Dynamical Systems on Sets of Holomorphic Functions

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Abstract

Two dynamical systems $u_t'(t,\cdot) = (Lu)(t,\cdot)$, $L = A, B$, $u(0,\cdot) = f$, are studied which can serve as toy models for infinite systems of interacting particles in continuum. Here $u(t,\cdot)$ and f are holomorphic functions in some $K \subset \mathbb{C}$ and A, B are linear operators in a certain Banach space $\mathcal{E}$ of such functions. It is proven that both A and B generate C_0 semigroups and hence the above Cauchy problems have solutions in $\mathcal{E}$. In some particular cases, ergodicity and reversibility are proven.

1 Motivation and posing of the problem

In recent years, there is a big activity in studying Markov processes with the state space Γ which can be thought of as the set of all locally finite configurations of 'particles' in $\mathbb{R}^d$, see [5, 6, 13, 14]. Here each $\gamma \in \Gamma$ is a function $\mathbb{R}^d \ni x \mapsto n_\gamma(x) \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, such that the total number of 'particles' located in a bounded Borel subset $\Lambda \subset \mathbb{R}^d$, that is $\gamma_\Lambda = \sum_{x\in\Lambda} n_\gamma(x)$, is finite. By letting $\mathcal{G}$ be the smallest σ-field of subsets of Γ such that for all bounded Λ, the map $\gamma \mapsto \gamma_\Lambda$ is $\mathcal{G}$-measurable, one turns the pair $(\Gamma, \mathcal{G})$ into a standard Borel space. Hence, one can speak about the set $\mathcal{P}(\Gamma)$ of all probability measures on $(\Gamma, \mathcal{G})$. Let also $C_0(\mathbb{R}^d)$ be the set of all continuous functions $\mathbb{R}^d \to \mathbb{R}$ with compact support, that is, each $f \in C_0(\mathbb{R}^d)$ vanishes outside a compact set Λ_f. A special element of $\mathcal{P}(\Gamma)$ is the Poisson measure $\pi_\varkappa$, $\varkappa > 0$, defined by its Laplace transform

$$\int_\Gamma \exp\left(\langle \gamma, f\rangle\right) \pi_\varkappa(d\gamma) = \exp\left(\varkappa \int_{\mathbb{R}^d} \left[e^{f(x)} - 1\right] dx\right), \qquad (1.1)$$

where $f \in C_0(\mathbb{R}^d)$ and $\langle \gamma, f\rangle = \sum_{x\in\Lambda_f} n_\gamma(x) f(x)$.

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In this setting, Markov processes with state space Γ correspond to the evolution $\mathbb{R}_+ \ni t \mapsto \mu_t \in \mathcal{P}(\Gamma)$, defined via the Kolmogorov equations by Markov generators. In an approach based on the harmonic analysis on Γ, this evolution is described by means of equations for correlation functions rather than for the corresponding measures, see e.g. Section 2 in [5]. Here one can employ generating functionals, for example,

$$F_\mu(\theta) = \int_\Gamma \exp\left(\langle \log(1+\theta), \gamma\rangle\right)\mu(d\gamma), \tag{1.2}$$

where $\theta \in C_0(\mathbb{R}^d)$ is such that $\theta(x) > -1$ for all $x \in \Lambda_\theta$. Traditionally, (1.2) is called the Bogoliubov functional for the measure $\mu \in \mathcal{P}(\Gamma)$, see Section 9 in [17]. By direct calculations, one gets

$$F_{\pi_\varkappa}(\theta) = \exp\left(\varkappa \int_{\mathbb{R}^d} \theta(x)dx\right), \tag{1.3}$$

which points to a possibility of continuing Bogoliubov functionals to holomorphic functions over the complex Banach space $L^1(\mathbb{R}^d)$. Then instead of the Kolmogorov equations one deals with evolution-type linear equations for such holomorphic functions of the following type

$$\frac{\partial}{\partial t} F(t,\cdot) = LF(t,\cdot). \tag{1.4}$$

In this paper, we study simplified (toy) versions of the following two models taken from [5], where one can find all the details. In the first model, there is a system of 'particles' which appear and disappear but do not move (a spatial birth-and-death system). 'Particles' disappear independently with constant rate, whereas the appearance of a new 'particle' is affected by the already existing 'particles'. To describe this influence one uses a function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, which is supposed to be measurable, finite for $x \neq 0$, below bounded, symmetric, and such that

$$\int_{\mathbb{R}^d} \left| e^{-\phi(x)} - 1 \right| dx < \infty.$$

For such ϕ and for $\theta \in C_0(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we set

$$\alpha_x(\theta)(y) = e^{-\phi(x-y)}\theta(y) + e^{-\phi(x-y)} - 1. \tag{1.5}$$

Then the 'operator' in (1.4) corresponding to the considered model has the form, see equation (3.21) in [5],

$$(LF)(t,\theta) = -\int_{\mathbb{R}^d} \theta(x)\left[\frac{\delta}{\delta\theta(x)} F(t,\theta) - \varkappa F(t, \alpha_x(\theta))\right] dx, \tag{1.6}$$

where $\delta F(t,\theta)/\delta\theta(x)$ is the Fréchet derivative of $F(t,\cdot)$ at θ, and $\varkappa > 0$ is a parameter. For $\phi \equiv 0$, we have $LF_{\pi_\varkappa}(t,\cdot) = 0$ for all $t \geq 0$, that is, the steady state in this case is the Poisson measure $\pi_\varkappa$, which corresponds to independent births and deaths. The second example which we mention here is the so called linear voter model, that can also describe dynamics of plants or similar objects. For such a model, the 'operator' in (1.4) has the form, see equation (3.24) in [5],

$$\begin{aligned}(LF)(t,\theta) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}[1+\theta(x)]\theta(y)\left[a_+(x,y)\frac{\delta}{\delta\theta(x)}F(t,\theta)\right. \qquad (1.7)\\ &\left. -a_-(x,y)\frac{\delta^2}{\delta\theta(x)\delta\theta(y)}F(t,\theta)\right]dxdy,\end{aligned}$$

where the functions $a_\pm(x,y) \geq 0$ are symmetric and measurable; they describe the influence of the 'particles' existing at y on the appearance – a_+, or disappearance – a_-, of a new 'particle' at x.

Our general aim is to elaborate tools for studying evolution equations like (1.4) in spaces of holomorphic functions of infinitely many variables. In this article, we consider a strongly simplified situation where all the 'particles' are located at one point, which can be realized by setting $d = 0$. In this case, the set of configurations turns into $\Gamma = \mathbb{N}_0$; hence, the Bogoliubov functional of a probability measure μ on such Γ is

$$F_\mu(z) = \sum_{n=0}^{\infty}(1+z)^n p_n^\mu, \quad p_n^\mu = \mu(\{n\}). \qquad (1.8)$$

As a result of such a simplification, one deals with holomorphic functions of a single variable and with two models, called model A and model B, the evolution of which is generated by the following operators[1]

$$\begin{aligned}(Af)(z) &= -zf'(z) + \varkappa z f(az+a-1), \quad \varkappa, a > 0, \qquad (1.9)\\ (Bf)(z) &= z(z+1)[\alpha f'(z) - \beta f''(z)], \quad \alpha, \beta > 0,\end{aligned}$$

where z belongs to a domain $K \subset \mathbb{C}$, in which f is supposed to be holomorphic. Our aim in this article is to elaborate tools for describing dynamical systems based on (1.4) with L being A and B given above. The main problems which we address here are: (a) the natural choice of the domain K with $K = \mathbb{C}$ as a preferable one; (b) the choice of the topological vector spaces (possibly Banach) in which the operators A and B are dissipative and hence generate C_0 contractive semigroups; (c) the asymptotic properties of the considered dynamical systems in the limit $t \to +\infty$.

[1] Evolution equations in spaces of entire functions were studied e.g. in [15].

2 The domain K and the space $\mathcal{E}$

As we are going to place the functions f in (1.9) into a space of holomorphic functions, we have to employ their representation in the form

$$f(z) = \sum_{n=0}^{\infty} (z+c)^n f_n w_n, \quad f_n \in \mathbb{C}, \quad w_n > 0, \tag{2.1}$$

where the center point $c \in \mathbb{C}$ and the weights w_n are to be properly chosen. Employing this expansion in (1.9) we can express the action of A and B in terms of the coefficients f_n, $n \in \mathbb{N}_0$, the sequences of which, i.e., $\mathsf{f} = \{f_n\}_{n\in\mathbb{N}_0}$ are supposed to belong to a certain linear space. Having this in mind we present here a fact, which can be useful in studying such actions. For $\mathsf{f} \in \mathbb{R}^\infty$, we let

$$(\mathcal{L}\mathsf{f})_n = b_{n-1} f_{n-1} + a_n f_n + d_{n+1} f_{n+1}, \quad n \in \mathbb{N}_0, \tag{2.2}$$

with the convention that $d_0 = b_{-1} = f_{-1} = 0$. Here b_n, a_n, and d_n, $n \in \mathbb{N}_0$ are real parameters, and for $\mathsf{g} = \{g_n\}_{n\in\mathbb{N}_0}$, we write $(\mathsf{g})_n = g_n$. Futhermore, let ℓ^1 be the real space of summable sequences f and

$$\mathcal{D} = \{\mathsf{f} \in \ell^1 \mid \mathcal{L}\mathsf{f} \in \ell^1\}. \tag{2.3}$$

In the sequel, by $\mathcal{L}$ we mean the linear operator $\mathcal{L} : \mathcal{D} \to \ell^1$ defined in (2.2). The following is known, see Theorem 2.1 in [2], which in this part is a refinement of Theorem 1 of [9].

Proposition 2.1 *Suppose that, for all $n \in \mathbb{N}_0$, one has $b_n, d_n \geq 0$ and*

$$b_n + a_n + d_n \leq 0, \tag{2.4}$$

$$b_{n-1} + a_n + d_{n+1} \leq M, \tag{2.5}$$

for some $M \in \mathbb{R}$. Then $\mathcal{L}$ is the generator of a positive C_0 semigroup of contractions in ℓ^1.

The proof of this statement is conducted along the following line of arguments. Let $\mathcal{L}_0 : \mathcal{D} \to \ell^1$ be defined by the diagonal part of (2.2). Then it is the generator of a positive C_0 semigroup of contractions in ℓ^1 as $a_n \leq 0$, $n \in \mathbb{N}_0$, by (2.4). Thereafter, perturbation arguments and (2.5) are employed to show that $\mathcal{L}$ also generates a C_0 semigroup, which is contractive by (2.4).

By direct calculations, one gets that the action of A in (1.9) in terms of the coefficients f_n can have the form of (2.2) only if $c = -1$. In the case of B, one can also take $c = 1$, but then the corresponding coefficients are positive and hence the first condition in (2.4) cannot be satisfied. Thus, for both operators we choose $c = -1$ and hence K is to be a circle centered at such c. In this case, for Bogoliubov functionals the coefficients in (2.1) have the meaning arising from the representation (1.8). There is one more argument in favor of this choice. The steady state equation $Af = 0$ can be written in the form, c.f. (1.5),

$$f'(z) = \varkappa f(\alpha(z)), \qquad \alpha(z) = az + a - 1. \tag{2.6}$$

Starting from the paper [10], such equations have been studied extensively, see also [3] for more recent development. In [16], a systematic study of the equation $f'(z) = F(z, f(z), f(g(z)))$ over the complex plane has been initiated. The main result of [16] is the existence and uniqueness of solutions holomorphic in the vicinity of the fixed point of g, which in (2.6) is $c = -1$.

Now let us choose the weights w_n. Inserting (2.1) with $c = -1$ into (1.9) we obtain

$$(Af)(z) = \sum_{n=0}^{\infty} (1+z)^n w_n g_n,$$

$$g_n = \frac{w_{n-1}}{w_n} \varkappa a^{n-1} f_{n-1} - (n + \varkappa a^n) f_n + \frac{w_{n+1}}{w_n}(n+1) f_{n+1}.$$

Then the conditions (2.4) and (2.5) take the form

$$\delta_n \varkappa a^n - \frac{n\delta_{n-1}}{1+\delta_{n-1}} \leq 0, \qquad n \in \mathbb{N}_0, \tag{2.7}$$

$$\delta_{n-1}\varkappa a^{n-1} - \frac{n\delta_n}{1+\delta_n} + (1-a)\varkappa a^{n-1} + \frac{1}{1+\delta_n} \leq M,$$

where

$$\delta_n = \frac{w_n}{w_{n+1}} - 1.$$

From the first inequality in (2.7) we see that $\delta_n \leq 0$ for all $n \in \mathbb{N}_0$. Furthermore, if $\delta_n < 0$ for some n, then $\delta_{n+k} < 0$ for all $k \in \mathbb{N}$. In this case, for $a \leq 1$ the left-hand side of the second line in (2.7) increases like $-n\delta_n$. Hence, for such a, the only solution is $\delta_n = 0$ for all n, for which the second condition in (2.7) takes the form

$$(1-a)\varkappa a^{n-1} + 1 \leq M, \tag{2.8}$$

which is satisfied with $M = 1 + \varkappa$ for all $a > 0$. Thus, for the operator A our choice of the weights is $w_n = 1$ for all $n \in \mathbb{N}_0$. In what follows, we set

$$K = \{z \in \mathbb{C} \mid |z+1| < 1\}. \tag{2.9}$$

Definition 2.2 *By $\mathcal{E}$ (resp. $\mathcal{E}^+$) we denote the set of all holomorphic functions $f : K \to \mathbb{C}$, which take real values at real $z \in K$ (resp. real positive values for real $z \in (-1, 0)$), and are such that*

$$f(z) = \sum_{n=0}^{\infty} (1+z)^n f_n, \qquad \mathsf{f} = \{f_n\}_{n\in\mathbb{N}_0} \in \ell^1. \tag{2.10}$$

The set $\mathcal{E}$ is endowed with the real linear operations and with the norm $\|f\| = \|\mathsf{f}\|_{\ell^1}$.

Clearly, $\mathcal{E}$ is a real Banach space. The series in (2.10) converges absolutely at any $|z+1| = 1$, that is, elements of $\mathcal{E}$ can be continuously extended to the boundary of K, which includes the origin $z = 0$. However, their derivatives, which are holomorphic functions in K, need not have such extensions.

A similar analysis performed for the operator B shows that for any choice of the weights the conditions (2.4) and (2.5) cannot be satisfied simultaneously. For $w_n = 1$, $n \in \mathbb{N}_0$, the condition (2.5) takes the form

$$-\alpha + 2n\beta \leq M, \tag{2.11}$$

which means that for such a choice of the weights Proposition 2.1 can be applied to the first model only, whuch we do in the next section. Model B will be considered in Section 4.

3 Model A

3.1 The general case

Set

$$\mathcal{E}^{+,1} = \left\{ f \in \mathcal{E}^+ \mid \sum_{n=0}^{\infty} f_n = 1 \right\}, \tag{3.1}$$

and

$$\mathcal{K} = \{f \in \mathcal{E} \mid f' \in \mathcal{E}\}, \qquad \mathcal{K}^{+,1} = \mathcal{K} \cap \mathcal{E}^{+,1}. \tag{3.2}$$

As was established above, both conditions of Proposition 2.1 are satisfied for all $a > 0$. Thus, as its direct corollary we have the following

Theorem 3.1 *For every $a > 0$, the operator $A : \mathcal{K} \to \mathcal{E}$, defined in (1.9), is the generator of a C_0 semigroup $\mathcal{T}_A = \{T_A(t)\}_{t\geq 0}$, $T_A(t) : \mathcal{E}^{+,1} \to \mathcal{E}^+$, of contractions.*

By Theorem 2.4, page 4, and Theorem 1.3, page 102, both in [18], and by the above statement, we have the following

Theorem 3.2 *For every $f \in \mathcal{K}$, the Cauchy problem*

$$\begin{cases} u_t'(t,z) = -zu_z'(t,z) + \varkappa z u(t, az + a - 1), \quad z \in K, \quad t > 0, \\ u(0,z) = f(z), \quad a > 0, \end{cases} \tag{3.3}$$

has a unique solution $u(t,\cdot) \in \mathcal{K}$, which is continuously differentiable on $[0, +\infty)$.

Let us remind that the mentioned function u has the representation

$$u(t,z) = \sum_{n=0}^{\infty} (1+z) u_n(t), \qquad \mathsf{u}(t) = \{u_n(t)\}_{n\in\mathbb{N}_0} \in \ell^1. \tag{3.4}$$

The solution of (3.3) such that $u(t,\cdot) = f$ for all $t \geq 0$ will be called steady state or equilibrium. It can be obtained from the equation (2.6). Clearly, for $a > 1$ and $\varkappa > 0$ there is no such solutions, c.f. Theorem 4.1 in [16]. For $a \leq 1$, we have the following

Theorem 3.3 *For $a \leq 1$, there exists an equilibrium solution of (3.3) $u(t,\cdot) \in \mathcal{K}^{+,1}$, which has the form*

$$u(t,z) = \psi(z) = \sum_{n=0}^{\infty} (1+z)^n \psi_n, \tag{3.5}$$

$$\psi_n = C \varkappa^n a^{n(n-1)/2}/n!, \quad C^{-1} = \sum_{n=0}^{\infty} \frac{\varkappa^n}{n!} a^{n(n-1)/2}.$$

If there exists $f \in \mathcal{K}^{+,1}$ such that the solution of (3.3) has the property

$$u(\tau,\cdot) = f, \tag{3.6}$$

for some $\tau > 0$, then $f = \psi$ and hence (3.6) holds for all $\tau \geq 0$.

The part of the theorem concerning the equilibrium solution is straightforward. The second part will be proven in Section 5 below.

3.2 Compactness for $a \leq 1$

Our main result here is the following statement the proof of which will be done in several steps.

Theorem 3.4 *For every $a \leq 1$ and $t > 0$, the operator $T_A(t)$ is compact.*

For $f \in \mathcal{E}$ and $g \in \mathcal{K}$, we denote

$$(Uf)(z) = \varkappa z f(az + a - 1), \qquad (Vg)(z) = -z g'(z), \tag{3.7}$$

which yields

$$\begin{aligned} (Uf)_n &= \varkappa a^{n-1} f_{n-1} - \varkappa a^n f_n, \quad n \in \mathbb{N}_0, \\ (Vg)_n &= -n g_n + (n+1) g_{n+1}. \end{aligned} \tag{3.8}$$

Thus, U is bounded and

$$\|U\| \leq 2\varkappa, \tag{3.9}$$

whereas $V : \mathcal{K} \to \mathcal{E}$ is closed. Furthermore, by Proposition 2.1 both U and V are the generators of positive C_0 semigroups of contractions. Let $\mathcal{T}_V = \{T_V(t)\}_{t\geq 0}$ be the one generated by V. By direct calculations, one readily gets

$$\begin{aligned} (T_V(t)g)(z) &= g(e^{-t}z), \qquad g \in \mathcal{E} \\ (T_V(t)g)_k &= \sum_{n=k}^{\infty} g_n \mathcal{B}_{n,k}(e^{-t}), \end{aligned} \tag{3.10}$$

where the Bernoulli coefficients

$$\mathcal{B}_{n,k}(p) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}, \quad p \in (0,1)$$

obey the estimate

$$\mathcal{B}_{n,k}(p) \leq C(p)/\sqrt{n}, \tag{3.11}$$

which holds for all $n \geq 1$, $k = 0, \ldots, n$, and some $C(p) > 0$.

Proposition 3.5 *For every $t > 0$, $T_V(t)$ is a compact operator in $\mathcal{E}$.*

Proof: We employ the following standard fact. Let $\{\mathsf{h}^m\}_{m\in\mathbb{N}_0} \subset \ell^1$ be such that: (a) $\|\mathsf{h}^m\|_{\ell^1} \leq C$ for some $C > 0$; (b) there exists a decreasing sequence $\{\lambda_k\}_{k\in\mathbb{N}_0}$, $\lambda_k \to 0$, such that $|h^m_k| \leq \lambda_k$ for all nonnegative integer m and k. Then $\{\mathsf{h}^m\}_{m\in\mathbb{N}_0}$ contains a convergent subsequence. By (3.10) and (3.11), for $t > 0$ and $k \in \mathbb{N}$, we have

$$|(T_V(t)g)_k| \leq \|g\| C(e^{-t})/\sqrt{k},$$

which together with the fact just mentioned yields the compactness to be proven. $\square$

Remark 3.6 *By the result just proven, the semigroup $\{T_V(t)\}_{t\geq 0}$ is continuous in the uniform topology for $t > 0$, see Theorem 3.3, page 48 in [18]. However, it is not differentiable, that is, the map*

$$g \mapsto \frac{d}{dt} T_V(t) g \tag{3.12}$$

is not bounded in $\mathcal{E}$. It would be bounded if on the right-hand side of (3.11) we had $C(p)/n$.

As is well-know, see e.g., page 20 of [18], the resolvent of V can be obtained from the Laplace transform

$$(R_\lambda(V)g)(z) = \int_0^\infty e^{-\lambda t} T_V(t) g(z) dt, \quad \lambda \in \varrho(V), \tag{3.13}$$

where $\varrho(V)$ is the corresponding resolvent set. By Theorem 3.3, page 48 in [18], and our Proposition 3.5 it follows that also $R_\lambda(V)$ is compact for all $\lambda \in \varrho(V)$. By the Lumer-Phillips theorem, page 14, and Theorem 5.2, page 19, both in [18], we have $\|R_\lambda(V)\| \leq 1/\lambda$ for all $\lambda > 0$. Then for $\lambda > 2\varkappa$, the series in

$$R_\lambda(A) = R_\lambda(V) \sum_{n=0}^{\infty} [U R_\lambda(V)]^n \qquad (3.14)$$

is norm-convergent. Hence $R_\lambda(A)$ is compact for such, and thereby for all, $\lambda \in \varrho(A)$.

Proof of Theorem 3.4: For $a \leq 1$, the operator U is bounded, see (3.8) and (3.9). Hence, the property stated follows by Proposition 3.5 and Proposition 1.4, page 79 in [18]. One observes that Remark 3.6 applies also to the semigroup $\{T_A(t)\}_{t\geq 0}$. $\square$

3.3 More on differentiability

For $n \in \mathbb{N}$, we set

$$\mathcal{E}_n = \{f \in \mathcal{E} \mid f^{(n)} \in \mathcal{E}\}, \qquad \mathcal{E}_n^{+,1} = \mathcal{E}_n \cap \mathcal{E}^{+,1}. \qquad (3.15)$$

If μ is a probability measure on $\mathbb{N}_0$ such that $F_\mu \in \mathcal{E}_n^{+,1}$, then μ has the n-th moment which can be expressed in terms of $F_\mu^{(k)}(0)$ with $k \leq n$. In $\mathcal{E}_n$, we introduce the norm

$$\|f\|_n = \sum_{k=0}^{\infty} \{|f_k| + k|f_{k-1}| + \cdots + k(k-1)\cdots(k-n+1)|f_{k-n}|\}, \qquad (3.16)$$

where f_k are the same as in (2.10). For $f \in \mathcal{E}_n^{+,1}$, this yields

$$\|f\|_n = \sum_{l=0}^{n} f^{(l)}(0). \qquad (3.17)$$

With this norm $\mathcal{E}_n$ turns into a Banach space. Note that the set $\mathcal{K}$ defined in (3.2) and $\mathcal{E}_1$ consist of the same functions, and the only difference is that they are equipped with different topologies. In the sequel, we will also use

$$\mathcal{K}_n = \{f \in \mathcal{E}_n \mid f' \in \mathcal{E}_n\}, \qquad n \in \mathbb{N}. \qquad (3.18)$$

Then the formulas (3.7) define the operators $U : \mathcal{E}_n \to \mathcal{E}_n$, $V : \mathcal{K}_n \to \mathcal{E}_n$, and hence $A = U + V : \mathcal{K}_n \to \mathcal{E}_n$. For any $n \in \mathbb{N}$, U is bounded whereas V and A are closed. Since the action of the operator $T_V(t)$ is given explicitly, see (3.10), one easily checks that $\{T_V(t)\}_{t\geq 0}$ is a positive C_0-semigroup of contractions. Indeed, by (3.10) $T_V(t)$ is positive. It is known, see e.g. Proposition 2.67, page 45 in [1], that in this case we have, see (3.17),

$$\begin{aligned}\|T_V(t)g\|_n &= \sup_{g\in\mathcal{E}_n^+} \|T_V(t)g\|_n/\|g\|_n \\ &= \sup_{g\in\mathcal{E}_n^+} \sum_{l=0}^{n} e^{-lt} g^{(l)}(0)/\|g\|_n \leq 1.\end{aligned}$$

Also by (3.10) we have $\|T_V(t)g - g\|_n \to 0$ as $t \downarrow 0$. Thus, by the Lumer-Phillips theorem and Theorem 5.2, page 19 in [18], it follows that the resolvent of V obeys the estimate

$$\|R_\lambda(V)\|_n \leq 1/\lambda, \qquad \lambda > 0. \tag{3.19}$$

Then for $m \in \mathbb{N}$, by (3.14) we obtain

$$\|[R_\lambda(A)]^m\|_n \leq \frac{1}{(\lambda-\gamma_n)^m}, \qquad \text{for} \quad \lambda > \gamma_n \stackrel{\text{def}}{=} \|U\|_n. \tag{3.20}$$

By (3.13) we have

$$\left(R_\lambda(V)g\right)^{(l)}(0) = g^{(l)}(0)/(\lambda + l), \quad l \in \mathbb{N}; \tag{3.21}$$

hence, $R_\lambda(V)$ maps every $\mathcal{K}_n$ into itself for any $\lambda > 0$. By (3.14) this also holds for $R_\lambda(A)$, $\lambda > \gamma_n$. Finally, the range of $R_\lambda(A)$ in $\mathcal{E}_n$ is $\mathcal{K}_n$, which is obviously dense in $\mathcal{E}_n$. All these facts allow us to apply Corollary 5.6, page 124 in [18], and obtain the following

Theorem 3.7 *For every $n \in \mathbb{N}$, the space $\mathcal{E}_n$ is A-admissible, which here means that that $\mathcal{K}_n \subset \mathcal{E}$ is an invariant subspace of $T_A(t)$ and the semigroup generated by A in $\mathcal{E}_n$ is a positive C_0-semigroup.*

It should be noted, however, that the semigroup just mentioned need not be contractive. Let us consider its action on the function $g(z) \equiv 1$, which is an element of $\mathcal{K}_n$ for any $n \in \mathbb{N}_0$. Set $u(t,\cdot) = T_A(t)g$. As we show in Proposition 4.2 below, $u(t,\cdot) \in \mathcal{E}^{+,1}$ and hence $u(t,0) = 1$ for all $t \geq 0$. The derivatives $u^{(l)}(t,0)$ can be found as solutions to the following chain of equations obtained from the problem (3.3)

$$\frac{d}{dt}u^{(l)}(t,0) = -lu^{(l)}(t,0) + \varkappa l a^{l-1} u^{(l-1)}(t,a-1), \quad l = 1,2,\ldots,$$

subject to the condition $u^{(l)}(0,0) = 0$ for all $l \geq 1$. Since $u(t,\cdot)$ is in $\mathcal{E}^{+,1}$, $u^{(l-1)}(t,a-1)$ exists for all l. Hence, the above chain can be solved in the form

$$u^{(l)}(t,0) = l\varkappa a^{l-1} \int_0^t e^{-l(t-s)} u^{(l-1)}(s,a-1)ds. \tag{3.22}$$

As

$$u(s, a-1) = \sum_{k\geq 0} a^k u_k(s) \leq \sum_{k\geq 0} u_k(s) = 1, \tag{3.23}$$

we have from (3.22) that

$$0 < u^{(1)}(t, 0) \leq \varkappa(1 - e^{-t}). \tag{3.24}$$

By the induction over l we then obtain therefrom, and from the estimate

$$u^{(l)}(t, a-1) = \sum_{k=l}^{\infty} a^{k-l} k(k-1)\cdots(k-l+1) u_{k-l}(t) \leq u^{(l)}(t, 0), \tag{3.25}$$

the following fact.

Proposition 3.8 *Let $u(t, \cdot) = T_A(t)g$ for $g(z) \equiv 1$. Then for any $l \in \mathbb{N}$ and $t > 0$, one has*

$$0 < u^{(l)}(t, 0) \leq \varkappa^l a^{l(l-1)/2}(1 - e^{-t})^l. \tag{3.26}$$

Therefore, the solution of the problem (3.3) with such g can be extended to an entire function of zeros order for $a < 1$, and of the first order for $a = 1$.

This also yields that for all $t > 0$ and all $n \in \mathbb{N}$, see (3.17), one has

$$\|u(t, \cdot)\|_n = 1 + \sum_{l=1}^{n} u^{(l)}(t, 0) > 1, \tag{3.27}$$

which means that $\|T_A(t)g\|_n > 1$ for any $t > 0$. It turns out that the above statement can be extended to all initial conditions.

Theorem 3.9 *Let the initial function of the problem (3.3), $f \in \mathcal{E}^{+,1}$, be such that $f^{(n)}(0)$ exists for some $n \in \mathbb{N}$. Then the solution of (3.3) is also n times differentiable at zero and its derivatives obey the estimate*

$$0 \leq u^{(l)}(t, 0) \leq v^{(l)}(t, 0), \qquad l = 1, \ldots, n, \tag{3.28}$$

where

$$v(t, z) = \tilde{\psi}\left((1 - e^{-t})z\right) f(e^{-t}z), \quad \tilde{\psi}(z) = \sum_{m=0}^{\infty} \frac{\varkappa^m}{m!} a^{m(m-1)/2} z^m. \tag{3.29}$$

In particular, if f is an entire function, then for every finite t, the solution $u(t, \cdot)$ is an entire function of the order which does not exceed the order of f.

Proof: The lower bound in (3.28) is readily obtained from (3.4) and the positivity of the semigroup $\{T_A(t)\}_{t\geq 0}$. To get the upper bound, we consider the following system of ordinary differential equations

$$\begin{cases} \frac{d}{dt}u^{(l)}(t,0) = -lu^{(l)}(t,0) + l\varkappa a^{l-1}u^{(l-1)}(t,a-1), \\ u^{(l)}(0,0) = f^{(l)}(0), \quad l = 1,\ldots,n, \end{cases} \tag{3.30}$$

which readily follows from (3.3). Let us first suppose that $f(z) = (1+z)^k$ for some $k \in \mathbb{N}_0$. Then we can integrate the system (3.30) and obtain

$$u^{(l)}(t,0) = \alpha_{k,l}e^{-lt} + l\varkappa a^{l-1}\int_0^t e^{-l(t-s)}u^{(l-1)}(s,a-1)ds, \tag{3.31}$$

where $\alpha_{k,l} = k!/(k-l)!$ for $l \leq k$, and $\alpha_{k,l} = 0$ otherwise. Similarly as in (3.22) we obtain

$$\begin{aligned} u^{(1)}(t,0) &= \alpha_{k,1}e^{-t} + \varkappa\int_0^t e^{-(t-s)}u(s,a-1)ds \\ &\leq \alpha_{k,1}e^{-t} + \varkappa(1-e^{-t}). \end{aligned}$$

Thereafter, we apply this estimate and (3.25) in (3.31) with $l = 2,3,\ldots,$ and obtain

$$u^{(l)}(t,0) \leq U_{k,l}(t), \tag{3.32}$$

$$U_{k,l}(t) = \sum_{m=0}^{\min\{k,l\}} \frac{k!l!\varkappa^{l-m}}{m!(l-m)!(k-m)!}a^{(l-m)(l+m-1)/2}e^{-mt}(1-e^{-t})^{l-m}.$$

Since the system (3.30) is linear, its solution with the initial

$$f(z) = \sum_{k\geq 0}(1+z)^k f_k \in \mathcal{E}^{+,1}$$

is the linear combination of the functions obeying (3.32), taken with the coefficients f_k. That is,

$$\begin{aligned}
u^{(l)}(t,0) &\leq \sum_{k=0}^{\infty} f_k U_{k,l}(t) \\
&= \sum_{m=0}^{l} \frac{l!\varkappa^{l-m}}{m!(l-m)!} a^{(l-m)(l-m-1)/2} a^{m(l-m)} e^{-mt}(1-e^{-t})^{l-m} \\
&\times \sum_{k=m}^{\infty} \frac{k!}{(k-m)!} f_k \\
&\leq \sum_{m=0}^{l} \frac{l!\varkappa^{l-m}}{m!(l-m)!} a^{(l-m)(l-m-1)/2} (1-e^{-t})^{l-m} e^{-mt} f^{(m)}(0) \\
&= v^{(l)}(t,0),
\end{aligned}$$

which completes the proof, see (3.29). □

Remark 3.10 *The result just proven allows one to consider the problem (3.3) in appropriate locally convex spaces spaces of entire functions, e.g., like the spaces introduced in [15].*

4 Associated Markov processes

In this section, we apply the results of [19] to both models, A and B, if the initial function f is the Bogoliubov functional of a probability measure on $\mathbb{N}_0$. We begin by presenting here some relevant facts. A time-homogeneous Markov process with state space $\mathbb{N}_0$ is defined by the transition probabilities $P_{kn}(t)$, $k, n \in \mathbb{N}_0$, $t \geq 0$, obeying the usual conditions by which they determine also a semigroup of contractions. Among such conditions one finds the following one

$$\sum_{n} P_{kn}(t) \leq 1, \quad t \geq 0.$$

According to [19], the process is called *honest* if the above sum is equal to one. In a more conventional form, the process $X(t)$ is defined by the probabilities

$$\begin{aligned}
&\mathrm{Prob}(X(t_1) = k_1; \ldots; X(t_n) = k_n) \qquad (4.1)\\
&= \sum_{k=0}^{\infty} \mu_0(\{k\}) P_{kl_1}(t_1) P_{l_1 l_2}(t_2 - t_1) \cdots P_{l_{n-1}l_n}(t_n - t_{n-1}),
\end{aligned}$$

for all $k_1, \ldots, k_n \in \mathbb{N}_0$ and $0 \leq t_1 \leq \cdots \leq t_n$. μ_0 is the initial distribution. The semigroup $\{P_{kn}(t)\}_{t\geq 0}$ has a generator, Q, considered as an infinite matrix, which is defined by means of the derivatives of $P_{kn}(t)$ at zero, that is, $Q_{kn} = P'_{kn}(0^+)$. Usually, it is supposed that $|Q_{nn}| < \infty$ for all n. Then $P(t)$ and Q obey the Kolmogorov equations

$$P'_{kn}(t) = \sum_l Q_{kl} P_{kn}(t), \tag{4.2}$$

$$P'_{kn}(t) = \sum_l P_{kl} Q_{ln}(t), \tag{4.3}$$

called also backward (4.2), and forward (4.3), equations. If Q has a three-diagonal form, the corresponding process is called a *birth and death* process. The main problem studied in [19], see also [7, 8], was to find the condition under which the matrix Q determines at least one process $P(t) = (P_{kn}(t))$, and when it is unique and honest. Such processes are called Q-processes.

Let μ_t be the probability distribution of the process at time t, and $f_n(t) = \mu_t(\{n\})$. By $\mathsf{f}(t)$ we denote the row-vector with components $f_n(t)$, $n \in \mathbb{N}_0$, and by $f(t,z)$ we write the Bogoliubov functional (1.8) of μ_t. Since in this section we deal with such objects only, we do not indicate explicitly their connection with this measure. By (4.3) we have

$$f'_n(t) = \sum_k f_k(t) Q_{kn}, \quad n \in \mathbb{N}_0. \tag{4.4}$$

Comparing this with (2.2) we see that Q corresponds to a birth-and-death process if for all k, n,

$$Q_{kn} = b_{n-1}\delta_{k,n-1} + a_n \delta_{k,n} + d_{n+1}\delta_{k,n+1}. \tag{4.5}$$

In this section, we assume that $\sum_n Q_{kn} = 0$ for all k, that is, we assume the equality in (2.4). In Section 8.4 of [19], a complete characterization of birth-and-death processes was given, see also [7, 8] and [12]. To formulate the corresponding results we introduce the following notations:

$$r_n = \frac{1}{b_n} + \frac{d_n}{b_n b_{n-1}} + \cdots + \frac{d_n \cdots d_1}{b_n \cdots b_2 b_1}, \quad R = \sum_{n=1}^{\infty} r_n, \tag{4.6}$$

$$s_n = \frac{1}{d_{n+1}} + \frac{b_n}{d_{n+1} d_n} + \cdots + \frac{b_n \cdots b_1}{d_{n+1} \cdots d_2}, \quad S = \sum_{n=1}^{\infty} s_n. \tag{4.7}$$

Then the relevant part of Theorem 11 of [19] reads as follows.

Proposition 4.1 *If $R = +\infty$, then there exists exactly one Q-process; it is honest and satisfies the forward equations. If $R < +\infty$ and $S = +\infty$, there exist infinitely many Q-processes. Only one of these processes satisfies the forward equations, but it is dishonest.*

Note that the uniqueness stated above was first proven by R. L. Dobrushin. The fact that the process satisfies the forward equations is important in view of the representation (4.4).

Let us now turn to our models A and B described by (1.9). For model A, we have $b_n^A = \varkappa a^n$ and $d_n^A = n$; hence,

$$r_n^A = \frac{n!}{\varkappa^{n+1} a^{n(n+1)/2}} \sum_{k=1}^{n} \frac{1}{k!} \varkappa^k a^{k(k-1)/2}, \tag{4.8}$$

and

$$s_n^A = \frac{\varkappa^{n+1} a^{n(n+1)/2}}{(n+1)!(n+1)} \sum_{k=1}^{n+1} k! \varkappa^{-k} a^{-k(k-1)/2}. \tag{4.9}$$

For $a \leq 1$, we use the estimate $r_n^A \geq n!/\varkappa^n a^{n(n+1)/2}$, by which $R = +\infty$. For $a > 1$, we use the recursion

$$r_n^A = (1 + n r_{n-1}^A)/\varkappa a^n,$$

which readily follows from (4.8), and obtain for $n > n_a$,

$$r_n^A \leq 1/(\varkappa a^n - n),$$

n_a being the least n for which $\varkappa a^n \geq n$. This yields $R < +\infty$ for this case. Likewise, from (4.9) we get $s_n^A \geq 1/(n+1)$, thus, $S = +\infty$ for all $a > 0$. For model B, we have

$$r_n^B = (n-1)! \alpha^{-1} \left(\frac{\beta}{\alpha}\right)^n \sum_{k=0}^{n-1} \frac{1}{k!} \left(\frac{\alpha}{\beta}\right)^k \geq (n-1)! \alpha^{-1} \left(\frac{\beta}{\alpha}\right)^n, \tag{4.10}$$

which yields $R = +\infty$, irrespectively to the values of α and β. We summarize the above analysis in the following statement.

Proposition 4.2 *For model A with $a \leq 1$ (resp. for model B with any positive α and β), there exists a unique Q^A-process (P_{kn}^A) (resp. Q^B-process (P_{kn}^B)), which satisfies the forward equations and is honest. For model A with $a > 1$, there exists a unique Q^A-process, which satisfies the forward equations, but is dishonest.*

For model A with $a > 1$, this proposition adds nothing new to what has been obtained in Theorems 3.1 and 3.2. However, for $a \leq 1$, Proposition 4.2 yields that $T_A(t)$ maps $\mathcal{E}^{+,1}$ into $\mathcal{E}^{+,1}$ and that its action on a given $f \in \mathcal{E}$ can be described as

$$(T_A(t) f)(z) = \sum_{n=0}^{\infty} (1+z)^n \sum_{k=0}^{\infty} f_k P_{kn}^A(t). \tag{4.11}$$

For model B, we can now formulate the following analog of Theorems 3.1 and 3.2.

Theorem 4.3 *For any positive α and β, the operator*

$$B : \{f \in \mathcal{E} \mid f'' \in \mathcal{E}\} \to \mathcal{E},$$

defined in (1.9), is a generator of a C_0 semigroup of contractions, which acts as

$$(T_B(t)f)(z) = \sum_{n=0}^{\infty}(1+z)^n \sum_{k=0}^{\infty} f_k P_{kn}^B(t) \tag{4.12}$$

For every $f \in \mathcal{E}$ such that $f'' \in \mathcal{E}$, the Cauchy problem

$$\begin{cases} u_t'(t,z) = z(1+z)\left[\alpha u_z'(t,z) - \beta u_{zz}''(t,z)\right], & \alpha, \beta > 0 \\ u(0,z) = f(z), \end{cases} \tag{4.13}$$

has a unique solution, which is continuously differentiable on $[0, +\infty)$.

5 Model A with $a \leq 1$: reversibility and ergodicity

Let us first show that the process $(P_{kn}^A(t))$ is reversible, i.e, the tuples $(X(t_1), X(t_2), \ldots, X(t_n))$ and $(X(\tau - t_1), X(\tau - t_2), \ldots, X(\tau - t_n))$ have the same distribution for all $t_1, \ldots, t_n$ and all $\tau > 0$, see (4.2) and [11]. The process is reversible if and only if

$$\psi_k P_{kn}^A(t) = \psi_n P_{nk}^A(t), \tag{5.1}$$

for all $k, n \in \mathbb{Z}_+$ and $t > 0$. Here ψ_n, $n \in \mathbb{N}_0$ are the same as in (3.5). By direct calculations, one can verify that

$$\psi_k L_{kn}^A = \psi_n L_{nk}^A. \tag{5.2}$$

By iterating the backward equation (4.2) one obtains $P_{kn}^A(t)$ in the form

$$P_{kn}^A(t) = 1 + \sum_{m=1}^{\infty} \frac{t^m}{m!} \sum_{l_1, \ldots, l_m \geq 0} Q_{kl_1}^A Q_{l_1 l_2}^A \cdots Q_{l_m n}^A. \tag{5.3}$$

Since each $Q_{ll'}^A$ has only three nonzero elements, the absolute value of the m-th term in (5.3) can be estimated from above by

$$\frac{C_{kn}(3t)^m (n+m-1)!}{m!} \leq \tilde{C}_{kn}(3t)^m (n+m)^n,$$

where C_{kn} and $\tilde{C}_{kn}$ are appropriate constants. Thus, the series in (5.3) converges, and hence (5.1) holds in view of (5.2), at least for small enough t. To bigger values of t, the validity of (5.1) can be extended by the Kolmogorov equation

$$P_{kn}^A(t+s) = \sum_l P_{kl}^A(t) P_{ln}^A(s).$$

Proof of Theorem 3.3: We prove here only the second part of the statement. Suppose that for a given $f \in \mathcal{K}^{+,1} = \mathcal{K} \cap \mathcal{E}^{+,1}$, the solution of (3.3) obeys (3.6) for some $\tau > 0$. Then by (4.11), for any $n \in \mathbb{N}_0$ we have

$$u(t,z) = \sum_{n=0}^{\infty}(1+z)^n u_n(t) = \sum_{n=0}^{\infty}(1+z)^n \sum_{k=0}^{\infty} f_k P_{kn}^A(t), \tag{5.4}$$

that is, $u_n(t) = \mathrm{Prob}[X(t) = n]$, and hence, for all $t \in [0, \tau]$,

$$u_n(t) = u_n(\tau - t), \tag{5.5}$$

which follows by the reversibility just proven. The evolution of $u_n(t)$, $n \in \mathbb{N}_0$, is described by the system of equations

$$\frac{d}{dt}u_n(t) = b_{n-1}^A u_{n-1}(t) + a_n^A u_n(t) + d_{n+1}^A u_{n+1}(t). \tag{5.6}$$

Integrating both sides we get, see (3.6),

$$b_{n-1}^A v_{n-1} + a_n^A v_n + d_{n+1}^A v_{n+1} = 0, \quad v_n \stackrel{\mathrm{def}}{=} \int_0^\tau u_n(t)dt.$$

Thus,

$$v_n = \psi_n, \quad n \in \mathbb{N}_0, \tag{5.7}$$

in view of which, we have for $\tilde{u}_n(t) \stackrel{\mathrm{def}}{=} u_n(t)/\psi_n$, the following analog of the system (5.6)

$$\frac{d}{dt}\tilde{u}_n(t) = d_n^A \tilde{u}_{n-1}(t) + a_n^A \tilde{u}_n(t) + b_n^A \tilde{u}_{n+1}(t), \quad n \in \mathbb{N}_0. \tag{5.8}$$

To solve it we use the Fourier transform

$$\begin{aligned} \tilde{u}_n(t) &= \sum_{m\in\mathbb{Z}} \hat{u}_n(m) \exp\left(\frac{2\pi i}{\tau} mt\right), \qquad (5.9)\\ \hat{u}_n(m) &= \frac{1}{\tau}\int_0^\tau \tilde{u}_n(t) \exp\left(-\frac{2\pi i}{\tau} mt\right) dt. \end{aligned}$$

As all $\tilde{u}_n(t)$ are differentiable, the Fourier coefficients obey the condition

$$\sum_{m\in\mathbb{Z}} m|\hat{u}_n(m)| < \infty.$$

In these coefficients, the system (5.8) takes the form

$$\frac{2\pi i}{\tau} m\hat{u}_n(m) = d_n^A \hat{u}_{n-1}(m) + a_n^A \hat{u}_n(m) + b_n^A \hat{u}_{n+1}(m), \quad n \in \mathbb{N}_0, \quad m \in \mathbb{Z}. \tag{5.10}$$

Since all $\tilde{u}_n(t)$ are real, we have $\hat{u}_n(-m) = \overline{\hat{u}_n(m)}$. At the same time, by (5.5) and (5.9), we get $\hat{u}_n(-m) = \hat{u}_n(m)$; hence all $\hat{u}_n(m)$ are real as well. This means that, for $m \neq 0$, the only solution of (5.10) is $\hat{u}_n(m) = 0$ for all such m and all n. This yields $\tilde{u}_n(t) = \hat{u}_n(0) = 1$, which yields in turn $u(t, \cdot) = \psi$ and hence $f = \psi$ by (3.6). □

The theorem just proven has two important corollaries.

Corollary 5.1 *For every $f \in \mathcal{K}^{+,1}$, the solution $u(t, \cdot)$ of the problem (3.3) tends to the equilibrium solution (3.5), as $t \to +\infty$. Here the convergence is in the norm of $\mathcal{E}$.*

Proof: For any $\tau > 0$, the sequence $\{T^A(n\tau)f\}_{n\in\mathbb{Z}_0} \subset \mathcal{K}_1^+$ is relatively compact in $\mathcal{E}$, see Theorem 3.4. By Theorem 3.3, its only accumulation point is ψ. As τ is arbitrary, this yields the result to be proven. □

Corollary 5.2 *The Markov process $(P^A_{kn}(t))$ is ergodic and hence irreducible.*

Proof: Let $u_n(t)$, $n \in \mathbb{Z}_+$ be as in (5.4). Then by Corollary 5.1, the vector function

$$[0, +\infty) \ni t \mapsto u(t) = (u_0(t), \dots, u_n(t), \dots) \in \ell^1$$

converges in ℓ^1 to the vector $(\psi_0, \psi_1, \dots)$. This certainly yields that $u_n(t) \to \psi_n$ for any $n \in \mathbb{Z}_+$. Now for a given $k \in \mathbb{Z}_+$, take $f_l^{(k)} = \delta_{kl}$. Hence

$$P^A_{kn}(t) \to \psi_n, \quad t \to +\infty,$$

which completes the proof. □

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