DELONE SETS WITH FINITE LOCAL COMPLEXITY: LINEAR REPETITIVITY VERSUS POSITIVITY OF WEIGHTS

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Abstract. We consider Delone sets with finite local complexity. We characterize validity of a subadditive ergodic theorem by uniform positivity of certain weights. The latter can be considered to be an averaged version of linear repetitivity. In fact, it is shown that linear repetitivity is equivalent to positivity of weights combined with a certain repulsion property of patterns and a balancedness of the shape of return patterns.

1. Introduction

Aperiodic point sets with long range order have attracted considerable attention in recent years (see e.g. the monographs and conference proceedings [1, 19, 21, 25]). On the one hand, this is due to the actual discovery of physical substances, later called quasicrystals, exhibiting such features [9, 24]. On the other hand this is due to intrinsic mathematical interest in describing the very border between crystallinity and aperiodicity. While there is no aximatic framework for aperiodic order yet, various types of order conditions in terms of local complexity functions have been studied [10, 11, 12]. Here, we are concerned with linear repetitivity. This condition is given by a linear bound on the growth rate of the repetitivity function. It has been brought forward by Lagarias/Pleasants in [11] as a characterization of perfectly ordered quasicrystals. In fact, as shown in [12], among the aperiodic repetitive Delone sets, linearly repetitive ones have the slowest possible growth rate of their repetitivity function.

Let us also note that indepentent of the work of Lagarias/Pleasants, the analogue notion for subshifts has received a thorough study in works of Durand [4, 5].

It turns out that linear repetitivity (or linear recurrence) implies validity of a subadditive ergodic theorem [3]. Such a subadditive ergodic theorem, is useful in application to Schrödinger operators and to lattice gas theory [6, 7, 8, 15]. Now, for subshifts it is possible to characterize validity of a certain subadditive ergodic theorem by positivity of certain weights as shown by one of the authors in [14]. This positivity of weights can be considered as an averaged version of linear repetitivity. This raises two questions:

Q1: Can validity of a subadditive ergodic theorem be characterized for Delone sets in terms of positivity of certain weights?

Q2: What is the relationship between the positivity of weights and linear repetitivity?

Our main results deal with these questions. Theorem 1 shows that positivity of certain weights is indeed equivalent to validity of a subadditive ergodic theorem. This answers Question 1. Theorem 2 characterizes linear repetitivity in terms of positivity of these weights combined with two further conditions. This gives an answer to Question 2. However, it is not completely satisfactory in that independence of the two further condition of positivity of weights is not shown. We consider this an interesting open question.

2. Notation and results

Fix $N \in \mathbb{N}$. We consider subsets of $\mathbb{R}^N$. The Euclidean distance on $\mathbb{R}^N$ is denoted by d. The closed ball around $p \in \mathbb{R}^N$ with radius R is denoted by $B_R(p)$. If p is the origin we just write B_R. A subset Λ of $\mathbb{R}^N$ is called uniformly discrete if its packing radius.

$$r_{\text{pack}}(\Lambda) := \inf\Big\{\frac{1}{2}\, d(x,y) : x, y \in \Lambda, x \neq y\Big\}$$

is positive. A subset Λ is called relatively dense if its covering radius.

$$r_{\text{cov}}(\Lambda) := \sup\{d(p,\Lambda) : p \in \mathbb{R}^N\}$$

is finite. A set which is both relatively dense and uniformly discrete is called a Delone set. A ball pattern P in Λ of radius $R = r(P)$ is a set of the form

$$(\Lambda - x) \cap B_R$$

for some $x \in \Lambda$. A Delone set is said to have finite local complexity if for any $R > 0$, there exist only finitely many ball patterns. For a Delone set Λ and a ball pattern P, the locater set L_P^Λ of P in Λ is defined by

$$L_P^\Lambda := \{x \in \Lambda : (\Lambda - x) \cap B_{r(P)} = P\}.$$

If the Delone set Λ is clear from the context we write only L_P. A Delone set Λ with finite local complexity is said to be repetitive if the locator set of any ball pattern is relatively dense. A Delone set Λ is said to be non-periodic if

$$\Lambda - x \neq \Lambda$$

for any $x \neq 0$.

We will deal exclusively with repetitive Delone sets of finite local complexity.

For a ball pattern P of Λ and $Q \subset \mathbb{R}^N$ we define the number of copies of P in Q by

$$\sharp_P Q := \sharp Q \cap L_P^\Lambda.$$

Here, $\sharp$ denotes the cardinality. We will also need the maximal number of disjoint copies. This is defined by

$$\sharp'_P Q := \max\{\sharp A : A \subset Q \cap L_P^\Lambda, r_{\text{pack}}(A) \geq r(P)\}.$$

Let $|\cdot|$ denotes the Lebesgue measure. A sequence (Q_n) of subsets of $\mathbb{R}^N$ is called van Hove sequence if

$$\lim_{n\to\infty} \frac{|\partial^R Q_n|}{|Q_n|} = 0$$

for any $R > 0$. Here, $\partial^R Q$ is the set of all points in R^N whose distance to the boundary of Q does not exceed R. A van Hove sequence is called cube like it there exists a sequence of cubes C_n and a $\delta > 0$ with $Q_n \subset C_n$ and $|Q_n| \geq \delta |C_n|$ for all $n \in \mathbb{N}$. A Delone set Λ is said to satisfy positivity of weights (PQ) if there exists a $w > 0$ with

$$\liminf_{n\to\infty} \frac{\sharp'_P Q_n}{|Q_n|} |P| \geq w \tag{PQ}$$

for any ball pattern P and any van Hove sequence (Q_n). A function

$$F\colon \{\text{Bounded measurable Borel sets in } \mathbb{R}^N\} \longrightarrow \mathbb{R}$$

is called subadditive if

$$F(Q_1 \cup Q_2) \leq F(Q_1) + F(Q_2),$$

whenever Q_1 and Q_2 are disjoint. Such a function F is said to be $\varLambda$-invariant if

$$F(Q) = F(t+Q) \quad \text{whenever} \quad t + Q \cap \varLambda = (t+Q) \cap \varLambda.$$

The Delone set $\varLambda$ is said to satisfy a subadditive ergodic theorem (SET) if the limit

$$\text{(SET)} \qquad \lim_{n\to\infty} \frac{F(Q_n)}{|Q_n|} \text{ exists}$$

for any subadditive invariant function F and any cube like sequence (Q_n).

Theorem 1. *Let $\varLambda$ be a repetitive non-periodic Delone set. Then* (SET) *is equivalent to* (PQ).

The Theorem has the following corollary.

Corollary 1. *If $\varLambda$ satisfies* (PQ)*, then the frequency*

$$\lim_{n\to\infty} \frac{\sharp_P(C_n)}{|C_n|}$$

exists for any ball pattern P along any sequence (C_n) of cubes with sidelength going to ∞.

Remark. The corollary implies that the dynamical system associated to a Λ satisfying (PQ) is uniquely ergodic [13, 18].

We will be concerned with further combinatorial quantities associated to Delone sets of finite local complexity. These quantities will be discussed next.

A Delone set $\varLambda$ is called linearly repetitive (LR) if there exists an $C > 0$ with

$$\text{(LR)} \qquad r_{\text{cov}}(L_P^{\varLambda}) \leq C\, r(P)$$

for any ball pattern P. A Delone set is said to satisfy the repulsion property (RP) if there exists a $c > 0$ with

$$\text{(RP)} \qquad r_{\text{pack}}(L_P^{\varLambda}) \geq c\, r(P)$$

for any ball pattern P. We finally need the notion of return pattern. For subshifts an intense study of this notion has been carried out recently in work of Durand [4, 5] . The analogue for Delone sets (or rather tilings) has then been investigated by Priebe [22] (see e.g. [23, 17] as well).

Given a Delone set Γ we define the Voronoi cell of an $x \in \Gamma$ by

$$V_x(\Gamma) := \{p \in R^N : d(p,x) \leq d(p,y) \text{ for all } y \in \Gamma\}.$$

Given a ball pattern P of the repetitive $\varLambda$, we thus obtain the collection

$$V_x(L_P),\ x \in L_P,$$

of Voronoi cells. By construction, these Voronoi cells are convex polytopes in $\mathbb{R}^N$. We define inner and outer radius $r_{\text{in}}(V_x)$ and $r_{\text{out}}(V_x)$ respectively of a Voronoi cell by

$$r_{\text{in}}(V_x) := \sup\{R > 0 : B_R(x) \subset V_x\}, \quad r_{\text{out}}(V_x) := \inf\{R > 0 : V_x \subset B_R(x)\}.$$

The Delone set Λ is said to satisfy uniformity of return words (U) it there exists an $\sigma > 0$ such that for any ball pattern P and any $x \in L_P$

$$r_{\rm out}(V_x) \leq \sigma\, r_{\rm in}(V_x). \tag{U}$$

Loosely speaking this means that the Voronoi cells are ball-like.

Theorem 2. *Let Λ be a non-periodic Delone set. Then, the following assertions are equivalent:*

(i) *The set Λ satisfies (LR).*
(ii) *The set Λ satisfies (PQ) and (U).*

We finish this section by discussing analogues of our results in the symbolic dynamics setting. Let $\mathcal{A}$ be a finite set. We will consider finite and two sided infinite words over $\mathcal{A}$ (see [20]). The length $|v|$ of a finite word $v = v_1 \cdots v_n$, $v_j \in \mathcal{A}$, is given by $|v| = n$. The number $\sharp_v w$ of copies of the finite word v in $w = w_1 \cdots v_m$ is given by

$$\sharp_v w := \sharp\{j : w_j \cdots w_{j+|v|-1} = v\}.$$

A finite word v is said to occur in the infinite word $\omega : \mathbb{Z} \longrightarrow \mathcal{A}$ if there exists a $j \in \mathbb{Z}$ with $\omega(j) \cdots \omega(j + |v| - 1) = v$. The set of all finite words occuring in ω is denoted by $W(\omega)$. An infinite word $\omega : \mathbb{Z} \longrightarrow \mathcal{A}$ is called linearly repetitive if there exists a $C > 0$ with

$$\sharp_v \omega(j) \cdots \omega(j + C|v| - 1) \geq 1$$

for any $v \in W(\omega)$. In the symbolic dynamical setting the obvious analog of (PQ) is actually equivalent to existence of a $c > 0$ with

$$\liminf_{|w|\to\infty} \frac{\sharp_v w}{|w|} |P| \geq c, \quad \text{for all } v \in W(\omega), \tag{PW}$$

where the lim inf is taken over words w occuring in ω (see [14] for details). A function $F\colon W(\omega) \longrightarrow \mathbb{R}$ is called subadditive if $F(vw) \leq F(v) + F(w)$, whenever $vw \in W(\omega)$ and (SET) now means existence of $\lim_{|w|\to\infty} F(w)/|w|$ for all subadditive functions. Now, one of the main results of [14] asserts the equivalence of (PW) and (SET) for symbolic dynamics. Our Theorem 1 above is thus the analogue of that result. Its proof is modelled after the proof in [14].

For the symbolic dynamical case the condition (U) becomes unnecessary. Thus, Theorem 2 has the following subshift analogue.

Theorem 3. *Let $\mathcal{A}$ be a finite set and ω an infinite word over $\mathcal{A}$. Then, the following assertions are equivalent:*

(i) *ω is linearly repetitive.*
(ii) *ω satisfies* (PW).

In Section 3, we provide a proof for Theorem 1. Section 4 deals with Theorem 2. We do not provide a separate proof of Theorem 3. We will just explain why the condition (U) becames unnecessary in this case.

3. A subadditive ergodic theorem

This section is concerned with Theorem 1 and its corollary. The proofs are rather similar to already existing proofs of related results. Therefore, we only shortly sketch them.

Proof of Theorem 1. The analogue result in the setting of symbolic dynamics is given in [14]. Here, we indicate the necessary modifications.

(PQ) $\Longrightarrow$ (SET): The proof given in [14] can be easily carried over to show that (PQ) implies convergence of $F(C_n)/|C_n|$ for subadditive F along sequences (C_n) consisting of cubes with sidelength going to ∞. (A similar reasoning can in fact also be found in the proof of (LR) $\Longrightarrow$ (SET) given in [3].) The extension to cube-like sequence (Q_n) can be performed as in [6].

(SET) $\Longrightarrow$ (PQ): Assume the contrary, i.e. assume that (SET) holds but (PQ) fails.

By dividing elements of a van Hove sequence (Q_n) into cubes, it is not hard to see that (PQ) is actually equivalent to existence of $w > 0$ with

$$\liminf_{n\to\infty} \frac{\sharp'_P C_n}{|C_n|}|P| \geq w$$

for any ball pattern P and any sequence (C_n) of cubes. The sidelength of a cube C is denoted by $s(C)$. For each ball pattern P let

$$\nu_P := \liminf_{s(C)\to\infty} \frac{\sharp'_P C}{|C|}|P|.$$

As, by assumption, (PQ) does not hold, there exists a sequence (P_n) of ball patterns with

$$\nu_{P_n} \longrightarrow 0, \;\; n \to \infty. \tag{1}$$

By repetitivity, each ν_P is positive. Thus, we can conlude from finite local complexity that

$$r(P_n) \longrightarrow \infty, \;\; n \to \infty. \tag{2}$$

Define the real valued function l_n on subsets of $\mathbb{R}^N$ by

$$l_n(Q) := \widetilde{\sharp}_{P_n}(Q)|P_n|.$$

Here, $\widetilde{\sharp}_P(Q)$ denotes the maximal number of disjoint copies of P, which are contained in Q. Obviously, $-l_n$ is subadditive and hence

$$\lim_{s(C)\to\infty} \frac{l_n(C)}{|C|} = \nu(P_n). \tag{3}$$

Set κ to be the ratio between the volume of a ball with radius 1 and a cube with sidelength 2. Then, (1) and (2) can be used to construct inductively a subsequence $(P_{n(k)})$ of (P_n) with $r(P_{n(k+1)}) > r(P_{n(k)})$ and

$$\frac{1}{C}\sum_{j=1}^{k} l_{n(j)}(C) < \frac{1}{2}\kappa$$

for any cube C with sidelength $s(C) \geq \frac{1}{2}r(P_{n(k+1)})$. Define the function l on the bounded measurable subsets of $\mathbb{R}^N$ by

$$l(Q) := \sum_{j=1}^{\infty} l_{n(j)}(Q).$$

Note that by (2), only finitely many terms in this sum do not vanish. As each $-l_n$ is subadditive, so is $-l$. Therefore, the limit

$$\overline{l} := \lim_{s(C)\to\infty} \frac{l(C)}{|C|}$$

exists by assumption (SET). On the other hand we have for any cube C with $\frac{1}{2}r(P_{n(k+1)}) \leq s(C)/2 < r(P_{n(k+1)})$

$$\frac{l(C)}{|C|} = \sum_{j=1}^{\infty} \frac{l_{n(j)}(C)}{|C|} = \sum_{j=1}^{k} \frac{l_{n(j)}(C)}{|C|} \leq \frac{1}{2}\kappa,$$

whereas a cube C with sidelength $2r(P_{n(k)})$ with center of mass $x \in L_{P_{n(k)}}$ satisfies

$$\frac{l(C)}{|C|} \geq \frac{l_{n(k)}(C)}{|C|} \geq \frac{|P_{n(k)}|}{|C|} \geq \kappa.$$

These inequalities give the contradiction

$$\kappa \leq \overline{l} < \frac{1}{2}\kappa.$$

This finishes the proof of Theorem 1. □

Proof of Corollary 1. Note that for any ball pattern P the function

$$F_P(Q) := -\sharp_P(Q)$$

is subadditive. Hence, the corollary follows from the theorem. □

4. linear repetitive Delone sets

In this section we give a proof for Theorem 2. We start with some preparation.

Proposition 4.1. *Let Λ be a repetitive non-periodic Delone set satsfying* (LR) *then it satisfies* (RP).

Proof. For linearly repetitive tilings this (and in fact a slightly stronger result) is proven in Lemma 2.4 of [26]. That proof carries easily over to Delone sets, as discussed in Lemma 2.1 in [14] (see [12] and [5] as well). □

Proposition 4.2. *Let Λ be a repetitive non-periodic Delone set satsfying* (LR) *then it satisfies* (U).

Proof. By Proposition 4.1, Λ satisfies (RP) with constant $c > 0$. For $x \in L_P$ we have

$$B\big(x, r_{\text{pack}}(L_P)\big) \subset V_x \text{ and } V_x \subset B\big(x, r_{\text{cov}}(L_P)\big),$$

see Corollary 5.2 in [25] for details. Hence we have

$$r_{\text{in}}(V_x) \geq r_{\text{pack}}(L_P) \geq c\ r(P)$$

and

$$r_{\text{out}}(V_x) \leq r_{\text{cov}}(L_P) \leq C\ r(P) \leq \frac{C}{c} r_{\text{in}}(V_x).$$

Taking $\sigma = \frac{C}{c}$ we obtain the desired result. □

Proposition 4.3. *Let Λ be a repetitive non-periodic Delone set satsfying* (LR) *then it satisfies* (PQ).

Proof. By definition, (LR) implies that a pattern P of size R occurs in every boxes with sidelength CR. We can now partition Q_n into cubes of sidelength $3CR$ up to the boundary of Q_n. Each of these boxes contains a cube of sidelength CR "in the middle", i.e., with distance CR to the boundary. Choosing a copy of P in each of these middle cubes, we easily obtain the statement. □

Lemma 4.4. *Let $m > n$ two integers then we have*

$$\sum_{k=n}^{m} \frac{1}{k} \geq \ln\Big(\frac{m}{n-1}\Big).$$

Proof. As the function $f(x) = \frac{1}{x}$ is decreasing on the the interval $]0, +\infty[$ then we get

$$\sum_{k=n}^{m} \frac{1}{k} \geq \int_{n-1}^{m} \frac{1}{x} dx = \ln \frac{m}{n-1}\,. \qquad \square$$

Proof of Theorem 2. Propositions 4.2, 4.3 show that a linearly repetitive non-periodic Delone set satisfies also (PQ) and (U).

It remains to show the inverse implication. Let Λ be a repetitive non-periodic Delone set satsfying (PQ) and (U). Consider a patch P which occurs in Λ, let $r(P) = R$. Let $x \in L_P^\Lambda$, and x' the nearest point to x in L_P^Λ. Let $\|x - x'\| = d$. For $t \in \mathbb{R}$ let $\mathsf{E}(t)$ denotes the greatest integer smaller than t. We have to consider the two following cases.

Case 1: $d \leq 4R$.

Case 2: $d > 4R$. Then consider the following patches

$$\begin{aligned} P_1 &= (\Lambda - x) \cap B_{3(\mathsf{E}(R)+1)}(x)\\ P_2 &= (\Lambda - x) \cap B_{3(\mathsf{E}(R)+2)}(x)\\ &\vdots\\ P_k &= (\Lambda - x) \cap B_{3(\mathsf{E}(R)+k)}(x)\\ &\vdots\\ P_{\mathsf{E}(\frac{d}{3})-\mathsf{E}(R)} &= (\Lambda - x) \cap B_{3(\mathsf{E}(\frac{d}{3}))}(x) \end{aligned}$$

For $i, j \in \{1, \dots,\ \mathsf{E}(\frac{d}{3}) - \mathsf{E}(R)\}$, if $i \neq j$ and $x \in L_{P_i} \cap L_{P_j}$ then it is clear that the sets $B_{\mathsf{E}(R)+i}(x) \setminus B_{\mathsf{E}(R)+i-1}(x)$ and $B_{\mathsf{E}(R)+j}(x) \setminus B_{\mathsf{E}(R)+j-1}(j)$ are disjoints.

Now where $i \geq j$, for arbitrary $x \in L_{P_i}$ and $y \in L_{P_j}$. As $L_{P_j} \subset L_P$ and there is no point of $L_P \setminus \{x\}$ in the ball $B_{2(\mathsf{E}(R)+i)}(x)$, then $\|y - x\| > 2(\mathsf{E}(R) + i)$. Thus, as $i \geq j$ the sets $B_{\mathsf{E}(R)+i}(x) \setminus B_{\mathsf{E}(R)+i-1}(x)$ and $B_{\mathsf{E}(R)+j}(y) \setminus B_{\mathsf{E}(R)+j-1}(y)$ are disjoints. Let (Q_n) be a Van Hove sequence, considering the last two facts we infer that for $n \in \mathbb{N}$

$$1 \geq \sum_{k=1}^{\mathsf{E}(\frac{d}{2})-\mathsf{E}(R)} \frac{\sharp_{P_k}(Q_{n,\mathsf{E}(\frac{d}{2})})}{|Q_n|} \big|B_{\mathsf{E}(R)+k} \setminus B_{\mathsf{E}(R)+k-1}\big|.$$

Using (PQ) when $n \to \infty$ we obtain

$$(4) \qquad 1 \geq \sum_{k=1}^{\mathsf{E}(\frac{d}{2})-\mathsf{E}(R)} \frac{w}{|P_k|} |B_{\mathsf{E}(R)+k} \setminus B_{\mathsf{E}(R)+k-1}| \geq \frac{w}{3^N} \sum_{s=\mathsf{E}(R)+1}^{\mathsf{E}(\frac{d}{2})} \frac{s^N - (s-1)^N}{s^N}.$$

Now as $\frac{s^N-(s-1)^N}{s^N} = 1-(1-1/s)^N = f(1) - f(1-1/s)$ where f is the function defined by $f(x) = x^N$. Now by the mean value theorem we have

$$f(1) - f(1-1/s) = \frac{1}{s} f'(t)$$

where t is some point between $1-1/s$ and 1, and f' is the derivative. Then for $s \geq 2$, we have $1/2 \leq t \leq 1$ and

$$\frac{s^N - (s-1)^N}{s^N} \geq \frac{1}{s} \frac{N}{2^{N-1}}.$$

Now by (4) and Lemma 4.4 we obtain

$$1 \geq \frac{w}{3^N} \frac{N}{2^{N-1}} \sum_{s=\mathsf{E}(R)+1}^{\mathsf{E}(\frac{d}{3})} \frac{1}{s} \geq \frac{2wN}{6^N} \ln \left(\frac{\mathsf{E}(\frac{d}{3})}{\mathsf{E}(R)} \right).$$

This gives us more or less

$$(5) \qquad 6 \exp\left(\frac{6^N}{2wN}\right) R \geq d$$

Set $c = \max\left(6 \exp\left(\frac{6^N}{2wN}\right), 4\right)$. Now for arbitrary $t \in \mathbb{R}^N$ let $x \in L_P$ such that $t \in V_x$, by (5) we infer that

$$r_{\text{in}}(V_x) \leq \frac{c}{2} R.$$

Now by (U) we have

$$r_{\text{out}}(V_x) \leq \frac{\sigma c}{2} R.$$

The ball of radius $r_{\text{out}}(V_x)$ around t contains x that's why

$$r_{\text{cov}}(L_P) \leq \frac{\sigma c}{2} R, \text{ whenever } R \geq 2.$$

Taking $C = \frac{\sigma c}{2}$ ends the proof. □

For the proof of Theorem 3, we can do the same strategy by replacing the nearest copy by the next copy and we show then that the distance between two successive copies of some word u is bounded by $C|u|$ for some constant C which not depend on u. This gives us the linear repetitivity.

Acknowledgements. This work was partially supported by the German Science Foundation (DFG). Part of it was done while one of the authors A.B. was visiting Chemnitz. He would like to thank the department for hospitality.

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