

# Additive derivations on generalized Arens algebras

S. Albeverio<sup>1</sup>, Sh.A. Ayupov<sup>2,\*</sup>, R.Z. Abdullaev<sup>3</sup>, K.K. Kudaybergenov<sup>4</sup>

October 7, 2010

## Abstract

Given a von Neumann algebra  $M$  with a faithful normal finite trace  $\tau$  denote by  $L^\Lambda(M, \tau)$  the generalized Arens algebra with respect to  $M$ . We give a complete description of all additive derivations on the algebra  $L^\Lambda(M, \tau)$ . In particular each additive derivation on the algebra  $L^\Lambda(M, \tau)$ , where  $M$  is a type II von Neumann algebra, is inner.

<sup>1</sup> Institut für Angewandte Mathematik, Universität Bonn, Wegelerstr. 6, D-53115 Bonn (Germany); SFB 611; HCM; BiBoS; IZKS; CERFIM (Locarno); Acc. Arch.(USI), e-mail address: *albeverio@uni-bonn.de*

<sup>2</sup> Institute of Mathematics and Information Technologies, Uzbekistan Academy of Sciences, Dormon Yoli str. 29, 100125, Tashkent (Uzbekistan), ICTP (Trieste, Italy), e-mail: *sh\_ayupov@mail.ru*

<sup>3</sup> Institute of Mathematics and Information Technologies, Uzbekistan Academy of Science, Dormon Yoli str. 29, 100125, Tashkent, (Uzbekistan) *arustambay@yandex.ru*

<sup>4</sup> Karakalpak state university, Ch. Abdirov str. 1, 142012, Nukus (Uzbekistan), e-mail: *karim2006@mail.ru*

**AMS Subject Classifications (2000):** 46L57, 46L50, 46L55, 46L60.

**Key words:** von Neumann algebras, measurable operator, generalized Arens algebra, additive derivation, inner derivation.

\* Corresponding author

## 1. Introduction

The present paper continues the series of papers [2]-[9] devoted to the study and a description of derivations on the algebra  $LS(M)$  of locally measurable operators affiliated with a von Neumann algebra  $M$  and on its various subalgebras.

Let  $\mathcal{A}$  be an algebra over the field complex number  $\mathbb{C}$ . A linear (additive) operator  $D : \mathcal{A} \rightarrow \mathcal{A}$  is called a linear (additive) *derivation* if it satisfies the identity  $D(xy) = D(x)y + xD(y)$  for all  $x, y \in \mathcal{A}$  (Leibniz rule). Each element  $a \in \mathcal{A}$  defines a linear derivation  $D_a$  on  $\mathcal{A}$  given as  $D_a(x) = ax - xa$ ,  $x \in \mathcal{A}$ . Such derivations  $D_a$  are said to be *inner derivations*. If the element  $a$  implementing the derivation  $D_a$  on  $\mathcal{A}$ , belongs to a larger algebra  $\mathcal{B}$ , containing  $\mathcal{A}$  (as a proper ideal as usual) then  $D_a$  is called a *spatial derivation*.

One of the main problems in the theory of derivations is to prove the automatic continuity, innerness or spatialness of derivations or to show the existence of non inner and discontinuous derivations on various topological algebras.

On this way A. F. Ber, F. A. Sukochev, V. I. Chilin [10] obtained necessary and sufficient conditions for the existence of non trivial derivations on commutative regular algebras. In particular they have proved that the algebra  $L^0(0, 1)$  of all (classes of equivalence of) complex measurable functions on the interval  $(0, 1)$  admits non trivial derivations. Independently A. G. Kusraev [16] by means of Boolean-valued analysis has also proved the existence of non trivial derivations and automorphisms on  $L^0(0, 1)$ . It is clear that these derivations are discontinuous in the measure topology, and therefore they are neither inner nor spatial. It was conjectured that the existence of such exotic examples of derivations deeply depends on the commutativity of the underlying von Neumann algebra  $M$ . In this connection we have initiated the study of the above problems in the non commutative case [2]-[6], by considering derivations on the algebra  $LS(M)$  of all locally measurable operators affiliated with a von Neumann algebra  $M$  and on various subalgebras of  $LS(M)$ . In [2] so-called noncommutative Arens algebras  $L^\omega(M, \tau) = \bigcap_{p \geq 1} L^p(M, \tau)$  and related algebras associated with a von Neumann algebra  $M$  and a faithful normal semi-finite trace  $\tau$  have been considered. It has been proved that every derivation on this algebra is spatial, and, if the trace  $\tau$  is finite, then all derivations are inner. In [5] and [6] the mentioned conjecture concerning derivations on

on the algebra  $LS(M)$  has been confirmed for type I von Neumann algebras.

Recently this conjecture was also independently confirmed for the type I case in the paper of A.F. Ber, B. de Pagter and A.F. Sukochev [11] by means of a representation of measurable operators as operator valued functions. Another approach to similar problems in the framework of type I  $AW^*$ -algebras has been outlined in the paper of A.F. Gutman, A.G.Kusraev and S.S. Kutateladze [13].

In [5] we considered derivations on the algebra  $LS(M)$  of all locally measurable operators affiliated with a type I von Neumann algebra  $M$ , and also on its subalgebras  $S(M)$  – of measurable operators and  $S(M, \tau)$  of  $\tau$ -measurable operators, where  $\tau$  is a faithful normal semi-finite trace on  $M$ . It was proved that an arbitrary derivation  $D$  on each of these algebras can be uniquely decomposed into the sum  $D = D_a + D_\delta$  where the derivation  $D_a$  is inner (for  $LS(M)$ ,  $S(M)$  and  $S(M, \tau)$ ) while the derivation  $D_\delta$  is an extension of a derivation  $\delta$  (possibly non trivial) on the center of the corresponding algebra.

In the present paper we consider additive derivations on generalized Arens algebras in the sense of Kunze [15] with respect to a von Neumann algebra with a faithful normal finite trace.

In section 1 we give some necessary properties of the generalized Arens algebra  $L^\Lambda(M, \tau)$ .

Section 2 is devoted to study of additive derivations on generalized Arens algebras. We prove that an arbitrary additive derivation  $D$  on the algebra  $L^\Lambda(M, \tau)$  can be uniquely decomposed into the sum  $D = D_a + D_\delta$ , where the derivation  $D_a$  is inner while the derivation  $D_\delta$  is an extension of some additive derivation  $\delta$  on the center of the algebra  $L^\Lambda(M, \tau)$ . In particular, if  $M$  is a type II von Neumann algebra then every additive derivation on the algebra  $L^\Lambda(M, \tau)$  is inner.

## 2. Generalized Arens algebras

Let  $H$  be a complex Hilbert space and let  $B(H)$  be the algebra of all bounded linear operators on  $H$ . Consider a von Neumann algebra  $M$  in  $B(H)$  with the operator norm  $\|\cdot\|_M$ . Denote by  $P(M)$  the lattice of projections in  $M$ .

A linear subspace  $\mathcal{D}$  in  $H$  is said to be *affiliated* with  $M$  (denoted as  $\mathcal{D}\eta M$ ), if

$u(\mathcal{D}) \subset \mathcal{D}$  for every unitary  $u$  from the commutant

$$M' = \{y \in B(H) : xy = yx, \forall x \in M\}$$

of the von Neumann algebra  $M$ .

A linear operator  $x$  on  $H$  with the domain  $\mathcal{D}(x)$  is said to be *affiliated* with  $M$  (denoted as  $x\eta M$ ) if  $\mathcal{D}(x)\eta M$  and  $u(x(\xi)) = x(u(\xi))$  for all  $\xi \in \mathcal{D}(x)$ .

Let  $\tau$  be a faithful normal semi-finite trace on  $M$ . We recall that a closed linear operator  $x$  is said to be  $\tau$ -*measurable* with respect to the von Neumann algebra  $M$ , if  $x\eta M$  and  $\mathcal{D}(x)$  is  $\tau$ -dense in  $H$ , i.e.  $\mathcal{D}(x)\eta M$  and given  $\varepsilon > 0$  there exists a projection  $p \in M$  such that  $p(H) \subset \mathcal{D}(x)$  and  $\tau(p^\perp) < \varepsilon$ . The set  $S(M, \tau)$  of all  $\tau$ -measurable operators with respect to  $M$  is a unital \*-algebra when equipped with the algebraic operations of strong addition and multiplication and taking the adjoint of an operator (see [18]).

Consider the topology  $t_\tau$  of convergence in measure or *measure topology* on  $S(M, \tau)$ , which is defined by the following neighborhoods of zero:

$$V(\varepsilon, \delta) = \{x \in S(M, \tau) : \exists e \in P(M), \tau(e^\perp) \leq \delta, xe \in M, \|xe\|_M \leq \varepsilon\},$$

where  $\varepsilon, \delta$  are positive numbers, and  $\|\cdot\|_M$  denotes the operator norm on  $M$ .

It is well-known [18] that  $S(M, \tau)$  equipped with the measure topology is a complete metrizable topological \*-algebra.

Recall [14] that  $\phi$  is a Young function, if

$$\phi(t) = \int_0^t \varphi(s) ds, \quad t \geq 0,$$

where the real-valued function  $\varphi$  defined on  $[0, \infty)$  has the following properties:

- (i)  $\varphi(0) = 0$ ,  $\varphi(s) > 0$  for  $s > 0$  and  $\lim_{s \rightarrow \infty} \varphi(s) = \infty$ ,
- (ii)  $\varphi$  is right continuous,
- (iii)  $\varphi$  is nondecreasing on  $(0, \infty)$ .

Every Young function is a continuous, convex and strictly increasing function. For every Young function  $\phi$  there is a complementary Young function  $\psi$  given by the density

$$\tilde{\psi}(t) = \sup\{s : \phi(s) \leq t\}.$$

The complement of  $\psi$  is  $\phi$  again. Further a Young function  $\phi$  is said to satisfy the  $\Delta_2$ -condition, shortly  $\phi \in \Delta_2$ , if there exists a  $k > 0$  and  $T \geq 0$  such that:

$$\phi(2t) \leq k\phi(t)$$

for all  $t \geq T$ .

Put

$$K_\phi = \{x \in S(M, \tau) : \tau(\phi(|x|)) \leq 1\}$$

and

$$L_\phi(M, \tau) = \bigcup_{n=1}^{\infty} nK_\phi.$$

It is known [17] (see also [15]) that  $L_\phi(M, \tau)$  is a Banach space with respect to the norm

$$\|x\|_\phi = \inf \left\{ \lambda > 0 : \frac{1}{\lambda}x \in K_\phi \right\}, \quad x \in L_\phi(M, \tau).$$

Recall [15] that  $\phi_1 \prec \phi_2$ , if there exist two nonnegative constants  $c$  and  $T$  such that  $\phi_1(t) \leq \phi_2(ct)$  for all  $t \geq T$ . Let  $\Lambda$  be a generating family of Young functions, i.e. for  $\phi_1, \phi_2 \in \Lambda$  there is a  $\psi \in \Lambda$  with  $\phi_1, \phi_2 \prec \psi$ . A generating family  $\Lambda$  of Young functions is said to be quadratic, if for any  $\phi \in \Lambda$  there is a  $\psi \in \Lambda$  such that the composition of  $\phi$  and the squaring function as a Young function is smaller than  $\psi$  regarding the partial order  $\prec$ , i.e. there are  $c > 0$  and  $T \geq 0$  with  $\phi(t^2) \leq \psi(ct)$  for all  $t \geq T$ . For a quadratic family  $\Lambda$  of Young functions we define

$$L^\Lambda(M, \tau) = \bigcap_{\phi \in \Lambda} L_\phi(M, \tau).$$

On the space  $L^\Lambda(M, \tau)$  one can consider the topology  $t_\Lambda$ , generated by the system of norms  $\{\|\cdot\|_\phi : \phi \in \Lambda\}$ .

It is known [15, Proposition 4.1] that if  $\Lambda$  is a quadratic family of Young functions, then  $(L^\Lambda(M, \tau), t_\Lambda)$  is a complete locally convex \*-algebra with jointly continuous multiplications.

Note that if  $\Lambda = \{t^p : p \geq 1\}$  we have that

$$L^\Lambda(M, \tau) = L^\omega(M, \tau) = \bigcap_{p \geq 1} L^p(M, \tau).$$

Non-commutative Arens algebras  $L^\omega(M, \tau)$  were introduced by Inoue [12] and their properties were investigated in [1]. Generalized Arens algebras were introduced by Kunze [15].

Let  $\varphi \in \Lambda$  be a Young function. Then there exists a Young function  $\phi \in \Lambda$  and  $k > 0$  such that

$$\|xy\|_\varphi \leq k\|x\|_\phi\|y\|_\phi \quad (1)$$

for all  $x, y \in L^\Lambda(M, \tau)$  (see [15]).

Let us remark that, if  $\tau$  is a finite trace, then  $t \prec \phi(t)$  for every Young function and for any quadratic family  $\Lambda$  of Young functions we obtain that

$$L^\Lambda(M, \tau) \subset L^\omega(M, \tau). \quad (2)$$

Further, if every  $\phi \in \Lambda$  satisfies the  $\Delta_2$ -condition then

$$L^\omega(M, \tau) \subset L^\Lambda(M, \tau).$$

It is known [15] that if  $N$  is a von Neumann subalgebra of  $M$  then

$$L_\phi(N, \tau_N) = S(N, \tau_N) \cap L_\phi(M, \tau),$$

where  $\tau_N$  is the restriction of the trace  $\tau$  onto  $N$ .

It should be noted, that if  $M$  is a finite von Neumann algebra with a faithful normal semi-finite trace  $\tau$ , then the restriction  $\tau_Z$  of the trace  $\tau$  onto the center  $Z(M)$  of  $M$  is also semi-finite.

Further we shall need the description of the center of the algebra  $L^\Lambda(M, \tau)$  for finite von Neumann algebras.

**Proposition 2.1.** *Let  $M$  be a von Neumann algebra with a faithful normal finite trace  $\tau$  and with the center  $Z(M)$ . Then*

$$Z(L^\Lambda(M, \tau)) = L^\Lambda(Z(M), \tau_Z).$$

Proof. Using the equality

$$L_\phi(N, \tau_N) = S(N, \tau_N) \cap L_\phi(M, \tau).$$

we obtain that

$$L^\Lambda(N, \tau_N) = S(N, \tau_N) \cap L^\Lambda(M, \tau).$$

Hence

$$\begin{aligned} L^\Lambda(Z(M), \tau_Z) &= S(Z(M), \tau_Z) \cap L^\Lambda(M, \tau) = \\ &= Z(S(M, \tau)) \cap L^\Lambda(M, \tau) = Z(L^\Lambda(M, \tau)), \end{aligned}$$

i.e.

$$Z(L^\Lambda(M, \tau)) = L^\Lambda(Z(M), \tau_Z).$$

The proof is complete. ■

### 3. Derivations on the generalized Arens algebras

In this section we give a complete description of all additive derivations on the algebra  $L^\Lambda(M, \tau)$ .

Let  $\mathcal{A}$  be an algebra with the center  $Z(\mathcal{A})$  and let  $D : \mathcal{A} \rightarrow \mathcal{A}$  be an additive derivation. Given any  $x \in \mathcal{A}$  and a central element  $a \in Z(\mathcal{A})$  we have

$$D(ax) = D(a)x + aD(x)$$

and

$$D(xa) = D(x)a + xD(a).$$

Since  $ax = xa$  and  $aD(x) = D(x)a$ , it follows that  $D(a)x = xD(a)$  for any  $a \in \mathcal{A}$ . This means that  $D(a) \in Z(\mathcal{A})$ , i.e.  $D(Z(\mathcal{A})) \subseteq Z(\mathcal{A})$ . Therefore given any additive derivation  $D$  on the algebra  $\mathcal{A}$  we can consider its restriction  $\delta : Z(\mathcal{A}) \rightarrow Z(\mathcal{A})$ .

We shall need some facts about additive derivations  $\delta : \mathbb{C} \rightarrow \mathbb{C}$ . Every such derivation vanishes at every algebraic number. On the other hand, if  $\lambda \in \mathbb{C}$  is transcendental then there is a additive derivation  $\delta : \mathbb{C} \rightarrow \mathbb{C}$  which does not vanish at  $\lambda$  (see [20]).

Let  $M_n(\mathbb{C})$  be the algebra of  $n \times n$  matrices over  $\mathbb{C}$ . If  $e_{i,j}$ ,  $i, j = \overline{1, n}$ , are the matrix units in  $M_n(\mathbb{C})$ , then each element  $x \in M_n(\mathbb{C})$  has the form

$$x = \sum_{i,j=1}^n \lambda_{ij} e_{ij}, \quad \lambda_{i,j} \in \mathbb{C}, \quad i, j = \overline{1, n}.$$

Let  $\delta : \mathbb{C} \rightarrow \mathbb{C}$  be an additive derivation. Setting

$$D_\delta \left( \sum_{i,j=1}^n \lambda_{ij} e_{ij} \right) = \sum_{i,j=1}^n \delta(\lambda_{ij}) e_{ij} \quad (3)$$

we obtain a well-defined additive operator  $D_\delta$  on the algebra  $M_n(\mathbb{C})$ . Moreover  $D_\delta$  is an additive derivation on the algebra  $M_n(\mathbb{C})$  and its restriction onto the center of the algebra  $M_n(\mathbb{C})$  coincides with the given  $\delta$ .

It is known [21, Theorem 2.2] that if  $M$  be a von Neumann factor of type  $I_n$ ,  $n \in \mathbb{N}$  then every additive derivation  $D$  on the algebra  $M$  can be uniquely represented as a sum

$$D = D_a + D_\delta,$$

where  $D_a$  is an inner derivation implemented by an element  $a \in M$  while  $D_\delta$  is the additive derivation of the form (3) generated by an additive derivation  $\delta$  on the center of  $M$  identified with  $\mathbb{C}$ .

Note that if  $M$  is a finite-dimensional von Neumann algebra then  $L^\Lambda(M, \tau) = M$  for any faithful normal finite trace  $\tau$ .

Now let  $M$  be an arbitrary finite-dimensional von Neumann algebra with the center  $Z(M)$ . There exist a family of mutually orthogonal central projections  $\{z_1, z_2, \dots, z_k\}$  from  $M$  with  $\bigvee_{i=1}^k z_i = \mathbf{1}$  and  $n_1, n_2, \dots, n_k \in \mathbb{N}$  such that the algebra  $M$  is \*-isomorphic with the  $C^*$ -product of von Neumann factors  $z_i M$  of type  $I_{n_i}$  respectively, i.e.

$$M \cong M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \dots \oplus M_{n_k}(\mathbb{C}).$$

Suppose that  $D$  is an additive derivation on  $M$ , and  $\delta$  is its restriction onto its center  $Z(M)$ . Since  $\delta(zx) = z\delta(x)$  for all central projection  $z \in Z(M)$  and  $x \in M$  then  $\delta$  maps each  $z_i Z(M) \cong \mathbb{C}$  into itself,  $\delta$  generates an additive derivation  $\delta_i$  on  $\mathbb{C}$  for each  $i = \overline{1, k}$ .

Let  $D_{\delta_i}$  be the additive derivation on the matrix algebra  $M_{n_i}(\mathbb{C})$ ,  $i = \overline{1, k}$ , defined as in (3). Put

$$D_\delta((x_i)_{i=1}^k) = (D_{\delta_i}(x_i)), (x_i)_{i=1}^k \in M. \quad (4)$$

Then the map  $D_\delta$  is an additive derivation on  $M$ .

**Lemma 3.1.** *Let  $M$  be a finite-dimensional von Neumann algebra. Each additive derivation  $D$  on the algebra  $M$  can be uniquely represented in the form*

$$D = D_a + D_\delta,$$

where  $D_a$  is an inner derivation implemented by an element  $a \in M$ , and  $D_\delta$  is an additive derivation given as (4).



Proof. Let  $D$  be an additive derivation on  $M$ , and let  $\delta$  be its restriction onto  $Z(M)$ . Consider an additive derivation  $D_\delta$  on  $Z(M)$  of the form (4), generated by an additive derivation  $\delta$ . Since an additive derivations  $D$  and  $D_\delta$  coincides on  $Z(M)$  then an additive derivation  $D - D_\delta$  is a linear derivation. Hence by Sakai's theorem [19, Theorem 4.1.6]  $D - D_\delta$  is an inner derivation. This means that there exists an element  $a \in M$  such that  $D_a = D - D_\delta$  and therefore  $D = D_a + D_\delta$ . The proof is complete. ■

Now let  $M$  be a commutative von Neumann algebra with a faithful normal finite trace  $\tau$ . Given an arbitrary additive derivation  $\delta$  on  $L^\Lambda(M, \tau)$  the element

$$z_\delta = \inf\{z \in P(M) : z\delta = \delta\}$$

is called the support of the derivation  $\delta$ .

Suppose that  $M$  is a commutative von Neumann algebra with a faithful normal finite trace  $\tau$  and  $q_1, q_2, \dots, q_k$  are atoms in  $M$ . Then

$$L^\Lambda(M, \tau) \cong q_1\mathbb{C} \oplus q_2\mathbb{C} \oplus \dots \oplus q_k\mathbb{C} \oplus pL^\Lambda(M, \tau),$$

where  $p = \mathbf{1} - \bigvee_{i=1}^k q_i$ .

Now if  $\delta_i : \mathbb{C} \rightarrow \mathbb{C}$  is an additive derivation then

$$\delta(x) = (\delta_1(q_1x), \dots, \delta_k(q_kx), 0), \quad x \in L^\Lambda(M, \tau) \tag{5}$$

is also an additive derivation. Note that  $z_\delta = \bigvee\{q_i : \delta_i \neq 0, 1 \leq i \leq k\}$ .

**Lemma 3.2.** *Let  $M$  be a commutative von Neumann algebra with a faithful normal finite trace  $\tau$ . For any non trivial additive derivation  $\delta : L^\Lambda(M, \tau) \rightarrow L^\Lambda(M, \tau)$  there exists a sequence  $\{a_n\}_{n=1}^\infty$  in  $M$  with  $|a_n| \leq \mathbf{1}$ ,  $n \in \mathbb{N}$ , such that*

$$|\delta(a_n)| \geq nz_\delta$$

for all  $n \in \mathbb{N}$ .

In [5, Lemma 2.6] (see also [11, Lemma 4.6]) this assertion was proved for linear derivations on the algebra  $S(M)$ , but the proof is the same for additive derivations on  $L^\Lambda(M, \tau)$ .

The following result shows that the above construction (5) is the general form of additive derivations on the generalized Arens algebras in the commutative case.

**Lemma 3.3.** *Let  $M$  be a commutative von Neumann algebra with a faithful normal finite trace  $\tau$  and let  $\delta$  be an additive derivation on the algebra  $L^\Lambda(M, \tau)$ . Then  $z_\delta M$  is a finite-dimensional algebra.*

Proof. Suppose that  $z_\delta M$  is infinite-dimensional. Then there exists a sequence of mutually orthogonal projections  $\{z_n\}_{n=1}^\infty$  in  $M$  such that  $\bigvee_{n=1}^\infty z_n = z_\delta$ . By Lemma 3.2 there exists a sequence  $\{a_n\}_{n=1}^\infty$  in  $M$  with  $|a_n| \leq \mathbf{1}$ ,  $n \in \mathbb{N}$ , such that

$$|\delta(a_n)| \geq 2^n \tau(z_n)^{-1} z_\delta \quad (6)$$

for all  $n \in \mathbb{N}$ . Put

$$a = \sum_{n=1}^{\infty} \frac{a_n z_n}{2^n}.$$

Then  $a \in M \subset L^\Lambda(M, \tau)$  and

$$\delta(a) = \delta \left( \sum_{n=1}^{\infty} \frac{a_n z_n}{2^n} \right) = \sum_{n=1}^{\infty} \frac{z_n}{2^n} \delta(a_n).$$

From (6) we obtain that

$$|\delta(a)| = \sum_{n=1}^{\infty} \frac{z_n}{2^n} |\delta(a_n)| \geq \sum_{n=1}^{\infty} \frac{z_n}{2^n} 2^n \tau(z_n)^{-1} z_\delta,$$

i.e.

$$|\delta(a)| \geq \sum_{n=1}^{\infty} \tau(z_n)^{-1} z_n.$$

Thus

$$\tau(|\delta(a)|) \geq \sum_{n=1}^{\infty} \tau(z_n)^{-1} \tau(z_n) = \sum_{n=1}^{\infty} 1 = \infty.$$

This means that  $\delta(a) \notin L^1(M, \tau)$ . Then by (2) we have that  $\delta(a) \notin L^\Lambda(M, \tau)$ . This contradiction implies that  $z_\delta M$  is a finite-dimensional algebra. The proof is complete.  $\blacksquare$

Lemma 3.3 implies the following

**Corollary 3.1.** *Let  $M$  be a commutative von Neumann algebra with a faithful normal finite trace  $\tau$  such that a Boolean algebra  $P(M)$  of all projections of  $M$  is continuous. Then every additive derivation on the algebra  $L^\Lambda(M, \tau)$  is zero.*

Note that properties of additive derivations on the algebras  $S(M, \tau)$  and  $L^\Lambda(M, \tau)$ , where  $M$  be a commutative von Neumann algebra with a faithful normal finite trace  $\tau$ , are quite opposite. Indeed, if the Boolean algebra  $P(M)$  is continuous then the algebra

$S(M, \tau)$  admits a non-zero linear, in particular additive, derivation, (see [10, Theorem 3.3]), whereas the algebra  $L^\Lambda(M, \tau)$  in this case does not admit a non-zero additive derivation (see Corollary 3.1).

Now we consider the noncommutative case.

We shall need following result ([7, Theorem 4.1], see also [9, Theorem 6.8]).

**Theorem 3.1.** *Let  $M$  be a von Neumann algebra with a faithful normal finite trace  $\tau$ . If  $A \subseteq L^\omega(M, \tau)$  is a solid  $*$ -subalgebra such that  $M \subseteq A$ , then every linear derivation on  $A$  is inner.*

The following theorem is one of the main results of this paper.

**Theorem 3.2.** *Let  $M$  be a type II von Neumann algebra with a faithful normal finite trace  $\tau$ . Then every additive derivation on the algebra  $L^\Lambda(M, \tau)$  is inner.*

The proof of the theorem 3.2 follows from Theorem 3.1 and the following assertion.

**Lemma 3.4.** *Let  $M$  be a type II von Neumann algebra with a faithful normal finite trace  $\tau$ , and suppose that  $D : L^\Lambda(M, \tau) \rightarrow L^\Lambda(M, \tau)$  is an additive derivation. Then  $D|_{Z(L^\Lambda(M, \tau))} \equiv 0$ , in particular,  $D$  is a linear.*

Proof. Let  $D$  be an additive derivation on  $L^\Lambda(M, \tau)$ , and let  $\delta$  be its restriction onto  $Z(L^\Lambda(M, \tau))$ .

Since  $M$  is of type II there exists a sequence of mutually orthogonal projections  $\{p_n\}_{n=1}^\infty$  in  $M$  with central covers  $\mathbf{1}$  (i.e. faithful projections). For any bounded sequence  $B = \{b_n\}_{n \in \mathbb{N}}$  in  $Z(M)$  define an operator  $x_B$  by

$$x_B = \sum_{n=1}^{\infty} b_n p_n.$$

Then

$$x_B p_n = p_n x_B = b_n p_n \tag{7}$$

for all  $n \in \mathbb{N}$ .

Take  $b \in Z(M)$  and  $n \in \mathbb{N}$ . From the identity

$$D(bp_n) = D(b)p_n + bD(p_n)$$

multiplying it by  $p_n$  from the both sides we obtain

$$p_n D(bp_n) p_n = p_n D(b) p_n + b p_n D(p_n) p_n.$$

Since  $p_n$  is a projection, one has that  $p_n D(p_n) p_n = 0$ , and since  $D(b) = \delta(b) \in Z(L^\Lambda(M, \tau))$ , we have

$$p_n D(b p_n) p_n = \delta(b) p_n. \quad (8)$$

Now from the identity

$$D(x_B p_n) = D(x_B) p_n + x_B D(p_n),$$

in view of (7) one has similarly

$$p_n D(b_n p_n) p_n = p_n D(x_B) p_n + b_n p_n D(p_n) p_n,$$

i.e.

$$p_n D(b_n p_n) p_n = p_n D(x_B) p_n. \quad (9)$$

Now (8) and (9) imply

$$p_n D(x_B) p_n = \delta(b_n) p_n. \quad (10)$$

Let  $\varphi \in \Lambda$ . By (1) there are  $\phi, \psi \in \Lambda$  and  $k > 0$  such that

$$\|x_1 x_2 x_3\|_\varphi \leq k \|x_1\|_\phi \|x_2\|_\phi \|x_3\|_\psi$$

for all  $x_1, x_2, x_3 \in L^\Lambda(M, \tau)$ . If we suppose that  $\delta \neq 0$  then  $z_\delta \neq 0$ . By Lemma 3.2 there exists a bounded sequence  $B = \{b_n\}_{n \in \mathbb{N}}$  in  $Z(M)$  such that

$$|\delta(b_n)| \geq n c_n z_\delta$$

for all  $n \in \mathbb{N}$ , where  $c_n = k \|p_n\|_\phi^2 \|p_n z_\delta\|_\varphi^{-1}$ . Then in view of (10) we obtain

$$\begin{aligned} k \|p_n\|_\phi \|D(x)\|_\psi \|p_n\|_\phi &\geq \|p_n D(x) p_n\|_\varphi = \\ &= \|\delta(b_n) p_n\|_\varphi \geq \|n c_n p_n z_\delta\|_\varphi = n c_n \|p_n z_\delta\|_\varphi, \end{aligned}$$

i.e.

$$\|D(x)\|_\psi \geq n c_n k^{-1} \|p_n\|_\phi^{-2} \|p_n z_\delta\|_\varphi.$$

Hence

$$\|D(x)\|_\psi \geq n$$

for all  $n \in \mathbb{N}$ . This contradiction implies that  $\delta \equiv 0$ , i.e.  $D$  is identically zero on the center of  $L^\Lambda(M, \tau)$ , and therefore it is linear. The proof is complete.  $\blacksquare$

Now consider an additive derivation  $D$  on  $L^\Lambda(M, \tau)$  and let  $\delta$  be its restriction onto its center  $Z(L^\Lambda(M, \tau))$ . By Lemma 3.3  $z_\delta M$  is a finite-dimensional and  $z_\delta^\perp \delta \equiv 0$ , i.e.  $\delta = z_\delta \delta$ .

Let  $D_\delta$  be the derivation on  $z_\delta L^\Lambda(M, \tau) = z_\delta M$  defined as in (4) and consider its extension  $D_\delta$  on  $L^\Lambda(M, \tau) = z_\delta L^\Lambda(M, \tau) \oplus z_\delta^\perp L^\Lambda(M, \tau)$  which is defined as

$$D_\delta(x_1 + x_2) := D_\delta(x_1), \quad x_1 \in z_\delta L^\Lambda(M, \tau), x_2 \in z_\delta^\perp L^\Lambda(M, \tau). \quad (11)$$

The following theorem is the main result of this paper, and gives the general form of derivations on the algebra  $L^\Lambda(M, \tau)$ .

**Theorem 3.3.** *Let  $M$  be a von Neumann algebra with faithful normal finite trace  $\tau$ . Each additive derivation  $D$  on  $L^\Lambda(M, \tau)$  can be uniquely represented in the form*

$$D = D_a + D_\delta$$

where  $D_a$  is an inner derivation implemented by an element  $a \in L^\Lambda(M, \tau)$ , and  $D_\delta$  is an additive derivation of the form (11), generated by an additive derivation  $\delta$  on the center of  $L^\Lambda(M, \tau)$ .

*Proof.* Let  $D$  be an additive derivation on  $L^\Lambda(M, \tau)$ , and let  $\delta$  be its restriction onto  $Z(L^\Lambda(M, \tau)) = L^\Lambda(Z(M), \tau_Z)$ . By Lemma 3.3  $z_\delta Z(M)$  is a finite-dimensional. Thus  $z_\delta M$  is a  $C^*$ -product of finite number of von Neumann factors of type  $I_n$  or  $II$ . Since by Lemma 3.4 any additive derivation on  $L^\Lambda(M, \tau)$ , where  $M$  is a type  $II$  algebra, is linear then by Theorem 3.2 it is inner. Therefore  $z_\delta M$  is a  $C^*$ -product of finite number of von Neumann factors of type  $I_n$ .

Now consider an additive derivation  $D_\delta$  on  $L^\Lambda(M, \tau)$  of the form (11), generated by a derivation  $\delta$ . Since the derivations  $D$  and  $D_\delta$  coincide on  $L^\Lambda(Z(M), \tau)$  then  $D - D_\delta$  is a linear derivation. Hence Theorem 3.2 implies that the derivation  $D - D_\delta$  is inner. This means that there exists an element  $a \in L^\Lambda(M, \tau)$  such that  $D_a = D - D_\delta$  and therefore  $D = D_a + D_\delta$ . The proof is complete. ■

**Corollary 3.2.** *Let  $M$  be a von Neumann algebra without type  $I_n$  direct summands and with a faithful normal finite trace  $\tau$ . Then each additive derivation on  $L^\Lambda(M, \tau)$  is inner.*

**Acknowledgments.** *The third named author would like to acknowledge the hospitality of the "Institut für Angewandte Mathematik", Universität Bonn (Germany). This work is supported in part by the German Academic Exchange Service - DAAD .*

## References

- [1] Abdullaev R.Z., The dual space for Arens algebra, *Uzbek. Math. Journal.* 2 (1997) 3–7.
- [2] Albeverio S., Ayupov Sh.A., Kudaybergenov K.K., Non commutative Arens algebras and their derivations, *J. Funct. Anal.* 253 (2007) 287–302.
- [3] Albeverio S., Ayupov Sh. A., Kudaybergenov K. K., Derivations on the algebra of measurable operators affiliated with a type I von Neumann algebra, *Siberian Adv. Math.* 18 (2008) 86–94.
- [4] Albeverio S., Ayupov Sh. A., Kudaybergenov K. K., Derivations on the algebra of  $\tau$ -compact operators affiliated with a type I von Neumann algebra, *Positivity.* 12 (2008) 375–386.
- [5] Albeverio S., Ayupov Sh. A., Kudaybergenov K. K., Structure of derivations on various algebras of measurable operators for type I von Neumann algebras, *J. Funct. Anal.* 256 (2009) 2917–2943.
- [6] Albeverio S., Ayupov Sh. A., Kudaybergenov K. K., Description of Derivations on Locally Measurable Operator Algebras of Type I, *Extracta Math.* 24 (2009) 1–15.
- [7] Ayupov Sh. A., Abdullaev R. Z., Kudaybergenov K. K., On a certain class of algebras and their derivations, ICTP, Preprint, No IC/2009/058, - Trieste, 2009. - 13 p. arXiv:0908.1203 (2009).
- [8] Ayupov Sh. A., Kudaybergenov K. K., Additive derivations on algebras of measurable operators, ICTP, Preprint, No IC/2009/059, - Trieste, 2009. - 16 p. arXiv:0908.1202 (2009).
- [9] Ayupov Sh. A., Kudaybergenov K. K., Derivations on algebras of measurable operators, *Infinite Dimensional Analysis, Quantum Probability and Related Topics.* 13 (2010) 305–337.
- [10] Ber A.F., Chilin V.I., Sukochev F.A., Non-trivial derivation on commutative regular algebras, *Extracta Math.* 21 (2006) 107–147.

- [11] Ber A. F., de Pagter B., Sukochev F. A., Derivations in algebras of operator-valued functions, *arXiv.math.OA.0811.0902*. 2008 (to appear in *J. Operator Theory*).
- [12] Inoue A., On a class of unbounded operators II, *Pacific J. Math.* 66 (1976) 411431.
- [13] Gutman A.E., Kusraev A.G., Kutateladze S.S., The Wickstead problem, *Siberian Electronic Math. Reports.* 5 (2008) 293–333.
- [14] Krasnosel'skii M.A., Rutickii Ya.B., Convex functions and Orlicz spaces, P. Noordhoff Ltd, Groningen, 1961.
- [15] Kunze W., Noncommutative Orlicz spaces and generalized Arens algebras, *Math. Nachr.* 147 (1990) 123–138.
- [16] Kusraev A.G., Automorphisms and derivations on a universally complete complex  $f$ -algebra, *Siberian Math. J.* 47 (2006) 77–85.
- [17] Muratov M.A., The Luxemburg norm in Orlicz spaces, *Dokl. Akad. Nauk UzSSR* 6 (1978) 11–13.
- [18] Muratov M.A., Chilin V.I., Algebras of measurable and locally measurable operators, Institute of Mathematics Ukrainian Academy of Sciences, Kiev 2007.
- [19] Sakai S.,  $C^*$ -algebras and  $W^*$ -algebras. Springer-Verlag, 1971.
- [20] Samuel P., Zariski O., Commutative algebra. Van Nostrand, New York, 1958.
- [21] Semrl P., Additive derivations on some operator algebras, *Illinois J. Math.* 35 (1991) 234–240.