Harnack Inequalities and Applications for Ornstein-Uhlenbeck Semigroups with Jump*

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Abstract

The Harnack inequality established in [11] for generalized Mehler semigroup is improved and generalized. As applications, the log-Harnack inequality, the strong Feller property, the hyper-bounded property, and some heat kernel inequalities are presented for a class of O-U type semigroups with jump. These inequalities and semigroup properties are indeed equivalent, and thus sharp, for the Gaussian case. As an application of the log-Harnack inequality, the HWI inequality is established for the Gaussian case. Perturbations with linear growth are also investigated.

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1 Introduction

In this paper we aim to establish Harnack inequalities and applications for a class of Ornstein-Uhlenbeck type SDEs driven by Lévy noises on Hilbert spaces. This problem has been investigated in [11] by using Mehler type formula for the associated semigroups and gradient estimates for dimension-free Harnack inequalities developed in [12]. In this paper, we shall adopt a measure transformation argument to derive a more general and sharper Harnack inequality, and to present finer estimates of heat kernels. This method

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was initiated in [1] by using coupling and Girsanov transformation to establish Harnack inequalities for diffusion semigroups on manifolds with unbounded below curvature, and has been applied in [13, 8, 3] for non-linear SPDEs driven by Gaussian noises and also in [10] for diffusions with singular drifts and multivalued stochastic evolution equations. In this paper we shall modify this argument to SPDEs with jumps.

Let us first recall the Harnack inequality derived in [11]. Let $\mathbb{H}$ be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Consider the following Lévy driven stochastic differential equation

$$\mathrm{d}X_t = AX_t\mathrm{d}t + \mathrm{d}Z_t, \quad X_0 = x \in \mathbb{H}, \tag{1.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$ on $\mathbb{H}$, $Z_t := \{Z_t^u,\ u \in \mathbb{H}\}$ is a cylindrical Lévy process with characteristic triplet (a, R, M) on some filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, that is, for every $u \in \mathbb{H}$ and $t \geq 0$

$$\begin{aligned}\mathbb{E}\exp(\mathrm{i}\langle Z_t, u\rangle) = \exp\Big[&\mathrm{i}t\langle a, u\rangle - \frac{t}{2}\langle Ru, u\rangle \\ &- \int_{\mathbb{H}} \left[1 - \exp(\mathrm{i}\langle x, u\rangle) + \mathrm{i}\langle x, u\rangle 1_{\{|x|\leq 1\}}(x)\right]\, M(\mathrm{d}x)\Big],\end{aligned}$$

where $a \in \mathbb{H}$, R is a symmetric linear operator on $\mathbb{H}$ such that

$$R_t := \int_0^t T_s R T_s^*\,\mathrm{d}s$$

is trace class for each $t > 0$, and M is a Lévy measure on $\mathbb{H}$. (For simplicity, we shall write $Z_t^u = \langle Z_t, u\rangle$ for every $u \in \mathbb{H}$.) In this case, (1.1) has a unique mild solution

$$X_t = T_t x + \int_0^t T_{t-s}\mathrm{d}Z_s, \quad t \geq 0.$$

Let

$$P_t f(x) = \mathbb{E}f(X_t), \quad x \in \mathbb{H},\ f \in \mathscr{B}_b(\mathbb{H}),$$

where $\mathscr{B}_b(\mathbb{H})$ is the space of all bounded measurable functions on $\mathbb{H}$. Similarly, let $\mathscr{B}_b^+(\mathbb{H})$, $\mathscr{C}_b(\mathbb{H})$, $\mathscr{C}^\infty(\mathbb{H})$ be the classes of bounded positive, bounded continuous, and smooth functions on $\mathbb{H}$ respectively. Let G be the orthogonal complement of $\mathrm{Ker}R^{1/2}$. Then the inverse $R^{-1/2}$ of $R^{1/2}$ is well defined from $R^{1/2}\mathbb{H}$ to G. The following is the main result derived in [11].

Theorem 1.1. ([11]) *Assume that there exists a sequence of eigenvectors of A^* separating the points of $\mathbb{H}$, R is of trace class, and $T_t R\mathbb{H} \subset R^{1/2}\mathbb{H}$ holds for all $t > 0$. If*

$$\|R^{-1/2}T_t Rx\| \leq \sqrt{h(t)}\,\|R^{1/2}x\|, \quad x \in \mathbb{H}, t \geq 0 \tag{1.2}$$

holds for some positive function $h \in C[0,\infty)$. Then for any $f \in \mathscr{B}_b^+(\mathbb{H})$,

$$(P_t f)^2(x) \leq \exp\left[\frac{\|R^{-1/2}(x-y)\|^2}{\int_0^t h(s)^{-1}\mathrm{d}s}\right] P_t f^2(y), \quad t > 0, x - y \in R^{1/2}\mathbb{H}. \tag{1.3}$$

If $M=0$, then for any $\alpha>1$ and $f\in\mathscr{B}_b^+(\mathbb{H})$,

$$(1.4)\qquad (P_tf)^\alpha(x)\le \exp\left[\frac{\alpha\|R^{-1/2}(x-y)\|^2}{2(\alpha-1)\int_0^t h(s)^{-1}\mathrm{d}s}\right]P_tf^\alpha(y),\quad t>0, x-y\in R^{1/2}\mathbb{H}.$$

We note that due to the absence of a chain rule, for the case with jump (i.e. $M\neq 0$), the Harnack inequality was proved in [11] only for $\alpha=2$ (i.e. (1.3)) by using gradient estimates.

To improve this result, we shall adopt a measure transformation argument and the null controllability of the associated deterministic equation (see Section 2). As a result, we obtain the Harnack inequality by using the image norm $\|\Gamma_t x\|$ of the operator

$$\Gamma_t := R_t^{-1/2}T_t \text{ with domain } \mathscr{D}(\Gamma_t) := \big\{x\in\mathbb{H}:\ T_tx\in R_t^{1/2}\mathbb{H}\big\}.$$

As explained above, $R_t^{-1/2}$ is defined from $R_t^{1/2}\mathbb{H}$ to the orthogonal complement of $\operatorname{Ker} R_t^{1/2}$. By letting $\|\Gamma_t x\|=\infty$ for $x\notin\mathscr{D}(\Gamma_t)$ and $\inf\emptyset=\infty$, we have

$$\|\Gamma_t x\| = \inf\big\{\|z\|:\ z\in\mathbb{H}, R_t^{1/2}z=T_tx\big\},\quad x\in\mathbb{H}.$$

Our first result is an improvement of Theorem 1.1: our Harnack inequality generalizes (1.3) without the assumptions of Theorem 1.1. Moreover, our argument also implies the following inequality (1.6), which in particular implies the strong Feller property (even $\|\Gamma_t\cdot\|$-Lipschitz strong Feller property) of P_t if Γ_t is bounded.

Theorem 1.2. *For any $\alpha>1$ and $f\in\mathscr{B}_b^+(\mathbb{H})$,*

$$(1.5)\qquad (P_tf(x))^\alpha\le\exp\Big[\frac{\alpha\|\Gamma_t(x-y)\|^2}{2(\alpha-1)}\Big]P_tf^\alpha(y),\quad x,y\in\mathbb{H}, t>0.$$

Consequently, (1.2) *implies* (1.4) *for any $f\in\mathscr{B}_b^+(\mathbb{H})$. Moreover, for any $f\in\mathscr{B}_b(\mathbb{H})$ and $x,y\in\mathbb{H}$,*

$$(1.6)\qquad \begin{aligned}&|P_tf(x)-P_tf(y)|^2\\ \le&\big(\mathrm{e}^{\|\Gamma_t(x-y)\|^2}-1\big)\min\big\{P_tf^2(x)-(P_tf(x))^2, P_tf^2(y)-(P_tf(y))^2\big\}.\end{aligned}$$

When $T_t\mathbb{H}\subset R_t^{1/2}\mathbb{H}$, Γ_t is a bounded operator by the closed graph theorem. In this case Theorem 1.2 implies the following result.

Theorem 1.3. *Let $t>0$. The following statements are gradually weaker, i.e. statement (i) implies statement $(i+1)$ for $1\le i\le 4$:*

(1) $T_t\mathbb{H}\subset R_t^{1/2}\mathbb{H}$;

(2) $\|\Gamma_t\|<\infty$ *and for any $\alpha>1$ and $f\in\mathscr{B}_b^+(\mathbb{H})$,*

$$(1.7)\qquad (P_tf(x))^\alpha\le\exp\Big[\frac{\alpha(\|\Gamma_t\|\cdot\|x-y\|)^2}{2(\alpha-1)}\Big]P_tf^\alpha(y),\quad x,y\in\mathbb{H};$$

(3) $\|\Gamma_t\| < \infty$ *and there exists* $\alpha > 1$ *such that* (1.7) *holds for all* $f \in \mathscr{B}_b^+(\mathbb{H})$;

(4) $\|\Gamma_t\| < \infty$ *and for any* $f \in \mathscr{B}_b^+(\mathbb{H})$ *with* $f \geq 1$,

$$P_t \log f(x) \leq \log P_t f(y) + \frac{\|\Gamma_t\|^2}{2}\|x-y\|^2, \quad x, y \in \mathbb{H}; \tag{1.8}$$

(5) P_t *is strong Feller.*

If, in particular, $M = 0$, *then all the above statements are equivalent.*

According to [6, Theorem 3.1], (P_t) has a unique invariant probability measure μ provided

(A) $\lim\limits_{t\to\infty} \|T_t x\| = 0$ *for* $x \in \mathbb{H}$; $\sup_{t>0} \operatorname{Tr} R_t < \infty$; $\int_0^\infty \mathrm{d}s \int_{\mathbb{H}} (1 \wedge \|T_s x\|^2) M(\mathrm{d}x) < \infty$;

$$\lim_{t\to\infty} \Big\{ \int_0^t T_s a \,\mathrm{d}s + \int_0^t \mathrm{d}s \int_{\mathbb{H}} T_s x \Big(\frac{1}{1 + \|T_s x\|^2} - \frac{1}{1 + \|x\|^2} \Big) M(\mathrm{d}x) \Big\}$$

exists in $\mathbb{H}$; *and* R *is of trace class.*

In this case, if P_t is strong Feller then it has a density $p_t(x, y)$ w.r.t. μ on $\operatorname{supp}\mu$, the support of μ. As observed in the recent paper [14], the Harnack inequality (1.7) and the log-Harnack inequality (1.8) are equivalent to the following inequalities for $p_t(x, y)$ respectively:

$$\int_{\mathbb{H}} p_t(x,z) \Big(\frac{p_t(x,z)}{p_t(y,z)} \Big)^{\frac{1}{\alpha-1}} \mu(\mathrm{d}z) \leq \exp\Big[\frac{\alpha(\|\Gamma_T\| \cdot \|x-y\|)^2}{2(\alpha-1)^2} \Big], \quad x, y \in \operatorname{supp}\mu; \tag{1.9}$$

$$\int_{\mathbb{H}} p_t(x,y) \log \frac{p_t(x,z)}{p_t(y,z)} \mu(\mathrm{d}z) \leq \frac{\|\Gamma_t\|^2}{2} \|x-y\|^2, \quad x, y \in \operatorname{supp}\mu. \tag{1.10}$$

Moreover, if $M = 0$, by e.g. [5, Theorem 10.3.5], Theorem 1.3 (1) implies that $P_t L^p(\mathbb{H}; \mu) \subset \mathscr{C}^\infty(\mathbb{H})$ for $p > 1$ and $t > 0$. So, we have the following consequence of Theorem 1.3.

Corollary 1.4. *Let* $M = 0$ *and assume that* (P_t) *has a invariant probability measure* μ *with full support. Then for any* $t > 0$, (1)–(5) *in Theorem* 1.3 *and the following statements are equivalent:*

(6) *For any* $\alpha > 1$, (1.9) *holds;*

(7) *For some* $\alpha > 1$, (1.9) *holds;*

(8) *The entropy inequality* (1.10) *holds;*

(9) *For any* $p > 1$, $P_t L^p(\mathbb{H}; \mu) \subset \mathscr{C}^\infty(\mathbb{H})$.

The following result is a standard consequence of the Harnack inequality (1.7), where (i) follows from [3, Proposition 4.1], (ii) follows from Lemma [11, Lemma 2.2], and the proof of (iii) is similar to the those of [11, Theorem 1.5 and Proposition 1.6] (see also [10] for details).

Corollary 1.5. *Assume that* (P_t) *has a invariant measure and that* Γ_t *is bounded for a fixed* $t > 0$*, and let* $p_t(x, y)$ *be the density of* P_t *w.r.t.* μ*. Then:*

(i) $P_t L^p(\mathbb{H}; \mu) \subset \mathscr{C}(\mathbb{H})$ *for any* $p > 1$*.*

(ii) For any $\alpha > 1$*,*

$$\|p_t(x, \cdot)\|_{L^{\alpha/(\alpha-1)}(\mathbb{H};\mu)} \le \left[\int_{\mathbb{H}} \exp\left(-\frac{\alpha\|\Gamma_t\|^2}{2(\alpha-1)}\|x-y\|^2\right) \mu(dy)\right]^{-1/\alpha}, \quad x \in \mathrm{supp}\mu.$$

(iii) If there exist some constants $\varepsilon > 0$ *and* $\alpha > 1$ *such that*

$$C(t, \alpha, \varepsilon) := \int_{\mathbb{H}} \left[\int_{\mathbb{H}} \exp\left(-\frac{\alpha\|\Gamma_t\|^2}{2(\alpha-1)}\|x-y\|^2\right) \mu(dy)\right]^{-(1+\varepsilon)} \mu(dx) < \infty,$$

then P_t *is hyper-bounded with*

$$\|P_t\|_{\alpha \to \alpha(1+\varepsilon)} \le C(t, \alpha, \varepsilon)^{\frac{1}{\alpha(1+\varepsilon)}}.$$

If $C(t, \alpha, 0) < \infty$ *then* P_t *is uniformly integrable in* $L^\alpha(\mathbb{H}; \mu)$ *and hence* P_s *is compact on* $L^\alpha(\mathbb{H}, \mu)$ *for every* $s > t$*.*

We shall prove Theorems 1.2 and 1.3 in the next section, and present in Section 3 applications of the log-Harnack inequality to cost-entropy inequalities of the semigroup and the HWI inequality in the Gaussian case. Finally, in Section 4 we investigate the Harnack inequality and strong Feller property for a class of semi-linear stochastic equations by using a perturbation argument.

2 Proofs of Theorems 1.2 and 1.3

As explained in the last section, Corollary 1.4 is a direct consequence of Theorem 1.3. Since (2) implying (3) is trivial, (3) implying (4) and (4) implying (5) have been proved in [14] for Markov semigroups on abstract Polish spaces, and (5) implying (1) follows from [4, Theorem 9.19], it suffices to prove Theorem 1.2.

Consider the following linear control system on $\mathbb{H}$

$$\mathrm{d}x_t = Ax_t\, dt + R^{1/2}u_t\, dt, \quad x_0 = x \in \mathbb{H}. \tag{2.1}$$

According [15, Part IV, Theorem 2.3] (ref. also the appendix of [4] or [5]),

$$\|\Gamma_t x\|^2 = \inf\left\{\int_0^t \|u_s\|^2 \mathrm{d}s : \ u \in L^2([0,t] \to \mathbb{H}; \mathrm{d}s), x_0 = x, x_t = 0\right\}. \tag{2.2}$$

This implies the following upper bounds of $\|\Gamma_t x\|$.

Proposition 2.1. *Let $t > 0$. Then for any strictly positive $\xi \in C([0,t])$,*

$$\|\Gamma_t x\|^2 \le \frac{\int_0^t \|R^{-1/2}T_s x\|^2 \, \xi_s^2 \mathrm{d}s}{\left(\int_0^t \xi_s \mathrm{d}s\right)^2}, \quad x \in \mathbb{H}, \tag{2.3}$$

where $\|R^{-1/2}x\| = \infty$ if $x \notin R^{1/2}\mathbb{H}$. Consequently, (1.2) *implies*

$$\|\Gamma_t x\|^2 \le \frac{\|R^{-1/2}x\|^2}{\int_0^t h(s)^{-1}\, ds}, \quad x \in \mathbb{H}. \tag{2.4}$$

Proof. We only need to consider the case that $T_s x \in R^{1/2}\mathbb{H}$ for a.e. $s \in [0,t]$ and $\{\xi_s R^{-1/2} T_s x\}_{s\in[0,t]} \in L^2([0,t] \to \mathbb{H}; \mathrm{d}s)$. In this case, for

$$u_s := -\frac{\xi_s}{\int_0^t \xi_r \, \mathrm{d}r} R^{-1/2} T_s x, \quad s \in [0,t],$$

one has a null control of the system (2.1); that is, $u \in L^2([0,t] \to \mathbb{H}; \mathrm{d}s)$ and

$$x_t := T_t x + \int_0^t T_{t-s} R^{1/2} u_s \mathrm{d}s = 0.$$

Then (2.3) follows from (2.2) by taking $\xi(s) = h(s)^{-1}$, $s \in [0,t]$. □

To prove the desired Harnack inequality, we adopt the following Girsanov theorem for Lévy processes. Let $\|\cdot\|_0$ be the norm on $\mathbb{H}_0 := R^{1/2}(\mathbb{H})$ with inner product $\langle x, y\rangle_0 := \langle R^{-1/2}x, R^{-1/2}y\rangle$ for $x, y \in \mathbb{H}_0$.

Proposition 2.2. *Let $t > 0$. Suppose that $(Z_s)_{0\le s\le t}$ is an $\mathbb{H}$-valued Lévy process on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_s)_{0\le s\le t}, \mathbb{P})$ with characteristic triplet (a, R, M). Denote by Z' the Gaussian part of Z. For any $\mathbb{H}_0$-valued predictable process ψ_s, independent of $Z_s - Z'_s$ such that*

$$s \mapsto \rho_s := \exp\left(\int_0^s \langle \psi_r, \mathrm{d}Z'_r\rangle_0 - \frac{1}{2}\int_0^s \|\psi_r\|_0^2 \, \mathrm{d}r\right)$$

is a $\mathscr{F}_s$-martingale, the process

$$[0,t] \ni s \mapsto \tilde{Z}_s := Z_s - \int_0^s \psi_r \, \mathrm{d}r$$

is also a Lévy process with characteristic triplet (a, R, M) under the probability measure $\mathrm{d}\tilde{\mathbb{P}} := \rho_t \mathrm{d}\mathbb{P}$.

Proof. We write

$$\mathbb{E}\exp(\mathrm{i}\langle Z_s, z\rangle) = \exp\left[-s\vartheta_1(z) - z\vartheta_2(z)\right], \quad z \in \mathbb{H},$$

where

$$\vartheta_1(z) := \frac{1}{2}\langle Rz, z\rangle$$

and

$$\vartheta_2(z) := -\mathrm{i}\langle z, a\rangle + \int_{\mathbb{H}} \left[1 - \exp(\mathrm{i}\langle z, x\rangle) + \mathrm{i}\langle z, x\rangle 1_{\{|x|\le 1\}}(x)\right] \, M(\mathrm{d}x).$$

Correspondingly, the process Z_s is decomposed by $Z_s = Z'_s + Z''_s$, where Z'_s is the Gaussian part of Z_s with symbol ϑ_1, and Z''_s is the jump process with symbol ϑ_2.

By the Girsanov theorem for Wiener processes on Hilbert space (see [4, Theorem 10.14]),

$$\widetilde{Z}'_s = Z'_s - \int_0^s \psi_r \, \mathrm{d}r, \quad 0 \le s \le t$$

is an R-Wiener process under the probability measure $\tilde{\mathbb{P}}$. Consequently, for all $0 \le s \le t$ and all $z \in \mathbb{H}$, by the martingale property of ρ_s we have

$$\mathbb{E}_{\tilde{\mathbb{P}}}\big[\exp(\mathrm{i}\langle z, \tilde{Z}'_s\rangle\big] = \mathbb{E}\big[\rho_s \exp(\mathrm{i}\langle z, \tilde{Z}'_s\rangle)\big] = \mathbb{E}\exp(\mathrm{i}\langle z, Z'_s\rangle) = \exp\left[-s\vartheta_1(z)\right],$$

where $\mathbb{E}_{\tilde{\mathbb{P}}}$ is the expectation taken for $\tilde{\mathbb{P}}$. Combining this with the independence of Z' and Z'', we obtain

$$\mathbb{E}_{\tilde{\mathbb{P}}} \exp\left(\mathrm{i}\langle z, \widetilde{Z}_s\rangle\right) = \left(\mathbb{E}\rho_s \exp\left[\mathrm{i}\left\langle z, \widetilde{Z}'_s\right\rangle\right]\right)\mathbb{E}\exp\left(\mathrm{i}\langle z, Z''_s\rangle\right) = \exp\left[-s\vartheta_1(z) - t\vartheta_2(z)\right].$$

Thus, under $\tilde{\mathbb{P}}$ the characteristic symbol of $\widetilde{Z}_s$ is also $\vartheta_1 + \vartheta_2$. This completes the proof. □

By Proposition 2.2, we are able to establish the Harnack inequality by using the null controllability of the deterministic equation (2.1).

Proposition 2.3. *Let $t > 0$ and $x, y \in \mathbb{H}$. Suppose that there exists $u \in L^2([0,t] \to \mathbb{H}; \mathrm{d}s)$ such that $x_t = 0$, where x_s solves (2.1) with $x_0 = y - x$. Then for any $\alpha > 1$,*

$$(P_t f)^\alpha(x) \le \exp\left(\frac{\alpha}{2(\alpha-1)}\int_0^t \|u_s\|^2 \, \mathrm{d}s\right) P_t f^\alpha(y), \quad f \in \mathscr{B}_b^+(\mathbb{H}). \tag{2.5}$$

Moreover, for any $f \in \mathscr{B}_b(\mathbb{H})$ and $x, y \in \mathbb{H}$,

$$|P_t f(x) - P_t f(y)|^2 \le \left(\mathrm{e}^{\int_0^t \|u_s\|^2 \mathrm{d}s} - 1\right)\left\{P_t f^2(y) - (P_t f(y))^2\right\}. \tag{2.6}$$

Proof. Let $(Z'_s)_{0\le s\le t}$ be the Gaussian part of the Lévy process Z_s, which is an R-Wiener process on $\mathbb{H}$. Let $\psi_s = R^{1/2}u_s \in \mathbb{H}_0$ for $s \in [0,t]$. Then by Proposition 2.2,

$$\widetilde{Z}_s := Z_s - \int_0^s \psi_r \, \mathrm{d}r, \quad 0 \le s \le t$$

is a Lévy process with characteristic triplet (a, R, M) under the probability measure $\tilde{\mathbb{P}}$.

Let

$$Y_t^y = S_t y + \int_0^t S_{t-s}\,\mathrm{d}Z_s,$$
$$X_t^x = S_t x + \int_0^t S_{t-s}\,\mathrm{d}\widetilde{Z}_s.$$

Then, by the definition of P_t and since Z_s and $\widetilde{Z_s}$ are cylindrical Lévy processes with characteristic triplet (a, R, M) under $\mathbb{P}$ and $\tilde{\mathbb{P}}$ respectively, we have

$$P_t f(y) = \mathbb{E} f(Y_t^y), \quad P_t f(x) = \mathbb{E}_{\tilde{\mathbb{P}}} f(X_t^x) = \mathbb{E}\big[\rho_t f(X_t^x)\big], \quad f \in \mathscr{B}_b(\mathbb{H}). \tag{2.7}$$

Moreover, it is easy to see that

$$X_s^x = Y_s^y - x_s, \quad s \in [0, t].$$

So, $X_t^x = Y_t^y$ due to $x_t = 0$. Combining this with (2.7), for any $f \in \mathscr{B}_b^+(\mathbb{H})$ we have

$$\begin{aligned}\mathbb{P}_t f(x) &= \mathbb{E}\big[\rho_t f(X_t^x)\big] = \mathbb{E}\big[\rho_t f(Y_t^y)\big] \\ &\le \big(\mathbb{E}\rho_t^{\alpha/(\alpha-1)}\big)^{(\alpha-1)/\alpha}\big(\mathbb{E} f^\alpha(Y_t^y)\big)^{1/\alpha} = \big(\mathbb{E}\rho_t^{\alpha/(\alpha-1)}\big)^{\alpha-1)/\alpha}\big(P_t f^\alpha(y)\big)^{1/\alpha}.\end{aligned}$$

This implies (2.5) by noting that

$$\mathbb{E}\rho_t^{\alpha/(\alpha-1)} = \exp\Big[\frac{\alpha}{2(\alpha-1)^2}\int_0^t \|\psi_s\|_0^2\,\mathrm{d}s\Big].$$

Similarly, since $\mathbb{E}\rho_t = 1$, we have

$$\begin{aligned}|P_t f(x) - P_t f(y)|^2 &= |\mathbb{E} f(\rho_t X_t^x) - \mathbb{E} f(Y_t^y)|^2 \\ &= \big|\mathbb{E}\big\{(\rho_t - 1)\big(f(Y_t^y) - P_t f(y)\big)\big\}\big|^2 \le \big\{P_t f^2(y) - (P_t f(y))^2\big\}\mathbb{E}|\rho_t - 1|^2.\end{aligned}$$

This implies (2.6) by noting that

$$\mathbb{E}(\rho_t - 1)^2 = \mathbb{E}\rho_t^2 - 1 = \mathrm{e}^{\int_0^t \|u_s\|^2 \mathrm{d}s} - 1.$$

□

Proof of Theorem 1.2. Combining (2.2) with (2.5), we obtain (1.5). If (1.2) holds, then (1.4) follows from (2.5) according to Proposition 2.1. Finally, (1.6) follows from (2.2) and (2.6), where the latter holds also by exchanging the positions of x and y. □

3 Application to the HWI inequality

The HWI inequality, established in [9] and reproved in [2] for symmetric diffusions on finite dimensional Riemannian manifolds, links the entropy, information and the transportation-cost. In this section, we shall prove it for the present non-symmetric infinite-dimensional model.

Throughout this section we assume that

(A′) P_t has an invariant probability measure μ.

This assumption follows from assumption **(A)** as explained in Section 1. We first observe that the log-Harnack inequality (1.8) implies an entropy-cost inequality for P_t^*, the joint operator of P_t on $L^2(\mathbb{H};\mu)$.

Proposition 3.1. *Assume (A'). Let P_t^* be the adjoint operator of P_t on $L^2(\mathbb{H};\mu)$. If $\|\Gamma_t\| < \infty$, then*

$$\mu((P_t^* f)\log P_t^* f) \le \frac{\|\Gamma_t\|^2}{2} W_2(f\mu,\mu)^2, \quad f \ge 0, \mu(f) = 1$$

holds, where W_2^2 is the Warsserstein distance induced by the cost-function $(x,y) \mapsto \|x - y\|^2$; that is,

$$W_2(f\mu,\mu)^2 = \inf_{\pi\in\mathscr{C}(f\mu,\mu)} \int_{\mathbb{H}\times\mathbb{H}} \|x-y\|^2 \pi(\mathrm{d}x,\mathrm{d}y)$$

for $\mathscr{C}(f\mu,\mu)$ the set of all couplings of $f\mu$ and μ. Consequently, (1.2) *implies*

$$\mu((P_t^* f)\log P_t^* f) \le \frac{W_2(f\mu,\mu)^2}{2\int_0^t h(s)^{-1}\mathrm{d}s}, \quad t > 0.$$

Proof. Due to Proposition 2.1, it suffices to prove the first assertion. We shall adopt an argument in [2] by using the log-Harnack inequality (1.8). Let $f \ge 0$ such that $\mu(f) = 1$. By an approximation argument, we may assume that f is bounded. So, by Theorem 1.3 we have

$$P_t(\log P_t^* f)(x) \le \log(P_t P_t^* f)(y) + \frac{\|\Gamma_t\|^2}{2}\|x-y\|^2, \quad x, y \in \mathbb{H}.$$

Integrating both sides w.r.t. $\pi \in \mathscr{C}(f\mu,\mu)$, and minimizing in π, we arrive at

$$\mu((P_t^* f)\log P_t^* f) \le \mu(\log(P_t P_t^* f)) + \frac{\|\Gamma_t\|^2}{2} W_2(f\mu,\mu)^2.$$

This completes the proof by noting that, since μ is invariant for P_t and P_t^*,

$$\mu(\log(P_t P_t^* f)) \le \log \mu(P_t P_t^* f) = \log 1 = 0.$$

□

According to the above result, to derive the entropy-cost inequality for P_t,we shall need the log-Harnack inequality for the adjoint semigroup P_t^*. To ensure that P_t^* is again an O-U type semigroup, we shall simply consider the Gaussian case (i.e. $M = 0$), and assume **(A)**. In this case, μ is a Gaussian measure with co-variance

$$R_\infty := \int_0^\infty T_s R T_s^* \mathrm{d}s.$$

To see that P_t^* as a generalized Mehler semigroup (in the sense of [6]), we assume further

(B) *$M = 0, R_\infty \mathbb{H} \subset \mathscr{D}(A^*)$, and the operator $\tilde{A} = R_\infty A^* R_\infty^{-1}$ with domain*

$$\mathscr{D}(\tilde{A}) := \left\{x \in R_\infty \mathbb{H} :\ R_\infty^{-1} x \in \mathscr{D}(A^*)\right\}$$

generates a C_0-semigroup $\tilde{T}_t$ on $\mathbb{H}$ such that

$$\tilde{R}_t = \int_0^t \tilde{T}_s R \tilde{T}_s^* \mathrm{d}s$$

is of trace class for $t > 0$.

In this case, P_t^* can be calculated explicitly as (see [5, Proposition 10.1.9])

$$P_t^* f(x) = \int_{\mathbb{H}} f(\tilde{T}_t x + y) N_{\tilde{R}_t}(dy), \quad f \in \mathscr{B}_b(\mathbb{H}),$$

where $N_{\tilde{R}_t}$ is the centered Gaussian measure with co-variance $\tilde{R}_t$. Thus, P_t^* is the transition semigroup of the solution to

$$\mathrm{d}\tilde{X}_t = \tilde{A}\tilde{X}_t \mathrm{d}t + R^{1/2}\mathrm{d}W_t$$

for W_t the cylindrical Brownian motion on $\mathbb{H}$. So, the following is a direct consequence of Theorems 1.2 and 1.3 and Proposition 3.1.

Proposition 3.2. *Assume* **(A), (B)**. *Let $\tilde{\Gamma}_t = \tilde{R}_t^{-1/2}\tilde{T}_t$. Then*

$$(P_t^* f)^\alpha(x) \le P_t^* f^\alpha(y) \exp\left(\frac{\alpha \|\tilde{\Gamma}_t(x-y)\|^2}{2(\alpha-1)}\right), \quad f \in \mathscr{B}_b^+(\mathbb{H}), x, y \in \mathbb{H}.$$

If $\tilde{\Gamma}_t$ is bounded, then (1.8) *holds for $P_t^*, \tilde{\Gamma}_t$ in place of P_t and Γ_t respectively. In particular,*

$$\mu((P_t f)\log P_t f) \le \frac{\|\tilde{\Gamma}_t\|^2}{2} W_2(\mu, f\mu)^2, \quad f \ge 0, \mu(f) = 1. \tag{3.1}$$

Let W_0 be the space of functions f of the form

$$f(x) = F(\langle \xi_1, x\rangle, \cdots, \langle \xi_m, x\rangle), \quad x \in \mathbb{H}$$

for some $m \ge 1$ and $F \in \mathscr{S}(\mathbb{R}^m, \mathbb{C})$ (i.e. the Schwartz space of complex-valued functions, "rapidly decreasing" at infinity as well as their derivatives). Let W be the real-valued elements of W_0. According to [7], W is dense in $L^p(\mu)$ for any $p \ge 1$ and is a core of $D(L)$, the $L^2(\mu)$-domain of the generator L of P_t. We are now able to present the following result on the HWI inequality.

Theorem 3.3. *Assume* **(A)** *and* **(B)**. *Assume further that* A^* *has a sequence of eigenvectors separating the points in* $\mathbb{H}$. *If* (1.2) *holds then*

$$\mu(f^2\log f^2) \le 2\mu(\langle \mathbb{R}Df, Df\rangle)\int_0^t h(s)\mathrm{d}s + \frac{\|\tilde\Gamma_t\|^2}{2}W_2(\mu, f^2\mu)^2, \quad t>0, f\in W, \mu(f^2)=1,$$

where Df *is the Fréchet derivative of* f.

Proof. Let $f\in W$ such that $\mu(f^2)=1$. By [11, Theorem 1.3(2)], we have

$$P_t(f^2\log f^2) \le (P_tf^2)\log P_tf^2 + 2\bigg(\int_0^t h(s)\mathrm{d}s\bigg)P_t\langle RDf, Df\rangle.$$

Integrating w.r.t. μ we obtain

$$\mu(f^2\log f^2) \le 2\mu(\langle \mathbb{R}Df, Df\rangle)\int_0^t h(s)\mathrm{d}s + \mu((P_tf^2)\log P_tf^2).$$

The proof is then completed by combining this with Proposition 3.2. □

If in particular P_t is symmetric (it is the case iff $AR^{1/2}=R^{1/2}A^*$), then (1.2) implies

$$\mu(f^2\log f^2) \le 2\mu(\langle \mathbb{R}Df, Df\rangle)\int_0^t h(s)\mathrm{d}s + \frac{1}{2\int_0^t h(s)^{-1}\mathrm{d}s}W_2(\mu, f^2\mu)^2$$

for all $f\in W, \mu(f^2)=1, t>0$.

4 Semi-linear stochastic equations

Consider the equation

$$\mathrm{d}X_t^x = AX_t^x\mathrm{d}t + F(X_t^x)\mathrm{d}t + R^{1/2}\mathrm{d}W_t, \quad X_0^x = x\in\mathbb{H}, \tag{4.1}$$

where F is a measurable map on $\mathbb{H}$ such that $F(\mathbb{H})\subset R^{1/2}\mathbb{H}$, and W_t is the cylindrical Brownian motion on $\mathbb{H}$. We shall establish the Harnack inequality for the associated semigroup by regarding (4.1) as a perturbation to (1.1) with $Z_t = R^{1/2}W_t$, i.e. $b=0, M=0$. Since we do not assume that F is dissipative, the study is not included in [3]. In general, this equation only admits a weak solution. In this paper we shall consider the weak solution for (4.1) constructed from (1.1) with $Z_t=R^{1/2}W_t$ by Girsanov transformations.

Let $\tilde X_t^x$ be the mild solution to

$$\mathrm{d}\tilde X_t^x = A\tilde X_t^x\mathrm{d}t + R^{1/2}\mathrm{d}W_t, \quad X_0^x = x.$$

We have $\tilde X_t^x = W_A(t)+T_tx$, where

$$W_A(t) = \int_0^t T_{t-s}R^{1/2}\mathrm{d}W_s.$$

Since $\int_0^t T_s R T_s^* \mathrm{d}s$ is of trace class, $W_A \in L^2([0,t];\mathbb{H})$ for any $t>0$. Let

$$\begin{aligned}
&\psi_x(t) = R^{-1/2}F(W_A(t)+T_t x),\\
&\tilde{W}_t^x = W_t - \int_0^t \psi_x(s)\mathrm{d}s,\\
&\rho_t^x = \exp\left(\int_0^t \langle \psi_x(s), \mathrm{d}W_s\rangle - \frac{1}{2}\int_0^t \|\psi_x(s)\|^2 \mathrm{d}s\right).
\end{aligned}$$

Assume that

$$\|R^{-1/2}F(x)\|^2 \le k_1 + k_2\|x\|^2, \quad x \in \mathbb{H} \tag{4.2}$$

holds for some $k_1, k_2 \ge 0$. Then by [4, Theorem 10.20] and its proof, $\mathbb{Q}_x := \rho_t^x \mathbb{P}$ is a probability measure and $\tilde{X}_t^x$ is a weak solution to (4.1) under $\mathbb{Q}_x$ with respect to the cylindrical Brownian motion $\tilde{W}_t^x$. Denote the corresponding "semigroup" by

$$P_t^F f(x) = \mathbb{E}_{\mathbb{Q}_x} f(\tilde{X}_t^x), \quad f \in \mathscr{B}_b(\mathbb{H}). \tag{4.3}$$

We note that due to the lack of uniqueness, in general P_t^F may not provide a semigroup (but cf. also [7]). Let P_t be the semigroup of $\tilde{X}_t^\cdot$ under $\mathbb{P}$. By Theorem 1.2 we have

$$(P_t f)^\alpha(x) \le P_t f^\alpha(y) \exp\left(\frac{\alpha\|\Gamma_t(x-y)\|^2}{2(\alpha-1)}\right), \quad f \in \mathscr{B}_b^+(\mathbb{H}), \tag{4.4}$$

where $\Gamma_t := R_t^{-1/2} T_t$. Moreover, by [4, (10.42)], for any $p>0$ there exists $t_p>0$ such that

$$C_{p,k_2}(t) := \mathbb{E}\exp\left(2p(2p+1)k_2 \int_0^t \|W_A(s)\|^2 \mathrm{d}s\right) < \infty, \quad t \in [0, t_p].$$

In particular, if $k_2 = 0$ then $C_{p,k_2}(t) = 1,\ t \ge 0$. More precisely, let

$$\theta = \mathrm{Tr}\int_0^1 T_s R T_s^* \mathrm{d}s.$$

We have

$$C_0 := \sup_{s\in[0,1]} \mathbb{E}\mathrm{e}^{\|W_A(s)\|^2/4\theta} < \infty.$$

Thus, for any $\lambda > 0$,

$$\begin{aligned}
\mathbb{E}\mathrm{e}^{\lambda\int_0^t \|W_A(s)\|^2 \mathrm{d}s} &= \mathbb{E}\mathrm{e}^{\frac{1}{t}\int_0^t \lambda t\|W_A(s)\|\mathrm{d}s} \le \frac{1}{t}\int_0^t \mathbb{E}\mathrm{e}^{\lambda t\|W_A(s)\|^2}\mathrm{d}s\\
&\le \frac{1}{t}\int_0^t \left(\mathbb{E}\mathrm{e}^{\|W_A(s)\|^2/4\theta}\right)^{4\theta\lambda t}\mathrm{d}s \le C_0^{4\theta\lambda t}, \quad t \in [0, 1\wedge(4\theta\lambda)^{-1}].
\end{aligned} \tag{4.5}$$

Combining this with (4.3) we obtain the following result.

Theorem 4.1. *If* (4.2) *holds, then for any* $t > 0$, $\alpha > 1$, $x, y \in \mathbb{H}$, $p, q > 1$ *with* $\alpha/(pq) > 1$, *and* $f \in \mathscr{B}_b^+(\mathbb{H})$

$$(P_t^F f)^\alpha(x) \le \Big(C_{\frac{p}{p-1},k_2}(t)\Big)^{\alpha p/(2(p-1))} \Big(C_{\frac{1}{q-1},k_2}(t)\Big)^{\alpha q/(2(q-1))} P_t^F f^\alpha(y) \exp\Bigg(\frac{\alpha q\|\Gamma_t(x-y)\|^2}{2(\alpha-q)}$$
$$+\alpha\left[\frac{p+1}{p-1} + \frac{q+1}{q(q-1)}\right]\int_0^t \big[k_1 + k_2(\|T_s x\|^2 + \|T_s y\|^2)\big]\,\mathrm{d}s\Bigg).$$

Consequently, if $\|\Gamma_t\| < \infty$ *for* $t > 0$, *then* P_t^F *is strong Feller provided it is a semigroup.*

Proof. For simplicity, we denote $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$, $\theta = \alpha/(pq)$.

$$\begin{aligned} P_t^F f(x) &= \mathbb{E}_{\mathbb{Q}_x} f(\tilde X_t^x) = \mathbb{E}\rho_t^x f(\tilde X_t^x) \le (\mathbb{E} f^p(\tilde X_t^x))^{1/p}(\mathbb{E}(\rho_t^x)^{p'})^{1/p'} \\ &= (P_t f^p(x))^{1/p}(\mathbb{E}(\rho_t^x)^{p'})^{1/p'} \\ &\le \left[P_t f^{\theta p}(y)\exp\left(\frac{\theta\|\Gamma_t(x-y)\|^2}{2(\theta-1)}\right)\right]^{1/(\theta p)} (\mathbb{E}(\rho_t^x)^{p'})^{1/p'}. \end{aligned}$$

On the other hand, for any $g \in \mathscr{C}_b^+(\mathbb{H})$,

$$P_t g(y) \le \mathbb{E}_{\mathbb{P}} g(\tilde X_t^y) = \mathbb{E}_{\mathbb{Q}_y} g(\tilde X_t^y)(\rho_t^y)^{-1} \le (P_t^F g^q(y))^{1/q}(\mathbb{E}(\rho_t^y)^{1-q'})^{1/q'}.$$

So by taking $g = f^{\theta p}$

$$(P_t^F f)^\alpha(x) \le P_t^F f^\alpha(y)\exp\left(\frac{\alpha\|\Gamma_t(x-y)\|^2}{2p(\theta-1)}\right)(\mathbb{E}(\rho_t^x)^{p'})^{\alpha/p'}(\mathbb{E}(\rho_t^y)^{1-q'})^{\alpha/q'}.$$

This implies the desired Harnack inequality according to the following Lemma 4.2.

Now, assume that Γ_t is bounded for $t > 0$ and assume that P_t^F is a semigroup. Let $f \in \mathscr{B}_b^+(\mathbb{H})$. By the first assertion and (4.5), for any $\alpha > 1$ there exist constants $t_\alpha, c_\alpha > 0$ and a positive function H_α on $(0, t_\alpha)$ such that

$$P_t^F f(x) \le (P_t^F f^\alpha(y))^{1/\alpha} \mathrm{e}^{c_\alpha t + \|x-y\|^2 H_\alpha(t)}, \quad t \in (0, t_\alpha]. \tag{4.6}$$

Then, for any $t > 0$,

$$\begin{aligned} \limsup_{x\to y} P_t^F f(x) &\le \limsup_{\alpha\to1}\limsup_{s\to0}\limsup_{x\to y}\big\{P_s^F(P_{t-s}^F f)^\alpha(y)\big\}^{1/\alpha}\mathrm{e}^{c_\alpha s + \|x-y\|^2 H_\alpha(s)} \\ &\le \limsup_{\alpha\to1}\limsup_{s\to0}\limsup_{x\to y}\big\{P_t^F f^\alpha(y)\big\}^{1/\alpha}\mathrm{e}^{c_\alpha s + \|x-y\|^2 H_\alpha(s)} = P_t^F f(y). \end{aligned}$$

On the other hand, (4.6) also implies

$$\begin{aligned} P_t^F f(x) &\ge \big\{P_s^F(P_{t-s}^F f)^{1/\alpha}(y)\big\}^\alpha \mathrm{e}^{-\alpha c_\alpha s - \alpha H_\alpha(s)\|x-y\|^2} \\ &\ge \big\{P_t^F f^{1/\alpha}(y)\big\}^\alpha \mathrm{e}^{-\alpha c_\alpha s - \alpha H_\alpha(s)\|x-y\|^2}, \quad s \in (0, t_\alpha). \end{aligned}$$

So, by first letting $x \to y$ then $s \to 0$ and finally $\alpha \to 1$ we arrive at

$$\liminf_{x\to y} P_t^F f(x) \ge P_t^F f(y).$$

Therefore, $P_t^F f$ is continuous on $\mathbb{H}$. $\square$

Lemma 4.2. *Assume* (4.2). *For any* $p > 1$, $\delta > 0$ *and* $x \in \mathbb{H}$, *then*

$$\mathbb{E}(\rho_t^x)^p \leq (C_{p,k_2}(t))^{1/2} \exp\Big(\frac{p(2p-1)}{2}\int_0^t (k_1 + 2k_2\|T_s x\|^2)\mathrm{d}s\Big)$$
$$\mathbb{E}(\rho_t^x)^{-\delta} \leq (C_{\delta,k_2}(t))^{1/2} \exp\Big(\frac{\delta(2\delta+1)}{2}\int_0^t (k_1 + 2k_2\|x\|^2)\mathrm{d}s\Big).$$

Proof. According to the proof of [4, Theorem 10.20], for any $\lambda \in \mathbb{R}$, the process

$$t \mapsto \exp\left[\lambda\int_0^t \langle\psi_x(s), dW_s\rangle - \frac{\lambda^2}{2}\int_0^t \|\psi_x(s)\|^2\mathrm{d}s\right]$$

is a martingale. So,

$$\begin{aligned}
&\mathbb{E}(\rho_t^x)^p \\
=&\mathbb{E}\exp\left(p\int_0^t \langle\psi_x(s), dW_s\rangle - p^2\int_0^t \|\psi_x(s)\|^2\mathrm{d}s\right)\exp\left(\frac{p(2p-1)}{2}\int_0^t \|\psi_x(s)\|^2\mathrm{d}s\right)\\
\leq&\left[\mathbb{E}\exp\left(2p\int_0^t \langle\psi_x(s), dW_s\rangle - 2p^2\int_0^t \|\psi_x(s)\|^2 ds\right)\right]^{1/2}\\
&\qquad\qquad\cdot\left[\mathbb{E}\exp\left(p(2p-1)\int_0^t \|\psi_x(s)\|^2 ds\right)\right]^{1/2}\\
=&\left[\mathbb{E}\exp\left(p(2p-1)\int_0^t \|\psi_x(s)\|^2 ds\right)\right]^{1/2}.
\end{aligned}$$

This implies the first inequality since (4.2) and the boundedness of T_s imply

$$\|\psi_x(s)\|^2 \leq k_1 + 2k_2\|W_A(s)\|^2 + 2k_2\|x\|^2.$$

Similarly, the second inequality follows by noting that

$$\begin{aligned}
&\mathbb{E}(\rho_t^x)^{-\delta} \\
=&\mathbb{E}\exp\left(-\delta\int_0^t \langle\psi_x(s), dW_s\rangle - \delta^2\int_0^t \|\psi_x(s)\|^2\mathrm{d}s\right)\exp\left(\frac{\delta(2\delta+1)}{2}\int_0^t \|\psi_x(s)\|^2\mathrm{d}s\right)\\
\leq&\left[\mathbb{E}\exp\left(-2\delta\int_0^t \langle\psi_x(s), dW_s\rangle - 2\delta^2\int_0^t \|\psi_x(s)\|^2 ds\right)\right]^{1/2}\\
&\qquad\qquad\cdot\left[\mathbb{E}\exp\left(\delta(2\delta+1)\int_0^t \|\psi_x(s)\|^2 ds\right)\right]^{1/2}\\
=&\left[\mathbb{E}\exp\left(\delta(2\delta+1)\int_0^t \|\psi_x(s)\|^2 ds\right)\right]^{1/2}.
\end{aligned}$$

□

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