STOCHASTIC TAMED 3D NAVIER-STOKES EQUATIONS: EXISTENCE, UNIQUENESS AND ERGODICITY

MICHAEL RÖCKNER[1,2], XICHENG ZHANG[1,3,4]

[1] *Fakultät für Mathematik, Universität Bielefeld*
Postfach 100131, D-33501 Bielefeld, Germany
[2] *Departments of Mathematics and Statistics, Purdue University*
W. Laffayette, IN, 47907, USA
[3] *School of Mathematics and Statistics*
The University of New South Wales, Sydney, 2052, Australia
[4] *Department of Mathematics, Huazhong University of Science and Technology*
Wuhan, Hubei 430074, P.R.China
M. Röckner: roeckner@math.uni-bielefeld.de
X. Zhang: XichengZhang@gmail.com

Abstract. In this paper, we prove the existence of a unique strong solution to a stochastic tamed 3D Navier-Stokes equation in the whole space as well as in the periodic boundary case. Then, we also study the Feller property of solutions, and prove the existence of invariant measures for the corresponding Feller semigroup in the case of periodic conditions. Moreover, in the case of periodic boundary and degenerated additive noise, using the notion of asymptotic strong Feller property proposed by Hairer and Mattingly [15], we prove the uniqueness of invariant measures for the corresponding transition semigroup.

Contents

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1. Introduction

The classical 3D Navier-Stokes equations (NSE) describe the time evolution of an incompressible fluid and are given by

$$\partial_t \mathbf{u}(t) = \nu\Delta\mathbf{u}(t) - (\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t) + \nabla p(t) + \mathbf{f}(t)$$

and

$$\mathrm{div}\mathbf{u}(t) = 0,$$

where $\mathbf{u}(t,x) = (u^1(t,x), u^2(t,x), u^3(t,x))$ represents the velocity field, ν is the viscosity constant, $p(t,x)$ denotes the pressure, and $\mathbf{f}$ is an external force field acting on the fluid. In [19], Leray initially constructed a weak solution for the Cauchy problem of NSE in the whole space, since then, it is still not known whether there exists a smooth solution existing for all times. In [27], we analyzed the following tamed scheme for the classical 3D NSE:

$$\partial_t \mathbf{u}(t) = \nu\Delta\mathbf{u}(t) - (\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t) + \nabla p(t) - g_N(|\mathbf{u}(t)|^2)\mathbf{u}(t) + \mathbf{f}(t),$$

where the taming function $g_N : \mathbb{R}_+ \to \mathbb{R}_+$ is smooth and satisfies for some $N \in \mathbb{N}$,

$$\begin{cases} g_N(r) = 0, & \text{if } r \leqslant N, \\ g_N(r) = (r-N)/\nu, & \text{if } r \geqslant N+1, \\ 0 \leqslant g_N'(r) \leqslant 2/(\nu\wedge 1), & r \geqslant 0. \end{cases} \tag{1.1}$$

Therein, we proved the existence of smooth solutions to this tamed equation when $\mathbf{f}$ and the initial velocity are smooth. The main feature of this tamed equation is that if there is a bounded smooth solution to the classical 3D NSE, then this smooth solution must satisfy our tamed equation for some N large enough. Moreover, we can let $N \to \infty$ to obtain the existence of suitable weak solutions (cf. [27]). In this sense, the above tamed scheme can be considered as a regularized equation for the classical equation.

Following the above tamed scheme, in the present paper we shall study the stochastic tamed 3D NSE. Let us now describe our model equation. Let $\mathbb{D} := \mathbb{R}^3$ or $\mathbb{T}^3$ (in the periodic case), where $\mathbb{T} = [0,1)$ is the unit circle. Note that any function from $\mathbb{T}^3$ to $\mathbb{R}^3$ can be identified with a periodic function from $\mathbb{R}^3$ to $\mathbb{R}^3$. We consider the following stochastic tamed 3D Navier-Stokes equation with $\nu = 1$ in $\mathbb{D}$:

$$\begin{aligned} \mathrm{d}\mathbf{u}(t) = &\Big[\Delta\mathbf{u}(t) - (\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t) + \nabla p(t) - g_N(|\mathbf{u}(t)|^2)\mathbf{u}(t) + \mathbf{f}(t,\mathbf{u}(t))\Big]\mathrm{d}t \\ &+ \sum_{k=1}^{\infty}\Big[(\sigma_k(t)\cdot\nabla)\mathbf{u}(t) + \nabla\tilde{p}_k(t) + \mathbf{h}_k(t,\mathbf{u}(t))\Big]\mathrm{d}W_t^k \end{aligned} \tag{1.2}$$

subject to the incompressibility condition

$$\mathrm{div}\mathbf{u}(t) = 0 \tag{1.3}$$

and the initial condition

$$\mathbf{u}(0) = \mathbf{u}_0, \tag{1.4}$$

where $p(t,x)$ and $\tilde{p}_k(t,x)$ are unknown scalar functions, $N > 0$ and the taming function $g_N : \mathbb{R}^+ \to \mathbb{R}^+$ as above satisfies (1.1), and $\{W_t^k; t \geqslant 0, k = 1,2,\cdots\}$ is a sequence of independent one dimensional standard Brownian motions on some complete filtration probability space $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\geqslant 0})$. The stochastic integral is understood as Itô's integral. The entries of the coefficients are given as follows:

$$\mathbb{R}_+ \times \mathbb{D} \times \mathbb{R}^3 \ni (t,x,\mathbf{u}) \to \mathbf{f}(t,x,\mathbf{u}) \in \mathbb{R}^3,$$

$$\mathbb{R}_+ \times \mathbb{D} \ni (t,x) \to \sigma(t,x) \in \mathbb{R}^3 \times l^2,$$

$$\mathbb{R}_+ \times \mathbb{D} \times \mathbb{R}^3 \ni (t,x,\mathbf{u}) \to \mathbf{h}(t,x,\mathbf{u}) \in \mathbb{R}^3 \times l^2,$$

where l^2 denotes the Hilbert space consisting of all sequences of square summable real numbers with standard norm $\|\cdot\|_{l^2}$. In the following, $\mathbf{f}, \sigma$ and $\mathbf{h}$ are always assumed to be measurable with respect to all their variables.

The study of stochastic Navier-Stokes equations (SNSE) began with the work of Bensoussan -Temam in [2]. Using Galerkin's approximation and compactness method, Flandoli and Gatarek in [10] proved the existence of martingale solutions and stationary solutions for any dimensional stochastic Navier-Stokes equations in a bounded domain. In particular, when the transition semigroup is well defined, the stationary martingale solutions will yield the existence of invariant measures. We remark that their results cannot be used in the case of whole space because of the absence of compact Sobolev embeddings. Recently, Mikulevicius and Rozovskii in [24] proved the existence of martingale solutions to SNSE in $\mathbb{R}^d$ ($d \geqslant 2$) under less assumptions on the coefficients (without the extra term g_N). To avoid the use of compact Sobolev embeddings, they used the approach of mollifying and cutting off the coefficients. In the case of two dimension, they also obtained the existence and pathwise uniqueness of L^2-continuous adapted solutions.

On the other hand, the ergodicity of invariant measures for 2D stochastic Navier-Stokes equations has been studied extensively (cf. [11, 22, 7, 15] and reference therein). Especially, Hairer and Mattingly [15] recently developed two important tools: the asymptotic strong Feller property and an approximative integration by parts formula in the Malliavin calculus, and then used them to derive an optimal ergodicity result for 2D SNSE in the sense that the random forces only has two modes. As pointed out in [15], the asymptotic strong Feller property is much weaker than the usual strong Feller property since many degenerated equations have the former property rather than the later one.

Up to now, to the best of our knowledge, most of the well known results about the stochastic Navier-Stokes equations such as the existence of invariant measures and the ergodicity under different conditions on the noise are for 2D SNSE. As for the three dimensional case, there are only a few results (cf. [4, 5, 1, 25, 12, 29]), of course, because of the lack of uniqueness. Recently, in [4, 5, 25], Da-Prato, Debussche and Odasso proved the existence and ergodicity of Markov solutions for 3D SNSE without the taming term g_N, which are obtained as limits of Galerkin's approximations. Similar results were obtained by Flandoli and Romito in [12, 29] for all Markov solutions. Moreover, using stochastic cascades, Bakhtin [1] explicitly constructed a stationary solution of 3D Navier-Stokes system and proved a uniqueness theorem.

In the present paper, we shall prove the existence of a unique strong solution to our stochastic tamed 3D Navier-Stokes equation (1.2) under some assumptions on $\mathbf{f}$, σ and $\mathbf{h}$. Here, the word "strong" means "strong" both in the sense of the theory of stochastic differential equations and the theory of partial differential equations. Let $\mathbb{H}^m$ denote the Sobolev space of divergence free vector fields (see (2.2) below). Instead of working on the evolution triple $\mathbb{H}^1 \subset \mathbb{H}^0 \subset \mathbb{H}^{-1}$, we shall work on the evolution triple $\mathbb{H}^2 \subset \mathbb{H}^1 \subset \mathbb{H}^0$. This will enable us to obtain the "strong" solution in the sense of partial differential equations. For the "strong" solution in the sense of stochastic differential equations, we shall use the famous Yamada-Watanabe theorem: the existence of martingale solutions plus pathwise uniqueness implies the existence of a unique strong solution. Different from the method in [24], we still use the classical Galerkin approximation to prove our existence of strong solutions. To overcome the absence of compact Sobolev embeddings, we shall use localization method to prove tightness. We think that it is of interest in itself and can be used in other cases. Moreover, as in the deterministic case, we can take limits $N \to \infty$ to prove the existence of weak solutions for the true stochastic Navier-Stokes equations (without taming term). This will be done in a further investigation.

After obtaining the existence of a unique strong solution to Eq. (1.2), we turn to the study of uniqueness of invariant measures in the case of periodic boundary conditions and degenerated additive noise. As a first step, we need to prove the Feller property and the existence of an invariant measure. Then, using the asymptotic strong Feller property and approximative integration by parts formula in [15], we can prove the uniqueness of invariant measures. As said above, since we shall work in the first order Sobolev space $\mathbb{H}^1$, all of our discussions will take place in $\mathbb{H}^1$. This requires some delicate analysis and calculations, and the special form of g_N plays an important role throughout this paper. It should be emphasized that the optimal results in [15] seem to depend strongly on the structure of 2D Navier-Stokes equations, we can not develop a similar non-adapted analysis along their lines to obtain some optimal result for our tamed 3D SNSE.

This paper is organized as follows: in Section 2, we give some preliminaries, that include some necessary estimates and a tightness result for later use. In Section 3, we shall prove the existence and uniquess result by Galerkin's approximation. In Section 4, we study the Feller property of the solutions to Eq. (1.2) and the existence of invariant measures for the Feller semigroup in the case of periodic boundary conditions. In Section 5, we study the ergodicity of invariant measures. In the Appendix, for the reader's convenience, the martingale characterization for weak solutions is proved, two necessary basic estimates are given, and the derivative flow equation is proved.

2. Preliminaries

2.1. **Notations and Assumptions.** Let $C_0^\infty(\mathbb{D};\mathbb{R}^3)$ denote the set of all smooth functions from $\mathbb{D}$ to $\mathbb{R}^3$ with compact supports. When $\mathbb{D}=\mathbb{T}^3$, a function $f\in C_0^\infty(\mathbb{D};\mathbb{R}^3)$ means that it is a smooth periodic function from $\mathbb{R}^3$ to $\mathbb{R}^3$. For $p\geqslant 1$, let $L^p(\mathbb{D};\mathbb{R}^3)$ be the vector valued L^p-space in which the norm is denoted by $\|\cdot\|_{L^p}$. For $m\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, let H^m be the usual Sobolev space on $\mathbb{D}$ with values in $\mathbb{R}^3$, i.e., the closure of $C_0^\infty(\mathbb{D};\mathbb{R}^3)$ with respect to the norm:

$$\|\mathbf{u}\|_{H^m}=\left(\int_{\mathbb{D}}|(I-\Delta)^{m/2}\mathbf{u}|^2\mathrm{d}x\right)^{1/2}.$$

Here as usual, $(I-\Delta)^{m/2}$ is defined by Fourier transformation. For two separable Hilbert spaces $\mathbb{K}$ and $\mathbb{H}$, $L_2(\mathbb{K};\mathbb{H})$ will denote the space of all Hilbert-Schmidt operators from $\mathbb{K}$ to $\mathbb{H}$ with norm $\|\cdot\|_{L_2(\mathbb{K};\mathbb{H})}$.

The following Gagliardo-Nirenberg interpolation inequality will be used frequently. It plays an essential role in the study of Navier-Stokes equations (cf. [33]). Let $q\in[1,\infty]$ and $m\in\mathbb{N}$. If

$$\frac{1}{q}=\frac{1}{2}-\frac{m\alpha}{3},\quad 0\leqslant\alpha\leqslant 1,$$

then for any $\mathbf{u}\in H^m$

$$\|\mathbf{u}\|_{L^q}\leqslant C_{m,q}\|\mathbf{u}\|_{H^m}^{\alpha}\|\mathbf{u}\|_{L^2}^{1-\alpha}. \tag{2.1}$$

Set for $m\in\mathbb{N}_0$

$$\mathbb{H}^m:=\{\mathbf{u}\in H^m:\mathrm{div}(\mathbf{u})=0\}. \tag{2.2}$$

Then $(\mathbb{H}^m,\|\cdot\|_{H^m})$ is a separable Hilbert space. We shall denote the norm $\|\cdot\|_{H^m}$ in $\mathbb{H}^m$ by $\|\cdot\|_{\mathbb{H}^m}$. We remark that $\mathbb{H}^0$ is a closed linear subspace of the Hilbert space $L^2(\mathbb{D};\mathbb{R}^3)=H^0$.

Let $\mathscr{P}$ be the orthogonal projection from $L^2(\mathbb{D};\mathbb{R}^3)$ to $\mathbb{H}^0$ (cf. [8, 20]). It is well known that $\mathscr{P}$ commutes with the derivative operators, and that $\mathscr{P}$ can be restricted to a bounded linear operator from H^m to $\mathbb{H}^m$. For any $\mathbf{u}\in\mathbb{H}^0$ and $\mathbf{v}\in L^2(\mathbb{D};\mathbb{R}^3)$, we have

$$\langle\mathbf{u},\mathbf{v}\rangle_{\mathbb{H}^0}:=\langle\mathbf{u},\mathscr{P}\mathbf{v}\rangle_{\mathbb{H}^0}=\langle\mathbf{u},\mathbf{v}\rangle_{L^2}.$$

Let $\mathcal{V}$ be defined by

$$\mathcal{V} := \{\mathbf{u} : \mathbf{u} \in C_0^\infty(\mathbb{D}; \mathbb{R}^3), \mathrm{div}(\mathbf{u}) = 0\}.$$

We have the following density result (cf. [27]).

Lemma 2.1. *$\mathcal{V}$ is dense in $\mathbb{H}^m$ for any $m \in \mathbb{N}_0$.*

We now introduce the following assumptions on the coefficients $\mathbf{f}, \sigma$ and $\mathbf{h}$:

(H1) For any $T > 0$, there exist a constant $C_{T,\mathbf{f}} > 0$ and a function $H_\mathbf{f}(t,x) \in L^1([0,T]\times\mathbb{D})$ such that for any $t \in [0,T], x \in \mathbb{D}, \mathbf{u} \in \mathbb{R}^3$ and $j = 1,2,3$

$$|\partial_{x^j}\mathbf{f}(t,x,\mathbf{u})|^2 + |\mathbf{f}(t,x,\mathbf{u})|^2 \leqslant C_{T,\mathbf{f}} \cdot |\mathbf{u}|^2 + H_\mathbf{f}(t,x),$$
$$|\partial_{u^j}\mathbf{f}(t,x,\mathbf{u})| \leqslant C_{T,\mathbf{f}}.$$

(H2) For any $T > 0$, there exists a constant $C_{\sigma,T} > 0$ such that

$$\sup_{t\in[0,T],x\in\mathbb{D}} \|\partial_{x_j}\sigma(t,x)\|_{l^2} \leqslant C_{\sigma,T}, \quad j = 1,2,3$$

and

$$\sup_{t\in\mathbb{R}_+,x\in\mathbb{D}} \|\sigma(t,x)\|_{l^2}^2 \leqslant 1/4. \tag{2.3}$$

(H3) For any $T > 0$, there exist a constant $C_{T,\mathbf{h}} > 0$ and a function $H_\mathbf{h}(t,x) \in L^1([0,T]\times\mathbb{D})$ such that for any $t \in [0,T], x \in \mathbb{D}, \mathbf{u} \in \mathbb{R}^3$ and $j = 1,2,3$

$$\|\partial_{x^j}\mathbf{h}(t,x,\mathbf{u})\|_{l^2}^2 + \|\mathbf{h}(t,x,\mathbf{u})\|_{l^2}^2 \leqslant C_{T,\mathbf{h}} \cdot |\mathbf{u}|^2 + H_\mathbf{h}(t,x),$$
$$\|\partial_{u^j}\mathbf{h}(t,x,\mathbf{u})\|_{l^2} \leqslant C_{T,\mathbf{h}}.$$

Remark 2.2. *The factor $\frac{1}{4}$ in (2.3) is related to the viscosity constant ν assumed to be* 1. *That is to say, the first order term appearing in diffusion coefficients will be absorbed by the Laplace term. Here, the factor $\frac{1}{4}$ is not optimal (see* [24]*).*

For any $\mathbf{u} \in \mathbb{H}^2$, define

$$A(\mathbf{u}) := \mathscr{P}\Delta\mathbf{u} - \mathscr{P}((\mathbf{u}\cdot\nabla)\mathbf{u}) - \mathscr{P}(g_N(|\mathbf{u}|^2)\mathbf{u}), \tag{2.4}$$

and for any $\mathbf{v} \in \mathbb{H}^2$, we write

$$[\![A(\mathbf{u}), \mathbf{v}]\!] := \langle A(\mathbf{u}), (I-\Delta)\mathbf{v}\rangle_{\mathbb{H}^0} = A_1(\mathbf{u},\mathbf{v}) + A_2(\mathbf{u},\mathbf{v}) + A_3(\mathbf{u},\mathbf{v}),$$

where

$$A_1(\mathbf{u},\mathbf{v}) := \langle \Delta\mathbf{u}, (I-\Delta)\mathbf{v}\rangle_{\mathbb{H}^0},$$
$$A_2(\mathbf{u},\mathbf{v}) := -\langle (\mathbf{u}\cdot\nabla)\mathbf{u}, (I-\Delta)\mathbf{v}\rangle_{\mathbb{H}^0},$$
$$A_3(\mathbf{u},\mathbf{v}) := -\langle g_N(|\mathbf{u}|^2)\mathbf{u}, (I-\Delta)\mathbf{v}\rangle_{\mathbb{H}^0}.$$

Below, for the sake of simplicity, the variable "x" in the coefficients will be dropped. Define for $k \in \mathbb{N}$

$$B_k(t,\mathbf{u}) := \mathscr{P}((\sigma_k(t)\cdot\nabla)\mathbf{u}) + \mathscr{P}\mathbf{h}_k(t,\mathbf{u}). \tag{2.5}$$

Letting the operator $\mathscr{P}$ act on both sides of equation (1.2), we can and shall consider the following equivalent abstract stochastic evolution equation in the sequel:

$$\begin{cases} \mathrm{d}\mathbf{u}(t) = \Big[A(\mathbf{u}(t)) + \mathscr{P}\mathbf{f}(t,\mathbf{u}(t))\Big]\mathrm{d}t + \sum_{k=1}^\infty B_k(t,\mathbf{u}(t))\mathrm{d}W_t^k, \\ \mathbf{u}(0) = \mathbf{u}_0 \in \mathbb{H}^1. \end{cases} \tag{2.6}$$

2.2. **Estimates on A and B.** We now prepare several important estimates for later use. In the sequel, we shall use the following convention: The letter C with subscripts will denote a constant depending on its subscripts and the coefficients. The letter C without subscripts will denote an absolute constant, i.e., its value does not depend on any data. All the constants may have different values in different places.

Lemma 2.3. *For any $\mathbf{u} \in \mathbb{H}^2$, we have*

$$\|A(\mathbf{u})\|_{\mathbb{H}^0} \leqslant C(1 + \|\mathbf{u}\|_{\mathbb{H}^0}^4 + \|\mathbf{u}\|_{\mathbb{H}^2}^2), \tag{2.7}$$

$$\langle A(\mathbf{u}), \mathbf{u}\rangle_{\mathbb{H}^0} = -\|\nabla \mathbf{u}\|_{\mathbb{H}^0}^2 - \| \sqrt{g_N(|\mathbf{u}|^2)} \cdot |\mathbf{u}|\|_{L^2}^2 \tag{2.8}$$

$$\leqslant -\|\nabla \mathbf{u}\|_{\mathbb{H}^0}^2 - \|\mathbf{u}\|_{L^4}^4 + C \cdot N\|\mathbf{u}\|_{\mathbb{H}^0}^2, \tag{2.9}$$

$$[\![A(\mathbf{u}), \mathbf{u}]\!] \leqslant -\frac{1}{2}\|\mathbf{u}\|_{\mathbb{H}^2}^2 - \frac{1}{2}\||\mathbf{u}| \cdot |\nabla \mathbf{u}|\|_{L^2}^2 + C \cdot N\|\nabla \mathbf{u}\|_{\mathbb{H}^0}^2 + \|\mathbf{u}\|_{\mathbb{H}^0}^2. \tag{2.10}$$

Proof. Estimate (2.7) is direct from (2.4) and the Sobolev inequality (2.1). Estimate (2.8) follows from

$$\langle (\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{u}\rangle_{\mathbb{H}^0} = 0.$$

For inequality (2.10), we have

$$A_1(\mathbf{u}, \mathbf{u}) = -\|(I - \Delta)\mathbf{u}\|_{\mathbb{H}^0}^2 + \langle \mathbf{u}, (I - \Delta)\mathbf{u}\rangle_{\mathbb{H}^0} = -\|\mathbf{u}\|_{\mathbb{H}^2}^2 + \|\nabla \mathbf{u}\|_{\mathbb{H}^0}^2 + \|\mathbf{u}\|_{\mathbb{H}^0}^2,$$

and by Young's inequality,

$$A_2(\mathbf{u}, \mathbf{u}) \leqslant \frac{1}{2}\|(I - \Delta)\mathbf{u}\|_{\mathbb{H}^0}^2 + \frac{1}{2}\|(\mathbf{u} \cdot \nabla)\mathbf{u}\|_{\mathbb{H}^0}^2 \leqslant \frac{1}{2}\|\mathbf{u}\|_{\mathbb{H}^2}^2 + \frac{1}{2}\||\mathbf{u}| \cdot |\nabla \mathbf{u}|\|_{\mathbb{H}^0}^2,$$

where

$$|\mathbf{u}|^2 = \sum_{k=1}^{3} |u^k|^2, \quad |\nabla \mathbf{u}|^2 = \sum_{k,i=1}^{3} |\partial_i u^k|^2.$$

Recalling $\nu = 1$, from (1.1), we also have

$$\begin{aligned}
A_3(\mathbf{u}, \mathbf{u}) &= -\langle \nabla(g_N(|\mathbf{u}|^2)\mathbf{u}), \nabla \mathbf{u}\rangle_{\mathbb{H}^0} - \langle g_N(|\mathbf{u}|^2)\mathbf{u}, \mathbf{u}\rangle_{\mathbb{H}^0} \\
&= -\sum_{k,i=1}^{3} \int_{\mathbb{D}} \partial_i u^k \cdot \partial_i(g_N(|\mathbf{u}|^2)u^k)\mathrm{d}x - \int_{\mathbb{D}} |\mathbf{u}|^2 \cdot g_N(|\mathbf{u}|^2)\mathrm{d}x \\
&\leqslant -\sum_{k,i=1}^{3} \int_{\mathbb{D}} \partial_i u^k \cdot \left(g_N(|\mathbf{u}|^2) \cdot \partial_i u^k - g_N'(|\mathbf{u}|^2)\partial_i|\mathbf{u}|^2 \cdot u^k\right) \mathrm{d}x \\
&= -\int_{\mathbb{D}} |\nabla \mathbf{u}|^2 \cdot g_N(|\mathbf{u}|^2)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{D}} g_N'(|\mathbf{u}|^2)|\nabla|\mathbf{u}|^2|^2\mathrm{d}x \\
&\leqslant -\int_{\mathbb{D}} |\nabla \mathbf{u}|^2 \cdot |\mathbf{u}|^2\mathrm{d}x + C \cdot N\|\nabla \mathbf{u}\|_{\mathbb{H}^0}^2.
\end{aligned}$$

Combining the above calculations yields (2.10). □

Lemma 2.4. *Let $\mathbf{v} \in \mathcal{V}$, and let the support of $\mathbf{v}$ be contained in $O := \{x \in \mathbb{D}, |x| \leqslant m\}$ for some $m \in \mathbb{N}$. Let $T > 0$. For any $\mathbf{u}, \mathbf{u}' \in \mathbb{H}^2$ and $t \in [0, T]$, we have*

$$|[\![A(\mathbf{u}), \mathbf{v}]\!]| \leqslant C_{\mathbf{v}} \cdot (1 + \|\mathbf{u}\|_{L^3(O)}^3), \tag{2.11}$$

$$\|\langle B_\cdot(t, \mathbf{u}), \mathbf{v}\rangle_{\mathbb{H}^1}\|_{l^2}^2 \leqslant C_{\mathbf{v},T} \cdot (1 + \|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})} + \|\mathbf{u}\|_{L^2(O)}^2) \tag{2.12}$$

and

$$|[\![A(\mathbf{u}) - A(\mathbf{u}'), \mathbf{v}]\!]| \leqslant C_{\mathbf{v}} \cdot \|\mathbf{u} - \mathbf{u}'\|_{L^2(O)} \cdot (1 + \|\mathbf{u}\|_{\mathbb{H}^1}^2 + \|\mathbf{u}'\|_{\mathbb{H}^1}^2). \tag{2.13}$$

Proof. For estimate (2.11), we have

$$A_1(\mathbf{u},\mathbf{v}) = \langle \mathbf{u}, (I-\Delta)\Delta\mathbf{v}\rangle_{\mathbb{H}^0} \leqslant C\|\mathbf{u}\|_{L^2(O)} \cdot \|\mathbf{v}\|_{\mathbb{H}^4},$$

$$A_2(\mathbf{u},\mathbf{v}) = \langle \mathbf{u}^* \cdot \mathbf{u}, \nabla(I-\Delta)\mathbf{v}\rangle_{\mathbb{H}^0} \leqslant C\|\mathbf{u}\|^2_{L^2(O)} \cdot \sup_{x\in\mathbb{D}} |\nabla(I-\Delta)\mathbf{v}(x)|,$$

where $\mathbf{u}^*$ denotes the transposition of the row vector $\mathbf{u}$, and

$$A_3(\mathbf{u},\mathbf{v}) \leqslant \|\mathbf{u}\|^3_{L^3(O)} \cdot \sup_{x\in\mathbb{D}} |(I-\Delta)\mathbf{v}(x)|.$$

Combining them gives (2.11).

For estimate (2.12), by **(H2)** and **(H3)**, we have

$$\begin{aligned}
\|\langle B_\cdot(t,\mathbf{u}),\mathbf{v}\rangle_{\mathbb{H}^1}\|^2_{l^2} &\leqslant C \sup_{t\in\mathbb{R}_+, x\in\mathbb{D}} \|\sigma(t,x)\|^2_{l^2} \cdot \sup_{x\in\mathbb{D}} |\nabla(I-\Delta)\mathbf{v}(x)|^2 \cdot \|\mathbf{u}\|^2_{L^2(O)} \\
&\quad + C \sup_{x\in\mathbb{D}} \|\nabla_x\sigma(t,x)\|^2_{l^2} \cdot \sup_{x\in\mathbb{D}} |(I-\Delta)\mathbf{v}(x)|^2 \cdot \|\mathbf{u}\|^2_{L^2(O)} \\
&\quad + C \sup_{x\in\mathbb{D}} |(I-\Delta)\mathbf{v}(x)|^2 \cdot \left(C_{T,\mathbf{h}} \cdot \|\mathbf{u}\|^2_{L^2(O)} + \|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})}\right) \\
&\leqslant C_{\mathbf{v},T} \cdot \left(1 + \|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})} + \|\mathbf{u}\|^2_{L^2(O)}\right).
\end{aligned}$$

We now look at (2.13). For A_1, we clearly have

$$|A_1(\mathbf{u},\mathbf{v}) - A_1(\mathbf{u}',\mathbf{v})| = |\langle(\mathbf{u}-\mathbf{u}')\cdot 1_O, (I-\Delta)\Delta\mathbf{v}\rangle_{\mathbb{H}^0}| \leqslant C_{\mathbf{v}} \cdot \|\mathbf{u}-\mathbf{u}'\|_{L^2(O)}.$$

For A_2, we have

$$\begin{aligned}
|A_2(\mathbf{u},\mathbf{v}) - A_2(\mathbf{u}',\mathbf{v})| &= |\langle \mathbf{u}^*\cdot\mathbf{u} - \mathbf{u}'^*\cdot\mathbf{u}', \nabla(I-\Delta)\mathbf{v}\rangle_{\mathbb{H}^0}| \\
&\leqslant C_{\mathbf{v}} \cdot \|\mathbf{u}-\mathbf{u}'\|_{L^2(O)} \cdot (\|\mathbf{u}\|_{\mathbb{H}^0} + \|\mathbf{u}'\|_{\mathbb{H}^0}).
\end{aligned}$$

For A_3, by Sobolev inequality (2.1) we similarly have

$$\begin{aligned}
|A_3(\mathbf{u},\mathbf{v}) - A_3(\mathbf{u}',\mathbf{v})| &\leqslant C_{\mathbf{v}} \cdot \|\mathbf{u}-\mathbf{u}'\|_{L^2(O)} \cdot (\|\mathbf{u}\|^2_{L^4} + \|\mathbf{u}'\|^2_{L^4}) \\
&\leqslant C_{\mathbf{v}} \cdot \|\mathbf{u}-\mathbf{u}'\|_{L^2(O)} \cdot (\|\mathbf{u}\|^2_{\mathbb{H}^1} + \|\mathbf{u}'\|^2_{\mathbb{H}^1}).
\end{aligned}$$

□

Lemma 2.5. *For any $T > 0$ and $\mathbf{u} \in \mathbb{H}^2$,*

$$\|B(t,\mathbf{u})\|^2_{L_2(l^2;\mathbb{H}^0)} \leqslant \frac{1}{2}\|\mathbf{u}\|^2_{\mathbb{H}^1} + C_T\|\mathbf{u}\|^2_{\mathbb{H}^0} + \|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})}, \tag{2.14}$$

$$\|B(t,\mathbf{u})\|^2_{L_2(l^2;\mathbb{H}^1)} \leqslant \frac{1}{2}\|\mathbf{u}\|^2_{\mathbb{H}^2} + C_T\|\mathbf{u}\|^2_{\mathbb{H}^1} + C\|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})}. \tag{2.15}$$

Proof. First of all, by **(H2)** and **(H3)**, we have

$$\begin{aligned}
\|B(t,\mathbf{u})\|^2_{L_2(l^2;\mathbb{H}^0)} &= \sum_{k=1}^{\infty} \int_{\mathbb{D}} |B_k(t,x,\mathbf{u}(x))|^2 \mathrm{d}x \leqslant \\
&\leqslant 2\int_{\mathbb{D}} \|\sigma(t,x)\|^2_{l^2} \cdot |\nabla\mathbf{u}(x)|^2 \mathrm{d}x + 2\int_{\mathbb{D}} (C_{T,\mathbf{h}}|\mathbf{u}(x)|^2 + H_{\mathbf{h}}(t,x))\mathrm{d}x \\
&\leqslant \frac{1}{2}\|\mathbf{u}\|^2_{\mathbb{H}^1} + C_T\|\mathbf{u}\|^2_{\mathbb{H}^0} + \|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})}.
\end{aligned}$$

Secondly, noting that

$$\|B(t,\mathbf{u})\|^2_{L_2(l^2;\mathbb{H}^1)} = \|B(t,\mathbf{u})\|^2_{L_2(l^2;\mathbb{H}^0)} + \|\nabla B(t,\mathbf{u})\|^2_{L_2(l^2;\mathbb{H}^0)}$$

and

$$\partial_{x^j} B_k(t,\mathbf{u}) = \mathscr{P}\partial_{x^j}((\sigma_k(t)\cdot\nabla)\mathbf{u}) + \mathscr{P}\partial_{x^j}\mathbf{h}_k(t,\mathbf{u})$$

$$= \mathscr{P}\Big((\partial_{x^j}\sigma_k(t)\cdot\nabla)\mathbf{u} + (\sigma_k(t)\cdot\nabla)\partial_{x^j}\mathbf{u}\Big)$$
$$+ \mathscr{P}\Big((\partial_{x^j}\mathbf{h}_k)(t,\mathbf{u}) + \sum_{i=1}^{3}\partial_{u^i}\mathbf{h}_k(t,\mathbf{u})\cdot\partial_{x^j}u^i\Big),$$

by **(H2)** and **(H3)**, we have

$$\|B(t,\mathbf{u})\|^2_{L_2(l^2;\mathbb{H}^1)} \leqslant \frac{1}{2}\|\mathbf{u}\|^2_{\mathbb{H}^2} + C_T\|\mathbf{u}\|^2_{\mathbb{H}^1} + C\|H_{\mathbf{h}}(t)\|_{L^1(\mathbb{D})}.$$

□

2.3. **Tightness Criterion.** In the following, we only give a tightness criterion in the case of $\mathbb{D} = \mathbb{R}^3$. When $\mathbb{D} = \mathbb{T}^3$, since $\mathbb{H}^1$ is compactly embedded in $\mathbb{H}^0$, the corresponding result is simple and well known.

By $\mathbb{H}^0_{loc}$ we denote the space of all locally L^2-integrable and divergence free vector fields endowed with the Fréchet metric: for $\mathbf{u},\mathbf{v} \in \mathbb{H}^0_{loc}$

$$\rho(\mathbf{u},\mathbf{v}) := \sum_{m=1}^{\infty} 2^{-m}\left(\left[\int_{|x|\leqslant m}|\mathbf{u}(x)-\mathbf{v}(x)|^2\mathrm{d}x\right]^{1/2}\wedge 1\right).$$

Thus, $(\mathbb{H}^0_{loc},\rho)$ is a Polish space and $\mathbb{H}^0 \subset \mathbb{H}^0_{loc}$.

Let $\mathbb{X} := C(\mathbb{R}_+;\mathbb{H}^0_{loc})$ denote the space of all continuous functions from $\mathbb{R}_+$ to $(\mathbb{H}^0_{loc},\rho)$ equipped with the metric

$$\rho_{\mathbb{X}}(\mathbf{u},\mathbf{v}) := \sum_{m=1}^{\infty} 2^{-m}\left(\sup_{t\in[0,m]}\rho(\mathbf{u}(t),\mathbf{v}(t))\wedge 1\right).$$

In the following, we shall fix a complete orthonormal basis $\mathscr{E} := \{\mathbf{e}_k, k\in\mathbb{N}\} \subset \mathcal{V}$ of $\mathbb{H}^1$ such that span$\{\mathscr{E}\}$ is a dense subset of $\mathbb{H}^3$ and, in the case of periodic boundary conditions, we also require that $\mathscr{E}$ is an orthogonal basis of $\mathbb{H}^0$. Moreover, for $\mathbf{u}\in\mathbb{H}^0$ and $\mathbf{v}\in\mathbb{H}^2$, the inner product $\langle\mathbf{u},\mathbf{v}\rangle_{\mathbb{H}^1}$ is taken in the generalized sense, i.e.,

$$\langle\mathbf{u},\mathbf{v}\rangle_{\mathbb{H}^1} = \langle\mathbf{u},(I-\Delta)\mathbf{v}\rangle_{\mathbb{H}^0}.$$

We need the following relative compactness result, which is essentially due to Ladyzhenskaya [18, Theorem 13].

Lemma 2.6. *Let $K\subset\mathbb{X}$. If for every $T>0$,*

(1^o) $\sup_{\mathbf{u}\in K}\sup_{s\in[0,T]}\|\mathbf{u}(s)\|_{\mathbb{H}^1} < +\infty$,

(2^o) $\lim_{\delta\to 0}\sup_{\mathbf{u}\in K}\sup_{t,s\in[0,T],|t-s|<\delta}|\langle\mathbf{u}(t)-\mathbf{u}(s),\mathbf{e}\rangle_{\mathbb{H}^1}| = 0$ *for any* $\mathbf{e}\in\mathscr{E}$,

then K is relatively compact in $\mathbb{X}$.

Proof. We only need to prove that K is relatively compact in $C([0,T];\mathbb{H}^0_{loc})$ for every $T>0$. Let $\{\mathbf{u}_n, n\in\mathbb{N}\}\subset K$ be any sequence of K. Define for $\mathbf{e}\in\mathscr{E}$,

$$G^{\mathbf{e}}_n(t) := \langle\mathbf{u}_n(t),\mathbf{e}\rangle_{\mathbb{H}^1}.$$

Then, by (1^o) and (2^o), the sequence $\{t\mapsto G^{\mathbf{e}}_n(t), n\in\mathbb{N}\}$ is uniformly bounded and equi-continuous on $[0,T]$. Hence, by Ascoli-Arzelà's lemma, there exist a subsequence n_l (depending on $\mathbf{e}$) and a continuous function $G^{\mathbf{e}}(t)$ such that $G^{\mathbf{e}}_{n_l}(t)$ uniformly converges to $G^{\mathbf{e}}(t)$ on $[0,T]$. Since $\mathscr{E}$ is countable, by a diagonalization method, we may further find a common subsequence (still denoted by n) such that for any $\mathbf{e}\in\mathscr{E}$,

$$\lim_{n\to\infty}\sup_{t\in[0,T]}|G^{\mathbf{e}}_n(t)-G^{\mathbf{e}}(t)| = 0.$$

Thus, by the weak compactness of closed balls in $\mathbb{H}^1$, there is a $\mathbf{u} \in L^\infty(0,T;\mathbb{H}^1)$ such that for any $\mathbf{e} \in \mathscr{E}$,

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |\langle \mathbf{u}_n(t) - \mathbf{u}(t), \mathbf{e}\rangle_{\mathbb{H}^1}| = 0.$$

By a simple approximation we further have for any $\mathbf{v} \in \mathbb{H}^1$,

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |\langle \mathbf{u}_n(t) - \mathbf{u}(t), \mathbf{v}\rangle_{\mathbb{H}^1}| = 0.$$

Note that $(I-\Delta)^{-1}\mathbf{v} \in \mathbb{H}^2$ for any $\mathbf{v} \in \mathbb{H}^0$. Hence, we also have for any $\mathbf{v} \in \mathbb{H}^0$,

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |\langle \mathbf{u}_n(t) - \mathbf{u}(t), \mathbf{v}\rangle_{\mathbb{H}^0}| = 0.$$

Hence, by Helmholtz-Weyl's decomposition (cf. [32, 13]),

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |\langle \mathbf{u}_n(t) - \mathbf{u}(t), \mathbf{v}\rangle| = 0 \tag{2.16}$$

for any $\mathbf{v} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$.

We now show that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \rho(\mathbf{u}_n(t), \mathbf{u}(t)) = 0.$$

It suffices to prove that for any $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \int_{|x|\leqslant m} |\mathbf{u}_n(t,x) - \mathbf{u}(t,x)|^2 \mathrm{d}x = 0,$$

which follows from (1^o), (2.16) and the following Friedrichs inequality (cf. [18, p.176]): Let $\mathcal{O} \subset \mathbb{R}^3$ be any bounded domain. For any $\epsilon > 0$, there exist $N_\epsilon \in \mathbb{N}$ and functions $h_i \in L^2(\mathcal{O})$, $i = 1, \cdots, N_\epsilon$ such that for any $w \in W_0^{1,2}(\mathcal{O})$,

$$\int_{\mathcal{O}} |w(x)|^2 \mathrm{d}x \leqslant \sum_{i=1}^{N_\epsilon} \left(\int_{\mathcal{O}} w(x) h_i(x) \mathrm{d}x \right)^2 + \epsilon \int_{\mathcal{O}} |\nabla w(x)|^2 \mathrm{d}x.$$

□

Lemma 2.7. *Let μ_n be a family of probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Assume that*

(1^o) For each $\mathbf{e} \in \mathscr{E}$ and any $\epsilon, T > 0$,

$$\lim_{\delta\downarrow 0} \sup_n \mu_n \Big\{ \mathbf{u} \in \mathbb{X} : \sup_{s,t\in[0,T], |s-t|\leqslant\delta} |\langle \mathbf{u}(t) - \mathbf{u}(s), \mathbf{e}\rangle_{\mathbb{H}^1}| > \epsilon \Big\} = 0.$$

(2^o) For any $T > 0$

$$\lim_{R\to\infty} \sup_n \mu_n \Big\{ \mathbf{u} \in \mathbb{X} : \sup_{s\in[0,T]} \|\mathbf{u}(s)\|_{\mathbb{H}^1} > R \Big\} = 0.$$

Then $\{\mu_n, n \in \mathbb{N}\}$ is tight on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$.

Proof. Fix $\eta > 0$. For any $l \in \mathbb{N}$, by (2^o) one can choose R_l sufficiently large such that

$$\sup_n \mu_n \Big\{ \mathbf{u} \in \mathbb{X} : \sup_{s\in[0,l]} \|\mathbf{u}(s)\|_{\mathbb{H}^1} > R_l \Big\} \leqslant \frac{\eta}{2^l}. \tag{2.17}$$

For $k, l \in \mathbb{N}$ and $\mathbf{e}_i \in \mathscr{E}$, by ($1^o$) one may choose $\delta_{k,i,l} > 0$ small enough such that

$$\sup_n \mu_n \Big\{ \mathbf{u} \in \mathbb{X} : \sup_{s,t\in[0,l], |s-t|\leqslant\delta_{k,i,l}} |\langle \mathbf{u}(t) - \mathbf{u}(s), \mathbf{e}_i\rangle_{\mathbb{H}^1}| > \frac{1}{k} \Big\} \leqslant \frac{\eta}{2^{k+i+l}}. \tag{2.18}$$

Now let us define

$$K_1 := \bigcap_{k,l\in\mathbb{N}, \mathbf{e}_i\in\mathscr{E}} \Big\{ \mathbf{u} \in \mathbb{X} : \sup_{s,t\in[0,l], |s-t|\leqslant\delta_{k,i,l}} |\langle \mathbf{u}(t) - \mathbf{u}(s), \mathbf{e}_i\rangle_{\mathbb{H}^1}| \leqslant \frac{1}{k} \Big\}$$

$$K_2 := \bigcap_{l\in\mathbb{N}} \Big\{ \mathbf{u} \in \mathbb{X} : \sup_{s\in[0,l]} \|\mathbf{u}(s)\|_{\mathbb{H}^1} \leqslant R_l \Big\}.$$

By Lemma 2.6, $K_1 \cap K_2$ is a relatively compact set in $\mathbb{X}$. By (2.17) and (2.18), we also have

$$\sup_n \mu_n(K_1^c \cup K_2^c) \leqslant 2\eta.$$

In view of the arbitrariness of η, $\{\mu_n, n \in \mathbb{N}\}$ is tight on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. □

3. Existence and Uniqueness of Strong Solutions

3.1. **Weak and Strong Solutions.** For a metric space $\mathbb{U}$, we use $\mathcal{P}(\mathbb{U})$ to denote the total of all probability measures on $\mathbb{U}$. We first introduce the following notion of weak solutions to Eq. (2.6).

Definition 3.1. *We say that Eq. (2.6) has a weak solution with initial law* $\vartheta \in \mathcal{P}(\mathbb{H}^1)$ *if there exist a stochastic basis* $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\geqslant 0})$*, an* $\mathbb{H}^1$*-valued* $(\mathcal{F}_t)$*-adapted process* $\mathbf{u}$ *and an infinite sequence of independent standard* $(\mathcal{F}_t)$*-Brownian motions* $\{W^k(t), t \geqslant 0, k \in \mathbb{N}\}$ *such that*

(i) $\mathbf{u}(0)$ *has law* ϑ *in* $\mathbb{H}^1$*;*

(ii) for almost all $\omega \in \Omega$ *and every* $T > 0$, $\mathbf{u}(\cdot, \omega) \in C([0,T]; \mathbb{H}^1) \cap L^2([0,T]; \mathbb{H}^2)$*;*

(iii) it holds that in $\mathbb{H}^0$

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \Big[A(\mathbf{u}(s)) + \mathscr{P}\mathbf{f}(s, \mathbf{u}(s)) \Big] \mathrm{d}s + \sum_{k=1}^{\infty} \int_0^t B_k(s, \mathbf{u}(s)) \mathrm{d}W_s^k,$$

for all $t \geqslant 0$*,* P*-a.s..*

This solution is denoted by $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\geqslant 0}; W; \mathbf{u})$.

Remark 3.2. *Under* **(H1)**-**(H3)***, by (2.7) the above integrals are meaningful.*

Definition 3.3. *(Pathwise Uniqueness) We say that the pathwise uniqueness holds for Eq. (2.6) if whenever we are given two weak solutions of Eq. (2.6) defined on the same probability space together with the same Brownian motion*

$$(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\geqslant 0}; W; \mathbf{u})$$
$$(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\geqslant 0}; W; \tilde{\mathbf{u}}),$$

the condition $P\{\mathbf{u}(0) = \tilde{\mathbf{u}}(0)\} = 1$ *implies* $P\{\omega : \mathbf{u}(t,\omega) = \tilde{\mathbf{u}}(t,\omega), \forall t \geqslant 0\} = 1$.

We have the following martingale characterization for the weak solution (cf. [31]). For the reader's convenience, a short proof is provided in the Appendix.

Proposition 3.4. *Let* $\mathscr{E}$ *be given in Subsection 2.3. For* $\vartheta \in \mathcal{P}(\mathbb{H}^1)$*, the following two statements are equivalent:*

(i) Eq. (2.6) has a weak solution with initial law ϑ*.*

(ii) There exists a probability measure $P_\vartheta \in \mathcal{P}(\mathbb{X})$ *such that for* P_ϑ*-almost all* $\mathbf{u} \in \mathbb{X}$ *and any* $T > 0$*,*

$$\mathbf{u} \in L^\infty([0,T]; \mathbb{H}^1) \cap L^2([0,T]; \mathbb{H}^2), \tag{3.1}$$

and for any $h \in C_0^\infty(\mathbb{R})$*, i.e., any smooth function with compact support, and any* $\mathbf{e} \in \mathscr{E}$*,*

$$M_{\mathbf{e}}^h(t, \mathbf{u}) := h(\langle \mathbf{u}(t), \mathbf{e} \rangle_{\mathbb{H}^1}) - h(\langle \mathbf{u}(0), \mathbf{e} \rangle_{\mathbb{H}^1})$$
$$- \int_0^t h'(\langle \mathbf{u}(s), \mathbf{e} \rangle_{\mathbb{H}^1}) \cdot [\![A(\mathbf{u}(s)), \mathbf{e}]\!] \mathrm{d}s$$

$$- \int_0^t h'(\langle \mathbf{u}(s), \mathbf{e}\rangle_{\mathbb{H}^1}) \cdot \langle \mathbf{f}(s, \mathbf{u}(s)), \mathbf{e}\rangle_{\mathbb{H}^1} \mathrm{d}s$$
$$- \frac{1}{2} \int_0^t h''(\langle \mathbf{u}(s), \mathbf{e}\rangle_{\mathbb{H}^1}) \cdot \|\langle B(s, \mathbf{u}(s)), \mathbf{e}\rangle_{\mathbb{H}^1}\|_{l^2}^2 \mathrm{d}s$$

is a continuous local martingale under P_ϑ with respect to $\mathcal{B}_t(\mathbb{X})$. Here and below, $\mathcal{B}_t(\mathbb{X})$ denotes the sub σ-algebra of $\mathbb{X}$ up to time t.

In order to introduce the notion of strong solutions to Eq. (2.6), we need a canonical realization of an infinite sequence of independent standard Brownian motions on a Polish space.

Let $C(\mathbb{R}_+; \mathbb{R})$ denote the space of all continuous functions defined on $\mathbb{R}_+$, which is equipped with the metric

$$\tilde{\rho}(w, w') = \sum_{k=1}^{\infty} 2^{-k} \left(\sup_{t \in [0,k]} |w(t) - w'(t)| \wedge 1 \right).$$

Define the product space $\mathbb{W} := \prod_{j=1}^{\infty} C(\mathbb{R}_+; \mathbb{R})$, which is endowed with the metric:

$$\rho_{\mathbb{W}}(w, w') = \sum_{j=1}^{\infty} 2^{-j} (\tilde{\rho}(w^j, w'^j) \wedge 1), \quad w = (w^1, w^2, \cdots), w' = (w'^1, w'^2, \cdots).$$

Then $(\mathbb{W}, \rho_{\mathbb{W}})$ is a Polish space. Let $\mathcal{B}_t(\mathbb{W}) \subset \mathcal{B}(\mathbb{W})$ be the σ-algebra up to time t. We endow $(\mathbb{W}, \mathcal{B}(\mathbb{W}))$ with the Wiener measure $\mathbb{P}$ such that the coordinate process

$$w(t) := (w^1(t), w^2(t), \cdots)$$

is an infinite sequence of independent standard $\mathcal{B}_t(\mathbb{W})$-Brownian motions on $(\mathbb{W}, \mathcal{B}(\mathbb{W}), \mathbb{P})$.

Let $\mathbb{B} := C(\mathbb{R}_+; \mathbb{H}^1)$ denote the space of all continuous functions from $\mathbb{R}_+$ to $\mathbb{H}^1$, which is endowed with the metric

$$\rho_{\mathbb{B}}(\mathbf{u}, \mathbf{v}) := \sum_{k=1}^{\infty} 2^{-k} \left(\sup_{t \in [0,k]} \|\mathbf{u}(t) - \mathbf{v}(t)\|_{\mathbb{H}^1} \wedge 1 \right).$$

In the following, $\mathcal{B}_t(\mathbb{B})$ denotes the sub σ-algebra of $\mathbb{B}$ up to time t. For a measure space $(S, \mathcal{S}, \lambda)$, $\overline{\mathcal{S}}^{\lambda}$ will denote the completion of $\mathcal{S}$ with respect to λ.

Definition 3.5. *Let $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\geqslant 0}; W; \mathbf{u})$ be a weak solution of Eq. (2.6) with initial distribution $\vartheta \in \mathcal{P}(\mathbb{H}^1)$. If there exists a $\overline{\mathcal{B}(\mathbb{H}^1) \times \mathcal{B}(\mathbb{W})}^{\vartheta\times\mathbb{P}} / \mathcal{B}(\mathbb{B})$-measurable functional $F_\vartheta : \mathbb{H}^1 \times \mathbb{W} \mapsto \mathbb{B}$, with the property that for every $t > 0$,*

$$F_\vartheta \in \hat{\mathcal{B}}_t / \mathcal{B}_t(\mathbb{B}); \quad \hat{\mathcal{B}}_t := \overline{\mathcal{B}(\mathbb{H}^1) \times \mathcal{B}_t(\mathbb{W})}^{\vartheta\times\mathbb{P}} \tag{3.2}$$

and such that

$$\mathbf{u}(\cdot) = F_\vartheta(\mathbf{u}(0), W(\cdot)), \quad P - a.s.,$$

we call $\mathbf{u}$ together with W a strong solution.

We shall say that Eq. (2.6) has a unique strong solution associated with $\vartheta \in \mathcal{P}(\mathbb{H}^1)$ if there exists a functional $F_\vartheta : \mathbb{H}^1 \times \mathbb{W} \mapsto \mathbb{B}$ with the same properties as above such that

(i) for any infinite sequence of independent standard $(\mathcal{F}_t)$-Brownian motions $\{W(t), t \geqslant 0\}$ on stochastic basis $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\geqslant 0})$, and any $\mathbb{H}^1$-valued random variable $\mathbf{u}_0 \in \mathcal{F}_0$ with distribution ϑ,

$$(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\geqslant 0}; W; F_\vartheta(\mathbf{u}_0, W(\cdot))) \text{ is a weak solution of Eq. (2.6)};$$

(ii) for any weak solution $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\geqslant 0}; W; \mathbf{u})$ *of Eq. (2.6) with initial law* ϑ,

$$\mathbf{u}(\cdot) = F_\vartheta(\mathbf{u}(0), W(\cdot)), \quad P - a.s..$$

The following Yamada-Watanabe theorem holds in this case (cf. [28]).

Theorem 3.6. *Existence of weak solutions plus pathwise uniqueness implies the existence of a unique strong solution.*

3.2. **Pathwise Uniqueness.** We first prove the following pathwise uniqueness result.

Theorem 3.7. *Under* **(H1)**-**(H3)***, pathwise uniqueness holds for Eq. (2.6).*

Proof. Let $\mathbf{u}$ and $\tilde{\mathbf{u}}$ be two weak solutions of Eq. (2.6) defined on the same probability space together with the same Brownian motion, and starting from the same initial value $\mathbf{u}_0$. For any $T > 0$ and $R > 0$, define the stopping time

$$\tau_R := \inf\{t \in [0, T] : \|\mathbf{u}(t)\|_{\mathbb{H}^1} \vee \|\tilde{\mathbf{u}}(t)\|_{\mathbb{H}^1} \geqslant R\}.$$

By the definition of weak solutions, one knows that $\tau_R \uparrow \infty$ as $R \uparrow \infty$.

Set

$$\mathbf{w}(t) := \mathbf{u}(t) - \tilde{\mathbf{u}}(t).$$

Then by Itô's formula, we have

$$\begin{aligned} \|\mathbf{w}(t)\|^2_{\mathbb{H}^0} &= 2\int_0^t \langle A(\mathbf{u}(s)) - A(\tilde{\mathbf{u}}(s)), \mathbf{w}(s)\rangle_{\mathbb{H}^0} \mathrm{d}s \\ &\quad + 2\int_0^t \langle \mathbf{f}(s, \mathbf{u}(s)) - \mathbf{f}(s, \tilde{\mathbf{u}}(s)), \mathbf{w}(s)\rangle_{\mathbb{H}^0} \mathrm{d}s \\ &\quad + 2\sum_{k=1}^\infty \int_0^t \langle B_k(s, \mathbf{u}(s)) - B_k(s, \tilde{\mathbf{u}}(s)), \mathbf{w}(s)\rangle_{\mathbb{H}^0} \mathrm{d}W_s^k \\ &\quad + \sum_{k=1}^\infty \int_0^t \|B_k(s, \mathbf{u}(s)) - B_k(s, \tilde{\mathbf{u}}(s))\|^2_{\mathbb{H}^0} \mathrm{d}s \\ &=: I_1(t) + I_2(t) + I_3(t) + I_4(t). \end{aligned} \tag{3.3}$$

By $|g_N(r) - g_N(r')| \leqslant |r - r'|$ and a simple calculation, it is easy to see that

$$\begin{aligned} I_1(t) &= -2\int_0^t \|\nabla \mathbf{w}(s)\|^2_{\mathbb{H}^0} \mathrm{d}s + 2\int_0^t \langle \nabla \mathbf{w}(s), (\mathbf{u}^*(s) \cdot \mathbf{u}(s) - \tilde{\mathbf{u}}^*(s) \cdot \tilde{\mathbf{u}}(s))\rangle_{\mathbb{H}^0} \mathrm{d}s \\ &\quad - 2\int_0^t \langle g_N(|\mathbf{u}(s)|^2)\mathbf{u}(s) - g_N(|\tilde{\mathbf{u}}(s)|^2)\tilde{\mathbf{u}}(s), \mathbf{w}(s)\rangle_{\mathbb{H}^0} \mathrm{d}s \\ &\leqslant -\int_0^t \|\nabla \mathbf{w}(s)\|^2_{\mathbb{H}^0} \mathrm{d}s + \int_0^t \|\mathbf{u}^*(s) \cdot \mathbf{u}(s) - \tilde{\mathbf{u}}^*(s) \cdot \tilde{\mathbf{u}}(s)\|^2_{\mathbb{H}^0} \mathrm{d}s \\ &\quad + 8\int_0^t \||\mathbf{w}(s)| \cdot (|\mathbf{u}(s)| + |\tilde{\mathbf{u}}(s)|)\|^2_{\mathbb{H}^0} \mathrm{d}s. \end{aligned}$$

Noting that by Sobolev inequality (2.1),

$$\begin{aligned} \|\mathbf{u}^*(s) \cdot \mathbf{u}(s) - \tilde{\mathbf{u}}^*(s) \cdot \tilde{\mathbf{u}}(s)\|^2_{\mathbb{H}^0} &\leqslant \||\mathbf{w}(s)|(|\mathbf{u}(s)| + |\tilde{\mathbf{u}}(s)|)\|^2_{\mathbb{H}^0} \\ &\leqslant 2\|\mathbf{w}(s)\|^2_{L^4}(\|\mathbf{u}(s)\|^2_{L^4} + \|\tilde{\mathbf{u}}(s)\|^2_{L^4}) \\ &\leqslant 2C^2_{1,4} \cdot \|\mathbf{w}(s)\|^{3/2}_{\mathbb{H}^1} \|\mathbf{w}(s)\|^{1/2}_{\mathbb{H}^0} (\|\mathbf{u}(s)\|^2_{\mathbb{H}^1} + \|\tilde{\mathbf{u}}(s)\|^2_{\mathbb{H}^1}), \end{aligned} \tag{3.4}$$

we have by Young's inequality,

$$
\begin{aligned}
I_1(t\wedge\tau_R) &\leqslant -\int_0^{t\wedge\tau_R}\|\nabla\mathbf{w}(s)\|_{\mathbb{H}^0}^2\mathrm{d}s + C_R\int_0^{t\wedge\tau_R}\|\mathbf{w}(s)\|_{\mathbb{H}^1}^{3/2}\|\mathbf{w}(s)\|_{\mathbb{H}^0}^{1/2}\mathrm{d}s\\
&\leqslant -\frac{1}{2}\int_0^{t\wedge\tau_R}\|\nabla\mathbf{w}(s)\|_{\mathbb{H}^0}^2\mathrm{d}s + C_R\int_0^{t\wedge\tau_R}\|\mathbf{w}(s)\|_{\mathbb{H}^0}^2\mathrm{d}s.
\end{aligned}
$$

Moreover, it is clear that

$$
I_2(t\wedge\tau_R)\leqslant C_T\int_0^{t\wedge\tau_R}\|\mathbf{w}(s)\|_{\mathbb{H}^0}^2\mathrm{d}s
$$

and by **(H3)**,

$$
I_4(t\wedge\tau_R)\leqslant \sup_{t\geqslant 0,x\in\mathbb{D}}\|\sigma(t,x)\|_{l^2}\cdot\int_0^{t\wedge\tau_R}\|\nabla\mathbf{w}(s)\|_{\mathbb{H}^0}^2\mathrm{d}s + C_T\int_0^{t\wedge\tau_R}\|\mathbf{w}(s)\|_{\mathbb{H}^0}^2\mathrm{d}s.
$$

Taking expectations for (3.3) and combining the above calculations as well as (2.3), we find that for any $t\in[0,T]$,

$$
\mathbb{E}\|\mathbf{w}(t\wedge\tau_R)\|_{\mathbb{H}^0}^2\leqslant C_{R,T}\cdot\mathbb{E}\left(\int_0^{t\wedge\tau_R}\|\mathbf{w}(s)\|_{\mathbb{H}^0}^2\mathrm{d}s\right)\leqslant C_{R,T}\int_0^t\mathbb{E}\|\mathbf{w}(s\wedge\tau_R)\|_{\mathbb{H}^0}^2\mathrm{d}s.
$$

By Gronwall's inequality, we get for any $t\in[0,T]$,

$$
\mathbb{E}\|\mathbf{w}(t\wedge\tau_R)\|_{\mathbb{H}^0}^2 = 0.
$$

Now the uniqueness follows by letting $R\uparrow\infty$ and Fatou's lemma. □

3.3. **Existence of Martingale Solutions.** We now prove the existence of a weak solution to Eq. (2.6).

Theorem 3.8. *Under* **(H1)-(H3)***, for any initial law $\vartheta\in\mathcal{P}(\mathbb{H}^1)$, there exists a weak solution for Eq. (2.6) in the sense of Definition 3.1.*

We shall use Galerkin's approximation to prove this theorem. In the following, we fix a stochastic basis $(\Omega,\mathcal{F},P;(\mathcal{F}_t)_{t\geqslant 0})$, and an infinite sequence of independent standard $(\mathcal{F}_t)$-Brownian motions $\{W^k(t),t\geqslant 0,k\in\mathbb{N}\}$, as well as an $\mathcal{F}_0$-measurable random variable $\mathbf{u}_0$ having law ϑ.

Recall that $\mathscr{E}=\{\mathbf{e}_i,i\in\mathbb{N}\}\subset\mathcal{V}$ is a complete orthonormal basis of $\mathbb{H}^1$. Set

$$
\mathbb{H}_n^1 := \mathrm{span}\{e_i, i=1,\cdots,n\}
$$

and for $\mathbf{u}\in\mathbb{H}^0$,

$$
\Pi_n\mathbf{u} := \sum_{i=1}^n\langle\mathbf{u},\mathbf{e}_i\rangle_{\mathbb{H}^1}\mathbf{e}_i = \sum_{i=1}^n\langle\mathbf{u},(I-\Delta)\mathbf{e}_i\rangle_{\mathbb{H}^0}\mathbf{e}_i.
$$

Consider the following finite dimensional stochastic ordinary differential equation in $\mathbb{H}_n^1$

$$
\left\{
\begin{array}{rcl}
\mathrm{d}\mathbf{u}_n(t) &=& [\Pi_nA(\mathbf{u}_n(t))+\Pi_n\mathbf{f}(t,\mathbf{u}_n(t))]\mathrm{d}t+\sum_k\Pi_nB_k(t,\mathbf{u}_n(t))\mathrm{d}W_t^k,\\
\mathbf{u}_n(0) &=& \Pi_n\mathbf{u}_0.
\end{array}
\right.
$$

By Lemmas 2.3 and 2.5, we have, for some $C_{n,N}>0$ and any $\mathbf{u}\in\mathbb{H}_n^1$,

$$
\begin{aligned}
\langle\mathbf{u},\Pi_nA(\mathbf{u})+\Pi_n\mathbf{f}(t,\mathbf{u})\rangle_{\mathbb{H}_n^1}&\leqslant C_{n,N}(\|\mathbf{u}\|_{\mathbb{H}_n^1}^2+1),\\
\|\Pi_nB(t,\mathbf{u})\|_{l^2\otimes\mathbb{H}_n^1}&\leqslant C_{n,N}(\|\mathbf{u}\|_{\mathbb{H}_n^1}^2+1).
\end{aligned}
$$

Moreover, by **(H1)-(H3)** it is easy to see that

$$
\mathbb{H}_n^1\ni\mathbf{u}\mapsto\Pi_nA(\mathbf{u})+\Pi_n\mathbf{f}(t,\mathbf{u})\in\mathbb{H}_n^1
$$

and

$$
\mathbb{H}_n^1\ni\mathbf{u}\mapsto\Pi_nB(t,\mathbf{u})\in l^2\times\mathbb{H}_n^1
$$

are locally Lipschitz continuous. Hence, by the theory of SDE (cf. [17, 26]), there is a unique continuous $(\mathcal{F}_t)$-adapted process $\mathbf{u}_n(t)$ satisfying

$$\mathbf{u}_n(t) = \mathbf{u}_n(0) + \int_0^t \Pi_n A(\mathbf{u}_n(s))\mathrm{d}s + \int_0^t \Pi_n \mathbf{f}(s, \mathbf{u}_n(s))\mathrm{d}s + \sum_{k=1}^{\infty} \int_0^t \Pi_n B_k(s, \mathbf{u}_n(s))\mathrm{d}W_s^k \tag{3.5}$$

and for any $n \geqslant i$,

$$\begin{aligned}\langle \mathbf{u}_n(t), \mathbf{e}_i\rangle_{\mathbb{H}^1} &= \langle \mathbf{u}_0, \mathbf{e}_i\rangle_{\mathbb{H}^1} + \int_0^t [\![A(\mathbf{u}_n(s)), \mathbf{e}_i]\!]\mathrm{d}s + \int_0^t \langle \mathbf{f}(s, \mathbf{u}_n(s)), \mathbf{e}_i\rangle_{\mathbb{H}^1}\mathrm{d}s \\ &\quad + \sum_{k=1}^{\infty} \int_0^t \langle B_k(s, \mathbf{u}_n(s)), \mathbf{e}_i\rangle_{\mathbb{H}^1}\mathrm{d}W_s^k. \end{aligned} \tag{3.6}$$

We now prove a series of lemmas.

Lemma 3.9. *For any $T > 0$, there exists a positive constant $C_{T,N} > 0$ such that for any $n \in \mathbb{N}$,*

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|\mathbf{u}_n(t)\|_{\mathbb{H}^1}^2\right) + \int_0^T \mathbb{E}\|\mathbf{u}_n(s)\|_{\mathbb{H}^2}^2\mathrm{d}s + \int_0^T \mathbb{E}\|\nabla|\mathbf{u}_n(s)|^2\|_{L^2}^2\mathrm{d}s \leqslant C_{T,N}, \tag{3.7}$$

and also in the periodic case

$$\int_0^T \mathbb{E}\|\mathbf{u}_n(s)\|_{L^4}^4\mathrm{d}s \leqslant C_{T,N}. \tag{3.8}$$

Proof. By Itô's formula and Lemmas 2.3 and 2.5, we have

$$\begin{aligned}\|\mathbf{u}_n(t)\|_{\mathbb{H}^1}^2 &= \|\mathbf{u}_0\|_{\mathbb{H}^1}^2 + 2\int_0^t [\![A(\mathbf{u}_n(s)), \mathbf{u}_n(s)]\!]\mathrm{d}s + 2\int_0^t \langle \mathbf{f}(s, \mathbf{u}_n(s)), \mathbf{u}_n(s)\rangle_{\mathbb{H}^1}\mathrm{d}s \\ &\quad + M(t) + \int_0^t \|B(s, \mathbf{u}_n(s))\|_{L_2(l^2;\mathbb{H}^1)}^2\mathrm{d}s \\ &\leqslant \|\mathbf{u}_0\|_{\mathbb{H}^1}^2 - \int_0^t \|\mathbf{u}_n(s)\|_{\mathbb{H}^2}^2\mathrm{d}s - \int_0^t \||\mathbf{u}_n(s)| \cdot |\nabla \mathbf{u}_n(s)|\|_{L^2}^2\mathrm{d}s \\ &\quad + C\cdot N\int_0^t \|\nabla \mathbf{u}_n(s)\|_{\mathbb{H}^0}^2\mathrm{d}s + 2\int_0^t \|\mathbf{u}_n(s)\|_{\mathbb{H}^0}^2\mathrm{d}s \\ &\quad + 2\int_0^t \|\mathbf{f}(s, \mathbf{u}_n(s))\|_{\mathbb{H}^0} \cdot \|\mathbf{u}_n(s)\|_{\mathbb{H}^2}\mathrm{d}s + M(t) \\ &\quad + \int_0^t \Big(\frac{1}{2}\|\mathbf{u}_n(s)\|_{\mathbb{H}^2}^2 + C_T\|\mathbf{u}_n(s)\|_{\mathbb{H}^1}^2 + C\|H_{\mathbf{h}}(s)\|_{L^1(\mathbb{D})}\Big)\mathrm{d}s, \end{aligned} \tag{3.9}$$

where $M(t)$ is a continuous martingale defined by

$$M(t) := 2\sum_{k=1}^{\infty} \int_0^t \langle B_k(s, \mathbf{u}_n(s)), \mathbf{u}_n(s)\rangle_{\mathbb{H}^1}\mathrm{d}W_s^k.$$

Taking expectations and by Young's inequality, one finds that for any $t \in [0, T]$,

$$\begin{aligned}\mathbb{E}\|\mathbf{u}_n(t)\|_{\mathbb{H}^1}^2 &\leqslant \mathbb{E}\|\mathbf{u}_0\|_{\mathbb{H}^1}^2 - \frac{1}{4}\int_0^t \mathbb{E}\|\mathbf{u}_n(s)\|_{\mathbb{H}^2}^2\mathrm{d}s - \int_0^t \mathbb{E}\||\mathbf{u}_n(s)| \cdot |\nabla \mathbf{u}_n(s)|\|_{L^2}^2\mathrm{d}s \\ &\quad + C\cdot N\int_0^t \mathbb{E}\|\nabla \mathbf{u}_n(s)\|_{\mathbb{H}^0}^2\mathrm{d}s + C_T\int_0^t \mathbb{E}\|\mathbf{u}_n(s)\|_{\mathbb{H}^0}^2\mathrm{d}s \\ &\quad + C_T\int_0^t \Big(\|H_{\mathbf{f}}(s)\|_{L^1(\mathbb{D})} + \|H_{\mathbf{h}}(s)\|_{L^1(\mathbb{D})}\Big)\mathrm{d}s. \end{aligned}$$

Hence, by Gronwall's inequality, we have for any $T > 0$,

$$\sup_{t\in[0,T]} \mathbb{E}\|\mathbf{u}_n(t)\|_{\mathbb{H}^1}^2 + \int_0^T \mathbb{E}\|\mathbf{u}_n(s)\|_{\mathbb{H}^2}^2 \mathrm{d}s + \int_0^T \mathbb{E}\|\nabla|\mathbf{u}_n(s)|^2\|_{L^2}^2 \mathrm{d}s \leqslant C_{T,N}. \tag{3.10}$$

Here, the constant $C_{T,N}$ is independent of n, and we have used that $|\nabla|\mathbf{u}|^2| \leqslant C|\mathbf{u}| \cdot |\nabla\mathbf{u}|$.

Furthermore, from (3.9) and using Burkholder's inequality, Young's inequality, Lemma 2.5 and (3.10), we have for any $T > 0$ and $\epsilon > 0$,

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[0,T]} \|\mathbf{u}_n(t)\|_{\mathbb{H}^1}^2\right) &\leqslant C_{T,N} + C\mathbb{E}\left(\int_0^T \|B(s,\mathbf{u}_n(s))\|_{L_2(l^2;\mathbb{H}^0)}^2 \cdot \|\mathbf{u}_n(s)\|_{\mathbb{H}^2}^2 \mathrm{d}s\right)^{1/2} \\ &\leqslant C_{T,N} + \epsilon \cdot \mathbb{E}\left(\sup_{t\in[0,T]} \|B(s,\mathbf{u}_n(s))\|_{L_2(l^2;\mathbb{H}^0)}^2\right) + C_\epsilon \int_0^T \mathbb{E}\|\mathbf{u}_n(s)\|_{\mathbb{H}^2}^2 \mathrm{d}s \\ &\leqslant C_{T,N,\epsilon} + \epsilon \cdot C_T \mathbb{E}\left(\sup_{t\in[0,T]} \|\mathbf{u}_n(t)\|_{\mathbb{H}^1}^2\right).\end{aligned}$$

Choosing ϵ small enough, we get

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|\mathbf{u}_n(t)\|_{\mathbb{H}^1}^2\right) \leqslant C_{T,N}.$$

In the periodic case, since $\mathscr{E}$ is also orthogonal in $\mathbb{H}^0$, we have by (2.9) and **(H1)-(H3)**,

$$\begin{aligned}\mathbb{E}\|\mathbf{u}_n(t)\|_{\mathbb{H}^0}^2 &= \mathbb{E}\|\mathbf{u}_0\|_{\mathbb{H}^0}^2 + 2\int_0^t \mathbb{E}\langle A(\mathbf{u}_n(s)), \mathbf{u}_n(s)\rangle_{\mathbb{H}^0} \mathrm{d}s \\ &\quad + 2\int_0^t \mathbb{E}\langle \mathbf{f}(s,\mathbf{u}_n(s)), \mathbf{u}_n(s)\rangle_{\mathbb{H}^0} \mathrm{d}s + \int_0^t \mathbb{E}\|B(s,\mathbf{u}_n(s))\|_{L_2(l^2;\mathbb{H}^0)}^2 \mathrm{d}s \\ &\leqslant \mathbb{E}\|\mathbf{u}_0\|_{\mathbb{H}^1}^2 - 2\int_0^t \mathbb{E}\|\nabla\mathbf{u}_n(s)\|_{\mathbb{H}^0}^2 \mathrm{d}s - 2\int_0^t \mathbb{E}\|\mathbf{u}_n(s)\|_{L^4}^4 \mathrm{d}s \\ &\quad + C_N \int_0^t \mathbb{E}\|\mathbf{u}_n(s)\|_{\mathbb{H}^0}^2 \mathrm{d}s + C_T,\end{aligned}$$

which yields (3.8) by Gronwall's lemma. □

Lemma 3.10. *Let μ_n be the law of $\mathbf{u}_n$ in $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then the family of probability measures $\{\mu_n, n \in \mathbb{N}\}$ is tight on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$.*

Proof. Set for $R > 0$

$$\tau_R^n := \inf\{t \geqslant 0 : \|\mathbf{u}_n(t)\|_{\mathbb{H}^1} \geqslant R\}.$$

Then, by (3.7) we have for any $T > 0$,

$$\sup_n P(\tau_R^n < T) = \sup_n P\left(\sup_{t\in[0,T]} \|\mathbf{u}_n(t)\|_{\mathbb{H}^1} > R\right) \leqslant \frac{C_{T,N}}{R^2}. \tag{3.11}$$

On the other hand, from (3.6) and using (2.11), Lemma 2.5 and Burkholder's inequality, we have for any $q \geqslant 2$ and $s, t \in [0,T]$, $\mathbf{e} \in \mathscr{E}$,

$$\begin{aligned}&\mathbb{E}|\langle \mathbf{u}_n(t\wedge\tau_R^n) - \mathbf{u}_n(s\wedge\tau_R^n), \mathbf{e}\rangle_{\mathbb{H}^1}|^q \\ &\leqslant C\mathbb{E}\left|\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n} [\![A(\mathbf{u}_n(s)), \mathbf{e}]\!] \mathrm{d}s\right|^q + C\mathbb{E}\left|\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n} \langle \mathbf{f}(s,\mathbf{u}_n(s)), \mathbf{e}\rangle_{\mathbb{H}^1} \mathrm{d}s\right|^q \\ &\quad + C\mathbb{E}\left|\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n} \langle B_k(s,\mathbf{u}_n(s)), \mathbf{e}\rangle_{\mathbb{H}^1} \mathrm{d}W_s^k\right|^q\end{aligned}$$

$$\leqslant C_{\mathbf{e}} \cdot \mathbb{E}\left|\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n}(1+\|\mathbf{u}_n(s)\|_{\mathbb{H}^1}^3)\mathrm{d}s\right|^q + C_{\mathbf{e}} \cdot \mathbb{E}\left|\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n}\|\mathbf{f}(s,\mathbf{u}_n(s))\|_{\mathbb{H}^0}\mathrm{d}s\right|^q$$

$$+ C_{\mathbf{e}} \cdot \mathbb{E}\left|\int_{s\wedge\tau_R^n}^{t\wedge\tau_R^n}\|B(s,\mathbf{u}_n(s))\|_{L_2(l^2;\mathbb{H}^0)}^2\mathrm{d}s\right|^{q/2} \leqslant C_{\mathbf{e},R,T}\cdot|t-s|^{q/2}.$$

By Kolomogorov's criterion (cf. [16]), we get for any $T>0$ and $0<\alpha<\frac{1}{2}$,

$$\mathbb{E}\left(\sup_{s,t\in[0,T],|t-s|\leqslant\delta}|\langle\mathbf{u}_n(t\wedge\tau_R^n)-\mathbf{u}_n(s\wedge\tau_R^n),\mathbf{e}\rangle_{\mathbb{H}^1}|\right)\leqslant C_{\mathbf{e},R,T}\cdot\delta^\alpha.$$

So, for any $\epsilon>0$ and $R>0$,

$$\sup_n P\left\{\sup_{s,t\in[0,T],|t-s|\leqslant\delta}|\langle\mathbf{u}_n(t)-\mathbf{u}_n(s),\mathbf{e}\rangle_{\mathbb{H}^1}|>\epsilon\right\}$$

$$\leqslant \sup_n P\left\{\sup_{s,t\in[0,T],|t-s|\leqslant\delta}|\langle\mathbf{u}_n(t)-\mathbf{u}_n(s),\mathbf{e}\rangle_{\mathbb{H}^1}|>\epsilon;\tau_R^n\geqslant T\right\}+\sup_n P\{\tau_R^n<T\}$$

$$\leqslant \frac{C_{\mathbf{e},R,T}\cdot\delta}{\epsilon}+\frac{C_{T,N}}{R^2},$$

which then gives that

$$\lim_{\delta\downarrow 0}\sup_n P\left\{\sup_{s,t\in[0,T],|t-s|\leqslant\delta}|\langle\mathbf{u}_n(t)-\mathbf{u}_n(s),\mathbf{e}\rangle_{\mathbb{H}^1}|>\epsilon\right\}=0. \tag{3.12}$$

The tightness of $\{\mu_n, n\in\mathbb{N}\}$ now follows from (3.11), (3.12) and Lemma 2.7. □

In the sequel, without loss of generality, we assume that μ_n weakly converges to $\mu\in\mathcal{P}(\mathbb{X})$. By Skorohod's embedding theorem (cf. [16]), there exist a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$ and $\mathbb{X}$-valued random variables $\tilde{\mathbf{u}}^n$ and $\tilde{\mathbf{u}}$ such that

(I) $\tilde{\mathbf{u}}^n$ has the same law as $\mathbf{u}^n$ in $\mathbb{X}$ for each $n\in\mathbb{N}$;

(II) $\tilde{\mathbf{u}}^n\to\tilde{\mathbf{u}}$ in $\mathbb{X}$, $\tilde{P}$-a.e., and $\tilde{\mathbf{u}}$ has law μ.

Moreover, by (3.7) and Fatou's lemma, we have for any $T>0$,

$$\mathbb{E}^\mu\left(\sup_{t\in[0,T]}\|\mathbf{u}(t)\|_{\mathbb{H}^1}^2\right)=\mathbb{E}^{\tilde{P}}\left(\sup_{t\in[0,T]}\|\tilde{\mathbf{u}}(t)\|_{\mathbb{H}^1}^2\right)<+\infty, \tag{3.13}$$

$$\int_0^T\mathbb{E}^\mu\|\mathbf{u}(s)\|_{\mathbb{H}^2}^2\mathrm{d}s=\int_0^T\mathbb{E}^{\tilde{P}}\|\tilde{\mathbf{u}}(s)\|_{\mathbb{H}^2}^2\mathrm{d}s<+\infty, \tag{3.14}$$

$$\int_0^T\mathbb{E}^\mu\|\nabla|\mathbf{u}(s)|^2\|_{L^2}^2\mathrm{d}s=\int_0^T\mathbb{E}^{\tilde{P}}\|\nabla|\tilde{\mathbf{u}}(s)|^2\|_{L^2}^2\mathrm{d}s<+\infty \tag{3.15}$$

and also in the periodic case

$$\int_0^T\mathbb{E}^\mu\|\mathbf{u}(s)\|_{L^4}^4\mathrm{d}s=\int_0^T\mathbb{E}^{\tilde{P}}\|\tilde{\mathbf{u}}(s)\|_{L^4}^4\mathrm{d}s<+\infty. \tag{3.16}$$

Let $h\in C_0^\infty(\mathbb{R})$ and $\mathbf{e}\in\mathscr{E}$. Define for any $t\geqslant 0$ and $\mathbf{u}\in\mathbb{X}$,

$$M_{\mathbf{e}}^h(t,\mathbf{u}):=I_1^h(t,\mathbf{u})-I_2^h(t,\mathbf{u})-I_3^h(t,\mathbf{u})-I_4^h(t,\mathbf{u})-I_5^h(t,\mathbf{u}),$$

where

$$I_1^h(t,\mathbf{u}):=h(\langle\mathbf{u}(t),\mathbf{e}\rangle_{\mathbb{H}^1}),$$

$$I_2^h(t,\mathbf{u}):=h(\langle\mathbf{u}(0),\mathbf{e}\rangle_{\mathbb{H}^1}),$$

$$I_3^h(t,\mathbf{u}) := \int_0^t h'(\langle \mathbf{u}(s), \mathbf{e}\rangle_{\mathbb{H}^1}) \cdot [\![A(\mathbf{u}(s)), \mathbf{e}]\!] \mathrm{d}s,$$
$$I_4^h(t,\mathbf{u}) := \int_0^t h'(\langle \mathbf{u}(s), \mathbf{e}\rangle_{\mathbb{H}^1}) \cdot \langle \mathbf{f}(s, \mathbf{u}(s)), \mathbf{e}\rangle_{\mathbb{H}^1} \mathrm{d}s,$$
$$I_5^h(t,\mathbf{u}) := \frac{1}{2}\int_0^t h''(\langle \mathbf{u}(s), \mathbf{e}\rangle_{\mathbb{H}^1}) \cdot \|\langle B(s, \mathbf{u}(s)), \mathbf{e}\rangle_{\mathbb{H}^1}\|_{l^2}^2 \mathrm{d}s.$$

Note that $\mathbf{e} \in \mathscr{E} \subset \mathcal{V}$ has compact support, there exists $m \in \mathbb{N}$ such that

$$\mathrm{supp}\{\mathbf{e}\} \subset O := \{x \in \mathbb{R}^3, |x| \leqslant m\}. \tag{3.17}$$

Lemma 3.11. *We have*

$$\sup_n \mathbb{E}^{\tilde{P}} |M_{\mathbf{e}}^h(t, \tilde{\mathbf{u}}_n)|^{4/3} + \mathbb{E}^{\tilde{P}} |M_{\mathbf{e}}^h(t, \tilde{\mathbf{u}})|^{4/3} < +\infty. \tag{3.18}$$

Proof. It is clear that $I_1^h(t, \tilde{\mathbf{u}}_n)$ and $I_2^h(t, \tilde{\mathbf{u}}_n)$ are bounded by some constant C_h. For I_3^h, noting that in the whole space case,

$$\|\mathbf{u}\|_{L^6} \leqslant C\|\nabla \mathbf{u}\|_{L^2},$$

by (2.11) and (3.7), we have

$$\begin{aligned}
\mathbb{E}^{\tilde{P}} |I_3^h(t, \tilde{\mathbf{u}}_n)|^{4/3} &\leqslant C_{T,h} \int_0^T \mathbb{E}^{\tilde{P}} \Big| [\![A(\tilde{\mathbf{u}}_n(s)), \mathbf{e}]\!] \Big|^{4/3} \mathrm{d}s \\
&\leqslant C_{T,h,\mathbf{e}} \int_0^T \mathbb{E}^{\tilde{P}} \left(1 + \|\tilde{\mathbf{u}}_n(s)\|_{L^3(O)}^4\right) \mathrm{d}s \\
&\leqslant C_{T,h,\mathbf{e}} \int_0^T \mathbb{E}^{\tilde{P}} \left(1 + \|\tilde{\mathbf{u}}_n(s)\|_{L^{12}(O)}^4\right) \mathrm{d}s \\
&= C_{T,h,\mathbf{e}} \int_0^T \mathbb{E}^{\tilde{P}} \left(1 + \||\tilde{\mathbf{u}}_n(s)|^2\|_{L^6}^2\right) \mathrm{d}s \\
&\leqslant C_{T,h,\mathbf{e}} \int_0^T \mathbb{E}^{\tilde{P}} \left(1 + \|\nabla |\tilde{\mathbf{u}}_n(s)|^2\|_{L^2}^2\right) \mathrm{d}s \\
&\leqslant C_{T,h,\mathbf{e},N}.
\end{aligned}$$

In the periodic case, by (2.11) and (3.8), we have

$$\begin{aligned}
\mathbb{E}^{\tilde{P}} |I_3^h(t, \tilde{\mathbf{u}}_n)|^{4/3} &\leqslant C_{T,h} \int_0^T \mathbb{E}^{\tilde{P}} \left(1 + \|\tilde{\mathbf{u}}_n(s)\|_{L^3(\mathbb{T}^3)}^4\right) \mathrm{d}s \\
&\leqslant C_{T,h,\mathbf{e}} \int_0^T \mathbb{E}^{\tilde{P}} \left(1 + \|\tilde{\mathbf{u}}_n(s)\|_{L^4(\mathbb{T}^3)}^4\right) \mathrm{d}s \\
&\leqslant C_{T,h,\mathbf{e},N}.
\end{aligned}$$

For I_5^h, by (2.12), we similarly have

$$\mathbb{E}^{\tilde{P}} |I_5^h(t, \tilde{\mathbf{u}}_n)|^2 \leqslant C_{T,h,\mathbf{e},N}.$$

For I_4^h, it is clear that

$$\mathbb{E}^{\tilde{P}} |I_4^h(t, \tilde{\mathbf{u}}_n)|^2 \leqslant C_{T,h,\mathbf{e},N}.$$

Moreover, by (3.13)-(3.16), we also have

$$\mathbb{E}^{\tilde{P}} |M_{\mathbf{e}}^h(t, \tilde{\mathbf{u}})|^{4/3} < +\infty.$$

The proof is thus complete. □

Lemma 3.12. *For any $t > 0$ and $\epsilon > 0$,*

$$\lim_{n\to\infty} \tilde{P}(|M^h_{\mathbf{e}}(t,\tilde{\mathbf{u}}_n) - M^h_{\mathbf{e}}(t,\tilde{\mathbf{u}})| > \epsilon) = 0. \tag{3.19}$$

That is, $M^h_{\mathbf{e}}(t,\tilde{\mathbf{u}}_n)$ converges to $M^h_{\mathbf{e}}(t,\tilde{\mathbf{u}})$ in probability $\tilde{P}$ as $n \to \infty$.

Proof. Recalling the definition of $\mathbb{X}$ in Subsection 2.2, by (II) we have

$$\lim_{n\to\infty} \int_O |\tilde{\mathbf{u}}_n(t,x,\tilde{\omega}) - \tilde{\mathbf{u}}(t,x,\tilde{\omega})|^2 \mathrm{d}x = 0, \quad \tilde{P} - a.a.\ \tilde{\omega} \in \tilde{\Omega},$$

where O is from (3.17). Thus, by the dominated convergence theorem, we have

$$\lim_{n\to\infty} \mathbb{E}^{\tilde{P}} |I^h_1(t,\tilde{\mathbf{u}}_n) - I^h_1(t,\tilde{\mathbf{u}})| = 0,$$
$$\lim_{n\to\infty} \mathbb{E}^{\tilde{P}} |I^h_2(t,\tilde{\mathbf{u}}_n) - I^h_2(t,\tilde{\mathbf{u}})| = 0.$$

ForI^h_3, define for any $R > 0$,

$$\tilde{\tau}^n_R := \inf\{t \geqslant 0 : \|\tilde{\mathbf{u}}_n(t)\|_{\mathbb{H}^1} \geqslant R\}.$$

Then, by (I) and (3.7), for any $T > 0$, we have

$$\sup_n \tilde{P}(\tilde{\tau}^n_R \leqslant T) \leqslant \frac{C_{T,N}}{R^2}.$$

Thus, by the dominated convergence theorem and Lemma 2.4, we have from the proof of Lemma 3.11,

$$\begin{aligned}
&\lim_{n\to\infty} \tilde{P}(|I^h_3(t,\tilde{\mathbf{u}}_n) - I^h_3(t,\tilde{\mathbf{u}})| > \epsilon) \\
&\quad \leqslant \lim_{R\to\infty}\lim_{n\to\infty} \tilde{P}(|I^h_3(t,\tilde{\mathbf{u}}_n) - I^h_3(t,\tilde{\mathbf{u}})| > \epsilon; \tilde{\tau}^n_R > t) + \lim_{R\to\infty}\sup_n \tilde{P}(\tilde{\tau}^n_R \leqslant T) \\
&\quad \leqslant \lim_{R\to\infty}\lim_{n\to\infty} \mathbb{E}^{\tilde{P}} \left(1_{\{\tilde{\tau}^n_R > t\}} \cdot |I^h_3(t,\tilde{\mathbf{u}}_n) - I^h_3(t,\tilde{\mathbf{u}})|\right)/\epsilon \\
&\quad \leqslant \lim_{R\to\infty} \mathbb{E}^{\tilde{P}}\Bigg(\int_0^t \overline{\lim_{n\to\infty}} \Big(1_{\{\tilde{\tau}^n_R > t\}} \cdot \Big| h'(\langle \tilde{\mathbf{u}}_n(s), \mathbf{e}\rangle_{\mathbb{H}^1}) \cdot [\![A(\tilde{\mathbf{u}}_n(s)), \mathbf{e}]\!] \\
&\qquad - h'(\langle \tilde{\mathbf{u}}(s), \mathbf{e}\rangle_{\mathbb{H}^1}) \cdot [\![A(\tilde{\mathbf{u}}(s)), \mathbf{e}]\!] \Big|\Big) \mathrm{d}s \Bigg)/\epsilon = 0.
\end{aligned}$$

Similarly, we also have

$$\lim_{n\to\infty} \tilde{P}(|I^h_4(t,\tilde{\mathbf{u}}_n) - I^h_4(t,\tilde{\mathbf{u}})| > \epsilon) = 0,$$
$$\lim_{n\to\infty} \tilde{P}(|I^h_5(t,\tilde{\mathbf{u}}_n) - I^h_5(t,\tilde{\mathbf{u}})| > \epsilon) = 0.$$

Combining the above calculations yields (3.19). □

We can now give the proof of Theorem 3.8.

Proof of Theorem 3.8: Now let $t > s$ and G be any bounded and real valued $\mathcal{B}_s(\mathbb{X})$-measurable continuous function on $\mathbb{X}$. Then by (3.18) and (3.19), we have

$$\begin{aligned}
\mathbb{E}^{\mu}\Big((M^h_{\mathbf{e}}(t,\mathbf{u}) - M^h_{\mathbf{e}}(s,\mathbf{u})) \cdot G(\mathbf{u})\Big) &= \mathbb{E}^{\tilde{P}}\Big((M^h_{\mathbf{e}}(t,\tilde{\mathbf{u}}) - M^h_{\mathbf{e}}(s,\tilde{\mathbf{u}})) \cdot G(\tilde{\mathbf{u}})\Big) \\
&= \lim_{n\to\infty} \mathbb{E}^{\tilde{P}}\Big((M^h_{\mathbf{e}}(t,\tilde{\mathbf{u}}_n) - M^h_{\mathbf{e}}(s,\tilde{\mathbf{u}}_n)) \cdot G(\tilde{\mathbf{u}}_n)\Big) \\
&= \lim_{n\to\infty} \mathbb{E}^{P}\Big((M^h_{\mathbf{e}}(t,\mathbf{u}_n) - M^h_{\mathbf{e}}(s,\mathbf{u}_n)) \cdot G(\mathbf{u}_n)\Big) = 0,
\end{aligned}$$

where the last step is due to the martingale property of $M^h_{\mathbf{e}}(t,\mathbf{u}_n)$ on $(\Omega,\mathcal{F},P;(\mathcal{F}_t)_{t\geqslant 0})$ and $G(\mathbf{u}_n)\in\mathcal{F}_s$. This means that $\{M^h_{\mathbf{e}}(t,\mathbf{u}),t\geqslant 0\}$ is a $\mathcal{B}_t(\mathbb{X})$-martingale. The existence of a weak solution to Eq. (2.6) now follows from Proposition 3.4.

Summarizing Theorems 3.7, 3.8 and 3.6, we have the following main result in the present paper.

Theorem 3.13. *Under* **(H1)**-**(H3)**, *for any* $\mathbf{u}_0\in\mathbb{H}^1$, *there exists a unique* $\mathbf{u}(t,x)$ *such that*
(1^o*)* $\mathbf{u}\in L^2(\Omega,P;C([0,T],\mathbb{H}^1))\cap L^2(\Omega,P;L^2([0,T],\mathbb{H}^2))$ *for any* $T>0$, *and*

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|\mathbf{u}(t)\|^2_{\mathbb{H}^1}\right)+\int_0^T\mathbb{E}\|\mathbf{u}(s)\|^2_{\mathbb{H}^2}\mathrm{d}s\leqslant C_T(1+\|\mathbf{u}_0\|^2_{\mathbb{H}^1})N; \tag{3.20}$$

(2^o*) it holds that in* $\mathbb{H}^0$,

$$\mathbf{u}(t)=\mathbf{u}_0+\int_0^t\Big[A(\mathbf{u}(s))+\mathscr{P}\mathbf{f}(s,\mathbf{u}(s))\Big]\mathrm{d}s+\sum_{k=1}^\infty\int_0^t B_k(s,\mathbf{u}(s))\mathrm{d}W^k_s,$$

for all $t\geqslant 0$, *P-a.s..*

Proof. We only need to prove estimate (3.20). By Itô's formula, (2.8) and Lemma 2.5, we have

$$\begin{aligned}\mathbb{E}\|\mathbf{u}(t)\|^2_{\mathbb{H}^0}&=\|\mathbf{u}_0\|^2_{\mathbb{H}^0}+2\int_0^t\mathbb{E}\langle A(\mathbf{u}(s)),\mathbf{u}(s)\rangle_{\mathbb{H}^0}\mathrm{d}s\\&\quad+2\int_0^t\mathbb{E}\langle\mathbf{f}(s,\mathbf{u}(s)),\mathbf{u}(s)\rangle_{\mathbb{H}^0}\mathrm{d}s+\int_0^t\mathbb{E}\|B(s,\mathbf{u}(s))\|^2_{L_2(l^2;\mathbb{H}^0)}\mathrm{d}s\\&\leqslant\|\mathbf{u}_0\|^2_{\mathbb{H}^0}+C-\frac{1}{2}\int_0^t\mathbb{E}\|\mathbf{u}(s)\|^2_{\mathbb{H}^1}\mathrm{d}s+C\int_0^t\mathbb{E}\|\mathbf{u}(s)\|^2_{\mathbb{H}^0}\mathrm{d}s.\end{aligned}$$

By Gronwall's inequality, we obtain

$$\sup_{t\in[0,T]}\mathbb{E}\|\mathbf{u}(t)\|^2_{\mathbb{H}^0}+\int_0^T\mathbb{E}\|\mathbf{u}(s)\|^2_{\mathbb{H}^1}\mathrm{d}s\leqslant C_T(\|\mathbf{u}_0\|^2_{\mathbb{H}^0}+1).$$

Using this estimate, as in the proof of (3.7), we obtain (3.20). □

4. Feller Properties and Invariant Measures

In the following, we consider the time homogenous case, i.e., the coefficients $\mathbf{f}$, σ and $\mathbf{h}$ are independent of t, and assume a stronger assumption than **(H3)**, namely:

(H3)′ There exist a constant $C_{\mathbf{h}}>0$ and a function $H_{\mathbf{h}}(x)\in L^1(\mathbb{D})$ such that for any $x\in\mathbb{D}$, $\mathbf{u},\mathbf{v}\in\mathbb{R}^3$ and $j=1,2,3$,

$$\begin{aligned}\|\partial_{x^j}\mathbf{h}(x,\mathbf{u})\|^2_{l^2}+\|\mathbf{h}(x,\mathbf{u})\|^2_{l^2}&\leqslant C_{\mathbf{h}}\cdot|\mathbf{u}|^2+H_{\mathbf{h}}(x),\\\|\partial_{x^j}\mathbf{h}(x,\mathbf{u})-\partial_{x^j}\mathbf{h}(x,\mathbf{v})\|_{l^2}&\leqslant C_{\mathbf{h}}\cdot|\mathbf{u}-\mathbf{v}|,\\\|\partial_{u^j}\mathbf{h}(x,\mathbf{u})\|_{l^2}&\leqslant C_{\mathbf{h}},\\\|\partial_{u^j}\mathbf{h}(x,\mathbf{u})-\partial_{v^j}\mathbf{h}(x,\mathbf{v})\|_{l^2}&\leqslant C_{\mathbf{h}}\cdot|\mathbf{u}-\mathbf{v}|.\end{aligned}$$

For fixed initial value $\mathbf{u}_0=\mathbf{v}\in\mathbb{H}^1$, we denote the unique solution in Theorem 3.13 by $\mathbf{u}(t;\mathbf{v})$. Then $\{\mathbf{u}(t;\mathbf{v}):\mathbf{v}\in\mathbb{H}^1,t\geqslant 0\}$ forms a strong Markov process with state space $\mathbb{H}^1$. We have:

Lemma 4.1. *For* $\mathbf{v},\mathbf{v}'\in\mathbb{H}^1$ *and* $R>0$, *define*

$$\tau^{\mathbf{v}}_R:=\inf\{t\geqslant 0:\|\mathbf{u}(t;\mathbf{v})\|_{\mathbb{H}^1}>R\}$$

and

$$\tau_R^{\mathbf{v},\mathbf{v}'} := \tau_R^{\mathbf{v}} \wedge \tau_R^{\mathbf{v}'}.$$

Assume **(H1)**, **(H2)** *and* **(H3)**$'$, *then*

$$\mathbb{E}\|\mathbf{u}(t \wedge \tau_R^{\mathbf{v},\mathbf{v}'}; \mathbf{v}) - \mathbf{u}(t \wedge \tau_R^{\mathbf{v},\mathbf{v}'}; \mathbf{v}')\|_{\mathbb{H}^1}^2 \leqslant C_{t,R} \cdot \|\mathbf{v} - \mathbf{v}'\|_{\mathbb{H}^1}^2.$$

Proof. Write $\mathbf{u}(t) := \mathbf{u}(t; \mathbf{v})$, $\tilde{\mathbf{u}}(t) := \mathbf{u}(t, \mathbf{v}')$ and

$$\mathbf{w}(t) := \mathbf{u}(t) - \tilde{\mathbf{u}}(t).$$

Set $t_R := \tau_R^{\mathbf{v},\mathbf{v}'} \wedge t$. By Itô's formula (cf. [30, 26]), we have

$$\begin{aligned}
\|\mathbf{w}(t_R)\|_{\mathbb{H}^1}^2 &= \|\mathbf{w}(0)\|_{\mathbb{H}^1}^2 + 2\int_0^{t_R} \langle A(\mathbf{u}(s)) - A(\tilde{\mathbf{u}}(s)), \mathbf{w}(s)\rangle_{\mathbb{H}^1} \mathrm{d}s \\
&\quad + 2\int_0^{t_R} \langle \mathbf{f}(s, \mathbf{u}(s)) - \mathbf{f}(s, \tilde{\mathbf{u}}(s)), \mathbf{w}(s)\rangle_{\mathbb{H}^1} \mathrm{d}s \\
&\quad + 2\sum_{k=1}^{\infty}\int_0^{t_R} \langle B_k(s, \mathbf{u}(s)) - B_k(s, \tilde{\mathbf{u}}(s)), \mathbf{w}(s)\rangle_{\mathbb{H}^1} \mathrm{d}W_s^k \\
&\quad + \sum_{k=1}^{\infty}\int_0^{t_R} \|B_k(s, \mathbf{u}(s)) - B_k(s, \tilde{\mathbf{u}}(s))\|_{\mathbb{H}^1}^2 \mathrm{d}s \\
&=: \|\mathbf{w}(0)\|_{\mathbb{H}^1}^2 + I_1(t_R) + I_2(t_R) + I_3(t_R) + I_4(t_R).
\end{aligned}$$

By $|g_N(r) - g_N(r')| \leqslant |r - r'|$ and Young's inequality, it is easy to see that

$$\begin{aligned}
I_1(t_R) &= -2\int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^2}^2 \mathrm{d}s + 2\int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^1}^2 \mathrm{d}s \\
&\quad + 2\int_0^{t_R} \langle ((\mathbf{u}(s)\cdot\nabla)\mathbf{u}(s) - (\tilde{\mathbf{u}}(s)\cdot\nabla)\tilde{\mathbf{u}}(s)), (I-\Delta)\mathbf{w}(s)\rangle_{\mathbb{H}^0} \mathrm{d}s \\
&\quad - 2\int_0^{t_R} \langle g_N(|\mathbf{u}(s)|^2)\mathbf{u}(s) - g_N(|\tilde{\mathbf{u}}(s)|^2)\tilde{\mathbf{u}}(s), (I-\Delta)\mathbf{w}(s)\rangle_{\mathbb{H}^0} \mathrm{d}s \\
&\leqslant -\int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^2}^2 \mathrm{d}s + 2\int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^1}^2 \mathrm{d}s \\
&\quad + C\int_0^{t_R} \|(\mathbf{w}(s)\cdot\nabla)\mathbf{u}(s)\|_{L^2}^2 \mathrm{d}s + C\int_0^{t_R} \|(\tilde{\mathbf{u}}(s)\cdot\nabla)\mathbf{w}(s)\|_{L^2}^2 \mathrm{d}s \\
&\quad + C\int_0^{t_R} \||\mathbf{w}(s)|\cdot(|\mathbf{u}(s)|^2 + |\tilde{\mathbf{u}}(s)|^2)\|_{L^2}^2 \mathrm{d}s.
\end{aligned}$$

By Hölder's inequality and the Sobolev inequality (2.1), we further have

$$\begin{aligned}
I_1(t_R) &\leqslant -\int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^2}^2 \mathrm{d}s + 2\int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^1}^2 \mathrm{d}s \\
&\quad + C_R\int_0^{t_R} \|\mathbf{w}(s)\|_{L^\infty}^2 \mathrm{d}s + C\int_0^{t_R} \|\tilde{\mathbf{u}}(s)\|_{L^6}^2 \cdot \|\nabla\mathbf{w}(s)\|_{L^3}^2 \mathrm{d}s \\
&\quad + C\int_0^{t_R} \|\mathbf{w}(s)\|_{L^6}^2 \cdot (\|\mathbf{u}(s)\|_{L^6}^2 + \|\tilde{\mathbf{u}}(s)\|_{L^6}^2)^2 \mathrm{d}s \\
&\leqslant -\int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^2}^2 \mathrm{d}s + C_R\int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^1}^2 \mathrm{d}s \\
&\quad + C_R\int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^2}^{3/2} \cdot \|\mathbf{w}(s)\|_{\mathbb{H}^0}^{1/2} \mathrm{d}s + C_R\int_0^{t_R} \|\mathbf{w}(s)\|_{\mathbb{H}^2} \cdot \|\mathbf{w}(s)\|_{\mathbb{H}^1} \mathrm{d}s
\end{aligned}$$

$$\leqslant -\frac{3}{4}\int_0^{t_R}\|\mathbf{w}(s)\|_{\mathbb{H}^2}^2 \mathrm{d}s + C_R\int_0^{t_R}\|\mathbf{w}(s)\|_{\mathbb{H}^1}^2 \mathrm{d}s.$$

By **(H1)**, **(H2)** and **(H3)**′, we similarly have

$$I_2(t_R) \leqslant \frac{1}{4}\int_0^{t_R}\|\mathbf{w}(s)\|_{\mathbb{H}^2}^2 \mathrm{d}s + C_R\int_0^{t_R}\|\mathbf{w}(s)\|_{\mathbb{H}^0}^2 \mathrm{d}s,$$
$$I_4(t_R) \leqslant \frac{1}{2}\int_0^{t_R}\|\mathbf{w}(s)\|_{\mathbb{H}^2}^2 \mathrm{d}s + C_R\int_0^{t_R}\|\mathbf{w}(s)\|_{\mathbb{H}^1}^2 \mathrm{d}s.$$

So,

$$\mathbb{E}\|\mathbf{w}(t\wedge\tau_R)\|_{\mathbb{H}^1}^2 \leqslant \|\mathbf{w}(0)\|_{\mathbb{H}^1}^2 + C_R\int_0^{t_R}\|\mathbf{w}(s)\|_{\mathbb{H}^1}^2 \mathrm{d}s$$
$$\leqslant \|\mathbf{v}-\mathbf{v}'\|_{\mathbb{H}^1}^2 + C_R\int_0^{t}\|\mathbf{w}(s\wedge\tau_R)\|_{\mathbb{H}^1}^2 \mathrm{d}s.$$

By Gronwall's inequality, we get the desired estimate. □

Let $C_b^{loc}(\mathbb{H}^1)$ denote the set of all bounded and locally uniformly continuous functions on $\mathbb{H}^1$. Then $C_b^{loc}(\mathbb{H}^1)$ is clearly a Banach space under the sup norm

$$\|\phi\|_\infty := \sup_{\mathbf{u}\in\mathbb{H}^1}|\phi(\mathbf{u})|.$$

For $t>0$, we define the semigroup $\mathbf{T}_t$ associated with $\{\mathbf{u}(t;\mathbf{v}) : \mathbf{v}\in\mathbb{H}^1, t\geqslant 0\}$ by

$$\mathbf{T}_t\phi(\mathbf{v}) := \mathbb{E}(\phi(\mathbf{u}(t;\mathbf{v}))), \quad \phi\in C_b^{loc}(\mathbb{H}^1).$$

We have:

Theorem 4.2. *Under* **(H1)**, **(H2)** *and* **(H3)**′, *for every $t>0$, $\mathbf{T}_t$ maps $C_b^{loc}(\mathbb{H}^1)$ into $C_b^{loc}(\mathbb{H}^1)$. That is, $(\mathbf{T}_t)_{t\geqslant 0}$ is a Feller semigroup on $C_b^{loc}(\mathbb{H}^1)$.*

Proof. Let $\phi\in C_b^{loc}(\mathbb{H}^1)$ be given. We want to prove that for any $t>0$ and $m\in\mathbb{N}$

$$\lim_{\delta\to 0}\sup_{\mathbf{v},\mathbf{v}'\in\mathbb{B}_m,\|\mathbf{v}-\mathbf{v}'\|_{\mathbb{H}^1}\leqslant\delta}|\mathbf{T}_t\phi(\mathbf{v})-\mathbf{T}_t\phi(\mathbf{v}')| = 0, \tag{4.1}$$

where $\mathbb{B}_m := \{\mathbf{v}\in\mathbb{H}^1 : \|\mathbf{v}\|_{\mathbb{H}^1}\leqslant m\}$ denotes the ball in $\mathbb{H}^1$.

For any $\mathbf{v},\mathbf{v}'\in\mathbb{B}_m$ and $R>m$, as in Lemma 4.1, define

$$\tau_R^{\mathbf{v}} := \{t\geqslant 0 : \|\mathbf{u}(t;\mathbf{v})\|_{\mathbb{H}^1} > R\}$$

and

$$\tau_R^{\mathbf{v},\mathbf{v}'} := \tau_R^{\mathbf{v}}\wedge\tau_R^{\mathbf{v}'}.$$

By (3.20), we have

$$\mathbb{E}|\phi(\mathbf{u}(t;\mathbf{v}))-\phi(\mathbf{u}(t\wedge\tau_R^{\mathbf{v},\mathbf{v}'};\mathbf{v}))| \leqslant 2\|\phi\|_\infty\cdot P(\tau_R^{\mathbf{v},\mathbf{v}'}<t) \leqslant 2\|\phi\|_\infty\cdot\sup_{\mathbf{v}\in\mathbb{B}_m}\mathbb{E}\left(\sup_{s\in[0,t]}\|\mathbf{u}(s;\mathbf{v})\|_{\mathbb{H}^1}^2\right)/R^2$$
$$\leqslant 2\|\phi\|_\infty\cdot C_{t,m,N}/R^2.$$

For any $\epsilon>0$, choose $R>m$ sufficiently large such that for any $\mathbf{v},\mathbf{v}'\in\mathbb{B}_m$

$$\mathbb{E}|\phi(\mathbf{u}(t;\mathbf{v}))-\phi(\mathbf{u}(t\wedge\tau_R^{\mathbf{v},\mathbf{v}'};\mathbf{v}))| \leqslant \epsilon, \tag{4.2}$$
$$\mathbb{E}|\phi(\mathbf{u}(t;\mathbf{v}'))-\phi(\mathbf{u}(t\wedge\tau_R^{\mathbf{v},\mathbf{v}'};\mathbf{v}'))| \leqslant \epsilon. \tag{4.3}$$

For this R, since ϕ is uniformly continuous on $\mathbb{B}_R$, one may choose $\eta>0$ such that for any $\mathbf{u},\mathbf{u}'\in\mathbb{B}_R$ with $\|\mathbf{u}-\mathbf{u}'\|_{\mathbb{H}^1}\leqslant\eta$

$$|\phi(\mathbf{u})-\phi(\mathbf{u}')| \leqslant \epsilon.$$

Thus, for any $\mathbf{v}, \mathbf{v}' \in \mathbb{B}_m$ with $\|\mathbf{v}-\mathbf{v}'\|_{\mathbb{H}^1} \leqslant \frac{\sqrt{\epsilon}\cdot\eta}{\sqrt{2C_\phi\cdot C_{t,R}}}$, by Lemma 4.1 we have

$$\begin{aligned}
&\mathbb{E}|\phi(\mathbf{u}(t\wedge\tau_R^{\mathbf{v},\mathbf{v}'};\mathbf{v})) - \phi(\mathbf{u}(t\wedge\tau_R^{\mathbf{v},\mathbf{v}'};\mathbf{v}'))| \\
&\qquad \leqslant \epsilon + 2C_\phi \cdot P(\|\mathbf{u}(t\wedge\tau_R^{\mathbf{v},\mathbf{v}'};\mathbf{v}) - \mathbf{u}(t\wedge\tau_R^{\mathbf{v},\mathbf{v}'};\mathbf{v}')\|_{\mathbb{H}^1} > \eta) \leqslant 2\epsilon.
\end{aligned} \tag{4.4}$$

Combining (4.2) (4.3) and (4.4), we get (4.1). □

In the periodic case, we have the following existence of invariant measures associated to $(\mathbf{T}_t)_{t\geqslant 0}$.

Theorem 4.3. *Under* **(H1)**, **(H2)** *and* **(H3)**′*, in the periodic case, there is an invariant measure* $\mu \in \mathcal{P}(\mathbb{H}^1)$ *associated to the semigroup* $(\mathbf{T}_t)_{t\geqslant 0}$ *such that for any* $t \geqslant 0$ *and* $\phi \in C_b^{loc}(\mathbb{H}^1)$,

$$\int_{\mathbb{H}^1} \mathbf{T}_t\phi(\mathbf{u})\mu(\mathrm{d}\mathbf{u}) = \int_{\mathbb{H}^1} \phi(\mathbf{u})\mu(\mathrm{d}\mathbf{u}).$$

Proof. In the following, we assume that $\mathbf{u}_0 = 0$. Using Itô's formula, we have by (2.9) and (2.14),

$$\begin{aligned}
\mathbb{E}\|\mathbf{u}(t)\|_{\mathbb{H}^0}^2 &= 2\int_0^t \mathbb{E}\langle A(\mathbf{u}(s)), \mathbf{u}(s)\rangle_{\mathbb{H}^0}\mathrm{d}s + 2\int_0^t \mathbb{E}\langle \mathbf{f}(\mathbf{u}(s)), \mathbf{u}(s)\rangle_{\mathbb{H}^0}\mathrm{d}s + \int_0^t \mathbb{E}\|B(\mathbf{u}(s))\|_{L_2(l^2;\mathbb{H}^0)}^2\mathrm{d}s \\
&\leqslant -\frac{3}{2}\int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{\mathbb{H}^1}^2\mathrm{d}s - 2\int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{L^4}^4\mathrm{d}s + C_{\mathbf{h},\mathbf{f},N}\int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{\mathbb{H}^0}^2\mathrm{d}s + C_{\mathbf{h},\mathbf{f}}\cdot t.
\end{aligned}$$

In the periodic case, noting that for any $\epsilon > 0$

$$\|\mathbf{u}\|_{\mathbb{H}^0}^2 \leqslant C\|\mathbf{u}\|_{L^4}^2 \leqslant \epsilon\|\mathbf{u}\|_{L^4}^4 + C_\epsilon,$$

we further have

$$\mathbb{E}\|\mathbf{u}(t)\|_{\mathbb{H}^0}^2 \leqslant -\frac{3}{2}\int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{\mathbb{H}^1}^2\mathrm{d}s - \int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{L^4}^4\mathrm{d}s + C_{\mathbf{h},\mathbf{f},N}\cdot t.$$

Hence, for any $t \geqslant 0$

$$\mathbb{E}\|\mathbf{u}(t)\|_{\mathbb{H}^0}^2 + \int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{\mathbb{H}^1}^2\mathrm{d}s + \int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{L^4}^4\mathrm{d}s \leqslant C_{\mathbf{h},\mathbf{f},N}\cdot t. \tag{4.5}$$

On the other hand, by Itô's formula again and (2.10), (2.15), as above we have

$$\begin{aligned}
\mathbb{E}\|\mathbf{u}(t)\|_{\mathbb{H}^1}^2 &= 2\int_0^t \mathbb{E}[\![A(\mathbf{u}(s)), \mathbf{u}(s)]\!]\mathrm{d}s + 2\int_0^t \mathbb{E}\langle \mathbf{f}(\mathbf{u}(s)), \mathbf{u}(s)\rangle_{\mathbb{H}^1}\mathrm{d}s + \int_0^t \mathbb{E}\|B(\mathbf{u}(s))\|_{L_2(l^2;\mathbb{H}^1)}^2\mathrm{d}s \\
&\leqslant -\frac{1}{4}\int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{\mathbb{H}^2}^2\mathrm{d}s + C_{\mathbf{h},\mathbf{f},N}\cdot\int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{\mathbb{H}^1}^2\mathrm{d}s + C_{\mathbf{h},\mathbf{f},N}\cdot t \\
&\leqslant -\frac{1}{4}\int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{\mathbb{H}^2}^2\mathrm{d}s + C_{\mathbf{h},\mathbf{f},N}\cdot t.
\end{aligned}$$

Therefore, for any $t \geqslant 0$

$$\frac{1}{t}\int_0^t \mathbb{E}\|\mathbf{u}(s)\|_{\mathbb{H}^2}^2\mathrm{d}s \leqslant C_{\mathbf{h},\mathbf{f},N}.$$

In the periodic case, since $\mathbb{H}^2$ is compactly embedded into $\mathbb{H}^1$, the existence of an invariant measure μ now follows from the classical Krylov-Bogoliubov method (cf. [3]). □

5. Ergodicity: Uniqueness of Invariant Measures

In the following, we shall work in the case of $\mathbb{D} = \mathbb{T}^3$, and suppose that for $\mathbf{f} \in \mathbb{H}^0$, the mean value of f on $\mathbb{T}^3$ vanishes, i.e.,

$$\int_{\mathbb{T}^3} \mathbf{f}(x)\mathrm{d}x = 0.$$

In this case, we assume that the orthonormal basis $\mathscr{E}$ of $\mathbb{H}^1$ consists of the eigenvectors of $\mathscr{P}\Delta$, i.e,

$$\mathscr{P}\Delta \mathbf{e}_i = -\lambda_i \mathbf{e}_i, \ \ \langle \mathbf{e}_i, \mathbf{e}_i\rangle_{\mathbb{H}^1} = 1, \ \ i = 1, 2, \cdots,$$

where $0 < \lambda_1 \leqslant \cdots \leqslant \lambda_n \uparrow \infty$. Recalling that the following Poincare inequality holds:

$$\|\mathbf{u}\|^2_{\mathbb{H}^0} \leqslant 1/\lambda_1 \|\nabla \mathbf{u}\|^2_{\mathbb{H}^0}, \tag{5.1}$$

two equivalent norms in $\mathbb{H}^1$ and $\mathbb{H}^2$ are given by

$$\|\mathbf{u}\|_{\mathbb{H}^1} := \|\nabla \mathbf{u}\|_{\mathbb{H}^0}, \ \ \|\mathbf{u}\|_{\mathbb{H}^2} := \|\Delta \mathbf{u}\|_{\mathbb{H}^0}.$$

We shall use these two norms in what follows.

For $m \in \mathbb{N}$, let $\Omega := C_0(\mathbb{R}_+; \mathbb{R}^m)$ denote the space of all continuous functions with initial values 0, P the standard Wiener measure on $\mathcal{F} := \mathcal{B}(C_0(\mathbb{R}_+; \mathbb{R}^m))$. Then, the coordinate process

$$W_t(\omega) := \omega(t), \ \ \omega \in \Omega,$$

is a standard Wiener process on $(\Omega, \mathcal{F}, P)$.

Consider the following stochastic tamed 3D Navier-Stokes equation:

$$\left\{ \begin{array}{lcl} \mathrm{d}\mathbf{u}(t) & = & A(\mathbf{u}(t))\mathrm{d}t + \mathrm{d}\mathbf{w}(t), \\ \mathbf{u}(0) & = & \mathbf{u}_0 \in \mathbb{H}^1, \end{array} \right. \tag{5.2}$$

where $\mathbf{w}(t) := QW_t$ is the noise, and the linear map $Q : \mathbb{R}^m \to \mathbb{H}^1$ is given by

$$Qe_i = q_i \mathbf{e}_i, \ \ q_i > 0, \ \ i = 1, \cdots, m.$$

Here, $\{e_i, i = 1, \cdots, m\}$ is the canonical basis of $\mathbb{R}^m$.

Set

$$\mathcal{E}_0 := \sum_{i=1}^m q_i^2/\lambda_i, \ \ \ \mathcal{E}_1 := \sum_{i=1}^m q_i^2.$$

Then the quadratic variation of $\mathbf{w}(t)$ in $\mathbb{H}^0$ and $\mathbb{H}^1$ are given respectively by

$$[\mathbf{w}.]_{\mathbb{H}^0}(t) = \mathcal{E}_0 t, \ \ [\mathbf{w}.]_{\mathbb{H}^1}(t) = \mathcal{E}_1 t.$$

We remark that $\mathcal{E}_0 \leqslant \mathcal{E}_1/\lambda_1$.

Our main result in this section is the following:

Theorem 5.1. *Let $(\mathbf{T}_t)_{t\geqslant 0}$ be the transition semigroup associated with (5.2). For any sufficiently large $m_* = m_*(\mathcal{E}_1, \lambda_1, N) \in \mathbb{N}$, there exists a unique invariant probability measure associated with $(\mathbf{T}_t)_{t\geqslant 0}$.*

We shall divide the proof into two parts. In the first part, we shall prove the asymptotic strong Feller property of $(\mathbf{T}_t)_{t\geqslant 0}$ (cf. [15, Proposition 3.12]). In the second part, we shall prove a support property of the invariant measure, namely that the origin 0 is contained in the support of each invariant measure (cf. [6]). By [15, Proposition 3.12 and Corollary 3.17], these two parts will imply Theorem 5.1.

5.1. **Asymptotic Strong Feller Property.** Let $\mathbf{u}(t,\omega;\mathbf{u}_0)$ be the unique solution of Eq. (5.2). For $0 \leqslant s < t$, let $\mathcal{J}_{s,t}$ denote the derivative flow of $\mathbf{u}(t,\omega;\mathbf{u}_0)$ between s and t with respect to the initial values $\mathbf{u}_0$, i.e., for every $\mathbf{v}_0 \in \mathbb{H}^1$, $\mathcal{J}_{s,t}\mathbf{v}_0 \in \mathbb{H}^1$ satisfies

$$\partial_t \mathcal{J}_{s,t}\mathbf{v}_0 = \Delta \mathcal{J}_{s,t}\mathbf{v}_0 + \mathcal{K}(\mathbf{u}(t,\omega;\mathbf{u}_0), \mathcal{J}_{s,t}\mathbf{v}_0), \quad \mathcal{J}_{s,s}\mathbf{v}_0 = \mathbf{v}_0, \tag{5.3}$$

where $\mathcal{K}$ is linear with respect to the second component and given by

$$\mathcal{K}(\mathbf{u},\mathbf{v}) := -\mathscr{P}((\mathbf{v}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{v}) - \mathscr{P}(g_N(|\mathbf{u}|^2)\mathbf{v} + 2g_N'(|\mathbf{u}|^2)\langle \mathbf{u},\mathbf{v}\rangle_{\mathbb{R}^3}\mathbf{u}).$$

In the Appendix, we shall prove that for each ω

$$(\mathcal{J}_{0,t}\mathbf{v}_0)(\omega) = \lim_{\epsilon\downarrow 0} \frac{\mathbf{u}(t,\omega;\mathbf{u}_0+\epsilon\mathbf{v}_0) - \mathbf{u}(t,\omega;\mathbf{u}_0)}{\epsilon} \quad \text{in } \mathbb{H}^1. \tag{5.4}$$

Let us now consider the Malliavin derivative of $\mathbf{u}(t,\omega;\mathbf{u}_0)$ with respect to ω. Let $\mathscr{H}$ be the Cameron-Martin space, i.e., all absolutely continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^m$ with locally square integrable derivative. For any $v \in \mathscr{H}$, the Malliavin derivative is defined by

$$D^v\mathbf{u}(t,\omega;\mathbf{u}_0) := \lim_{\epsilon\to 0} \frac{\mathbf{u}(t,\omega+\epsilon v;\mathbf{u}_0) - \mathbf{u}(t,\omega;\mathbf{u}_0)}{\epsilon}, \quad P-a.s.. \tag{5.5}$$

Notice that v can be random and possibly nonadapted to the filtration generated by W. For the sake of simplicity, we write $\mathcal{A}_t v := D^v\mathbf{u}(t,\omega;\mathbf{u}_0)$. Then

$$\partial_t \mathcal{A}_t v = \Delta \mathcal{A}_t v + \mathcal{K}(\mathbf{u}(t,\omega;\mathbf{u}_0), \mathcal{A}_t v) + Q\dot{v}(t), \quad \mathcal{A}_0 v = 0, \tag{5.6}$$

where $\dot{v}(t)$ is the derivative of $v(t)$ with respect to t.

By the formula of variation of constants, it is easy to see that

$$\mathcal{A}_t v = \int_0^t \mathcal{J}_{s,t} Q\dot{v}(s)\mathrm{d}s.$$

Moreover, for any $\mathbf{v}_0 \in \mathbb{H}^1$ and $v \in \mathscr{H}$, set

$$\mathbf{v}(t) := \mathcal{J}_{0,t}\mathbf{v}_0 - \mathcal{A}_t v.$$

Then

$$\partial_t \mathbf{v}(t) = \Delta\mathbf{v}(t) + \mathcal{K}(\mathbf{u}(t),\mathbf{v}(t)) - Q\dot{v}(t), \quad \mathbf{v}(0) = \mathbf{v}_0. \tag{5.7}$$

As done in [15], our main aim is to construct a suitable v such that $\mathbf{v}(t)$ exponentially decays to zero in some sense as $t\to\infty$. We first introduce some necessary notations and prove some preparing lemmas.

Let $\mathbb{H}^1_\ell$ denote the following finite dimensional subspace of $\mathbb{H}^1$ (called low mode space)

$$\mathbb{H}^1_\ell := \mathrm{span}\{\mathbf{e}_1,\cdots,\mathbf{e}_m\}.$$

Then we have the following direct sum decomposition:

$$\mathbb{H}^1 = \mathbb{H}^1_\ell \oplus \mathbb{H}^1_\hbar$$

and for any $\mathbf{u}\in\mathbb{H}^1$,

$$\mathbf{u} = \mathbf{u}_\ell + \mathbf{u}_\hbar, \quad \mathbf{u}_\ell \in \mathbb{H}^1_\ell,\ \mathbf{u}_\hbar \in \mathbb{H}^1_\hbar.$$

The co-dimensional space $\mathbb{H}^1_\hbar$ is also called high mode space. For any $\mathbf{v}\in\mathbb{H}^0$, we define

$$\Pi_\ell \mathbf{v} := \sum_{i=1}^m \langle (-\Delta)\mathbf{e}_i, \mathbf{v}\rangle_{\mathbb{H}^0}\mathbf{e}_i \in \mathbb{H}^1_\ell$$

and

$$\Pi_\hbar \mathbf{v} := \mathbf{v} - \Pi_\ell\mathbf{v} \in \mathbb{H}^0.$$

In what follows, we shall always write $\mathbf{v}_\ell := \Pi_\ell\mathbf{v}$ and $\mathbf{v}_\hbar := \Pi_\hbar\mathbf{v}$.

The following lemma is immediate.

Lemma 5.2. *For any* $\mathbf{u} \in \mathbb{H}^2$
$$\|\Delta \Pi_\hbar \mathbf{u}\|^2_{\mathbb{H}^0} \geqslant \lambda_m \|\nabla \Pi_\hbar \mathbf{u}\|^2_{\mathbb{H}^0}.$$

We also need the following lemma. Recall that $\|\mathbf{u}\|_{\mathbb{H}^2} = \|\Delta \mathbf{u}\|_{\mathbb{H}^0}$.

Lemma 5.3. *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{H}^2$*, set*
$$\mathcal{N}(\mathbf{u}) := \|\mathbf{u}\|^2_{\mathbb{H}^2} + \||\mathbf{u}| \cdot |\nabla \mathbf{u}|\|^2_{L^2}. \tag{5.8}$$
Then
$$\langle \mathbf{v}_\hbar, \mathcal{K}(\mathbf{u}, \mathbf{v})\rangle_{\mathbb{H}^1} \leqslant \frac{1}{2}\|\Delta \mathbf{v}_\hbar\|^2_{\mathbb{H}^0} + C_N \cdot \mathcal{N}(\mathbf{u}) \cdot (\|\mathbf{v}_\hbar\|^2_{\mathbb{H}^1} + \|\mathbf{v}_\ell\|^2_{\mathbb{H}^1})$$
and
$$\|\Pi_\ell \mathcal{K}(\mathbf{u}, \mathbf{v})\|^2_{\mathbb{H}^1} \leqslant C_m \|\mathbf{v}\|^2_{\mathbb{H}^0} \cdot (1 + \|\mathbf{u}\|^4_{\mathbb{H}^1}),$$
where the constant C_N *(resp.* C_m*) only depends on* N *(resp.* m*).*

Proof. For the first, we write
$$\langle \mathbf{v}_\hbar, \mathcal{K}(\mathbf{u}, \mathbf{v})\rangle_{\mathbb{H}^1} = I_1 + I_2 + I_3 + I_4,$$
where
$$\begin{aligned}
I_1 &:= -\langle \mathbf{v}_\hbar, \mathscr{P}((\mathbf{v} \cdot \nabla)\mathbf{u})\rangle_{\mathbb{H}^1},\\
I_2 &:= -\langle \mathbf{v}_\hbar, \mathscr{P}((\mathbf{u} \cdot \nabla)\mathbf{v})\rangle_{\mathbb{H}^1},\\
I_3 &:= -\langle \mathbf{v}_\hbar, \mathscr{P}(g_N(|\mathbf{u}|^2)\mathbf{v})\rangle_{\mathbb{H}^1},\\
I_4 &:= -2\langle \mathbf{v}_\hbar, \mathscr{P}(g'_N(|\mathbf{u}|^2)\langle \mathbf{u}, \mathbf{v}\rangle_{\mathbb{R}^3}\mathbf{u})\rangle_{\mathbb{H}^1}.
\end{aligned}$$
For I_1, by Young's inequality and the Sobolev inequality (2.1) we have
$$\begin{aligned}
I_1 &\leqslant \frac{1}{8}\|\Delta \mathbf{v}_\hbar\|^2_{\mathbb{H}^0} + 2\||\mathbf{v}| \cdot |\nabla \mathbf{u}|\|^2_{L^2} \leqslant \frac{1}{8}\|\Delta \mathbf{v}_\hbar\|^2_{\mathbb{H}^0} + 2\|\mathbf{v}\|^2_{L^6} \cdot \|\nabla \mathbf{u}\|^2_{L^3} \leqslant\\
&\leqslant \frac{1}{8}\|\Delta \mathbf{v}_\hbar\|^2_{\mathbb{H}^0} + C\|\mathbf{v}_\hbar\|^2_{\mathbb{H}^1} \cdot \|\mathbf{u}\|^2_{\mathbb{H}^2} + C\|\mathbf{v}_\ell\|^2_{\mathbb{H}^1} \cdot \|\mathbf{u}\|^2_{\mathbb{H}^2}.
\end{aligned}$$
For I_2, we have
$$\begin{aligned}
I_2 &\leqslant \frac{1}{8}\|\Delta \mathbf{v}_\hbar\|^2_{\mathbb{H}^0} + 2\||\nabla \mathbf{v}| \cdot |\mathbf{u}|\|^2_{L^2} \leqslant \frac{1}{8}\|\Delta \mathbf{v}_\hbar\|^2_{\mathbb{H}^0} + 2\|\mathbf{v}\|^2_{\mathbb{H}^1} \cdot \|\mathbf{u}\|^2_{L^\infty} \leqslant\\
&\leqslant \frac{1}{8}\|\Delta \mathbf{v}_\hbar\|^2_{\mathbb{H}^0} + C\|\mathbf{v}_\hbar\|^2_{\mathbb{H}^1} \cdot \|\mathbf{u}\|^2_{\mathbb{H}^2} + C\|\mathbf{v}_\ell\|^2_{\mathbb{H}^1} \cdot \|\mathbf{u}\|^2_{\mathbb{H}^2}.
\end{aligned}$$
For I_3, we have
$$\begin{aligned}
I_3 &= -\langle \nabla \mathbf{v}_\hbar, g_N(|\mathbf{u}|^2)\nabla \mathbf{v}\rangle_{\mathbb{H}^0} - \langle \nabla \mathbf{v}_\hbar, g'_N(|\mathbf{u}|^2)\nabla|\mathbf{u}|^2\mathbf{v}\rangle_{\mathbb{H}^0}\\
&\leqslant \|\mathbf{u}\|^2_{L^\infty} \cdot \||\nabla \mathbf{v}_\hbar| \cdot |\nabla \mathbf{v}|\|_{L^1} + \|\nabla \mathbf{v}_\hbar\|_{L^6} \cdot \|\mathbf{v}\|_{L^3} \cdot \|\nabla|\mathbf{u}|^2\|_{L^2}\\
&\leqslant C\|\mathbf{u}\|^2_{\mathbb{H}^2} \cdot (\|\mathbf{v}_\hbar\|^2_{\mathbb{H}^1} + \|\mathbf{v}_\ell\|^2_{\mathbb{H}^1}) + \frac{1}{8}\|\mathbf{v}_\hbar\|^2_{\mathbb{H}^2} + C\|\mathbf{v}\|^2_{\mathbb{H}^1} \cdot \||\mathbf{u}| \cdot |\nabla \mathbf{u}|\|^2_{L^2}.
\end{aligned}$$
For I_4, noting that
$$|g''_N(r)| \leqslant C \cdot 1_{\{N<r<N+1\}},$$
we similarly have
$$I_4 \leqslant C\|\mathbf{u}\|^2_{\mathbb{H}^2} \cdot (\|\mathbf{v}_\hbar\|^2_{\mathbb{H}^1} + \|\mathbf{v}_\ell\|^2_{\mathbb{H}^1}) + \frac{1}{8}\|\mathbf{v}_\hbar\|^2_{\mathbb{H}^2} + C_N\|\mathbf{v}\|^2_{\mathbb{H}^1} \cdot \||\mathbf{u}| \cdot |\nabla \mathbf{u}|\|^2_{L^2}.$$
Combining the above calculations, we obtain the first estimate.

As for the second one, we may write

$$\|\Pi_\ell \mathcal{K}(\mathbf{u},\mathbf{v})\|_{\mathbb{H}^1}^2 = \sum_{i=1}^m \langle \mathbf{e}_i, \mathcal{K}(\mathbf{u},\mathbf{v})\rangle_{\mathbb{H}^1}^2 = \sum_{i=1}^m \left(\sum_{j=1}^4 J_{ij}\right)^2,$$

where

$$\begin{aligned}
J_{i1} &:= -\langle \mathbf{e}_i, \mathscr{P}((\mathbf{v}\cdot\nabla)\mathbf{u})\rangle_{\mathbb{H}^1},\\
J_{i2} &:= -\langle \mathbf{e}_i, \mathscr{P}((\mathbf{u}\cdot\nabla)\mathbf{v})\rangle_{\mathbb{H}^1},\\
J_{i3} &:= -\langle \mathbf{e}_i, \mathscr{P}(g_N(|\mathbf{u}|^2)\mathbf{v})\rangle_{\mathbb{H}^1},\\
J_{i4} &:= -2\langle \mathbf{e}_i, \mathscr{P}(g_N'(|\mathbf{u}|^2)(\mathbf{u}\cdot\mathbf{v})\mathbf{u})\rangle_{\mathbb{H}^1}.
\end{aligned}$$

For J_{i1}, we have

$$J_{i1} = \langle \Delta\mathbf{e}_i, (\mathbf{v}\cdot\nabla)\mathbf{u}\rangle_{\mathbb{H}^0} = -\langle \nabla\Delta\mathbf{e}_i, \mathbf{v}\otimes\mathbf{u}\rangle_{\mathbb{H}^0} \leqslant \|\nabla\Delta\mathbf{e}_i\|_{L^\infty}\||\mathbf{v}|\cdot\mathbf{u}\|_{L^1} \leqslant C_{\mathbf{e}_i}\|\mathbf{v}\|_{\mathbb{H}_0}\|\mathbf{u}\|_{\mathbb{H}_0}.$$

Similarly, we have

$$\begin{aligned}
J_{i2} &\leqslant C_{\mathbf{e}_i}\|\mathbf{v}\|_{\mathbb{H}_0}\|\mathbf{u}\|_{\mathbb{H}_0},\\
J_{i3} &\leqslant C_{\mathbf{e}_i}\|\mathbf{v}\|_{\mathbb{H}_0}\|\mathbf{u}\|_{L^4}^2,\\
J_{i4} &\leqslant C_{\mathbf{e}_i}\|\mathbf{v}\|_{\mathbb{H}_0}\|\mathbf{u}\|_{L^4}^2.
\end{aligned}$$

Summarizing the above calculations and by the Sobolev embedding theorem, we obtain the second estimate. □

We now prove the following crucial estimate about the solution $\mathbf{u}(t)$.

Lemma 5.4. *(i) For any $\eta > 0$, there exist constants $C_\eta, C_{\mathcal{E}_1,\lambda_1,N,\eta} > 0$ such that for any $t > 0$ and $\mathbf{u}_0 \in \mathbb{H}^1$*

$$\mathbb{E}\exp\left\{\eta\int_0^t \mathcal{N}(\mathbf{u}(s;\mathbf{u}_0))\mathrm{d}s\right\} \leqslant \exp\{C_\eta\|\mathbf{u}_0\|_{\mathbb{H}^1}^2 + C_{\mathcal{E}_1,\lambda_1,N,\eta}t\},$$

where $\mathcal{N}(\mathbf{u})$ is defined by (5.8).

(ii) There exist constants $C_N, C_{\mathcal{E}_1,\lambda_1,N} > 0$ such that for any $t > 0$ and $\mathbf{u}_0 \in \mathbb{H}_1$

$$\mathbb{E}\|\mathbf{u}(t;\mathbf{u}_0)\|_{\mathbb{H}^1}^2 \leqslant \|\mathbf{u}_0\|_{\mathbb{H}^1}^2(C_N\cdot t + 1)e^{-t/2} + C_{\mathcal{E}_1,\lambda_1,N}.$$

Proof. By Itô's formula, we have

$$\mathrm{d}\|\mathbf{u}(t)\|_{\mathbb{H}^0}^2 = 2\langle \mathbf{u}(t), A(\mathbf{u}(t))\rangle_{\mathbb{H}^0}\mathrm{d}t + 2\langle \mathbf{u}(t), \mathrm{d}\mathbf{w}(t)\rangle_{\mathbb{H}^0} + \mathcal{E}_0\mathrm{d}t. \tag{5.9}$$

By (2.9) and Young's inequality, we know

$$\begin{aligned}
\langle \mathbf{u}(t), A(\mathbf{u}(t))\rangle_{\mathbb{H}^0} &\leqslant -\|\nabla\mathbf{u}(t)\|_{\mathbb{H}^0}^2 - \|\mathbf{u}(t)\|_{L^4}^4 + N\|\mathbf{u}(t)\|_{\mathbb{H}^0}^2\\
&\leqslant -\|\mathbf{u}(t)\|_{\mathbb{H}^1}^2 - \frac{1}{2}\|\mathbf{u}(t)\|_{\mathbb{H}^0}^4 + \frac{N^2}{2}.
\end{aligned} \tag{5.10}$$

Using Lemma 6.2 in the Appendix, we get for any $t, \eta > 0$

$$\mathbb{E}\exp\left\{\eta\int_0^t \|\mathbf{u}(s)\|_{\mathbb{H}^1}^2\mathrm{d}s\right\} \leqslant \exp\{\eta\|\mathbf{u}_0\|_{\mathbb{H}^0}^2 + C_{\mathcal{E}_0,N,\eta}t\}. \tag{5.11}$$

Again, by Itô's formula and (2.10), we have

$$\begin{aligned}
\mathrm{d}\|\mathbf{u}(t)\|_{\mathbb{H}^1}^2 &= 2[\![\mathbf{u}(t), A(\mathbf{u}(t))]\!]\mathrm{d}t + 2\langle \mathbf{u}(t), \mathrm{d}\mathbf{w}(t)\rangle_{\mathbb{H}^1} + \mathcal{E}_1\mathrm{d}t\\
&\leqslant (-\mathcal{N}(\mathbf{u}(t)) + C_N\|\mathbf{u}(t)\|_{\mathbb{H}^1}^2)\mathrm{d}t + 2\langle \mathbf{u}(t), \mathrm{d}\mathbf{w}(t)\rangle_{\mathbb{H}^1} + \mathcal{E}_1\mathrm{d}t.
\end{aligned} \tag{5.12}$$

As in the proof of Lemma 6.2 in the Appendix, using (5.11) and exponential martingales, we then get the first estimate.

On the other hand, from (5.9) and (5.10), we have

$$\mathrm{d}\|\mathbf{u}(t)\|_{\mathbb{H}^0}^2 \leqslant -\frac{1}{2}\|\mathbf{u}(t)\|_{\mathbb{H}^0}^2\mathrm{d}t + 2\langle \mathbf{u}(t), \mathrm{d}\mathbf{w}(t)\rangle_{\mathbb{H}^0} + (\mathcal{E}_0 + N^2 + \frac{1}{2})\mathrm{d}t.$$

It is direct by Gronwall's inequality that

$$\mathbb{E}\|\mathbf{u}(t;\mathbf{u}_0)\|_{\mathbb{H}^0}^2 \leqslant \|\mathbf{u}\|_{\mathbb{H}^0}^2 e^{-t/2} + 2(\mathcal{E}_0 + N^2 + 1).$$

Thus, thanks to

$$\|\mathbf{u}(t)\|_{\mathbb{H}^1}^2 \leqslant \|\mathbf{u}(t)\|_{\mathbb{H}^2}\|\mathbf{u}(t)\|_{\mathbb{H}^0}, \tag{5.13}$$

we obtain

$$\begin{aligned}
\mathrm{d}(e^{t/2}\|\mathbf{u}(t)\|_{\mathbb{H}^1}^2) &= e^{t/2}(2[\![\mathbf{u}(t), A(\mathbf{u}(t))]\!] + \mathcal{E}_1 + \frac{1}{2}\|\mathbf{u}(t)\|_{\mathbb{H}^1}^2)\mathrm{d}t + 2e^{t/2}\langle \mathbf{u}(t), \mathrm{d}\mathbf{w}(t)\rangle_{\mathbb{H}^1} \\
&\leqslant e^{t/2}(-\|\mathbf{u}(t)\|_{\mathbb{H}^2}^2 + C_N\|\mathbf{u}(t)\|_{\mathbb{H}^1}^2 + \mathcal{E}_1)\mathrm{d}t + 2e^{t/2}\langle \mathbf{u}(t), \mathrm{d}\mathbf{w}(t)\rangle_{\mathbb{H}^1} \\
&\leqslant e^{t/2}(C_N\|\mathbf{u}(t)\|_{\mathbb{H}^0}^2 + \mathcal{E}_1)\mathrm{d}t + 2e^{t/2}\langle \mathbf{u}(t), \mathrm{d}\mathbf{w}(t)\rangle_{\mathbb{H}^1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
e^{t/2}\mathbb{E}\|\mathbf{u}(t;\mathbf{u}_0)\|_{\mathbb{H}^1}^2 &\leqslant \|\mathbf{u}_0\|_{\mathbb{H}^1}^2 + C_N\int_0^t e^{s/2}\mathbb{E}\|\mathbf{u}(s;\mathbf{u}_0)\|_{\mathbb{H}^0}^2\mathrm{d}s + 2\mathcal{E}_1 e^{t/2} \\
&\leqslant \|\mathbf{u}_0\|_{\mathbb{H}^1}^2 + C_N\int_0^t (\|\mathbf{u}_0\|_{\mathbb{H}^0}^2 + 2e^{s/2}(\mathcal{E}_0 + N^2 + 1))\mathrm{d}s + 2\mathcal{E}_1 e^{t/2} \\
&\leqslant \|\mathbf{u}_0\|_{\mathbb{H}^1}^2 + C_N\|\mathbf{u}_0\|_{\mathbb{H}^0}^2 t + e^{t/2}(C_N(\mathcal{E}_0 + N^2 + 1) + 2\mathcal{E}_1),
\end{aligned}$$

which then gives the second estimate. □

Based on the previous discussions and lemmas, we can now prove the following proposition, which will imply the asymptotic strong Feller property of $(\mathbf{T}_t)_{t\geqslant 0}$ according to [15, Proposition 3.12].

Proposition 5.5. *Let $(\mathbf{T}_t)_{t\geqslant 0}$ be the semigroup associated with (5.2). There exist a constant $m_* := m_*(\mathcal{E}_1, N) \in \mathbb{N}$ and constants $C_0, C_1, \gamma > 0$ such that for any $t > 0$, $\mathbf{u}_0 \in \mathbb{H}^1$, and any Fréchet differentiable function φ on $\mathbb{H}^1$ with $\|\varphi\|_\infty, \|\nabla\varphi\|_\infty < +\infty$,*

$$\|\nabla \mathbf{T}_t\varphi(\mathbf{u}_0)\|_{\mathbb{H}^1} \leqslant C_0 \cdot \exp\{C_1\|\mathbf{u}_0\|_{\mathbb{H}^1}^2\} \cdot (\|\varphi\|_\infty + e^{-\gamma t}\|\nabla\varphi\|_\infty).$$

Proof. For any $\mathbf{v}_0 \in \mathbb{H}^1$ with $\|\mathbf{v}_0\|_{\mathbb{H}^1} = 1$, define

$$\mathbf{v}_\ell(t) := \begin{cases} \mathbf{v}_{0\ell}\cdot(1 - t/(2\|\mathbf{v}_{0\ell}\|_{\mathbb{H}^1})), & t \in [0, 2\|\mathbf{v}_{0\ell}\|_{\mathbb{H}^1}] \\ 0, & t \in (2\|\mathbf{v}_{0\ell}\|_{\mathbb{H}^1}, \infty). \end{cases}$$

Let $\mathbf{v}_\hbar(t)$ solve the following linear evolution equation:

$$\partial_t\mathbf{v}_\hbar(t) = \Delta\Pi_\hbar\mathbf{v}_\hbar(t) + \Pi_\hbar\mathcal{K}(\mathbf{u}(t), \mathbf{v}_\hbar(t) + \mathbf{v}_\ell(t)), \ \ \mathbf{v}_\hbar(0) = \mathbf{v}_{0\hbar}.$$

Set

$$\mathbf{v}(t) := \mathbf{v}_\ell(t) + \mathbf{v}_\hbar(t)$$

and

$$\dot{v}(t) := Q^{-1}\left(\frac{\mathbf{v}_\ell \cdot 1_{\{t<2\|\mathbf{v}_\ell\|_{\mathbb{H}^1}\}}}{2\|\mathbf{v}_\ell\|_{\mathbb{H}^1}} + \Delta\mathbf{v}_\ell(t) + \Pi_\ell\mathcal{K}(\mathbf{u}(t), \mathbf{v}(t))\right).$$

Then $v(t) \in \mathscr{H}$ is a continuous adapted process. From the construction, one finds that $\mathbf{v}(t)$ together with $v(t)$ solves the equation (5.7).

Thus, we have

$$\begin{aligned}
\langle \nabla\mathbf{T}_t\varphi(\mathbf{u}_0), \mathbf{v}_0\rangle_{\mathbb{H}^1} &= \mathbb{E}\langle(\nabla\varphi)(\mathbf{u}(t;\mathbf{u}_0)), \mathcal{J}_{0,t}\mathbf{v}_0\rangle_{\mathbb{H}^1} \\
&= \mathbb{E}\langle(\nabla\varphi)(\mathbf{u}(t;\mathbf{u}_0)), \mathcal{A}_t v(t)\rangle_{\mathbb{H}^1} + \mathbb{E}\langle(\nabla\varphi)(\mathbf{u}(t;\mathbf{u}_0)), \mathbf{v}(t)\rangle_{\mathbb{H}^1}
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}(D^v(\varphi(\mathbf{u}(t;\mathbf{u}_0)))) + \mathbb{E}\langle(\nabla\varphi)(\mathbf{u}(t;\mathbf{u}_0)),\mathbf{v}(t)\rangle_{\mathbb{H}^1}\\
&= \mathbb{E}\left(\varphi(\mathbf{u}(t;\mathbf{u}_0))\cdot\int_0^t \dot v(s)\mathrm{d}W_s\right) + \mathbb{E}\langle(\nabla\varphi)(\mathbf{u}(t;\mathbf{u}_0)),\mathbf{v}(t)\rangle_{\mathbb{H}^1}\\
&\leqslant \|\varphi\|_\infty\left(\int_0^t \mathbb{E}|\dot v(s)|^2\mathrm{d}s\right)^{1/2} + \|\nabla\varphi\|_\infty\mathbb{E}\|\mathbf{v}(t)\|_{\mathbb{H}^1},
\end{aligned}
\tag{5.14}
$$

where the last equality is due to the integration by parts formula in the Malliavin calculus (cf. [21]).

By the chain rule and Lemmas 5.3 and 5.2, we have

$$
\begin{aligned}
\partial_t\|\mathbf{v}_\hbar(t)\|^2_{\mathbb{H}^1} &= -2\|\Delta\Pi_\hbar\mathbf{v}_\hbar(t)\|^2_{\mathbb{H}^0} + 2\langle\mathbf{v}_\hbar(t),\Pi_\hbar\mathcal{K}(\mathbf{u}(t),\mathbf{v}(t))\rangle_{\mathbb{H}^1}\\
&\leqslant -\|\Delta\Pi_\hbar\mathbf{v}_\hbar(t)\|^2_{\mathbb{H}^0} + C_N\cdot\mathcal{N}(\mathbf{u}(t))\cdot(\|\mathbf{v}_\hbar(t)\|^2_{\mathbb{H}^1}+\|\mathbf{v}_\ell(t)\|^2_{\mathbb{H}^1})\\
&\leqslant (-\lambda_m + C_N\cdot\mathcal{N}(\mathbf{u}(t)))\cdot\|\mathbf{v}_\hbar(t)\|^2_{\mathbb{H}^1} + C_N\cdot\mathcal{N}(\mathbf{u}(t))\cdot\|\mathbf{v}_\ell(t)\|^2_{\mathbb{H}^1}.
\end{aligned}
$$

Noting that $v(t)=0$ for $t\geqslant 2$, by Gronwall's inequality we get

$$
\begin{aligned}
\|\mathbf{v}_\hbar(t)\|^2_{\mathbb{H}^1} \leqslant\ & \|\mathbf{v}_\hbar(0)\|^2_{\mathbb{H}^1}\exp\left\{-\lambda_m t + C_N\int_0^t\mathcal{N}(\mathbf{u}(s))\mathrm{d}s\right\}\\
&+\exp\left\{-\lambda_m(t-2)+C_N\int_0^t\mathcal{N}(\mathbf{u}(s))\mathrm{d}s\right\}\int_0^2\|\mathbf{v}_\ell(s)\|^2_{\mathbb{H}^1}\mathrm{d}s.
\end{aligned}
$$

By (i) of Lemma 5.4, since $\lambda_m\uparrow\infty$ as $m\to\infty$, there exist constants $\gamma>0$ and $m_*=m_*(\mathcal{E}_1,\lambda_1,N)\in\mathbb{N}$ such that for all $t\geqslant 0$,

$$
\mathbb{E}\|\mathbf{v}_\hbar(t)\|^4_{\mathbb{H}^1}\leqslant C_{\mathcal{E}_1,\lambda_1,N}\cdot e^{C_N\|\mathbf{u}_0\|^2_{\mathbb{H}^1}-\gamma t}.
$$

Hence, for any $t\geqslant 2$,

$$
\mathbb{E}\|\mathbf{v}(t)\|_{\mathbb{H}^1}\leqslant C_{\mathcal{E}_1,\lambda_1,N}\cdot e^{C_N\|\mathbf{u}_0\|^2_{\mathbb{H}^1}-\gamma t}. \tag{5.15}
$$

On the other hand, by Lemma 5.3, we have

$$
\begin{aligned}
\mathbb{E}|\dot v(t)|^2 &\leqslant C_m\left(1+\mathbb{E}(\|\mathbf{v}(t)\|^2_{\mathbb{H}^0}(1+\|\mathbf{u}(t)\|^4_{\mathbb{H}^1}))\right)\\
&\leqslant C_m\left(1+(\mathbb{E}\|\mathbf{v}(t)\|^4_{\mathbb{H}^0})^{1/2}(1+\mathbb{E}\|\mathbf{u}(t)\|^8_{\mathbb{H}^1})^{1/2}\right).
\end{aligned}
\tag{5.16}
$$

Using Itô's formula and (2.8), as in the proof of Theorem 4.3, we have

$$
\mathbb{E}\|\mathbf{u}(t)\|^{2p}_{\mathbb{H}^0}\leqslant C\|\mathbf{u}_0\|^{2p}_{\mathbb{H}^0}(1+t),
$$

and also by (2.10), $\|\mathbf{u}\|^2_{\mathbb{H}^1}\leqslant\|\mathbf{u}\|_{\mathbb{H}^0}\|\mathbf{u}\|_{\mathbb{H}^2}$ and Young's inequality,

$$
\begin{aligned}
\mathbb{E}\|\mathbf{u}(t)\|^{2p}_{\mathbb{H}^1} &\leqslant \|\mathbf{u}_0\|^{2p}_{\mathbb{H}^1} - p\mathbb{E}\int_0^t\|\mathbf{u}(s)\|^{2(p-1)}_{\mathbb{H}^1}\|\mathbf{u}(s)\|^2_{\mathbb{H}^2}\mathrm{d}s + C_N\mathbb{E}\int_0^t\|\mathbf{u}(s)\|^{2p}_{\mathbb{H}^1}\mathrm{d}s + Ct\\
&\leqslant \|\mathbf{u}_0\|^{2p}_{\mathbb{H}^1} - \frac{p}{2}\mathbb{E}\int_0^t\|\mathbf{u}(s)\|^{2(p-1)}_{\mathbb{H}^1}\|\mathbf{u}(s)\|^2_{\mathbb{H}^2}\mathrm{d}s + C_N\mathbb{E}\int_0^t\|\mathbf{u}(s)\|^{2(p-1)}_{\mathbb{H}^1}\|\mathbf{u}(s)\|^2_{\mathbb{H}^0}\mathrm{d}s + Ct\\
&\leqslant \|\mathbf{u}_0\|^{2p}_{\mathbb{H}^1} + C_N\mathbb{E}\int_0^t\|\mathbf{u}(s)\|^{2p}_{\mathbb{H}^0}\mathrm{d}s + Ct\leqslant C\|\mathbf{u}_0\|^{2p}_{\mathbb{H}^1}(1+t^2).
\end{aligned}
$$

Thus, integrating both sides of (5.16) and using (5.15), we obtain

$$
\int_0^\infty\mathbb{E}|\dot v(t)|^2\mathrm{d}t\leqslant C_{m,\mathcal{E}_1,\lambda_1,N,\gamma}\cdot e^{C_N\|\mathbf{u}_0\|^2_{\mathbb{H}^1}}\cdot\left(1+\int_0^\infty e^{-\gamma t}(1+t)\mathrm{d}t\right)\leqslant C_{m,\mathcal{E}_1,\lambda_1,N,\gamma}\cdot e^{C_N\|\mathbf{u}_0\|^2_{\mathbb{H}^1}}. \tag{5.17}
$$

The proof is thus completed by combining (5.14)-(5.17). □

5.2. **A Support Property of Invariant Measures.**

Proposition 5.6. *The point* 0 *belongs to the support of any invariant measure of* $(\mathbf{T}_t)_{t\geqslant 0}$.

For the proof we need the following lemma, whose proof in turn is inspired by [6].

Lemma 5.7. *For any* $r_1, r_2 > 0$, *there exists* $T > 0$ *such that*

$$\inf_{\|\mathbf{u}_0\|_{\mathbb{H}^1}\leqslant r_1} P\{\omega : \|\mathbf{u}(T,\omega;\mathbf{u}_0)\|_{\mathbb{H}^1} \leqslant r_2\} > 0.$$

Proof. Set

$$\mathbf{v}(t) := \mathbf{u}(t) - \mathbf{w}(t).$$

Then

$$\mathbf{v}'(t) = A(\mathbf{v}(t) + \mathbf{w}(t)), \quad \mathbf{v}(0) = \mathbf{u}_0.$$

Let $T > 0$ and $\epsilon \in (0,1)$, to be determined below. We assume that

$$\sup_{t\in[0,T]} \|\mathbf{w}(t)\|_{\mathbb{H}^6} < \epsilon. \tag{5.18}$$

First of all, by the chain rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}(t)\|^2_{\mathbb{H}^0} = J_1 + J_2 + J_3 + J_4,$$

where

$$\begin{aligned}
J_1 &:= -2\|\nabla\mathbf{v}(t)\|^2_{\mathbb{H}^0} + 2\langle\Delta\mathbf{w}(t), \mathbf{v}(t)\rangle_{\mathbb{H}^0},\\
J_2 &:= -2\langle\mathbf{v}(t), ((\mathbf{v}(t)+\mathbf{w}(t))\cdot\nabla)(\mathbf{v}(t)+\mathbf{w}(t))\rangle_{\mathbb{H}^0},\\
J_3 &:= -2\langle\mathbf{v}(t)+\mathbf{w}(t), g_N(|\mathbf{v}(t)+\mathbf{w}(t)|^2)(\mathbf{v}(t)+\mathbf{w}(t))\rangle_{\mathbb{H}^0},\\
J_4 &:= 2\langle\mathbf{w}(t), g_N(|\mathbf{v}(t)+\mathbf{w}(t)|^2)(\mathbf{v}(t)+\mathbf{w}(t))\rangle_{\mathbb{H}^0}.
\end{aligned}$$

For J_1, by (5.18) we have

$$J_1 \leqslant -2\|\nabla\mathbf{v}(t)\|^2_{\mathbb{H}^0} + C\epsilon\|\mathbf{v}(t)\|_{\mathbb{H}^0}.$$

Here and below, C denotes an absolute constant.

For J_2, by the Sobolev inequality (2.1) and (5.18) we have

$$\begin{aligned}
J_2 &= -2\langle\mathbf{w}(t), ((\mathbf{v}(t)+\mathbf{w}(t))\cdot\nabla)(\mathbf{v}(t)+\mathbf{w}(t))\rangle_{\mathbb{H}^0}\\
&\leqslant 2\|\nabla\mathbf{w}(t)\|_{L^\infty}\|\mathbf{v}(t)+\mathbf{w}(t)\|^2_{\mathbb{H}^0}\\
&\leqslant C\epsilon\cdot\|\mathbf{v}(t)\|^2_{\mathbb{H}^0} + C\epsilon.
\end{aligned}$$

For J_3, we obviously have

$$J_3 \leqslant 0.$$

For J_4, by (2.1) and Young's inequality we have

$$\begin{aligned}
J_4 &\leqslant 2\|\mathbf{w}(t)\|_{L^\infty}\cdot\|\mathbf{v}(t)+\mathbf{w}(t)\|^3_{L^3} \leqslant C\epsilon\cdot\|\mathbf{v}(t)\|^3_{L^3} + C\epsilon^4 \leqslant\\
&\leqslant C\epsilon\cdot\|\nabla\mathbf{v}(t)\|^{3/2}_{\mathbb{H}^0}\|\mathbf{v}(t)\|^{3/2}_{\mathbb{H}^0} + C\epsilon^4\\
&\leqslant \|\nabla\mathbf{v}(t)\|^2_{\mathbb{H}^0} + C\epsilon^4\cdot\|\mathbf{v}(t)\|^6_{\mathbb{H}^0} + C\epsilon^4.
\end{aligned}$$

Combing the above calculations gives that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}(t)\|^2_{\mathbb{H}^0} &\leqslant -\|\nabla\mathbf{v}(t)\|^2_{\mathbb{H}^0} + C\epsilon\cdot\|\mathbf{v}(t)\|^6_{\mathbb{H}^0} + C\epsilon\\
&\leqslant -\frac{1}{\lambda_1}\|\mathbf{v}(t)\|^2_{\mathbb{H}^0} + C\epsilon\cdot\|\mathbf{v}(t)\|^6_{\mathbb{H}^0} + C\epsilon,
\end{aligned}$$

where the second step is due to the Poincare inequality (5.1).

Note that $\|\mathbf{v}(t)\|^2_{\mathbb{H}^0}$ depends on ϵ through (5.18). By Lemma 6.1 in the Appendix, for any $\delta, h > 0$, we may choose a $T_0 > 0$ sufficiently large and an ϵ small enough such that

$$\sup_{t\in[0,T_0]} \|\mathbf{v}(t)\|_{\mathbb{H}^0} \leqslant 2r_1 \tag{5.19}$$

and

$$\sup_{t\in[T_0,T_0+h]} \|\mathbf{v}(t)\|_{\mathbb{H}^0} < \delta. \tag{5.20}$$

Let us now turn to the estimate of the first order Sobolev norm of $\mathbf{v}(t)$. By the chain rule again, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}(t)\|^2_{\mathbb{H}^1} = 2[\![\mathbf{v}(t), A(\mathbf{v}(t)+\mathbf{w}(t))]\!] = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}
I_1 &:= 2[\![\mathbf{v}(t)+\mathbf{w}(t), A(\mathbf{v}(t)+\mathbf{w}(t))]\!],\\
I_2 &:= -2\langle \Delta^2\mathbf{w}(t), \mathbf{v}(t)+\mathbf{w}(t)\rangle_{\mathbb{H}^0},\\
I_3 &:= 2\langle \Delta\mathbf{w}(t), ((\mathbf{v}(t)+\mathbf{w}(t))\cdot\nabla)(\mathbf{v}(t)+\mathbf{w}(t))\rangle_{\mathbb{H}^0},\\
I_4 &:= 2\langle \Delta\mathbf{w}(t), g_N(|\mathbf{v}(t)+\mathbf{w}(t)|^2)(\mathbf{v}(t)+\mathbf{w}(t))\rangle_{\mathbb{H}^0}.
\end{aligned}$$

For I_1, by (2.10) and (5.13) we have

$$\begin{aligned}
I_1 &\leqslant -\|\mathbf{v}(t)+\mathbf{w}(t)\|^2_{\mathbb{H}^2} + C_N\|\mathbf{v}(t)+\mathbf{w}(t)\|^2_{\mathbb{H}^1}\\
&\leqslant -\frac{1}{2}\|\mathbf{v}(t)\|^2_{\mathbb{H}^2} + 4N\|\mathbf{v}(t)\|^2_{\mathbb{H}^1} + C_N\epsilon\\
&\leqslant -\frac{1}{4}\|\mathbf{v}(t)\|^2_{\mathbb{H}^2} + C_N\|\mathbf{v}(t)\|^2_{\mathbb{H}^0} + C_N\epsilon.
\end{aligned}$$

Here and below, C_N denotes a constant only depending on N.

For I_2, we have

$$I_2 \leqslant C\epsilon + C\epsilon\|\mathbf{v}(t)\|_{\mathbb{H}^0}.$$

For I_3, we have

$$I_3 \leqslant 2\|\nabla\Delta\mathbf{w}(t)\|_{L^\infty}\|\mathbf{v}(t)+\mathbf{w}(t)\|^2_{\mathbb{H}^0} \leqslant C\epsilon + C\epsilon\|\mathbf{v}(t)\|^2_{\mathbb{H}^0}.$$

For I_4, we have

$$I_4 \leqslant C\epsilon + C\epsilon\|\mathbf{v}(t)\|^3_{L^3} \leqslant \frac{1}{8}\|\mathbf{v}(t)\|^2_{\mathbb{H}^2} + C\epsilon + C\epsilon\|\mathbf{v}(t)\|^6_{\mathbb{H}^0}.$$

Combing the above calculations gives that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}(t)\|^2_{\mathbb{H}^1} &\leqslant -\frac{1}{8}\|\mathbf{v}(t)\|^2_{\mathbb{H}^2} + C_N\cdot\|\mathbf{v}(t)\|^2_{\mathbb{H}^0} + C\epsilon\cdot\|\mathbf{v}(t)\|^6_{\mathbb{H}^0} + C\epsilon\\
&\leqslant -C_0\|\mathbf{v}(t)\|^2_{\mathbb{H}^1} + C_N\cdot\|\mathbf{v}(t)\|^2_{\mathbb{H}^0} + C\epsilon\|\mathbf{v}(t)\|^6_{\mathbb{H}^0} + C\epsilon.
\end{aligned}$$

By Gronwall's inequality, for any $0 < t_1 < t_2$ we have

$$\|\mathbf{v}(t_2)\|^2_{\mathbb{H}^1} \leqslant e^{-C_0(t_2-t_1)}\|\mathbf{v}(t_1)\|^2_{\mathbb{H}^1} + \frac{1}{C_0}\Big(C_N\cdot\sup_{t\in[t_1,t_2]}\|\mathbf{v}(t)\|^2_{\mathbb{H}^0} + C\epsilon\cdot\sup_{t\in[t_1,t_2]}\|\mathbf{v}(t)\|^6_{\mathbb{H}^0} + C\epsilon\Big).$$

Firstly, letting $t_1 = 0$ and $t_2 = T_0$ and by (5.19), we find

$$\|\mathbf{v}(T_0)\|^2_{\mathbb{H}^1} \leqslant r_1^2 + \frac{1}{C_0}\Big(C_N\cdot\sup_{t\in[0,T_0]}\|\mathbf{v}(t)\|^2_{\mathbb{H}^0} + C\epsilon\cdot\sup_{t\in[0,T_0]}\|\mathbf{v}(t)\|^6_{\mathbb{H}^0} + C\epsilon\Big) \leqslant C_{N,C_0}(r_1^6+1).$$

Secondly, letting $t_1 = T_0$ and $t_2 = T_0 + h$ yields

$$\|\mathbf{v}(T_0+h)\|_{\mathbb{H}^1}^2 \leqslant e^{-C_0 h} C_{N,C_0}(r_1^6+1) + \frac{1}{C_0}\Big(C_N \cdot \sup_{t\in[T_0,T_0+h]} \|\mathbf{v}(t)\|_{\mathbb{H}^0}^2 + C\epsilon \cdot \sup_{t\in[T_0,T_0+h]} \|\mathbf{v}(t)\|_{\mathbb{H}^0}^6 + C\epsilon\Big),$$

which together with (5.20) implies that for some T large enough and $\epsilon > 0$ small enough

$$\|\mathbf{v}(T)\|_{\mathbb{H}^1} \leqslant r_2/2.$$

Therefore, there exist T sufficiently large and ϵ small enough such that for any $\|\mathbf{u}_0\|_{\mathbb{H}^1} \leqslant r_1$

$$\|\mathbf{u}(T,\omega;\mathbf{u}_0)\|_{\mathbb{H}^1} \leqslant r_2.$$

That is, if we set

$$\Omega_\epsilon := \left\{\omega : \sup_{t\in[0,T]} \|\mathbf{w}(t,\omega)\|_{\mathbb{H}^6} < \epsilon\right\},$$

then

$$\Omega_\epsilon \subset \cap_{\|\mathbf{u}_0\|_{\mathbb{H}^1}\leqslant r_1}\{\omega : \|\mathbf{u}(T,\omega;\mathbf{u}_0)\|_{\mathbb{H}^1} \leqslant r_2\}.$$

The desired estimate now follows from the fact that Ω_ϵ is an open subset of Ω and $P(\Omega_\epsilon) > 0$. □

Proof of Proposition 5.6: For $r > 0$, let $\mathbb{B}_r := \{\mathbf{u}_0 \in \mathbb{H}^1 : \|\mathbf{u}_0\|_{\mathbb{H}^1} \leqslant r\}$ be the ball in $\mathbb{H}^1$. For each invariant measure μ, we can choose some $r_1 > 0$ such that

$$\mu(\mathbb{B}_{r_1}) \geqslant 1/2.$$

By Lemma 5.7, we further have for any $r_2 > 0$ and some $t > 0$

$$\mu(\mathbb{B}_{r_2}) = \int_{\mathbb{H}^1} (\mathbf{T}_t 1_{\mathbb{B}_{r_2}})(\mathbf{u}_0)\mu(\mathrm{d}\mathbf{u}_0) \geqslant \int_{\mathbb{B}_{r_1}} (\mathbf{T}_t 1_{\mathbb{B}_{r_2}})(\mathbf{u}_0)\mu(\mathrm{d}\mathbf{u}_0) \geqslant \mu(\mathbb{B}_{r_1}) \cdot \inf_{\mathbf{u}_0\in\mathbb{B}_{r_1}} (\mathbf{T}_t 1_{\mathbb{B}_{r_2}})(\mathbf{u}_0) > 0,$$

which means that 0 belongs to the support of μ.

Proof of Theorem 5.1: The assertion follows from Propositions 5.5 and 5.6 due to [15, Proposition 3.12, Corollary 3.17].

6. Appendix

6.1. **Proof of Proposition 3.4.**

In this subsection, we prove the martingale characterization of weak solutions.

First of all, (i)⟹(ii) is direct by Itô's formula. Let us prove (ii)⟹(i). Define for $\mathbf{e} \in \mathscr{E}$ (see Subsection 2.3 for the notation $\mathscr{E}$)

$$M_{\mathbf{e}}(t,\mathbf{u}) := \langle \mathbf{u}(t) - \mathbf{u}(0), \mathbf{e}\rangle_{\mathbb{H}^1} - \int_0^t [\![A(\mathbf{u}(s)), \mathbf{e}]\!]\mathrm{d}s - \int_0^t \langle \mathbf{f}(s,\mathbf{u}(s)), \mathbf{e}\rangle_{\mathbb{H}^1}\mathrm{d}s.$$

Using (ii) and by simple approximations as in [31], one knows that $\{M_{\mathbf{e}}(t,\mathbf{u}), t \geqslant 0\}$ is a continuous local martingale under P_ϑ with respect to $\mathcal{B}_t(\mathbb{X})$, and its quadratic variation process is given by

$$[M_{\mathbf{e}}](t,\mathbf{u}) = \int_0^t \|\langle B(s,\mathbf{u}(s)), \mathbf{e}\rangle_{\mathbb{H}^1}\|_{l^2}^2 \mathrm{d}s.$$

Set

$$M(t,\mathbf{u}) := \sum_{j=1}^\infty M_{\mathbf{e}_j}(t,\mathbf{u})\mathbf{e}_j. \tag{6.1}$$

Then $t \mapsto M(t, \mathbf{u})$ is an $\mathbb{H}^1$-valued continuous local martingale under P_ϑ with respect to $\mathcal{B}_t(\mathbb{X})$. Indeed, for any $R > 0$, define the stopping time

$$\tau_R(\mathbf{u}) := \inf\left\{ t \geqslant 0 : \int_0^t \|B(s, \mathbf{u}(s))\|^2_{L_2(l^2;\mathbb{H}^1)} \mathrm{d}s \geqslant R \right\}.$$

Then by (3.1) and (2.15) we have

$$\tau_R(\mathbf{u}) \uparrow \infty, \quad P_\vartheta - a.a.\ \mathbf{u}, \quad \text{as } R \to \infty.$$

Set

$$M^{R,n}(t, \mathbf{u}) := \sum_{j=1}^n M_{\mathbf{e}_j}(t \wedge \tau_R, \mathbf{u}) \mathbf{e}_j.$$

It is clear that $M^{R,n}(t, \mathbf{u})$ is an $\mathbb{H}^1$-valued continuous martingale with

$$\begin{aligned} \ll M^{R,n} \gg_{\mathbb{H}^1} (t, \mathbf{u}) &= \sum_{i,j=1}^n [M_{\mathbf{e}_i}, M_{\mathbf{e}_j}](t \wedge \tau_R, \mathbf{u}) \cdot \mathbf{e}_i \otimes \mathbf{e}_j \\ &= \sum_{i,j=1}^n \int_0^{t\wedge\tau_R} \langle \langle B(s, \mathbf{u}(s)), \mathbf{e}_j \rangle_{\mathbb{H}^1}, \langle B(s, \mathbf{u}(s)), \mathbf{e}_j \rangle_{\mathbb{H}^1} \rangle_{l^2} \cdot \mathbf{e}_i \otimes \mathbf{e}_j \mathrm{d}s, \end{aligned}$$

where $\ll \cdot \gg_{\mathbb{H}^1}$ denotes the square variation of M^R in $\mathbb{H}^1$. Moreover, by Burkholder's inequality we have, for any $T > 0$

$$\mathbb{E}^{P_\vartheta}\left(\sup_{t\in[0,T]} \|M^{R,n}(t, \mathbf{u}) - M^{R,m}(t, \mathbf{u})\|^2_{\mathbb{H}^1} \right) \leqslant C \sum_{j=n}^m \mathbb{E}^{P_\vartheta}\left(\int_0^{T\wedge\tau_R} \|\langle B(s, \mathbf{u}(s)), \mathbf{e}_j \rangle_{\mathbb{H}^1}\|^2_{l^2} \mathrm{d}s \right) \to 0$$

as $n, m \to \infty$. Hence, the series in (6.1) converges in $C([0, T]; \mathbb{H}^1)$, P_ϑ-a.s., and $M^R(t, \mathbf{u}) := M(t \wedge \tau_R, \mathbf{u})$ is an $\mathbb{H}^1$-valued continuous square integrable martingale with

$$\begin{aligned} \ll M^R \gg_{\mathbb{H}^1} (t, \mathbf{u}) &= \sum_{i,j=1}^\infty [M_{\mathbf{e}_i}, M_{\mathbf{e}_j}](t \wedge \tau_R, \mathbf{u}) \cdot \mathbf{e}_i \otimes \mathbf{e}_j \\ &= \sum_{i,j=1}^\infty \int_0^{t\wedge\tau_R} \langle \langle B(s, \mathbf{u}(s)), \mathbf{e}_j \rangle_{\mathbb{H}^1}, \langle B(s, \mathbf{u}(s)), \mathbf{e}_j \rangle_{\mathbb{H}^1} \rangle_{l^2} \cdot \mathbf{e}_i \otimes \mathbf{e}_j \mathrm{d}s. \end{aligned}$$

Letting $R \to \infty$ we obtain the desired property of $M(t, \mathbf{u})$.

In particular, the following equality holds in $\mathbb{H}^0$

$$\mathbf{u}(t) = \mathbf{u}(0) + \int_0^t A(\mathbf{u}(s)) \mathrm{d}s + \int_0^t \mathbf{f}(s, \mathbf{u}(s)) \mathrm{d}s + M(t, \mathbf{u}), \quad P_\vartheta - a.s..$$

By Itô's formula (cf. [30, 26]), we obtain that $P_\vartheta(C([0, \infty), \mathbb{H}^1)) = 1$. The existence of weak solutions now follows from the representation theorem for martingales (cf. [23, Lemma 3.2] or [3, Theorem 8.2]).

6.2. **Two Basic Estimates.** In this subsection, we prove two basic estimates used in Section 5.

Lemma 6.1. *Let* $\{\varphi_\epsilon(\cdot, r_0), \epsilon \in (0, 1), r_0 \geqslant 0\}$ *be a family of positive real functions on* $\mathbb{R}_+$ *with* $\varphi_\epsilon(0, r_0) = r_0$. *Suppose that for some* $p > 1$, $C_0, C_1, C_2 > 0$, $C_3 \geqslant 0$ *and any* $\epsilon \in (0, 1)$ *and* $t \geqslant 0$

$$\varphi'_\epsilon(t, r_0) \leqslant -C_0 \varphi_\epsilon(t, r_0) + C_1 \epsilon \cdot \varphi_\epsilon(t, r_0)^p + C_2 \epsilon + C_3.$$

Then: (i) For any $T > 0$ *and* $R > 0$, *there exists* $\epsilon_0 > 0$ *such that*

$$\sup_{t\in[0,T],\epsilon\in[0,\epsilon_0],r_0\in[0,R]} \varphi_\epsilon(t, r_0) \leqslant 2R + 2C_3/C_0.$$

(ii) If $C_3 = 0$, then for any $\delta > 0$ and $R, h > 0$, there exist $T > 0$ and $\epsilon_0 > 0$ such that

$$\sup_{t\in[T,T+h],\epsilon\in[0,\epsilon_0],r_0\in[0,R]} \varphi_\epsilon(t, r_0) \leqslant \delta.$$

Proof. Let $C_4^\epsilon := (C_2\epsilon + C_3)/C_0$ and set

$$\phi(t) := e^{C_0 t}(\varphi_\epsilon(t, r_0) - C_4^\epsilon).$$

Then for fixed $T > 0$ and any $t \in [0, T]$

$$\phi'(t) \leqslant C_1 e^{C_0 t}\epsilon \cdot \varphi_\epsilon(t, r_0)^p \leqslant C_1\epsilon \cdot (\phi(t) + C_4^\epsilon \cdot e^{C_0 T})^p.$$

Solving this differential inequality gives that

$$\phi(T) \leqslant \Big[(\phi(0) + C_4^\epsilon \cdot e^{C_0 T})^{1-p} + C_1(1-p)\epsilon T\Big]^{\frac{1}{1-p}} - C_4^\epsilon \cdot e^{C_0 T}.$$

Hence,

$$\begin{aligned}\varphi_\epsilon(T, r_0) &\leqslant e^{-C_0 T}\Big[(r_0 + C_4^\epsilon \cdot (e^{C_0 T} - 1))^{1-p} + C_1(1-p)\epsilon T\Big]^{\frac{1}{1-p}}\\ &\leqslant \Big[(e^{-C_0 T}R + C_4^\epsilon)^{1-p} + C_1(1-p)\epsilon T e^{(p-1)C_0 T}\Big]^{\frac{1}{1-p}}.\end{aligned}$$

Now the assertions easily follow by suitable choices of ϵ and T. □

We now prove the following exponential estimate.

Lemma 6.2. *Let X_t be a positive Itô process of the form*

$$X_t = x_0 + \int_0^t M_s \mathrm{d}W_s + \int_0^t N_s \mathrm{d}s, \tag{6.2}$$

where $s \mapsto M_s, N_s$ are two measurable adapted processes. Suppose that there exist a positive process Y_s and some $\alpha > 1$ and $C_0, C_1, C_2, C_3 > 0$ such that for any $s \geqslant 0$

$$N_s \leqslant -C_0 X_s^\alpha - Y_s + C_1, \quad |M_s|^2 \leqslant C_2 X_s + C_3. \tag{6.3}$$

Then for any $t, \eta > 0$

$$\mathbb{E}e^{\eta X_t} \leqslant C_{\alpha,\eta} \cdot \exp\{e^{-C_0 t/2}\eta x_0\} \tag{6.4}$$

and

$$\mathbb{E}\exp\left\{\eta \int_0^t (\frac{C_0}{2}X_s^\alpha + Y_s)\mathrm{d}s\right\} \leqslant \exp\{\eta x_0 + C_{\alpha,\eta} t\}. \tag{6.5}$$

Proof. Let us first prove that for any $t, \eta > 0$

$$\mathbb{E}e^{\eta X_t} < +\infty. \tag{6.6}$$

Set for $R > 0$

$$\tau_R := \inf\{t \geqslant 0 : |X_t| \geqslant R\}.$$

By Itô's formula, (6.3) and Young's inequality, we have

$$\begin{aligned}\mathrm{d}e^{\eta X_t} &= \eta e^{\eta X_t} M_t \mathrm{d}W_t + \eta e^{\eta X_t} N_t \mathrm{d}t + \frac{\eta^2}{2} e^{\eta X_t}|M_t|^2 \mathrm{d}t\\ &\leqslant \eta e^{\eta X_t} M_t \mathrm{d}W_t + \eta e^{\eta X_t}(-C_0 X_t^\alpha + C_1 + \frac{\eta}{2}(C_2 X_t + C_3))\mathrm{d}t\\ &\leqslant \eta e^{\eta X_t} M_t \mathrm{d}W_t + \eta e^{\eta X_t}(-\frac{C_0}{2}X_t^\alpha + C_{\alpha,\eta})\mathrm{d}t\\ &\leqslant \eta e^{\eta X_t} M_t \mathrm{d}W_t + \eta e^{\eta X_t}(-\frac{C_0}{2}X_t + C_{\alpha,\eta})\mathrm{d}t.\end{aligned}$$

Set

$$f_R(t) := \mathbb{E}e^{\eta X_{t\wedge\tau_R}}.$$

Then

$$f_R'(t) \leqslant C_{\alpha,\eta} f_R(t).$$

Hence

$$f_R(t) = \mathbb{E}e^{\eta X_{t\wedge\tau_R}} \leqslant e^{\eta x_0} e^{C_{\alpha,\eta} t}.$$

By Fatou's lemma, we obtain (6.6).

We now set

$$f(t) := \mathbb{E}e^{\eta X_t}.$$

Then by Jensen's inequality, we obtain

$$f'(t) \leqslant -\frac{C_0}{2} f(t)\log f(t) + C_{\alpha,\eta} f(t).$$

Solving this differential equality gives the first estimate (6.4).

On the other hand, for any $t, \eta > 0$, we have by (6.2) and (6.3)

$$\eta \int_0^t (\frac{C_0}{2} X_s^\alpha + Y_s)\mathrm{d}s \leqslant \eta x_0 + \eta \int_0^t M_s \mathrm{d}W_s + \int_0^t (-\frac{\eta C_0}{2} X_s^\alpha + C_1\eta)\mathrm{d}s.$$

Noting that by (6.3) and (6.4)

$$\mathbb{E}\exp\left\{\frac{\eta^2}{2}\int_0^t |M_s|^2 \mathrm{d}s\right\} \leqslant \frac{1}{t}\int_0^t \mathbb{E}\exp\{t\eta^2|M_s|^2/2\}\mathrm{d}s < +\infty,$$

we know by Novikov's criterion that

$$t \mapsto \exp\left\{\eta\int_0^t M_s \mathrm{d}W_s - \frac{\eta^2}{2}\int_0^t |M_s|^2\mathrm{d}s\right\} =: \mathcal{E}(M)(t)$$

is an exponential martingale. Moreover, by (6.3) and Young's inequality

$$\frac{\eta^2}{2}|M_s|^2 - \frac{\eta C_0}{2} X_s^\alpha + C_1\eta \leqslant C_{\alpha,\eta}.$$

Therefore,

$$\mathbb{E}\exp\left\{\eta\int_0^t (\frac{C_0}{2} X_s^\alpha + Y_s)\mathrm{d}s\right\} \leqslant e^{\eta x_0}\mathbb{E}\left(\mathcal{E}(M)(t)\cdot\exp\{C_{\alpha,\eta} t\}\right) = e^{\eta x_0}\cdot\exp\{C_{\alpha,\eta} t\}.$$

The proof is thus complete. □

6.3. **Proof of the Derivative Flow Equation.** In this subsection, we prove (5.3). Note that (5.5) can be proved similarly.

Lemma 6.3. *For any $T > 0$, there exists a constant $C_{N,T} > 0$ such that for each ω and $\mathbf{u}_0 \in \mathbb{H}^1$*

$$\sup_{t\in[0,T]} \|\mathbf{u}(t,\omega)\|_{\mathbb{H}^1}^2 + \int_0^T \|\mathbf{u}(t,\omega)\|_{\mathbb{H}^2}^2 \leqslant C_{N,T}\Big(1 + \|\mathbf{u}_0\|_{\mathbb{H}^1}^6 + \sup_{t\in[0,T]}\|\mathbf{w}(t,\omega)\|_{\mathbb{H}^5}^{12}\Big).$$

Proof. Following the proof of Lemma 5.7, let us give different estimates for J_i, $i = 1, 2, 3, 4$.

For J_1, by (5.18) we have

$$J_1 \leqslant -2\|\nabla\mathbf{v}(t)\|_{\mathbb{H}^0}^2 + 2\|\Delta\mathbf{w}(t)\|_{\mathbb{H}^0}\cdot\|\mathbf{v}(t)\|_{\mathbb{H}^0}.$$

For J_2, by the Sobolev inequality (2.1) and Young's inequality we have

$$\begin{aligned} J_2 &= -2\langle\mathbf{w}(t), ((\mathbf{v}(t)+\mathbf{w}(t))\cdot\nabla)(\mathbf{v}(t)+\mathbf{w}(t))\rangle_{\mathbb{H}^0} \\ &= 2\langle\nabla\mathbf{w}(t), (\mathbf{v}(t)+\mathbf{w}(t))\otimes(\mathbf{v}(t)+\mathbf{w}(t))\rangle_{\mathbb{H}^0} \end{aligned}$$

$$\leqslant 2\|\nabla \mathbf{w}(t)\|_{L^\infty}\|\mathbf{v}(t)+\mathbf{w}(t)\|^2_{\mathbb{H}^0}$$
$$\leqslant C\|\nabla \mathbf{w}(t)\|_{L^\infty}\|\mathbf{v}(t)+\mathbf{w}(t)\|^2_{L^4}$$
$$\leqslant \|\mathbf{v}(t)+\mathbf{w}(t)\|^4_{L^4}+C\|\nabla \mathbf{w}(t)\|^2_{L^\infty}.$$

For J_3, we have

$$J_3 \leqslant -2\|\mathbf{v}(t)+\mathbf{w}(t)\|^4_{L^4}+N\|\mathbf{v}(t)+\mathbf{w}(t)\|^2_{\mathbb{H}^0}.$$

For J_4, by (2.1) and Young's inequality we have

$$\begin{aligned} J_3 &\leqslant 2\|\mathbf{w}(t)\|_{L^\infty}\cdot\|\mathbf{v}(t)+\mathbf{w}(t)\|^3_{L^3} \\ &\leqslant 2\|\mathbf{w}(t)\|_{L^\infty}\cdot\|\mathbf{v}(t)+\mathbf{w}(t)\|^3_{L^4} \\ &\leqslant -\|\mathbf{v}(t)+\mathbf{w}(t)\|^4_{L^4}+C\|\mathbf{w}(t)\|^4_{L^\infty}. \end{aligned}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}(t)\|^2_{\mathbb{H}^0} \leqslant C_N\|\mathbf{v}(t)\|^2_{\mathbb{H}^0}+C(\|\mathbf{w}(t)\|^4_{\mathbb{H}^2}+\|\mathbf{w}(t)\|^2_{\mathbb{H}^3}).$$

By Gronwall's inequality, we get

$$\sup_{t\in[0,T]}\|\mathbf{v}(t)\|^2_{\mathbb{H}^0} \leqslant C_{N,T}\Big(\|\mathbf{v}_0\|^2_{\mathbb{H}^0}+\sup_{t\in[0,T]}(\|\mathbf{w}(t)\|^4_{\mathbb{H}^2}+\|\mathbf{w}(t)\|^2_{\mathbb{H}^3})\Big). \tag{6.7}$$

Using the similar calculations as in the proof of Lemma 5.7, one finds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}(t)\|^2_{\mathbb{H}^1} \leqslant -\frac{1}{8}\|\mathbf{v}(t)\|^2_{\mathbb{H}^2}+C_N(1+\|\mathbf{w}(t)\|^4_{\mathbb{H}^5})\cdot(1+\|\mathbf{v}(t)\|^6_{\mathbb{H}^0}),$$

which together with (6.7) gives the desired estimate. □

For $\mathbf{v}_0 \in \mathbb{H}^1$, let us consider a small perturbation of the initial values given by $\mathbf{u}_\epsilon(0)=\mathbf{u}_0+\epsilon\mathbf{v}_0$. The corresponding solution of Eq. (5.2) is denoted by $\mathbf{u}_\epsilon(t)$.

Set

$$\mathbf{v}_\epsilon(t) := (\mathbf{u}_\epsilon(t)-\mathbf{u}(t))/\epsilon.$$

Then $\mathbf{v}_\epsilon(t)$ satisfies

$$\begin{aligned} \mathbf{v}'_\epsilon(t) = \Delta\mathbf{v}_\epsilon(t) &- \mathscr{P}[(\mathbf{u}_\epsilon(t)\cdot\nabla)\mathbf{v}_\epsilon(t)] - \mathscr{P}[(\mathbf{v}_\epsilon(t)\cdot\nabla)\mathbf{u}(t)] \\ &- \mathscr{P}[g_N(|\mathbf{u}_\epsilon(t)|^2)\mathbf{v}_\epsilon(t)] - \mathscr{P}[(g_N(|\mathbf{u}_\epsilon(t)|^2)-g_N(|\mathbf{u}(t)|^2))/\epsilon\cdot\mathbf{u}(t)], \end{aligned}$$

with initial value $\mathbf{v}_\epsilon(0)=\mathbf{v}_0$.

We have:

Lemma 6.4. *For any $T>0$, there is a constant $C_{N,T}>0$ such that for any $\epsilon\in(0,1)$*

$$\sup_{t\in[0,T]}\|\mathbf{v}_\epsilon(t)\|^2_{\mathbb{H}^1}+\int_0^T\|\mathbf{v}_\epsilon(t)\|^2_{\mathbb{H}^2}\mathrm{d}t \leqslant C_{T,N}.$$

Proof. As in the proof of Lemma 4.1, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}_\epsilon(t)\|^2_{\mathbb{H}^1} &\leqslant -\|\mathbf{v}_\epsilon(t)\|^2_{\mathbb{H}^2}+2\|\mathbf{v}_\epsilon(t)\|^2_{\mathbb{H}^1}+C\|\mathbf{u}_\epsilon(t)\|^2_{L^6}\cdot\|\nabla\mathbf{v}_\epsilon(t)\|^2_{L^3} \\ &\quad + C\|\mathbf{v}_\epsilon(t)\|^2_{L^\infty}\cdot\|\mathbf{u}(t)\|^2_{\mathbb{H}^1}+C\|\mathbf{v}_\epsilon(t)\|^2_{L^6}\cdot(\|\mathbf{u}_\epsilon(t)\|^4_{L^6}+\|\mathbf{u}(t)\|^4_{L^6}) \\ &\leqslant -\frac{1}{2}\|\mathbf{v}_\epsilon(t)\|^2_{\mathbb{H}^2}+C(\|\mathbf{u}_\epsilon(t)\|^4_{\mathbb{H}^1}+\|\mathbf{u}(t)\|^4_{\mathbb{H}^1}+1)\cdot\|\mathbf{v}_\epsilon(t)\|^2_{\mathbb{H}^1}, \end{aligned}$$

which together with Lemma 6.3 gives the desired estimate. □

We are now in a position to prove (5.4). Set

$$\mathbf{j}_\epsilon(t) := \mathbf{v}_\epsilon(t) - \mathcal{J}_{0,t}\mathbf{v}_0,$$

where $\mathcal{J}_{0,t}\mathbf{v}_0$ satisfies (5.3).

By Taylor's formula, we have

$$\begin{aligned} g_N(|\mathbf{u}_\epsilon(t)|^2) - g_N(|\mathbf{u}(t)|^2) &= g'_N(|\mathbf{u}(t)|^2)(|\mathbf{u}_\epsilon(t)|^2 - |\mathbf{u}(t)|^2) + g''_N(\theta)(|\mathbf{u}_\epsilon(t)|^2 - |\mathbf{u}(t)|^2)^2/2 \\ &= \epsilon^2 \cdot g'_N(|\mathbf{u}(t)|^2)|\mathbf{v}_\epsilon(t)|^2 + 2g'_N(|\mathbf{u}(t)|^2)\langle \mathbf{v}_\epsilon(t), \mathbf{u}(t)\rangle_{\mathbb{R}^3} \\ &\quad + g''_N(\theta)(|\mathbf{u}_\epsilon(t)|^2 - |\mathbf{u}(t)|^2)^2/2, \end{aligned}$$

where θ takes some value between $|\mathbf{u}(t)|^2$ and $|\mathbf{u}_\epsilon(t)|^2$.

Thus, it is not hard to see that $\mathbf{j}_\epsilon(t)$ satisfies

$$\mathbf{j}'_\epsilon(t) = \Delta\mathbf{j}_\epsilon(t) - \sum_{i=1}^{8} J_i(t),$$

where

$$\begin{aligned} J_1(t) &:= \epsilon \cdot \mathscr{P}[(\mathbf{v}_\epsilon(t) \cdot \nabla)\mathbf{v}_\epsilon(t)], \\ J_2(t) &:= \mathscr{P}[(\mathbf{u}(t) \cdot \nabla)\mathbf{j}_\epsilon(t)], \\ J_3(t) &:= \mathscr{P}[(\mathbf{j}_\epsilon(t) \cdot \nabla)\mathbf{u}(t)], \\ J_4(t) &:= \mathscr{P}[(g_N(|\mathbf{u}_\epsilon(t)|^2) - g_N(|\mathbf{u}(t)|^2)) \cdot \mathbf{v}_\epsilon(t)], \\ J_5(t) &:= \mathscr{P}[g_N(|\mathbf{u}(t)|^2) \cdot \mathbf{j}_\epsilon(t)], \\ J_6(t) &:= \epsilon \cdot \mathscr{P}[g'_N(|\mathbf{u}(t)|^2)|\mathbf{v}_\epsilon(t)|^2 \cdot \mathbf{u}(t)], \\ J_7(t) &:= 2\mathscr{P}[g'_N(|\mathbf{u}(t)|^2)\langle \mathbf{j}_\epsilon(t), \mathbf{u}(t)\rangle_{\mathbb{R}^3} \cdot \mathbf{u}(t)], \\ J_8(t) &:= \mathscr{P}[g''_N(\theta)(|\mathbf{u}_\epsilon(t)|^2 - |\mathbf{u}(t)|^2)^2 \cdot \mathbf{u}(t)/\epsilon]. \end{aligned}$$

By the chain rule and Young's inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{j}_\epsilon(t)\|^2_{\mathbb{H}^1} \leqslant -\|\mathbf{j}_\epsilon(t)\|^2_{\mathbb{H}^2} + 2\|\mathbf{j}_\epsilon(t)\|^2_{\mathbb{H}^0} + C\sum_{i=1}^{8} \|J_i(t)\|^2_{\mathbb{H}^0}.$$

Here and below, the constant C is independent of ϵ.

For $J_1(t)$, we have

$$\|J_1(t)\|^2_{\mathbb{H}^0} \leqslant C\epsilon^2 \cdot \|\mathbf{v}_\epsilon(t)\|^2_{\mathbb{H}^1} \cdot \|\mathbf{v}_\epsilon(t)\|^2_{\mathbb{H}^2}.$$

For $J_2(t)$, we have

$$\|J_2(t)\|^2_{\mathbb{H}^0} \leqslant C\|\mathbf{u}(t)\|^2_{\mathbb{H}^2} \cdot \|\mathbf{j}_\epsilon(t)\|^2_{\mathbb{H}^1}.$$

For $J_3(t)$, we have

$$\begin{aligned} \|J_3(t)\|^2_{\mathbb{H}^0} &\leqslant \|\mathbf{u}(t)\|^2_{\mathbb{H}^1} \cdot \|\mathbf{j}_\epsilon(t)\|^2_{L^\infty} \leqslant C\|\mathbf{u}(t)\|^2_{\mathbb{H}^1} \cdot \|\mathbf{j}_\epsilon(t)\|_{\mathbb{H}^1} \cdot \|\mathbf{j}_\epsilon(t)\|_{\mathbb{H}^2} \leqslant \\ &\leqslant C\|\mathbf{u}(t)\|^4_{\mathbb{H}^1} \cdot \|\mathbf{j}_\epsilon(t)\|^2_{\mathbb{H}^1} + \frac{1}{4}\|\mathbf{j}_\epsilon(t)\|^2_{\mathbb{H}^2}. \end{aligned}$$

For $J_4(t)$, we have

$$\begin{aligned} \|J_4(t)\|^2_{\mathbb{H}^0} &\leqslant C\epsilon^2 \cdot \|\mathbf{v}_\epsilon(t)\|^4_{L^6} \cdot (\|\mathbf{u}_\epsilon(t)\|^2_{L^6} + \|\mathbf{u}(t)\|^2_{L^6}) \\ &\leqslant C\epsilon^2 \cdot \|\mathbf{v}_\epsilon(t)\|^4_{\mathbb{H}^1} \cdot (\|\mathbf{u}_\epsilon(t)\|^2_{\mathbb{H}^1} + \|\mathbf{u}(t)\|^2_{\mathbb{H}^1}). \end{aligned}$$

For $J_5(t)$, we have

$$\|J_5(t)\|^2_{\mathbb{H}^0} \leqslant C\|\mathbf{u}(t)\|^4_{L^6} \cdot \|\mathbf{j}_\epsilon(t)\|^2_{L^6} \leqslant C\|\mathbf{u}(t)\|^4_{\mathbb{H}^1} \cdot \|\mathbf{j}_\epsilon(t)\|^2_{\mathbb{H}^1}.$$

For $J_6(t)$, we have

$$\|J_6(t)\|^2_{\mathbb{H}^0} \leqslant C\epsilon^2 \cdot \|\mathbf{v}_\epsilon(t)\|^2_{\mathbb{H}^1} \cdot \|\mathbf{u}(t)\|^2_{\mathbb{H}^1}.$$

For $J_7(t)$, we have

$$\|J_7(t)\|^2_{\mathbb{H}^0} \leqslant C \cdot \|\mathbf{j}_\epsilon(t)\|^2_{\mathbb{H}^1} \cdot \|\mathbf{u}(t)\|^4_{\mathbb{H}^1}.$$

For $J_8(t)$, we have

$$\begin{aligned}\|J_8(t)\|^2_{\mathbb{H}^0} &\leqslant C\epsilon^2 \cdot \||\mathbf{v}_\epsilon(t)|^2 \cdot (|\mathbf{u}_\epsilon(t)|^2 + |\mathbf{u}(t)|^2)\|^2_{\mathbb{H}^0}\\ &\leqslant C\epsilon^2 \cdot \|\mathbf{v}_\epsilon(t)\|^4_{L^4} \cdot (\|\mathbf{u}_\epsilon(t)\|^4_{L^4} + \|\mathbf{u}(t)\|^4_{L^4})\\ &\leqslant C\epsilon^2 \cdot \|\mathbf{v}_\epsilon(t)\|^4_{\mathbb{H}^1} \cdot (\|\mathbf{u}_\epsilon(t)\|^4_{\mathbb{H}^1} + \|\mathbf{u}(t)\|^4_{\mathbb{H}^1}).\end{aligned}$$

Combining the above calculations and Lemmas 6.3 and 6.4 yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{j}_\epsilon(t)\|^2_{\mathbb{H}^1} \leqslant C\epsilon^2(1 + \|\mathbf{v}_\epsilon(t)\|^2_{\mathbb{H}^2}) + C(1 + \|\mathbf{u}(t)\|^2_{\mathbb{H}^2}) \cdot \|\mathbf{j}_\epsilon(t)\|^2_{\mathbb{H}^1}.$$

By Gronwall's inequality, we get

$$\|\mathbf{j}_\epsilon(t)\|^2_{\mathbb{H}^1} \leqslant C\epsilon^2\left(1 + \int_0^t \|\mathbf{v}_\epsilon(s)\|^2_{\mathbb{H}^2}\mathrm{d}s\right) \cdot \exp\left\{C + C\int_0^t \|\mathbf{u}(s)\|^2_{\mathbb{H}^2}\mathrm{d}s\right\},$$

which together with Lemmas 6.3 and 6.4 clearly gives (5.4).

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