Erratum and Addentum to: “Singular dissipative stochastic equations in Hilbert spaces”

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Abstract

An error in the above paper is corrected. All main results of the paper remain correct except for one: we get a slightly modified statement concerning the path regularity of the constructed Markov process at $t = 0$.

1 Introduction

We place ourselves entirely into the framework of the above paper [DPR02] and assume the reader to be familiar with the notation introduced there.

There is an error in the proof of Lemma 5.5 which seems impossible to be corrected without a further assumption. This slightly affects the path continuity at $t = 0$ of the Markov process constructed subsequently. However, Theorem 7.4, which is the only main result relying on Lemma 5.5, remains correct just by changing the topology on the state space to the one naturally given by the Markov process in question. More precisely, the Markov process constructed in Theorem 7.4 which solves the desired martingale problem has the following property for its path space measures $\mathbb{P}_x$:

$$\mathbb{P}_x[C([0,\infty);(H_0,\tau_{S_2})] = 1, \quad \forall x \in H_0 \tag{1.1}$$

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where $H_0 :=$ supp ν, with ν as in [DPR02, Hypothesis 2.1], is equipped with the topology τ_{S_2} generated by the cone S_2 of α-supermedian continuous functions on H_0 defined in [DPR02, Section 5].

If H_0 is equipped with the norm topology induced by H, we only have

$$\mathbb{P}_x[C((0,\infty);H_0)] = 1 \quad \forall x \in H_0, \tag{1.2}$$

The original slightly stronger claim in [DPR02, Theorem 7.4] that

$$\mathbb{P}_x[C([0,\infty);H_0)] = 1 \tag{1.3}$$

for $x \in H_0$ remains unproven in general, unless $\dim H < \infty$. In this note we prove (1.1) and (1.2) (cf. Theorem 7.4' below) and, under the assumption that $\dim H < \infty$, also (1.3) (see Proposition 2.1).

Finally, we take the opportunity to correct a misprint in the statement of [DPR02, Lemma 5.6] and present a polished version of the proof of the crucial [DPR02, Proposition 5.7]. Both are included in the appendix. All applications in [DPR02, Section 9] are still valid, but if $\dim H = \infty$, path continuity at $t = 0$ is to be understood with respect to the topology τ_{S_2}.

2 Description of the problem

In the proof of Lemma 5.5 it cannot be concluded from the last inequality that $x_n \to x$ in H_0. Hence $\{g_n | n \in \mathbb{N}\}$, though being point separating on H_0, may not generate the topology on H_0 inherited from H. The proof, however, can trivially be modified to prove that $\{g_n | n \in \mathbb{N}\}$ generates this topology if the following condition holds:

$$\lim_{m\to\infty} mR_m f_k = f_k \quad \text{uniformly on } H_0 \text{ for all } k \in \mathbb{N}, \tag{C.1}$$

with f_k as defined in [DPR02, (5.5)]. This condition on the other hand can be hard to check, in particular, if H is infinite dimensional.

Let us now list all places where Lemma 5.5 is mentioned or used in the subsequent part of the paper. In the proof of Lemma 5.6 the map $i : H_0 \to i(H_0)$ is clearly continuous, but Lemma 5.5 is quoted there to conclude that it is a homeomorphism. Fortunately, only the continuity of i is used in the rest of the proof, so Lemma 5.5 is not used here. In Corollary 6.4 assertion (i) relies on Lemma 5.5 and should, therefore, be deleted.

The proof of Proposition 7.2 uses Lemma 5.5 in its last line. Its assertion must be modified. The correct formulation is as follows.

Proposition 7.2'. *Let $x \in H_0$. Then $\mathbb{P}_x$-a.s.*

$$\lim_{\substack{t\to 0\\ t>0}} g_n(X_t^0) = g_n(x) \quad \forall n \in \mathbb{N}. \tag{2.1}$$

Because the g_n do not generate the topology on H_0 induced by the norm $|\cdot|$ on H, we cannot in general conclude from here that

$$\lim_{\substack{t\to 0\\ t>0}} X_t^0 = x \quad \mathbb{P}_x\text{-a.s.}. \tag{2.2}$$

(So, only the very last line of the proof of Proposition 7.2 is erroneous.) (2.2) is, however, true, if $\dim H < \infty$.

Proposition 2.1. *Assume*

$$\dim H < \infty, \tag{C.2}$$

and let $x \in H_0$. Then (2.2) *holds and thus [DPR02, Theorem 7.4] holds.*

Proof. Suppose we can prove that there exists $V : H_0 \to \mathbb{R}_+$ with relatively compact level sets (with respect to $|\cdot|$) which for some $\alpha > 0$ is α-supermedian for $(p)_{t>0}$, i.e. $e^{\alpha t} p_t V \le V$ on $H_0 \forall t > 0$. Then as in the first part of the proof of [DPR02, Proposition 7.2] the martingale convergence theorem implies that $\mathbb{P}_x$-a.s.

$$\lim_{\substack{t\to 0\\ t>0}} V(X_t^0) \text{ exists in } \mathbb{R}.$$

Hence for $\mathbb{P}_x$-a.e. $\omega \in \Omega$, $X_t^0(\omega)$ is in a compact subset of H_0 $\forall t \ge t(\omega)$. But all its accumulation point must coincide by (2.1) since the g_n are point separating. Hence (2.2) follows.

Claim. $V := (1 + |\cdot|)^{1/2}$ is α-supermedian for $(p_t)_{t>0}$ for some $\alpha > 0$.

Indeed, for $n \in \mathbb{N}$ let $\chi_n \in C_b^2(\mathbb{R})$, $0 \le \chi_n' \le 1, \chi_n(s) = s \,\forall s \in [-n, n]$, $|\chi_n(s)| = n+1$ if $|s| \ge n+2$, $\sup_n \chi_n'' =: c < \infty$, $\chi_n \le \chi_{n+1}$. Then it is easy to see that $\chi_n(|\cdot|) \in D(N_2)$ and that for ν-a.e. $y \in H$.

$$\begin{aligned} N_2\chi_N(|\cdot|^2)(y) &= \chi_n'(|y|^2) N_2|\cdot|^2(y) + \chi_n''(|y|^2)|C^{1/2}D|\cdot|^2|^2(y) \\ &\le (\mathrm{Tr}\ C + 2\langle Ay, y\rangle + 2\langle F_0(y), y\rangle) + c4\|C\|\,|y|^2 \le \alpha u \end{aligned} \tag{2.3}$$

where $u := 1 + |\cdot|^2$ ($\in L^2(H_0, \nu)$) and $\alpha := \max(\mathrm{Tr}\ C + |F_0(0)|^2, 4c\|C\| + 1)$ and we used both Hypothesis 1.1(i) (which can, however, be avoided since

$\dim H < \infty$, so A is bounded) and the dissipativity of F_0. Hence for $u_n := 1 + \chi_n(|\cdot|^2)$ and all $\lambda > 0$ we have ν-a.e.

$$\begin{aligned}
\lambda R_{\lambda+\alpha} u_n &= \lambda R_{\lambda+\alpha}(R_\alpha(\alpha - N_2)u_n) \\
&= (R_\alpha - R_{\lambda+\alpha})((\alpha - N_2)u_n) \\
&= u_n - R_{\lambda+\alpha}(\alpha u_n - N_2 u_n) \\
&= u_n - R_{\lambda+\alpha}(\alpha u - N_2 u_n) - \alpha R_{\lambda+\alpha} u_n + \alpha R_{\lambda+\alpha} u \\
&\leq u_n + \alpha R_{\lambda+\alpha} u - \alpha R_{\lambda+\alpha} u_n,
\end{aligned}$$

where we used (2.3) in the last step. Consequently,

$$(\lambda + \alpha) R_{\lambda+\alpha} u_n \leq u_n + \alpha R_{\lambda+\alpha} u, \quad \nu - \text{a.e.}$$

und letting $n \to \infty$ by monotone convergence

$$(\lambda + \alpha) R_{\lambda+\alpha} u \leq u + \alpha R_{\lambda+\alpha} u,$$

i.e. (since all involved functions are finite ν-a.e.)

$$\lambda R_{\lambda+\alpha} u \leq u \quad \nu\text{-a.e..}$$

We conclude that by Jensen's inequality ν-a.e.

$$\lambda R_{\lambda+\alpha} u^{1/2} \leq \frac{\lambda}{\lambda+\alpha}((\lambda+\alpha)R_{\lambda+\alpha}u)^{1/2} = \left(\frac{\lambda}{\lambda+\alpha}\right)^{1/2} (\lambda R_{\lambda+\alpha} u)^{1/2} \leq u^{1/2}.$$

But since $u^{1/2} \in \mathrm{Lip}(H)$ and since it easily follows from (5.3) by approximation that $R_{\lambda+\alpha} f \in \mathrm{Lip}(H_0)$ for all $f \in \mathrm{Lip}(H)$, it follows by continuity that

$$\lambda R_{\lambda+\alpha} u^{1/2} \leq u^{1/2} \quad \text{(everywhere) on } H_0.$$

Clearly, this is equivalent with the claim. □

3 The correctly modified version of Theorem 7.4

In this section we will state and prove the correctly modified version of [DPR02, Theorem 7.4].

We first recall that our standing assumptions in [DPR02], namely Hypotheses 1.1 and 1.2, are still in force as well as Hypothesis 7.3 (i.e., A is self-adjoint) and the assumption that $C^{-1} \in L(H)$. Furthermore, we consider the countable cone S_2 of Lipschitz-continuous bounded functions, which are α-supermedian (for $(R_\lambda)_{\lambda>0}$) for some $\alpha \in \mathbb{Q}_+ \setminus \{0\}$, introduced in preparation of [DPR02, Lemma 5.6]. By Proposition 5.2 we know that

$$\lim_{\lambda \to \infty} \lambda R_{\lambda+\alpha} f(x) = f(x) \quad \forall x \in H_0,\ f \in S_2$$

and this limit is in fact a supremum by the resolvent equation. Such α-supermedian functions are called α-excessive functions, so S_2 consists of α-excessive (with respect to $(p_t)_{t>0}$ or equivalently to $(R_\lambda)_{\lambda>0}$) functions. Let τ_{S_2} be the topology generated by S_2 on H_0 ($:=$ supp ν with ν as in Hypothesis 1.2).

Theorem 7.4'. *(i) There exists a conservative (normal) strong Markov process* $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in H_0})$ *with* τ_{S_2}*-continuous sample paths having transition semigroup* $(p_t)_{t\geq 0}$ *(as defined in [DPR02, Proposition 5.7(iii)]). In particular,* ν *is an invariant measure for* ν*.*

(ii) For $x \in \mathbb{P}_x$ *the paths* $t \mapsto X_t$ *are* $|\cdot|$*-continuous on* $(0,\infty)$ $\mathbb{P}_x$*-a.s..*

(iii) For every $x \in H_0$*,* $\mathbb{P}_x$ *solves the martingale problem for* N_2 *with test function space*

$$D_0 := \{\varphi \in D(N_2) \cap C_b(H) |\ N_2\varphi \in L^\infty(H,\nu)\}$$

and initial condition x*, i.e., under* $\mathbb{P}_x$

$$\varphi(X_t) - \int_0^t N_2\varphi(X_s)ds, \quad t \geq 0, \tag{3.1}$$

is an $(\mathcal{F}_t)$*–martingale with* $X_0 = x$ *for all* $\varphi \in D_0$*.*

Proof. Let D denote the dyadics and $\Omega := H_0^D$. Replacing the metric d in [DPR02, Section 7] by the one generated by $|\cdot|$ and realizing that by the same arguments as in the first part of the proof of [DPR02, Proposition 7.2] we obtain that for all $f \in S_2$

$$\lim_{t\to 0} f(X_t^0) = f(x) \quad \mathbb{P}_x\text{-a.s. } \forall x \in H_0,$$

the same proof as that of [DPR02, Theorem 7.4] implies (i) - (iii).

□

Remark 3.1. *It is possible to show that the set of all $x \in H_0$, for which the process X_t, $t \geq 0$, is not weakly continuous at $t = 0$ $\mathbb{P}_x$-a.s., is polar, i.e. is not hit by X_t, $t \geq 0$, $\mathbb{P}_y$-a.s. for all $y \in H_0$. This fact will be proved in a forthcoming paper.*

Appendix

Finally, in [DPR02] there was a mistake in the statement of Lemma 5.6. The correct formulation and what was actually proved and used in the subsequent part of the paper is the following

Lemma 5.6'. *Let $f \in S$. Then there exists a ν–version $\overline{p}_t f$ of $P_t f$, $t > 0$, such that for all $x \in H_0$*

$$t \mapsto \overline{p}_t f(x) \text{ is right continuous on } [0, +\infty),$$

and for $\lambda > 0$

$$\int_0^\infty e^{-t\lambda} \overline{p}_t f(x) dt = R_\lambda f(x). \tag{A.1}$$

We also include a polished version of the proof of [DPR02, Proposition 5.7]:

Proof. (iii) and (iv) follow from $(i), (ii)$ by exactly the same arguments used in the proofs of [DPR02, Proposition 5.2] and [DPR02, Corollaries 5.3, 5.4]. So, we only have to prove (i),(ii).

(i) Let $N \in \mathbb{N}$ and let Y_N denote the closed ball of radius $\sqrt{N}\ \|f\|_0$ in $L^2([0, N], ds)$ equipped with the weak topology. So, Y_N is compact. Let $\{l_n |\ n \in \mathbb{N}\}$ be a dense set in $L^2([0, N], ds)$ consisting of bounded functions. Then

$$\begin{aligned} d_{Y_N}(h_1, h_2) : \ &= \ \mathrm{d}s \textstyle\sum_{n=1}^{\infty} 2^{-n} \left(\|l_n\|_{L^\infty([0,N],ds)} + \|l_n\|_{L^2([0,N],ds)} + 1 \right)^{-1} \\ & \quad \inf \left(|\textstyle\int_0^N l_n(s)(h_1(s) - h_2(s)) ds|, 1 \right),\ h_1.h_2 \in Y_N, \end{aligned}$$

defines a metric on Y_N generating its topology, which is complete, since Y_N is compact.

Now consider the maps $\Lambda_N^{\alpha,\beta} : H \to Y_N$ defined for $\alpha, \beta > 0$ by

$$\Lambda_N^{\alpha,\beta}(x) := \left(s \to P_s^{\alpha,\beta} f(x),\ s \in [0, N]\right),\ x \in H.$$

Then for all $x, y \in H$, $\alpha, \beta > 0$, by [DPR02, (4.7)]

$$d_{Y_N}(\Lambda_N^{\alpha,\beta}(x), \Lambda_N^{\alpha,\beta}(y)) \leq \int_0^N s^{-1/2} ds \|C^{-1}\|^{1/2} \|f\|_0 |x - y|. \tag{A.2}$$

Since ν is a probability measure on a polish space there exist $\tilde{K}_n \subset H_0,\ n \in \mathbb{N}$, compact and increasing, such that

$$\lim_{n\to\infty} \nu(H_0 \backslash \tilde{K}_n) = 0.$$

Defining

$$K_n := \text{supp}\ [1_{\tilde{K}_n} \nu],\ n \in \mathbb{N},$$

it is easy to check (cf. the proof of [MR92, Chapter III, Proposition 3.8]), that $K_n \subset \tilde{K}_n,\ n \in \mathbb{N}$, and still

$$\lim_{n\to\infty} \nu(H_0 \backslash K_n) = 0$$

and that, in addition,

$$K_n \cap U \neq \emptyset \Rightarrow \nu(K_n \cap U) > 0,\ \forall \text{ open sets } U \subset H_0,\ \forall\, n \in \mathbb{N}. \tag{A.3}$$

By [DPR02, Proposition 4.1] we can find α_n,$\beta_n > 0$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} R(\lambda, N_2^{\alpha_n,\beta_n}) f = R(\lambda, N_2) f,\ \forall\, \lambda > 0\ \text{ in } L^2(H, \nu) \text{ and } \nu - \text{a.e..} \tag{A.4}$$

Applying the Ascoli theorem and a diagonal argument, selecting a subsequence if necessary, we obtain that there exists a map $\Lambda : \cup_n K_n \to L^\infty([0, N], ds)$ such that for all $N \in \mathbb{N}$

$$\Lambda(x)|_{[0,N]} = \lim_{n\to\infty} \Lambda_N^{\alpha_n,\beta_n}(x) \text{ uniformly in } x \in K_n,\ \forall\, n \in \mathbb{N}. \tag{A.5}$$

We now show that for all $\lambda > 0$

$$\int_0^\infty e^{-\lambda s} \Lambda(\cdot)(s) ds \text{ is a } \ \nu - \text{version of } \ R(\lambda, N_2) f. \tag{A.6}$$

To prove (A.6) let $\lambda > 0$. Then by (A.4), (A.5) and dominated convergence for all $g \in L^\infty(H,\nu)$

$$\mathrm{d}s \int_0^\infty e^{-\lambda s} \int_H g(x) P_s f(x) \nu(dx) = \int_H g(x) R(\lambda, N_2) f(x) \nu(dx) ds$$

$$\mathrm{d}s = \lim_{n\to\infty} \int_H g(x) \int_0^\infty e^{-\lambda s} P_s^{\alpha_n,\beta_n} f(x) ds \; \nu(dx)$$

$$\mathrm{d}s = \int_H \int_0^\infty g(x) e^{-\lambda s} \Lambda(x)(s) ds \; \nu(dx) = \int_0^\infty e^{-\lambda s} \int_H g(x) \Lambda(x)(s) \; \nu(dx) ds,$$

where the interchange of limits is justified, since $|P_s^{\alpha_n,\beta_n} f(x)| \leq \|f\|_0$ and hence $|\Lambda(x)(s)| \leq \|f\|_0$ for ds-a.e. $s \in [0,\infty)$ and all $x \in \bigcup_n K_n$. So, (A.6) follows. Obviously, by (A.5)

$$x \mapsto \int_0^\infty e^{-\lambda s} \Lambda(x)(s) \, \mathrm{d}s$$

is continuous on each K_n, $n \in \mathbb{N}$. Hence by (A.1) in Lemma 5.6' and (A.6), (A.3) for all $f \in S$,

$$\int_0^\infty e^{-\lambda s} \Lambda(x)(s) \, \mathrm{d}s = \int_0^\infty e^{-\lambda s} \overline{p}_s f(x) \, \mathrm{d}s \quad \text{for all } x \in \bigcup_{n\in\mathbb{N}} K_n.$$

Hence by the uniqueness of the Laplace transform

$$\Lambda(x)(t) = \overline{p}_t f(x) \text{ for a.e. } t \text{ and all } x \in \bigcup_{n\in\mathbb{N}} K_n. \tag{A.7}$$

So, if $f \in S$, and $\delta_k \in C_0^\infty(\mathbb{R})$, $k \in \mathbb{N}$, approximate the identity, we obtain for all $x, y \in \bigcup_{n\in\mathbb{N}} K_n$, that for some subsequence $\{k_l\}$ and a.e. $t \in (0, N)$

$$\overline{p}_t f(x) - \overline{p}_t f(y) = \lim_{l\to\infty} \int_0^N \delta_{k_l}(t-s)(\overline{p}_s f(x) - \overline{p}_s f(y)) ds. \tag{A.8}$$

But for $l \in \mathbb{N}$ the integral in (A.8) is by (A.7) and (A.5) equal to

$$\lim_{n\to\infty} \int_0^N \delta_{k_l}(t-s)(P_s^{\alpha_n,\beta_n} f(x) - P_s^{\alpha_n,\beta_n} f(y)) ds,$$

which by [DPR02, (4.7)] is dominated by

$$\int_0^N \delta_{k_l}(t-s) s^{-1/2} ds \; \|C^{-1}\|^{1/2} \|f\|_0 |x-y|$$
$$\to t^{-1/2} \; \|C^{-1}\| \|f\|_0 |x-y|, \quad \text{as } l \to \infty.$$

Since $t \to \overline{p}_t f(x)$ is right continuous for all $x \in H_0$, [DPR02, (5.11)] follows if $f \in S$. Since S is a vector lattice containing the constants and generating $\mathcal{B}(H_0)$, it follows that S is dense in $L^2(H_0, \nu)$. Now [DPR02, (5.11)] follows for all $f \in B_b(H_0)$ and thus all $f \in B_b(H)$ by approximation since P_t is continuous on $L^2(H_0, \nu)$.

(ii). Let $f \in S$. Then [DPR02, (5.12)] follows by exactly the same arguments as above, but employing [DPR02, (4.8)] instead of [DPR02, (4.7)]. If $f \in S_0$, then $mR_m f \in S$, $m \in \mathbb{N}$, $\|mR_m f\|_0 \leq \|f\|_0$ and by [DPR02, Proposition 5.2], $\lim_{m\to\infty} mR_m f(x) = f(x)$ for all $x \in H_0$ and

$$\|mR_m f\|_{Lip} \leq \|f\|_{Lip}, \ \forall\, m \in \mathbb{N}.$$

Hence [DPR02, (5.12)] follows by approximation for $f \in S_0$. Consequently, using [DPR02, (5.8)] we can approximate again to obtain [DPR02, (5.12)] for all $f \in Lip_b(H)$. □

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