Probabilistic representation for solutions of an irregular porous media type equation.

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Summary: We consider a porous media type equation over all of $\mathbb{R}^d$ with $d = 1$, with monotone discontinuous coefficients with linear growth and prove a probabilistic representation of its solution in terms of an associated microscopic diffusion. This equation is motivated by some singular behaviour arising in complex self-organized critical systems. One of the main analytic ingredients of the proof, is a new result on uniqueness of distributional solutions of a linear PDE on $\mathbb{R}^1$ with non-continuous coefficients.

Key words: singular porous media type equation, probabilistic representation, self-organized criticality (SOC).

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1 Introduction

We are interested in the probabilistic representation of the solution to a porous media type equation given by

$$\begin{cases} \partial_t u &= \frac{1}{2}\partial_{xx}^2(\beta(u)),\ t \in [0,\infty[\\ u(0,x) &= u_0(x),\ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

in the sense of distributions, where u_0 is an initial probability density. We look for a solution of (1.2) with time evolution in $L^1(\mathbb{R})$.

We will always suppose that $\beta : \mathbb{R} \to \mathbb{R}$ is monotone. We may always assume that $\beta(0) = 0$; otherwise we add a constant which will not change (1.1). We also suppose

$$|\beta(u)| \leq \text{const}|u|. \tag{1.2}$$

In particular, β is continuous at zero. Moreover we will assume the existence of λ positive such that $(\beta + \lambda id)(\mathbb{R}) = \mathbb{R}$, $id(x) \equiv x$. Since β is monotone, (1.2) implies $\beta(u) = \Phi^2(u)u$, Φ being a non-negative bounded Borel function. We recall that when $\beta(u) = |u|u^{m-1}$, $m > 1$, (1.1) is nothing else but the classical *porous media equation.*

One of our targets was to consider Φ as continuous except for a possible jump at one positive point, say $e_c > 0$. A typical example is

$$\Phi(u) = H(u - e_c), \tag{1.3}$$

H being the Heaviside function.

The analysis of (1.1) and its probabilistic representation can be done in the framework of monotone partial differential equations (PDE) allowing multi-valued coefficients and will be discussed in detail in the main body of the paper. In this introduction, for simplicity, we restrict our presentation to the single-valued case.

Definition 1.1 *We will say that equation (1.1) or Φ is* **non-degenerate** *if there is a constant $c > 0$ such that $\Phi \geq c$.*

Of course, Φ in (1.3) is not non-degenerate. In order to have Φ non-degenerate, one needs to add a positive constant to it.

Several contributions were made in this framework starting from [9] for existence, [15] for uniqueness in the case of bounded solutions and [10] for continuous dependence on the coefficients. The authors consider the case where β is continuous, even if their arguments allow some extensions for the discontinuous case.

As mentioned in the abstract, the first motivation of this paper was to discuss a continuous time model of self-organized criticality (SOC), which are described by equations of type (1.3).

SOC is a property of dynamical systems which have a critical point as an attractor, see [2] for a significant monography on the subject. SOC is typically observed in slowly-driven out-of-equilibrium systems with threshold dynamics relaxing through a hierarchy of avalanches of all sizes. We, in particular, refer to the interesting physical papers [3] and [16]. The second makes reference to a system whose evolution is similar to the evolution of a "snow layer" under the influence of an "avalanch effect" which starts when the top of the layer attains a critical value e_c. Adding a stochastic noise should describe other contingent effects. For instance, an additive perturbation by noise could describe the regular effect of "snow falling". Taking inspiration from the discrete models, a possible representation of that phenomenon is equation (1.1) perturbed by a space time noise $\xi(t, x)$.

Since the "avalanche effect" happens on a much shorter time scale than the "snow falling effect" it seems to be reasonnable to analyze the phenomenon in two separate phases, i.e. avalanche and snow falling. This paper concentrates on the avalanche phase and therefore it investigates the (unperturbed) equation discussing existence, uniqueness and a probabilistic representation.

The singular non-linear diffusion equation (1.1) models the *macroscopic* phenomenon for which we try to give a *microscopic* probabilistic representation, via a non-linear stochastic differential equation (NLSDE) modeling the evolution of a single point on the layer.

Even if the irregular diffusion equation (1.1) can be shown to be well-posed, up to now we can only prove existence (but not yet uniqueness) of solutions to the corresponding NLSDE. On the other hand if $\Phi \geq c > 0$, then uniqueness can be proved. For our applications, this will solve the case $\Phi(u) = H(x -$

$e_c) + \varepsilon$ for some positive ε. The main novelty with respect to the literature is the fact that Φ can be irregular with jumps.

To the best of our knowledge the first author who considered a probabilistic representation (of the type studied in this paper) for the solutions of a non-linear deterministic PDE was McKean [23], particularly in relation with the so called propagation of chaos. In his case, however, the coefficients were smooth. From then on the literature steadily grew and nowadays there is a vast amount of contributions to the subject, especially when the non-linearity is in the first order part, as e.g. in Burgers equation. We refer the reader to the excellent survey papers [30] and [20].

A probabilistic interpretation of (1.1) when $\beta(u) = |u|u^{m-1}, m > 1$, was provided for instance in [8]. For the same β, though the method could be adapted to the case where β is Lipschitz, in [21] the author has studied the evolution equation (1.1) when the initial condition and the evolution takes values in the class of probability distribution functions on $\mathbb{R}$. Therefore, instead of an evolution equation in $L^1(\mathbb{R})$, he considers a state space of functions vanishing at $-\infty$ and with value 1 at $+\infty$. He studies both the probabilistic representation and propagation of chaos.

Let us now describe the principle of the mentioned probabilistic representation. The stochastic differential equation (in the weak sense) rendering the probabilistic representation is given by the following (random) non-linear diffusion:

$$\begin{cases} Y_t &= Y_0 + \int_0^t \Phi(u(s, Y_s))dW_s \\ \text{Law density}(Y_t) &= u(t, \cdot), \end{cases} \tag{1.4}$$

where W is a classical Brownian motion. The solution of that equation may be visualised as a continuous process Y on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ equipped with a Brownian motion W. By looking at a properly chosen version, we can and shall assume that $u : [0, T] \times \Omega \to \mathbb{R}_+$ is $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable. Of course, we can only have (weak) uniqueness for (1.4) fixing the initial distribution, i.e. we have to fix the distribution (density) u_0 of Y_0.

The connection with (1.1) is then given by the following result.

Theorem 1.2 *(i) Let us assume the existence of a solution Y for (1.4). Then $u : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ provides a solution in the sense of distributions of (1.1) with $u_0 := u(0,\cdot)$.*

(ii) Let u be a solution of (1.1) in the sense of distributions and let Y solve the first equation in (1.4) with law density $v(t,\cdot)$ and initial law density $u_0 = u(0,\cdot)$. Then

$$\partial_t v = \frac{1}{2}\partial_{xx}^2(\Phi^2(u)v), \tag{1.5}$$

in the sense of distributions. In particular, if v is the unique solution of (1.5), with $v(0,\cdot) = u_0$, then $v = u$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R})$, Y be a solution to the first line of (1.5) such that $v(t,\cdot)$ is the law density Y_t, for positive t. We apply Itô's formula to $\varphi(Y)$, to obtain

$$\varphi(Y_t) = \varphi(Y_0) + \int_0^t \varphi'(Y_s)\Phi(u(s,Y_s))\, dW_s + \frac{1}{2}\int_0^t \varphi''(Y_s)\Phi^2(u(s,Y_s))\, ds$$

Taking expectation we obtain

$$\int_{\mathbb{R}} \varphi(y)v(t,y)dy = \int_{\mathbb{R}} \varphi(y)u_0(y)dy + \frac{1}{2}\int_0^t ds \int_{\mathbb{R}} \varphi''(y)\Phi^2(u(s,y))v(s,y)\, dy.$$

At this point both conclusions (i) and (ii) follow. ■

Remark 1.3 *An immediate consequence of the associated solution of (1.1) is its positivity at any time if it starts with an initial value u_0 which is positive. Also the mass 1 of the initial condition is in this case conserved.*

The main purpose of the paper is to show existence and uniqueness in law of the probabilistic representation equation (1.4), in the case that Φ is non-degenerate and not necessarily continuous. In addition, we prove existence for (1.4), in some particular degenerate cases.

Let us now briefly explain the points that we are able to treat and the difficulties which naturally appear in the probabilistic representation.

For simplicity we do this for β being single-valued (and) continuous. However, with some technical complications this generalizes to the multi-valued case, as spelt out in the subsequent sections.

1. Monotonicity methods allow us to show existence and uniqueness of solutions to (1.1) in the sense of distributions under the assumption that β is monotone, that there exists $\lambda > 0$ with $(\beta + \lambda id)(\mathbb{R}) = \mathbb{R}$ and that β is continuous at zero, see Proposition 3.2 below. We emphasize that for uniqueness no surjectivity for $\beta + \lambda id$ is required, see Remark 3.4 below.

2. Let $a : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a strictly positive bounded Borel function. Let $\mathcal{M}(\mathbb{R})$ be the set of all signed measure on $\mathbb{R}$ with finite total variation. We prove uniqueness of solutions of

$$\begin{cases} \partial_t v & = & \partial_{xx}^2 (av) \\ v(0,x) & = & u_0(x), \end{cases} \tag{1.6}$$

as an evolution problem in $\mathcal{M}(\mathbb{R})$, at least under an additional assumption (A), see Theorem 3.6 below.

3. If Φ is *non-degenerate*, we can construct a unique (weak) solution Y to the non-linear SDE corresponding to (1.4), for any intial bounded probability density u_0 on $\mathbb{R}$, see Theorem 4.3 below. For this construction, items 1. and 2. above are used in a crucial way.

4. Suppose Φ is possibly degenerate. We fix a bounded probability density u_0. We set $\Phi_\varepsilon = \Phi + \varepsilon$ and consider the weak solution Y^ε of

$$Y_t^\varepsilon = \int_0^t \Phi_\varepsilon(u^\varepsilon(s, Y_s^\varepsilon)) dW_s, \tag{1.7}$$

where $u^\varepsilon(t,\cdot)$ is the law of $Y_t^\varepsilon, t \geq 0$ and Y_0^ε is distributed according to $u_0(x)dx$.

The sequence of laws of the processes (Y^ε) are tight. However, the limiting processes of convergent subsequences may in general not solve the SDE

$$Y_t = \int_0^t \Phi(u(s, Y_s)) dW_s. \tag{1.8}$$

However, under some additional assumptions, see Properties 4.7 and 4.8 below, it will be the case. The analysis of the degenerate case in greater generality (including case (1.3)) will be the subject of the forthcoming paper [6].

In this paper, we proceed as follows. Section 2 is devoted to preliminaries about elliptic PDEs satisfying monotonicity conditions. In Section 3, we first state a general existence and uniqueness result (Proposition 3.2) for equation (1.1) and provide its proof, see item 1. above.

The rest of the section is devoted to the study of the uniqueness of a deterministic, time inhomogeneous singular linear equation with evolution in the space of probabilities on $\mathbb{R}$. This will be applied for studying the uniqueness of equation (1.6) in item 2. above. This is only possible in the non-degenerate case, see Theorem 3.6; in the degenerate case we give a counterexample in Remark 3.9.

Section 4 is devoted to the probabilistic representation (1.4). In particular, in the non-degenerate (however not smooth) case, Theorem 4.3 gives existence and uniqueness of the non-linear diffusion (1.4) which represents probabilistically (1.1). In the degenerate case, Proposition 4.10 gives an existence result.

2 Preliminaries

We start with some basic analytical framework.

If $f : \mathbb{R} \to \mathbb{R}$ is a bounded function we will denote $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$. By $C_b(\mathbb{R})$ we denote the space of bounded continuous real functions and by $S(\mathbb{R})$ the space of rapidly decreasing infinitely differentiable functions $\varphi : \mathbb{R} \to \mathbb{R}$, by $S'(\mathbb{R})$ its dual (the space of tempered distributions).

Let K_ε be the Green function of $\varepsilon - \Delta$, that is the kernel of the operator $(\varepsilon - \Delta)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. So, for all $\varphi \in L^2(\mathbb{R})$, we have

$$B_\varepsilon(\varphi) := (\varepsilon - \Delta)^{-1}\varphi(x) = \int_{\mathbb{R}} K_\varepsilon (x - y) \varphi(y)dy. \tag{2.9}$$

The next lemma provides us with an explicit expression of the kernel function K_ε.

Lemma 2.1

$$K_\varepsilon (x) = \frac{1}{2\sqrt{\varepsilon}} e^{-\sqrt{\varepsilon}|x|}, \tag{2.10}$$

Proof. From Def. 6.27 in [28], we get

$$K_\varepsilon(x) = \frac{1}{(4\pi)^{1/2}} \int_0^\infty t^{-\frac{1}{2}} e^{-\frac{|x|^2}{4t} - \varepsilon t} dt \tag{2.11}$$

If $x = 0$, the calculation to show $K_\varepsilon(0) = \frac{1}{2\sqrt{\varepsilon}}$ can be easily performed setting $u = \sqrt{t}$. So let us suppose $x \neq 0$.

Using (2.11) we get

$$\begin{aligned} K_\varepsilon(x) &= \frac{1}{\sqrt{4\pi}} \sqrt{\frac{2\sqrt{\varepsilon}}{|x|}} \int_0^\infty \left(\frac{2\sqrt{\varepsilon}}{|x|} t\right)^{-\frac{1}{2}} e^{-\frac{\sqrt{\varepsilon}|x|}{2}} \left(\frac{1}{\frac{2\sqrt{\varepsilon}}{|x|} t} + \frac{2\sqrt{\varepsilon} t}{|x|} \right) dt \\ &= \frac{1}{\sqrt{4\pi}} \sqrt{\frac{|x|}{2\sqrt{\varepsilon}}} \int_0^\infty u^{-\frac{1}{2}} e^{-\frac{\sqrt{\varepsilon}|x|}{2}(\frac{1}{u}+u)} du, \end{aligned}$$

where for the last equality we have set $u = \frac{2\sqrt{\varepsilon}}{|x|} t$. Consequently, setting $t = \sqrt{u}$ we get

$$\begin{aligned} K_\varepsilon(x) &= 2\left(\frac{|x|}{8\pi\sqrt{\varepsilon}}\right)^{\frac{1}{2}} \int_0^\infty e^{-\frac{\sqrt{\varepsilon}|x|}{2}(\frac{1}{t^2}+t^2)} dt \\ &= 2\left(\frac{|x|}{8\pi\sqrt{\varepsilon}}\right)^{\frac{1}{2}} e^{-\sqrt{\varepsilon}|x|} \int_0^\infty e^{-\frac{\sqrt{\varepsilon}|x|}{2}(\frac{1}{t}-t)^2} dt = \left(\frac{|x|}{8\pi\sqrt{\varepsilon}}\right)^{\frac{1}{2}} e^{-\sqrt{\varepsilon}|x|} \\ &\quad \left(\int_0^\infty t^{-2} e^{-\frac{\sqrt{\varepsilon}|x|}{2}(\frac{1}{t}-t)^2} dt + \int_0^\infty e^{-\frac{\sqrt{\varepsilon}|x|}{2}(\frac{1}{t}-t)^2} dt \right) \end{aligned}$$

where, in the last equality, we have set $t \to \frac{1}{t}$ in one integral. Changing variables $t \to t - \frac{1}{t}$ we obtain

$$\begin{aligned} K_\varepsilon(x) &= \left(\frac{|x|}{8\pi\sqrt{\varepsilon}}\right)^{\frac{1}{2}} e^{-\sqrt{\varepsilon}|x|} \int_{-\infty}^\infty e^{-\frac{\sqrt{\varepsilon}|x|}{2} t^2} dt \\ &= \left(\frac{|x|}{8\pi\sqrt{\varepsilon}}\right)^{\frac{1}{2}} e^{-\sqrt{\varepsilon}|x|} \sqrt{\frac{2\pi}{\sqrt{\varepsilon}|x|}} = \frac{1}{2\sqrt{\varepsilon}} e^{-\sqrt{\varepsilon}|x|}. \end{aligned}$$

This concludes the proof of the lemma. ■

Clearly, if $\varphi \in C^2(\mathbb{R}) \bigcap \mathcal{S}'(\mathbb{R})$, then $(\varepsilon - \Delta)\varphi$ coincides with the classical associated PDE operator.

Remark 2.2 *An obvious consequence of Lemma (2.1) is that K_ε is a bounded continuous function.*

We need to write carefully some results about elliptic linear problems.

Lemma 2.3 *Let $\varepsilon > 0, m \in \mathcal{M}(\mathbb{R})$. There is a unique solution $v_\varepsilon \in C_b(\mathbb{R}) \bigcap L^p(\mathbb{R})$ for every $p \geq 1$ of*

$$\varepsilon v_\varepsilon - \Delta v_\varepsilon = m \tag{2.12}$$

in the sense of distributions. Moreover it fulfills

$$\sup_x \sqrt{\varepsilon} |v_\varepsilon(x)| \leq \frac{\|m\|_{\mathrm{var}}}{2}, \tag{2.13}$$

where $\|m\|_{\mathrm{var}}$ denotes the total variation norm. In addition, the derivative v'_ε has a bounded cadlag version which is locally of bounded variation.

In the sequel, as announced, that solution will be denoted by $B_\varepsilon m$.

Proof. Uniqueness follows from an obvious application of Fourier transform. In fact, uniqueness holds in $\mathcal{S}'(\mathbb{R})$.

Existence is easy and well-known. One simply sets

$$v_\varepsilon(x) = \int_{\mathbb{R}} K_\varepsilon(x - y) dm(y), \quad x \in \mathbb{R}. \tag{2.14}$$

Then v_ε clearly satisfies (2.12) in the sense of distributions. By Lebesgue's dominated convergence theorem and Remark 2.2 1., it follows that $v_\varepsilon \in C_b(\mathbb{R})$.

By Lemma 2.1 we have

$$\sup_x |v_\varepsilon(x)| \leq \frac{1}{2\sqrt{\varepsilon}} \|m\|_{\mathrm{var}}, \tag{2.15}$$

By Fubini's theorem and (2.10), it follows that $v_\varepsilon \in L^1(\mathbb{R})$. Hence $v_\varepsilon \in L^p(\mathbb{R}), \forall p \geq 1$, because v_ε is bounded.

Since v''_ε equals $\varepsilon v_\varepsilon - m$ in the sense of distributions, after integration, we can show that

$$v'_\varepsilon(x) = \mathrm{const} + \varepsilon \int_{-\infty}^x v_\varepsilon(y) dy + m(] - \infty, x]),$$

for dx-a.e. $x \in \mathbb{R}$. In particular, v_ε has a bounded cadlag version which is locally of bounded variation and

$$\|v'_\varepsilon\|_\infty \leq \text{const} + \varepsilon\|v\|_{L^1(\mathbb{R})} + \|m\|_{\text{var}}.$$

$\blacksquare$

We now recall some basic notations about analysis related to monotonicity methods. More information can also be found for instance in [27]. See also [5, 14].

Let E be a Banach space with dual space E^*.

One of the most basic notions of this paper is the one of a multivalued function (graph). A **multivalued function** (graph) β on E will be a subset of $E \times E$. It can be seen, either as a family of couples $(e, f), e, f \in E$ and we will write $f \in \beta(e)$ or as a function $\beta : E \to \mathcal{P}(E)$.

We start with a definition concerning $E = \mathbb{R}$.

Definition 2.4 *A multivalued function β defined on $\mathbb{R}$ with values in subsets of $\mathbb{R}$ is said to be* **monotone** *if given $x_1, x_2 \in \mathbb{R}$, $(x_1-x_2)(\beta(x_1)-\beta(x_2)) \geq 0$.*

We say that β is **maximal monotone** *if is monotone and it there exists $\lambda > 0$ such that $\beta + \lambda id$ is surjective, i.e.*

$$\mathcal{R}(\beta + \lambda id) := \bigcup_{x\in\mathbb{R}} (\beta(x) + \lambda x) = \mathbb{R}.$$

We recall that one motivation of this paper is the case where $\beta(u) = H(u - e_c)u$.

Let us consider a monotone function ψ. Then all the discontinuities are of jump type. At every discontinuity point x of ψ, it is possible to *complete* ψ by setting $\psi(x) = [\psi(x-), \psi(x+)]$.

Since ψ is a monotone function, the corresponding multivalued function will be, of course, also monotone.

Now we come back to the case of our general Banach space E with norm $\|\cdot\|$. An operator $T : E \to E$ is said to be a **contraction** if it is Lipschitz of norm less or equal to 1 and $T(0) = 0$.

Definition 2.5 *A map $A : E \to E$, or more generally a multivalued map $A : E \to \mathcal{P}(E)$ is said to be* **monotone** *or* **accretive** *if for any $f_1, f_2, g_1, g_2 \in E$ such that $g_i \in Af_i, i = 1, 2$, we have*

$$\|f_1 - f_2\| \leq \|f_1 - f_2 + \lambda(g_1 - g_2)\|,$$

for any $\lambda > 0$.

This is equivalent to saying the following: for any $\lambda > 0$, $(1 + \lambda A)^{-1}$ is a contraction for any λ greater than 0 on $Rg(I + \lambda A)$. We remark that a contraction is necessarily single-valued.

Remark 2.6 *Suppose that E is a Hilbert space equipped with the scalar product $(\, , \,)_H$. Then A is monotone if and only if $(f_1 - f_2, g_1 - g_2)_H \geq 0$ for any $f_1, f_2, g_1, g_2 \in E$ such that $g_i \in Af_i, i = 1, 2$, see Corollary 1.3 of [27].*

Definition 2.7 *A monotone map $A : E \to E$ (possiblly multivalued) is said to be* **maximal monotone** *or* **m-accretive** *if for some $\lambda > 0$, $A + \lambda Id$ is surjective (as a graph in $E \times E$).*

Remark 2.8 *A monotone map $A : E \to E$ is maximal monotone if and only if $A + \lambda Id$ is surjective for any $\lambda > 0$.*

So, A is maximal monotone, if and only if for all λ strictly positive, $(I + \lambda A)^{-1}$ is a contraction on E.

Now, let us consider the case $E = L^1(\mathbb{R})$, so $E^* = L^\infty(\mathbb{R})$. The following is taken from [10], section 1.

Remark 2.9 *Let $\beta : \mathbb{R} \to \mathbb{R}$ be a monotone (possibly multi-valued) map such that the corresponding graph is maximal monotone. Suppose that $\beta(0) = 0$.*

Let $f \in E = L^1(\mathbb{R})$.

1. *There is a unique $u \in L^1(\mathbb{R})$ for which there is $w \in L^1_{\text{loc}}(\mathbb{R})$ such that*

$$u - \Delta w = f \quad \text{in} \quad \mathcal{D}'(\mathbb{R}), \quad w(x) \in \beta(u(x)), \quad \text{for a.e.} \quad x \in \mathbb{R}, \tag{2.16}$$

   see Proposition 2 of [10].

2. *It is then possible to define a (multivalued) operator* $A := A_\beta : E \to E$ *where* $D(A)$ *is the set of* $u \in L^1(\mathbb{R})$ *for which there is* $w \in L^1_{\mathrm{loc}}(\mathbb{R})$ *such that* $w(x) \in \beta(u(x))$ *for a.e.* $x \in \mathbb{R}$ *and* $\Delta w \in L^1(\mathbb{R})$. *For* $u \in D(A)$, *we set*
$$Au = \{-w | w \text{ as in definition of } D(A)\}.$$
This is a consequence of the remarks following Theorem 1 in [10].

   In particular, if β *is single-valued,* $Au = -\Delta\beta(u)$. *We will adopt this notation also if* β *is multi-valued.*

3. *The operator* A *defined in 2. above is maximal monotone on* $E = L^1(\mathbb{R})$, *see Proposition 2 of [10].*

4. *We set* $J_\lambda = (I + \lambda A)^{-1}$, *which is a single-valued operator. If* $f \in L^\infty(\mathbb{R})$, *then* $\||J_\lambda f\|_\infty \le \|f\|_\infty$, *see Proposition 2 (iii) of [10]. In particular, for every positive integer* n, $\|J_\lambda^n f\|_\infty \le \|f\|_\infty$.

Let us summarize some important results of the theory of non-linear semi-groups, see for instance [18, 4, 5, 9] or the more recent monograph [27], which we shall use below. Let $A : E \to E$ be a (possibly multivalued) monotone operator. We consider the equation

$$0 \in u'(t) + A(u(t)), \quad 0 \le t \le T. \tag{2.17}$$

A function $u : [0, T] \to E$ which is absolutely continuous such that for a.e. t, $u(t, \cdot) \in D(A)$ and fulfills (2.17) in the following sense is called **strong solution**.

There exists $\eta : [0, T] \to E$, Bochner integrable, such that $\eta(t) \in A(u(t))$ for a.e. $t \in [0, T]$ and

$$u(t) = u_0 + \int_0^t \eta(s)ds, \quad 0 < t \le T.$$

A weaker notion for (2.17) is the so-called C^0**- solution**, see chapter IV.8 of [27]. In order to introduce it, one first defines the notion of ε-solution related to (2.17).

An ε**-solution** is a discretization

$$\mathcal{D} = \{0 = t_0 < t_1 < \ldots < t_N = T\}$$

and an E-valued step function

$$u^\varepsilon(t) = \begin{cases} u_0 & : \quad t = t_0 \\ u_j \in D(A) & : \quad t \in]t_{j-1}, t_j] \end{cases}$$

for which $t_j - t_{j-1} \leq \varepsilon$ for $1 \leq j \leq N$, and

$$0 \in \frac{u_j - u_{j-1}}{t_j - t_{j-1}} + Au_j, 1 \leq j \leq N.$$

We remark that, since A is maximal monotone, u^ε is determined by $\mathcal{D}$ and u_0, see Remark 2.9 1.

Definition 2.10 *A* C^0*-* **solution** *of (2.17) is an* $u \in C([0,T];E)$ *such that for every* $\varepsilon > 0$*, there is an* ε*-solution* u^ε *of (2.17) with*

$$\|u(t) - u^\varepsilon(t)\| \leq \varepsilon, \quad 0 \leq t \leq T.$$

Proposition 2.11 *Let* A *be a maximal monotone (multivalued) operator on a Banach space* E*. We set again* $J_\lambda := (I + \lambda A)^{-1}, \lambda > 0$*. Suppose* $u_0 \in \overline{D(A)}$*. Then:*

1. *There is a unique* C^0*- solution* $u : [0,T] \to E$ *of (2.17)*
2. $u(t) = \lim_{n\to\infty} J^n_{\frac{t}{n}} u_0$ *uniformly in* $t \in [0,T]$.

Proof.

1) is stated in Corollary IV.8.4. of [27] and 2) is contained in Theorem IV 8.2 of [27]. ∎

The complications coming from the definition of C^0-solution arise because the dual E^* of $E = L^1(\mathbb{R})$ is not uniformly convex. Otherwise, the following properties would hold. According to Theorem IV 7.1 of [27], for a given $u_0 \in D(A)$, there would exist a (strong) solution $u : [0,T] \to E$ to (2.17). Moreover Theorem 1.2 of [17] says the following. Given $u_0 \in \overline{D(A)}$ and given a sequence (u_0^n) in $D(A)$ converging to u_0, then, the sequence of the corresponding strong solutions (u_n) would converge to the unique C^0-solution of the same equation.

3 A porous media equation with singular coefficients

In this section, we will provide first an existence and uniqueness result for solutions to the parabolic deterministic equation (1.1) in the sense of distributions for multi-valued maximal monotone β. The proof is partly based on the theory of non-linear semigroups, see [10] for the case when β is continuous.

However, the most important result of this section, is an existence and uniqueness result for a "non-degenerate" linear equation for measures, see (1.6). This technical result will be crucial for identifying the law of the process appearing in the probabilistic representation (1.4).

We suppose that β has the same properties as those given in the introduction. However, β is allowed to be multi-valued, hence maximal monotone, as a graph, in the sense of Definition 2.4. Furthermore, generalizing (1.2) we assume that for some constant $c > 0$

$$w \in \beta(u) \Rightarrow |w| \leq c|u|. \tag{3.1}$$

In particular, $\beta(0) = 0$ and β is continuous at zero.

We use again the representation $\beta(u) = \Phi^2(u)u$ with Φ being a non-negative bounded multi-valued map $\Phi : \mathbb{R} \to \mathbb{R}$.

Remark 3.1 *As mentioned before, if $\beta : \mathbb{R} \to \mathbb{R}$ is monotone (possibly discontinuous), it is possible to complete β into a monotone graph.*

For instance, if $\Phi(x) = H(x - e_c)$, then

$$\beta(x) = \begin{cases} 0 & : \ x < e_c \\ [0, e_c] & : \ x = e_c \\ x & : \ x > e_c \end{cases}$$

Since the function β is monotone, the corresponding graph is monotone. Moreover $\beta + id$ is surjective so that, by definition, β is maximal monotone.

Proposition 3.2 *Let $u_0 \in L^1(\mathbb{R}) \bigcap L^\infty(\mathbb{R})$ Then there is a unique solution in the sense of distributions $u \in (L^1 \bigcap L^\infty)([0,T] \times \mathbb{R})$ of*

$$\left\{ \begin{array}{rcl} \partial_t u & \in & \frac{1}{2}\partial_{xx}^2(\beta(u)), \\ u(t,x) & = & u_0(x), \end{array} \right. \tag{3.2}$$

that is, there exists a unique couple $(u, \eta_u) \in ((L^1 \bigcap L^\infty)([0,T] \times \mathbb{R}))^2$ such that

$$\begin{aligned} \int u(t,x)\varphi(x)dx &= \int u_0(x)\varphi(x)dx + \frac{1}{2}\int_0^t ds \int \eta_u(s,x)\varphi''(x)dx, \\ & \quad \forall \varphi \in \mathcal{S}(\mathbb{R}) \quad \text{and} \qquad (3.3) \\ \eta_u(t,x) &\in \beta(u(t,x)) \text{ for } dt \otimes dx - \text{a.e. } (t,x) \in [0,t] \times \mathbb{R}. \end{aligned}$$

Furthermore, $\|u(t,\cdot)\|_\infty \leq \|u_0\|_\infty$ for every $t \in [0,T]$ and there is a unique version of u such that $u \in C([0,T]; L^1(\mathbb{R}))$ $(\subset L^1([0,T] \times \mathbb{R}))$.

Remark 3.3 1. *We remark that, the uniqueness of u determines the uniqueness of $\eta \in \beta(u)$ a.e. In fact, for $s,t \in [0,T]$, we have*

$$\left(\frac{1}{2}\int_s^t \eta_u(r,\cdot)dr\right)'' = u(t,\cdot) - u(s,\cdot), \text{ a.e.} \tag{3.4}$$

Since $\eta_u \in L^1([0,T] \times \mathbb{R})$, this implies that the function η_u is $dt \otimes dx$-a.e. uniquely determined. Furthermore, since $\beta(0) = 0$ and because β is monotone, for $dt \otimes dx$ a.e. $(t,x) \in [0,T] \times \mathbb{R}$ we have

$$u(t,x) = 0 \Rightarrow \eta_u(t,x) = 0$$

and

$$u(t,x)\eta_u(t,x) \geq 0.$$

2. *If β is continuous then we can take $\eta_u(s,x) = \beta(u(s,x))$.*

3. *This result applies in the Heaviside case where $\Phi(x) = H(x - e_c)$ and in the non-degenerate case $\Phi(x) = H(x - e_c) + \varepsilon$. In Chapter 4, we will give a probabilistic representation in the sense of (1.4). In the non-degenerate case the probabilistic representation is even unique.*

Proof (of Proposition 3.2).

We first recall that by our assumptions, we have $(\beta + \lambda id)(\mathbb{R}) = \mathbb{R}$ for every $\lambda > 0$.

1. The first step is to prove the existence of a C^0-solution of the evolution problem (2.17) in $E = L^1(\mathbb{R})$, with A and $D(A)$ as defined in Remark 2.9 2. Suppose $\overline{D(A)} = L^1(\mathbb{R})$. Then, the existence of a C^0-solution $u \in C([0,T]; L^1(\mathbb{R}))$ is a consequence of Remark 2.9 3. and Proposition 2.11 1. In particular, u belongs to $L^1([0,T] \times \mathbb{R})$.

2. We prove now thet $D(A)$ is dense in in $E = L^1(\mathbb{R})$.

   Let $u \in E$. We have to show the existence of a sequence (u_n) in $D(A)$ converging to u in E. We set $u_\lambda = (1 + \lambda A)^{-1} u$, so that $u \in u_\lambda - \lambda \Delta \beta(u_\lambda)$. The result follows if we are able to show that

   $$\lim_{\lambda \to 0} u_\lambda = u, \quad \text{weakly in} \quad E, \tag{3.5}$$

   because then $D(A)$ is weakly dense in $L^1(\mathbb{R})$, hence by the Hahn-Banach theorem strongly dense in $L^1(\mathbb{R})$.

   Being $(1 + \lambda A)^{-1}$ a contraction on E, then $u_\lambda \in E$ and the sequence (u_λ) is bounded in $L^1(\mathbb{R})$. Since $u_\lambda \in D(A)$, by definition, there exists $w_\lambda \in L^1_{\text{loc}}(\mathbb{R})$ such that $w_\lambda(x) \in \beta(u_\lambda(x))$ for dx-a.e. $x \in \mathbb{R}$, $\Delta w_\lambda \in L^1(\mathbb{R})$ and $u = u_\lambda - \lambda \Delta w_\lambda$. Since β has linear growth, w_λ also belongs to E for every $\lambda > 0$ and the sequence w_λ is bounded in E. Consequently, λw_λ converges to zero in E when $\lambda \to 0$ and it follows that $\lambda \Delta w_\lambda$ converges to zero in the sense of distributions, hence $u_\lambda \to u$ again in the sense of distributions. Because (u_λ) is bounded in $L^1(\mathbb{R})$, it follows that $u_\lambda \to u$ weakly in $E = L^1(\mathbb{R})$, as $\lambda \to 0$.

3. The third step consists in showing that a C^0-solution is a solution in the sense of distributions of (3.2).

   Let $\varepsilon > 0$ and consider a sequence $u^\varepsilon : [0,T] \to E$ being ε-solutions. Note that for $u_0^\varepsilon := u_0$ and for $1 \le j \le N$, for A as in Remark 2.9 2., we recursively have

   $$u_j^\varepsilon = (1 - (t_j^\varepsilon - t_{j-1}^\varepsilon) A)^{-1} u_{j-1}^\varepsilon, \tag{3.6}$$

hence

$$\Delta w_j^\varepsilon = -\frac{u_j^\varepsilon - u_{j-1}^\varepsilon}{t_j^\varepsilon - t_{j-1}^\varepsilon}$$

for some $w_j^\varepsilon \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that $w_j^\varepsilon \in \beta(u_j^\varepsilon)$, dx -a.e. Hence, for $t \in]t_{j-1}^\varepsilon, t_j^\varepsilon]$, we have

$$u^\varepsilon(t,\cdot) = u^\varepsilon(t_{j-1}^\varepsilon,\cdot) + \int_{t_{j-1}^\varepsilon}^{t_j^\varepsilon} \Delta w^\varepsilon(s,\cdot)ds.$$

where $w^\varepsilon(t) = w_j^\varepsilon, \quad t \in]t_{j-1}^\varepsilon, t_j^\varepsilon]$.

Consequently, summing up, if $t \in]t_{j-1}^\varepsilon, t_j^\varepsilon]$, then

$$u^\varepsilon(t,\cdot) = u_0 + \int_0^t \Delta w^\varepsilon(s,\cdot)ds + (t_j^\varepsilon - t)\Delta w^\varepsilon(t_j^\varepsilon,\cdot).$$

We integrate against a test function $\alpha \in \mathcal{S}(\mathbb{R})$ and get

$$\begin{aligned} \int_{\mathbb{R}} u^\varepsilon(t,x)\alpha(x)dx &= \int_{\mathbb{R}} u_0(x)\alpha(x)dx + \int_0^t \int_{\mathbb{R}} w^\varepsilon(s,x)\alpha''(x)dxds \\ &+ (t - t_j^\varepsilon)\int_{\mathbb{R}} w^\varepsilon(t_j^\varepsilon,x)\alpha''(x)dx. \end{aligned} \tag{3.7}$$

Letting ε go to zero we use the fact that $u^\varepsilon \to u$ uniformly in t in $L^1(\mathbb{R})$. (u^ε) converges in particular to $u \in L^1([0,T]\times\mathbb{R})$ when $\varepsilon \to 0$.

The third term in the right-hand side of (3.7) converges to zero since $t - t_j^\varepsilon$ is smaller than the mesh ε of the subdivision.

Consequently, (3.7) implies

$$\int_{\mathbb{R}} u(t,x)\alpha(x)dx = \int_{\mathbb{R}} u_0(x)\alpha(x)dx + \lim_{\varepsilon\to 0}\int_0^t \int_{\mathbb{R}} w^\varepsilon(s,x)\alpha''(x)dxds. \tag{3.8}$$

According to our assumption on β, there is a constant $c > 0$ such that $|w^\varepsilon| \le c|u^\varepsilon|$. Therefore the sequence (w^ε) is uniformly integrable on $[0,T]\times\mathbb{R}$. So, there is a sequence (ε_n) such that w^{ε_n} converges to some $\eta_u \in L^1([0,T]\times\mathbb{R})$ in $\sigma(L^1, L^\infty)$. Taking (3.8) into account, it remains to see that $\eta_u(t,x) \in \beta(u(t,x))$ a.e. $dt\otimes dx$, in order to prove that u solves (3.3).

Let $K > 0$. Using Remark 2.9 4., by (3.6) we conclude that $\|u^\varepsilon(t,\cdot)\|_\infty \leq \|u_0\|_\infty$. Consequently for any $K > 0$, the dominated convergence theorem, implies that the sequence u^{ε_n} restricted to $[0,T] \times [-K,K]$ converges to u restricted to $[0,T] \times [-K,K]$ in $L^2([0,T] \times [-K,K])$ and w^{ε_n} restricted to $[0,T] \times [-K,K]$, being bounded by $c|u^{\varepsilon_n}|$, converges (up to a subsequence) weakly in L^2, necessarily to η_u restricted to $[0,T] \times [-K,K]$. The map $v \to \beta(v)$ on $L^2([0,T] \times [-K,K])$ is an m-accretive multi-valued map, so it is weakly-strongly closed because of [5] p. 37, Proposition 1.1 (i) and (ii). Finally the result follows.

4. The fourth step consists in showing that the obtained solution is in $L^\infty([0,T] \times \mathbb{R})$.

   Point 2. of Proposition 2.11 tells us that

$$u(t,\cdot) = \lim_{n\to+\infty} J^n_{\frac{t}{n}} u_0$$

   in $L^1(\mathbb{R})$. Hence, for every $t \in]0,T]$ and for some subsequence (n_k) depending on t,

$$|u(t,\cdot)| = \lim_{k\to\infty} |J^{n_k}_{\frac{t}{n_k}} u_0| \leq \|u_0\|_\infty, \quad dx\text{−a.e.},$$

   where we used again Remark 2.9 4). It follows by Fubini's theorem that $|u(t,x)| \leq \|u_0\|_\infty$, for $dt \otimes dx$-a.e. $(t,x) \in [0,T] \times \mathbb{R}$.

5. Finally, uniqueness of the equation in $\mathcal{D}'([0,T] \times \mathbb{R})$ follows from Theorem 1 and Remark 1.20 of [15].

$\blacksquare$

Remark 3.4 *Theorem 1 and Remark 1.20 of [15] apply if β is continuous, to give the uniqueness in point 5. above. However, Remark 1.21 of [15] says that this holds true even if $\beta(0) = 0$ and β is only continuous in zero and possibly multi-valued. This case applies for instance when $\Phi(x) = H(x - e_c)$, $e_c > 0$.*

In order to establish the well-posedness for the related probabilistic representation one will need a uniqueness result for the evolution of probability measures. This will be the subject of Theorem 3.6 below. But as will turn out, it will require some global L^2-integrability for the solutions.

A first step in this direction was Corollary 3.2 of [12], that we quote here for the comfort of the reader.

Lemma 3.5 *Let $\kappa \in]0,T[$. Let μ be a finite Borel measure on $[\kappa,T]\times\mathbb{R}$; let $a,b \in L^1([\kappa,T]\times\mathbb{R};\mu)$. We suppose thet*

$$\int_{[\kappa,T]\times\mathbb{R}} \left(\partial_t\varphi(t,x) + a^2(t,x)\partial^2_{xx}\varphi(t,x) + b(t,x)\partial_x\varphi(t,x)\right)\mu(dtdx) = 0,$$

for all $\varphi \in C_0^\infty(]0,+\infty[\times\mathbb{R})$. Then, there is $\rho \in L^2_{\mathrm{loc}}([\kappa,T]\times\mathbb{R})$ such that

$$a(t,x)d\mu(t,x) = \rho(t,x)dtdx.$$

We denote the subset of positive measures in $\mathcal{M}(\mathbb{R})$ by $\mathcal{M}_+(\mathbb{R})$.

Theorem 3.6 *Let a be a Borel non negative bounded function on $[0,T]\times\mathbb{R}$. Let $z_i : [0,T] \to \mathcal{M}_+(\mathbb{R})$, $i=1,2$, be continuous with respect to the weak topology of finite measures on $\mathcal{M}(\mathbb{R})$.*

Let z^0 be an element of $\mathcal{M}_+(\mathbb{R})$. Suppose that both z_1 and z_2 solve the problem $\partial_t z = \partial^2_{xx}(az)$ in the sense of distributions with initial condition $z(0) = z^0$.

More precisely,

$$\int_{\mathbb{R}} \varphi(x)z(t)(dx) = \int_{\mathbb{R}} \varphi(x)z^0(dx) + \int_0^t ds \int_{\mathbb{R}} \varphi''(x)a(s,x)z(s)(dx) \qquad (3.9)$$

for every $t \in [0,T]$ and any $\varphi \in \mathcal{S}(\mathbb{R})$.

Then $(z_1 - z_2)(t)$ is identically zero for every t, if $\tilde{z} := z_1 - z_2$, satisfies the following.

ASSUMPTION (A): There is $\rho : [0,T]\times\mathbb{R} \to \mathbb{R}$ belonging to $L^2([\kappa,T]\times\mathbb{R})$ for every $\kappa > 0$ such that $\rho(t,\cdot)$ is the density of $\tilde{z}(t)$ for almost all $t \in]0,T]$.

Remark 3.7 *If $a \geq$ const > 0, then ρ such that $\rho(t,\cdot)$ is a density of $\tilde{z}(t)$ for almost all $t > 0$, always exists, via Lemma 3.5. It remains to know if it is indeed square integrable on every $[\kappa,T]\times\mathbb{R}$.*

Remark 3.8 *The weak continuity of* $z(t,\cdot)$ *implies that*

$$\sup_{t\in[0,T]} \|z(t)\|_{\mathrm{var}}$$

is finite. Indeed, if this were not true, we could find $t_n \in [0,T]$, *such that* $\|z(t_n)\|_{\mathrm{var}}$ *diverges to infinity. We may assume that* $\lim_{n\to\infty} t_n = t_0 \in [0,T]$. *Then*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)z(t_n)(dx) = \int_{\mathbb{R}} f(x)z(t_0)(dx)$$

for all $f \in C_b(\mathbb{R})$, *hence by the uniform boundedness principle one gets the contradiction that*

$$\sup_n \|z(t_n)\|_{\mathrm{var}} < \infty.$$

Remark 3.9 *Theorem 3.6 does not hold without that Assumption (A) even in the time-homogeneous case.*

To explain this, let $\Phi : \mathbb{R} \to \mathbb{R}_+$ *be continuous bounded such that* $\Phi(0) = 0$ *and* Φ *is strictly positive outside zero. We also suppose that* $\frac{1}{\Phi^2}$ *is integrable in a neighborhood of zero. We define* $a(y) = \Phi^2(y)$.

We set $z^0 = \delta_0$, *i.e. the Dirac delta-function at zero. It is then possible to exhibit two different solutions to the considered problem with initial condition* z^0.

We justify this in the following lines using probabilistic representation. Let Y_0 *be identically zero.*

According to the Engelbert-Schmidt criterion, see e. g. Theorem 5.4 and Remark 5.6 of Chapter 5, [22], it is possible to construct two solutions (in law) to the SDE

$$Y_t = \int_0^t \Phi(Y_s)dW_s. \tag{3.10}$$

where W *is a Brownian motion on some filtered probability space.*

One solution Y^1 *will be identically zero. The second one* Y^2 *will be a non-constant martingale starting from zero. We recall the construction of* Y^2, *since it is of independent interest.*

Let B *be a classical Brownian motion and we set*

$$T_t = \int_0^t \frac{du}{\Phi^2(B_u)}.$$

Problem 6.30 of [22] says that the increasing process (T_t) diverges to infinity when t goes to infinity. We define pathwise (A_t) as the inverse of (T_t) and we set $M_t = B_{A_t}$. M is a martingale since it is a time change of Brownian motion. One one hand we have $[M]_t = A_t$. But pathwise we have

$$A_t = \int_0^{A_t} \Phi^2(B_u) dT_u = \int_0^t \Phi^2(B_{A_v}) dv,$$

through a change of variables $u = A_v$. Consequently we get

$$A_t = \int_0^t \Phi^2(M_v) dv.$$

Theorem 4.2 of Ch. 3 of [22] says there is a Brownian motion $\tilde{W}$ on a suitable filtered probability space and an adapted process (ρ_t) so that $M_t = \int_0^t \rho d\tilde{W}$. We have $[M]_t = \int_0^t \rho_s^2 ds = \int_0^t \Phi^2(M_s) ds$, for all $t \geq 0$, hence $\rho_t^2 = \Phi^2(M_t)$ and so $\Phi(M_t)\mathrm{sign}(\rho_t) = \rho_t$.

We define

$$W_t = \int_0^t \mathrm{sign}(\rho_v) d\tilde{W}_v.$$

Clearly $[W]_t = t$. By Lévy's characterization theorem of Brownian motion, W is a standard Brownian motion. Moreover, we obtain $M_t = \int_0^t \Phi(M_s) dW_s$ so that $Y^2 := M$ solves the stochastic differential equation (3.10). Now Y_t^1 and Y_t^2 have not the same marginal laws $v_i(t, \cdot), i = 1, 2$. In fact $v_1(t, \cdot)$ is constantly Dirac measure at zero.

Using Itô's formula it is easy to show that the law $v(t, \cdot)$ of a solution Y of (3.10) solves the PDE in Theorem 3.6 with initial condition δ_0. This constitutes a counterexample to Theorem 3.6 without Assumption (A).

Proof (of Theorem 3.6).

In this proof given a locally integrable function $(t, x) \to u(t, x)$, u' (resp. u'') stands for the first (resp. second) distributional derivative with respect to the second variable x.

In the first part of the proof we do not use Assumption (A). We will explicitly state it from where it is needed.

Let z^1, z^2 be two solutions to (3.9) and we set $z = z^1 - z^2$. We will study the quantity

$$g_\varepsilon(t) = \int_{\mathbb{R}} B_\varepsilon z(t)(x) z(t)(dx),$$

where $B_\varepsilon z(t) \in (L^1 \bigcap L^\infty)(\mathbb{R})$ is the continuous function v_ε defined in Lemma 2.3, taking $m = z(t)$. $g_\varepsilon(t)$ is well-defined, since

$$g_\varepsilon(t) \le \|z(t)\|_{\text{var}} \sup_x |B_\varepsilon z(t)(x)|, \text{ for all } t \in [0,T].$$

Assume we can show that

$$\lim_{\varepsilon \to 0} g_\varepsilon(t) = 0, \text{ for all } t \in [0,T]. \tag{3.11}$$

Then we are able to prove that $z(t) \equiv 0$ for all $t \in [0,T]$.

Lemma 2.3 says that $B_\varepsilon z(t)'$ is bounded, with a version locally of bounded variation and that $B_\varepsilon z(t) \in C_b(\mathbb{R}) \bigcap L^p(\mathbb{R})$ for all $p \ge 1$.

Let now $c, \tilde{c}$ be positive real constants. Then, since all terms in (2.12) are signed measures of finite total variation, (2.12) implies that

$$\begin{aligned} \int_{]-\tilde{c},c]} B_\varepsilon z(t)(x) z(t)(dx) &= \varepsilon \int_{]-\tilde{c},c]} (B_\varepsilon z(t)(x))^2 dx \\ &\quad - \int_{]-\tilde{c},c]} B_\varepsilon z(t)(x) B_\varepsilon z(t)''(dx). \end{aligned} \tag{3.12}$$

If F, G are functions of locally bounded variation, F continuous, G right-continuous, classical Lebesgue-Stieltjes calculus implies that

$$\int_{]-\tilde{c},c]} F dG = FG(c) - FG(-\tilde{c}) - \int_{]-\tilde{c},c]} G dF. \tag{3.13}$$

Setting $F = B_\varepsilon z(t), G(x) = B_\varepsilon z(t)'$, we get

$$\begin{aligned} - \int_{]-\tilde{c},c]} B_\varepsilon z(t)(x) B_\varepsilon z(t)''(dx) &= -B_\varepsilon z(t)(c) B_\varepsilon z(t)'(c) + B_\varepsilon z(t)(-\tilde{c}) B_\varepsilon z(t)'(-\tilde{c}) \\ &\quad + \int_{]-\tilde{c},c]} (B_\varepsilon z(t)'(x))^2 dx. \end{aligned}$$

Since $B_\varepsilon z(t) \in L^1(\mathbb{R})$, we can choose sequences $(c_n), (\tilde{c}_n)$ converging to infinity such that $B_\varepsilon z(t)(c_n) \to 0, \quad B_\varepsilon z(t)(-\tilde{c}_n) \to 0$ as $n \to \infty$. Then, letting

$n \to \infty$ and using the fact that $B_\varepsilon z(t)$ and $B_\varepsilon z(t)'$ are bounded, by the monotone and Lebesgue dominated convergence theorems, we conclude that

$$-\int B_\varepsilon z(t)(x) B_\varepsilon z(t)''(dx) = \int (B_\varepsilon z(t)'(x))^2 dx.$$

In particular, $B_\varepsilon z(t)' \in L^2(\mathbb{R})$. Consequently, (3.12) implies that

$$\begin{aligned} g_\varepsilon(t) &= \int B_\varepsilon z(t)(x) z(t)(dx) = \varepsilon \int (B_\varepsilon z(t)(x))^2 dx \\ &+ \int (B_\varepsilon z(t)'(x))^2 dx. \end{aligned}$$

In particular, the left-hand side is positive. Therefore, if for all $t \in [0, T]$, $g_\varepsilon(t) \to 0$, as $\varepsilon \to 0$, then

$$\begin{aligned} \sqrt{\varepsilon} B_\varepsilon z(t) &\to 0 \\ B_\varepsilon z(t)' &\to 0 \end{aligned}$$

in $L^2(\mathbb{R})$, as $\varepsilon \to 0$, and so, for all $t \in [0, T]$,

$$z(t) = \varepsilon B_\varepsilon z(t) - B_\varepsilon z(t)'' \to 0$$

in the sense of distributions. Therefore, $z \equiv 0$. It remains to prove (3.11). Let $\delta > 0$ and $\phi_\delta \in C_\circ^\infty(\mathbb{R}), \phi_\delta \geq 0$, symmetric, with $\int_{\mathbb{R}} \phi_\delta(x) dx = 1$ weakly approximating the Dirac-measure with mass in $x = 0$. Set

$$z_\delta(t, x) := (\phi_\delta \star z(t))(x) := \int_{\mathbb{R}} \phi_\delta(x - y) z(t)(dy), \quad x \in \mathbb{R}, t \in [0, T].$$

We define $h : [0, T] \to \mathcal{M}(\mathbb{R})$ by $h(t)(dx) = a(t, x) z(t, dx)$. Note that by (3.9), since $\phi_\delta(x - \cdot) \in \mathcal{S}(\mathbb{R}), \forall x \in \mathbb{R}$, we have

$$z_\delta(t, x) = \int_0^t \int_{\mathbb{R}} \phi''_\delta(x - y) h(s)(dy) ds = \int_0^t (\phi''_\delta \star h(s))(x) ds, \forall t \in [0, T], x \in \mathbb{R}, \tag{3.14}$$

where we used that $z_\delta(0) = 0$, because $z(0) = 0$, and that $x \mapsto z_\delta(t, x)$ is continuous for all $t \in [0, T]$. In fact, one can easily prove that z_δ is continuous and bounded on $[0, T] \times \mathbb{R}$. Now, $B_\varepsilon z(0) = 0$ since $z(0) = 0$. Therefore, by (3.9), since $K_\varepsilon(x - \cdot) \in \mathcal{S}(\mathbb{R}), \forall x \in \mathbb{R}$, we likewise obtain

$$B_\varepsilon z(t)(x) = \int_0^t \int_{\mathbb{R}} K''_\varepsilon(x - y) h(s)(dy) ds = \int_0^t K''_\varepsilon \star h(s)(x) ds, \quad \forall t \in [0, T], x \in \mathbb{R}. \tag{3.15}$$

But, by Lemma 2.3 and Fubini's theorem, for all $\gamma \in \mathcal{S}(\mathbb{R})$ and $s \in [0,T]$ it follows

$$\begin{aligned}\int_{\mathbb{R}} \gamma(x) \int_{\mathbb{R}} K_\varepsilon''(x-y)h(s)(dy)dx &= \int_{\mathbb{R}} \gamma''(x)B_\varepsilon h(s)(x)dx \\ &= \int_{\mathbb{R}} \gamma(x)\left(\varepsilon B_\varepsilon h(s)(x)dx - h(s)(dx)\right),\end{aligned}$$

hence, by approximation,

$$\begin{aligned}\int_{\mathbb{R}} w(x) \int_{\mathbb{R}} K_\varepsilon''(x-y)h(s)(dy)dx &= \int_{\mathbb{R}} w(x)(\varepsilon B_\varepsilon h(s)(x)dx - h(s)(dx)), \\ & \quad \forall w \in C_b(\mathbb{R}), s \in [0,T].\end{aligned}$$

Consequently, by (3.15) and Fubini's theorem

$$\begin{aligned}\int_{\mathbb{R}} w(x)B_\varepsilon z(t)(x)dx &= \int_0^t \int_{\mathbb{R}} w(x)(\varepsilon B_\varepsilon h(s)(x)dx - h(s)(dx))ds \\ & \forall \quad w \in C_b(\mathbb{R}), t \in [0,T].\end{aligned} \tag{3.16}$$

As a consequence of (3.14) and (3.16) and again using Fubini's theorem, for all $t \in [0,T]$ we obtain

$$\begin{aligned}
g_{\varepsilon,\delta}(t) \quad &:= \int_{\mathbb{R}} z_\delta(t,x)B_\varepsilon z(t)(x)dx \\
&\underbrace{=}_{(3.16)} \int_0^t \int_{\mathbb{R}} z_\delta(t,x)(\varepsilon B_\varepsilon h(s)(x)dx - h(s)(dx))ds \\
&\underbrace{=}_{3.14)} \int_0^t \int_{\mathbb{R}} z_\delta(s,x)(\varepsilon B_\varepsilon h(s)(x)dx - h(s)(dx))ds \\
&+ \int_0^t \int_{\mathbb{R}} \int_s^t (\phi_\delta'' \star h(r))(x)dr(\varepsilon B_\varepsilon h(s)(x)dx - h(s)(dx))ds \\
&= \int_0^t \int_{\mathbb{R}} z_\delta(s,x)(\varepsilon B_\varepsilon h(s)(x)dx - h(s)(dx))ds \\
&+ \int_0^t \int_0^r \int_{\mathbb{R}} (\phi_\delta'' \star h(r))(x)(\varepsilon B_\varepsilon h(s)(x)dx - h(s)(dx))dsdr \\
&\underbrace{=}_{3.16)} \int_0^t \int_{\mathbb{R}} z_\delta(s,x)(\varepsilon B_\varepsilon h(s)(x)dx - h(s)(dx))ds \\
&+ \int_0^t \int_{\mathbb{R}} (\phi_\delta'' \star h(r))(x)B_\varepsilon z(r)(x)dxdr
\end{aligned}$$

The application of Fubini's theorem above is justified since a is bounded, $\sup_{t\in[0,T]}\|z(t)\|_{\text{var}} < \infty$, and $K_\varepsilon, \phi_\delta \in \mathcal{S}'(\mathbb{R})$. But the last term is equal to

$$
\begin{aligned}
& \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_\delta''(x-y) B_\varepsilon z(r)(x) dx h(r)(dy) dr \\
= & \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_\delta(x-y))(\varepsilon B_\varepsilon z(r)(x)dx - z(r)(dx))h(r)(dy)dr \\
= & \int_0^t \int_{\mathbb{R}} \varepsilon B_\varepsilon z(r)(x)(\phi_\delta \star h(r))(x) dx dr - \int_0^t \int_{\mathbb{R}} z_\delta(r,y) h(r)(dy) dr,
\end{aligned}
$$

where we could use Lemma 2.3 in the first step, since $\phi_\delta(\cdot - y) \in \mathcal{S}(\mathbb{R}), \forall y \in \mathbb{R}$. Hence, for all $t \in [0,T]$,

$$
\begin{aligned}
g_{\varepsilon,\delta}(t) = & \int_0^t \int_{\mathbb{R}} z_\delta(s,x) \varepsilon B_\varepsilon h(s)(x) dx ds \\
+ & \int_0^t \int_{\mathbb{R}} \varepsilon B_\varepsilon z(s)(x)(\phi_\delta \star h(s))(x) dx ds \\
- & 2\int_0^t \int_{\mathbb{R}} z_\delta(s,x) h(s)(dx) ds.
\end{aligned} \tag{3.17}
$$

For a signed measure ν, we denote its absolute value by $|\nu|$. By Lemma 2.3 we have

$$\sup_{s\in[0,T]} \int_{\mathbb{R}} (|z(s)| \star \phi_\delta)(x) \varepsilon B_\varepsilon |h(s)|(x) dx \le C\sqrt{\varepsilon},$$

where

$$C = \frac{1}{2}\|a\|_\infty \sup_{s\in[0,T]} \|z(s)\|_{\text{var}}^2,$$

and likewise the integrand of the second integral in (3.17) is bounded by the same constant independent of δ. Hence, as $\varepsilon \to 0$, the first and second term in the right-hand side of (3.17) converges to zero uniformly in δ and uniformly in $t \in [0,T]$. Now, we use Assumption (A), namely that $z \in L^2([\kappa,T] \times \mathbb{R})$ for all $\kappa > 0$. Then, since $B_\varepsilon z(t) \in L^2(\mathbb{R}), \forall t \in [\kappa, T]$, and $\|a\|_\infty < \infty$, (3.17) implies that $\forall \kappa > 0, t \in [\kappa, T]$,

$$
\begin{aligned}
g_\varepsilon(t) - g_\epsilon(\kappa) = & \lim_{\delta\to 0}(g_{\varepsilon,\delta}(t) - g_{\varepsilon,\delta}(\kappa)) \\
\le & 2\sqrt{\varepsilon}TC - 2\int_\kappa^t \int_{\mathbb{R}} z^2(s,x) a(s,x) dx ds \\
\le & 2\sqrt{\varepsilon}TC.
\end{aligned} \tag{3.18}
$$

Now, $\lim_{\kappa\to 0} g_\varepsilon(\kappa) = 0$. In fact $z(\kappa,\cdot) \to z(0,\cdot) = 0$ weakly, according to the assumption of Theorem 3.6. According to Theorem 8.4.10, page 192, of [11], the tensor product $z(\kappa,\cdot) \otimes z(\kappa,\cdot)$ converges weakly to zero. On the other hand $(x,y) \to K_\varepsilon(x-y)$ is bounded and continuous on $\mathbb{R}^2$. By Fubini's theorem

$$g_\varepsilon(\kappa) = \int_{\mathbb{R}^2} z(\kappa)(dx) z(\kappa)(dy) K_\varepsilon(x-y) \to 0.$$

So, letting first $\kappa \to 0$ in (3.18) and then $\varepsilon \to 0$, (3.11) follows since $g_\varepsilon(t) \geq 0$ for all $t \in [0,T]$. In fact, we even proved that the convergence in (3.11) is uniformly in $t \in [0,T]$.

■

Theorem 3.6 has a direct application which will be useful for the probabilistic representation of the solution of (3.3) when Φ is non-degenerate.

4 The probabilistic representation of the deterministic equation

Despite the fact that β is multi-valued, by its monotonicity and because of (3.1), it is still possible to find a multi-valued map $\Phi : \mathbb{R} \to \mathbb{R}_+$ such that

$$\beta(u) = \Phi^2(u)u, \quad u \in \mathbb{R},$$

which is bounded, i.e.

$$\sup_{u\in\mathbb{R}} \sup \Phi(u) < \infty.$$

We start with the case where when Φ is non-degenerate, i. e. there exists some constant $c > 0$ such that $y \in \Phi(u) \Rightarrow y \geq c$.

In the degenerate case, we can prove existence in some cases, see Subsection 4.2 below.

4.1 The non-degenerate case

We suppose in this subsection Φ to be non-degenerate.

First of all we need to show that solutions of the linear PDE (3.9), which are laws of solutions to an SDE, are space-time square integrable.

Proposition 4.1 *Suppose $a : [0,T] \times \mathbb{R} \to \mathbb{R}$ to be a bounded measurable function which is bounded below by a strictly positive constant on any compact set.*

We consider a stochastic process $Y = (Y_t, t \in [0,T])$ on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, being a weak solution of the SDE

$$Y_t = Y_0 + \int_0^t \sqrt{2a(s, Y_s)} dW_s,$$

where W is a standard $(\mathcal{F}_t)$-Brownian motion. For $t \in [0,T]$, let $z(t)$ be the law of Y_t and set $z^0 := z(0)$.

1. *Then z solves equation (3.9) with z^0 as initial condition.*

2. *There is $\rho \in L^2(]0,T] \times \mathbb{R})$ such that $\rho(t,\cdot)$ is the density of $z(t)$ for all $t \in [0,T]$.*

3. *z is the unique solution of (3.9)with initial condition z^0 having the property described in item 2. above.*

Remark 4.2 *A necessary and sufficient condition for the existence and uniquess in law of the equation in Proposition 4.1, is that Y solves the martingale problem of Stroock-Varadhan, see chap.6 of [29], related to $L_t f = a(t,x) f''$. In our case, existence and uniqueness follow for instance from [29], exercises 7.3.2-7.3.4, see also [22], Refinements 4.32, chap. 5. We remark that the coefficients are not continuous but only measurable, so that space dimension 1 (or possibly 2), is essential.*

The reader can also consult [25, 26] for more refined conditions to be able to construct a weak solution; however those do not apply in our case.

Proof (of the Proposition 4.1).

1. The first point follows from a direct application of Itô formula to $\varphi(Y_t)$, $\varphi \in \mathcal{S}(\mathbb{R})$, cf. the proof of Theorem 1.2.

2. We first suppose that $Y_0 = x_0$ where $x_0 \in \mathbb{R}$. In this case its law z^0 equals $\delta(dx - x_0)$. In Exercise 7.3.3 of [29], the following Krylov type estimate is provided:

$$\left| E\left(\int_0^T f(t, Y_t)dt\right)\right| \leq \text{const}\|f\|_{L^2([0,T]\times\mathbb{R})},$$

for every smooth function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ with compact support . Lemma 3.5 implies the existence of a density $y \mapsto p_t(x_0, y)$ for the law of Y_t for $t > 0$. This yields

$$\left|\int_{[0,T]\times\mathbb{R}} f(t,y)p_t(x_0,y)dtdy\right| \leq \text{const}\|f\|_{L^2([0,T]\times\mathbb{R})},$$

and the constant const does not depend on x_0, but only on lower and upper bounds of a. This obviously implies that

$$\sup_{x_0\in\mathbb{R}} \int_{[0,T]\times\mathbb{R}} p_t^2(x_0,y)dtdy < \infty.$$

This implies assertion 2. when Y_0 is deterministic.

If the initial condition Y_0 is any law $z^0(dx)$, then clearly the density of Y_t is $z_t(dy) = \rho(t,y)dy$ where $\rho(t,y) = \int_{\mathbb{R}} u_0(dx)p_t(x,y)$.

Consequently, by Jensen's inequality and Fubini's Theorem,

$$\int_{[0,T]\times\mathbb{R}} \rho^2(t,y)dtdy \leq \int_{\mathbb{R}} u_0(dx) \int_{[0,T]\times\mathbb{R}} p_t^2(t,x,y)dtdy < \infty.$$

3. The final assertion follows by 2. from Theorem 3.6.

■

Now we come back to the probabilistic representation of equation (1.1).

Let us consider the solution $u \in (L^1 \bigcap L^\infty)([0,T]\times\mathbb{R})$ from Proposition 3.2, that is, u solves equation (3.2), in the sense of (3.3), assuming the initial condition u_0 is an a.e. bounded probability density. Define

$$\chi_u(t,x) := \sqrt{\frac{\eta_u(t,x)}{u(t,x)}} \tag{4.1}$$

and note that, because Φ is non-degenerate and $\frac{\eta_u(t,x)}{u(t,x)} \in \Phi^2(u(t,x))$ for(t,x), $dt \otimes dx$-a.e., we have $\chi_u \geq c > 0$, $dt \otimes dx$-a.e. Since χ_u is only defined $dt \otimes dx$-a.e, let us fix a Borel version. According to Remark 4.2, it is possible to construct a (unique in law) process Y which is the weak solution of

$$Y_t = Y_0 + \int_0^t \chi_u(s, Y_s) dW_s \tag{4.2}$$

where W is a classical Brownian motion on some filtered probability space and Y_0 is a random variable distributed, so that u_0 is its density.

Consider now the law $v(t,\cdot)$ of the process Y_t. We set $a(t,x) = \frac{\chi_u^2(t,x)}{2}$. Since $a \geq c > 0$, Proposition 4.1 implies that $v \in L^2([0,T] \times \mathbb{R})$ and it solves the equation

$$\left\{ \begin{array}{rcl} \partial_t v & = & \partial_{xx}^2 (av) \\ v(0,x) & = & u_0(x). \end{array} \right. \tag{4.3}$$

On the other hand u itself, which is a solution to (3.2) (in the sense of (3.3)), is another solution of equation (4.3). So, being in $(L^1 \bigcap L^\infty)([0,T] \times \mathbb{R})$, u is also square integrable. Setting $z_1 = v, z_2 = u$, Theorem 3.6 implies that $v = u$, $dt \otimes dx$-a.e.

Since $u \in C([0,T], L^1(\mathbb{R})$ and Y has continuous sample paths, it follows that $u(t,\cdot) = v(t,\cdot)$, dx-a.e. for all $t \in [0,T]$.

The considerations above allow us to state the following representation theorem, at least in the non-degenerate case.

Theorem 4.3 *Let $u_0 \in L^1 \bigcap L^\infty$ such that $u_0 \geq 0$ and $\int_{\mathbb{R}} u_0(x) dx = 1$. Suppose the multi-valued map Φ is non-degenerate and bounded. Then there is a process Y, unique in law, such that there exists $\chi \in (L^1 \bigcap L^\infty)([0,T] \times \mathbb{R})$ with*

$$\left\{ \begin{array}{ccc} Y_t & = & Y_0 + \int_0^t \chi(s, Y_s) dW_s \text{ (weakly)} \\ \chi(t,x) & \in & \Phi(u(t,x)), \text{ for } dt \otimes dx\text{-a.e. } (t,x) \in [0,T] \times \mathbb{R} \\ \text{Law density of } Y_t & = & u(t,\cdot) \\ u(0,\cdot) & = & u_0, \end{array} \right. \tag{4.4}$$

with $u \in C([0,T]; L^1(\mathbb{R})) \bigcap L^\infty([0,T] \times \mathbb{R})$.

Remark 4.4 *If Φ is single-valued then $\chi_u \equiv \Phi(u)$.*

Proof. Existence has been established above. Concerning uniqueness, given two solutions $Y^i, i = 1,2$ of (4.4) i.e. (1.4). We denote by $u_i(t,\cdot), i = 1,2$ the law densities of respectively $Y^i, i = 1,2$ with corresponding χ_1 and χ_2.

The multi-valued version of Theorem 1.2 says that u_1 and u_2 solve equation (1.1) in the sense of distributions. so that by Proposition 3.2 (uniqueness for (3.3)) we have $u_1 = u_2$, and also $\chi_1 = \chi_2$ a.e.

We note that, since Y_t^i has a law density for all $t > 0$, the stochastic integrals in (4.4) are independent of the chosen Borel version of χ. Remark 4.2 now implies that the laws of Y^1 and Y^2 (on path space) coincide. ∎

Corollary 4.5 *Let $v_0 \in L^1 \bigcap L^\infty$ be such that $v_0 \geq 0$. Suppose Φ is Borel, non-degenerate and bounded. The unique solution v to equation (3.2) with initial condition v_0 is non negative for any $t \geq 0$. Moreover, the mass $\int_{\mathbb{R}} v(t,x)dx$ does not depend on t.*

Proof. Setting $\mu_0 = \int_{\mathbb{R}} v_0(y)dy$ that we can suppose to be greater than 0. Then the function $u(t,x) = \frac{v(t,x)}{\mu_0}$ solves equation (3.2)

$$\left\{ \begin{array}{rcl} \partial_t u &=& \frac{1}{2}\partial_{xx}^2\left(\frac{\beta(\mu_0 u)}{\mu_0}\right), \\ u(0,\cdot) &=& \frac{v_0}{\mu_0}. \end{array} \right. \tag{4.5}$$

Hence, the result follows from Theorem 4.3. ∎

Remark 4.6 *We note that if Φ is merely bounded below by a strictly positive constant on every compact set and if the solutions u are continuous on $[0,T] \times \mathbb{R}$, then Theorem 4.3 and Corollary 4.5 still hold.*

4.2 The degenerate case

The non-degenerate case is much more difficult and will be analyzed in detail in the forthcoming paper [6]. In this subsection we only take the first two steps in the special case where our β of Section 1 is of the form $\beta(u) = \Phi^2(u)u$ and the following properties hold:

Property 4.7 *$\Phi : \mathbb{R} \to \mathbb{R}$ is single-valued, continuous on $\mathbb{R} - \{0\}$ and lower semi-continuous in zero.*

We furthermore assume that the initial condition u_0 and Φ are such that we have for the corresponding solution u to (1.1) (in the sense of Proposition 3.2) the following:

Property 4.8 *$\Phi^2(u(t,\cdot)) : \mathbb{R} \to \mathbb{R}$ is Lebesgue almost everywhere continuous for dt a.e $t \in [0,T]$.*

Remark 4.9 *As will be shown in [6] condition (4.8) is fulfilled in many interesting cases, for a large class of intial conditions. In fact, we expect to be able to show that $u(t,\cdot)$ is even locally of bounded variation if so is u_0.*

Proposition 4.10 *Suppose that Property 4.7 holds. Let $u^0 \geq 0$ be a bounded integrable real function such that $\int_{\mathbb{R}} u_0(x)dx = 1$ and the corresponding solution u to (1.1) satisfies Property 4.8. Then, there is at least one process Y such that*

$$\left\{ \begin{array}{ccc} Y_t & = & Y_0 + \int_0^t \Phi(u(s,Y_s))dW_s \text{ in law} \\ \text{Law} \quad \text{density}(Y_t) & = & u(t,\cdot), \\ u(0,\cdot) & = & u_0 \end{array} \right. \tag{4.6}$$

Corollary 4.11 *Suppose that Property 4.7 holds. Let $u_0 \in L^1 \bigcap L^\infty$ be such that $u_0 \geq 0$ and that the corresponding solution u to (1.1) satisfies Property 4.8. The unique solution u to equation (3.2) is non-negative for any $t \geq 0$. Moreover, the mass $\int_{\mathbb{R}} u(t,x)dx$ is constant.*

Proof (of Proposition 4.10). We denote the solution to equation (3.3), by $u = u(t,x)$.

Let $\varepsilon \in]0,1]$ and set $\beta_\varepsilon(u) = (\Phi(u)+\varepsilon)^2 u, \quad \Phi_\varepsilon(u) = \Phi(u)+\varepsilon, u \in \mathbb{R}$. Proposition 3.2 provides the solution $u = u^\varepsilon$ to the deterministic PDE equation (3.3)

$$\left\{ \begin{array}{ccc} \partial_t u & = & \frac{1}{2}\partial^2_{xx}(\beta_\varepsilon(u)), \\ u(0,x) & = & u_0(x). \end{array} \right.$$

We consider the unique solution $Y = Y^\varepsilon$ in law of

$$\begin{cases} Y_t & = & Y_0 + \int_0^t \Phi_\varepsilon(u(s, Y_s))dW_s \\ \text{Law} \quad \text{density}(Y_t) & = & u^\varepsilon(t, \cdot) \\ u^\varepsilon(0, \cdot) & = & u_0. \end{cases} \tag{4.7}$$

Since $\Phi + \varepsilon$ is non-degenerate, this is possible through Theorem 4.3.

Since Φ is bounded, using Burkholder-Davies-Gundy inequality one obtains

$$\mathbb{E}\{Y_t^\varepsilon - Y_s^\varepsilon\}^4 \leq \text{const}(t-s)^2, \forall \varepsilon > 0. \tag{4.8}$$

where const does not depend on ε. Using the Garsia-Rodemich-Rumsey lemma (see for instance [7], (3.b), p. 203), we obtain that

$$\sup_{\varepsilon > 0} E \left(\sup_{s,t \in [0,T]} \frac{|Y_t^\varepsilon - Y_s^\varepsilon|^4}{|t-s|} \right) < \infty.$$

Consequently, using Chebyshev's inequality

$$\lim_{\delta \to 0} \sup_{\varepsilon > 0} P(\{ \sup_{s,t \in [0,T], |t-s| \leq \delta} |Y_t^\varepsilon - Y_s^\varepsilon| > \lambda \}) = 0, \ \forall \lambda > 0.$$

This implies condition (4.7) of theorem 4.10 in Section 2.4 of [22]. Condition (4.6) of the same theorem requires

$$\lim_{\lambda \to +\infty} \sup_{\varepsilon > 0} P\{|Y_0^\varepsilon| \geq \lambda\} = 0.$$

This is here trivially satisfied since the law of Y_0^ε is the same for all ε. Thus the same theorem implies that the family of laws of $Y^\varepsilon, \varepsilon > 0$, is tight.

Consequently, there is a subsequence $Y^n := Y^{\varepsilon_n}$ converging in law (as $C[0,T]$-valued random elements) to some process Y. We set $\Phi_n := \Phi_{\varepsilon_n}$ and $u^n := u^{\varepsilon_n}$ where we recall that $u^n(t, \cdot)$ is the law of Y_t^n.

We also set $X_t^n = Y_t^n - Y_0^n$. Since

$$[X^n]_t = \int_0^t \Phi_n^2(u^n(s, Y_s^n))ds,$$

and $E([X^n]_T)$ is finite, Φ being bounded, the continuous local martingales X^n are indeed martingales.

By Skorokhod's theorem there is a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and processes $\tilde{Y}^n$, with the same distribution as Y^n so that $\tilde{Y}^n$ converge $\tilde{P}$-a.e. to some process $\tilde{Y}$, of course distributed as Y, as $C([0,T])$- random elements. In particular, the processes $\tilde{X}^n := \tilde{Y}^n - \tilde{Y}^n_0$ remain martingales with respect to the filtrations generated by them $\tilde{Y}^n$. We denote the sequence $\tilde{Y}^n$ (resp. $\tilde{Y}$), again by Y^n (resp. Y).

Remark 4.12 *We observe that, for each $t \in [0,T]$, $u(t,\cdot)$ is the law density of Y_t. Indeed, for any $t \in [0,T]$, Y^n_t converges in probability to Y_t; on the other hand $u^n(t,\cdot)$, which is the law of Y^n_t, converges to $u(t,\cdot)$ in $L^1(\mathbb{R})$ uniformly in t, cf. [10].*

Remark 4.13 *Let $\mathcal{Y}^n$ (resp. $\mathcal{Y}$) be the canonical filtration associated with Y^n (resp. Y).*

We set

$$W^n_t = \int_0^t \frac{1}{\Phi_n(u^n(s, Y^n_s))} dY^n_s.$$

Those processes W^n are standard $(\mathcal{Y}^n_t)$ -Wiener processes since $[W^n]_t = t$ and because of Lévy's characterization theorem of Brownian motion. Then one has

$$Y^n_t = Y^n_0 + \int_0^t \Phi_n(u^n(s, Y^n_s)) dW^n_s.$$

We aim to prove first that

$$Y_t = Y_0 + \int_0^t \Phi(u(s, Y_s)) dW_s. \tag{4.9}$$

Once the previous equation is established for the given u, the statement of Proposition 4.10 would be completely proven because of Remark 4.12. In fact, that Remark shows in particular the third line of (4.6).

We consider the stochastic process X (vanishing at zero) defined by $X_t = Y_t - Y_0$. We also set again $X^n_t = Y^n_t - Y^n_0$.

Taking into account Theorem 4.2 of Ch. 3 of [22], as in Remark 3.9, to establish (4.9) it will be enough to prove that X is a $\mathcal{Y}$- martingale with quadratic variation $[X]_t = \int_0^t \Phi^2(u(s, Y_s)) ds$.

Let $s,t \in [0,T]$ with $t > s$ and Θ a bounded continuous function from $C([0,s])$ to $\mathbb{R}$.

In order to prove the martingale property for X, we need to show that

$$E\left((X_t - X_s)\Theta(Y_r, r \le s)\right) = 0.$$

But this follows because $Y^n \to Y$ a.s. (so $X^n \to X$ a.s.) as $C([0,T])$-valued process; so for each $t \ge 0$, $X_t^n \to X_t$ in $L^1(\Omega)$ since $(X_t^n, n \in \mathbb{N})$ is bounded in $L^2(\Omega)$ and

$$E\left((X_t^n - X_s^n)\Theta(Y_r^n, r \le s)\right) = 0.$$

It remains to show that $X_t^2 - \int_0^t \Phi^2(u(s,Y_s))ds, t \in [0,T]$, defines a $\mathcal{Y}$-martingale, that is, we need to verify

$$E\left((X_t^2 - X_s^2 - \int_s^t \Phi^2(u(r,Y_r))dr)\Theta(Y_r, r \le s)\right) = 0.$$

The left-hand side decomposes into $2(I^1(n) + I^2(n) + I^3(n))$ where

$$\begin{aligned} I^1(n) &= E\left((X_t^2 - X_s^2 - \int_s^t \Phi^2(u(r,Y_r))dr)\Theta(Y_r, r \le s)\right) \\ &- E\left(\left((X_t^n)^2 - (X_s^n)^2 - \int_s^t \Phi^2(u(r,Y_r^n))dr\right)\Theta(Y_r^n, r \le s)\right), \\ I^2(n) &= E\left(\left((X_t^n)^2 - (X_s^n)^2 - \int_s^t \Phi_n^2(u^n(r,Y_r^n))dr\right)\Theta(Y_r^n, r \le s)\right), \end{aligned}$$

and

$$I^3(n) = E\left(\int_s^t \left(\Phi_n^2(u^n(r,Y_r^n)) - \Phi^2(u(r,Y_r^n))\right) dr\Theta(Y_r^n, r \le s)\right).$$

We start by showing the convergence of $I^3(n)$. Now $\Theta(Y_r^n, r \le s)$ is dominated by a constant. Therefore, since Φ_n, Φ are uniformly bounded and $a^2 - b^2 = (a-b)(a+b)$, by the Cauchy-Schwarz inequality, it suffices to consider the expectation of

$$\int_s^t \left(\Phi_n(u^n(r,Y_r^n)) - \Phi(u(r,Y_r^n))\right)^2 dr \tag{4.10}$$

which is equal to

$$\begin{aligned} \int_s^t E(\Phi_n(u^n(r,Y_r^n)) &- \Phi(u(r,Y_r^n)))^2 dr \\ &= \int_s^t dr \int_{\mathbb{R}} \left(\Phi_n(u^n(r,y)) - \Phi(u(r,y))\right)^2 u^n(r,y)dy. \end{aligned}$$

This equals $J_1(n) + J_2(n) - 2J_3(n)$ where

$$\begin{aligned}
J_1(n) &= \int_s^t dr \int_{\mathbb{R}} \Phi_n^2(u^n(r,y))u^n(r,y)dy \\
J_2(n) &= \int_s^t dr \int_{\mathbb{R}} \Phi^2(u(r,y))u^n(r,y)dy \\
J_3(n) &= \int_s^t dr \int_{\mathbb{R}} \Phi_n(u^n(r,y))\Phi(u(r,y))u^n(r,y)dy.
\end{aligned}$$

Define

$$J := \int_s^t \int_{\mathbb{R}} \Phi^2(u(r,y))u(r,y)dy = \int_s^t \int_{\mathbb{R}} \beta(u(r,y))dy$$

To show that $I^3(n) \to 0$ as $n \to \infty$, it suffices to show that

$$\lim_{n\to\infty} J_1(n) = \lim_{n\to\infty} J_2(n) = J \tag{4.11}$$

and

$$\liminf_{n\to\infty} J_3(n) \geq J. \tag{4.12}$$

Now repeating exactly the same arguments as in Point 3. of Proposition 3.2, it follows that $\Phi_n^2(u_n)u_n \to \Phi^2(u)u$ in $\sigma(L^1, L^\infty)$ as $n \to \infty$ which immediately implies (4.11).

Furthermore, by Fatou's lemma and since $\Phi_n \geq \Phi$,

$$\liminf_{n\to\infty} J_3(n) \geq \int_0^t \int_{\mathbb{R}} \liminf_{n\to\infty} \Phi(u^n(r,y))\Phi(u(r,y))u(r,y)dydr$$

which by the lower semicontinuity of Φ, implies (4.12).

Now we go on with the analysis of $I^2(n)$ and $I^1(n)$. $I^2(n)$ equals zero because X^n is a martingale with quadratic variation given by $[X^n]_t = \int_0^t \Phi_n^2(u^n(r,Y_r^n))dr$.

We treat finally $I^1(n)$. We recall that $X^n \to X$ a. s. as a random element in $C([0,T])$ and that the sequence $E\left((X_t^n)^4\right)$ is bounded, so $(X_t^n)^2$ are uniformly integrable. Therefore, we have

$$E\left((X_t^n)^2 - (X_s^n)^2)\Theta(Y_r^n, r \leq s)\right) - E\left((X_t^2 - X_s^2)\Theta(Y_r, r \leq s)\right) \to 0,$$

when $n \to \infty$. It remains to prove that

$$\int_s^t E\left(\Phi^2(u(r,Y_r)) - \Phi^2(u(r,Y_r^n))\Theta(Y_r^n, r \leq s)dr\right) \to 0. \tag{4.13}$$

Now, for fixed dr-a.e. $r \in [0,T]$, $\Phi(u(r,\cdot))$ has a Lebesgue zero set of discontinuities. Moreover, the law of Y_r has a density. So, let $N(r)$ be the null event of all $\omega \in \Omega$ such that $Y_r(\omega)$ is a point of discontinuity of $\Phi(u(r,\cdot))$. For $\omega \notin N(r)$ we have

$$\lim_{n\to\infty} \Phi^2(u(r, Y_r^n(\omega))) = \Phi^2(u(r, Y_r(\omega))).$$

Hence Lebesgue dominated convergence theorem implies (4.13).

So equation (4.9) is shown. The last point concerns the question whether $u(t,\cdot)$ is the law of Y_t. We recall that for any t, Y_t^n converges (even in probability) to Y_t and $u^n(t,\cdot)$, which is the law density of Y_t^n goes to $u(t,\cdot)$ in $L^1(\mathbb{R})$. By obvious identification $u(t\cdot)$ is the law density of Y_t. ■

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