TIME INHOMOGENEOUS GENERALIZED MEHLER SEMIGROUPS AND SKEW CONVOLUTION EQUATIONS

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ABSTRACT. A time inhomogeneous generalized Mehler semigroup on a real separable Hilbert space $\mathbb{H}$ is defined through

$$p_{s,t}f(x) = \int_{\mathbb{H}} f(U(t,s)x + y)\, \mu_{t,s}(dy), \quad s,t \in \mathbb{R},\ t \geq s,\ x \in \mathbb{H},$$

for every bounded measurable function f on $\mathbb{H}$, where $(U(t,s))_{t\geq s}$ is an evolution family of bounded operators on $\mathbb{H}$ and $(\mu_{t,s})_{t\geq s}$ is a family of probability measures on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$ satisfying the following time inhomogeneous skew convolution equations

$$\mu_{t,s} = \mu_{t,r} * \left(\mu_{r,s} \circ U(t,r)^{-1}\right), \quad t \geq r \geq s.$$

This kind of semigroups typically arise as the "transition semigroups" of non-autonomous (possibly non-continuous) Ornstein-Uhlenbeck processes driven by some proper additive process. Suppose that $\mu_{t,s}$ converges weakly to δ_0 as $t \downarrow s$ or $s \uparrow t$. We show that $\mu_{t,s}$ has further weak continuity properties in t and s. As a consequence, we prove that for every $t \geq s$, $\mu_{t,s}$ is infinitely divisible. Natural stochastic processes associated with $(\mu_{t,s})_{t\geq s}$ are constructed and are applied to get probabilistic proofs for the weak continuity and infinite divisibility. Then we analyze the structure, existence and uniqueness of the corresponding evolution systems of measures (=space-time invariant measures) of $(p_{s,t})_{t\geq s}$. We also establish a dimension free Harnack inequality for $(p_{s,t})_{t\geq s}$ and present some of its applications.

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1. Introduction

We start the introduction of time inhomogeneous generalized Mehler semigroups and skew convolution equations from the well studied time homogeneous case.

Let $\mathbb{H}$ be a real separable Hilbert space with norm and inner product denoted by $|\cdot|$ and $\langle\cdot,\cdot\rangle$ respectively. Let $\mathscr{B}(\mathbb{H})$ be the space of Borel measurable subsets of $\mathbb{H}$, and let $B_b(\mathbb{H})$ be the space of all bounded Borel measurable functions on $\mathbb{H}$.

A time homogeneous generalized Mehler semigroup $(p_t)_{t\geq 0}$ on $\mathbb{H}$ is defined by

$$p_t f(x) = \int_{\mathbb{H}} f(T_t x + y)\,\mu_t(dy), \quad t\geq 0,\ x\in\mathbb{H},\ f\in B_b(\mathbb{H}). \tag{1.1}$$

Here $(T_t)_{t\geq 0}$ is a strongly continuous semigroup on $\mathbb{H}$ and $(\mu_t)_{t\geq 0}$ is a family of probability measures on $(\mathbb{H},\mathscr{B}(\mathbb{H}))$ satisfying the following skew convolution (semigroup) equation

$$\mu_{t+s} = \mu_s * (\mu_t\circ T_s^{-1}), \quad s,t\geq 0. \tag{1.2}$$

Recall that for any two positive Borel measures μ and ν on $\mathbb{H}$, the convolution $\mu*\nu$ of μ and ν is a Borel measure on $(\mathbb{H},\mathscr{B}(\mathbb{H}))$ such that

$$\mu*\nu(B) := \int_{\mathbb{H}}\int_{\mathbb{H}} \mathbb{1}_B(x+y)\,\mu(dx)\nu(dy) = \int_{\mathbb{H}} \mu(B-x)\,\nu(dx), \quad B\in\mathscr{B}(\mathbb{H}).$$

Condition (1.2) is necessary and sufficient for the semigroup property of $(p_t)_{t\geq 0}$ (and the Markov property of the corresponding stochastic process respectively) to hold. That is (see [22]),

(1.2) holds if and only if for all $t,s\geq 0,\ p_t p_s = p_{t+s}$ on $B_b(\mathbb{H})$.

The semigroup (1.1) is a generalization of the classical Mehler formula for the transition semigroup of an Ornstein-Uhlenbeck process driven by a Wiener process. The second named author and his coauthors studied this generalization for the Gaussian case in [8, 9] as well as the non-Gaussian case in [22]. Indeed, under some mild conditions there is a one-to-one correspondence between generalized Mehler semigroups and transition semigroups of Ornstein-Uhlenbeck processes driven by Lévy processes (cf. [9, 22]).

Generalized Mehler semigroups and skew convolution equations have been extensively studied. For instance, Schmuland and Sun [51] investigated the infinite divisibility of μ_t ($t\geq 0$) and the continuity of $t\mapsto \log\hat{\mu}_t$ (here $\hat{\mu}_t$ denotes the Fourier transformation of μ_t, see also (2.11)); Lescot and Röckner considered in [33] and [34] the generator and perturbations of $(p_t)_{t\geq 0}$ respectively; Wang and Röckner established some useful functional inequalities for $(p_t)_{t\geq 0}$; van Neerven [53], Li and his coauthors [18, 19] studied the representation of $\hat{\mu}_t$ ($t\geq 0$). For more literature on this topic we refer to [2, 3, 4, 30, 36] and the references therein.

Recently, much work, for instance [13, 14, 23, 32, 57], has been devoted to the study of non-autonomous Ornstein-Uhlenbeck processes which are solutions to linear stochastic partial differential equations (SPDE) with time-dependent drifts. The noise in these equations is modeled by a stationary process such as a Wiener process or Lévy process. To get fully inhomogeneous Ornstein-Uhlenbeck processes, it is natural to consider more general noise modeled by (time) non-stationary processes such as additive processes.

Let $(A(t), \mathscr{D}(A(t)))_{t\in\mathbb{R}}$ be a family of linear operators on the space $\mathbb{H}$ with dense domains. Suppose that the non-autonomous Cauchy problem

$$\begin{cases} dx_t = A(t)x_t dt, \quad t \geq s, \\ x_s = x \end{cases}$$

is well posed (cf. [44]). That is, there exists an evolution family of bounded operators $(U(t,s))_{t\geq s}$ on $\mathbb{H}$ associated with $A(t)$ such that for every $x \in \mathscr{D}(A(s))$, $x(t) = U(t,s)x$ is a unique classical solution of this Cauchy problem.

Typical examples for $A(t)$, $t \in \mathbb{R}$, are partial differential operators e.g. of divergence type on $\mathbb{H} = L^2(U)$ with U an open bounded domain in $\mathbb{R}^d$. So, for example

$$A(t) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right),$$

where $a_{i,j} \colon U \times \mathbb{R} \to \mathbb{R}$ are bounded measurable functions such that for some $c \in (0,\infty)$,

$$\sum_{i,j=1}^{d} a_{ij}(x,t)\xi_i\xi_j \geq c|\xi|^2_{\mathbb{R}^d}, \quad \xi \in \mathbb{R}^d.$$

For details we refer to [44] and [32].

A family of bounded linear operators $(U(t,s))_{t\geq s}$ on $\mathbb{H}$ is said to be a (strongly continuous) evolution family if:

(1) For every $s \in \mathbb{R}$, $U(s,s)$ is the identity operator and for all $t \geq r \geq s$,

$$U(t,r)U(r,s) = U(t,s).$$

(2) For every $x \in \mathbb{H}$, the map $(t,s) \mapsto U(t,s)x$ is strongly continuous on $\{(t,s) \in \mathbb{R}^2 \colon t \geq s\}$.

An evolution family is also called evolution system, propagator etc.. For more details we refer e.g. to [17, 21, 44].

Let $(Z_t)_{t\in\mathbb{R}}$ be an additive process taking values in $\mathbb{H}$, i.e. an $\mathbb{H}$-valued stochastically continuous stochastic process with independent increments. We shall consider stochastic integrals $\int_s^t \Phi(r)dZ_r$ on $[s,t]$ of non-random functions $\Phi(t)$, $t \in \mathbb{R}$, which takes values in the space of all linear operators on $\mathbb{H}$.

A direct way to define the stochastic integral is to regard it as the limit (e.g. in the sense of convergence in probability) of "Riemann sums". We refer to [38] to get some idea. For a more general definition (using so called independently scattered random measures), we refer to [46] for the infinite dimensional case and [48, 49] for the finite dimensional case. A work [41] is in preparation to extend the main results in [48, 49] by the first named author.

Consider the following stochastic differential equation

$$\begin{cases} dX_t = A(t)X_t dt + dZ_t, \\ X_s = x. \end{cases} \tag{1.3}$$

Suppose that $U(t,\cdot)$ is integrable on $[s,t]$ for every $s \le t$ with respect to Z_t. Hence the following stochastic convolution integrals

$$X_{t,s} := \int_s^t U(t,\sigma)\, dZ_\sigma, \quad t \ge s, \tag{1.4}$$

are well defined. Then

$$X(t,s,x) = U(t,s)x + \int_s^t U(t,r)\, dZ_r, \quad t \ge s,\ x \in \mathbb{H} \tag{1.5}$$

is called the mild solution of (1.3). For all $t \ge s$, let $\bar{\mu}_{t,s}$ denote the distribution of $X_{t,s}$. Obviously the transition semigroup of $X(t,s,x)$ is given by

$$P_{s,t}f(x) = \mathbb{E}f(X(t,s,x)) = \int_{\mathbb{H}} f\left(U(t,s)x + y\right) \bar{\mu}_{t,s}(dy) \tag{1.6}$$

for all $x \in \mathbb{H}$, $f \in B_b(\mathbb{H})$ and $t \ge s$. It is easy to check that $(\bar{\mu}_{t,s})_{t\ge s}$ satisfies the following time inhomogeneous skew convolution equation

$$\bar{\mu}_{t,s} = \bar{\mu}_{t,r} * \left(\bar{\mu}_{r,s} \circ U(t,r)^{-1}\right), \quad t \ge r \ge s$$

The aim of the present paper is to adopt the axiomatic approach in [9] and [22] to study the non-autonomous process (1.5) through its transition semigroup (1.6). That is, we shall start from (1.6) and define for a given strongly continuous evolution family $(U(t,s))_{t\ge s}$,

$$p_{s,t}f(x) := \int_{\mathbb{H}} f(U(t,s)x + y)\, \mu_{t,s}(dy), \quad x \in \mathbb{H},\ f \in B_b(\mathbb{H}),\ t \ge s. \tag{1.7}$$

Here $(\mu_{t,s})_{t\ge s}$ is a family of probability measures on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$. In order that the family $(p_{s,t})_{t\ge s}$ satisfies the Chapman-Kolmogorov equations (flow property), we shall assume that $(\mu_{t,s})_{t\ge s}$ satisfies the flowing time inhomogeneous skew convolution equations

$$\mu_{t,s} = \mu_{t,r} * \left(\mu_{r,s} \circ U(t,r)^{-1}\right), \quad t \ge r \ge s. \tag{1.8}$$

In this case we call $(p_{s,t})_{t\ge s}$ a time inhomogeneous generalized Mehler semigroup. Clearly, (1.7) and (1.8) are time inhomogeneous analogs of (1.1) and (1.2).

We would like to point out that $(p_{s,t})_{t\ge s}$ is of course not a "transition semigroup" although we call it (time inhomogeneous generalized Mehler) semigroup. In this two parameter case, some authors call it hemigroup, see for example [6, 25] and [10, 27] where (1.7) and (1.8) have been studied respectively in some special or different situations. So, one may also call (1.7) a (time inhomogeneous) generalized Mehler hemigroup. Similarly, one may call (1.8) a (time inhomogeneous) skew convolution hemigroup.

As a toy example one may consider the following time inhomogeneous one dimensional stochastic equation

$$\begin{cases} dx_t = a(t)x_t dt + db_t, \\ x_s = x, \end{cases} \tag{1.9}$$

where $(b_t)_{t\in\mathbb{R}}$ is a standard one-dimensional Brownian motion in $\mathbb{R}$ and $a(t)$ is a continuous function on $\mathbb{R}$. Then $(\exp(\int_s^t a(r)\, dr))_{t\ge s}$ is the evolution family associated with $a(t)$. The mild solution of (1.9) is given by

$$x(t,s,x) = \exp\left(\int_s^t a(r)\, dr\right) x + \int_s^t \exp\left(\int_u^t a(r)\, dr\right) db_u, \quad t \ge s,\ x \in \mathbb{H}. \tag{1.10}$$

Clearly

$$\int_s^t \exp\left(\int_u^t a(r)\,dr\right)\,db_u \sim N(0,\sigma_{t,s}),$$

i.e. a Gaussian measure with mean 0 and variance $\sigma_{t,s}$ given by

$$\sigma_{t,s} = \int_s^t \exp\left(2\int_u^t a(r)\,dr\right)d\,u.$$

It is easy to check that (cf. (3.52))

$$\sigma_{t,s} = \sigma_{t,r} + \sigma_{r,s}\cdot\exp\left(2\int_r^t a(r)\,dr\right), \quad t \geq r \geq s.$$

This means that (1.8) holds. The corresponding time inhomogeneous generalized Mehler semigroup is given by

$$p_{s,t}f(x) := \int_{\mathbb{H}} f\left(\exp\left(\int_s^t a(r)\,dr\right)x + y\right) N(0,\sigma_{t,s})(dy), \quad x \in \mathbb{H},\ f \in B_b(\mathbb{H}),\ t \geq s.$$

For further examples we refer to Section 5, where we consider time inhomogeneous SPDEs driven by Wiener processes in infinite dimension (in particular see Example 5.3). For corresponding examples, where the Wiener process is replaced by a Lévy process we refer e.g. to [32].

The present paper is organized as follows.

In Section 2 the main result is Proposition 2.1, which shows that the flow property for $(p_{s,t})_{t\geq s}$ holds if and only if we have (1.8) for $(\mu_{t,s})_{t\geq s}$.

In Section 3 we concentrate on the skew convolution equations (1.8). In Subsection 3.1 we give some preliminaries and motivations. In Subsection 3.2 we introduce Assumption 3.7 and show some results on the weak continuity of $\mu_{t,s}$ in t and s. In Subsection 3.3, we prove that for every $t \geq s$, $\mu_{t,s}$ is infinitely divisible. In Subsection 3.4 we show that there exists a natural stochastic process associated with $(\mu_{t,s})_{t\geq s}$. This allows us to get probabilistic proofs of the results on the weak continuity and infinite divisibility of $\mu_{t,s}$ proved in previous subsections. We shall study the spectral representation of $\mu_{t,s}$ in another work.

In Section 4 we study evolution systems of measures (i.e. space-time invariant measures) for the semigroup $(p_{s,t})_{t\geq s}$. We first show some basic properties of the evolution systems of measures. Then we give sufficient and necessary conditions for the existence and uniqueness of evolution systems of measures.

In Section 5 we prove a (dimension independent) Harnack inequality for $(p_{s,t})_{t\geq s}$ using a simple argument. As applications of the Harnack inequality, we prove that null controllability implies the strong Feller property and that for the Gaussian case, null controllability, Harnack inequality and strong Feller property are in fact equivalent to each other as in the time homogeneous case.

The semigroup $(p_{s,t})_{t\geq s}$ is called strongly Feller if for every bounded measurable function f and every $t \geq s$, $p_{s,t}f$ is a bounded continuous function. In the time homogeneous case, the strong Feller property has been investigated frequently. One of the reasons is that if a transition semigroup is strong Feller and irreducible, then it has a unique invariant measure, see e.g. [11].

As an application of the Harnack inequality, we e.g. look at the hypercontractivity of $(p_{s,t})_{t\geq s}$. Hypercontractivity of semigroup such as (1.6) is closely related to functional inequalities, spectral theory etc. for the Kolmogorov operator (or generator) corresponding to SDE (1.3), see e.g. [54].

In Section 6 we append a brief introduction to the control theory of non-autonomous linear control systems and null controllability. The minimal energy representation is useful for the estimate of the constant in the Harnack inequality obtained in Section 5.

2. Generalized Mehler semigroups and skew convolution equations

First we characterize when the Chapman-Kolmogorov equations hold for $(p_{s,t})_{t\geq s}$ in (1.7).

Proposition 2.1. *For all $s \leq r \leq t$,*

$$p_{s,r}p_{r,t} = p_{s,t} \quad \text{(“Chapman-Kolmogorov equations”)} \tag{2.1}$$

holds on $B_b(\mathbb{H})$ if and only if (1.8) *holds for all $s \leq r \leq t$.*

We shall prove this proposition in a more general framework. This is inspired by the fact that a generalized Mehler semigroup is a special case of the so called skew convolution semigroups (see [35]).

Let $(u_{s,t})_{t\geq s}$ be a family of Borel Markov transition functions on $\mathbb{H}$, i.e. each $u_{s,t}$ is a probability kernel on $\mathscr{B}(\mathbb{H})$ and the following Chapman-Kolmogorov equations

$$u_{s,t}f(x) = u_{s,r}(u_{r,t}f)(x), \quad x \in \mathbb{H},\ f \in B_b(\mathbb{H}) \tag{2.2}$$

hold for all $t \geq r \geq s$. Here $B_b(\mathbb{H})$ denote the space of all bounded Borel measurable functions on $\mathbb{H}$. Writing (2.2) in integral form, we have

$$\int_{\mathbb{H}^2} f(z)u_{r,t}(y,dz)u_{s,r}(x,dy) = \int_{\mathbb{H}} f(z)u_{s,t}(x,dz). \tag{2.3}$$

Assume that

$$u_{s,t}(x+y,\cdot) = u_{s,t}(x,\cdot) * u_{s,t}(y,\cdot) \tag{2.4}$$

for every $t \geq s$ and $x, y \in \mathbb{H}$. Clearly, Equation (2.4) implies

$$u_{s,t}(0,\cdot) = \delta_0. \tag{2.5}$$

For every probability measure μ on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$ we associate with $u_{s,t}$ $(t \geq s)$ a new probability measure $\mu u_{s,t}$ by

$$\mu u_{s,t}(A) = \int_{\mathbb{H}} u_{s,t}(x, A)\,\mu(dx), \quad A \in \mathscr{B}(\mathbb{H})$$

for every $t \geq s$.

Let $(\mu_{t,s})_{t\geq s}$ be a family of probability measures on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$. For all $t \geq s$, define a family of functions

$$q_{s,t}(\cdot,\cdot)\colon \mathbb{H} \times \mathscr{B}(\mathbb{H}) \to \mathbb{R}$$

by

$$q_{s,t}(x,\cdot) = u_{s,t}(x,\cdot) * \mu_{t,s}(\cdot), \quad x \in \mathbb{H}.$$

Associated with $q_{s,t}(\cdot,\cdot)$ we define an operator $q_{s,t}$ on $B(\mathbb{H})$ by

$$q_{s,t}f(x) = \int_{\mathbb{H}} f(y)\, q_{s,t}(x, dy), \quad x \in \mathbb{H},\ f \in B_b(\mathbb{H}).$$

We have the following characterization of the Chapman-Kolmogorov equations for $(q_{s,t})_{t\geq s}$.

Proposition 2.2. *The family of operators $(q_{s,t})_{t\geq s}$ satisfies*

$$q_{s,t} = q_{s,r}q_{r,t}, \quad t \geq r \geq s \tag{2.6}$$

if and only if

$$\mu_{t,s} = \mu_{t,r} * (\mu_{r,s}u_{r,t}), \quad t \geq r \geq s. \tag{2.7}$$

Proof. For every $f \in B_b(\mathbb{H})$, $x \in \mathbb{H}$, we have

$$\begin{aligned}
&q_{s,r}q_{r,t}f(x)\\
&= \int_{\mathbb{H}} q_{r,t}f(y)q_{s,r}(x, dy)\\
&= \int_{\mathbb{H}^2} q_{r,t}f(y_1 + y_2)u_{s,r}(x, dy_1)\mu_{r,s}(dy_2)\\
&= \int_{\mathbb{H}^4} f(z)q_{r,t}(y_1 + y_2, dz)u_{s,r}(x, dy_1)\mu_{r,s}(dy_2)\\
&= \int_{\mathbb{H}^4} f(z_1 + z_2)u_{r,t}(y_1 + y_2, dz_1)\mu_{t,r}(dz_2)u_{s,r}(x, dy_1)\mu_{r,s}(dy_2)\\
&= \int_{\mathbb{H}^5} f(z_{11} + z_{12} + z_2)u_{r,t}(y_1, dz_{11})u_{r,t}(y_2, dz_{12})\mu_{t,r}(dz_2)u_{s,r}(x, dy_1)\mu_{r,s}(dy_2)\\
&= \int_{\mathbb{H}^4} f(z_{11} + z_{12} + z_2)u_{s,t}(x, dz_{11})u_{r,t}(y_2, dz_{12})\mu_{t,r}(dz_2)\mu_{r,s}(dy_2)\\
&= \int_{\mathbb{H}^3} f(z_{11} + z_{12} + z_2)u_{s,t}(x, dz_{11})(\mu_{r,s}u_{r,t})(dz_{12})\mu_{t,r}(dz_2)\\
&= \int_{\mathbb{H}} f(z)(u_{s,t}(x,\cdot) * (\mu_{r,s}u_{r,t}) * \mu_{t,r})(dz).
\end{aligned} \tag{2.8}$$

In the calculation above we have used (2.3) and (2.4) to get the fifth and sixth identity respectively. If (2.7) holds, then by (2.8) we obtain

$$q_{s,r}q_{r,t}f(x) = \int_{\mathbb{H}} f(z)[u_{s,t}(x,\cdot) * \mu_{t,s}](dz) = q_{s,t}f(x).$$

That is, (2.6) holds.

Conversely, if (2.6) holds, then we have

$$q_{s,t}f(0) = q_{s,r}q_{r,t}f(0) \tag{2.9}$$

for all $f \in B_b(\mathbb{H})$. By taking $x = 0$ in (2.8) and using (2.5), we get

$$q_{s,r}q_{r,t}f(0) = \int_{\mathbb{H}} f(z)\left((\mu_{r,s}u_{r,t}) * \mu_{t,r}\right)(dz).$$

On the other hand, we have

$$q_{s,t}f(0) = \int_{\mathbb{H}} f(z)\mu_{t,s}(dz).$$

Hence

$$\int_{\mathbb{H}} f(z)\left((\mu_{r,s}u_{r,t}) * \mu_{t,r}\right)(dz) = \int_{\mathbb{H}} f(z)\mu_{t,s}(dz)$$

for all $f \in B_b(\mathbb{H})$. This implies (2.7). So the proof is complete. □

Proof of Proposition 2.1. Let

$$u_{s,t}(x,\cdot) = \delta_{U(t,s)x}(\cdot)$$

for every $t \geq s$ and $x \in \mathbb{H}$. Then by property (1) of the evolution family $(U(t,s))_{t\geq s}$, $(u_{s,t})_{t\geq s}$ is a Markov transition function satisfying (2.2). Note that we can rewrite the inhomogeneous generalized Mehler semigroup (1.7) as

$$p_{s,t}f(x) = (\delta_{U(t,s)x} * \mu_{t,s})f, \quad x \in \mathbb{H},\ f \in B_b(\mathbb{H}). \tag{2.10}$$

Hence it is clear that $(q_{s,t})_{t\geq s}$ coincides with $(p_{s,t})_{t\geq s}$. Therefore, the equivalence of (2.6) and (2.7) in Proposition 2.2 is exactly the equivalence of (2.1) and (1.8) in Proposition 2.1. The latter is thus proved. □

For a linear bounded operator U on $\mathbb{H}$, let U^* denote its adjoint. Clearly, U^* is also bounded. Let $\hat{\mu}$ denote the Fourier transform (or the characteristic functional) of a probability measure μ on $\mathbb{H}$, i.e.

$$\hat{\mu}(\xi) = \int_{\mathbb{H}} \mathrm{e}^{i\langle x,\xi\rangle}\,\mu(dx), \quad \xi \in \mathbb{H}. \tag{2.11}$$

Probability measures on Hilbert spaces are determined by their characteristic functionals (see e.g. [52, Section IV.2.2, Theorem 2.2]). In particular, (1.8) holds if and only if

$$\hat{\mu}_{t,s}(\xi) = \hat{\mu}_{t,r}(\xi)\hat{\mu}_{r,s}(U(t,r)^*\xi), \quad \xi \in \mathbb{H}. \tag{2.12}$$

3. Time inhomogeneous skew convolution equations

In this section, we concentrate on (1.8). We shall study the weak continuity, infinite divisibility and stochastic process associated with $(\mu_{t,s})_{t\geq s}$.

3.1. **Preliminaries and motivations.**

In this subsection we fix some notations and present some basic results and motivations. In particular, we give some results on the weak convergence of measures satisfying convolution equations such as (1.8).

Convergence of probability measures. We recall that a sequence of probability measures $(\mu_n)_{n\geq 1}$ on $\mathbb{H}$ converges weakly to a probability measure μ on $\mathbb{H}$, written as

$$\mu_n \Rightarrow \mu \quad \text{as } n \to \infty,$$

if for every $f \in C_b(\mathbb{H})$

$$\lim_{n\to\infty} \int_{\mathbb{H}} f(x)\,\mu_n(dx) = \int_{\mathbb{H}} f(x)\,\mu(dx).$$

Here $C_b(\mathbb{H})$ denotes the space of all bounded continuous functions on $\mathbb{H}$. Sometimes we also write

$$\lim_{n\to\infty} \mu_n = \mu.$$

A sequence of $\mathbb{H}$-valued random variables $(X_n)_{n\geq 1}$ converges stochastically, or converges in probability, to an $\mathbb{H}$-valued random variable X, written as

$$X_n \xrightarrow{\Pr} X,$$

if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0.$$

Let μ_n and μ denote the distributions of X_n and X respectively. Then it is well known that as $n \to \infty$, $X_n \xrightarrow{\Pr} X$ implies $\mu_n \Rightarrow \mu$ (in other words, X_n converges to X in distribution). On the other hand, if in particular, $X = x \in \mathbb{H}$ is deterministic, then $\mu_n \Rightarrow \delta_x$ implies $X_n \xrightarrow{\Pr} X$. Therefore, we have $X_n \xrightarrow{\Pr} x$ if and only if $\mu_n \Rightarrow \delta_x$.

Additive processes, Lévy processes and convolution equations. Let $(X_t)_{t\in\mathbb{R}}$ be a stochastic process taking values in $\mathbb{H}$. Assume that $X_0 = 0$. The process $(X_t)_{t\in\mathbb{R}}$ is called an additive process if it has independent increments, i.e. if for any $t > s$, $X_t - X_s$ is independent of $\sigma(\{X_r\colon r \leq s\})$. Let $\mu_{t,s}$ denote the distribution of $X_t - X_s$. For all $t \geq r \geq s$, we have

$$X_t - X_s = (X_t - X_r) + (X_r - X_s).$$

This implies

$$\mu_{t,s} = \mu_{t,r} * \mu_{r,s}, \quad t \geq r \geq s \tag{3.1}$$

since $X_t - X_r$, $X_r - X_s$ are independent.

Usually one requires that an additive process is stochastically continuous, i.e. for every $t \in \mathbb{R}$ and $\varepsilon > 0$,

$$\lim_{h\to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0. \tag{3.2}$$

This condition means that

$$\begin{aligned} \mu_{t,s} &\Rightarrow \delta_0 \quad \text{as } s \uparrow t, \\ \mu_{t,s} &\Rightarrow \delta_0 \quad \text{as } t \downarrow s. \end{aligned} \tag{3.3}$$

If in addition $(X_t)_{t\in\mathbb{R}}$ has stationary increments, i.e. if for any $t > s$ the distribution of $X_t - X_s$ only depends on $t - s$, then it is called a Lévy process. In this case we shall only consider X_t for $t \geq 0$. For each $t \geq 0$ let μ_t denote the distribution of X_t. Then obviously we have the following convolution equations

$$\mu_{t+s} = \mu_t * \mu_s, \quad t, s \geq 0. \tag{3.4}$$

The stochastic continuity condition (3.2) is reduced to

$$\lim_{t\downarrow 0} \mathbb{P}(|X_t| \geq \varepsilon) = 0. \tag{3.5}$$

This is equivalent to

$$\mu_t \Rightarrow \delta_0 \quad \text{as } t \downarrow 0.$$

Infinitely divisible probability measures. A probability measure μ on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$ is said to be infinitely divisible if for any $n \in \mathbb{N}$, there exists a probability measure μ_n on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$ such that

$$\mu = \mu_n^{*n} := \underbrace{\mu_n * \mu_n * \cdots * \mu_n}_{n \text{ times}}.$$

Let $(\mu_t)_{t\geq 0}$ be a family of probability measures. If it satisfies (3.4), then obviously for every $t \geq 0$, μ_t is infinitely divisible. If $(\mu_t)_{t\geq 0}$ satisfies the skew convolution equations (1.2)

$$\mu_{t+s} = \mu_s * (\mu_t \circ T_s^{-1}), \quad s, t \geq 0,$$

then it is proved in [51] that for every $t \geq 0$, μ_t is also infinitely divisible.

Now we look at the two-parameter convolution equations (3.1). First we consider the finite dimensional case when $\mathbb{H} = \mathbb{R}^d$. It is known (see [29] or [47, Theorem 9.1 and Theorem 9.7]) that if $(\mu_{t,s})_{t\geq s}$ satisfies (3.3), then for any $t \geq s$, $\mu_{t,s}$ is infinitely divisible. The idea of the proof given in [47] can be described as follows.

First one shows that $(\mu_{t,s})_{t\geq s}$ is locally uniformly weakly continuous by using (3.3). That is, for every $\varepsilon > 0$ and $\eta > 0$, there is $\delta > 0$ such that for all s and t in $[s_0, t_0]$ satisfying $0 \leq t - s \leq \delta$, we have (cf. Lemma 3.8)

$$\mu_{t,s}(|x| > \varepsilon) < \eta.$$

Then by the celebrated Kolmogorov-Khintchine limit theorems on sums of independent random variables (see [47, Theorem 9.3]), we obtain that μ_{t_0,s_0} is infinitely divisible.

The uniform weak continuity of $(\mu_{t,s})_{t\geq s}$ on $[s_0, t_0]$ can be proved by constructing a stochastically continuous additive process $(X_t)_{t\in\mathbb{R}}$ such that for any $t \geq s$ the increment $X_t - X_s$ has the distribution $\mu_{t,s}$ (see [47, Theorem 9.7 (ii)] and [47, Lemma 9.6]).

In Subsection 3.3 we shall modify the arguments above to study the infinite divisibility of $(\mu_{t,s})_{t\geq s}$ satisfying equation (1.8).

Weak convergence of measures satisfying convolution equations. The results in this part will be used in Section 4. First of all we include here two results from [43]. Recall that a set M of probability measures on $\mathbb{H}$ is said to be shift (relatively) compact if for every sequence $(\mu_n)_{n\geq 1}$ in M there is a sequence $(\nu_n)_{n\geq 1}$ such that

1. $(\nu_n)_{n\geq 1}$ is a translate of $(\mu_n)_{n\geq 1}$. That is, there exists a sequence $(x_n)_{n\geq 1}$ in $\mathbb{H}$ such that $\nu_n = \mu_n * \delta_{x_n}$ for all $n \geq 1$.
2. $(\nu_n)_{n\geq 1}$ has a convergent subsequence.

Theorem 3.1. *Let $(\sigma_n)_{n\geq 1}$, $(\mu_n)_{n\geq 1}$ and $(\nu_n)_{n\geq 1}$ be three sequences of measures on $\mathbb{H}$ such that $\sigma_n = \mu_n * \nu_n$ for all $n \in \mathbb{N}$.*

(1) ([43, Theorem III.2.1]) *If the sequences $(\sigma_n)_{n\geq 1}$ and $(\mu_n)_{n\geq 1}$ both are relatively compact, then so is the sequence $(\nu_n)_{n\geq 1}$.*

(2) ([43, Theorem III.2.2]) *If the sequence $(\sigma_n)_{n\geq 1}$ is relatively compact then the sequences $(\mu_n)_{n\geq 1}$ and $(\nu_n)_{n\geq 1}$ are shift compact, respectively.*

Now we can show the following lemma.

Lemma 3.2. *Let μ_n, ν_n, σ_n with $n \geq 1$, and μ, ν, σ be measures on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$ such that $\sigma_n = \mu_n * \nu_n$.*

(1) If $\mu_n \Rightarrow \mu$ and $\nu_n \Rightarrow \nu$ as $n \to \infty$, then $\sigma_n \Rightarrow \mu * \nu$ as $n \to \infty$.

(2) Suppose that $\sigma_n \Rightarrow \sigma$ and $\mu_n \Rightarrow \mu$ as $n \to \infty$. Then there exists a probability measure ν such that

$$\sigma = \mu * \nu. \tag{3.6}$$

If ν is the unique measure such that (3.6) *holds, then $\nu_n \Rightarrow \nu$ as $n \to \infty$.*

Proof. The first conclusion says that the convolution operation preserves weak continuity. The proof can be found, for example, in [28, Proposition 2.3] or [43, Theorem III.1.1].

Now we show the second assertion. Take an arbitrary subsequence $(\nu_{n_i})_{i\geq 1}$ from $(\nu_n)_{n\geq 1}$ and consider

$$\sigma_{n_i} = \mu_{n_i} * \nu_{n_i}, \quad i \geq 1.$$

Since $\sigma_n \Rightarrow \sigma$ and $\mu_n \Rightarrow \mu$ as $n \to \infty$, both $(\sigma_{n_i})_{i\geq 1}$ and $(\mu_{n_i})_{i\geq 1}$ are relatively compact. By Theorem 3.1, the sequence $(\nu_{n_i})_{i\geq 1}$ is also relatively compact. Let $(\nu_{n_i'})_{i\geq 1}$ be a weakly convergent subsequence of $(\nu_{n_i})_{i\geq 1}$ with limit ν'. Then by the first assertion of this theorem, we have

$$\sigma_{n_i'} = \mu_{n_i'} * \nu_{n_i'} \Rightarrow \mu * \nu', \quad n_i' \to \infty.$$

Since we also have $\sigma_{n_i'} \Rightarrow \sigma$ as $n_i' \to \infty$, we get $\sigma = \mu * \nu'$. This shows that there exists a probability measure $\nu = \nu'$ such that (3.6) holds. If there is only one measure ν such that (3.6) holds, then the discussion above shows that any subsequence of $(\nu_n)_{n\geq 1}$ contains a further subsequence converging weakly to ν. This is sufficient to conclude that $(\nu_n)_{n\geq 1}$ converges weakly to ν (cf. [7, Theorem 2.6]). Hence the proof is complete. □

Remark 3.3. In the second part of the previous theorem, the assumption that ν is the unique solution to the convolution equation $\sigma = \mu * \nu$ amounts to saying that the following cancellation law for convolution operation holds: Let ν, ν' be two measures on $\mathbb{H}$, if

$$\mu * \nu = \mu * \nu', \tag{3.7}$$

then $\nu = \nu'$. It is obvious that this cancellation law holds provided $\hat{\mu}$ has no zeros. Indeed, from (3.7) we get $\hat{\mu}\hat{\nu} = \hat{\mu}\hat{\nu'}$. If $\hat{\mu} \neq 0$, then $\hat{\nu} = \hat{\nu'}$. So $\nu = \nu'$. It is well known that if μ is an infinitely divisible distribution, then $\hat{\mu}$ has no zeros.

Remark 3.4. After the proof of the second assertion of Lemma 3.2, we found that there is a similar result in [26, Corollary 2.2.4] where the condition that $\hat{\mu}$ has no zeros is used. Our proof is different. The example after [37, Theorem 5.1.1] shows that there exist probability measures μ, ν and ν' on $\mathbb{R}$ with $\nu \neq \nu'$ such that (3.7) holds. It is called Khintchine phenomenon in the literature. So if (3.7) holds, then it is necessary to require $\nu = \nu'$ for the second assertion of Lemma 3.2. Otherwise, if $\nu \neq \nu'$, then $(\mu * \nu_n)_{n\geq 1}$ with $\nu_{2k-1} = \nu$ and $\nu_{2k} = \nu'$ for all $k \geq 1$ converges weakly, but $(\nu_n)_{n\geq 1}$ does not converge weakly.

As a summary of the discussion above, we have the following result.

Corollary 3.5. *Let μ_n, ν_n, σ_n with $n \geq 1$ and σ, μ be measures on $\mathbb{H}$ with the following properties*

(1) For all $n \geq 1$, $\sigma_n = \mu_n * \nu_n$;
(2) As $n \to \infty$, $\sigma_n \Rightarrow \sigma$ and $\mu_n \Rightarrow \mu$.

If μ is an infinitely divisible distribution, then the sequence $(\nu_n)_{n\geq 1}$ converges weakly to some measure ν on $\mathbb{H}$ such that $\sigma = \mu * \nu$ as $n \to \infty$.

In particular, we have the following result.

Corollary 3.6. *Let μ_n, ν_n, σ_n with $n \geq 1$, and σ be measures on $\mathbb{H}$. Suppose that for all $n \geq 1$, $\sigma_n = \mu_n * \nu_n$. If $\sigma_n \Rightarrow \sigma$ and $\mu_n \Rightarrow \delta_0$ as $n \to \infty$, then $\nu_n \Rightarrow \sigma$ as $n \to \infty$.*

3.2. **Weak continuity.** We shall use the following assumption.

Assumption 3.7. For all $s \in \mathbb{R}$

$$\lim_{t\downarrow s} \mu_{t,s} = \lim_{t\uparrow s} \mu_{s,t} = \delta_0. \tag{3.8}$$

Let us explain that the weak limit δ_0 in (3.8) is natural. Taking $s = r = t$ in (1.8) we obtain

$$\mu_{t,t} = \mu_{t,t} * \mu_{t,t}, \quad t \in \mathbb{R}. \tag{3.9}$$

That is, for every $t \in \mathbb{R}$, $\mu_{t,t}$ is an idempotent probability measure on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$. By [52, Section I.4.3, Proposition 4.7, Page 67, see also Section IV.2.2, Corollary 1, Page 203] (or [43, Section III.3, Theorem 3.1, Page 62] and noting that there is no nontrivial compact subgroup in $\mathbb{H}$), the trivial measure δ_0 is the only idempotent measure on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$. Hence

$$\mu_{t,t} = \delta_0. \tag{3.10}$$

Now Assumption 3.7 means that $\mu_{t,s}$ is weakly continuous on the diagonal $\{(t,s) \colon (t,s) \in \mathbb{R}^2,\ t = s\}$ in two directions. We aim to show that, combined with the skewed convolution equation (1.8), this assumption in fact implies more about the weak continuity of $\mu_{t,s}$ in t and s.

We need the following simple fact.

Lemma 3.8. *Let $(\mu_n)_{n\geq 1}$ be a sequence of probability measures on $\mathbb{H}$. Then $\mu_n \Rightarrow \delta_0$ as $n \to \infty$ if and only if for all $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mu_n(\{x \in \mathbb{H} \colon |x| > \varepsilon\}) = 0. \tag{3.11}$$

Proof. Suppose that $\mu_n \Rightarrow \delta_0$ as $n \to \infty$. Then by the Portmanteau theorem (see for instance [7, Theorem 2.1]),

$$\limsup_{n\to\infty} \mu_n(F) \leq \delta_0(F)$$

for all closed sets F in $\mathbb{H}$. Obviously $\{x \in \mathbb{H} \colon |x| \geq \varepsilon\}$ is closed. Hence

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \mu_n(\{x \in \mathbb{H} \colon |x| > \varepsilon\}) \\ &\leq \limsup_{n\to\infty} \mu_n(\{x \in \mathbb{H} \colon |x| \geq \varepsilon\}) \leq \delta_0(\{x \in \mathbb{H} \colon \{|x| \geq \varepsilon\}) = 0. \end{aligned}$$

So (3.11) holds.

Now we assume that (3.11) holds for all $\varepsilon > 0$. Let f be a continuous bounded function on $\mathbb{H}$. Define

$$M := \sup |f| + 1.$$

We are going to show $\mu_n(f) \to \delta_0(f)$ as $n \to \infty$.

Since f is continuous, for any $\eta > 0$, there exists a constant $\varepsilon_0 > 0$ such that for all $|x| \leq \varepsilon_0$,

$$|f(x) - f(0)| < \frac{\eta}{2}. \tag{3.12}$$

By (3.11) there exists a constant $N > 0$ such that for all $n > N$,

$$\mu_n(\{x \in \mathbb{H} \colon |x| > \varepsilon_0\}) < \frac{\eta}{4M}. \tag{3.13}$$

Combining (3.12) and (3.13) we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{H}} f\, d\mu_n - \int_{\mathbb{H}} f\, d\delta_0\right| \\
\leq &\int_{\mathbb{H}} |f(x) - f(0)|\, d\mu_n \\
= &\int_{\{x\in\mathbb{H}\colon |x|\geq\varepsilon_0\}} |f(x) - f(0)|\, d\mu_n + \int_{\{x\in\mathbb{H}\colon |x|<\varepsilon_0\}} |f(x) - f(0)|\, d\mu_n \\
\leq &2M\mu_n(\{x\in\mathbb{H}\colon |x| > \varepsilon_0\}) + \sup_{|x|<\varepsilon_0} |f(x) - f(0)| \cdot \mu_n(\{x\in\mathbb{H}\colon |x| \leq \varepsilon_0\}) \\
< &2M \cdot \frac{\eta}{4M} + \frac{\eta}{2} = \eta.
\end{aligned}$$

This completes the proof. □

By Lemma 3.8 we have the following equivalent description of Assumption 3.7.

Proposition 3.9. *Equation* (3.8) *is equivalent to*

$$\lim_{t\downarrow s} \mu_{t,s}(\{x\in\mathbb{H}\colon |x| > \varepsilon\}) = \lim_{s\uparrow t} \mu_{t,s}(\{x\in\mathbb{H}\colon |x| > \varepsilon\}) = 0 \tag{3.14}$$

for all $\varepsilon > 0$*. More precisely, they are equivalent to the following two conditions: for every* $\varepsilon, \eta > 0$*, and for every* $u \in \mathbb{R}$*, there exists a constant* δ_u *such that:*

(1) For every $t \in (u, u + \delta_u)$*,*

$$\mu_{t,u}(\{x\in\mathbb{H}\colon |x| > \varepsilon\}) < \eta. \tag{3.15}$$

(2) For every $s \in (u - \delta_u, u)$*,*

$$\mu_{u,s}(\{x\in\mathbb{H}\colon |x| > \varepsilon\}) < \eta. \tag{3.16}$$

We shall use the following two lemmas.

Lemma 3.10. *For every* $s_0 < t_0$*, there exists some constant* $c \geq 1$ *such that for all* $s_0 \leq s \leq t \leq t_0$*,*

$$|U(t,s)x| \leq c|x|, \quad x\in\mathbb{H},\ s_0 \leq s \leq t \leq t_0. \tag{3.17}$$

Proof. For every $x \in \mathbb{H}$, $|U(t,s)x|$ is a continuous function of (t,s) on $\Lambda_{t_0,s_0} := \{(t,s)\colon s_0 \leq s \leq t \leq t_0\}$. Hence $|U(t,s)x|$ is uniformly bounded on Λ_{t_0,s_0} for every $x \in \mathbb{H}$. By the Banach-Steinhaus theorem we have

$$\sup_{(t,s)\in\Lambda_{t_0,s_0}} \|U(t,s)\| < \infty.$$

That is, there exists some $c > 0$ such that (3.17) holds. □

Lemma 3.11. *Let* T *be a bounded linear operator on* $\mathbb{H}$*, hence for all* $x \in \mathbb{H}$*,* $|Tx| \leq c|x|$ *for some constant* $c > 0$*. Let* μ *be a measure on* $\mathbb{H}$ *and* $\varepsilon > 0$ *be any constant. Then we have*

$$\mu\circ T^{-1}(\{x\in\mathbb{H}\colon |x| > \varepsilon\}) \leq \mu(\{x\in\mathbb{H}\colon |x| > \varepsilon/c\}). \tag{3.18}$$

Proof. By assumption we have

$$\begin{aligned}\mu\circ T^{-1}(\{x\in\mathbb{H}\colon |x|>\varepsilon\}) =&\mu(\{x\in\mathbb{H}\colon |Tx|>\varepsilon\})\\ \leq&\mu(\{x\in\mathbb{H}\colon |x|>\varepsilon/c\}).\end{aligned}$$

□

Now we can prove the following result.

Theorem 3.12. *Suppose that Assumption 3.7 and* (1.8) *hold for a family of probability measures* $(\mu_{t,s})_{t\geq s}$. *Then:*

(1) For every $t\in\mathbb{R}$, *the map* $s\mapsto\mu_{t,s}$ *with* $s\leq t$ *is weakly continuous.*

(2) For every $t,s\in\mathbb{R}$ *with* $t\geq s$ *we have*

$$\mu_{t+\varepsilon,s}\Rightarrow\mu_{t,s}\quad \textit{as } \varepsilon\downarrow 0, \tag{3.19}$$

where $\varepsilon\downarrow 0$ *means* $\varepsilon\geq 0$ *and* $\varepsilon\to 0$.

(3) For every $t_0,t,s\in\mathbb{R}$ *with* $t_0\geq t>s$,

$$\mu_{t-\varepsilon,s}\circ U(t_0,t-\varepsilon)^{-1}\Rightarrow\mu_{t,s}\circ U(t_0,t)^{-1}\quad \textit{as } \varepsilon\downarrow 0. \tag{3.20}$$

Proof. (1) Let $s<t$. We need to show

$$\mu_{t,s-\varepsilon}\Rightarrow\mu_{t,s}\quad \text{as } \varepsilon\downarrow 0 \tag{3.21}$$

and

$$\mu_{t,s+\varepsilon}\Rightarrow\mu_{t,s}\quad \text{as } \varepsilon\downarrow 0. \tag{3.22}$$

Equation (1.8) implies that for every $\varepsilon\in(0,t-s)$

$$\mu_{t,s-\varepsilon}=\mu_{t,s}*\left(\mu_{s,s-\varepsilon}\circ U(t,s)^{-1}\right) \tag{3.23}$$

and

$$\mu_{t,s}=\mu_{t,s+\varepsilon}*\left(\mu_{s+\varepsilon,s}\circ U(t,s+\varepsilon)^{-1}\right). \tag{3.24}$$

By Lemma 3.10 there exists some constant $c\geq 1$ such that for all $\varepsilon\in(0,t-s)$, we have

$$\|U(t,s+\varepsilon)\|\leq c.$$

Hence by Lemma 3.11 we have for all $\eta>0$

$$\mu_{s,s-\varepsilon}\circ U(t,s)^{-1}(\{x\in\mathbb{H}\colon |x|>\eta\})\leq\mu_{s,s-\varepsilon}(\{x\in\mathbb{H}\colon |x|>\eta/c\}). \tag{3.25}$$

Because $\mu_{s,s-\varepsilon}\Rightarrow\delta_0$ as $\varepsilon\downarrow 0$, by Lemma 3.8 we get

$$\lim_{\varepsilon\downarrow 0}\mu_{s,s-\varepsilon}(\{x\in\mathbb{H}\colon |x|>\eta/c\})=0.$$

Hence it follows from (3.25) that

$$\lim_{\varepsilon\downarrow 0}\mu_{s,s-\varepsilon}\circ U(t,s)^{-1}(\{x\in\mathbb{H}\colon |x|>\eta\})=0.$$

By Lemma 3.8, we obtain

$$\mu_{s,s-\varepsilon}\circ U(t,s)^{-1}\Rightarrow\delta_0,\quad \varepsilon\downarrow 0.$$

Therefore, applying the first result of Lemma 3.2 to (3.23) we get (3.21).

By the same arguments, it is easy to show that

$$\mu_{s+\varepsilon,s}\circ U(t,s+\varepsilon)^{-1}\Rightarrow\delta_0\quad \text{as } \varepsilon\downarrow 0.$$

Then by Corollary 3.6, (3.22) follows from (3.24).

(2) According to (1.8) we have for all $t \geq s$, $\varepsilon \geq 0$,

$$\mu_{t+\varepsilon,s} = \mu_{t+\varepsilon,t} * (\mu_{t,s} \circ U(t+\varepsilon,t)^{-1}).$$

By assumption we have $\mu_{t+\varepsilon,t} \Rightarrow \delta_0$ as $\varepsilon \downarrow 0$. Hence by applying the first assertion of Lemma 3.2, we get (3.19) by proving

$$\mu_{t,s} \circ U(t+\varepsilon,t)^{-1} \Rightarrow \mu_{t,s}, \quad \text{as } \varepsilon \downarrow 0. \tag{3.26}$$

Now we show (3.26). Let f be a continuous and bounded function on $\mathbb{H}$. For every $\varepsilon > 0$ set

$$f_\varepsilon(x) := f\left(U(t+\varepsilon,t)x\right), \quad x \in \mathbb{H}.$$

It is clear that f_ε converges to f pointwise as $\varepsilon \downarrow 0$. Moreover, since f is bounded, we know that f_ε is bounded. Hence by Lebesgue's dominated convergence theorem we have

$$\lim_{\varepsilon\downarrow 0} \int_{\mathbb{H}} f(x)\, d\mu_{t,s} \circ U(t+\varepsilon,t)^{-1}(x) = \lim_{\varepsilon\downarrow 0} \int_{\mathbb{H}} f_\varepsilon(x)\, d\mu_{t,s}(x) = \int_{\mathbb{H}} f(x)\, d\mu_{t,s}(x).$$

This proves (3.26).

(3) We first show in particular the following result:

$$\mu_{t-\varepsilon,s} \circ U(t,t-\varepsilon)^{-1} \Rightarrow \mu_{t,s} \quad \text{as } \varepsilon \downarrow 0. \tag{3.27}$$

By (1.8) we have for all $t \geq t-\varepsilon > s$,

$$\mu_{t,s} = \mu_{t,t-\varepsilon} * (\mu_{t-\varepsilon,s} \circ U(t,t-\varepsilon)^{-1}).$$

Since $\mu_{t,t-\varepsilon} \Rightarrow \delta_0$ as $\varepsilon \downarrow 0$, by Corollary 3.6, we get (3.27).

Now let us show (3.20). By (3.27), we have for any bounded continuous function f on $\mathbb{H}$

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \int_{\mathbb{H}} f(x)\, d\mu_{t-\varepsilon,s} \circ U(t_0,t-\varepsilon)^{-1}(x) \\
=&\lim_{\varepsilon\downarrow 0} \int_{\mathbb{H}} f\left(U(t_0,t-\varepsilon)x\right)\, d\mu_{t-\varepsilon,s}(x) \\
=&\lim_{\varepsilon\downarrow 0} \int_{\mathbb{H}} f\left(U(t_0,t)U(t,t-\varepsilon)x\right)\, d\mu_{t-\varepsilon,s}(x) \\
=&\lim_{\varepsilon\downarrow 0} \int_{\mathbb{H}} f\left(U(t_0,t)y\right)\, d\mu_{t-\varepsilon,s} \circ U(t,t-\varepsilon)^{-1}(y) \\
=&\int_{\mathbb{H}} f\left(U(t_0,t)y\right)\, d\mu_{t,s}(y) \\
=&\int_{\mathbb{H}} f(y)\, d\mu_{t,s} \circ U(t_0,t)^{-1}(y).
\end{aligned}$$

This proves (3.20). □

Concerning the space-homogeneous case, we have the following result.

Theorem 3.13. *Let $(\tilde{\mu}_{t,s})_{t\geq s}$ be a family of probability measures on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$ satisfying*

$$\tilde{\mu}_{t,s} = \tilde{\mu}_{t,r} * \tilde{\mu}_{r,s}, \quad t \geq r \geq s \tag{3.28}$$

and

$$\lim_{t\downarrow s} \tilde{\mu}_{t,s} = \lim_{s\uparrow t} \tilde{\mu}_{t,s} = \delta_0.$$

Then $\tilde{\mu}_{t,s}$ is weakly continuous in t and s with $t \geq s$.

Proof. Below we always assume that we have $t' \geq s'$ when we write $\tilde{\mu}_{t',s'}$. Let ε_n and δ_n be two nonnegative sequences converging to 0 as $n \to \infty$. Recall that by Theorem 3.12,

$$\tilde{\mu}_{t,s\pm\delta_n} \Rightarrow \tilde{\mu}_{t,s}, \quad \text{as } n \to \infty. \tag{3.29}$$

By (3.28) we have

$$\tilde{\mu}_{t,s\pm\delta_n} = \tilde{\mu}_{t,t-\varepsilon_n} * \tilde{\mu}_{t-\varepsilon_n,s\pm\delta_n}. \tag{3.30}$$

Note that as $n \to \infty$, we have $\tilde{\mu}_{t,t-\varepsilon_n} \Rightarrow \delta_0$ by assumption. Hence by (3.29) and by applying Corollary 3.6 to Equation (3.30) we obtain

$$\tilde{\mu}_{t-\varepsilon_n,s\pm\delta_n} \Rightarrow \tilde{\mu}_{t,s}, \quad \text{as } n \to \infty. \tag{3.31}$$

Now by (3.28) we also have

$$\tilde{\mu}_{t+\varepsilon_n,s\pm\delta_n} = \tilde{\mu}_{t+\varepsilon_n,t} * \tilde{\mu}_{t,s\pm\delta_n}. \tag{3.32}$$

By assumption we have $\tilde{\mu}_{t+\varepsilon_n,t} \Rightarrow \delta_0$ as $n \to \infty$. Hence by using the weak continuity of the convolution operator and by applying (3.29) to (3.32), we get

$$\tilde{\mu}_{t+\varepsilon_n,s\pm\delta_n} \Rightarrow \tilde{\mu}_{t,s}, \quad \text{as } n \to \infty. \tag{3.33}$$

Combining (3.31) and (3.33) we get

$$\tilde{\mu}_{t\pm\varepsilon_n,s\pm\delta_n} \Rightarrow \tilde{\mu}_{t,s}, \quad \text{as } n \to \infty$$

and hence the proof is complete. □

Remark 3.14. We provide also probabilistic proofs of Theorem 3.12 and Theorem 3.13 in Subsection 3.4 below.

For every $t \geq s$, it is clear that $p_{s,t}$ is Feller, i.e. $p_{s,t}(C_b(\mathbb{H})) \subset C_b(\mathbb{H})$. Now we look at the continuity of the map $(s,x) \mapsto p_{s,t}f(x)$ for every f in $C_b(\mathbb{H})$. The proposition below is a direct generalization of [9, Lemma 2.1]. The proof is similar to the proof in [9].

Proposition 3.15. *Let $s_n, t_n \in \mathbb{R}$, $x_n \in \mathbb{H}$, $s_n \leq t_n$ with $n \geq 1$ such that $(s_n,t_n) \to (s,t) \in \mathbb{R}^2$ and $x_n \to x \in \mathbb{H}$ as $n \to \infty$. If $\mu_{t_n,s_n} \Rightarrow \mu_{t,s}$ as $n \to \infty$, then for any $f \in C_b(\mathbb{H})$, $p_{s_n,t_n}f(x_n) \to p_{s,t}f(x)$ as $n \to \infty$.*

Proof. Since $\mu_{t_n,s_n} \Rightarrow \mu_{t,s}$ as $n \to \infty$, by Prohorov's theorem, for every $\varepsilon > 0$, there exists a compact set $K \subset \mathbb{H}$ such that

$$\mu_{r,\sigma}(K) \geq 1 - \varepsilon, \quad \text{for all } (r,\sigma) \in \{(t,s),(t_n,s_n) \colon n \in \mathbb{N}\}. \tag{3.34}$$

For abbreviation, we set $z_n = U(t_n,s_n)x_n$ and $z = U(t,s)x$. By the strong continuity of the evolution family $(U(t,s))_{t\geq s}$, the set $Z := \{z, z_n \colon n \in \mathbb{N}\}$ is compact. Hence $Z + K$ is also compact. So there exists an $N \in \mathbb{N}$ such that for any $n > N$ and for any $y \in K$,

$$|f(z_n+y) - f(z+y)| < \varepsilon, \tag{3.35}$$

since f is uniformly continuous on compacts.

Because $\mu_{t_n,s_n} \Rightarrow \mu_{t,s}$ as $n \to \infty$, (taking N larger if necessary) we have for all $n > N$

$$\left| \int_{\mathbb{H}} f(z+y)\, \mu_{t_n,s_n}(dy) - \int_{\mathbb{H}} f(z+y)\, \mu_{t,s}(dy) \right| < \varepsilon. \tag{3.36}$$

From (3.34), (3.35) and (3.36) we get

$$\begin{aligned}&\left|\int_{\mathbb{H}} f(z_n+y)\,\mu_{t_n,s_n}(dy) - \int_{\mathbb{H}} f(z+y)\,\mu_{t,s}(dy)\right|\\ \leq&\left|\int_{\mathbb{H}} f(z+y)\,\mu_{t_n,s_n}(dy) - \int_{\mathbb{H}} f(z+y)\,\mu_{t,s}(dy)\right|\\ &\qquad\qquad + \int_K |f(z_n+y)-f(z+y)|\;\mu_{t_n,s_n}(dy) + 2\|f\|_\infty \mu(\mathbb{H}\setminus K)\\ <&2\varepsilon(1+\|f\|_\infty).\end{aligned}$$

Hence the result is proved since ε was arbitrary.

□

3.3. **Infinite divisibility.** The main result of this section is the following theorem.

Theorem 3.16. *Suppose that Assumption 3.7 and* (1.8) *hold for* $(\mu_{t,s})_{t\geq s}$. *Then for every* $t \geq s$, $\mu_{t,s}$ *is infinitely divisible.*

We shall use a similar method as indicated in Subsection 3.1. The main difficulty is proving the following lemma, i.e. showing $(\mu_{t,s}\circ U(t_0,t)^{-1})_{t\geq s}$ is uniformly weakly continuous on $[s_0,t_0]$. In this subsection we prove it analytically. In Subsection 3.4 we present a probabilistic proof of it by constructing an associated stochastically continuous additive process.

Lemma 3.17. *Suppose that* $(\mu_{t,s})_{t\geq s}$ *satisfies* (1.8) *and Assumption 3.7. Then on every compact interval* $[s_0,t_0]$, *for all* $\varepsilon,\eta>0$, *there exists a constant* $\delta>0$ *such that for all* $s,t\in[s_0,t_0]$ *with* $0\leq t-s<\delta$,

$$\mu_{t,s}\circ U(t_0,t)^{-1}(\{x\in\mathbb{H}\colon |x|>\varepsilon\})<\eta. \tag{3.37}$$

Proof. It is trivial to see that (3.37) holds for the case when $t=s$. So we shall assume $t>s$. By Lemma 3.10, there exists a constant $c\geq 1$ such that

$$|U(t,s)x|\leq c|x|, \quad x\in\mathbb{H},\ s_0\leq s<t\leq t_0. \tag{3.38}$$

Let us set

$$\varepsilon'=\varepsilon/c$$

and

$$A(r):=\{x\in\mathbb{H}\colon |x|>r\}, \quad r>0.$$

By Assumption 3.7 and Equations (3.15), (3.16) in Proposition 3.9, for every $\varepsilon,\eta>0$, $t\in[s_0,t_0]$, there exists a constant $\delta_t\geq 0$ such that

$$\mu_{t,s}\left(A\left(\frac{\varepsilon'}{2c}\right)\right)<\eta/2, \quad s\in(t-\delta_t,t) \tag{3.39}$$

and

$$\mu_{r,t}\left(A\left(\frac{\varepsilon'}{2c}\right)\right)<\eta/2, \quad r\in(t,t+\delta_t). \tag{3.40}$$

Since $c\geq 1$ we have $\frac{\varepsilon'}{2c}\leq\frac{\varepsilon'}{2}$. Hence from estimates (3.39) and (3.40) it follows that

$$\mu_{t,s}\left(A\left(\varepsilon'/2\right)\right)<\eta/2, \quad s\in(t-\delta_t,t) \tag{3.41}$$

and

$$\mu_{r,t}\left(A\left(\varepsilon'/2\right)\right) < \eta/2, \quad r \in (t, t+\delta_t). \tag{3.42}$$

Moreover, according to Lemma 3.11 and (3.38), from estimates (3.39) and (3.40) we obtain

$$\mu_{t,s} \circ U(t', t)^{-1}\left(A\left(\varepsilon'/2\right)\right) < \eta/2, \quad t-\delta_t \le s \le t,\ t \le t' \le t_0 \tag{3.43}$$

and

$$\mu_{r,t} \circ U(r', r)^{-1}\left(A\left(\varepsilon'/2\right)\right) < \eta/2, \quad t \le r \le t+\delta_t,\ r \le r' \le t_0. \tag{3.44}$$

For every $t \in [s_0, t_0]$, let

$$I_t := (t-\delta_t, t+\delta_t).$$

Obviously $\{I_t \colon t \in [s_0, t_0]\}$ covers the interval $[s_0, t_0]$. Hence there is a finite sub–covering $\{I_{t_j} \colon j = 1, 2 \cdots, n\}$ of $[s_0, t_0]$. Then for every $t \in [s_0, t_0]$, we have $t \in I_{t_j}$ for some $j \in \{1, 2, \cdots, n\}$. Let

$$\delta := \min\left\{\frac{\delta_{t_j}}{2} \colon j = 1, 2, \ldots, n\right\}.$$

For every $s \in [s_0, t_0]$ such that $0 < t - s < \delta$, we have

$$|s - t_j| \le |s - t| + |t - t_j| < \delta + \delta_{t_j}/2 \le \delta_{t_j}.$$

Therefore, both t and s are in the same sub-interval I_{t_j}. We need to consider the following three cases respectively: 1. $s \le t_j < t$; 2. $s < t \le t_j$; 3. $t_j < s < t$.

Case 1 ($s \le t_j < t$). Note that for all $x, y \in \mathbb{H}$, if $|x + y| > \varepsilon'$, then either $|x| > \varepsilon'/2$ or $|y| > \varepsilon'/2$. That is, the following inequality holds

$$\mathbb{1}_{A(\varepsilon')}(x+y) \le \mathbb{1}_{A(\varepsilon'/2)}(x) + \mathbb{1}_{A(\varepsilon'/2)}(y). \tag{3.45}$$

By (1.8), (3.42), (3.43) and (3.45) we have

$$\begin{aligned}
\mu_{t,s}(A(\varepsilon')) &= \mu_{t,t_j} * (\mu_{t_j,s} \circ U(t,t_j)^{-1})(A(\varepsilon')) \\
&= \int_{\mathbb{H}}\int_{\mathbb{H}} \mathbb{1}_{A(\varepsilon')}(x+y)\, \mu_{t,t_j}(dx)(\mu_{t_j,s} \circ U(t,t_j)^{-1})(dy) \\
&\le \int_{\mathbb{H}}\int_{\mathbb{H}} (\mathbb{1}_{A(\varepsilon'/2)}(x) + \mathbb{1}_{A(\varepsilon'/2)}(y))\, \mu_{t,t_j}(dx)(\mu_{t_j,s} \circ U(t,t_j)^{-1})(dy) \\
&= \mu_{t,t_j}(A(\varepsilon'/2)) + (\mu_{t_j,s} \circ U(t,t_j)^{-1})(A(\varepsilon'/2)) \\
&< \frac{\eta}{2} + \frac{\eta}{2} = \eta.
\end{aligned}$$

Therefore, by Lemma 3.11 and (3.38) we have

$$\mu_{t,s} \circ U(t_0, t)^{-1}(\{x \in \mathbb{H} \colon |x| > \varepsilon\}) < \eta.$$

Case 2 ($s < t \le t_j$). We first show that

$$(\mu_{t,s} \circ U(t_j, t)^{-1})(A(\varepsilon')) < \eta$$

by contradiction. If otherwise, the following inequality

$$(\mu_{t,s} \circ U(t_j, t)^{-1})(A(\varepsilon')) \ge \eta \tag{3.46}$$

holds. Then by (1.8), (3.41) and (3.46) we obtain

$$
\begin{aligned}
\frac{\eta}{2} &> \mu_{t_j,s}(A(\varepsilon'/2)) = \mu_{t_j,t} * (\mu_{t,s} \circ U(t_j,t)^{-1})(A(\varepsilon'/2)) \\
&= \int_{\mathbb{H}} \int_{\mathbb{H}} \mathbb{1}_{A(\varepsilon'/2)}(x+y)\, \mu_{t_j,t}(dx)(\mu_{t,s} \circ U(t_j,t)^{-1})(dy) \\
&\geq \int_{\mathbb{H}} \int_{\mathbb{H}} \mathbb{1}_{A(\varepsilon'/2)^c}(x) \cdot \mathbb{1}_{A(\varepsilon')}(y)\ \mu_{t_j,t}(dx)(\mu_{t,s} \circ U(t_j,t)^{-1})(dy) \\
&= \mu_{t_j,t}(A(\varepsilon'/2)^c) \cdot (\mu_{t,s} \circ U(t_j,t)^{-1})(A(\varepsilon')) \\
&\geq \eta \left(1 - \frac{\eta}{2}\right) = \eta - \frac{\eta^2}{2}.
\end{aligned}
$$

Here we have used the fact that for all $x, y \in \mathbb{H}$, $|x+y| \geq |y| - |x| > \frac{\varepsilon'}{2}$ if $|y| > \varepsilon'$ and $|x| \leq \varepsilon'/2$. Now we have

$$
\frac{\eta}{2} > \eta - \frac{\eta^2}{2},
$$

consequently, $\eta > 1$. By (3.46) this means that

$$
(\mu_{t,s} \circ U(t_j,t)^{-1})(A(\varepsilon')) > 1.
$$

This is impossible since $\mu_{t,s}$ is a probability measure.

Then by Lemma 3.11 and (3.38) we have

$$
\begin{aligned}
&\mu_{t,s} \circ U(t_0,t)^{-1}(\{x \in \mathbb{H}\colon |x| > \varepsilon\}) \\
=& \left(\mu_{t,s} \circ U(t_j,t)^{-1}\right) \circ U(t_0,t_j)^{-1}(\{x \in \mathbb{H}\colon |x| > \varepsilon\}) \\
\leq& \mu_{t,s} \circ U(t_j,t)^{-1}(\{x \in \mathbb{H}\colon |x| > \varepsilon/c\}) \\
=& \mu_{t,s} \circ U(t_j,t)^{-1}(A(\varepsilon')) < \eta.
\end{aligned}
$$

Case 3 ($t_j < s < t$). Similar to Case 1 we only need to show $\mu_{t,s}(A(\varepsilon')) < \eta$ whose proof turns out to be similar to the proof in Case 2. Indeed, if

$$
\mu_{t,s}(A(\varepsilon')) \geq \eta, \tag{3.47}
$$

then by (1.8), (3.42) and (3.44)

$$
\begin{aligned}
\frac{\eta}{2} &> \mu_{t,t_j}(A(\varepsilon'/2)) = \mu_{t,s} * (\mu_{s,t_j} \circ U(t,s)^{-1})(A(\varepsilon'/2)) \\
&= \int_{\mathbb{H}} \int_{\mathbb{H}} \mathbb{1}_{A(\varepsilon'/2)}(x+y)\, \mu_{t,s}(dx)(\mu_{s,t_j} \circ U(t,s)^{-1})(dy) \\
&\geq \mu_{t,s}(A(\varepsilon')) \cdot (\mu_{s,t_j} \circ U(t,s)^{-1})(A(\varepsilon'/2)^c) \\
&\geq \eta \left(1 - \frac{\eta}{2}\right).
\end{aligned}
$$

This implies $\eta > 1$ which contradicts (3.47) because $\mu_{t,s}$ is a probability measure.

Combining the three cases discussed above, we obtain (3.37) and hence the proof is complete. □

Now we are ready to prove Theorem 3.16.

Proof of Theorem 3.16. For simplicity we only show that $\mu_{1,0}$ is infinitely divisible. The proof for the case $\mu_{t,s}$ with arbitrary $t \geq s$ is similar.

First of all, we shall show by induction that for every $m \in \mathbb{N}$,

$$\mu_{1,0} = \prod_{j=0}^{2^m-1}{}^{*} \mu_{\frac{j+1}{2^m},\frac{j}{2^m}} \circ U\left(1, \frac{j+1}{2^m}\right)^{-1}. \tag{3.48}$$

Here $\prod^*$ denotes the convolution product.

By (1.8) we have

$$\mu_{1,0} = \mu_{1,\frac{1}{2}} * \left(\mu_{\frac{1}{2},0} \circ U\left(1, \frac{1}{2}\right)^{-1}\right).$$

So (3.48) holds for $m = 1$. Now we assume that (3.48) holds for some $m \geq 1$. Then by (1.8) we have for all $j = 0, 1, \cdots, 2^m - 1$,

$$\mu_{\frac{j+1}{2^m},\frac{j}{2^m}} = \mu_{\frac{(2j+1)+1}{2^{m+1}},\frac{2j+1}{2^{m+1}}} * \left(\mu_{\frac{2j+1}{2^{m+1}},\frac{2j}{2^{m+1}}} \circ U\left(\frac{(2j+1)+1}{2^{m+1}}, \frac{2j+1}{2^{m+1}}\right)^{-1}\right).$$

For any probability measures μ, ν on $\mathbb{H}$ and measurable map T on $\mathbb{H}$, it is easy to check that

$$(\mu * \nu) \circ T^{-1} = (\mu \circ T^{-1}) * (\nu \circ T^{-1}). \tag{3.49}$$

So,

$$\begin{aligned}
&\mu_{\frac{j+1}{2^m},\frac{j}{2^m}} \circ U\left(1, \frac{j+1}{2^m}\right)^{-1} \\
=&\mu_{\frac{(2j+1)+1}{2^{m+1}},\frac{2j+1}{2^{m+1}}} \circ U\left(1, \frac{(2j+1)+1}{2^{m+1}}\right)^{-1} * \left(\mu_{\frac{2j+1}{2^{m+1}},\frac{2j}{2^{m+1}}} \circ U\left(1, \frac{2j+1}{2^{m+1}}\right)^{-1}\right) \\
=&\prod_{k=2j}^{2j+1}{}^{*} \mu_{\frac{k+1}{2^{m+1}},\frac{k}{2^{m+1}}} \circ U\left(1, \frac{k+1}{2^{m+1}}\right)^{-1}.
\end{aligned}$$

Therefore, by assumption we have

$$\begin{aligned}
\mu_{1,0} =& \prod_{j=0}^{2^m-1}{}^{*} \prod_{k=2j}^{2j+1}{}^{*} \mu_{\frac{k+1}{2^{m+1}},\frac{k}{2^{m+1}}} \circ U\left(1, \frac{k+1}{2^{m+1}}\right)^{-1} \\
=& \prod_{k=0}^{2^{m+1}-1}{}^{*} \mu_{\frac{k+1}{2^{m+1}},\frac{k}{2^{m+1}}} \circ U\left(1, \frac{k+1}{2^{m+1}}\right)^{-1}.
\end{aligned}$$

This proves (3.48) for all $m \geq 1$.

By (3.48) and Lemma 3.17, $\mu_{1,0}$ is the limit of an infinitesimal triangular array. Hence $\mu_{1,0}$ is infinitely divisible according to [43, Corollary VI.6.2]. □

Now we assume that for every $t \geq s$, the measure $\mu_{t,s}$ is infinitely divisible. Then by the Lévy-Khintchine theorem [43, Theorem VI.4.10], there exists a negative definite, Sazonov continuous function $\psi_{t,s}$ on $\mathbb{H}$ such that

$$\hat{\mu}_{t,s}(\xi) = \exp(-\psi_{t,s}(\xi)), \quad \xi \in \mathbb{H}$$

and $\psi_{t,s}$ has the following form

$$\begin{aligned}\psi_{t,s}(\xi) = -i\langle a_{t,s}, \xi\rangle + \frac{1}{2}\langle \xi, R_{t,s}\xi\rangle \\ - \int_{\mathbb{H}} \left(e^{i\langle \xi, x\rangle} - 1 - \frac{i\langle \xi, x\rangle}{1+|x|^2}\right) m_{t,s}(dx), \quad \xi \in \mathbb{H},\end{aligned} \tag{3.50}$$

where $a_{t,s} \in \mathbb{H}$, $R_{t,s}$ is a nonnegative definite, symmetric trace class operator on $\mathbb{H}$, and $m_{t,s}$ is a Lévy measure on $\mathbb{H}$. We shall write

$$\mu_{t,s} = [a_{t,s}, R_{t,s}, m_{t,s}], \quad t \geq s.$$

In terms of the characteristic exponent $\psi_{t,s}$ of $\mu_{t,s}$, condition (1.8) is equivalent to

$$\psi_{t,s}(\xi) = \psi_{t,r}(\xi) + \psi_{r,s}(U(t,r)^*\xi), \quad \xi \in \mathbb{H} \tag{3.51}$$

for every $t \geq r \geq s$.

According to (3.50) the right hand side of (3.51) is given by

$$\begin{aligned}&\psi_{t,r}(\xi) + \psi_{r,s}(U(t,r)^*\xi) \\ =& - i\langle a_{t,r}, \xi\rangle + \frac{1}{2}\langle \xi, R_{t,r}\xi\rangle - \int_{\mathbb{H}} \left(e^{i\langle \xi, x\rangle} - 1 - \frac{i\langle \xi, x\rangle}{1+|x|^2}\right) m_{t,r}(dx) \\ &- i\langle U(t,r)a_{r,s}, \xi\rangle + \frac{1}{2}\langle \xi, U(t,r)R_{r,s}U(t,r)^*\xi\rangle \\ &\qquad - \int_{\mathbb{H}} \left(e^{i\langle \xi, U(t,r)x\rangle} - 1 - \frac{i\langle \xi, U(t,r)x\rangle}{1+|x|^2}\right) m_{r,s}(dx) \\ =& - i\langle a_{t,r} + U(t,r)a_{r,s}, \xi\rangle + \frac{1}{2}\langle \xi, (R_{t,r} + U(t,r)R_{r,s}U(t,r)^*)\xi\rangle \\ &- \int_{\mathbb{H}} \left(e^{i\langle \xi, x\rangle} - 1 - \frac{i\langle \xi, x\rangle}{1+|x|^2}\right) (m_{t,r} + m_{r,s}\circ U(t,r)^{-1})(dx) \\ &+ \int_{\mathbb{H}} \frac{i\langle \xi, U(t,r)x\rangle}{1+|x|^2} m_{r,s}(dx) - \int_{\mathbb{H}} \frac{i\langle \xi, U(t,r)x\rangle}{1+|U(t,r)x|^2} m_{r,s}(dx).\end{aligned}$$

Therefore, by the uniqueness of the canonical representation for infinitely divisible distributions we have the following identities (cf. also the proof of [40, Corollary 1.4.11]): for every $t \geq r \geq s$,

$$\begin{aligned}a_{t,s} &= a_{t,r} + U(t,r)a_{r,s} \\ &\qquad + \int_{\mathbb{H}} U(t,r)x \left(\frac{1}{1+|U(t,r)x|^2} - \frac{1}{1+|x|^2}\right) m_{r,s}(dx), \\ R_{t,s} &= R_{t,r} + U(t,r)R_{r,s}U(t,r)^*, \\ m_{t,s} &= m_{t,r} + m_{r,s}\circ U(t,r)^{-1}\end{aligned} \tag{3.52}$$

In particular, from (3.52) (or directly from (3.10)) we have

$$a_{t,t} = 0, \quad R_{t,t} = 0, \quad m_{t,t} = 0, \quad t \in \mathbb{R}. \tag{3.53}$$

3.4. **Associated stochastic processes.** There are natural Lévy processes and additive processes associated with the convolution equations (3.4) and (3.1) respectively. We refer to [47, Theorem 7.10 (ii) and Theorem 9.7 (ii)] for details. The following theorem shows that in some sense there is also a natural additive process associated with the family of measures satisfying the skewed convolution equations (1.8).

Theorem 3.18. *Let $t_0 \in \mathbb{R}$ and $(\mu_{t,s})_{t_0 \geq t \geq s}$ be a system of probability measures on $\mathbb{H}$ such that for all $s \leq r \leq t \leq t_0$,*

$$\mu_{t,s} = \mu_{t,r} * (\mu_{r,s} \circ U(t,r)^{-1}) \tag{3.54}$$

and

$$\begin{aligned} \mu_{t,s} &\Rightarrow \delta_0 \quad \text{as } s \uparrow t, \\ \mu_{t,s} &\Rightarrow \delta_0 \quad \text{as } t \downarrow s. \end{aligned} \tag{3.55}$$

Set for all $s \leq t \leq t_0$

$$\tilde{\mu}_{t,s} := \mu_{t,s} \circ U(t_0, t)^{-1}.$$

Then

(1) For all $s \leq r \leq t \leq t_0$,

$$\tilde{\mu}_{t,s} = \tilde{\mu}_{t,r} * \tilde{\mu}_{r,s}, \tag{3.56}$$

$$\tilde{\mu}_{s,s} = \delta_0 \tag{3.57}$$

and

$$\begin{aligned} \tilde{\mu}_{t,s} &\Rightarrow \delta_0 \quad \text{as } s \uparrow t, \\ \tilde{\mu}_{t,s} &\Rightarrow \delta_0 \quad \text{as } t \downarrow s. \end{aligned} \tag{3.58}$$

(2) There is a stochastically continuous additive process $(X_t)_{t_0 \geq t}$ satisfying the following conditions:

(a) For all $t \leq t_0$, X_t has the distribution $\mu_{t_0,t}$. In particular, $X_{t_0} = 0$ almost surely.

(b) For all $t_0 \geq t \geq s$, the increment $X_s - X_t$ has the distribution $\tilde{\mu}_{t,s}$.

(c) For all $t_0 \geq t_1 > t_2 > \cdots > t_n$, the increments $X_{t_j} - X_{t_{j-1}}$ with $j = 1, 2, \cdots, n$ are independent.

Proof. (1) By (3.54) we have

$$\begin{aligned} \tilde{\mu}_{t,s} &= \mu_{t,s} \circ U(t_0, t)^{-1} \\ &= \big(\mu_{t,r} * \big(\mu_{r,s} \circ U(t,r)^{-1}\big)\big) \circ U(t_0,t)^{-1} \\ &= \big(\mu_{t,r} \circ U(t_0,t)^{-1}\big) * \big(\mu_{r,s} \circ U(t_0, r)^{-1}\big) \\ &= \tilde{\mu}_{t,r} * \tilde{\mu}_{r,s}. \end{aligned}$$

This proves (3.56). Hence for all $s \leq t_0$, we have $\tilde{\mu}_{ss} = \tilde{\mu}_{ss} * \tilde{\mu}_{ss}$. Since the unique idempotent measure on a Hilbert space is the Dirac measure δ_0, (3.57) follows immediately.

Fix some $s_0 < t_0$. By Lemma 3.10 there exists some constant $c > 0$ such that for all $x \in \mathbb{H}$ and $s_0 \leq s \leq t \leq t_0$, $|U(t,s)x| \leq c|x|$. Hence for any $\varepsilon > 0$, as in Lemma 3.11 we have

$$\tilde{\mu}_{t,s}(\{x \in \mathbb{H} \colon |x| > \varepsilon\}) \leq \mu_{t,s}(\{x \in \mathbb{H} \colon |x| > \varepsilon/c\}).$$

Therefore, by Lemma 3.8 we obtain (3.58) from (3.55).

(2) For any $n \in \mathbb{N}$, $t_0 \geq t_1 > t_2 > \cdots > t_n$, let $\Upsilon_{t_1,t_2,\cdots,t_n}$ be the probability measure defined on $(\mathbb{H}^{\otimes n}, \mathscr{B}(\mathbb{H}^{\otimes n}))$ in the following way:

$$\begin{aligned}&\Upsilon_{t_1,t_2\cdots,t_n}(B_1 \times B_2 \times \cdots \times B_n)\\ =&\int_{\mathbb{H}} \mathbb{1}_{B_1}(y_1)\tilde{\mu}_{t_0,t_1}(dy_1)\int_{\mathbb{H}} \mathbb{1}_{B_2}(y_1+y_2)\tilde{\mu}_{t_1,t_2}(dy_2)\\ &\qquad\times\cdots\times\int_{\mathbb{H}} \mathbb{1}_{B_n}(y_1+y_2+\cdots+y_n)\tilde{\mu}_{t_{n-1},t_n}(dy_n),\end{aligned} \tag{3.59}$$

where $B_j \in \mathscr{B}(\mathbb{H})$ for $j = 1, 2, \cdots, n$.

By using (3.56), the family of probability measures $(\Upsilon_{t_1,t_2,\cdots,t_n})_{t_0 \geq t_1 > t_2 > \cdots > t_n}$ satisfies the consistency condition. Therefore, by Kolmogorov's extension theorem there is a unique probability measure $\mathbb{P}$ on the path space $(\Omega, \mathscr{F}) := \big(\mathbb{H}^{(-\infty,t_0]}, \mathcal{B}(\mathbb{H}^{(-\infty,t_0]})\big)$ such that for all $B_j \in \mathscr{B}(\mathbb{H})$, $j = 1, 2, \cdots, n$,

$$\mathbb{P}(X_{t_1} \in B_1, X_{t_2} \in B_2, \cdots, X_{t_n} \in B_n) = \Upsilon_{t_1,t_2,\cdots,t_n}(B_1 \times B_2 \times \cdots \times B_n). \tag{3.60}$$

Here X_t is the canonical process on $(\Omega, \mathscr{F})$ defined by $X_t(\omega) = \omega(t)$, $t \leq t_0$.

Note that for any $f \in B_b(\mathbb{H}^{\otimes n})$, (3.59) and (3.60) imply

$$\begin{aligned}&\mathbb{E}[f(X_{t_1}, X_{t_2}, \cdots, X_{t_n})]\\ =&\int_{\mathbb{H}^{\otimes n}} f(y_1, y_1+y_2, \cdots, y_1+y_2+\cdots+y_n)\,\tilde{\mu}_{t_0,t_1}(dy_1)\\ &\qquad\times\tilde{\mu}_{t_1,t_2}(dy_2)\times\cdots\times\tilde{\mu}_{t_{n-1},t_n}(dy_n).\end{aligned} \tag{3.61}$$

In particular, from (3.61) we get that for every $t \leq t_0$, X_t is distributed as $\tilde{\mu}_{t_0,t} = \mu_{t_0,t}$. Hence $\mathbb{P}(X_{t_0} = 0) = 1$ since $X_{t_0} \sim \tilde{\mu}_{t_0,t_0} = \delta_0$.

Let $z_1, \cdots, z_n \in \mathbb{H}$. It follows from (3.61) that

$$\begin{aligned}&\mathbb{E}\left[\exp\left(i\sum_{j=1}^n \left\langle z_j, X_{t_j} - X_{t_{j-1}}\right\rangle\right)\right]\\ =&\int_{\mathbb{H}^{\otimes n}} \exp\left(i\sum_{j=1}^n \langle z_j, y_j\rangle\right)\tilde{\mu}_{t_0,t_1}\tilde{\mu}_{t_1,t_2}(dy_2)\cdots\tilde{\mu}_{t_{n-1},t_n}(dy_n)\\ =&\prod_{j=1}^n \int_{\mathbb{H}} \exp(i\langle z_j, y_j\rangle)\,\tilde{\mu}_{t_{j-1},t_j}(dy_j).\end{aligned}$$

This implies for every $j = 1, 2, \ldots, n$,

$$\mathbb{E}\left[\exp\left(i\langle z_j, X_{t_j} - X_{t_{j-1}}\rangle\right)\right] = \int_{\mathbb{H}} \exp\left(i\langle z_j, y_j\rangle\right)\tilde{\mu}_{t_j,t_{j-1}}(dy_j) \tag{3.62}$$

and

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^n \langle z_j, X_{t_j} - X_{t_{j-1}}\rangle\right)\right] = \prod_{j=1}^n \mathbb{E}\left[\exp\left(i\langle z_j, X_{t_j} - X_{t_{j-1}}\rangle\right)\right]. \tag{3.63}$$

Equation (3.62) shows that $X_{t_j} - X_{t_{j-1}}$ has distribution $\tilde{\mu}_{t_j,t_{j-1}}$, while Equation (3.63) shows that $(X_t)_{t_0 \geq t}$ has independent increments.

For any $t_0 \geq t \geq s$, the increment $X_s - X_t$ has distribution $\tilde{\mu}_{t,s}$. It follows from (3.58) that $X_s - X_t$ converges to 0 in probability as t tends to s or s tends to t. This proves that $(X_t)_{t_0 \geq t}$ is stochastically continuous. $\square$

There is another way to construct a stochastic process associated with $(\mu_{t,s})_{t\geq s}$ satisfying (1.8).

Theorem 3.19. *Let $s_0 \in \mathbb{R}$ and $(\mu_{t,s})_{t\geq s\geq s_0}$ be a system of probability measures on $\mathbb{H}$ such that for all $s_0 \leq s \leq r \leq t$,*

$$\mu_{t,s} = \mu_{t,r} * (\mu_{r,s} \circ U(t,r)^{-1}). \tag{3.64}$$

Then there is a stochastically continuous process $(Y_s)_{s\geq s_0}$ satisfying the following conditions:

(1) For every $s_0 \leq s$, Y_s has the distribution μ_{s,s_0}. In particular, $Y_{s_0} = 0$ almost surely.
(2) For every $s_0 \leq s \leq t$, the increment $Y_t - U(t,s)Y_s$ has the distribution $\mu_{t,s}$. Moreover, $Y_t - U(t,s)Y_s$ and Y_s are independent.

Suppose that for every $s \geq s_0$, $\mu_{s+\varepsilon,s} \Rightarrow \delta_0$ as $\varepsilon \downarrow 0$. Then $Y_{s+\varepsilon}$ converges in probability to Y_s as $\varepsilon \downarrow 0$. So $\mu_{s+\varepsilon,s_0}$ converges weakly to μ_{s,s_0} as $\varepsilon \downarrow 0$.

Proof. Let $(\Omega, \mathscr{F}) = \left(\mathbb{H}^{[s_0,-\infty)}, \mathcal{B}(\mathbb{H}^{[s_0,-\infty)})\right)$ and let $(Y_s)_{s\geq s_0}$ be the canonical process on $(\Omega, \mathscr{F})$ defined by

$$Y_s(\omega) = \omega(s), \quad s \geq s_0.$$

For any $n \in \mathbb{N}$, $s_0 \leq s_1 < s_2 < \cdots < s_n$, let $\tau_{s_1,s_2,\cdots,s_n}$ be the probability measure defined on $(\mathbb{H}^{\otimes n}, \mathscr{B}(\mathbb{H}^{\otimes n}))$ by

$$\begin{aligned}
&\tau_{s_1,s_2,\cdots,s_n}(B_1 \times B_2 \times \cdots \times B_n) \\
&= \int_{\mathbb{H}} \mathbb{1}_{B_1}(y_1)\mu_{s_1,s_0}(dy_1) \int_{\mathbb{H}} \mathbb{1}_{B_1}\left(U(s_2,s_1)y_1 + y_2\right)\mu_{s_2,s_1}(dy_2) \\
&\quad \times \int_{\mathbb{H}} \mathbb{1}_{B_3}(U(s_3,s_1)y_1 + U(s_3,s_2)y_2 + y_3)\mu_{s_3,s_2}(dy_3) \\
&\quad \times \cdots \times \int_{\mathbb{H}} \mathbb{1}_{B_n}\left(\sum_{j=1}^{n-1} U(s_n,s_j)y_j + y_n\right)\mu_{s_n,s_{n-1}}(dy_n),
\end{aligned} \tag{3.65}$$

where $B_j \in \mathscr{B}(\mathbb{H})$, $j = 1, 2, \cdots, n$.

Equation (3.64) implies that the family of probability measures $(\tau_{s_1,s_2,\cdots,s_n})_{s_0\leq s_1<s_2<\cdots<s_n}$ satisfies the consistency condition. Hence by Kolmogorov's extension theorem, there is a unique probability measure $\mathbb{P}$ on $(\Omega, \mathscr{F})$ such that for all $B_j \in \mathscr{B}(\mathbb{H})$, $j = 1, 2, \cdots, n$, and $s_0 \leq s_1 < s_2 < \cdots < s_n$,

$$\mathbb{P}(Y_{s_1} \in B_1, Y_{s_2} \in B_2, \cdots, Y_{s_n} \in B_n) = \tau_{s_1,s_2,\cdots,s_n}(B_1 \times B_2 \times \cdots \times B_n). \tag{3.66}$$

Hence for every $f \in B_b(\mathbb{H}^{\otimes 2})$, $s_0 \leq s < t$, we have

$$\mathbb{E}[f(Y_s, Y_t)] = \int_{\mathbb{H}\times\mathbb{H}} f\left(y_1, U(t,s)y_1 + y_2\right)\, \mu_{s,s_0}(dy_1)\mu_{t,s}(dy_2). \tag{3.67}$$

Therefore, for every $z_1, z_2 \in \mathbb{H}$, $s_0 \le s \le t$, we have

$$\begin{aligned}&\mathbb{E}\left[\exp\left(i\left\langle z_1, X_s\right\rangle + i\langle z_2, X_t - U(t,s)X_s\rangle\right]\right.\\ &= \int_{\mathbb{H}} \exp(i\left\langle z_1, y_1\right\rangle)\mu_{s,s_0}(dy_1)\int_{\mathbb{H}} \exp(i\left\langle z_2, y_2\right\rangle)\mu_{t,s}(dy_2)\end{aligned}$$

This implies that X_s and $X_t - U(t,s)X_s$ are independent and they have distributions μ_{s,s_0} and $\mu_{t,s}$ respectively. In particular, the distribution of X_{s_0} is given by δ_0. So, $X_{s_0} = 0$ almost surely. Thus, (1) and (2) are proved.

Now for every $s \ge s_0$ and $\varepsilon > 0$, we have

$$X_{s+\varepsilon} - X_s = (X_{s+\varepsilon} - U(s+\varepsilon, s)X_s) + (U(s+\varepsilon,s) - I)\, X_s.$$

Since X_s and $X_{s+\varepsilon} - U(s+\varepsilon,s)X_s$ are independent, the distribution of $X_{s+\varepsilon} - X_s$ is given by

$$\mu_{s+\varepsilon,s} * \left(\mu_{s,s_0} \circ (U(s+\varepsilon,s) - I)^{-1}\right).$$

It is obvious that as $\varepsilon \downarrow 0$, $\mu_{s,s_0} \circ (U(s+\varepsilon,s) - I)^{-1}$ converges weakly to δ_0. Indeed, for every continuous bounded function f on $\mathbb{H}$, we have

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\int_{\mathbb{H}} f(x)\, d\mu_{s,s_0} \circ (U(s+\varepsilon,s) - I)^{-1}(x)\\ &= \int_{\mathbb{H}} \lim_{\varepsilon\downarrow 0} f\left((U(s+\varepsilon,s) - I)x\right)\, d\mu_{s,s_0}(x)\\ &= \int_{\mathbb{H}} f(0)\, d\mu_{s,s_0}(x) = f(0) = \delta_0(f).\end{aligned}$$

Suppose in addition that $\mu_{s+\varepsilon,s}$ converges weakly to δ_0 as $\varepsilon \downarrow 0$. Then we have

$$\mu_{s+\varepsilon,s} * \left(\mu_{s,s_0} \circ (U(s+\varepsilon,s) - I)^{-1}\right) \Rightarrow \delta_0, \quad \text{as } \varepsilon \downarrow 0.$$

Hence $X_{s+\varepsilon}$ converges in probablity to X_s as $\varepsilon \downarrow 0$. This implies that $X_{s+\varepsilon}$ converges in distribution to X_s as $\varepsilon \downarrow 0$. That is, $\mu_{s+\varepsilon,s_0}$ converges weakly to μ_{s,s_0} as $\varepsilon \downarrow 0$. □

In the following example, we construct concrete stochastic processes $(X_t)_{t\le t_0}$ and $(Y_s)_{s\le s_0}$ that satisfy the conditions in Theorem 3.18 and Theorem 3.19 respectively.

Example 3.20. Let $(U(t,s))_{t\ge s}$ be an evolution system of bounded operators on $\mathbb{H}$ and let $(Z_t)_{t\in\mathbb{R}}$ be a stochastically continuous additive process on some probability space $(\Omega, \mathbb{P})$ taking values in $\mathbb{H}$. As in (1.4), we assume that the following stochastic convolution integrals

$$X_{t,s} := \int_s^t U(t,\sigma)\, dZ_\sigma, \quad t \ge s$$

are well defined. Let $\mu_{t,s}$ be the distribution of $X_{t,s}$.

(1) For fixed $t_0 \in \mathbb{R}$, consider the stochastic process $(X_t)_{t_0\ge t} := (X_{t_0,t})_{t_0\ge t}$. It is clear that for all $t \le t_0$, X_t has distribution $\mu_{t_0,t}$. Moreover, $(\mu_{t,s})_{t_0\ge t\ge s}$ fulfills condition (3.54). Note that for all $t_0 \ge t > s$, the increment $X_{t_0,s} - X_{t_0,t}$ has distribution

$$\tilde{\mu}_{t,s} = \mu_{t,s} \circ U(t_0,t)^{-1}.$$

First, we note that (cf. [41])

$$X_{t_0,s} - X_{t_0,t} = \int_s^{t_0} U(t_0,\sigma)\, dZ_\sigma - \int_t^{t_0} U(t_0,\sigma)\, dZ_\sigma = U(t_0,t)\int_s^t U(t,\sigma)\, dZ_\sigma.$$

So, for all $B \in \mathscr{B}(\mathbb{H})$ we have

$$\begin{aligned}\mathbb{P}(X_{t_0,s} - X_{t_0,t} \in B) &= \mathbb{P}\left(\int_s^t U(t,\sigma)\, dZ_\sigma \in U(t_0,t)^{-1}B\right) \\ &= \mu_{t,s}(U(t_0,t)^{-1}B) = \tilde{\mu}_{t,s}(B).\end{aligned}$$

Clearly, the increments of $(X_{t_0,t})_{t_0\geq t}$ are independent. Hence

$$X_{t_0,s} - X_{t_0,t} = (X_{t_0,s} - X_{t_0,r}) + (X_{t_0,r} - X_{t_0,t}), \quad s \leq r \leq t \leq t_0,$$

implies (3.56).

Suppose that (3.55) holds. Then it follows that the process $(X_{t_0,t})_{t_0\geq t}$ is stochastically continuous. Hence (3.58) follows.

(2) For fixed $s_0 \in \mathbb{R}$, consider the stochastic process $(Y_s)_{s\geq s_0} := (X_{s,s_0})_{s\geq s_0}$. Clearly for every $s \geq s_0$, Y_s has distribution μ_{s,s_0}. Moreover, for every $t \geq s \geq s_0$, we have

$$\begin{aligned}Y_t - U(t,s)Y_s &= \int_{s_0}^t U(t,\sigma)\, dZ_\sigma - U(t,s)\int_{s_0}^s U(s,\sigma)\, dZ_\sigma \\ &= \int_{s_0}^t U(t,\sigma)\, dZ_\sigma - \int_{s_0}^s U(t,\sigma)\, dZ_\sigma \\ &= \int_s^t U(t,\sigma)\, dZ_\sigma.\end{aligned} \tag{3.68}$$

This shows that $Y_t - U(t,s)Y_s$ has distribution $\mu_{t,s}$ and that $Y_t - U(t,s)Y_s$ is independent of Y_s.

Let $s \geq s_0$ and $\varepsilon > 0$. By (3.68), it is clear that

$$Y_{s+\varepsilon} - Y_s = X_{s+\varepsilon,s} + (U(s+\varepsilon,s) - I)\, X_{s,s_0}.$$

Note that $X_{s+\varepsilon,s}$ and X_{s,s_0} are independent. So the distribution of $Y_{s+\varepsilon} - Y_s$ is given by

$$\mu_{s+\varepsilon,s} * \left(\mu_{s,s_0} \circ (U(s+\varepsilon,s) - I)^{-1}\right).$$

Suppose that $X_{s+\varepsilon,s}$ converges in probability to 0 (equivalently, $\mu_{s+\varepsilon,s}$ converges weakly to δ_0) as $\varepsilon \downarrow 0$. Since we also know that $\mu_{s,s_0} \circ (U(s+\varepsilon,s) - I)^{-1}$ converges weakly to δ_0, we obtain that $Y_{s+\varepsilon}$ converges in probability to Y_s as $\varepsilon \downarrow 0$. Therefore, $Y_{s+\varepsilon}$ converges in distribution to Y_s as $\varepsilon \downarrow 0$, i.e. $\mu_{s+\varepsilon,s_0}$ converges weakly to μ_{s,s_0} as $\varepsilon \downarrow 0$.

Using the stochastic processes constructed in Theorem 3.18 and Theorem 3.19, we have probabilistic proofs of Theorem 3.12, Theorem 3.13 and Lemma 3.17, as we shall show now:

Another proof of Theorem 3.12. Since Part (2) has been shown in Theorem 3.19, it remains to show (1) and (3). By Theorem 3.18 there is a stochastically continuous additive process $(X_t)_{t_0\geq t}$ such that for all $t_0 \geq t$, the distribution of X_t is given by $\mu_{t_0,t}$, and for all $t_0 \geq t \geq s$, the distribution of the increment $X_s - X_t$ is given by $\tilde{\mu}_{t,s} = \mu_{t,s} \circ U(t_0,t)$. Hence for every $\delta > 0$, $t_0 > s$, and $\varepsilon \in \mathbb{R}$ we have

$$\lim_{\varepsilon\to 0} \mathbb{P}(|X_{s+\varepsilon} - X_s| \geq \delta) = 0.$$

This means that $X_{s+\varepsilon}$ converges in probability to X_s as $\varepsilon \to 0$. This implies that $X_{s+\varepsilon}$ converges in distribution to X_s as $\varepsilon \to 0$. That is, $\mu_{t_0,s+\varepsilon}$ converges weakly to $\mu_{t_0,s}$ as

$\varepsilon \to 0$. Since t_0 was arbitrary, we have $\mu_{t,s+\varepsilon}$ converges weakly to $\mu_{t,s}$ as $\varepsilon \to 0$. So (1) is proved.

Now for every $t_0 \geq t \geq s$, we have

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(|(X_s - X_{t-\varepsilon}) - (X_s - X_t)| \geq \delta) = 0.$$

So $X_s - X_{t-\varepsilon}$ converges in distribution to $X_s - X_t$ as $\varepsilon \downarrow 0$. This proves (3). □

Another proof of Theorem 3.13. Similar to the proof of Theorem 3.18, there is a stochastically continuous additive process $(X_t)_{t\in\mathbb{R}}$ such that $\tilde{\mu}_{t,s}$ is the distribution of the increment $X_t - X_s$ for all $t \geq s$. Hence for every $\delta > 0$, we have

$$\lim_{\varepsilon \to 0, \eta \to 0} \mathbb{P}(|(X_{t+\varepsilon} - X_{s+\eta}) - (X_t - X_s)| \geq \delta) = 0.$$

This implies that $X_{t+\varepsilon} - X_{s+\eta}$ converges in distribution to $X_t - X_s$. Hence $\tilde{\mu}_{t+\varepsilon,s+\eta}$ converges weakly to $\tilde{\mu}_{t,s}$ as $\varepsilon \to 0$ and $\eta \to 0$. □

To give a probabilistic proof of Lemma 3.17, let us recall the following lemma which has been proved in [47, Lemma 9.6] for the finite dimensional case. The proof for the infinite dimensional case is the same.

Lemma 3.21. *A stochastically continuous process $(X_t)_{t\in\mathbb{R}}$ taking values in $\mathbb{H}$ is uniformly stochastically continuous on any finite interval. That is, for every $s_0 < t_0$, for every $\varepsilon > 0$ and $\eta > 0$, there is $\delta > 0$ such that for all s and t in $[s_0, t_0]$ with $|t-s| < \delta$, we have*

$$\mathbb{P}(|X_t - X_s| > \varepsilon) < \eta.$$

Another proof of Lemma 3.17. By Theorem 3.18 there is a stochastically continuous additive process $(X_t)_{t_0 \geq t \geq s_0}$ such that $\tilde{\mu}_{t,s}$ is the distribution of the increment $X_s - X_t$ for all $t_0 \geq t \geq s \geq s_0$. By Lemma 3.21 we obtain that $(X_t)_{t_0 \geq t \geq s_0}$ is uniformly stochastically continuous. This means that for every $\varepsilon > 0$ and $\eta > 0$, there is a $\delta > 0$ such that for all $s, t \in [s_0, t_0]$ satisfying $|t - s| < \delta$, we have

$$\mathbb{P}(|X_s - X_t| > \varepsilon) < \eta.$$

In other words

$$\tilde{\mu}_{t,s}(|x| > \varepsilon) < \eta.$$

This proves (3.37) since $\tilde{\mu}_{t,s} = \mu_{t,s} \circ U(t_0, t)^{-1}$ by definition. □

4. EVOLUTION SYSTEMS OF MEASURES

In general, we cannot expect a stationary invariant measure for the time inhomogeneous generalized Mehler semigroup $(p_{s,t})_{t\geq s}$ defined in (1.7). So, we shall look for a family of probability measures $(\nu_t)_{t\in\mathbb{R}}$ on $\mathbb{H}$ such that

$$\int_{\mathbb{H}} p_{s,t} f(x)\, \nu_s(dx) = \int_{\mathbb{H}} f(x)\, \nu_t(dx), \quad s \leq t \tag{4.1}$$

for all $f \in B_b(\mathbb{H})$. Such a family of probability measures is called an evolution system of measures for $(p_{s,t})_{t\geq s}$ in [14] (and entrance law in [20]). It can be regarded as a space-time invariant measure for $(p_{s,t})_{t\geq s}$.

We shall first show some properties of evolution systems of measures for $(p_{s,t})_{t\geq s}$ and then study their existence and uniqueness.

4.1. Some properties.

Lemma 4.1. *A family of probability measures* $(\nu_t)_{t\in\mathbb{R}}$ *on* $\mathbb{H}$ *is an evolution system of measures for* $(p_{s,t})_{t\geq s}$ *if and only if for every* $t \geq s$,

$$\mu_{t,s} * \big(\nu_s \circ U(t,s)^{-1}\big) = \nu_t, \tag{4.2}$$

or equivalently, for every $t \geq s$,

$$\hat{\mu}_{t,s}(\xi)\hat{\nu}_s\big(U(t,s)^*\xi\big) = \hat{\nu}_t(\xi), \quad \xi \in \mathbb{H}. \tag{4.3}$$

Proof. Note that for all $f \in B_b(\mathbb{H})$ and $s \leq t$, we have

$$\begin{aligned}
&\int_{\mathbb{H}} p_{s,t}f(x)\,\nu_s(dx)\\
&= \int_{\mathbb{H}}\int_{\mathbb{H}} f(U(t,s)x+y)\,\mu_{t,s}(dy)\nu_s(dx)\\
&= \int_{\mathbb{H}}\int_{\mathbb{H}} f(x+y)\,\mu_{t,s}(dy)(\nu_s\circ U(t,s)^{-1})(dx)\\
&= \int_{\mathbb{H}} f(z)\,(\mu_{t,s} * (\nu_s\circ U(t,s)^{-1})(dz).
\end{aligned}$$

So, (4.1) holds if and only if for all $f \in B_b(\mathbb{H})$,

$$\int_{\mathbb{H}} f(z)\,(\mu_{t,s} * (\nu_s\circ U(t,s)^{-1})(dz) = \int_{\mathbb{H}} f(z)\,\nu_t(dz). \tag{4.4}$$

Thus, the proof is complete by noting that (4.4) holds if and only if (4.2) holds. □

Remark 4.2. A probability measure μ on $\mathbb{H}$ is said to be operator self-decomposable if

$$\mu = (\mu\circ T_t^{-1}) * \mu_t, \quad t \geq 0, \tag{4.5}$$

holds for a family of semigroups $(T_t)_{t\geq 0}$ and measures $(\mu_t)_{t\geq 0}$. Operator self-decomposability has been studied very well, see for example, [2, 5, 31, 50, 56] and the references therein. In the setting of (1.1), any solution μ to the convolution equation (4.5) is just an invariant measure for the generalized Mehler semigroup (1.1). Obviously, Equation (4.5) is the homogeneous version of Equation (4.2).

Proposition 4.3. *Let* $(\nu_t)_{t\in\mathbb{R}}$ *and* $(\mu_{t,s})_{t\geq s}$ *be families of probability measures on* $\mathbb{H}$. *Let* $(U(t,s))_{t\geq s}$ *be an evolution family of operators on* $\mathbb{H}$. *Suppose that* (4.2) *holds and for all* $\xi \in \mathbb{H}$ *and all* $t,s\in\mathbb{R}, t\geq s$, $\hat{\nu}_t(U(t,s)^*\xi)\neq 0$. *Then* $(\mu_{t,s})_{t\geq s}$ *satisfies* (1.8).

Proof. For any $t \geq r \geq s$, by (4.2) and (3.49) we have

$$\begin{aligned}
\mu_{t,s} * \big(\nu_s\circ U(t,s)^{-1}\big) &= \nu_t\\
&= \mu_{t,r} * \big(\nu_r\circ U(t,r)^{-1}\big)\\
&= \mu_{t,r} * \big([\mu_{r,s} * (\nu_s\circ U(r,s)^{-1})]\circ U(t,r)^{-1}\big)\\
&= \mu_{t,r} * \big([\mu_{r,s}\circ U(t,r)^{-1}] * [(\nu_s\circ U(r,s)^{-1})\circ U(t,r)^{-1}]\big)\\
&= \mu_{t,r} * \big(\mu_{r,s}\circ U(t,r)^{-1}\big) * \big(\nu_s\circ U(t,s)^{-1}\big).
\end{aligned}$$

So, for all $\xi \in \mathbb{H}$, we have

$$\hat{\mu}_{t,s}(\xi)\cdot\hat{\nu}_s(U(t,s)^*\xi) = \hat{\mu}_{t,r}(\xi)\cdot\hat{\mu}_{r,s}(U(r,s)^*\xi)\cdot\hat{\nu}_s(U(t,s)^*\xi).$$

Since $\hat{\nu}_t(U(t,s)^*\xi) \neq 0$ by assumption, we have

$$\hat{\mu}_{t,s}(\xi) = \hat{\mu}_{t,r}(\xi) \cdot \hat{\mu}_{r,s}(U(r,s)^*\xi).$$

This proves (1.8). □

Similar as in Theorem 3.19, there exists a stochastic process associated with $(\nu_t)_{t\in\mathbb{R}}$.

Theorem 4.4. *Let $(\nu_t)_{t\in\mathbb{R}}$ and $(\mu_{t,s})_{t\geq s}$ be families of probability measures on $\mathbb{H}$. Let $(U(t,s))_{t\geq s}$ be an evolution family of operators on $\mathbb{H}$. Suppose that* (1.8) *and* (4.2) *hold. Then there is a stochastic process $(X_{t,-\infty})_{t\in\mathbb{R}}$, such that for every $t \geq s$, $X_{t,-\infty} - U(t,s)X_{s,-\infty}$ and $X_{s,-\infty}$ are independent and have distributions $\mu_{t,s}$ and ν_s respectively.*

Proof. Let $\Omega = \mathbb{H}^{(-\infty,\infty)}$ be the collection of all functions $\omega = (\omega(t))_{t\in(-\infty,\infty)}$ from $(-\infty,\infty)$ into $\mathbb{H}$. Let

$$X_{t,-\infty}(\omega) = \omega(t), \quad t \in (-\infty,\infty),$$

be the canonical process on Ω. Let $\mathscr{F}$ be the Borel σ-algebra generated by cylinder sets on Ω. For any $n \in \mathbb{N}$, $-\infty < t_1 \leq t_2 \leq \cdots \leq t_n < \infty$, $B_j \in \mathscr{B}(\mathbb{H})$, $j = 1, 2, \cdots, n$, define

$$\begin{aligned}
&\nu_{t_1,t_2,\cdots,t_n}(B_1 \times B_2 \times \cdots \times B_n) \\
=& \int_{\mathbb{H}} \mathbb{1}_{B_1}(y_1)\nu_{t_1}(dy_1) \int_{\mathbb{H}} \mathbb{1}_{B_2}(U(t_2,t_1)y_1 + y_2)\mu_{t_2,t_1}(dy_2) \\
&\times \int_{\mathbb{H}} \mathbb{1}_{B_3}(U(t_3,t_1)y_1 + U(t_3,t_2)y_2 + y_3)\mu_{t_3,t_2}(dy_3) \times \cdots \\
&\times \int_{\mathbb{H}} \mathbb{1}_{B_n}\left(\sum_{j=1}^{n-1} U(t_n,t_j)y_j + y_n\right)\mu_{t_n,t_{n-1}}(dy_n).
\end{aligned} \tag{4.6}$$

Then we extend $\nu_{t_1,t_2,\cdots,t_n}$ to a probability measure on $(\mathbb{H}^{\otimes n}, \mathscr{B}(\mathbb{H}^{\otimes n}))$. Then it is easy to check that the family of probability measures $\{\nu_{t_1,t_2,\cdots,t_n}\}_{t_1\leq t_2\leq\cdots\leq t_n}$ satisfies the consistency condition. Therefore, by Kolmogorov's extension theorem there is a unique probability measure $\mathbb{P}$ on $(\Omega,\mathscr{F})$ such that for all $-\infty < t_1 \leq t_2 \leq \cdots \leq t_n < \infty$, and $B_j \in \mathscr{B}(\mathbb{H})$, $j = 1, 2, \cdots, n$, we have

$$\begin{aligned}
&\mathbb{P}(X_{t_1,-\infty} \in B_1, X_{t_2,-\infty} \in B_2, \cdots, X_{t_n,-\infty} \in B_n) \\
=&\nu_{t_1,t_2,\cdots,t_n}(B_1 \times B_2 \times \cdots \times B_n).
\end{aligned} \tag{4.7}$$

Similar to the proof of Theorem 3.19, for every $z_1, z_2 \in \mathbb{H}$, $s \leq t$, we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(i\langle z_1, X_{s,-\infty}\rangle + i\langle z_2, X_{t,-\infty} - U(t,s)X_{s,-\infty}\rangle\right]\right. \\
=& \int_{\mathbb{H}} \exp(i\langle z_1, y_1\rangle)\nu_s(dy_1) \int_{\mathbb{H}} \exp(i\langle z_2, y_2\rangle)\mu_{t,s}(dy_2).
\end{aligned}$$

This implies that $X_{s,-\infty}$ and $X_t - U(t,s)X_{s,-\infty}$ are independent and they have distributions ν_s and $\mu_{t,s}$ respectively. So, the proof is complete. □

Example 4.5. As in Example 3.20 let $(U(t,s))_{t\geq s}$ be an evolution system of bounded operators on $\mathbb{H}$ and let $(Z_t)_{t\in\mathbb{R}}$ be an additive process taking values in $\mathbb{H}$. Suppose that for all $s < t$, $U(t,\cdot)$ is integrable with respect to $(Z_t)_{t\in\mathbb{R}}$ on $[s,t]$. That is, the following stochastic convolution integrals

$$X_{t,s} := \int_s^t U(t,\sigma)\, dZ_\sigma, \quad t \geq s$$

are well defined. Suppose that for all $t \in \mathbb{R}$, $X_{t,s}$ converges in probability to $X_{t,-\infty}$ as $s \to -\infty$. That is, we assume that the following improper integral

$$X_{t,-\infty} := \int_{-\infty}^{t} U(t,\sigma)\, dZ_\sigma, \quad t \in \mathbb{R}$$

exists (cf. [41]). For every $t \geq s$, let $\mu_{t,s}$ be the distribution of $X_{t,s}$. For every $t \in \mathbb{R}$, let ν_t denote the distribution of $X_{t,-\infty}$.

Note that for all $t \geq s$,

$$X_{t,-\infty} = U(t,s)X_{s,-\infty} + X_{t,s}. \tag{4.8}$$

So, $X_{t,-\infty} - U(t,s)X_{s,-\infty}$ has distribution $\mu_{t,s}$. Since $X_{s,-\infty}$ and $X_{t,s}$ are independent, we obtain that $X_{s,-\infty}$ and $X_{t,-\infty} - U(t,s)X_{s,-\infty}$ are independent. By (4.8) we get $\nu_t = \mu_{t,s} * \big(\nu_s \circ U(t,s)^{-1}\big)$. This proves that $(\nu_t)_{t\in\mathbb{R}}$ is an evolution system of measures for the transition function $(p_{s,t})_{t\geq s}$ of $X(t,s,x) := U(t,s)x + X_{t,s}$, $x \in \mathbb{H}$, $t \geq s$.

Concerning the infinite divisibility of ν_t, we have the following simple result.

Proposition 4.6. *Let $(\nu_t)_{t\in\mathbb{R}}$ be an evolution system of measure for $(p_{s,t})_{t\geq s}$. Suppose that for some $s_0 \in \mathbb{R}$ the measures ν_{s_0} and $(\mu_{t,s_0})_{t\geq s_0}$ are infinitely divisible, then for all $t \geq s_0$, ν_t is infinitely divisible.*

Proof. According to (4.2), for all $t \geq s_0$, ν_t is the convolution of μ_{t,s_0} and $\nu_{s_0} \circ U(t,s)^{-1}$. Since μ_{t,s_0} is infinitely divisible, we only need to show that $\nu_{s_0} \circ U(t,s_0)^{-1}$ is also infinitely divisible.

Because ν_{s_0} is infinitely divisible, for any $n \in \mathbb{N}$, there is some probability measure $\nu_{s_0}^{(n)}$ on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$ such that $\nu_{s_0} = (\nu_{s_0}^{(n)})^{*n}$. So by (3.49), we have

$$\nu_{s_0} \circ U(t,s_0)^{-1} = (\nu_{s_0}^{(n)} \circ U(t,s_0)^{-1})^{*n}.$$

for all $t \geq s_0$. This proves that $\nu_{s_0} \circ U(t,s_0)^{-1}$ is also infinitely divisible. □

Theorem 4.7. *Let $(\nu_t^{(1)})_{t\in\mathbb{R}}$ be an evolution system of measures for $(p_{s,t})_{t\geq s}$. Let $(\nu_t^{(2)})_{t\in\mathbb{R}}$ and $(\sigma_t)_{t\in\mathbb{R}}$ be two families of probability measures on $\mathbb{H}$ such that*

$$\nu_t^{(2)} = \nu_t^{(1)} * \sigma_t, \quad t \in \mathbb{R} \tag{4.9}$$

and

$$\sigma_t = \sigma_s \circ U(t,s)^{-1}, \quad t \geq s.$$

Then $(\nu_t^{(2)})_{t\in\mathbb{R}}$ is also an evolution system of measures for $(p_{s,t})_{t\geq s}$.

Proof. For every $\xi \in \mathbb{H}$, by (4.3) and (4.9), we have

$$\begin{aligned}\hat{\nu}_t^{(2)}(\xi) &= \hat{\nu}_t^{(1)}(\xi)\hat{\sigma}_t(\xi) = \hat{\mu}_{t,s}(\xi)\hat{\nu}_s^{(1)}\big(U(t,s)^*\xi\big)\hat{\sigma}_t(\xi)\\ &= \hat{\mu}_{t,s}(\xi)\hat{\nu}_s^{(1)}\big(U(t,s)^*\xi\big)\hat{\sigma}_s(U(t,s)^*\xi) = \hat{\mu}_{t,s}(\xi)\hat{\nu}_s^{(2)}\big(U(t,s)^*\xi\big).\end{aligned}$$

Hence the assertion follows by Lemma 4.1. □

4.2. **Existence and uniqueness.** For special cases of our $(p_{s,t})_{t\geq s}$ in this paper, existence and uniqueness of evolution systems of measures have been studied in [13, 32, 57] etc. in different settings. Our framework is more general. We emphasize that Theorem 4.9, Theorem 4.10 and Corollary 4.12 below not only generalize the corresponding results in [57] for finite dimensional Lévy driven non-autonomous Ornstein-Uhlenbeck processes, but also contain some new results even in the finite dimension case.

From now on we always assume that

(1) The family $(\mu_{t,s})_{t\geq s}$ satisfies (1.8). That is, for all $s \leq r \leq t$,

$$\mu_{t,s} = \mu_{t,r} * \left(\mu_{r,s} \circ U(t,r)^{-1}\right).$$

(2) For every $t \geq s$, $\mu_{t,s}$ is infinitely divisible with representation (as e.g. in the case it satisfies Assumption 3.7)

$$\mu_{t,s} = [a_{t,s}, R_{t,s}, \mathrm{m}_{t,s}],$$

where $a_{t,s} \in \mathbb{H}$, $R_{t,s}$ is a nonnegative definite, self-adjoint trace class operator on $\mathbb{H}$, and $\mathrm{m}_{t,s}$ is a Lévy measure on $\mathbb{H}$.

By (3.52), for every fixed $t \in \mathbb{R}$, $(\mathrm{m}_{t,s})_{t\geq s}$ is a family of Lévy measures decreasing in s in the sense that for all $A \in \mathscr{B}(\mathbb{H} \setminus \{0\})$, and all $s \leq r \leq t$,

$$\mathrm{m}_{t,s}(A) \geq \mathrm{m}_{t,r}(A),$$

which allows us to define $\mathrm{m}_{t,-\infty}$ for every $t \in \mathbb{R}$ by setting $\mathrm{m}_{t,-\infty}(\{0\}) = 0$ and

$$\mathrm{m}_{t,-\infty}(A) = \lim_{s\to-\infty} \mathrm{m}_{t,s}(A), \quad A \in \mathscr{B}(\mathbb{H} \setminus \{0\}).$$

Conditions under which $\mathrm{m}_{t,-\infty}$ is a Lévy measure will be given later in Theorem 4.9.

Lemma 4.8. *Suppose that for every $t \in \mathbb{R}$,*

$$\sup_{s\leq t} \operatorname{tr} R_{t,s} < \infty. \tag{4.10}$$

Then there is a trace class operator $R_{t,-\infty}$ on $\mathbb{H}$ such that for all $x, y \in \mathbb{H}$,

$$\langle R_{t,-\infty}x, y\rangle = \lim_{s\to-\infty} \langle R_{t,s}x, y\rangle. \tag{4.11}$$

Proof. By (3.52), for every $x \in \mathbb{H}$ and $t \in \mathbb{R}$, $\langle R_{t,s}x, x\rangle$ is decreasing in s. More precisely, for every $x \in \mathbb{H}$ and $s \leq r \leq t$, we have

$$\langle R_{t,s}x, x\rangle = \langle R_{t,r}x, x\rangle + \langle R_{r,s}U(t,r)^*x, U(t,r)^*x\rangle \geq \langle R_{t,r}x, x\rangle.$$

It follows that for every $s \leq r \leq t$,

$$\operatorname{tr} R_{t,r} \leq \operatorname{tr} R_{t,s}.$$

Therefore, as s tends to $-\infty$, the limit of $\langle R_{t,s}x, x\rangle$ exists.

By (4.10), for every $t \in \mathbb{R}$, there exists a constant $C_t > 0$ such that

$$\sup_{s\leq t} \langle R_{t,s}x, x\rangle \leq C_t|x|^2$$

holds for every $x \in \mathbb{H}$. By the polarization identity, for every $t \in \mathbb{R}$, $x, y \in \mathbb{H}$, $\lim_{s\to-\infty}\langle R_{t,s}x, y\rangle$ exists. Fixing $x \in \mathbb{H}$ and letting $y \in \mathbb{H}$ vary, we get a functional $\lim_{s\to-\infty}\langle R_{t,s}x, \cdot\rangle$. So, by

Riesz's representation theorem, for every $x \in \mathbb{H}$, there exists an element $x_t^* \in \mathbb{H}$ for every $t \in \mathbb{R}$ such that for all $y \in \mathbb{H}$,

$$\lim_{s\to-\infty} \langle R_{t,s}x, y\rangle = \langle x_t^*, y\rangle.$$

Let $R_{t,-\infty}$ denote the map $x \mapsto x_t^*$. By (4.10), it is clear that $R_{t,-\infty}$ is a trace class operator. This proves (4.11). □

Now we are ready to show the following result on the existence of evolution system of measures for $(p_{s,t})_{t\geq s}$.

Theorem 4.9. *Suppose that for every $t \in \mathbb{R}$, the following three hypotheses hold:*

(H1) $\sup_{s\leq t} \operatorname{tr} R_{t,s} < \infty$*;*
(H2) $\sup_{s\leq t} \int_{\mathbb{H}} (1 \wedge |x|^2)\, \mathrm{m}_{t,s}(dx) < \infty$*;*
(H3) *For every $t \in \mathbb{R}$, $a_{t,-\infty} := \lim_{s\to-\infty} a_{t,s}$ exists and is finite.*

Then for every $t \in \mathbb{R}$, $\mathrm{m}_{t,-\infty}$ is a Lévy measure, $R_{t,-\infty}$ is a nonnegative definite, self-adjoint trace class operator such that

$$\operatorname{tr} R_{t,-\infty} = \sup_{s\leq t} R_{t,s} < \infty.$$

Moreover, the system of measures $(\nu_t)_{t\in\mathbb{R}}$ given by

$$\nu_t = [a_{t,-\infty}, R_{t,-\infty}, \mathrm{m}_{t,-\infty}], \quad t \in \mathbb{R}$$

is an evolution system of measures for $(p_{s,t})_{t\geq s}$.

Proof. Suppose that (H1), (H2) and (H3) hold. By Lemma 4.8, for every $t \in \mathbb{R}$, $R_{t,-\infty}$ is a nonnegative definite, self-adjoint trace class operator satisfying

$$\operatorname{tr} R_{t,-\infty} = \sup_{s\leq t} R_{t,s} < \infty.$$

For each $t \in \mathbb{R}$,

$$\int_{\mathbb{H}} (1 \wedge |y|^2)\, \mathrm{m}_{t,-\infty}(dy) = \sup_{s\leq t} \int_{\mathbb{H}} (1 \wedge |x|^2)\, \mathrm{m}_{t,s}(dx) < \infty.$$

This shows that $\mathrm{m}_{t,-\infty}$ is a Lévy measure.

Now we show that $(\nu_t)_{t\in\mathbb{R}}$ is an evolution system of measures for $(p_{s,t})_{t\geq s}$. By (1.8), for every $t \geq s \geq r$,

$$\mu_{t,s} * \left(\mu_{s,r} \circ U(t,s)^{-1}\right) = \mu_{t,r}. \tag{4.12}$$

Note that $\mu_{t,s} = [a_{t,s}, R_{t,s}, \mathrm{m}_{t,s}]$ converges weakly to $[a_{t,-\infty}, R_{t,-\infty}, \mathrm{m}_{t,-\infty}] = \nu_t$ as $s \to -\infty$ (cf. [22, Lemma 3.4]). Hence letting $r \to -\infty$ on both sides of (4.12) we obtain

$$\mu_{t,s} * \left(\nu_s \circ U(t,s)^{-1}\right) = \nu_t.$$

By Lemma 4.1 this proves that $(\nu_t)_{t\in\mathbb{R}}$ is an evolution system of measures for $(p_{s,t})_{t\geq s}$. □

The following theorem is the converse to Theorem 4.9. It also gives some sufficient conditions for the uniqueness of the evolution system of measures.

Theorem 4.10. *Suppose that there is an evolution system of measures $(\tilde{\nu}_t)_{t\in\mathbb{R}}$ for $(p_{s,t})_{t\geq s}$. Then*

(1) Hypotheses (H1) *and* (H2) *hold and for every* $t \in \mathbb{R}$

$$[0, R_{t,s}, \mathrm{m}_{t,s}] \Rightarrow [0, R_{t,-\infty}, \mathrm{m}_{t,-\infty}] \quad \text{as } s \to -\infty. \tag{4.13}$$

(2) There exists a family of probability measures $(\tilde{\sigma}_t)_{t\in\mathbb{R}}$ *such that for every* $t \in \mathbb{R}$,

$$\delta_{a_{t,s}} * \big(\tilde{\nu}_s \circ U(t,s)^{-1}\big) \Rightarrow \tilde{\sigma}_t \quad \text{as } s \to -\infty \tag{4.14}$$

and

$$\tilde{\nu}_t = [0, R_{t,-\infty}, \mathrm{m}_{t,-\infty}] * \tilde{\sigma}_t, \quad t \in \mathbb{R}. \tag{4.15}$$

(3) Suppose that the following hypothesis holds:

(H4) *For every* $t \in \mathbb{R}$, *as* $s \to -\infty$, $\tilde{\nu}_s \circ U(t,s)^{-1}$ *converges weakly to some probability measure* σ_t *on* $\mathbb{H}$.

If for each $t \in \mathbb{R}$, σ_t *is infinitely divisible, then the limit in* (H3) *exists. Moreover, for all* $t \in \mathbb{R}$ *we have*

$$\tilde{\sigma}_t = \delta_{a_{t,-\infty}} * \sigma_t, \tag{4.16}$$

$$\tilde{\nu}_t = \nu_t * \sigma_t \tag{4.17}$$

and

$$\sigma_t = \sigma_s \circ U(t,s)^{-1}, \quad t \geq s. \tag{4.18}$$

Here $(\nu_t)_{t\in\mathbb{R}}$ *is as defined in Theorem 4.9.*

(4) If the limit in (H3) *exists, then the limit in* (H4) *exists. Hence* (4.16), (4.17) *and* (4.18) *hold.*

Proof. (1) For every $t \geq s$, we set

$$N_{R_{t,s}} := [0, N_{R_{t,s}}, 0], \quad M_{t,s} := [0, 0, \mathrm{m}_{t,s}].$$

Since $(\tilde{\nu}_t)_{t\in\mathbb{R}}$ is an evolution system of measures for $(p_{s,t})_{t\geq s}$, by Lemma 4.1 we have for all $t \geq s$,

$$\begin{aligned}\tilde{\nu}_t = \mu_{t,s} * \big(\tilde{\nu}_s \circ U(t,s)^{-1}\big) &= [a_{t,s}, R_{t,s}, \mathrm{m}_{t,s}] * \big(\tilde{\nu}_s \circ U(t,s)^{-1}\big) \\ &= \delta_{a_{t,s}} * N_{R_{t,s}} * M_{t,s} * \big(\tilde{\nu}_s \circ U(t,s)^{-1}\big).\end{aligned} \tag{4.19}$$

Applying Theorem 3.1 to (4.19), the sequence of probability measure $(\delta_{a_{t,-n}} * N_{R_{t,-n}} * M_{t,-n})_{n\geq 1}$ is shift compact. That is, for every $t \in \mathbb{R}$, there exists a sequence $(y_{t,-n})_{n\geq 1}$ in $\mathbb{H}$ such that the following sequence of probability measures

$$\delta_{y_{t,-n}} * (\delta_{a_{t,-n}} * N_{R_{t,-n}} * M_{t,-n}) = [y_{t,-n} + a_{t,-n}, R_{t,-n}, \mathrm{m}_{t,-n}], \quad n \geq 1,$$

is weakly relatively compact. This implies (see [43, Theorem VI.5.3])

$$\sup_{n\in\mathbb{N}} \mathrm{m}_{t,-n}(\{x \in \mathbb{H} \colon |x| \geq 1\}) < \infty \tag{4.20}$$

and

$$\sup_{n\in\mathbb{N}} \left(\operatorname{tr} R_{t,-n} + \int_{|x|<1} |x|^2 \, \mathrm{m}_{t,-n}(dx) \right) < \infty. \tag{4.21}$$

It follows from (4.21) that

$$\sup_{s\leq t} \operatorname{tr} R_{t,s} < \infty.$$

Combining (4.20) and (4.21) we obtain

$$\sup_{s\leq t} \int_{\mathbb{H}} (1 \wedge |x|^2) \, \mathrm{m}_{t,s}(dx) < \infty.$$

Therefore, for every $t \in \mathbb{R}$ we obtain by taking limits as before a Lévy measure $\mathrm{m}_{t,-\infty}$ and a trace class operator $R_{t,-\infty}$. Hence (4.13) follows.

(2) By (4.19) we have for all $t \geq s$

$$\tilde{\nu}_t = \big(\delta_{a_{t,s}} * \big(\tilde{\nu}_s \circ U(t,s)^{-1}\big)\big) * N_{R_{t,s}} * M_{t,s}.$$

We have shown in (4.13) that $N_{R_{t,s}} * M_{t,s}$ converges weakly to an infinitely divisible measure $[0, R_{t,-\infty}, \mathrm{m}_{t,-\infty}]$ as $s \to -\infty$. Therefore by Corollary 3.5, the measures $\delta_{a_{t,s}} * \big(\tilde{\nu}_s \circ U(t,s)^{-1}\big)$ converges weakly as $s \to -\infty$. So, (4.14) and (4.15) are proved.

(3) Applying Corollary 3.5 to (4.14), using (H4) and the assumption that σ_t is infinitely divisible for every $t \in \mathbb{R}$, the limit of $a_{t,s}$ as $s \to -\infty$ exists. So, (H3) and hence (4.16) hold. By (4.15) and (4.16), we have for every $t \in \mathbb{R}$,

$$\tilde{\nu}_t = [0, R_{t,-\infty}, \mathrm{m}_{t,-\infty}] * \tilde{\sigma}_t = [a_{t,-\infty}, R_{t,-\infty}, \mathrm{m}_{t,-\infty}] * \sigma_t = \nu_t * \sigma_t.$$

This proves (4.17).

Now we show (4.18). For every $u \leq s \leq t$ and $\xi \in \mathbb{H}$, we have

$$\hat{\tilde{\nu}}_u(U(t,u)^*\xi) = \hat{\tilde{\nu}}_u\big(U(s,u)^*U(t,s)^*\xi\big). \tag{4.22}$$

Letting $u \to -\infty$ in (4.22) and using (H4), we get $\hat{\sigma}_t(\xi) = \hat{\sigma}_s\big(U(t,s)^*\xi\big)$. This is equivalent to (4.18).

(4) If (H3) holds, then $\delta_{a_{t,s}} \Rightarrow \delta_{a_{t,-\infty}}$ as $s \to -\infty$. Note that any Dirac measure is infinitely divisible. So, by applying Corollary 3.5 to (4.14), we see that the limit in (H4) exists and hence (4.17) and (4.18) hold.

□

Remark 4.11. As is known, any invariant measure ν for a time homogeneous Gaussian Ornstein-Uhlenbeck semigroup is of the form $\nu * \mu_\infty$, where ν is a measure on $\mathbb{H}$ that is invariant under the action of a semigroup, and μ_∞ is a Gaussian measure. We refer to [24, Theorem 5.22] for details. We emphasize that the structure of $\nu * \mu_\infty$ is analogous to (4.17).

By Theorem 4.10 we have the following result on the uniqueness.

Corollary 4.12. *Let $(\tilde{\nu}_t)_{t\in\mathbb{R}}$ be an evolution system of measures for $(p_{s,t})_{t\geq s}$. Suppose that for every $t \in \mathbb{R}$, there is a sequence $(s_n)_{n\geq 1}$ bounded above by t such that the following conditions are fulfilled:*

(1) As $n \to \infty$, $s_n \to -\infty$.

(2) There exist some constants $M, \omega > 0$ such that

$$\|U(t,s_n)\| \leq M\,\mathrm{e}^{-\omega(t-s_n)}. \tag{4.23}$$

(3) The sequence of probability measures $(\tilde{\nu}_{s_n})_{n\geq 1}$ is uniformly tight.

Then (H1), (H2) *and* (H3) *hold. Hence $([a_{t,-\infty}, R_{t,-\infty}, \mathrm{m}_{t,-\infty}])_{t\in\mathbb{R}}$ exists and it is the unique evolution system of measures for $(p_{s,t})_{t\geq s}$, hence equal to $(\tilde{\nu}_t)_{t\in\mathbb{R}}$.*

Proof. By the proof of Theorem 4.10, it is sufficient to show

$$\hat{\tilde{\nu}}_{s_n} \circ U(t,s_n)^{-1} \Rightarrow \delta_0 \quad \text{as } n \to \infty. \tag{4.24}$$

Let $\varepsilon, \eta > 0$. Since $(\tilde{\nu}_{s_n})_{n\geq 1}$ is uniformly tight, there is a compact set $K_\eta \subset \mathbb{H}$ such that for all $n \geq 1$,

$$\tilde{\nu}_{s_n}(\mathbb{H} \setminus K_\eta) < \eta. \tag{4.25}$$

Set for all $n \geq 1$, $C_n = M^{-1} \mathrm{e}^{\omega(t-s_n)}$. Since $s_n \to -\infty$ as $n \to \infty$, also $C_n \to \infty$ as $n \to \infty$. So, there exists some $N_0 > 0$ such that the compact set K_η is contained in $\{x \in \mathbb{H}\colon |x| \leq \varepsilon C_n\}$ for all $n \geq N_0$. Therefore, for all $n \geq N_0$, we have

$$\tilde{\nu}_{s_n}(\{x \in \mathbb{H}\colon |x| > \varepsilon C_n\}) \leq \tilde{\nu}_{s_n}(\{\mathbb{H} \setminus K_\eta\}) < \eta. \tag{4.26}$$

Because

$$\|U(t, s_n)\| \leq M \mathrm{e}^{-\omega(t-s_n)} = 1/C_n,$$

by (4.26) we have for all $n \geq N_0$,

$$\tilde{\nu}_{s_n} \circ U(t, s_n)^{-1}(\{x \in \mathbb{H}\colon |x| > \varepsilon\}) \leq \tilde{\nu}_{s_n}(\{x \in \mathbb{H}\colon |x| > \varepsilon C_n) < \eta.$$

By Lemma 3.8 this implies (4.24). Thus, the proof is complete. □

As an application of Corollary 4.12, we consider the uniqueness of periodic evolution systems of measures.

Corollary 4.13. *Suppose that* (H1), (H2) *and* (H3) *hold and that* $(\mu_{t,s})_{t\geq s}$ *is periodic with period* $T > 0$, *i.e. for every* $t \geq s$, $\mu_{t+T,s+T} = \mu_{t,s}$. *Then* $(\nu_t)_{t\in\mathbb{R}} = ([a_{t,-\infty}, R_{t,-\infty}, \mathrm{m}_{t,-\infty}])_{t\in\mathbb{R}}$ *exists and is a periodic evoluton system of measures for* $(p_{s,t})_{t\geq s}$ *with period* T. *That is, for every* $t \in \mathbb{R}$, $\nu_{t+T} = \nu_t$. *Suppose in addition that there exist some constants* $M, \omega > 0$ *such that*

$$\|U(t, s)\| \leq M \mathrm{e}^{-\omega(t-s)}.$$

Then $(\nu_t)_{t\in\mathbb{R}}$ *is the unique periodic evolution system of measures with period* T *for* $(p_{s,t})_{t\geq s}$.

Proof. By Theorem 4.9, $(\nu_t)_{t\in\mathbb{R}} = ([a_{t,-\infty}, R_{t,-\infty}, \mathrm{m}_{t,-\infty}])_{t\in\mathbb{R}}$ exists. For every $t \geq s$ we have

$$\mu_{t+T,s+T} = \mu_{t,s}. \tag{4.27}$$

By letting $s \to -\infty$ on both sides of (4.27) we obtain $\nu_{t+T} = \nu_t$. This shows that $(\nu_t)_{t\in\mathbb{R}}$ is periodic with period T.

Now it remains to show the uniqueness. Take any $s_0 \leq t$ and set $s_n = s_0 - nT$ for all $n \geq 1$. Then $\nu_{s_n} \equiv \nu_{s_0}$ for all $n \geq 1$. So it is obvious that $(\nu_{s_n})_{n\geq 1}$ is uniformly tight. By Corollary 4.12, $(\nu_t)_{t\in\mathbb{R}}$ is the unique evolution system of measures for $(p_{s,t})_{t\geq s}$. □

Remark 4.14. Existence and uniqueness of evolution systems of measures have been studied for stochastic evolution equations with time dependent periodic coefficients driven by Gaussian and Lévy processes in [13] and [32] respectively. Clearly, Corollary 4.13 applies to these cases. More generally, one can apply it to study stochastic evolution equations with time dependent periodic coefficients driven by so called semi-Lévy processes. A stochastic process $(Z_t)_{t\in\mathbb{R}}$ is called a semi-Lévy process with period $T > 0$ if it is an additive process such that for all $t \geq s$, $Z_{t+T} - Z_{s+T}$ has the same distribution as $Z_t - Z_s$ as in [39].

In [39] it is shown that for the finite dimensional case, under some conditions, ν_0 is semi-self-decomposable. Moreover, this is closely related to the so called semi-selfsimilar and semi-stationary processes. One may study similar self-decomposibility and semi-stationarity in the infinite dimensional case as in [1, 39].

5. HARNACK INEQUALITIES AND APPLICATIONS

Harnack inequalities for generalized Mehler semigroups or Ornstein-Uhlenbeck semigroup driven by Lévy processes were proved in [32, 40, 42, 45]. The method in [32] and [45] relies on taking the derivative of a proper functional; the method in [40, 42] is based on coupling of stochastic processes and Girsanov transformation. Here we shall establish a Harnack inequality for $(p_{s,t})_{t\geq s}$ defined by (1.7) by a much simper method.

Suppose that for all $t \geq s$, $\mu_{t,s} = [a_{t,s}, R_{t,s}, \mathrm{m}_{t,s}]$ is an infinitely divisible measure on $\mathbb{H}$ satisfying (1.8).

For each $t \geq s$, set

$$\mu^g_{t,s} = [0, R_{t,s}, 0], \quad \mu^j_{t,s} = [a_{t,s}, 0, \mathrm{m}_{t,s}]$$

and for every $f \in B_b(\mathbb{H})$, $x \in \mathbb{H}$, set

$$p^g_{s,t}f(x) := (\mu^g_{t,s} * \delta_{U(t,s)x})(f) = \int_{\mathbb{H}} f(U(t,s)x + y)\, \mu^g_{t,s}(dy),$$

$$p^j_{s,t}f(x) := (\mu^j_{t,s} * \delta_x)(f) = \int_{\mathbb{H}} f(x + y)\, \mu^j_{t,s}(dy).$$

With these notations, we have the following decomposition for $p_{s,t}$ which plays a key role.

Proposition 5.1. *For every $t \geq s$, $x \in \mathbb{H}$ and $f \in B_b(\mathbb{H})$, we have*

$$p_{s,t}f(x) = p^g_{s,t}(p^j_{s,t})f(x).$$

Proof. Since $\mu_{t,s} = \mu^g_{t,s} * \mu^j_{t,s}$, we get

$$\begin{aligned} p_{s,t}f(x) =& (\mu_{t,s} * \delta_{U(t,s)x})(f) = (\mu^g_{t,s} * \mu^j_{t,s} * \delta_{U(t,s)x})(f) \\ =& \big((\mu^g_{t,s} * \delta_{U(t,s)x}) * \mu^j_{t,s}\big)(f) \\ =& \int_{\mathbb{H}} \mu^g_{t,s} * \delta_{U(t,s)x}(dy) \int_{\mathbb{H}} f(y+z)\, \mu^j_{t,s}(dz) \\ =& \big(\mu^g_{t,s} * \delta_{U(t,s)x}\big)(p^j_{s,t}f) = p^g_{s,t}(p^j_{s,t}f)(x). \end{aligned}$$

This completes the proof. □

Define for every $t \geq s$,

$$\Gamma_{t,s} := R^{-1/2}_{t,s} U(t,s) \tag{5.1}$$

with domain

$$\mathscr{D}(\Gamma_{t,s}) := \{x \in \mathbb{H} \colon U(t,s)x \in R^{1/2}_{t,s}(\mathbb{H})\}.$$

If $x \notin \mathscr{D}(\Gamma_{t,s})$ then we set $|\Gamma_{t,s}x| := \infty$. Let $B^+_b(\mathbb{H})$ denote the space of all bounded positive measurable functions on $\mathbb{H}$.

Theorem 5.2. *For every $\alpha > 1$, $t \geq s$ and $f \in B^+_b(\mathbb{H})$, we have*

$$(p_{s,t}f(x))^\alpha \leq \exp\left(\frac{\alpha|\Gamma_{t,s}(x-y)|^2}{2(\alpha-1)}\right) p_{s,t}f^\alpha(y), \quad x, y \in \mathbb{H}. \tag{5.2}$$

Proof. We only need to consider the case when $U(t,s)(\mathbb{H}) \in R_{t,s}^{1/2}(\mathbb{H})$. Otherwise, the right hand side of (5.2) is infinite by the definition of $|\Gamma_{t,s}(\cdot)|$ and hence the inequality (5.2) becomes trivial.

Let us first show that it is sufficient to have the following Harnack inequality for $p^g_{s,t}$:

$$(p^g_{s,t} f(x))^\alpha \le \exp\left(\frac{\alpha|\Gamma_{t,s}(x-y)|^2}{2(\alpha-1)}\right) p^g_{s,t} f^\alpha(y), \quad x,y \in \mathbb{H}. \tag{5.3}$$

Indeed, by Proposition 5.1, we have $p_{s,t} = p^g_{s,t} p^j_{s,t}$. So by applying inequality (5.3) to $p^g_{s,t}$ and Jensen's inequality to $p^j_{s,t}$ we obtain

$$\begin{aligned}
\big(p_{s,t} f(x)\big)^\alpha =& \big(p^g_{s,t}(p^j_{s,t} f)(x)\big)^\alpha \\
\le& \exp\left(\frac{\alpha|\Gamma_{t,s}(x-y)|^2}{2(\alpha-1)}\right) \big(p^g_{s,t}(p^j_{s,t} f)^\alpha\big)(y) \\
\le& \exp\left(\frac{\alpha|\Gamma_{t,s}(x-y)|^2}{2(\alpha-1)}\right) \big(p^g_{s,t}(p^j_{s,t} f^\alpha)\big)(y) \\
=& \exp\left(\frac{\alpha|\Gamma_{t,s}(x-y)|^2}{2(\alpha-1)}\right) \big(p_{s,t} f^\alpha\big)(y).
\end{aligned}$$

This proves (5.3)

Now it remains to show (5.3). The method is the same as in the case of a one dimensional (time homogeneous) Ornstein-Uhlenbeck process, see [40, Page 69].

Let $N(m,Q)$ denote the Gaussian measure on $\mathbb{H}$ with mean $m \in \mathbb{H}$ and covariance operator Q on $\mathbb{H}$. By the Cameron-Martin formula for Gaussian measures (see [15, Theorem 2.21]), we have

$$\begin{aligned}
\rho_{t,s}(x-y,z) :=& \frac{dN(U(t,s)(x-y), R_{t,s})}{dN(0,R_{t,s})}(z) \\
=& \exp\left(\left\langle R_{t,s}^{-1/2} U(t,s)(x-y), R_{t,s}^{-1/2} z\right\rangle - \frac{1}{2}|R_{t,s}^{-1/2} U(t,s)(x-y)|^2\right).
\end{aligned} \tag{5.4}$$

Moreover, for any $h \in \mathbb{H}$, we have (see [16, Proposition 1.2.5])

$$\int_{\mathbb{H}} \exp(\langle h, x\rangle)\, d\mu^g_{t,s}(x) = \exp\left(\frac{1}{2}|R_{t,s}^{1/2} h|^2\right). \tag{5.5}$$

By changing variables and using (5.4) we obtain

$$
\begin{aligned}
&p^g_{s,t}f(x)\\
&=\int_{\mathbb{H}} f(U(t,s)x+z)\,\mu^g_{t,s}(dz)\\
&=\int_{\mathbb{H}} f(U(t,s)y+U(t,s)(x-y)+z)\,N(0,R_{t,s})(dz)\\
&=\int_{\mathbb{H}} f(U(t,s)y+z')\frac{dN(U(t,s)(x-y),R_{t,s})}{dN(0,R_{t,s})}(z')\,N(0,R_{t,s})(dz')\\
&=\int_{\mathbb{H}} f(U(t,s)y+z')\rho_{t,s}(x-y,z')\,\mu^g_{t,s}(dz')\\
&=\exp\left(-\frac{1}{2}|\Gamma_{t,s}(x-y)|^2\right)\\
&\qquad\cdot\int_{\mathbb{H}} f(U(t,s)y+z')\exp\left(\left\langle R_{t,s}^{-1/2}U(t,s)(x-y),R_{t,s}^{-1/2}z\right\rangle\right)\,\mu^g_{t,s}(dz')
\end{aligned}
\tag{5.6}
$$

By Hölder's inequality and (5.5) we have

$$
\begin{aligned}
&\int_{\mathbb{H}} f(U(t,s)y+z')\exp\left(\left\langle R_{t,s}^{-1/2}U(t,s)(x-y),R_{t,s}^{-1/2}z\right\rangle\right)\,\mu^g_{t,s}(dz')\\
&\le\left(\int_{\mathbb{H}} f^\alpha(U(t,s)y+z')\,\mu^g_{t,s}(dz')\right)^{1/\alpha}\\
&\qquad\cdot\left(\int_{\mathbb{H}}\exp\left(\frac{\alpha}{\alpha-1}\langle R_{t,s}^{-1/2}R_{t,s}^{-1/2}U(t,s)(x-y),z'\rangle\right)\,\mu^g_{t,s}(dz')\right)^{(\alpha-1)/\alpha}\\
&\le\left(p^g_{s,t}f^\alpha(y)\right)^{1/\alpha}\cdot\left(\exp\left(\frac{1}{2}\frac{\alpha^2}{(\alpha-1)^2}|R_{t,s}^{-1/2}U(t,s)(x-y)|^2\right)\right)^{(\alpha-1)/\alpha}\\
&=\left(p^g_{s,t}f^\alpha(y)\right)^{1/\alpha}\exp\left(\frac{\alpha}{2(\alpha-1)}|\Gamma_{t,s}(x-y)|^2\right).
\end{aligned}
\tag{5.7}
$$

Combining (5.6) and (5.7) we get

$$
\begin{aligned}
&p^g_{s,t}f(x)\\
&\le\exp\left(-\frac{1}{2}|\Gamma_{t,s}(x-y)|^2\right)\left(p^g_{s,t}f^\alpha(y)\right)^{1/\alpha}\exp\left(\frac{\alpha}{2(\alpha-1)}|\Gamma_{t,s}(x-y)|^2\right)\\
&=\exp\left(\frac{1}{2(\alpha-1)}|\Gamma_{t,s}(x-y)|^2\right)\left(p^g_{s,t}f^\alpha(y)\right)^{1/\alpha}.
\end{aligned}
$$

This proves (5.3). Hence the proof is complete as we have argued at the beginning that (5.3) is sufficient. □

Example 5.3. Suppose that $\mathbb{H}=L^2(0,1)$. Let $\varDelta$ denote the Dirichlet Laplacian on $\mathbb{H}$. Let e_n, $n\ge 1$, be the eigenbasis of $\varDelta$ with respective eigenvalues $-n^2\pi^2$, $n\ge 1$. Let $a(t)$ be a continuous function on $\mathbb{R}$. Then

$$
a(t)\varDelta e_n=-a(t)n^2\pi^2 e_n,\quad n=1,2,\cdots.
$$

The evolution family $(U(t,s))_{t\geq s}$ associated with $a(t)\Delta$ is given by

$$U(t,s)x = \sum_{n=1}^{\infty} \exp\left(-n^2\pi^2 \int_s^t a(r)\,dr\right) \langle x, e_n\rangle e_n, \quad x \in \mathbb{H}.$$

Let $(W_t)_{t\in\mathbb{R}}$ be a R-Wiener process taking values in $\mathbb{H}$. We assume that there exist a sequence of nonnegative real numbers $(\lambda_n)_{n\geq 1}$ such that

$$Re_n = \lambda_n e_n$$

and a sequence $(b_n)_{n\geq 1}$ of real independent Brownian motion such that

$$\langle W_t, x\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\langle x, e_n\rangle b_n(t), \quad x \in \mathbb{H},\ t \in \mathbb{R}.$$

Now let us consider the following SPDE on $\mathbb{H}$.

$$\begin{cases} dX_t = a(t)\Delta X_t dt + dW_t, \\ X_s = x. \end{cases} \tag{5.8}$$

Assume that for all $t \geq s$,

$$r_{t,s}\colon = \sum_{n=0}^{\infty} \lambda_n \int_s^t \exp\left(-2n^2\pi^2 \int_s^t a(r)\,dr\right) du < \infty.$$

Then the following stochastic convolution

$$W_U(t,s) := \int_s^t U(t,r)\,dW_r$$

is well defined. The distribution of $W_U(t,s)$ is given by $\mu_{t,s} \sim N(0, R_{t,s})$, where

$$R_{t,s} := \int_s^t U(t,r)RU(t,r)^*\,dr.$$

Clearly $\operatorname{tr} R_{t,s} = r_{t,s}$. By [15, Chapter 5], the mild solution of (5.8) is given by

$$X(t,s,x) = U(t,s)x + W_U(t,s), \quad x \in \mathbb{H},\ t \geq s.$$

The generalized Mehler semigroup of $X(t,s,x)$ is given by

$$p_{s,t}f(x) = \int_{\mathbb{H}} f\left(U(t,s)x + y\right)\mu_{t,s}(dy) \tag{5.9}$$

for all $x \in \mathbb{H}$, $f \in B_b(\mathbb{H})$ and $t \geq s$.

We are going to look for the Harnack inequality for $p_{s,t}$.

Let $\Gamma_{t,s} = R_{t,s}^{-1/2}U(t,s)$. Note that for all $n \geq 1$,

$$R_{t,s}e_n = \lambda_n \left[\int_s^t \exp\left(-2n^2\pi^2 \int_s^t a(r)\,dr\right) du\right] e_n.$$

We have

$$R_{t,s}^{-1/2}U(t,s)e_n = \frac{\exp\left(-n^2\pi^2 \int_s^t a(r)\,dr\right)}{\left[\lambda_n \int_s^t \exp\left(-2n^2\pi^2 \int_s^t a(r)\,dr\right) du\right]^{1/2}}\, e_n$$

Therefore for every $x \in \mathbb{H}$, we have

$$\begin{aligned}\|\Gamma_{t,s}x\|^2 &= \langle R_{t,s}^{-1/2}U(t,s)x, R_{t,s}^{-1/2}U(t,s)x\rangle \\ &= \sum_{n=1}^{\infty} \frac{\exp\left(-2n^2\pi^2 \int_s^t a(r)\,dr\right)}{\lambda_n \int_s^t \exp\left(-2n^2\pi^2 \int_s^t a(r)\,dr\right) du} \langle x, e_n\rangle^2.\end{aligned}$$

Hence we obtain the following Harnack inequality for $p_{s,t}$:

$$(p_{s,t}f(x))^\alpha \le \exp\left(\frac{\alpha}{2(\alpha-1)} \sum_{n=1}^{\infty} \frac{\exp\left(-2n^2\pi^2 \int_s^t a(r)\,dr\right) \langle x-y, e_n\rangle^2}{\lambda_n \int_s^t \exp\left(-2n^2\pi^2 \int_s^t a(r)\,dr\right) du}\right) p_{s,t}f^\alpha(y),$$

for every $\alpha > 1$, $x, y \in \mathbb{H}$, $t \ge s$ and $f \in B_b^+(\mathbb{H})$.

Applying Theorem 5.2, we have the following result.

Theorem 5.4. *Fix $t \ge s$. The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ of the following statements hold:*

(1) one has

$$U(t,s)(\mathbb{H}) \subset R_{t,s}^{1/2}(\mathbb{H}), \tag{5.10}$$

(2) $\|\Gamma_{t,s}\| < \infty$ and for every $\alpha > 1$, $f \in B_b^+(\mathbb{H})$,

$$(p_{s,t}f(x))^\alpha \le \exp\left(\frac{\alpha(\|\Gamma_{t,s}\| \cdot |x-y|)^2}{2(\alpha-1)}\right) p_{s,t}f^\alpha(y), \quad x, y \in \mathbb{H}; \tag{5.11}$$

(3) $\|\Gamma_{t,s}\| < \infty$ and there exists some $\alpha > 1$ such that (5.11) holds for all $f \in B_b^+(\mathbb{H})$;

(4) $\|\Gamma_{t,s}\| < \infty$ and for every $f \in B_b^+(\mathbb{H})$ with $f > 1$,

$$p_{s,t}\log f(x) \le \log p_{s,t}f(y) + \frac{\|\Gamma_{t,s}\|^2}{2}|x-y|^2, \quad x, y \in \mathbb{H}; \tag{5.12}$$

(5) $p_{s,t}$ is strong Feller, i.e. for every $f \in B_b(\mathbb{H})$, $p_{s,t}f \in C_b(\mathbb{H})$.

In particular, if $\mathrm{m}_{t,s} \equiv 0$, then all these statements are equivalent.

Proof. If (5.10) holds, then $\Gamma_{t,s}$ is a bounded linear operator on $\mathbb{H}$. Hence by Theorem 5.2, we get (2) from (1). That (2) implies (3) is trivial. The implications (3)⇒(4)⇒(5) are consequences of Harnack inequalities, as proved in [55].

Now we show that if $\mathrm{m}_{t,s} \equiv 0$, then (5) implies (1). Note that

$$p_{s,t}f(x) = \int_{\mathbb{H}} f(y)\, N(U(t,s)x, R_{t,s})(dy).$$

If (5.10) does not hold, then there exists an $x_0 \in \mathbb{H}$ such that $U(t,s)x_0 \notin R_{t,s}^{1/2}(\mathbb{H})$. Take

$$x_n = \frac{1}{n}x_0 \in \mathbb{H}, \quad n = 1, 2, \ldots.$$

By the Cameron-Martin theorem (see e.g. [15]), for each $n = 1, 2, \cdots$, the Gaussian measure $\mu_n := N(U(t,s)x_n, R_{t,s})$ is orthogonal to $\mu_0 := N(0, R_{t,s})$, since $U(t,s)x_n \notin R_{t,s}^{1/2}(\mathbb{H})$. It means that for every $n = 1, 2, \cdots$, there exists a set $A_n \in \mathscr{B}(\mathbb{H})$ such that $\mu_n(A_n) = 1$ and $\mu_0(A_n) = 0$. Let $A := \cup_{n\ge1}A_n$. Then we have $\mu_0(A) = 0$, $\mu_n(A) = 1$ since $\mu_0(A) \le \sum_{n=1}^\infty \mu_0(A_n) = 0$ and $\mu_n(A) \ge \mu_n(A_n) = 1$.

Take $f = \mathbb{1}_A$. Because x_n tends to 0 as $n \to \infty$ and $p_{s,t}$ is strong Feller, $p_{s,t}f(x_n)$ converges to $p_{s,t}f(0)$ as $n \to \infty$. But this is impossible because we have $p_{s,t}f(x_n) = 1$ for all $n \geq 1$ and $p_{s,t}f(0) = 0$. So, (5.10) must hold. □

Remark 5.5. If $R_{t,s}$ has the form (6.2), then (5.10) is equivalent to the null controllability of a non-autonomous control system (6.1) (see Section 6 for details). For this reason, condition (5.10) is also called null-controllability condition. This gives an equivalent description of the strong Feller property in the Gaussian case.

Remark 5.6. In [12] it is shown that the null controllability implies the strong Feller property for autonomous Ornstein-Uhlenbeck processes with deterministic perturbation driven by a Wiener process. Obviously, our result generalizes this result.

In fact (5.10) implies more. Let $UC^\infty(\mathbb{H})$ denote the space of all infinitely Fréchet differentiable functions with uniform continuous derivatives on $\mathbb{H}$.

Proposition 5.7. *Suppose that* (5.10) *holds. Then for every* $f \in B_b(\mathbb{H})$ *and every* $t > s$, $p_{s,t}f \in UC^\infty(\mathbb{H})$.

Proof. In view of the decomposition $p_{s,t} = p^g_{s,t}p^j_{s,t}$ shown in Proposition 5.1, we only need to show that $p^g_{s,t} \in UC^\infty(\mathbb{H})$ for every $g \in B_b(\mathbb{H})$. The rest of the proof is the same as in [16, Theorem 6.2.2]. □

We have the following quantitative estimate for the strong Feller property. This result is shown in [42] for Lévy driven Ornstein-Uhlenbeck process by a coupling method.

Proposition 5.8. *Let* $t > s$ *and* $x, y \in \mathbb{H}$. *Then*

$$\begin{aligned} &|p_{s,t}f(x) - p_{s,t}f(y)|^2 \\ &\leq \left(\mathrm{e}^{|\Gamma_{t,s}(x-y)|^2} - 1\right) \min\left\{p_{s,t}f^2(z) - (p_{s,t}f(z))^2 \colon z = x, y\right\}. \end{aligned} \tag{5.13}$$

Proof. Let $h = p^j_{s,t}f$. Then by Proposition 5.1 we have $p_{s,t}f = p^g_{s,t}h$. Moreover,

$$h^2 = (p^j_{s,t}f)^2 \leq p^j_{s,t}f^2$$

by Jensen's inequality. So, for every $z \in \mathbb{H}$, we have

$$\begin{aligned} &p^g_{s,t}h^2(z) - \left(p^g_{s,t}h(z)\right)^2 \\ \leq &p^g_{s,t}p^j_{s,t}f^2(z) - \left(p^g_{s,t}p^j_{s,t}f(z)\right)^2 = p_{s,t}f^2(z) - (p_{s,t}f(z))^2. \end{aligned} \tag{5.14}$$

Note also that (5.13) is symmetric in x and y. So, according to (5.14) we only need to show the following inequality

$$|p^g_{s,t}h(x) - p^g_{s,t}h(y)|^2 \leq \left(\mathrm{e}^{|\Gamma_{t,s}(x-y)|^2} - 1\right)\left(p^g_{s,t}h^2(y) - (p^g_{s,t}h(y))^2\right). \tag{5.15}$$

Using the notation in (5.4) we have

$$p^g_{s,t}h(x) = \int_{\mathbb{H}} h(U(t,s)x + z)\,\mu^g_{t,s}(dz) = \int_{\mathbb{H}} \rho_{t,s}(x-y,z)h(U(t,s)y + z)\,\mu^g_{t,s}(dz).$$

Therefore we obtain

$$\begin{aligned}
&|p^g_{s,t}h(x) - p^g_{s,t}h(y)|^2 \\
&= \left(\int_{\mathbb{H}} [\rho_{t,s}(x-y,z) - 1] \cdot [h(U(t,s)y+z) - p^g_{s,t}h(y)] \, \mu^g_{t,s}(dz) \right)^2 \\
&\le \int_{\mathbb{H}} (\rho_{t,s}(x-y,z)-1)^2 \, \mu^g_{t,s}(dz) \int_{\mathbb{H}} \big[h(U(t,s)y+z) - p^g_{s,t}h(y)\big]^2 \, \mu^g_{t,s}(dz) \\
&= \left(\int_{\mathbb{H}} \rho^2_{t,s}(x-y,z) \, \mu^g_{t,s}(dz) - 1 \right) \cdot \left(\int_{\mathbb{H}} h^2(U(t,s)y+z) \, \mu^g_{t,s}(dz) - (p^g_{s,t}h(y))^2 \right) \\
&= \left(\mathrm{e}^{|\Gamma_{t,s}(x-y)|^2} - 1 \right) \left(p^g_{s,t}h^2(y) - (p^g_{s,t}h(y))^2 \right).
\end{aligned}$$

Note that here we have used (5.5) to obtain

$$\int_{\mathbb{H}} \rho^2_{t,s}(x-y,z) \, \mu^g_{t,s}(dz) = \mathrm{e}^{|\Gamma_{t,s}(x-y)|^2}.$$

□

Now we apply the Harnack inequality (5.2) to study the hyperboundedness of the transition function $p_{s,t}$. In [23] hypercontractivity is studied for the Gaussian case via log-Sobolev inequality.

Theorem 5.9. *Let $(\nu_t)_{t\in\mathbb{R}}$ be an evolution system of measures for $(p_{s,t})_{t\ge s}$. For every $s \le t$, $\alpha > 1$, and $\varepsilon \ge 0$, let*

$$C_{s,t}(\alpha,\varepsilon) := \int_{\mathbb{H}} \left[\int_{\mathbb{H}} \exp\left(-\frac{\alpha |\Gamma_{t,s}(x-y)|^2}{2(\alpha-1)} \right) \nu_s(dy) \right]^{-(1+\varepsilon)} \nu_s(dx).$$

Then for all $f \in L^\alpha(\mathbb{H},\nu_t)$,

$$\|p_{s,t}f\|_{L^{\alpha(1+\varepsilon)}(\mathbb{H},\nu_s)} \le C_{s,t}(\alpha,\varepsilon)^{-\alpha(1+\varepsilon)} \|f\|_{L^\alpha(\mathbb{H},\nu_t)}. \tag{5.16}$$

Proof. From the Harnack inequality (5.2) we have

$$(p_{s,t}f(x))^\alpha \exp\left[-\frac{\alpha |\Gamma_{t,s}(x-y)|^2}{2(\alpha-1)} \right] \le p_{s,t}f^\alpha(y), \quad x,y \in \mathbb{H}.$$

Integrating both sides of the inequality above with respect to $\nu_s(dy)$ and using the fact that $(\nu_t)_{t\in\mathbb{R}}$ is an evolution system of measures, we obtain

$$(p_{s,t}|f|)^\alpha(x) \int_{\mathbb{H}} \exp\left(-\frac{\alpha |\Gamma_{t,s}(x-y)|^2}{2(\alpha-1)} \right) \nu_s(dy) \le \int_{\mathbb{H}} |f|^\alpha(y) \, \nu_t(dy).$$

Hence

$$(p_{s,t}|f|)^{\alpha(1+\varepsilon)}(x) \le \left[\int_{\mathbb{H}} \exp\left(-\frac{\alpha |\Gamma_{t,s}(x-y)|^2}{2(\alpha-1)} \right) \nu_s(dy) \right]^{-(1+\varepsilon)} \|f\|^{\alpha(1+\varepsilon)}_{L^\alpha(\mathbb{H},\nu_t)}.$$

Integrating both sides of the inequality above with respect to $\nu_s(dx)$, we get (5.16). □

6. APPENDIX: NULL CONTROLLABILITY

Consider the following non-autonomous linear control system

$$\begin{cases} dz(t) = A(t)z(t)dt + C(t)u(t)\,dt, \\ z(s) = x, \end{cases} \tag{6.1}$$

where $(A(t))_{t\in\mathbb{R}}$ is a family of linear operators on $\mathbb{H}$ with dense domains and $(C(t))_{t\in\mathbb{R}}$ is a family of bounded linear operators on $\mathbb{H}$. Let $(U(t,s))_{t\geq s}$ be an evolution family on $\mathbb{H}$ associated with $(A(t))_{t\in\mathbb{R}}$. Consider the mild solution of (6.1)

$$z(t,s,x) = U(t,s)x + \int_s^t U(t,r)C(r)u(r)\,dr. \quad x\in\mathbb{H},\ t\geq s.$$

In control theory, $z(t,s,x)$ is interpreted as the state of the system and u as a strategy to control the system. If there exists some $u\in L^2([s,t],\mathbb{H})$ such that $z(t,s,x)=0$, then we say the system (6.1) can be transferred to 0 at time t from the initial state $x\in\mathbb{H}$ at time s. If for every initial state $x\in\mathbb{H}$ the system (6.1) can be transferred to 0 then we say the system (6.1) is null controllable at time t. We refer to [58] (see also [15, Appendix B]) for the details on null controllability of autonomous control systems.

Set for every $t\geq s$

$$\Pi_{t,s}x := \int_s^t U(t,r)C(r)C(r)^*U(t,r)^*\,dr, \quad x\in\mathbb{H}. \tag{6.2}$$

Proposition 6.1. *Let $x\in\mathbb{H}$ and $t\geq s$. The system* (6.1) *can be transferred to 0 at time t from x if and only if $U(t,s)x\in\Pi_{t,s}^{1/2}(\mathbb{H})$. Moreover, the minimal energy among all strategies transferring x to 0 at time t is given by $|\Pi_{t,s}^{-1/2}U(t,s)x|^2$, i.e.*

$$\begin{aligned} &|\Pi_{t,s}^{-1/2}U(t,s)x|^2 \\ &= \inf\left\{\int_s^t |u(r)|^2\,dr\colon z(t,s,x)=0, z(s,s,x)=x, u\in L^2([s,t],\mathbb{H})\right\}. \end{aligned} \tag{6.3}$$

Proof. For every $t\geq s$, define a linear operator

$$L_{t,s}\colon L^2([s,t],\mathbb{H})\to\mathbb{H}, \quad u\mapsto L_{t,s}u := \int_s^t U(t,r)C(r)u(r)\,dr.$$

The adjoint $L_{t,s}^*$ of $L_{t,s}$ is given by

$$(L_{t,s}^*x)(r) = C^*(r)U(t,r)^*x, \quad x\in\mathbb{H},\ r\in[s,t].$$

It is easy to check that

$$\Pi_{t,s} = L_{t,s}L_{t,s}^*.$$

Then by [15, Corollary B.4], $L_{t,s}(L^2([s,t],\mathbb{H}) = \Pi_{t,s}(\mathbb{H})$. Hence the first assertion of the theorem is proved, since the initial state x can be transferred to 0 if and only if $U(t,s)x$ is contained in the image space of $L_{t,s}$ due to the fact that $z(t,s,x) = U(t,s)x + L_{t,s}u$.

By [15, Corollary B.4] we also get

$$|\Pi_{t,s}^{-1/2}y| = |L_{t,s}^{-1}y|, \quad y\in L_{t,s}(L^2([s,t],\mathbb{H})). \tag{6.4}$$

Here the inverse is understood as a pseudo–inverse. Taking $y=U(t,s)x$ in (6.4), we obtain (6.3). □

From Proposition 6.1, we get the following corollary.

Corollary 6.2. *The system* (6.1) *is null controllable at time* t *if and only if*

$$U(t,s)(\mathbb{H}) \subset \Pi_{t,s}^{1/2}(\mathbb{H}). \tag{6.5}$$

From (6.3), it is easy to get upper bounds of $|\Pi_{t,s}^{-1/2}U(t,s)x|^2$ by choosing proper null control functions u. The following proposition is analogous to [42, Proposition 2.1].

Proposition 6.3. *Let* $t > s$. *Assume that for every* $r \in [s,t]$, *the operator* $C(r)$ *is invertible. Then for every strictly positive function* $\xi \in C([s,t])$,

$$|\Pi_t^{-1/2}U(t,s)x|^2 \le \frac{\int_s^t |C(r)^{-1}U(r,s)x|^2\,\xi_r^2\,dr}{\left(\int_s^t \xi_r\,dr\right)^2}, \quad x \in \mathbb{H}. \tag{6.6}$$

In particular, if $C(r) \equiv C$ *and* $|C^{-1}U(r,s)x|^2 \le h(r)|C^{-1}x|^2$ *for every* $x \in \mathbb{H}$, *then*

$$|\Pi_t^{-1/2}U(t,s)x|^2 \le \frac{|C^{-1}x|^2}{\int_s^t h(r)^{-1}\,dr}, \quad x \in \mathbb{H}. \tag{6.7}$$

Proof. We only need to consider the case when $U(t,s)x \in \Pi_{t,s}^{1/2}(\mathbb{H})$ and the function $[s,t] \ni r \mapsto \xi_r C(r)^{-1}U(r,s)x$ belongs to $L^2([0,t],\mathbb{H})$. Then the following function

$$u(r) := -\frac{\xi_r}{\int_s^t \xi_r\,dr}C(r)^{-1}U(r,s)x, \quad r \in [s,t],$$

is a null control of the system (6.1). And hence the estimate (6.6) follows from (6.3). The second estimate (6.7) follows by taking $\xi(r) = h(r)^{-1}$ for all $r \in [s,t]$. □

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