Stochastic nonlinear diffusion equations with singular diffusivity

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Abstract

This paper is concerned with the stochastic diffusion equation $dX(t) = \mathrm{div}[\mathrm{sgn}(\nabla(X(t))]dt + \sqrt{Q}\ dW(t)$ in $(0,\infty)\times\mathscr{O}$ where $\mathscr{O}$ is a bounded open subset of $\mathbb{R}^d$, $d = 1, 2$, $W(t)$ is a cylindrical Wiener process on $L^2(\mathscr{O})$ and $\mathrm{sgn}(\nabla X) = \nabla X/|\nabla X|_d$ if $\nabla X \neq 0$ and sgn $(0) = \{v \in \mathbb{R}^d : |v|_d \leq 1\}$. The multivalued and highly singular diffusivity term $\mathrm{sgn}(\nabla X)$ describes interaction phenomena and the solution $X = X(t)$ might be viewed as the stochastic flow generated by the gradient of the total variation $\|DX\|$. Our main result says

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that this problem is well posed in the space of processes with bounded variation in the spatial variable ξ. The above equation is relevant for modeling crystal growth as well as for total variation based techniques in image restoration.

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1 Introduction

We are concerned here with the following stochastic diffusion equation on $H = L^2(\mathscr{O})$

$$\begin{cases} dX(t) = \text{div}[\text{sgn}\ (\nabla(X(t))]dt + \sqrt{Q}\, dW(t) \text{ in } (0,\infty)\times\mathscr{O} \\ \\ X(t) = 0 \quad \text{on } \partial\mathscr{O}\times(0,T) \\ \\ X(0) = x \quad \text{in } \mathscr{O}, \end{cases} \tag{1.1}$$

where $\mathscr{O}$ is a bounded open subset of $\mathbb{R}^d$, $d = 1,2$, $W(t)$ is a cylindrical Wiener process on $L^2(\mathscr{O})$ of the form

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t)e_k, \quad t \geq 0,$$

where $\{\beta_k\}$ is a sequence of mutually independent real Brownian motions on a filtered probability spaces $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ (see [8]) and $\{e_k\}$ is an orthonormal basis in $H = L^2(\mathscr{O})$. We set

$$A = -\Delta, \quad D(A) = H^2(\mathscr{O}) \cap H_0^1(\mathscr{O}). \tag{1.2}$$

The operator $Q \in L(H)$ is symmetric, self-adjoint, nonnegative. For simplicity we assume that A and Q have a common eigenbasis e_n, $n \in \mathbb{N}$, with eigenvalues λ_n and μ_n, $n \in \mathbb{N}$, respectively. We shall assume that

$$\sum_{n=1}^{\infty} \lambda_n^{1+\kappa}\mu_n < \infty, \tag{1.3}$$

for some $\kappa > 0$. For example, we can take $Q = A^{-1-\delta}$ where $\delta > \frac{1}{2} + \kappa$ if $d = 1$, $\delta > 1 + \kappa$ if $d = 2$ and

The multi-valued function $u \to \text{sgn } u$ from $\mathbb{R}^d$ into $2^{\mathbb{R}^d}$ is defined by

$$\text{sgn } u = \begin{cases} \frac{u}{|u|_d}, & \text{if } u \neq 0, \\ \\ \{v \in \mathbb{R}^d : \ |v|_d \leq 1\}, & \text{if } u = 0. \end{cases}$$

(Here $|\cdot|_d$ is the Euclidean norm and $\langle \cdot, \cdot \rangle_d$ is the Euclidean inner product.)

Equation (1.1) is relevant in material science to describe the motion of grain boundaries and in image processing. The first model is concerned with facet growth of cristals derived from the spatially homogeneous energy

$$E(X) = \int_{\mathscr{O}} |\nabla X|_d \, d\xi,$$

which formally leads to the gradient system

$$dX(t) = -\text{div} \left(\frac{\nabla X(t)}{|\nabla X(t)|_d} \right) dt, \quad t \geq 0, \tag{1.4}$$

or to (1.1) in presence of the Gaussian perturbation $\sqrt{Q} \ dW$. (We refer to [11], [12], [17] for the presentation and treatment of the corresponding deterministic models.)

The total variation based image restoration model based on $E(X)$ has been proposed in [19] (see also [7], [8], [16], [17]), i.e., as the solution to the minimization problem,

$$\min \int_{\mathscr{O}} \left(|\nabla X|_d + \frac{1}{2} \, |X - f|^2 \right) d\xi, \tag{1.5}$$

where f is the given image and X is the restored image. The minimization problem (1.5) leads to a flow $X = X(t)$ generated by the evolution equation

$$dX(t) = -\text{div} \left(\frac{\nabla X(t)}{|\nabla X(t)|_d} \right) dt - (X(t) - f(t))dt, \quad t \geq 0, \tag{1.6}$$

which perturbed by a Gaussian process leads to equation (1.1). This restoration model was designed with the explicit aim to preserve edges and sharp discontinuities of the image.

In both equations (1.4) and (1.6) the discontinuous map $u \to \frac{u}{|u|_d}$ should be replaced of course by its multi-valued maximal monotone graph $u \to$ sign u obtained by filling the jumps. It should also be mentioned that equation (1.1) (as well as the deterministic version (1.4) or (1.6)) is highly nonlinear. For instance in 1-D equation (1.1) has the form

$$dX(t) = -\delta(\nabla X(t)) \Delta X(t) dt + \sqrt{Q} \ dW(t), \quad t \geq 0, \tag{1.7}$$

where δ is the Dirac measure at zero on $\mathscr{O}$. Of course this is only a formal representation because the multiplier $\delta(\nabla X(t))$ is not well defined and so (1.7) does not make sense.

These equations derived from the mathematical description of diffusion phenomena with non differentiable energy are modeling non local interactions via singular diffusivity (see [12]).

The main result established here (see Theorem 3.2 below) is concerned, however, with existence and uniqueness of a variational solution for $d = 1, 2$ in the space of functions with bounded variation in the spatial variable $\xi \in \mathscr{O}$. A similar result is proved in Section 6 for equation (1.1) with linear multiplicative noise along with positivity of solutions.

It should be noted that, though equation (1.1) arises in a variational setting, its existence theory is not covered by the classical results of E. Pardoux [14] or N. Krylov and B. Rozovskii [13] (see also [15], [18]). Indeed, a general stochastic equation of the form

$$\begin{cases} dX(t) = \text{div}\,(a(\nabla(X(t)))dt + \sqrt{Q}\,dW(t) \text{ in } (0,\infty)\times\mathscr{O} \\ X(t) = 0 \quad \text{on } \partial\mathscr{O}\times(0,T) \\ X(0) = x \quad \text{in } \mathscr{O}, \end{cases} \tag{1.8}$$

where $a : \mathbb{R}^d \to \mathbb{R}^d$ is a monotonically increasing, continuous and coercive vector field with polynomial growth can be solved in the abstract variational setting

$$\begin{cases} dX(t) + \tilde{A}X(t)dt = \sqrt{Q}\,dW(t) \\ X(0) = x \end{cases} \tag{1.9}$$

where $\tilde{A} : V \to V'$ is a nonlinear monotone and demi-continuous operator (see [2]) such that

$$(\tilde{A}x, x) \geq \omega\|x\|_V^p - \omega_1|x|_H^2,$$
$$\|\tilde{A}x\|_{V'} \leq C_1\|x\|_V^{p-1} + C_2,$$

where $\omega > 0$, $p > 1$ and $\omega_1, C_1, C_2 \in \mathbb{R}$. (Here $V \subset H \subset V'$ is a classical variational Gelfand triple.)

This is exactly the variational stochastic framework developed in [14], [13], which however does not apply in this situation. As a matter of fact, the situation considered here is a limit case of (1.8)-(1.9) and this fact will be exploited later to obtain existence of solutions for (1.1).

Notations. Everywhere in the following H is the Hilbert space $L^2(\mathscr{O})$ with the scalar product $(\cdot,\cdot)$ and the norm $|\cdot|)$. $L^p(\mathscr{O})$, $p \geq 1$ and $W_0^{1,p}(\mathscr{O})$ are the standard spaces of integrable functions and Sobolev spaces on $\mathscr{O}$

with Dirichlet boundary conditions. We set $H_0^1(\mathscr{O}) := W_0^{1,2}(\mathscr{O})$ and by $L^2_W(0,T;H)$ (resp. $C_W([0,T];H)$) we shall denote the space of all square integrable (resp. all continuous) functions from $[0,T]$ to $L^2(\Omega;H)$ which are adapted to $\{\mathscr{F}_t\}_{t\geq 0}$. The spatial variables in $\mathscr{O}$ are denoted by ξ.

2 Preliminaries

Here we shall recall a few standard results on functions of bounded variation on $\mathscr{O}$ for later use. A function $f \in L^1(\mathscr{O})$ is said to be of bounded variation on $\mathscr{O}$ if

$$\|Df\| := \sup\left\{\int_{\mathscr{O}} f \text{ div } \psi \, d\xi : \ \psi \in C_0^\infty(\mathscr{O};\mathbb{R}^d), \ |\psi|_\infty \leq 1\right\} < +\infty. \quad (2.1)$$

In the following we shall denote the gradient of f in the sense of distributions by Df, which by (2.1) is a vector valued measure of bounded variation.

The space of all functions of bounded variation on $\mathscr{O}$ will be denoted by $BV(\mathscr{O})$. It is a Banach space with the norm

$$\|f\|_{BV(\mathscr{O})} = |f|_{L^1(\mathscr{O})} + \|Df\|.$$

Let $f \in BV(\mathscr{O})$. Then there is a Radon measure μ_f on $\overline{\mathscr{O}}$ and a μ_f-measurable function $\sigma_f : \mathscr{O} \to \mathbb{R}^d$ such that $|\sigma_f(x)| = 1$, μ_f a.e. and

$$\int_{\mathscr{O}} f \text{ div } \psi \, d\xi = -\int_{\mathscr{O}} \psi \cdot \sigma_f \, d\mu_f, \quad \forall\, \psi \in C_0^1(\mathscr{O};\mathbb{R}^d). \quad (2.2)$$

For each $f \in BV(\mathscr{O})$ there is the trace $\gamma(f)$ on $\partial\mathscr{O}$ (assumed sufficiently smooth) defined by

$$\int_{\mathscr{O}} f \text{ div } \psi \, d\xi = -\int_{\mathscr{O}} \psi \cdot \sigma_f \, d\mu_f + \int_{\partial\mathscr{O}} \gamma(f)\psi \cdot \nu \, dH^{d-1}, \quad \forall\, \psi \in C^1(\overline{\mathscr{O}};\mathbb{R}^d), \quad (2.3)$$

where ν is the outward normal and dH^{d-1} is the Hausdorff measure on $\partial\mathscr{O}$. We have that $|\gamma(f)|_d \in L^1(\partial\mathscr{O}; H^{d-1})$ (See [1]).

In the following we shall denote by $BV^0(\mathscr{O})$ the space of all $BV(\mathscr{O})$ functions with vanishing trace on $\partial\mathscr{O}$. By the Poincaré inequality it follows that on $BV^0(\mathscr{O})$, $\|Df\|$ is norm equivalent with $\|f\|_{BV^0(\mathscr{O})}$.

Consider the function $\Phi : L^1(\mathscr{O}) \to \bar{R} = (-\infty, +\infty]$

$$\Phi(x) = \begin{cases} \|Dx\| & \text{if } x \in BV^0(\mathscr{O}), \\ +\infty & \text{if } x \in L^1(\mathscr{O}) \setminus BV^0(\mathscr{O}) \end{cases} \quad (2.4)$$

Obviously, Φ is convex on $BV^0(\mathscr{O})$ and lower semicontinuous on $L^1(\mathscr{O})$, hence on every $L^p(\mathscr{O})$, $p \geq 1$.

Lemma 2.1 *Assume that* $d = 1$ *or* 2. *Then*

$$BV(\mathscr{O}) \subset L^{\frac{d}{d-1}}(\mathscr{O}) \tag{2.5}$$

compactly.

Proof. See e.g. [1, Corollary 3.49]. □

By $\partial\Phi : L^2(\mathscr{O}) \to L^2(\mathscr{O})$ we shall denote the sub-differential of Φ, i.e.,

$$\partial\Phi(x) \;=\; \left\{\eta \in L^2(\mathscr{O}) : \; \Phi(x) - \Phi(y) \leq \int_{\mathscr{O}} \eta(x-y)d\xi, \; \forall\, y \in \; BV^0(\mathscr{O})\right\},$$

$$\forall\, x \in BV^0(\mathscr{O}).$$

It is clear that if $x \in W_0^{1,1}(\mathscr{O})$ then $x \in BV^0(\mathscr{O})$ and $\|Dx\| = |\nabla x|_{L^1(\mathscr{O})}$ and so if

$$\eta = -\text{div}\left(\frac{\nabla x}{|\nabla x|_d}\right) \in L^2(\mathscr{O}), \tag{2.6}$$

then $\eta \in \partial\Phi(x)$. A precise description of $\partial\Phi$ is hard to get. We note, however, that according to the general theory of sub-differential mappings (see e.g. [2], [6]) the domain

$$D(\partial\Phi) = \{x \in BV^0(\mathscr{O}) : \; \partial\Phi(x) \neq \varnothing\}$$

of $\partial\Phi$ is dense in

$$D(\Phi) = \{x \in L^2(\mathscr{O}) : \; \Phi(x) < \infty\} = BV^0(\mathscr{O}).$$

Remark 2.2 With the above notation and because of (2.6) we can (informally) rewrite (1.1) as

$$\begin{cases} dX(t) + \partial\Phi(X(t))dt = \sqrt{Q}\, dW(t) \text{ in } (0,\infty) \times \mathscr{O} \\ X(t) = 0 \quad \text{on } \partial\mathscr{O} \times (0,T) \\ X(0) = x \quad \text{in } \mathscr{O}. \end{cases} \tag{2.7}$$

3 Definition of a strong solution to equation (1.1) and the main result

Definition 3.1 *A stochastic process $X = X(t,x)$ with $\mathbb{P}$-a.s. continuous sample paths in H is said to be a strong solution to equation* (1.1) *if*

$$X \in C_W([0,T];H) \cap L^1((0,T)\times\Omega, BV^0(\mathscr{O})), \quad X(0) = x \in H$$

and

$$\begin{aligned}&\frac{1}{2}\,|X(t)-Y(t)|^2 + \int_0^t (\Phi(X(s)) - \Phi(Y(s)))ds \\ &\leq \frac{1}{2}\,|x - Y(0)|^2 + \int_0^t (G(s), X(s) - Y(s))ds, \quad t \in [0,T],\end{aligned} \tag{3.1}$$

for all $G \in L^2_W(0,T;H)$ and $Y \in C_W([0,T];H) \cap L^1((0,T)\times\Omega; BV^0(\mathscr{O}))$ satisfying the equation

$$dY(t) + G(t)dt = \sqrt{Q}\, dW(t), \quad t \in [0,T]. \tag{3.2}$$

We recall that $H = L^2(\mathscr{O})$ and $(\cdot,\cdot)$ is its scalar product. Definition 3.1 generalizes the usual definition of solution, since if $\partial\Phi$ is regular, by Itô's formula a solution of (2.7), satisfies (3.1). In this sense equation (3.1) is a variational version of problem (2.7). The main point is, of course, to show uniqueness for the process X satisfying (3.1) which we shall do below in our case.

Definition 3.1 resembles the classical definition of a mild (integral) solution to deterministic variational inequalities (see e.g. [2],[6]) and in a slightly different version it was used in [18], but in a different context.

Theorem 3.2 below is the main result of this paper.

Theorem 3.2 *Assume that $d = 1$ or $d = 2$. Then there is a unique strong solution*

$$X \in C_W([0,T];H) \cap L^1((0,T)\times\Omega; BV^0(\mathscr{O})),$$

to equation (1.1) *for each $x \in H$. Furthermore, for all $x, y \in H$, $T > 0$*

$$\sup_{t\in[0,T]} |X(t,x) - X(t,y)| \leq |x-y|. \tag{3.3}$$

A similar result in the case of equation (1.1) with linear multiplicative noise holds (see Theorem 6.2 below).

The solution X is the limit in $C_W([0,T];H)$ of solutions X_ϵ to an approximative diffusion equation with diffusivity β_ϵ which is a Lipschitz continuous approximation of the sign function as a multi-valued function. In particular; this implies that the solution X in Definition 3.1 keeps most of the nice features of the approximating solution X_ϵ (for instance a Lipschitz dependence on initial data or positivity in the case of linear multiplicative noise.)

Remark 3.3 As it will become clear from the proof, Theorem 3.2 remains true for equation (1.1) with Neumann boundary conditions

$$\frac{\nabla X(t)}{|\nabla X(t)|_d} \cdot n(X(t)) = 0 \quad \text{on } \partial\mathscr{O}, \tag{3.4}$$

where $n(X(t))$ is the normal to $\partial\mathscr{O}$. Definition 3.1 is the same in this case except that the space $BV^0(\mathscr{O})$ must be replaced by $BV(\mathscr{O})$.

4 Proof of Theorem 3.2

We start with the approximating equation

$$\begin{cases} dX_\epsilon(t) = -A^\epsilon X_\epsilon(t)dt + \sqrt{Q}\, dW(t), \\ X_\epsilon(0) = x \quad \text{in } \mathscr{O}, \end{cases} \tag{4.1}$$

where

$$\beta_\epsilon(u) = \begin{cases} \dfrac{u}{\epsilon}, & \text{if } |u|_d \le \epsilon, \\ \\ \dfrac{u}{|u|_d}, & \text{if } |u|_d > \epsilon, \end{cases}$$

and the operator $A^\epsilon : H \to H$ is defined by

$$A^\epsilon u = -(1+\epsilon A)^{-1} \operatorname{div}[\beta_\epsilon(\nabla(1+\epsilon A)^{-1}u)], \quad \forall\, u \in H. \tag{4.2}$$

where (cf. (1.2))

$$A = -\Delta, \quad D(A) = H^2(\mathscr{O}) \cap H_0^1(\mathscr{O}).$$

It is easily seen that A^ϵ is Lipschitzian in H.

We see that $\beta_\epsilon = \nabla j_\epsilon$ where

$$j_\epsilon(u) = \begin{cases} \dfrac{|u|_d^2}{2\epsilon}, & \text{if } |u|_d \le \epsilon, \\ \\ |u|_d - \dfrac{\epsilon}{2}, & \text{if } |u|_d > \epsilon. \end{cases} \tag{4.3}$$

In other words β_ϵ is the Yosida approximation of sgn, i.e.,

$$\beta_\epsilon(u) = \frac{1}{\epsilon}\,(u - (1+\epsilon \text{ sgn })^{-1}u) \in \text{ sgn } (1+\epsilon \text{ sgn })^{-1}u, \quad \forall\, u \in \mathbb{R}^d, \tag{4.4}$$

where 1 denotes the identity function on $\mathbb{R}^d$.

By standard existence results equation (4.1) has a unique solution $X_\epsilon \in C_W([0,T];H)$ which is path-wise continuous in H $\mathbb{P}$-a.s.(see [9],[10]).

Existence. Because $|\beta_\epsilon|_\infty \le 1$ we have

$$|\beta_\epsilon(\nabla(1+\epsilon A)^{-1}X_\epsilon)|_{L^\infty(\Omega\times(0,T)\times\mathscr{O})} \le C, \quad \forall\, \epsilon > 0. \tag{4.5}$$

We set $\tilde X_\epsilon = (1+\epsilon A)^{-1}X_\epsilon$, $\tilde X_\lambda = (1+\lambda A)^{-1}X_\lambda$. Then by (4.2) we have

$$\begin{aligned}&\frac{1}{2}\,|X_\epsilon(t) - X_\lambda(t)|^2\\ &\quad + \int_0^t \int_{\mathscr{O}} \langle \beta_\epsilon(\nabla \tilde X_\epsilon(s)) - \beta_\lambda(\nabla \tilde X_\lambda(s)), \nabla \tilde X_\epsilon(s) - \nabla \tilde X_\lambda(s)\rangle_d d\xi\, ds = 0.\end{aligned}$$

But, setting $\beta(x) = \text{sign }(x)$, $u_\epsilon = \nabla X_\epsilon$ and $J_\epsilon = (1+\epsilon\ \beta)^{-1}$ by (4.4) we have that $\beta_\epsilon(u) \in \beta(J_\epsilon(u))$, so by the monotonicity of β

$$\begin{aligned}&\langle \beta_\epsilon(u_\epsilon) - \beta_\lambda(u_\lambda), u_\epsilon - u_\lambda\rangle_d\\ &= \langle \beta_\epsilon(u_\epsilon) - \beta_\lambda(u_\lambda), J_\epsilon(u_\epsilon) - J_\lambda(u_\lambda)\rangle_d + \langle \beta_\epsilon(u_\epsilon) - \beta_\lambda(u_\lambda), \epsilon\beta_\epsilon(u_\epsilon) - \lambda\beta_\lambda(u_\lambda)\rangle_d\\ &\qquad \ge \langle \beta_\epsilon(u_\epsilon) - \beta_\lambda(u_\lambda), \epsilon\beta_\epsilon(u_\epsilon) - \lambda\beta_\lambda(u_\lambda)\rangle_d\end{aligned} \tag{4.6}$$

Therefore

$$\begin{aligned}&\frac{1}{2}\,|X_\epsilon(t) - X_\lambda(t)|^2\\ &+ \int_0^t \int_{\mathscr{O}} \langle \beta_\epsilon(\nabla \tilde X_\epsilon(s)) - \beta_\lambda(\nabla \tilde X_\lambda(s)), \epsilon\beta_\epsilon(\nabla \tilde X_\epsilon(s)) - \lambda\beta_\lambda(\nabla \tilde X_\lambda(s))\rangle_d d\xi\, ds \le 0.\end{aligned}$$

Since $|\beta_\epsilon| \le 1$ we deduce that

$$|X_\epsilon(t) - X_\lambda(t)|^2 \le C(\epsilon + \lambda), \quad \forall\, \epsilon, \lambda > 0,\ t \in [0,T],\ \mathbb{P}\text{-a.s.}. \tag{4.7}$$

Hence there is a continuous H-valued process $X \in C_W([0,T];H)$ such that

$$\lim_{\epsilon\to 0} \sup_{t\in[0,T]} |X_\epsilon(t) - X(t)| = 0, \qquad \mathbb{P}\text{-a.s.}. \tag{4.8}$$

Repeating the above argument for $X_\epsilon(t) = X_\epsilon(t,x)$ and $X_\epsilon(t,y)$ for some $y \in H$, we obtain that

$$|X_\epsilon(t,x) - X_\epsilon(t,y)|^2 \le C\epsilon + |x-y|^2, \quad \forall\, \epsilon > 0,\ t \in [0,T],\ \mathbb{P}\text{-a.s.}. \tag{4.9}$$

Hence

$$\sup_{t\in[0,T]} |X(t,x)-X(t,y)| \le |x-y|, \quad \forall\, x,y\in H,\ \mathbb{P}\text{-a.s.}. \tag{4.10}$$

Taking into account that

$$\langle u, \beta_\epsilon(u)\rangle_d \ge j_\epsilon(u) \quad \forall\, u\in\mathbb{R}^d,\ \epsilon>0,$$

(4.1) and Itô's formula imply that

$$\begin{aligned} &\frac{1}{2}\,|X_\epsilon(t)|^2 + \int_0^t \int_{\mathscr{O}} j_\epsilon(\nabla(1+\epsilon A)^{-1}X_\epsilon(s))d\xi ds \\ &\qquad \le \frac{1}{2}\,|x|^2 + \frac{t}{2}\ \mathrm{Tr}\ Q + \int_0^t (X_\epsilon(s), \sqrt{Q}\ dW(s)), \quad t\in[0,T],\ \epsilon>0. \end{aligned} \tag{4.11}$$

Clearly by (4.3)

$$\begin{aligned} &\int_{\mathscr{O}} j_\epsilon(\nabla(1+\epsilon A)^{-1}u)d\xi - \frac{3}{2}\,\epsilon|\mathscr{O}| \le |\nabla(1+\epsilon A)^{-1}u|_{L^1(\mathscr{O})} \\ &\qquad \le \int_{\mathscr{O}} j_\epsilon(\nabla(1+\epsilon A)^{-1}u)d\xi + \frac{3}{2}\,\epsilon|\mathscr{O}|, \quad \forall\, u\in L^1(\mathscr{O}),\ \epsilon>0, \end{aligned} \tag{4.12}$$

where $|\mathscr{O}| = \int_{\mathscr{O}} d\xi$. (4.8), (4.11) and(4.12) imply that

$$\sup_{\epsilon\in(0,1]} \mathbb{E}\int_0^T |\nabla(1+\epsilon A)^{-1}X_\epsilon(s)|_{L^1(\mathscr{O})}ds \le C(1+|x|^2+\ \mathrm{Tr}\ Q)$$

and that for some subsequence $\epsilon_n\to 0$

$$\sup_{n\in\mathbb{N}} \int_0^T \int_{\mathscr{O}} j_{\epsilon_n}(\nabla(1+\epsilon_n A)^{-1}X_{\epsilon_n}(s))d\xi ds < \infty, \quad \mathbb{P}\text{-a.s.} \tag{4.13}$$

Since we also have that by (4.8)

$$\begin{aligned} &\lim_{\epsilon\to 0} |(1+\epsilon A)^{-1}X_\epsilon(t) - X(t)| \le \lim_{\epsilon\to 0} |X_\epsilon(t)) - X(t)| \\ &\qquad + \lim_{\epsilon\to 0} |(1+\epsilon A)^{-1}X(t) - X(t)| = 0, \quad \mathbb{P}\text{-a.s.},\ \forall\, t\in[0,T], \end{aligned} \tag{4.14}$$

we conclude by the lower semicontinuity of Φ, Fatou's lemma and (4.12) that

$$\int_0^T \Phi(X(s))ds \le \liminf_{n\to\infty} \int_0^T \int_{\mathscr{O}} j_{\epsilon_n}(\nabla(1+\epsilon_n A)^{-1}X_{\epsilon_n}(s))d\xi ds, \quad \mathbb{P}\text{-a.s.}.$$

$$(4.15)$$

Now it also follows that $X \in L^1((0,T)\times\Omega; BV(\mathscr{O}))$. In fact, since $(1+\epsilon A)^{-1}X_\epsilon(t) \in H_0^1(\mathscr{O})$ and as seen above it is a.e. convergent to $X(t)$ it follows by (4.12), (4.13) that $X(t)\in BV^0(\mathscr{O}), a.e.$ and so $X \in L^1((0,T)\times\Omega; BV^0(\mathscr{O}))$.

We set

$$\Phi_\epsilon(u) = \int_{\mathscr{O}} j_\epsilon(\nabla(1+\epsilon A)^{-1}u)d\xi, \quad \forall\, u \in H. \tag{4.16}$$

Then

$$\partial\Phi_\epsilon(u) = \nabla\Phi_\epsilon(u) = A^\epsilon(u), \quad \forall\, u\in H. \tag{4.17}$$

Let $Y \in C_W([0,T];H)\cap L^1((0,T)\times\Omega; BV^0(\mathscr{O}))$ and $G \in L^2_W(0,T;H)$ be such that

$$dY(t) + G(t)dt = \sqrt{Q}\, dW(t),$$

i.e.,

$$Y(t) = Y(0) - \int_0^t G(s)ds + \sqrt{Q}\, W(t), \quad \forall\, t\in[0,T],\ \mathbb{P}\text{-a.s.}.$$

By (4.1) and (4.17) we have that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|X_\epsilon(t)-Y(t)|^2\\ &+\int_{\mathscr{O}}\langle\beta_\epsilon(\nabla(1+\epsilon A)^{-1}X_\epsilon(t), \nabla(1+\epsilon A)^{-1}X_\epsilon(t)-\nabla(1+\epsilon A)^{-1}Y(t)\rangle_d d\xi\\ &= (G(t), X_\epsilon(t)-Y(t)).\end{aligned} \tag{4.18}$$

On the other hand, since j_ϵ is convex we have that

$$\langle\beta_\epsilon(u), u-v\rangle_d \ge j_\epsilon(u) - j_\epsilon(v), \quad \forall\, u,v\in\mathbb{R}^d.$$

Consequently, taking $u = \nabla X_\epsilon(t)$, $v = \nabla Y(t)$ in (4.18), we deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|X_\epsilon(t)-Y(t)|^2+\int_{\mathscr{O}}(j_\epsilon(\nabla(1+\epsilon A)^{-1}X_\epsilon(t))-j_\epsilon(\nabla(1+\epsilon A)^{-1}Y(t)))d\xi\\ \le (G(t), X_\epsilon(t)-Y(t)),\end{aligned}$$

which is equivalent to

$$\frac{1}{2}\frac{d}{dt}|X_\epsilon(t)-Y(t)|^2 + \Phi_\epsilon(X_\epsilon(t)) - \Phi_\epsilon(Y(t)) \le (G(t), X_\epsilon(t)-Y(t)).$$

Integrating with respect to t yields

$$\frac{1}{2}\,|X_\epsilon(t)-Y(t)|^2+\int_0^t(\Phi_\epsilon(X_\epsilon(s))-\Phi_\epsilon(Y(s))ds$$

$$\le\int_0^t(G(s),X_\epsilon(s)-Y(s))ds+\frac{1}{2}\,|x-Y(0)|^2,\quad \forall\, t\in[0,T],\ \mathbb{P}\text{-a.s..}$$

Furthermore, since $(1+\epsilon A)^{-1}$ is a contraction on $L^\infty(\mathscr{O})$, for all $y\in L^1(\mathscr{O})$ we have

$$|\nabla(1+\epsilon A)^{-1}y|_{L^1(\mathscr{O})}\le\|Dy\|=\Phi(y). \tag{4.19}$$

In particular, this holds for $y=Y(s),\ s\in[0,T]$. Hence replacing ϵ by ϵ_n and letting $n\to\infty$, by (4.8), (4.12), (4.14) and (4.16) we obtain that

$$\frac{1}{2}\,|X(t)-Y(t)|^2+\int_0^t\Phi(X(s))ds\le\int_0^t\Phi(Y(s))ds$$

$$+\int_0^t(G(s),X(s)-Y(s))ds+\frac{1}{2}\,|x-Y(0)|^2,\quad \forall\, t\in[0,T],\ \mathbb{P}\text{-a.s..}$$

Hence X is a solution to (3.1) as claimed. This completes the proof of existence.

Uniqueness. Let $Z,\ Z(0)=x$, be an arbitrary solution in the sense of Definition 3.1. Let X_ϵ be the solution to (4.1). We set

$$G_\epsilon(t)=\int_0^t(A_\epsilon X_\epsilon(s),X_\epsilon(s)-(1+\epsilon A)^{-2}X_\epsilon(s)-(1+\epsilon A)^{-1}\eta_\epsilon(s))ds,$$

where $d\eta_\epsilon=(-(1+\epsilon A)^{-1}+1)\sqrt{Q}\;dW,\ \eta_\epsilon(0)=0$. We note that, as shown above, $A^\epsilon X_\epsilon \in L^2_W(0,T;H)$ and

$$\begin{aligned}
&\frac{1}{2}\,\mathbb{E}|Z(t)-(1+\epsilon A)^{-1}X_\epsilon(s)-\eta_\epsilon(s)|^2\\
&+\mathbb{E}\int_0^t(\Phi(Z(s))-\Phi((1+\epsilon A)^{-1}X_\epsilon(s)+\eta_\epsilon(s)))ds\\
&\le \mathbb{E}\int_0^t((1+\epsilon A)^{-1}A^\epsilon X_\epsilon(s),Z(s)-(1+\epsilon A)^{-1}X_\epsilon(s)-\eta_\epsilon(s))ds\\
&=\mathbb{E}\int_0^t(A^\epsilon X_\epsilon(s),(1+\epsilon A)^{-1}Z(s)-X_\epsilon(s))ds+\mathbb{E}\int_0^t G_\epsilon(s)ds\\
&\le \mathbb{E}\int_0^t(\Phi_\epsilon((1+\epsilon A)^{-1}Z(s))-\Phi_\epsilon(X_\epsilon(s)))ds+\mathbb{E}\int_0^t G_\epsilon(s)ds,\quad \forall\, t\in[0,T].
\end{aligned}\tag{4.20}$$

(Here we have applied (3.1) with $Y=(1+\epsilon A)^{-1}X_\epsilon(s)+\eta_\epsilon(s)$ and $G=(1+\epsilon A)^{-1}A^\epsilon X_\epsilon$, and(4.16)). On the other hand, $\eta_\epsilon(t)\in H^1_0(\mathscr{O})$ $\mathbb{P}$-a.s. (see (4.21) below) and hence by (4.3),(4.16) we have

$$\begin{aligned}
&\Phi((1+\epsilon A)^{-1}X_\epsilon+\eta_\epsilon)-\Phi_\epsilon(X_\epsilon)\\
&=\int_{\mathscr{O}}|\nabla(1+\epsilon A)^{-1}X_\epsilon+\nabla\eta_\epsilon|_d d\xi-\int_{\mathscr{O}}j_\epsilon(\nabla(1+\epsilon A)^{-1}X_\epsilon)d\xi\\
&\le\int_{\{|\nabla(1+\epsilon A)^{-1}X_\epsilon|_d\le\epsilon\}}|\nabla(1+\epsilon A)^{-1}X_\epsilon|_d\left(1-\frac{|\nabla(1+\epsilon A)^{-1}X_\epsilon|_d}{2\epsilon}\right)d\xi\\
&+\int_{\mathscr{O}}\left(|\nabla\eta_\epsilon|_d+\frac{\epsilon}{2}\right)d\xi\le\frac{3\epsilon}{2}\int_{\mathscr{O}}d\xi+\int_{\mathscr{O}}|\nabla\eta_\epsilon|_d d\xi,\quad \mathbb{P}\text{-a.s.}.
\end{aligned}$$

It remains to estimate $G_\epsilon(t)$ and $\eta_\epsilon(t)$. We have

$$\begin{aligned}
\mathbb{E}|\eta_\epsilon(t)|^2_{H^1_0(\mathscr{O})}&\le \mathrm{Tr}\,[A(1-(1+\epsilon A)^{-1})^2Q]t\\
&=t\sum_{n=1}^\infty\lambda_n\mu_n\left(\frac{\epsilon\lambda_n}{1+\epsilon\lambda_n}\right)^2\le t\epsilon^\kappa\sum_{n=1}^\infty\lambda_n^{1+\kappa}\mu_n.
\end{aligned}\tag{4.21}$$

By (1.3) it follows that

$$\lim_{\epsilon\to0}\mathbb{E}|\eta_\epsilon(t)|^2_{H^1_0(\mathscr{O})}=0.$$

In particular, this implies also that

$$\lim_{\epsilon\to 0}\mathbb{E}\int_0^t (A^\epsilon X_\epsilon(s),(1+\epsilon A)^{-1}\eta_\epsilon(s))ds=0,$$

because

$$(A^\epsilon X_\epsilon(s),(1+\epsilon A)^{-1}\eta_\epsilon(s))=\int_{\mathscr{O}}(\beta^\epsilon(\nabla(1+\epsilon A)^{-1}X_\epsilon(s)),\nabla(1+\epsilon A)^{-2}\eta_\epsilon(s))d\xi$$

and $\{\beta^\epsilon\}$ is bounded.

Finally, applying Itô's formula in (4.1) with $\varphi_\epsilon(x)=\frac{1}{2}\,|x|^2-\frac{1}{2}\,(x,(1+\epsilon A)^{-2}x)$ we obtain that for $\epsilon\in(0,1]$

$$\begin{aligned}&\mathbb{E}\int_0^t (A^\epsilon X_\epsilon(s),X_\epsilon(s)-(1+\epsilon A)^{-2}X_\epsilon(s))ds\\ &\qquad\le \frac{1}{2}\,\mathrm{Tr}\,[(1-(1+\epsilon A)^{-2})Q]+\varphi_\epsilon(x)\le\frac{3}{2}\,\epsilon\sum_{n=1}^\infty\lambda_n\mu_n+\varphi_\epsilon(x).\end{aligned}$$

Hence

$$\limsup_{\epsilon\to 0}\mathbb{E}\int_0^t G_\epsilon(s)ds\le 0.$$

Now we take $\limsup_{\epsilon\to0}$ on both sides of (4.20). Recalling that $\lim_{\epsilon\to0}X_\epsilon(t)=X(t)$, $\mathbb{P}$-a.s. and that by (4.12) and (4.19) $\limsup_{\epsilon\to0}\Phi_\epsilon((1+\epsilon A)^{-1}Z(s))\le\Phi(Z(s))$, $\mathbb{P}$-a.s. for all $s\in[0,T]$, letting $\epsilon\to0$ in (4.20), by (4.14) we see that $Z(t)=X(t)=\lim_{\epsilon\to0}X_\epsilon(t)$ where X is the solution of (3.1) from the previous step. This completes the proof of Theorem 3.2. □

Remark 4.1 As follows from the proof we even obtain that for every $T>0$ there exists a constant $C>0$ such that

$$\sup_{t\in[0,T]}|X(t)-X_\epsilon(t)|\le C\epsilon,\quad \forall\,\epsilon\in(0,1],\ \mathbb{P}\text{-a.s.}.$$

Remark 4.2 By the previous proof it follows that the existence part of Theorem 3.2 requires instead of (1.3) the weaker assumption Tr $Q<\infty$.

5 Invariant measure

Let $P_t:C_b(H)\to C_b(H)$ be the transition semigroup associated with equation (1.1), i.e.,

$$P_t\varphi(x)=\mathbb{E}[\varphi(X(t,x))],\quad \forall\,t\ge0,\ \varphi\in C_b(H)\tag{5.1}$$

where $X(t,x)$ is the solution given by Definition 3.1 (Theorem 3.2). Here $C_b(H)$ is the space of all uniformly continuous bounded functions on H.

Theorem 5.1 *There is at least one invariant measure ν for the semigroup P_t, i.e.*

$$\int_H P_t\varphi(x)\nu(dx) = \int_H \varphi(x)\nu(dx), \quad \forall\, t \geq 0,\ \varphi \in C_b(H) \tag{5.2}$$

and $\nu(BV^0(\mathscr{O})) = 1$.

Proof. By estimates (4.11) and (4.15) it follows for $\epsilon \to 0$ that

$$\mathbb{E}|X(t)|^2 + 2\mathbb{E}\int_0^t \Phi(X(s))ds \leq |x|^2 + t\ \mathrm{Tr}\ Q, \quad \forall\, t \geq 0,\ x \in H. \tag{5.3}$$

We set

$$\mu_T := \frac{1}{T}\int_0^T \pi_{t,x}dt, \quad \forall\, T \geq 0,$$

where $\pi_{t,x}$ is the law of $X(t,x)$. Let

$$\Gamma_R = \{x \in BV^0(\mathscr{O}) :\ \|Dx\| \leq R\}.$$

Since the embedding of $BV^0(\mathscr{O})$ into $L^2(\mathscr{O}) = H$ is compact we infer that Γ_R is a compact subset of H. On the other hand, by (5.3) we see that

$$\mu_T(\Gamma_R^c) \leq \frac{|x|^2}{2TR^2} + \frac{\mathrm{Tr}\ Q}{2R^2}, \quad \forall\, T \geq 0. \tag{5.4}$$

Hence the set $\{\mu_T\}_{T>0}$ is tight and so by the Krylov-Bogoliubov theorem it is weakly convergent along a sequence $\{T_n\} \to \infty$ to an invariant measure ν on $L^2(\mathscr{O})$ of P_t.

Now by (5.4) it follows that

$$\nu(\Gamma_R^c) \leq \frac{\mathrm{Tr}\ Q}{2R^2}, \quad \forall\, R > 0.$$

Taking into account that $BV^0(\mathscr{O}) = \bigcup_{R>0}\Gamma_R$, we infer that $\nu(BV^0(\mathscr{O})) = 1$ as claimed. $\square$

Remark 5.2 Taking into account that the diffusion effect of equation (1.1) is non local, one might suspect that the measure ν is not full, i.e. it may be zero on a non-empty open set. Also for the same reason (the gradient operator $\partial\Phi$ is not strongly monotone) perhaps we do not have uniqueness of the invariant measure.

6 Equation (1.1) with linear multiplicative noise

Consider the problem

$$\begin{cases} dX(t) = \text{div}\,(\text{sgn}\,(\nabla(X(t)))dt + \sigma(X(t))\,dW(t) \text{ in } (0,\infty)\times\mathscr{O} \\ X(t) = 0 \quad \text{on } \partial\mathscr{O}\times(0,T) \\ X(0) = x \quad \text{in } \mathscr{O}, \end{cases} \tag{6.1}$$

where

$$\sigma(X(t))\,dW(t) = \sum_{k=1}^{\infty} Xe_k\mu_k d\beta_k(t), \tag{6.2}$$

$\{e_k\}$ is the eigenbasis of A and $\{\mu_k\}$ is a sequence of positive numbers.

We need the linear map $x \mapsto \sigma(x) = \sum_{k=1}^{\infty}\mu_k(e_k,\cdot)xe_k$ to be continuous (hence Lipschitz) from $L^2(\mathscr{O})$ to $L_2(L^2(\mathscr{O}),L^2(\mathscr{O}))$ (the space of all Hilbert–Schmidt operators on $L^2(\mathscr{O})$) which is the case if

$$\sum_{k=1}^{\infty}\mu_k^2|e_k|_{\infty}^2 < \infty. \tag{6.3}$$

By Sobolev embedding (6.3) holds if

$$\sum_{k=1}^{\infty}\mu_k^2\lambda_k^2 < +\infty. \tag{6.4}$$

By standard fixed point arguments it follows that the equation

$$\begin{cases} dY_\epsilon(t) + A^\epsilon Y_\epsilon(t)dt = \sigma(Y_\epsilon(t))dW(t) \\ Y_\epsilon(0) = x, \end{cases} \tag{6.5}$$

where A^ϵ is as in Section 4, has a unique solution $Y_\epsilon \in C_W([0,T];H)$ which is path-wise H continuous (See [9], [10]).

Definition 6.1 *A stochastic process $X = X(t,x)$ with $\mathbb{P}$-a.s. continuous sample paths in H is said to be a strong solution to equation* (6.1) *if*

$$X \in C_W([0,T];H)\cap L^1((0,T)\times\Omega;BV^0(\mathscr{O})), \quad X(0) = x \tag{6.6}$$

and

$$\begin{aligned} &\frac{1}{2}\,\mathbb{E}|X(t)-Y(t)|^2 + \mathbb{E}\int_0^t(\Phi(X(s))-\Phi(Y(s)))ds \\ &\le \frac{1}{2}\,\mathbb{E}|x-Y(0)|^2 + \mathbb{E}\int_0^t(G(s),X(s)-Y(s))ds \\ &+\frac{1}{2}\,\mathbb{E}\sum_{k=1}^{\infty}\mu_k^2\int_0^t|(X(s)-Y(s))e_k|^2ds, \quad t\in[0,T], \end{aligned} \tag{6.7}$$

for all $G \in L^2_W(0,T;H)$ *and* $Y \in C_W([0,T];H) \cap L^1((0,T)\times H; BV^0(\mathscr{O}))$ *satisfying the equation*

$$dY(t) + G(t)dt = \sigma(Y)dW(t), \quad t \in [0,T]. \tag{6.8}$$

Theorem 6.2 *For each* $x \in H$ *there is at least one strong solution*

$$X \in L^1(\Omega \times (0,T); BV^0(\mathscr{O})) \cap L^2_W(\Omega; C([0,T];H)),$$

to equation (6.1)*. Moreover, we have for every* $T > 0$

$$\lim_{\epsilon\to 0} \mathbb{E} \sup_{t\in[0,T]} |X(t) - Y_\epsilon(t)|^2 = 0 \tag{6.9}$$

and for some constant $C > 0$

$$\mathbb{E} \sup_{t\in[0,T]} |X(t,x) - X(t,y)| \le C|x-y|, \quad \forall\, x,y \in H. \tag{6.10}$$

Proof. We proceed in the same way as for Theorem 3.2, so the proof will be sketched only. By (6.5) and by Itô's formula we have by (4.6) that

$$\begin{aligned}
&\frac{1}{2}\,|Y_\epsilon(t) - Y_\lambda(t)|^2 \\
&+ \int_0^t \int_{\mathscr{O}} \langle \beta_\epsilon(\nabla \tilde{Y}_\epsilon(s)) - \beta_\lambda(\nabla \tilde{Y}_\lambda(s)), (\epsilon\beta_\epsilon(\nabla \tilde{Y}_\epsilon(s)) - \lambda\beta_\lambda(\nabla \tilde{Y}_\lambda(s))\rangle_d d\xi ds \\
&\le \int_0^t \langle (\sigma(Y_\epsilon(s)) - \sigma(Y_\lambda(s))dW(s), Y_\epsilon(s) - Y_\lambda(s)\rangle ds \\
&+\frac{1}{2} \sum_{k=1}^\infty \mu_k^2 |e_k|_\infty^2 \int_0^t |Y_\epsilon(s) - Y_\lambda(s)|^2 ds, \quad \mathbb{P}\text{-a.s.}.
\end{aligned}$$

Here $\tilde{Y}_\epsilon = (1+\epsilon A)^{-1} Y_\epsilon$.

Now arguing as in the proof of Theorem 3.2 it follows, by the Burkholder-Davis-Gundy inequality, that

$$\mathbb{E} \sup_{t\in[0,T]} |Y_\epsilon(t) - Y_\lambda(t)|^2 \le C(\epsilon + \lambda).$$

Hence there exists $X \in C_W([0,T];H))$ with $\mathbb{P}$-a.s. H-continuous sample paths satisfying (6.9). Similarly as we proved (3.3) we obtain (6.10).

If G, Y are as in Definition 6.1 and Φ_ϵ is defined by (4.16) we have by Itô's formula that

$$\frac{1}{2}\,\mathbb{E}|Y_\epsilon(t)-Y(t)|^2+\mathbb{E}\int_0^t(\Phi_\epsilon(Y_\epsilon(s))-\Phi_\epsilon(Y(s)))ds$$

$$\le\frac{1}{2}\,|x-Y(0)|^2+\frac{1}{2}\,\mathbb{E}\sum_{k=1}^{\infty}\mu_k^2\,|e_k|_\infty^2\,\int_0^t|(Y_\epsilon(s)-Y(s))|^2ds$$

$$+\mathbb{E}\int_0^t(G(s),Y_\epsilon(s)-Y(s))ds,$$

for all $t\in[0,T]$, which implies (6.7) letting $\epsilon\to 0$ by the same arguments as in the proof of Theorem 3.2. This completes the proof. □

Theorem 6.3 *Assume that $x\in L^2(\mathscr{O})$, $x\ge 0, \mathbb{P}$-a.s. in $\mathscr{O}$. Then the solution X given by Theorem 6.2 satisfies*

$$X(t,x)\ge 0\quad \mathbb{P}\text{-a.s. in }(0,T)\times\mathscr{O}.$$

Proof. The proof is very similar to that of [4, Theorem 2.2] so, we only give a sketch. By virtue of (6.9) it suffices to prove that $Y_\epsilon\ge 0$ and by (6.10) without loss of generality we may assume that $x\in L^4(\mathscr{O})$. Then we apply Itô's formula to

$$\phi(x)=\frac{1}{4}\int_{\mathscr{O}}(x^-)^4d\xi$$

in equation (6.5). Since $D\phi(x)=-(x^-)^3$, we get

$$\begin{aligned}&\mathbb{E}\phi(Y_\epsilon(t))-3\mathbb{E}\int_0^t\int_{\mathscr{O}}\left\langle\beta_\epsilon(\nabla Y_\epsilon(s)),\nabla Y_\epsilon^-(s)\right\rangle_d(Y_\epsilon^-(s))^2d\xi ds\\&+\frac{3}{2}\,\epsilon\mathbb{E}\int_0^t\int_{\mathscr{O}}|\nabla Y_\epsilon^-(s)|_d^2|Y_\epsilon^-(s)|^2\xi ds\\&\le\frac{3}{2}\,\mathbb{E}\int_0^t\int_{\mathscr{O}}\sum_{k=1}^{\infty}\mu_k^2|Y_\epsilon^-(s)e_k|^2\,|Y_\epsilon^-(s)|^2d\xi ds\\&\le C\mathbb{E}\int_0^t\int_{\mathscr{O}}|Y_\epsilon^-(s)|^4d\xi ds,\quad\forall\,\epsilon>0,\ t\in[0,T].\end{aligned}\tag{6.11}$$

Of course, this calculation is formal. But for making it rigorous one regularizes ϕ in exactly the same way as in [4, Lemma 3.5]. Now, taking into account that

$$\left\langle\beta_\epsilon(\nabla Y_\epsilon(s)),\nabla Y_\epsilon^-(s)\right\rangle_d=-\left\langle\beta_\epsilon(\nabla Y_\epsilon^-(s)),\nabla Y_\epsilon^-(s)\right\rangle_d\le 0,$$

it follows by (6.11) that Y_ϵ^- is identically equal to zero in $(0,T) \times \Omega \times \mathscr{O}$. as claimed. For details we refer to the part of the proof of [4, Theorem 2.2] after the proof of [4, Lemma 3.5]. □

Remark 6.4 The uniqueness of solution X remains open and the arguments of the proof of Theorem 3.2 can not be used here without imposing some strong assumptions on $\sigma(x)$.

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