\circ U_\beta \leq \mu\circ U_\beta\}.$$

In particular, a set $N\in\mathcal{B}$ *will be* μ*-polar and* μ*-negligible if and only if* $c_\mu(N)=0$.

Proof. (*i*) Recall that by Proposition 2.1 it follows that $(H4)$ holds and by Proposition 2.2 there exists a Ray cone $\mathcal{R}$ such that the Ray topology $\mathcal{T}(\mathcal{R})$ generated by $\mathcal{R}$ is smaller than $\mathcal{T}(\mathcal{C})$. Consequently, we have $\mathcal{T}(\mathcal{R})|_{K_n} = \mathcal{T}|_{K_n} = \mathcal{T}(\mathcal{C})|_{K_n}$ for all n. Let $\mathcal{F}$ be a countable subset of $\mathcal{R}\cap\mathcal{C}$ which is dense in $\mathcal{R}$ in the supremum norm, such that $1\in\mathcal{F}$ and $U_\beta(\mathcal{F})\subset\mathcal{F}$; see the proof of Proposition 2.2 for the existence of such a set. As a consequence $(H6)$ is satisfied and we claim that $(H7)$ also hold. Clearly $M=\emptyset$ and let $\xi,\eta\in \mathsf{Exc}(\mathcal{U}_\beta)$ be such that $L_\beta(\xi,u) = L_\beta(\eta,u) < \infty$ for all $u\in\mathcal{F}$. Using $(2.1a)$, we see that for all $u\in\mathcal{F}$

$$\mu(u) = L_\beta(\xi, U_\beta u) = L_\beta(\eta, U_\beta u) = \nu(u).$$

Since ξ and η are finite measures and $\mathcal{F}$ is dense in $\mathcal{R}$ in the supremum norm, it follows that $\mu(u)=\nu(u)$ for all $u\in\mathcal{R}$ and we conclude that $\mu=\nu$ by a monotone class argument.

(*ii*) By (*i*), X may be taken such that it is a right process having càdlàg trajectories with respect to $\mathcal{T}$. Let $(\tau_n)_n$ be an increasing sequence of stopping times and $\tau = \lim_{n\to\infty}\tau_n$. Let

$$V := \begin{cases} \lim\limits_{n\to\infty} X_{\tau_n} \text{ on } [\tau<\zeta], \text{ if the limit exists} \\ 0 (\in E) \text{ else.} \end{cases}$$

If $f, g \in \mathcal{C}$, since X possesses left limits in E P^x-a.e. on $[0, \zeta)$, and using the fact that $U_\alpha g$ is a continuous function on E, arguing as in the proof of Lemma IV.3.21 from [MaRö 92], one obtains for all $x \in E$

$$E^x[f(X_\tau)U_\alpha g(V)] = E^x[f(X_\tau)U_\alpha g(X_\tau)].$$

Multiplying by α and letting $\alpha \to \infty$, by $(H3)$ we get

$$E^x[f(X_\tau)g(V)] = E^x[f(X_\tau)g(X_\tau)].$$

Using a monotone class argument we deduce that for all $h \in p\mathcal{B}(E \times E)$

$$E^x[1_{[\tau<\zeta]}h(X_\tau, V)] = E^x[1_{[\tau<\zeta]}h(X_\tau, X_\tau)]$$

and taking as h the indicator function of the diagonal of $E \times E$ we conclude that $X_\tau = V$ P^x-a.e. on $[\tau < \zeta]$.

(iii) Note that by the hypothesis on μ we get that every λ-nest is a μ-nest. We observed in the proof of (i) that there exists a Ray cone $\mathcal{R}$ such that $\mathcal{T}(\mathcal{R})|_{K_n} = \mathcal{T}|_{K_n}$ for all n. On the other hand by Proposition 1.6.3 and Proposition 1.6.4 in [BeBo 04] we obtain the claimed equalities, in the case when the topology $\mathcal{T}$ is a Ray one. Consequently we have $c_\mu(N \cap K_n) = \mu(R_\beta^{N\cap K_n} u_o)$ for all n. The assertion follows now since $c_\mu(N) = \sup_n c_\mu(N \cap K_n)$. □

Remark. (i) By Theorem 5.5 and Proposition 4.1 the following assertion holds: *If $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ such that conditions $(H1)$, $(H2)$, and $(H3)$ are verified, E is endowed with the topology $\mathcal{T}(\mathcal{C})$, assumed to be metrizable Lusin with $\mathcal{B}(\mathcal{T}(\mathcal{C})) = \mathcal{B}$, and there exists a compact Lyapunov function which is finite λ-a.e., then there exists a standard process with state space E, whose resolvent equals $\mathcal{U}$ λ-quasi everywhere.*

(ii) The above assertion (i) was used in [BeCoRö 10] to prove the càdlàg property of the restriction to an invariant set of the Lévy processes on Hilbert spaces.

(iii) The hypothesis of Theorem 5.5 is closed to the conditions imposed in [St83] to a resolvent of kernels on a locally compact space with a countable base in order to produce a Hunt process.

The following corollary shows that in the case of the Ray topology, conditon $(H5)$ alone (the existence of a nest of Ray compacts) is sufficent for the existence of an associated right process which is in addition standard; see also [BeBo 05], Theorem 1.3 and [BeBoRö 06a], Lemma 3.5.

Corollary 5.6. *Let $\mathcal{U}$ be a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ and assume that for some $\beta > 0$ there exists a Ray cone $\mathcal{R}$ associated with $\mathcal{U}_\beta$. If condition $(H5)$ is satisfied with $\mathcal{T} = \mathcal{T}(\mathcal{R})$, then there exists a standard process with state space E such that its resolvent equals $\mathcal{U}$ λ-quasi everywhere.*

Proof. From Corollary 2.3 and Proposition 2.2(i) conditions $(H1)$, $(H2)$, and $(H3)$ are satisfied, taking $\mathcal{C} = \mathcal{R} - \mathcal{R}$. The assertion follows now by Theorem 5.5. □

The next result offers the main tools of an approach to the infinite dimensional Brownian motion, using the existence of the compact Lyapunov functions (cf. Theorem 2.9 in [BeCoRö 10]).

Corollary 5.7. *Assume that conditions* $(H1)$, $(H2)$, *and* $(H3)$ *are satisfied. Then the following assertions hold.*

(i) *Suppose that for some* $\beta > 0$ *and every* $z \in E$ *we have:*

$(*)$ if $\xi \in \mathsf{Exc}(\mathcal{U}_\beta)$ and $\xi \le U_\beta(z,\cdot)$ then $\xi \in \mathsf{Pot}(\mathcal{U}_\beta)$;

here $U_\beta(z,\cdot)$ *denotes the measure* $f \longmapsto U_\beta f(z)$*. Then the resolvent* $\mathcal{U}$ *is associated with a right process with state space* E *endowed with any natural topology.*

(ii) *Endow* E *with the topology* $\mathcal{T}(\mathcal{C})$*, assume that this topology is natural with respect to* $\mathcal{U}$ *and that there exists a* $\mathcal{U}_\beta$*-excessive function* v *such that the set* $[v \le n]$ *is a relatively* $\mathcal{T}(\mathcal{C})$*-compact subset of* E *for all* $n \in \mathbb{N}$*. Then the following assertions hold.*

$(ii.1)$ *The restriction of* $\mathcal{U}$ *to* $E_o := [v < \infty]$ *is the resolvent of a right process with state space* E_o *and condition* $(*)$ *holds for all* $z \in E_o$*.*

$(ii.2)$ *Assume in addition that the set* $[v \le n]$ *is* $\mathcal{T}(\mathcal{C})$*-closed for all* n*. Then the process on* E_o *given by* $(ii.1)$ *is standard.*

Proof. (i) According with (2.3), to get that $\mathcal{U}$ is the resolvent of a right process with state space E, we have to prove that the set $E_1 \setminus E$ is polar with respect to $\mathcal{U}^1$. From $(*)$ and Proposition 1.7.6 in [BeBo 04] we get that the set $E_1 \setminus E$ is $U_\beta(z,\cdot)$-polar for all $z \in E$. Hence $U_{\beta+\alpha}(\widehat{R_\beta^{E_1\setminus E}}1)(z) \le U_\beta(\widehat{R_\beta^{E_1\setminus E}}1)(z) = 0$, for all $\alpha > 0$ and $z \in E$, therefore

$$\widehat{R_\beta^{E_1\setminus E}}1(z) = \lim_{\alpha\to\infty} \alpha U_{\beta+\alpha}(\widehat{R_\beta^{E_1\setminus E}}1)(z) = 0,$$

and so $\widehat{R_\beta^{E_1\setminus E}}1 = 0$ on E and thus on E_1.

$(ii.1)$ We proceed as in the proof of Proposition 4.1, the implication $(ii) \Longrightarrow (i)$. Let $K_n := \overline{[v \le n]}$. It turns out that $(K_n)_n$ is a ε_z-nest of $\mathcal{T}(\mathcal{C})$-compact set for every $z \in E_o$. By Proposition 2.2 there exists a Ray cone $\mathcal{R}$ such that the Ray topology generated by $\mathcal{R}$ is smaller than $\mathcal{T}(\mathcal{C})$. It follows that $E_1 \setminus K_n$ is a finely open subset of E_1 and we obtain on E

$$R_\beta^{E_1\setminus E_o}1 \le R_\beta^{E_1\setminus K_n}1 = R_\beta^{E\setminus K_n}1 \le R_\beta^{[1\le \frac{v}{n}]}1.$$

Hence on E_o we have

$$R_\beta^{E_1\setminus E_o}1 \le \inf_n R_\beta^{[1\le \frac{v}{n}]}1 \le \inf_n \frac{v}{n} = 0.$$

We conclude that the set E_o belongs to $\mathcal{A}(\mathcal{U}^1)$.

Because $\mathcal{U}^1$ is the resolvent of a right process with state space E_1, by Corollary 3.2 we deduce that the restriction $\mathcal{U}|_{E_o}$ of $\mathcal{U}$ to E_o is the resolvent of a right process with state space E_o as claimed. The fact that condition $(*)$ holds for all $z \in E_o$ follows now by (2.3a) and Lemma 3.1 b).

$(ii.2)$ The additional assumption of $(ii.2)$ implies that $K_n = [v \le n] \subset E_o$ for all n. Note that $\mathcal{U}|_{E_o}$ satisfies $(H1)$, $(H2)$, and $(H3)$ for $\mathcal{C}|_{E_o}$ (as we already remarked in Section 3) and $(K_n)_n$ is an increasing sequence of $\mathcal{T}(\mathcal{C}|_{E_o})$-compact subsets of E_o such that $\inf_n R_\beta^{E_o\setminus K_n}1 = 0$ on E_o. Hence $(H5)$ is verfied by $\mathcal{U}|_{E_o}$ for every finite measure λ on E_o. Applying Theorem 5.5(i) on E_o, it follows that $(H6)$ and $(H7)$ are also verified on E_o for every finite measure λ with u_φ and v_φ real-valued. By Corollary 5.3(ii) $\mathcal{U}|_{E_o}$ is the resolvent of a càdlàg Markov process with state space E_o. Reasoning as in the proof of assertion (ii) of Theorem 5.5, we conclude that the process on E_o is standard, hence $(ii.2)$ holds. □

6 Applications

6.1 Lévy processes on infinite dimensional spaces

As already stressed at the beginning of Subsection 5.3 the results on standardness of Markov processes have already been applied to show this property for the first time for Lévy processes on Hilbert spaces in [BeCoRö 10]. For details we refer to that paper.

6.2 Weak solutions for singular SDE on Hilbert spaces

As already mentioned in the introduction, one motivation of this paper is to develop techniques to control (e.g. describe explicitly) the exceptional (polar) sets, where usually in the procedure of constructing a Markov process, given its resolvent, the latter has to be modified and the Markov process is defined trivially as being stuck when started at any point of this exceptional set. When one wants to apply such process constructions to obtain solutions to SDE, this is important because it gives information which initial data, i.e. starting points "are allowed", to solve the SDE. In this subsection we describe a class of SDE on Hilbert spaces with non-continuous drift, for which so far no existence results are known, whereas our techniques, more precisely Corollary 5.3, imply existence of (in the probabilistic sense) weak solutions for such SDE for all starting points, i.e. there is no such exceptional set mentioned above. In fact these solutions are also unique strong solutions (see Remark 6.6 below).

To this end consider the stochastic equation

$$\begin{cases} dX(t) = (AX(t) + F(X(t)))dt + (-A)^{-\frac{\gamma}{2}} dW(t) \\ X(0) = x \in H. \end{cases} \tag{6.1}$$

Here H is a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$, norm $|\cdot|$ and Borel σ-algebra $\mathcal{B}(H)$, $W = W(t)$, $t \geq 0$ is a cylindrical Brownian motion on H defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and the coefficients satisfy the following hypotheses:

(A) $(A, D(A))$ is a self-adjoint operator such that for some $\omega \in (0, \infty)$

$$\langle Ax, x\rangle \leq -\omega |x|^2 \text{ for all } x \in D(A)$$

and $Tr(-A)^{-\gamma} < \infty$ for some $\gamma \in (0, 1]$.

$(F1)$ $F : D(F) \subset H \to 2^H$ is an m-dissipative map (i.e.,

$$\langle u - v, x - y\rangle \leq 0 \text{ for all } x, y \in D(F),\ u \in F(x),\ v \in F(y),$$

("dissipativity") and

$$\text{Range } (I - F) := \bigcup_{x\in D(F)} (x - F(x)) = H)$$

such that $0 \in D(F)$ and $0 \in F(0)$.

Furthermore, let $F_0(x) \in F(x)$, $x \in D(F)$, be such that

$$|F_0(x)| = \min_{y\in F(x)} |y|.$$

Here we recall that for F as in $(F1)$ we have that $F(x)$ is closed, non empty and convex.

The corresponding Kolmogorov operator is then given as follows: Let $\mathcal{E}_A(H)$ denote the linear span of all real parts of functions of the form $\varphi = e^{i\langle h,\cdot\rangle}$, $h \in D(A)$, and define for any $x \in D(F) = D(F_0)$,

$$L_0\varphi(x) = \frac{1}{2}Tr((-A)^{-\gamma}D^2\varphi(x)) + \langle x, AD\varphi(x)\rangle + \langle F_0(x), D\varphi(x)\rangle,\ \varphi \in \mathcal{E}_A(H).$$

Additionally, we assume:

(λ) There exists a probability measure λ on H (equipped with its Borel σ-algebra $\mathcal{B}(H)$) such that

(i) $\lambda(D(F)) = 1$,

(ii) $\int_H (1 + |x|^2)(1 + |F_0(x)|)\lambda(dx) < \infty$,

(iii) $\int_H L_0\varphi\, d\lambda = 0$, for all $\varphi \in \mathcal{E}_A(H)$.

For examples of F ensuring that hypotheses (λ) holds, we refer to [BoRö 01], which, however, we do not use in Example 6.7 below. Let $P(H)$ denote the set of all probability measures on H and define $\mathcal{M}$ to be the subset of all $\nu \in P(H)$ satisfying (i) and (ii) in Hypothesis (λ) with ν replacing λ and for which there exists $\alpha(\nu) \in (0, \infty)$ such that

$$\int L_0\varphi\, d\nu \le \alpha(\nu) \int \varphi\, d\nu \ \text{ for all } \varphi \in \mathcal{E}_A(H), \varphi \ge 0.$$

Then we have the following two theorems:

Theorem 6.1. *Assume that Hypotheses (A) and $(F1)$ hold. Then for any $\nu \in \mathcal{M}$ the operator $(L_0, \mathcal{E}_A(H))$ is quasi-dissipative on $L^1(H, \nu)$, hence closable. Its closure $(L_\nu, D(L_\nu))$ generates a C_0-semigroup P_t^ν, $t \ge 0$, on $L^1(H, \nu)$ which is Markovian, i.e., $P_t^\nu 1 = 1$ and $P_t^\nu f \ge 0$ for all nonnegative $f \in L^1(H, \nu)$ and all $t > 0$. Furthermore,*

$$\int_H P_t^\nu f d\nu \le e^{\alpha(\nu)t}\int_H f d\nu \text{ for all } f \in L^1(H, \nu), t > 0.$$

Below $C_b(H)$ denotes the set of all bounded continuous functions from H to $\mathbb{R}$ and $||\cdot||$ denotes the usual norm on $L(H)$:= all bounded linear operators on H.

Theorem 6.2. *Assume that Hypotheses (A), $(F1)$ and (λ) hold. Let $H_0 := supp(\lambda)$ (:= smallest closed set of H whose complement is a λ-zero set). Then there exists a semigroup $p_t(x, dy)$, $x \in H_0$, $t > 0$, of Markovian kernels such that $p_t f$ is a λ-version of $P_t^\lambda f$ for all $f \in b\mathcal{B}(H)$, $t > 0$, where as usual*

$$p_t f(x) = \int_H f(y)p_t(x, dy),\ x \in H_0.$$

Furthermore, for all $t > 0$, $x, y \in H_0$, we have $p_t(x, H \setminus H_0) = 0$ and

$$|p_t f(x) - p_t f(y)| \le \frac{1}{\sqrt{t \wedge 1}}||f||_0\, ||(-A)^{-\frac{\gamma}{2}}||\, |x - y| \text{ for all } f \in b\mathcal{B}(H),$$

in particular, p_t, $t > 0$, is strongly Feller, i.e. $p_t(b\mathcal{B}(H)) \subset C_b(H)$, for all $t > 0$. In addition, for all $f \in Lip_b(H)$ (:=all bounded Lipschitz functions on H)

$$|p_t f(x) - p_t f(y)| \le ||f||_{Lip}|x - y| \text{ for all } t > 0, x, y \in H_0,$$

and

$$\lim_{t\to 0} p_t f(x) = f(x) \text{ for all } x \in H_0.$$

(Here $||f||_0$, $||f||_{Lip}$ denote the supremum, Lipschitz norm of f respectively.) Finally, λ is p_t-invariant.

Proof of Theorem 6.1. By [Eb 99], Lemma 1.8, p. 36, $(L_0 - \alpha(\nu), \mathcal{E}_A(H))$ is dissipative on $L^1(H, \nu)$. Furthermore, because of [DaRö 02], Remark 4.4, it is then easy to check that the proofs of [DaRö 02], Theorems 2.3 and Corollary 2.5, easily generalize to all ν in our more general class of measures $\mathcal{M}$ replacing that particular ν in [DaRö 02], Hypothesis 1.2.

Proof of Theorem 6.2. As above one easily checks that the proof of [DaRö 02], Proposition 5.7, extends to the more general λ in Hypothesis (λ) replacing the measure ν in [DaRö 02].

Remark 6.3. (i) *We stress that since F_0 is merely measurable, even under the additional assumption $(F2)$ on F_0 introduced below, it is not known whether a (strong or weak) solution exists for SDE* (6.1)*. One could try to apply the general theory of stochastic evolution equations with monotone coefficients, to solve* (6.1)*. But one would need additional conditions on $D(F)$ and the solution would not solve* (6.1)*, but a variant of it with $F_0(X(t))$ replaced by a section of the multivalued process $F(X(t))$ with no information whether it coincides with the minimal section $F_0(X(t))$. Furthermore, we note that also applying the standard method based on Girsanov's Theorem does not work in the case of* (6.1)*, since F_0 does not take values in the image of $(-A)^{\frac{\gamma}{2}}$.*

(ii) *We also want to emphasize that the existence of a weak solution to SDE* (6.1) *also does not follow from [DaRö 02], Theorem 7.4. It was pointed out in [DaRö 09] that because of an error (in [DaRö 02], Lemma 5.5) that theorem only holds if either* $\dim H < \infty$ *or only guarantees the existence of a weak solution which has paths only continuous in H on* $(0, \infty)$ *(so not in $t = 0$) or only continuous on* $[0, \infty)$ *in another (though natural) topology on H, different from the norm or weak topology on H.*

Consider the resolvent $\mathcal{U} = (U_\alpha)_{\alpha>0}$ corresponding to p_t, $t > 0$, from Theorem 6.2, i.e. for all $\alpha > 0$

$$U_\alpha f(x) := \int_0^\infty e^{-\alpha t} p_t f(x) dt \quad \text{for all } f \in b\mathcal{B}(H), x \in H_0. \tag{6.2}$$

Then clearly $(U_\alpha)_{\alpha>0}$ is also strongly Feller and we have the following result:

Proposition 6.4. *Assume that Hypotheses (A), $(F1)$, and (λ) hold and let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be as above. Then:*

(i) $(H1), (H2), (H3)$ *(and hence $(H4)$) from Subsection 2.1 hold with $\mathcal{C} := Lip_b(H)|_{H_0}$.*

(ii) *λ is a reference measure for $\mathcal{U}$, i.e. $U_\alpha(x, dy) \ll \lambda(dy)$ for all $\alpha > 0$, $x \in H_0$.*

Proof. (i) Since H_0 is a closed subset of H, it is well-known that $Lip_b(H_0) = Lip_b(H)|_{H_0}$. Hence the assertion follows from Theorem 6.2.

(ii) Since p_t, $t > 0$, is strongly Feller, by Theorem 6.2 it trivially follows that $p_t(x, dy) \ll \lambda(dy)$ for all $t > 0$, $x \in H_0$, hence by (6.2), the same holds for each U_α. □

Our aim is to apply Corollary 5.3 with $\mathcal{T}$:=weak topology on H to obtain the desired weak solution to (6.1), which, however, will then only have weakly continuous sample paths.

Exactly analogous to the proof of [DaRö 02], Theorem 2.3, one shows that for each $\nu \in \mathcal{M}$ we have

(6.3) $\quad (1 - L_0)(\mathcal{E}_A(H))$ is dense in $L^1(H, \nu)$.

By a simple approximation argument it follows that $\mathcal{F}C_b^2 \subset D(L_\nu)$ and that $L_\nu = L_0$ on $\mathcal{F}C_b^2$. Hence by (6.3)

$$(1 - L_0)(\mathcal{F}C_b^2) \text{ is dense in } L^1(H, \nu),$$

where $\mathcal{F}C_b^2$ denotes the set of all functions $\varphi : H \longrightarrow \mathbb{R}$ such that for some $N \in \mathbb{N}$, and $\varphi^{(N)} \in C_b^2(\mathbb{R}^N)$,

$$\varphi(x) = \varphi^{(N)}(\langle e_1, x\rangle, \ldots, \langle e_N, x\rangle) \text{ for all } x \in H.$$

Since $C_b^2(\mathbb{R}^N)$ with norm $||\cdot||_0 + ||D\cdot||_0 + ||D^2\cdot||_0$ is separable, there exists a countable set $\mathcal{F} \subset \mathcal{F}C_0^2$ such that

(6.4) $\quad (1 - L_0)(\mathcal{F})$ is dense in $L^1(H, \nu)$ for all $\nu \in \mathcal{M}$.

Now for $\varphi \in \mathcal{F}$ define the β-excessive functions

$$u_\varphi := U_\beta(((\beta - L_0)\varphi)^+),\ v_\varphi := U_\beta(((\beta - L_0)\varphi)^-),$$

where $\beta := \sum_{i=1}^\infty \alpha_i^{-\gamma}$ (see below). Then u_φ, v_φ are 1-excessive and since for all $u \in b\mathcal{B}(H)$ by Theorem 6.2, $U_\beta u$ is a λ-version of $(\beta - L_\lambda)^{-1}u$ we have

(6.5) $\quad \varphi(x) = u_\varphi(x) - v_\varphi(x)$ for λ-a.e. $x \in H_0$.

We shall see below, however, that this equality holds for every $x \in H_0$.

To verify this and the other assumptions in assertion (i) of Corollary 5.3 we need a further hypothesis. To this end let e_i, $i \in \mathbb{N}$, be an eigenbasis of A (which exists since $A^{-\frac{\gamma}{2}}$ is Hilbert-Schmidt) and $-\alpha_i$, $i \in \mathbb{N}$, $\alpha_i \in [\omega, \infty)$, be the corresponding eigenvalues. Let P_N be the orthogonal projection onto the linear span of $\{e_1, \ldots, e_N\}$, $N \in \mathbb{N}$. Define $V : H \longrightarrow [0, \infty)$ and $\Theta : H \longrightarrow [0, \infty]$ by

$$V(x) := \sum_{i=1}^\infty \langle e_i, x\rangle^2,\ \Theta(x) := \sum_{i=1}^\infty \alpha_i \langle e_i, x\rangle^2,\ x \in H.$$

Then the further hypothesis reads:

$(F2)$ $\quad$ There exist $m \in \mathbb{N}$ and $c \in (0, \infty)$ such that

$$|F_0| \le c(1 + V^m\Theta) \quad \lambda\text{-a.e.}$$

Proposition 6.5. *Suppose Hypotheses (A), $(F1)$, and (λ) hold.*

(i) Let $\beta := \sum_{i=1}^\infty \alpha_i^{-\gamma}$. Then $V|_{H_0}$ is $\mathcal{U}_\beta$-excessive. In particular,

$$K_n := \{x \in H_0 : |x| \le n\},\ n \in \mathbb{N},$$

forms a λ-nest.

(ii) $U_\beta(V^m\Theta)$, $m \in \mathbb{N} \cup \{0\}$, are (real-valued) $|\cdot|$-continuous functions on H_0.

(iii) If in addition $(F2)$ holds, then $U_\beta(((\beta - L_0)\varphi)^+)$, $U_\beta(((\beta - L_0)\varphi)^-)$, $\varphi \in \mathcal{F}C_b^2$, are also (real-valued) $|\cdot|$-continuous functions on H_0. In particular, (6.5) holds on all of H_0.

Proof. Let $N, M, m \in \mathbb{N}$ and $\chi_M \in C_b^2(\mathbb{R})$ such that $\chi_M(r) = r$ for all $r \in [-M, M]$, $\chi_M(r) = (M + \frac{1}{2})\,\mathrm{sign}\, r$, if $|r| \geq M + 1$, $0 \leq \chi_M' \leq 1$, $\chi_M'' \leq 0$ on $[0, \infty)$. Define

$$V_N(x) := |P_N x|^2,\ x \in H.$$

Then an easy computation gives that

$$(6.6)\qquad L_0(\chi_M(V_N^m)) = mV_N^{m-1}\chi_M'(V_N^m)L_0V_N +$$

$$2mV_N^{m-2}\left[mV_N^m\chi_M''(V_N^m) + (m-1)\chi_M'(V_N^m)\right] \cdot \sum_{i=1}^{N} \alpha_i^{-\gamma}\langle e_i, \cdot\rangle^2$$

and

$$(6.7)\qquad L_0V_N = \sum_{i=1}^{N}\alpha_i^{-\gamma} - \sum_{i=1}^{N}\alpha_i\langle e_i, \cdot\rangle^2 + \sum_{i=1}^{N}\langle e_i, F_0\rangle\langle e_i, \cdot\rangle.$$

Claim: $V^m\Theta \in L^1(H, \lambda)$ for all $m \in \mathbb{N} \cup \{0\}$.

Integrating (6.6) and using Hypothesis (λ), part (iii), we obtain for $C := \sum_{i=1}^{\infty}\alpha_i^{-\gamma}$:

$$(6.8)\qquad \int V_N^{m-1}\chi_M'(V_N^m)\left(\sum_{i=1}^{N}\alpha_i\langle e_i, \cdot\rangle^2 - C\right)d\lambda \leq$$

$$\int V_N^{m-1}\chi_M'(V_N^m)\sum_{i=1}^{N}\langle e_i, F_0\rangle\langle e_i, \cdot\rangle d\lambda + \frac{2}{\omega^\gamma}(m-1)\int V_N^{m-2}\sum_{i=1}^{N}\alpha_i\langle e_i, \cdot\rangle^2 d\lambda.$$

So, if $m = 1$ the last term vanishes and, since $|\cdot|\,|F_0| \in L^1(H, \lambda)$, we can let first $N \to \infty$, and the first term in the right hand side of (6.8) converges to a negative number, since $\langle F_0(x), x\rangle \leq 0$ for all $x \in D(F)$. Finally, letting $M \to \infty$ we conclude by monotone convergence that $\Theta \in L^1(H, \lambda)$. Now we can proceed by induction to prove the claim as follows: Suppose $V^{m-2}\Theta \in L^1(H, \lambda)$ for some $m \in \mathbb{N},\ m \geq 2$. Then letting $N \to \infty$ we obtain by (6.8) that:

$$(6.9)\qquad \int V^{m-1}\chi_M'(V^m)\,(\Theta - C)\,d\lambda \leq \frac{2}{\omega^\gamma}(m-1)\int V^{m-2}\Theta d\lambda < \infty.$$

Since $V \leq \omega\Theta$ and thus $V^{m-1}\chi_M'(V^m)\,(\Theta - C)^- \leq \omega^{m-1}C^m$, we can let $M \to \infty$ in (6.9) to obtain that $V^{m-1}\Theta \in L^1(H, \lambda)$ which proves the claim.

Clearly, by Theorem 6.2, for all $f \in b\mathcal{B}(H),\ \alpha > 0,\ U_\alpha f$ is a λ-version of $(\alpha - L_\lambda)^{-1}f$. Hence (6.6) and (6.7) imply that λ-a.e.

$$\chi_M(V_N^m) = \alpha U_\alpha(\chi_M(V_N^m)) - U_\alpha(mV_N^m\chi_M'(V_N^m)[\sum_{i=1}^{N}\alpha_i^{-\gamma} - \sum_{i=1}^{N}\alpha_i\langle e_i, \cdot\rangle^2 +$$

$$\sum_{i=1}^{N}\langle e_i, F_0\rangle\langle e_i, \cdot\rangle]) - U_\alpha(2mV_N^{m-2}[mV_N^m\chi_M''(V_N^m) + (m-1)\chi_M'(V_N^m)]\sum_{i=1}^{N}\alpha_i^{-\gamma}\langle e_i, \cdot\rangle^2).$$

Since by the claim above the functions under U_α converge in $L^1(H,\lambda)$ and since U_α is continuous on $L^1(H,\lambda)$, letting first $N \to \infty$ and then $M \to \infty$ we obtain that λ-a.e for $\beta_m := m \sum_{i=1}^{\infty} \alpha_i^{-\gamma}$

$$(6.10) \qquad V^m = U_{\beta_m}(mV^m[\Theta - \langle F_0, \cdot\rangle]) - U_{\beta_m}(2m(m-1)V^{m-2}\sum_{i=1}^{\infty} \alpha_i^{-\gamma}\langle e_i, \cdot\rangle^2).$$

We note that by Hypothesis $(F1)$ in particular $\langle F_0, \cdot\rangle \le 0$. Hence for $m = 1$, one easily deduces that for all $t > 0$:

$$(6.11) \qquad p_t V(x) \le e^{-\beta_1 t} V(x) \text{ for } \lambda\text{-a.e. } \ x \in H_0.$$

By the same method as in [DaRöWa 09], but instead of the Galerkin approximation used there, one applies [Liu 09], Theorem 1.2, to the Yoshida approximation in [DaRöWa 09], Section 2, we obtain that each p_t satisfies Wang's dimension free Harnack inequality. In particular, we have by [DaRöWa 09], Proposition 4.1, that

$$p_t(L^{1+\varepsilon}(H,\lambda)) \subset C(H_0).$$

Hence (6.11) holds for all $x \in H_0$ and thus V is $\mathcal{U}_{\beta_1}$-excessive, and hence (i) is proved.

To prove (ii) we return to (6.10) and note that, since $U_{\beta_m}$is strongly Feller, both summands on the right hand side of (6.10) are lower semicontinuous and thus for $m = 1$ we see that $U_{\beta_1}(V\Theta)$, $U_{\beta_1}(-V\langle F_0, \cdot\rangle)$ are both real valued. By induction we then easily deduce that both summands in (6.10) are real-valued for all $m \in \mathbb{N}$. But note that all functions in (6.10) are $\mathcal{U}_\beta$-excessive, hence finely continuous and that by (2.4), λ as reference measure has also fine full support, i.e., $\lambda(G) > 0$ for every non-empty finely open set $G \subset H_0$. Therefore, (6.10) holds on all of H_0. Using the fact that if a sum of two lower semicontinuous real valued functions is continuous, each summand must be continuous, again by induction we deduce from (6.10) (now valid on all of H_0) assertion (ii).

(iii) is proved by applying the above fact again, which is possible since for all $\varphi \in \mathcal{F}C_b^2$ by Hypothesis $(F2)$

$$((\beta - L_0)\varphi)^{\pm} \le \beta|\varphi|_\infty + |\ \frac{1}{2} Tr A^{-\gamma} D^2\varphi\ |_\infty + [c_\varphi V^{1/2} + C(1 + V^m\Theta)]\ |D\varphi|_\infty$$

for some $c_\varphi \in (0,\infty)$ and since U_β applied to the right hand side is (real-valued) continuous by (ii). □

Proposition 6.5 implies that, except for $(H7')$, all conditions in Corollary 5.3 hold. So, it remains to check condition $(H7')$. For the function g in $(H7')$ we take the function from Hypothesis (λ), part (ii), i.e.:

$$g := (1 + V)(1 + |F_0|).$$

Then it follows from $(F2)$ and Proposition 6.5 that $U_\beta g < \infty$ on H_0. Now let ξ, η be as in $(H7')$. So, by (2.1a) and Proposition 6.5(iii) we have:

$$(6.12) \qquad \int (\beta - L_0)\varphi d\xi = L_\beta(\xi, u_\varphi - v_\varphi) = L_\beta(\eta, u_\varphi - v_\varphi) = \int (\beta - L_0)\varphi d\eta \ \text{ for all } \varphi \in \mathcal{F}.$$

Let $f \in C_b(H_0)$. Note that $\xi, \eta \in \mathcal{M}$ and so is $\nu := \frac{1}{2}\xi + \frac{1}{2}\eta$. By (6.4) (and the resolvent equation):

$$(\beta - L_0)(\mathcal{F}) \text{ is dense in } L^1(H, \nu).$$

So, there exist $\varphi_n \in \mathcal{F}$ such that $(\beta - L_0)\varphi_n \to f$ as $n \to \infty$ in $L^1(H, \nu)$. Since ξ, η have bounded Radon-Nikodym densities with respect to ν, this also holds both in $L^1(H, \xi)$ and in $L^1(H, \eta)$. Hence by (6.12)

$$\int f d\xi = \int f d\eta.$$

Since $f \in C_b(H_0)$ was arbitrary, we conclude that $\xi = \eta$. Hence we have altogether proved that Corollary 5.3 applies. But we emphazise that in our case since $N := H_0 \setminus \cup_{n \geq 1} K_n = \emptyset$, no modification of the resolvent is necessary. In particular, for an $x \in H_0$ the associated strong Markov process P^x-a.s. has right-weakly continuous sample paths from $[0, \infty)$ to H_0. Since by [DaRö 09], Theorem 7.4'(ii), we already know the $|\cdot|$-continuity of the sample paths on $(0, \infty)$, we obtain that the sample paths are weakly continuous on $[0, \infty)$.

Remark 6.6. *In fact the strong Markov process constructed above is a unique strong solution to SDE* (6.1) *for all* $x \in H_0$. *This can be shown as in [DaRöWa 09], Proof of Corollary 1.10. The details will be done in a forthcoming paper.*

Example 6.7. Let $H \in L^2(0,1)$, $Ax = \Delta x$, $x \in D(A) := H^2(0,1) \cap H_0^1(0,1)$. Let $f : \mathbb{R} \to \mathbb{R}$ be decreasing such that $f(0) = 0$ and for some $c_1 > 0, m \in \mathbb{N}$,

$$|f(s)| \leq c_1(1 + |s|^m) \text{ for all } s \in \mathbb{R}.$$

Let $s_i \in \mathbb{R}, i \in \mathbb{N}$, be the set of all arguments where f is not continuous and define

$$\bar{f}(s) = \begin{cases} [f(s_i+), f(s_i-)], & \text{if } s = s_i \text{ for some } i \in \mathbb{N}, \\ f(s) \text{ else.} \end{cases}$$

Define

$$F : D(F) \subset H \longrightarrow 2^H, x \longmapsto \bar{f} \circ x,$$

where

$$D(F) = \{x \in H : \bar{f} \circ x \subset H\}.$$

Then F is m-dissipative, hence Hypothesis $(F1)$ is fulfilled. Let F_0 be defined as in Section 2. Then $F_0(x) = f_0 \circ x$, with

$$f_0(s) = \begin{cases} f(s_i-), & \text{if } s = s_i > 0 \text{ for some } i \in \mathbb{N}, \\ f(s_i+), & \text{if } s = s_i < 0 \text{ for some } i \in \mathbb{N}, \\ f(s) \text{ else.} \end{cases}$$

Obviously, Hypothesis (A) is also fulfilled with $\gamma \in (\frac{1}{2}, 1]$. To check Hypothesis (λ) we first recall that if $f \equiv 0$, then by [Da 04], Theorem 2.34 and Proposition 2.47, the unique measure λ_0 satisfying (iii) is the centered Gaussian measure on H with covariance operator $(-A)^{-(1+\gamma)}$, i.e.

$$\lambda_0 := N(0, (-A)^{-(1+\gamma)}).$$

It is well-known that (see e.g. [RöSo 06], Theorem 2.4, (2.20), in the case $\Phi \equiv \Psi \equiv 0$) for all $N \in \mathbb{N}$:

$$\int_H | \, x \, |^{2N}_{L^{2N}(0,1)} \, \lambda_0(dx) < \infty. \tag{6.13}$$

Define $\Phi : \mathbb{R} \to \mathbb{R}_+$

$$\Phi(r) = -\int_0^r f(s)ds, \; r \in \mathbb{R},$$

and the probability measure λ on H by

$$\lambda(dx) := Z_\lambda^{-1} e^{-|\Phi(x)|_{L^1(0,1)}} \lambda_0(dx),$$

where Z_λ is a normalization constant. Then by (6.13) λ is equivalent to λ_0 which has a full support on H. So, also $supp(\lambda) = H$, i.e. $H_0 = H$ in this case. Then again by (6.13) parts (i) and (ii) of Hypothesis (λ) hold and one can check that also part (iii) holds, by using the well-known integration by parts formula for Gaussian measures (see, e.g., [Da 04], Lemma 1.5) and the fact that this integration by parts formula extends for partial derivatives $\dfrac{\partial}{\partial e_i}$ (i.e., in directions e_i in the eigenbasis of A) to λ. The latter is sufficient since we only have to check (λ)(iii) for $\varphi \in \mathcal{E}_A(H)$. Finally, we check our last Hypothesis $(F2)$. For this we need to assume that

$$m \leq 5.$$

Then for all $x \in L^5(0,1)$ (which has λ-measure 1 by (6.13)) we have

$$|F_0(x)| \leq c_1(1 + |x|^5_{L^{10}(0,1)}).$$

But by Sobolev embedding (see, e.g. [Ta 95], Theorem 2.21)

$$|x|_{L^{10}(0,1)} \leq const \, (|x|^{2/5}_{H_0^1} \, |x|_2^{3/5} + |x|_2)$$

and $(F2)$ follows. So, all hypotheses imposed above are fulfilled.

Remark 6.8. *(i) The restriction $m \leq 5$ in the previous example can be dropped. However, one has to consider other Lyapunov functions than V, namely instead, as in [RöSo 06] one has to consider:*

$$\widetilde{V}(x) := e^{K|x|^2_{L^2(0,1)}}(1 + |x|^p_{L^p(0,1)}), \; x \in L^p(0,1),$$

for some suitable $p \in [2, \infty)$, $K \in (0, \infty)$. But then one cannot start at every point in $H = L^2(0,1)$ (see [RöSo 06], Theorems 2.2 and 2.3).
(ii) Example 6.7 above improves [RöSo 06], Theorem 2.3 (with $\Psi \equiv 0$), where $m < 5$, and the continuity of f had to be assumed, to be able to start the process at every $x \in L^2(0,1)$. In addition, no strong Feller property of $p_t, t > 0$, was proved in [RöSo 06].

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