Applications of Compact Superharmonic Functions: Path Regularity and Tightness of Capacities

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Abstract. We establish relations between the existence of the $\mathcal{L}$-superharmonic functions that have compact level sets ($\mathcal{L}$ being the generator of a right Markov process), the path regularity of the process, and the tightness of the induced capacities. We present several examples in infinite dimensional situations, like the case when $\mathcal{L}$ is the Gross-Laplace operator on an abstract Wiener space and a class of measure-valued branching process associated with a nonlinear perturbation of $\mathcal{L}$.

Mathematics Subject Classification (2000). Primary 60J45, 31C15; Secondary 47D07, 60J40, 60J35.

Keywords. resolvent of kernels, tight capacity, Lyapunov function, right process, standard process.

1. Introduction

A major difficulty in the stochastic analysis in infinite dimensions is the lack of the local compactness of the state space. The developments from the last two decades, in particular the theory of quasi-regular Dirichlet forms (cf. [14]), indicated, as an adequate substitute, the existence of a nest of compact sets, or equivalently, the tightness property of the associated capacity. However, in applications it is sometimes not easy to verify such a property, for example, because the capacity is not expressed directly in terms of the infinitesimal generator $\mathcal{L}$ of the theory. It happens that a more convenient property is the existence of an $\mathcal{L}$-superharmonic function having compact level sets; such a function was called compact Lyapunov.

The aim of this paper is to give an overview on the relations between the existence of a compact Lyapunov function, the path regularity of the process, and the tightness of the induced capacities. We collect relevant examples where such functions were constructed and used further as main investigation tools.

The paper is organized as follows. In Section 2 we first precise the setting and recall some preliminary results. We then expose the general results relating the tightness property of the capacity, the existence of the compact Lyapunov functions, and the path regularity of the associated Markov process. In Section 3 we present as application three situations with explicit constructions of compact Lyapunov functions.

The first example is a method of finding martingale solutions of stochastic partial differential equations on Hilbert spaces (cf. [4]). The square of the norm is the initial natural candidate, but the convenient function is obtained by approximation.

The second application is given in the frame of the Lévy processes in infinite dimensions (see [7] for details). It turns out that the square of the continuous linear functionals are $\mathcal{L}$-subharmonic functions ($\mathcal{L}$ denoting the infinitesimal generator of the Lévy process). The desired function having compact level sets is then obtained from the potentials of these functions.

The last application is the construction of a compact Lyapunov function for a measure-valued branching process. This is a main argument in the proof in [1] of the càdlàg property of the paths of the measure-valued process

2. The general frame

Let E be a metrizable Lusin topological space (i.e., it is homeomorphic to a Borel subset of a compact metrizable space) endowed with the Borel σ-algebra $\mathcal{B} = \mathcal{B}(E)$.

We consider a sub-Markovian resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ and we denote by $\mathcal{E}(\mathcal{U})$ the set of all $\mathcal{B}$-measurable *$\mathcal{U}$-excessive functions:* $u \in \mathcal{E}(\mathcal{U})$ if and only if u is a positive numerical $\mathcal{B}$-measurable function, $\alpha U_\alpha u \leq u$ for all $\alpha > 0$ and $\lim_{\alpha\to\infty} \alpha U_\alpha u(x) = u(x)$ for all $x \in E$.

If $\beta > 0$ we denote by $\mathcal{U}_\beta$ the sub-Markovian resolvent of kernels $(U_{\beta+\alpha})_{\alpha>0}$. If v is a *$\mathcal{U}_\beta$-supermedian function* (i.e., $\alpha U_{\beta+\alpha} v \leq v$ for all $\alpha > 0$), its $\mathcal{U}_\beta$-excessive regularization $\widehat{v}$ is given by $\widehat{v}(x) = \sup_\alpha \alpha U_{\beta+\alpha} v(x)$, $x \in E$.

We assume that:

(2.1) $\sigma(\mathcal{E}(\mathcal{U}_\beta)) = \mathcal{B}$ and all the points of E are non-branch points with respect to $\mathcal{U}_\beta$, that is $1 \in \mathcal{E}(\mathcal{U}_\beta)$ and if $u, v \in \mathcal{E}(\mathcal{U}_\beta)$ then for all $x \in E$ we have $\inf(u,v)(x) = \widehat{\inf(u,v)}(x)$.

Recall that a σ-finite measure ξ on $(E, \mathcal{B})$ is called *$\mathcal{U}$-excessive* provided that $\xi \circ \alpha U_\alpha \leq \xi$ for all $\alpha > 0$.

Remark 2.0.1. (i) Assume that (2.1) holds, let m be a fixed $\mathcal{U}$-excessive measure, and $p \in \mathbb{R}$, $p > 1$. For every $\alpha > 0$ we denote by V_α the linear operator on $L^p(E, m)$ induced by U_α and notice that V_α is sub-Markovian, i.e., if $f \in L^p(E, m)$, $0 \leq f \leq 1$ the $0 \leq V_\alpha f \leq 1$. It turns out that the family $(V_\alpha)_{\alpha>0}$ is a C_0-resolvent of contractions on $L^p(E, m)$.

(ii) Conversely, assume that m is a σ finite measure on $(E,\mathcal{B})$ and $(V_\alpha)_{\alpha>0}$ is a C_0-resolvent of sub-Markovian contractions on $L^p(E,m)$, $p\geq 1$. Then there exists sub-Markovian resolvent of kernels $\mathcal{U}=(U_\alpha)_{\alpha>0}$ on $(E,\mathcal{B})$ which satisfies (2.1) and such that $U_\alpha=V_\alpha$ as operators on $L^p(E,m)$ for all $\alpha>0$; see Remark 2.3 in [5].

(iii) Let $(V_\alpha)_{\alpha>0}$ and $\mathcal{U}$ be as above, and $(\mathcal{L},D(\mathcal{L}))$ be the infinitesimal generator of $(V_\alpha)_{\alpha>0}$: $D(\mathcal{L})=V_\alpha(L^p(E,m))$, $\mathcal{L}(V_\alpha f)=\alpha V_\alpha f-f$ for all $f\in L^p(E,m)$. If $u\in D(\mathcal{L})$ and $\beta>0$ then: $\mathcal{L}u\leq\beta u$ if and only if there exists a m-version of u which is a $\mathcal{U}_\beta$-excessive function.

Let $\mathsf{Exc}(\mathcal{U})$ be the set of all $\mathcal{U}$-excessive measures on E and denote by $\mathsf{Pot}(\mathcal{U})$ the set of all *potential* $\mathcal{U}$-excessive measures: if $\xi\in\mathsf{Exc}(\mathcal{U})$ then $\xi\in\mathsf{Pot}(\mathcal{U})$ if $\xi=\mu\circ U$, where μ is a σ-finite measure on $(E,\mathcal{B})$. If $\beta>0$ then the *energy functional* $L^\beta:\mathsf{Exc}(\mathcal{U}_\beta)\times\mathcal{E}(\mathcal{U}_\beta)\longrightarrow\overline{\mathbb{R}}_+$ is defined by

$$L^\beta(\xi,u):=\sup\{\nu(u):\,\mathsf{Pot}(\mathcal{U}_\beta)\ni\nu\circ U_\beta\leq\xi\}.$$

Let E_1 be the set of all extreme points of the set $\{\xi\in\mathsf{Exc}(\mathcal{U}_\beta):\,L^\beta(\xi,1)=1\}$, endowed with the σ-algebra $\mathcal{B}_1$ generated by the functionals $\widetilde{u}$, $\widetilde{u}(\xi):=L^\beta(\xi,u)$ for all $\xi\in\mathsf{Exc}(\mathcal{U}_\beta)$ and $u\in\mathcal{E}(\mathcal{U}_\beta)$. Then $(E_1,\mathcal{B}_1)$ is a Lusin measurable space, the map $x\longmapsto\varepsilon_x\circ U_\beta$ identifies E with a subset of E_1, $E\in\mathcal{B}_1$, $\mathcal{B}=\mathcal{B}_1|_E$ and there exists a Markovian resolvent of kernels $\mathcal{U}^1=(U^1_\alpha)_{\alpha>0}$ on $(E_1,\mathcal{B}_1)$ such that $\sigma(\mathcal{E}(\mathcal{U}^1_\beta))=\mathcal{B}_1$, every point of E_1 is a non-branch point with respect to $\mathcal{U}^1_\beta$, $U^1_\beta(1_{E_1\setminus E})=0$, and $\mathcal{U}$ is the restriction of $\mathcal{U}^1$ to E.

If $M\in\mathcal{B}$ and $u\in\mathcal{E}(\mathcal{U}_\beta)$, then the *reduced function* (with respect to $\mathcal{U}_\beta$) *of u on M* is the function $R^M_\beta u$ defined by

$$R^M_\beta u:=\inf\left\{v\in\mathcal{E}(\mathcal{U}_\beta):\,v\geq u\text{ on }M\right\}.$$

The reduced function $R^M_\beta u$ is a universally $\mathcal{B}$-measurable $\mathcal{U}_\beta$-supermedian function.

In the sequel λ will be a finite measure on $(E,\mathcal{B})$.

Let

$$u_o:=U_\beta f_o,$$

where f_o is a bounded, strictly positive $\mathcal{B}$-measurable function. Consider the functional $M\longmapsto c_\lambda(M)$, $M\subset E$, defined as

$$c_\lambda(M):=\inf\{\lambda(R^G_\beta u_o):\,G\in\mathcal{T},M\subset G\}.$$

By [2] it follows that c_λ is a Choquet capacity on E.

The *fine topology* on E is the topology generated by $\mathcal{E}(\mathcal{U}_\beta)$. We assume that the Lusin topology of E is smaller than the fine topology.

2.1. Compact Lyapunov functions and tightness of capacities

We assume for simplicity that there exists a strictly positive constant k such that $k \leq u_o$. Clearly, this happens if the resolvent $\mathcal{U}$ is Markovian, taking $u_o = 1$.

Proposition 2.1.1. *The following assertions are equivalent.*

(i) The capacity c_λ is tight, *i.e., there exists an increasing sequence $(K_n)_n$ of compact sets such that $\inf_n c_\lambda(E \setminus K_n) = 0$.*

(ii) There exists a strictly positive $\mathcal{U}_\beta$-excessive u such that for every increasing sequence $(D_n)_n$ of open sets with $\bigcup_n D_n = E$ we have $\inf_n R_\beta^{E\setminus D_n} u = 0$ λ-a.e.

(iii) There exists a $\mathcal{U}_\beta$-excessive function v which is λ-integrable, such that the set $[v \leq \alpha]$ is relatively compact for all $\alpha > 0$. Such a function v is called compact Lyapunov function.

Proof. For the proof of the equivalence between (i) and (ii) see [2] and [3], while for the proof of $(i) \Longleftrightarrow (iii)$ see [8] and Remark 3.3 from [4]. □

Remark. The function u in assertion (ii) of the above Proposition 2.1.1 may be considered as the analogue, in this general frame, of a potential from the $\mathfrak{P}$-harmonic space context (in the sense of [10]); cf. the discussion in [3].

2.2. Path regularity and compact Lyapunov functions

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X_t, P^x)$ be a fixed Borel right Markov process with state space E and $(P_t)_{t\geq 0}$ be the transition semigroup of X,

$$P_t f(x) = E^x(f \circ X_t; t < \zeta)$$

(see, e.g., [15]). Notice that if $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is the resolvent of kernels associated with the process X, i.e.,

$$U_\alpha f = \int_0^\infty e^{-\alpha t} P_t f dt \qquad \text{for all } f \in p\mathcal{B} \text{ and } \alpha > 0,$$

then condition (2.1) is verified; here p$\mathcal{B}$ denotes the set of all positive numerical $\mathcal{B}$-measurable functions on E. In this case, by a theorem of Hunt we have in addition

$$R_\beta^M u(x) = E^x(e^{-\beta D_M} u \circ X_{D_M} ; D_M < \infty),$$

$$\widehat{R_\beta^M} u(x) = E^x(e^{-\beta T_M} u \circ X_{T_M} ; T_M < \infty),$$

where $D_M(\omega) := \inf\{t \geq 0 : X_t(\omega) \in M\}$, $T_M(\omega) := \inf\{t > 0 : X_t(\omega) \in M\}$, $\omega \in \Omega$.

The right process X is called λ-*standard* if:

– X has *càdlàg trajectories* P^λ-a.e., i.e., it possesses left limits in E P^λ-a.e. on $[0, \zeta)$; ζ is the life time of X;

– X is *quasi-left continuous up to* ζ P^λ-a.e., i.e., for every increasing sequence $(T_n)_n$ of stopping times with $T_n \nearrow T$ we have $X_{T_n} \longrightarrow X_T$ P^λ-a.e. on $[T < \zeta]$.

The right process X is called *standard* if it is λ-standard for every finite measure λ.

The set $M \in \mathcal{B}$ is called *polar* (resp. *μ-polar*; where μ is a σ-finite measure on $(E, \mathcal{B})$) if $\widehat{R_\beta^M} 1 = 0$ (resp. $\widehat{R_\beta^M} 1 = 0$ μ-a.e.).

(2.2) By Sections 1.7 and 1.8 in [2] and Theorem 1.3 in [4] we get that the following assertions are equivalent:

(2.2*a*) Every $\mathcal{U}_\beta$-excessive measure dominated by a potential is also a potential.

(2.2*b*) There exists a right process with state space E having $\mathcal{U}$ as associated resolvent.

If $\beta > 0$ then a *Ray cone* associated with $\mathcal{U}_\beta$ is a cone $\mathcal{R}$ of bounded $\mathcal{U}_\beta$-excessive functions such that: $U_\alpha(\mathcal{R}) \subset \mathcal{R}$ for all $\alpha > 0$, $U_\beta\big((\mathcal{R} - \mathcal{R})_+\big) \subset \mathcal{R}$, $\sigma(\mathcal{R}) = \mathcal{B}$, it is min-stable, separable in the supremum norm and $1 \in \mathcal{R}$. A *Ray topology* on E is the topology generated by a Ray cone.

Remark 2.2.1. Any Ray topology is smaller than the fine topology and the resolvent $\mathcal{U}^1$ is always the resolvent of a right process with state space E_1 endowed with a Ray topology; see [2].

The following two propositions indicate the connection between the path regularity of a right Markov process and the tightness of the associated capacities (or equivalently, the existence of the compact Lyapunov functions; cf. Proposition 2.1.1).

Proposition 2.2.2. *Suppose that $\mathcal{U}$ is the resolvent family of a right process which has càdlàg trajectories P^λ-a.e. Then there exists a compact Lyapunov function which is finite λ-a.e.*

Proof. By [13] (see also [3] for a different approach) the càdlàg property of the trajectories implies that tightness property of the capacity c_λ. The assertion follows now using Proposition 2.1.1, the implication $(i) \Longrightarrow (iii)$. □

The next result shows that a converse of the assertion in Proposition 2.2.2 holds. It is a particular case of Theorem 5.2 from [8] and a version of Proposition 2.2 in [9].

Proposition 2.2.3. *Assume that E is endowed with a Ray topology and there exists a compact Lyapunov function which is finite λ-a.e. Then there exists a λ-standard process with state space E such that its resolvent equals $\mathcal{U}$ λ-quasi everywhere.*

Sketch of the proof. By Proposition 2.1.1, the implication $(iii) \Longrightarrow (i)$, the capacity c_λ is tight with respect to the Ray topology. By Lemma 3.5 from [4] it follows that the set $E_1 \setminus E$ is λ-polar. The claimed process is the restriction to E of the right process with state space E_1 given by Remark 2.2.1.

3. Applications: explicit constructions of Lyapunov functions

3.1. Martingale solutions for stochastic PDE on Hilbert spaces

We consider the stochastic differential equation on a Hilbert space H (with inner product $\langle \ , \ \rangle$ and norm $|\cdot|$) of type

(3.1.1) $$\mathrm{d}X(t) = \big[AX(t) + F_0\big(X(t)\big)\big]dt + \sqrt{C}dW(t) \ ;$$

see [11] and Section 5 in [4] for a complete treatment. Here $W(t)$, $t \geq 0$, is a cylindrical Brownian motion on H, C is a positive definite self-adjoint linear operator on H and $A : D(A) \subset H \to H$ the infinitesimal generator of a C_0-semigroup on H. Furthermore, $F_0(x) := y_0$, $x \in D(F)$, where $y_0 \in F(x)$ such that $|y_0| = \min\limits_{y\in F(x)} |y|$, and $F : D(F) \subset H \to 2^H$ is an m-dissipative map. This means that $D(F)$ is a Borel set in H and $\langle u - v, x - y\rangle \leq 0$ for all $x, y \in D(F)$, $u \in F(x)$, $v \in F(y)$, and $\mathrm{Range}(I - F) := \bigcup_{x\in D(F)}\big(x - F(x)\big) = H$. Since for any $x \in D(F)$ the set $F(x)$ is closed, non-empty and convex, F_0 is well-defined.

Construction of a Markovian C_0-resolvent on an L^p space. Let us first write the underlying Kolmogorov operator $\mathcal{L}_0$. A heuristic application of Itô's formula to a solution of (3.1.1) implies that the Kolmogorov operator on test functions $\varphi \in \mathcal{E}_A(H) := \mathrm{lin.\,span}\big\{\sin\langle h, x\rangle,\ \cos\langle h, x\rangle \ \big|\ h \in D(A^*)\big\}$ has the following form:

$$\mathcal{L}_0\varphi(x) = \frac{1}{2}\cdot \mathrm{Tr}\big[CD^2\varphi(x)\big] + \big\langle x, A^*D\varphi(x)\big\rangle + \big\langle F_0(x), D\varphi(x)\big\rangle\ , \quad x \in H,$$

where $D\varphi(x)$, $D^2\varphi(x)$ denote the first and second Fréchet derivatives of φ at $x \in H$ considered as an element in H and as an operator on H, respectively. We note that by the chain rule $D\varphi(x) \in D(A^*)$ for all $\varphi \in \mathcal{E}_A(H)$, $x \in H$. Clearly, $\mathcal{L}_0$ is well-defined for all φ of the form $\varphi(x) = f\big(\langle h_1, x\rangle, \dots, \langle h_M, x\rangle\big)$, $x \in H$, with $f \in C^2(\mathbb{R}^M)$, $M \in \mathbb{N}$, $h_1, \dots, h_M \in D(A^*)$. As in [11], we make the following assumptions:

$(H1)$ (i) A is the infinitesimal generator of a strongly continuous semigroup e^{tA}, $t \geq 0$, on H, and there exists a constant $\omega > 0$ such that $\langle Ax, x\rangle \leq -\omega\,|x|^2$ for all $x \in D(A)$.
(ii) C is self-adjoint, nonnegative definite and such that $\mathrm{Tr}\,Q < \infty$, where $Qx := \int_0^\infty e^{tA}\,C\,e^{tA^*}\,xdt$, $x \in H$.

$(H2)$ There exists a probability measure μ on the Borel σ-algebra $\mathcal{B}(H)$ of H such that
(i) $\displaystyle\int_{D(F)} \big(|x|^{2p} + |F_0(x)|^p + |x|^{2p}\cdot|F_0(x)|^p\big)\,\mu(\mathrm{d}x) < \infty.$
(ii) For all $\varphi \in \mathcal{E}_A(H)$ we have $\mathcal{L}_0\varphi \in L^p(H, \mu)$ and $\int \mathcal{L}_0\varphi d\mu = 0$ ('infinitesimal invariance').
(iii) $\mu\big(D(F)\big) = 1$.

In the sequel, for simplicity, we shall treat only the case $p = 2$.

By assumption $(H2)$ (ii) it is easy to prove that $\big(\mathcal{L}_0, \mathcal{E}_A(H)\big)$ is dissipative on $L^2(H, \mu)$ (cf. [11, Proposition 2.1]), hence closable. Let $\big(\mathcal{L}, D(\mathcal{L})\big)$ denote its closure. The first main result in [11], however, is that $(H1)$ and $(H2)$ imply that $\big(\mathcal{L}, D(\mathcal{L})\big)$ is m-dissipative (cf. [11, Theorem 2.3]), hence generates a C_0-semigroup $P_t := e^{t\mathcal{L}}$, $t \geq 0$, on $L^2(H,\mu)$. By [11, Corollary 2.5], $(P_t)_{t\geq 0}$ is Markovian, i.e. positivity preserving and $P_t 1 = 1$ for all $t \geq 0$. Clearly, μ is invariant for $(P_t)_{t\geq 0}$, i.e. $\int P_t f d\mu = \int f d\mu$ for all $t \geq 0$, $f \in L^2(H,\mu)$. For $f \in L^2(H,\mu)$ and $\alpha > 0$ we define $V_\alpha f := \int_0^\infty e^{-\alpha t} P_t f dt$. Then $(V_\alpha)_{\alpha>0}$ is a Markovian C_0-resolvent of contractions on $L^2(H,\mu)$.

Compact Lyapunov functions constructed by approximation. We need the following additional condition:
$(H3)$ (i) There exists an orthonormal basis $\{e_j \mid j \in \mathbb{N}\}$ of H so that $\bigcup_{N\in\mathbb{N}} E_N$ with $E_N := \operatorname{lin.span}\{e_j \mid 1 \leq j \leq N\}$ is dense in $D(A^*)$ with respect to $|\cdot|_{A^*}$ and such that for the orthogonal projection P_N onto E_N in H we have that the function $H \ni x \mapsto \langle P_N x, A^* P_N x\rangle$ converges in $L^1(H,\mu)$ to $H \ni x \mapsto \langle x, A^* x\rangle$ (defined to be $+\infty$ if $x \in H \setminus D(A^*)$).
(ii) There exist two increasing functions $\varrho_1, \varrho_2 : [1,\infty) \to (0,\infty)$ such that $\big|F_0(x)\big|^2 \leq \varrho_1\big(|x|\big) + \varrho_2\big(|x|\big)\, \big|\langle x, A^* x\rangle\big|$ for all $x \in H$, and the function on the right hand side is in $L^1(H,\mu)$.

By [11, Theorem 2.3] the set $(1-\mathcal{L}_0)\,\mathcal{E}_A(H)$ is dense in $L^2(H,\mu)$, hence also in $L^1(H,\mu)$. Therefore the closure $\big(\mathcal{L}_1, D(\mathcal{L}_1)\big)$ of $\big(\mathcal{L}_0, \mathcal{E}_A(H)\big)$ (which exists since $\big(\mathcal{L}_0, \mathcal{E}_A(H)\big)$ is also dissipative on $L^1(H,\mu)$) also generates a C_0-semigroup $P_t^{(1)} := e^{t\mathcal{L}_1}$, $t \geq 0$, on $L^1(H,\mu)$. Let $(V_\alpha^{(1)})_{\alpha>0}$, denote the corresponding resolvent. For $\alpha > 0$ by definition $V_\alpha = V_\alpha^{(1)}$ on $(\lambda - \mathcal{L}_0)\,\big(\mathcal{E}_A(H)\big)$, hence $V_\alpha f = V_\alpha^{(1)} f$ for all $f \in L^2(H,\mu)$ by continuity.

According with assertion (ii) of Remark 2.0.1, the C_0-resolvent $(V_\alpha^{(1)})_{\alpha>0}$ on $L^1(H,\mu)$ is generated by a resolvent of kernels.

Lemma 3.1.1. *Let $u, g : H \longrightarrow \mathbb{R}_+$, $u, g \in L^1(H,\mu)$, such that u has compact level sets and assume that there exist two sequences, $(u_N)_N \subset \mathcal{D}(\mathcal{L})$ and $(g_N)_N \subset L^1(H,\mu)$, such that $(\beta - \mathcal{L})u_N \leq g_N$ for all $N \in \mathbb{N}$, the sequence $(u_N)_N$ converges μ-a.e to u, and the sequence $(g_N)_N$ converges in $L^1(H,\mu)$ to g as $N \to \infty$. Then $v = V_\beta^{(1)} g$ is a compact Lyapunov function.*

Proof. The assertion follows since $u_N \leq V_\beta^{(1)} g_N$ for all N and passing to the limit we get $u \leq v$. □

Remark. (i) We can apply Lemma 3.1.1 taking $g(x) := 2\,|x|^2 + \big(2+\varrho_2(|x|)\big)\,\big|\langle x, A^* x\rangle\big| + \varrho_1\big(|x|\big)$, $u(x) := |x|^2$, $x \in H$; $u_N := u \circ P_N$, $g_N := g \circ P_N$, $N \in \mathbb{N}$. Notice that by $(H2)$ (i) and $(H3)$ (ii) we have $g \in L^1(H,\mu)$ and the required convergence hypotheses are ensured also by $(H3)$.
(ii) Using the Lyapunov function constructed above, Proposition 2.2.3, and assuming $(H1)-(H3)$, it was proved in [4], Section 5, that there exists a right process

X with state space H (endowed with the weak topology) having $(V_\alpha)_{\alpha>0}$ as associate resolvent and X is a martingale solution of (3.1.1), i.e., for every $f \in D(\mathcal{L})$, the process $t \longmapsto f(X_t) - \int_0^t \mathcal{L}f(X_s)ds$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale under P^μ.

3.2. Lévy processes on Hilbert spaces

Let us now informally describe how one can apply the results in Section 2 in concrete situations. There is usually given a "candidate" of a generator $\mathcal{L}$ of a resolvent $(U_\alpha)_{\alpha>0}$ which was the object we started with in our considerations in the first section, e.g. a closed linear operator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ on $C_u(E)$ (the space of all bounded uniformly continuous real valued functions on the Lusin topological space E) such that

$$(\alpha - \mathcal{L})^{-1} = U_\alpha, \quad \alpha > 0.$$

Generally, $\mathcal{L}$ is initially only known for "nice" functions $u : E \longrightarrow \mathbb{R}$. Let us look at the case where E is real separable Banach space with topological dual E'. Then often one can check that the given $\mathcal{L}$ is defined on the set $\mathcal{P}$ of polynomials of elements in E', i.e. function $u : E \longrightarrow \mathbb{R}$ of type

$$u(x) = p(l_1(x), l_2(x), \ldots, l_m(x)), \quad x \in E$$

with $m \in \mathbb{N}$ arbitrary and $l_1, \ldots l_m \in E'$, p a polynomial in m variables. Furthermore, let us assume that $\mathcal{L}$ is a diffusion operator, that is it satisfies the Leibniz rule and that $\mathcal{L}\, l = 0$ for all $l \in E'$. This is e.g. the case when $\mathcal{L}$ is a differential operator only involving second derivatives. More precisely, for $u \in \mathcal{P}$, $u = p(l_1, l_2, \ldots, l_m)$,

(3.2.1) $\quad \mathcal{L}u(x) = \sum_{i,j=1}^{\infty} {}_{E'}\langle l_j, A(x)l_i\rangle_E\, \partial_i\partial_j\, p(l_1, \ldots, l_m)$

where ∂_i denotes derivative with respect to the i-th variable and for $x \in E$, $A(x) : E' \longrightarrow E$, linear, bounded and nonnegative, continuous in x. Note that the sum in (3.2.1) is finite. Then we have for the corresponding semigroup $(P_t)_{t\geq 0}$,

$$P_t u - u = \int_0^t P_s \mathcal{L}u\, ds, \quad t \geq 0,$$

hence in particular for $l \in E'$, $t \geq 0$,

(3.2.2) $\quad P_t l^2 - l^2 = \int_0^t P_s(\mathcal{L}\, l^2)ds = \int_0^t P_s(2l\mathcal{L}\, l + 2{}_{E'}\langle l, A(x)l\rangle_E)ds \geq 0.$

It follows that

(3.2.3) $\quad P_t l^2 \geq l^2,\;$ for all $t > 0$ and $l \in E'$.

Hence l^2 is $\mathcal{L}$-subharmonic. Then in many cases it is possible to construct a Lyapunov function, which is the crucial ingredient for the process construction presented in Section 2, in terms of $l \in E'$. In [7], for simplicity in the concrete case of Brownian motion on abstract Wiener space, the above ideas are implemented in a rigorous way, in particular the mentioned construction of the Lyapunov function in terms of $l \in E'$. For this we only use (3.2.3) and not (3.2.2), more precisely we argue as follows: the function $U_\beta(l^2)$ is $(\mathcal{L} - \beta)$-superharmonic and it has compact level sets because by (3.2.3) we have

$$\beta U_\beta(l^2) \geq l^2.$$

Taking this into account one can extend the method to the case where $\mathcal{L}$ does no longer satisfy the Leibniz rule, so it is maybe not a local operator. We therefore give a new construction of Lévy process in infinite dimensions, based on our results in Section 2. As a byproduct, using Proposition 2.2.3, we obtain in particular that infinite dimensional Lévy processes are quasi-left continuous.

3.3. Measure-valued branching processes

The frame we consider in this subsection is as in [12] and [1]. More precisely, we assume that $\mathcal{U}$ is the resolvent of a right Markov process X with state space E, called *spatial motion.*

We fix a *branching mechanism*, that is, a function $\Phi : E \times [0,\infty) \longrightarrow \mathbb{R}$ of the form

$$\Phi(x,\lambda) = -b(x)\lambda - c(x)\lambda^2 + \int_0^\infty (1 - e^{-\lambda s} - \lambda s) N(x, ds)$$

where $c \geq 0$ and b are bounded $\mathcal{B}$-measurable functions and $N : \mathrm{p}\mathcal{B}((0,\infty)) \longrightarrow \mathrm{p}\mathcal{B}(E)$ is a kernel such that $N(u \wedge u^2) \in \mathrm{bp}\mathcal{B}$. Recall that examples of branching mechanisms are $\Phi(\lambda) = -\lambda^\alpha$ for $1 < \alpha \leq 2$ and $\Phi(\lambda) = \lambda^\alpha$ for $0 < \alpha < 1$.

We present now briefly (cf. [12] and [1]) the construction of the measure-valued branching Markov process associated with X and Φ, the (X,Φ)-superprocess, a Borel right process with state space $M(E)$, the space of all positive finite measures on $(E,\mathcal{B})$, endowed with the weak topology.

For each $f \in \mathrm{bp}\mathcal{B}$ the equation

$$v_t(x) = P_t f(x) + \int_0^t P_s(x, \Phi(\cdot, v_{t-s}))ds, \quad t \geq 0, \quad x \in E,$$

has a unique solution $(t,x) \longmapsto V_t f(x)$ jointly measurable in (t,x) such that $\sup_{0 \leq s \leq t} ||v_s||_\infty < \infty$ for all $t > 0$. The mappings $f \longmapsto V_t f$ form a nonlinear semigroup of operators on $\mathrm{bp}\mathcal{B}$. Notice that the above equation is formally equivalent with

$$\begin{cases} \frac{d}{dt} v_t(x) = \mathcal{L} v_t(x) + \Phi(x, v_t(x)) \\ v_0 = f, \end{cases}$$

where $\mathcal{L}$ is the infinitesimal generator of the spatial motion X.

For a function $f \in \mathrm{bp}\mathcal{B}$ we shall consider the mappings $l_f : M(E) \longrightarrow \mathbb{R}$ and $e_f : M(E) \longrightarrow [0,1]$ defined by

$$l_f(\mu) := \langle \mu, f \rangle := \int f d\mu, \ \mu \in M(E), \quad e_f := exp(-l_f).$$

$M(E)$ is endowed with the σ-algebra $\mathcal{M}(E)$ generated by $\{l_f | f \in \mathrm{bp}\mathcal{B}\}$.

For each $t \geq 0$ there exists a unique kernel Q_t on $(M(E), \mathcal{M}(E))$ such that

$$Q_t(e_f) = e_{V_t f}, \quad f \in \mathrm{bp}\mathcal{B}.$$

Since the family $(V_t)_{t \geq 0}$ is a (nonlinear) semigroup on $\mathrm{bp}\mathcal{B}$, $(Q_t)_{t \geq 0}$ is a (linear) semigroup of kernels on $(M(E), \mathcal{M}(E))$.

Let $\overline{\mathcal{U}} = (\overline{U}_\alpha)_{\alpha>0}$ be the Markovian resolvent of kernels on $(M(E), \mathcal{M}(E))$ generated by the semigroup $(Q_t)_{t\geq 0}$. It turns out that all the points of $M(E)$ are non-branch points with respect to $\overline{\mathcal{U}}$; cf. Proposition 4.5 from [1]. Let

$$\beta := ||b^-||_\infty \text{ and } b' := b + \beta' \text{ with } \beta' \geq \beta.$$

Then $b' \geq 0$ and the resolvent $\mathcal{U}_{b'}$ generated by $(P_t^{b'})_{t\geq 0}$ is sub-Markovian and bounded if $\beta' > \beta$. In this case $(P_t^{b'})_{t\geq 0}$ is the transition function of a right Markov process with state space E (having $\mathcal{L} - b'$ as infinitesimal generator). If $u \in \mathrm{bp}\mathcal{B}$ then by Corollary 4.3 in [1] we have

$$u \in \mathcal{E}(\mathcal{U}_{b'}) \Longleftrightarrow l_u \in \mathcal{E}(\overline{\mathcal{U}}_{\beta'}). \tag{3.3.1}$$

A consequence of the equivalence (3.3.1) is the next result on the existence of the compact Lyapunov functions for the (X, Φ)-superprocess.

Proposition 3.3.1. *Assume that X is a Hunt process (i.e., it is quasi-left-continuous on $[0, \infty)$). Then for every $\lambda \in M(E)$ there exists a compact Lyapunov function F with respect to the (X, Φ)-superprocess, such that $F(\lambda) < \infty$.*

Sketch of the proof; see Step III of the proof of Theorem 4.9 in [1] for details. Consider a Ray topology which is finer than the original topology. It follows that X has càdlàg trajectories in the Ray topology and by Proposition 2.1.1 we deduce that there exists a function $v \in \mathcal{E}(\mathcal{U}_{b'}) \cap L^1(E, \lambda)$ having Ray-compact level sets. Taking $F = l_v$ we get by (3.3.1) that $F \in \mathcal{E}(\overline{\mathcal{U}}_{\beta'})$, clearly we have $F(\lambda) < \infty$, and one can show that it has compact level sets in the weak topology.

Remark. The existence of the compact Lyapunov functions given by Proposition 3.3.1 is the main step in [1] for the proof of the càdlàg property of the paths of the measure-valued (X, Φ)-superprocess, as Proposition 2.2.3 indicates.

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Acknowledgment

Financial support by the Deutsche Forschungsgemeinschaft, through CRC 701 is gratefully acknowledged. The first named author gratefully acknowledges support from the Romanian Ministry of Education, Research, and Innovation (PN II Program, CNCSIS code 1186/2008).

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