ON STOCHASTIC EVOLUTION EQUATIONS WITH NON-LIPSCHITZ COEFFICIENTS

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Abstract. In this paper, we study the existence and uniqueness of solutions for several classes of stochastic evolution equations with non-Lipschitz coefficients, that contains backward stochastic evolution equations, stochastic Volterra type evolution equations and stochastic functional evolution equations. In particular, the results can be used to treat a large class of quasi-linear stochastic equations, which includes the reaction diffusion and porous medium equations.

Contents

1. Introduction

Let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}^d$. Consider the following stochastic porous medium equation with Dirichlet boundary condition:

$$\left\{\begin{array}{l} \mathrm{d}u_t = |w_t| \cdot \Delta(|u_t|^{p-2}u_t)\mathrm{d}t + \mathrm{d}w_t, \\ u_t(x) = 0, \quad x \in \partial\mathcal{O}, t > 0 \\ u_0 = \phi \in L^p(\mathcal{O}), \end{array}\right. \tag{1}$$

where $p \geqslant 2$, Δ is the usual Laplace operator, and $\{w_t, t \geqslant 0\}$ is a one dimensional standard Brownian motion. This is a degenerate non-linear stochastic partial differential equation. Notice that the degeneracy may be caused by $w_t = 0$ and $u_t = 0$. In the deterministic case, it is well known that porous medium equations can be written as abstract monotone operator equations(cf. [36] [31]). Thus, in the stochastic case, it can

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fall into a class of stochastic evolution equations studied by Krylov-Rozovskii [18]. More discussions about the stochastic porous medium equation are referred to [8] [27] [24].

On the other hand, let us consider the following stochastic reaction diffusion equation:

$$\left\{\begin{array}{l} \mathrm{d}u_t = |w_t| \cdot (\Delta u_t - |u_t|^{p-2} \cdot u_t)\mathrm{d}t + w_t \cdot u_t \mathrm{d}w_t, \\ u_t(x) = 0, \quad x \in \partial\mathcal{O}, t > 0 \\ u_0 = \phi \in L^2(\mathcal{O}), \end{array}\right. \tag{2}$$

where $p \geqslant 2$. Usually, one wants to find an adapted process u such that for almost all w

$$u_\cdot(w) \in L^2([0,T], H_0^1(\mathcal{O})) \cap L^p([0,T] \times \mathcal{O}) \cap C([0,T], L^2(\mathcal{O})),$$

and (2) holds in the generalized sense, where $H_0^1(\mathcal{O})$ is the usual Sobolev space.

However, from the well known results, it seems that one cannot solve Eq.(1) and Eq.(2) because of the presence of $|w_t|$ in front of the Laplace operator. One of the main purposes in this paper is to extend the well known results in [18] [11] so that we can solve Eq.(1) and Eq.(2) in the generalized sense for almost all path w_t.

In the present paper, we shall work on the framework of evolution triple. This is crucial for treating a wide class of quasi-linear stochastic partial differential equations(including reaction diffusion equations and porous medium equations). We now recall some well-known results in this direction. In [21] [22], Pardoux considered linear stochastic partial differential equations(SPDEs) using the monotonicity method. In [18], basing on their established Itô's formula, Krylov and Rozovskii proved a more general result under some monotonicity or dissipative conditions. This classic work was later extended in several aspects: to stochastic evolution equations(SEEs) driven by general (discontinuous) martingales in [10], to SEEs with coercivity constants depending on t in [11], to SEEs related to some Orlicz spaces in [27]. All these works are based on Galerkin's approximation. It should be remarked that the semigroup method is another main tool in the theory of semi-linear SPDEs (cf. [9] [6] [7] [17] [13] [37] [38], etc.). In order to solve Eq.(1), we need to deal with SEEs with random coercivity coefficients. This is our first goal, and will be done in Section 3 after some preliminaries of Section 2. Here, some stopping time techniques will be used.

The second aim is to prove the existence and uniqueness of solutions to backward stochastic evolution equations. Since Pardoux and Peng in [23] proved the existence and uniqueness of solutions to nonlinear backward stochastic differential equations(BSDEs), the theory of BSDEs has already been developed extensively. It is well known that BSDEs can be applied to the studies of stochastic controls, mathematics finances, deterministic PDEs, etc.. Meanwhile, backward SPDEs have also been studied in [14] [26] etc.. In these works, the authors mainly concentrated on semilinear BSPDEs. The second aim in this paper is to prove the existence and uniqueness of solutions to BSEEs with non-Lipschitz coefficients in the framework of evolution triple. Thus, it can be used to deal with a large class of quasi linear BSPDE. We remark that Mao in [19] has already studied the BSDEs with non-Lipschitz coefficients, and the authors in [1] also investigated the BSDEs with monotone and arbitrary growth coefficients. This is the content of Section 4.

The third aim is to study the stochastic functional integral evolution equations with non-Lipschitz coefficients, which in particular includes a class of stochastic Volterra type evolution equations. Stochastic Volterra equations driven by Brownian motion were first studied by Berger-Mizel [3]. Later, Protter [25] proved the existence and uniqueness of stochastic Volterra equations driven by general semimartingales. Recently, Wang in [34] studied the the existence and uniqueness of stochastic Volterra equations with singular kernels and non-Lipschitz coefficients. About the stochastic functional differential equations, Mohammend's book [20] is one of the main references. In [32], using the evolution semigroup approach, the authors studied the existence, uniqueness and asymptotic behavior of mild solutions to stochastic semilinear functional differential equations in Hilbert spaces. In our proof of Section 5, the main tool is the usual Picard iteration. As above, the results in Section 5 can be also used to treat a class of quasi linear stochastic functional partial differential equations.

Lastly, in Section 6 we shall present two applications for our abstract results: stochastic porous medium equations and stochastic reaction diffusion equations. In particular, Eq.(1) and Eq.(2) will be two special cases. It is worthy to say that the two examples given in Section 6 have stochastic non-linear second order terms. Moreover, we may also consider the corresponding backward and functional stochastic partial differential equations with a slight modification.

2. Framework and Preliminaries

In this section we present a general setting in which we can deal with a large class of non-linear stochastic partial differential equations, and also recall the powerful Itô formula and a nonlinear Gronwall type inequality (Bihari's inequality) for treating non-Lipschitz equations.

Let $\mathbb{X}$ be a reflexive and separable Banach space, which is densely injected in a separable Hilbert space $\mathbb{H}$. Identifying $\mathbb{H}$ with its dual we get

$$\mathbb{X} \subset \mathbb{H} \simeq \mathbb{H}^* \subset \mathbb{X}^*,$$

where the star '*' denotes the topological dual space.

Assume that the norm in $\mathbb{X}$ is given by

$$\|x\|_{\mathbb{X}} := \|x\|_{1,\mathbb{X}} + \|x\|_{2,\mathbb{X}}, \quad x \in \mathbb{X}.$$

Denote by $\mathbb{X}_i$, $i = 1, 2$ the completions of $\mathbb{X}$ with respect to the norms $\|\cdot\|_{i,\mathbb{X}} =: \|\cdot\|_{\mathbb{X}_i}$. Then $\mathbb{X} = \mathbb{X}_1 \cap \mathbb{X}_2$. Let us also assume that both spaces are reflexive and embedded in $\mathbb{H}$. Thus, we get two triples:

$$\mathbb{X}_1 \subset \mathbb{H} \simeq \mathbb{H}^* \subset \mathbb{X}_1^*, \quad \mathbb{X}_2 \subset \mathbb{H} \simeq \mathbb{H}^* \subset \mathbb{X}_2^*.$$

Noticing that $\mathbb{X}_1^*$ and $\mathbb{X}_2^*$ can be thought as subspaces of $\mathbb{X}^*$, one may define a Banach space $\mathbb{Y} := \mathbb{X}_1^* + \mathbb{X}_2^* \subset \mathbb{X}^*$ as follows: $f \in \mathbb{Y}$ if and only if $f = f_1 + f_2$, $f_i \in \mathbb{X}_i^*, i = 1, 2$ and the norm of f is defined by

$$\|f\|_{\mathbb{Y}} = \inf_{f = f_1 + f_2} (\|f_1\|_{\mathbb{X}_1^*} + \|f_2\|_{\mathbb{X}_2^*}).$$

In the following, the dual pairs of $(\mathbb{X}, \mathbb{X}^*)$ and $(\mathbb{X}_i, \mathbb{X}_i^*), i = 1, 2$ are denoted respectively by

$$[\cdot, \cdot]_{\mathbb{X}}, \quad [\cdot, \cdot]_{\mathbb{X}_i}, \quad i = 1, 2.$$

Then, for any $x \in \mathbb{X}$ and $f = f_1 + f_2 \in \mathbb{Y} \subset \mathbb{X}^*$,

$$[x, f]_{\mathbb{X}} = [x, f_1]_{\mathbb{X}_1} + [x, f_2]_{\mathbb{X}_2}.$$

We remark that if $f \in \mathbb{H}$ and $x \in \mathbb{X}$, then

$$[x, f]_{\mathbb{X}} = [x, f]_{\mathbb{X}_1} = [x, f]_{\mathbb{X}_2} = \langle x, f \rangle_{\mathbb{H}}, \tag{3}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ stands for the inner product in $\mathbb{H}$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, P)$ be a complete separable filtration probability space, and Q a nonnegative definite and symmetric bounded linear operator on another Hilbert space $\mathbb{U}$. A cylindrical Q-Wiener process $\{W(t), t \geqslant 0\}$ defined on $(\Omega, \mathcal{F}, P)$ is given and assumed to be adapted to $(\mathcal{F}_t)_{t \geqslant 0}$(cf. [9]). In the following we shall only consider the case of $Q \equiv I$ for simplicity. Let $L_2(\mathbb{U}, \mathbb{H})$ denote the Hilbert space consisting of all Hilbert-Schmidt operators from $\mathbb{U}$ to $\mathbb{H}$, where the norm is denoted by $\|\cdot\|_{L_2(\mathbb{U},\mathbb{H})}$, and the inner product by $\langle\cdot,\cdot\rangle_{L_2(\mathbb{U},\mathbb{H})}$.

Fix $T>0$. Let $\mathcal{M}$ be the total of progressively measurable subsets of $[0,T]\times\Omega$. The following Itô's formula is taken from Gyöngy-Krylov [12].

Theorem 2.1. *Let X_0 be an $\mathcal{F}_0$-measurable $\mathbb{H}$-valued random variable. Let*

$$Y_i:[0,T]\times\Omega\to\mathbb{X}_i^*\in\mathcal{M}/\mathcal{B}(\mathbb{X}_i^*),\quad i=1,2,$$

and M an $\mathbb{H}$-valued continuous locally square integrable martingale starting form zero. Let λ_1,λ_2 be two $\mathcal{M}/\mathcal{B}(\mathbb{R})$-measurable real valued processes such that for $(\mathrm{d}t\times\mathrm{d}P)$-almost all (t,ω), $\lambda_1(t,\omega),\lambda_2(t,\omega)>0$. Assume that for some $q_1,q_2>1$ and for almost all ω,

$$\lambda_i(\cdot,\omega)\in L^1([0,T],\mathrm{d}t),\quad Y_i(\cdot,\omega)\cdot\lambda_i^{-\frac{1}{q_i}}(\cdot,\omega)\in L^{\frac{q_i}{q_i-1}}([0,T],\mathrm{d}t;\mathbb{X}^*),\quad i=1,2.$$

Define an $\mathbb{X}^*$-valued process by

$$X(t):=X_0+\int_0^t Y_1(s)\mathrm{d}s+\int_0^t Y_2(s)\mathrm{d}s+M(t).$$

If there exists a $(\mathrm{d}t\times\mathrm{d}P)$-version $\tilde{X}$ of X such that for almost all ω,

$$\tilde{X}(\cdot,\omega)\cdot\lambda_i^{\frac{1}{q_i}}(\cdot,\omega)\in L^{q_i}([0,T],\mathrm{d}t;\mathbb{X}_i),\quad i=1,2,$$

then for almost all ω,

(i) $[0,T]\ni t\mapsto X(t,\omega)\in\mathbb{H}$ is continuous;

(ii) for all $t\in[0,T]$

$$\begin{aligned}\|X(t,\omega)\|_{\mathbb{H}}^2 &= \|X_0(\omega)\|_{\mathbb{H}}^2+2\int_0^t[\tilde{X}(s,\omega),(Y_1+Y_2)(s,\omega)]_{\mathbb{X}}\mathrm{d}s\\ &\quad +2\int_0^t\langle X(s),\mathrm{d}M(s)\rangle_{\mathbb{H}}(\omega)+\langle M\rangle(t,\omega),\end{aligned}$$

where $\langle\cdot\rangle$ denotes the quadratic variation of $\mathbb{H}$-valued martingale.

Proof. Set $N_i(t):=\int_0^t\lambda_i^{\frac{1}{q_i}}(s)\mathrm{d}s$ and $\tilde{Y}_i(t):=Y_i(t)\cdot\lambda_i^{-\frac{1}{q_i}}(s), i=1,2$. Then

$$X(t)=X_0+\int_0^t\tilde{Y}_1(s)\mathrm{d}N_1(s)+\int_0^t\tilde{Y}_2(s)\mathrm{d}N_2(s)+M(t).$$

By the assumptions and Hölder's inequality, we have for almost all ω,

$$\begin{aligned}\tilde{Y}_i(\cdot,\omega) &\in L^1([0,T],\mathrm{d}N_i;\mathbb{X}_i^*),\quad i=1,2,\\ \tilde{X}(\cdot,\omega) &\in \cap_{i=1,2}L^1([0,T],\mathrm{d}N_i;\mathbb{X}_i).\end{aligned}$$

Moreover, by Hölder's inequality we have for almost all ω and $i=1,2$

$$\begin{aligned}&\int_0^T\|\tilde{Y}_i(t,\omega)\|_{\mathbb{X}_i^*}\cdot\|\tilde{X}(t,\omega)\|_{\mathbb{X}_i}\mathrm{d}N_i(t)\\ =&\int_0^T\|Y_i(t,\omega)\|_{\mathbb{X}_i^*}\lambda_i^{-\frac{1}{q_i}}(t)\cdot\|\tilde{X}(t,\omega)\|_{\mathbb{X}_i}\lambda_i^{\frac{1}{q_i}}(t)\mathrm{d}t\\ \leqslant&\left(\int_0^T\|Y_i(t,\omega)\|_{\mathbb{X}_i^*}^{\frac{q_i}{q_i-1}}\lambda_i^{-\frac{1}{q_i-1}}(t)\mathrm{d}t\right)^{\frac{q_i-1}{q_i}}\\ &\times\left(\int_0^T\|\tilde{X}(t,\omega)\|_{\mathbb{X}_i}^{q_i}\lambda_i(t)\mathrm{d}t\right)^{\frac{1}{q_i}}<+\infty.\end{aligned}$$

Thus, we can prove this Theorem along the same lines as in the proof of [12, Theorem 2] (see also [18] [30] [24]). We omit the details. $\square$

We now recall the following Bihari's inequality(cf. [4]). A multi-parameter version with jump was proved in [40].

Lemma 2.2. *Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and non-decreasing function. Let $g(s)$ and $\lambda(s)$ be two strictly positive functions on $\mathbb{R}^+$ such that for some $g_0 > 0$*

$$g(t) \leqslant g_0 + \int_0^t \lambda(s) \cdot \rho(g(s))\mathrm{d}s, \quad t \geqslant 0.$$

If λ is locally integrable, then

$$g(t) \leqslant G^{-1}\left(G(g_0) + \int_0^t \lambda(s)\mathrm{d}s\right),$$

where $G(x) := \int_{x_0}^x \frac{1}{\rho(y)}\mathrm{d}y$ is well defined for some $x_0 > 0$, and G^{-1} is the inverse function of G.

In particular, if $g_0 = 0$ and for some $\varepsilon > 0$

$$\int_0^\varepsilon \frac{1}{\rho(x)}\mathrm{d}x = +\infty, \tag{4}$$

then $g(t) \equiv 0$.

Remark 2.3. *The typical concave functions satisfying (4) are given by $\rho_k(x)$, $k = 1, 2, \cdots$,*

$$\rho_k(x) := \begin{cases} c_0 \cdot x \cdot \Pi_{j=1}^k \log^j x^{-1}, & x \leqslant \eta \\ c_0 \cdot \eta \cdot \Pi_{j=1}^k \log^j \eta^{-1} + c_0 \cdot \rho_k'(\eta-) \cdot (x - \eta), & x > \eta, \end{cases} \tag{5}$$

where $\log^j x^{-1} := \log\log\cdots\log x^{-1}$ and $c_0 > 0$, $0 < \eta < 1/e^k$.

In the sequel, we use the following convention: $c_0, c_1, \cdots$ will denote positive constants whose values may change in different occasions. Moreover, the following Young's inequality will be used frequently: Let $a, b > 0$ and $\alpha, \beta > 1$ satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then for any $\varepsilon > 0$

$$ab \leqslant \varepsilon a^\alpha + \frac{b^\beta}{(\alpha\varepsilon)^{\beta/\alpha}\beta}. \tag{6}$$

For simplicity of notation, we also write

$$\mathfrak{A} := ([0, T] \times \Omega, \mathcal{B}([0, T]) \times \mathcal{F}, \mathrm{d}t \times \mathrm{d}P)$$

and

$$\mathfrak{A}_a := ([0, T] \times \Omega, \mathcal{M}, \mathrm{d}t \times \mathrm{d}P).$$

We now introduce three evolution operators used in the present paper:

$$\begin{aligned} A_i : [0, T] \times \Omega \times \mathbb{X}_i \to \mathbb{X}_i^* &\in \mathcal{M} \times \mathcal{B}(\mathbb{X}_i)/\mathcal{B}(\mathbb{X}_i^*), \quad i = 1, 2, \\ B : [0, T] \times \Omega \times \mathbb{X} \to L_2(\mathbb{U}, \mathbb{H}) &\in \mathcal{M} \times \mathcal{B}(\mathbb{X})/\mathcal{B}(L_2(\mathbb{U}, \mathbb{H})). \end{aligned}$$

In the following, for the sake of simplicity, we write

$$A = A_1 + A_2 \in \mathbb{Y} \subset \mathbb{X}^*.$$

Assume that

(H1) (Hemicontinuity) For any $(t,\omega)\in[0,T]\times\Omega$ and $x,y,z\in\mathbb{X}$, the mapping

$$[0,1]\ni\varepsilon\mapsto[x,A(t,\omega,y+\varepsilon z)]_{\mathbb{X}}$$

is continuous.

(H2) (Weak monotonicity) There exists $0\leqslant\lambda_0\in L^1(\mathfrak{A}_a)$ such that for all $x,y\in\mathbb{X}$ and $(t,\omega)\in[0,T]\times\Omega$

$$\begin{aligned}2[x-y,A(t,\omega,x)-A(t,\omega,y)]_{\mathbb{X}}+\|B(t,\omega,x)-B(t,\omega,y)\|^2_{L_2(\mathbb{U},\mathbb{H})}\\ \leqslant\lambda_0(t,\omega)\cdot\|x-y\|^2_{\mathbb{H}}.\end{aligned}$$

(H3) (Weak coercivity) There exist $q_1,q_2>1,c_1>0$ and positive functions $\lambda_1,\lambda_2,\xi\in L^1(\mathfrak{A})$, $\lambda_3\in L^1(\mathfrak{A}_a)$ such that for all $x\in\mathbb{X}$ and $(t,\omega)\in[0,T]\times\Omega$

$$\begin{aligned}&2[x,A(t,\omega,x)]_{\mathbb{X}}+\|B(t,\omega,x)\|^2_{L_2(\mathbb{U},\mathbb{H})}\\ \leqslant\ &-\sum_{i=1,2}\left(\lambda_i(t,\omega)\cdot\|x\|^{q_i}_{\mathbb{X}_i}\right)+\lambda_3(t,\omega)\cdot\|x\|^2_{\mathbb{H}}+\xi(t,\omega)\end{aligned}$$

and for $(\mathrm{d}t\times\mathrm{d}P)$-almost all (t,ω), $\lambda_1(t,\omega),\lambda_2(t,\omega)>0$ and

$$0\leqslant\lambda_0(t,\omega)\leqslant c_1\lambda_3(t,\omega), \tag{7}$$

where λ_0 is same as in **(H2)**.

(H4) (Boundedness) There exist $c_{A_i}>0$ and $0\leqslant\eta_i\in L^{\frac{q_i}{q_i-1}}(\mathfrak{A})$, $i=1,2$ such that for all $x\in\mathbb{X}$ and $(t,\omega)\in[0,T]\times\Omega$

$$\|A_i(t,\omega,x)\|_{\mathbb{X}_i^*}\leqslant\eta_i(t,\omega)\cdot\lambda_i^{\frac{1}{q_i}}(t,\omega)+c_{A_i}\cdot\lambda_i(t,\omega)\cdot\|x\|^{q_i-1}_{\mathbb{X}_i},\quad i=1,2,$$

where q_1,q_2 and λ_1,λ_2 are same as in **(H3)**.

In order to emphasize λ_i,ξ and q_i,η_i below, we shall say that

$$(A,B)\text{ satisfies }\mathscr{H}(\lambda_0,\lambda_1,\lambda_2,\lambda_3,\xi,\eta_1,\eta_2,q_1,q_2).$$

Remark 2.4. *By* **(H3)**, **(H4)** *and Young's inequality (6), it follows that for any $x\in\mathbb{X}$ and $(t,\omega)\in[0,T]\times\Omega$*

$$\begin{aligned}\|B(t,\omega,x)\|^2_{L_2(\mathbb{U},\mathbb{H})}\ &\leqslant\ 2\sum_{i=1,2}\left(c_{A_i}\cdot\lambda_i(t,\omega)\cdot\|x\|^{q_i}_{\mathbb{X}_i}+\eta_i(t,\omega)\cdot\lambda_i^{\frac{1}{q_i}}(t,\omega)\cdot\|x\|_{\mathbb{X}_i}\right)\\ &\quad+\lambda_3(t,\omega)\cdot\|x\|^2_{\mathbb{H}}+\xi(t,\omega)\\ &\leqslant\ \sum_{i=1,2}\left(c_B\cdot\lambda_i(t,\omega)\cdot\|x\|^{q_i}_{\mathbb{X}_i}+\eta_i^{\frac{q_i}{q_i-1}}(t,\omega)\right)\\ &\quad+\lambda_3(t,\omega)\cdot\|x\|^2_{\mathbb{H}}+\xi(t,\omega),\end{aligned}$$

where $c_B>1$ only depends on c_{A_i} and q_i, $i=1,2$.

The following lemma is well known(cf. [18]).

Lemma 2.5. *Let $(A,0)$ satisfy $\mathscr{H}(\lambda_0,\lambda_1,\lambda_2,\lambda_3,\xi,\eta_1,\eta_2,q_1,q_2)$, and $0\leqslant\tau\leqslant T$ a bounded random variable. Let X and $Y_i (i=1,2)$ be respectively $\mathbb{X}$ and $\mathbb{X}_i^*$-valued measurable processes with*

$$1_{[0,\tau]}\cdot\lambda_i^{\frac{1}{q_i-1}}\cdot X\in L^{q_i-1}(\mathfrak{A};\mathbb{X}_i),\quad 1_{[0,\tau]}\cdot Y_i\in L^1(\mathfrak{A};\mathbb{X}_i^*),\quad i=1,2.$$

Let Λ be a strictly positive and real-valued bounded measurable process. Assume that for any $\mathbb{X}$-valued measurable process Φ satisfying

$$1_{[0,\tau]}\cdot\lambda_i^{\frac{1}{q_i-1}}\cdot\Phi\in L^{q_i-1}(\mathfrak{A};\mathbb{X}_i),\quad i=1,2,$$

it holds

$$
\begin{aligned}
& \mathbb{E}\left(\int_0^\tau \Lambda(s)\cdot[X(s)-\Phi(s),Y(s)-A(s,\Phi(s))]_{\mathbb{X}}\mathrm{d}s\right)\\
\leqslant\ & \mathbb{E}\left(\int_0^\tau \Lambda(s)\cdot\lambda_0(s)\cdot\|X(s)-\Phi(s)\|_{\mathbb{H}}^2\mathrm{d}s\right),
\end{aligned}
\tag{8}
$$

where $Y=Y_1+Y_2\in\mathbb{Y}\subset\mathbb{X}^*$.

Then $Y(t,\omega)=A(t,\omega,X(t,\omega))$ *for almost all* $(t,\omega)\in\{(t,\omega):t\in[0,\tau(\omega)]\}$.

Proof. For any $\varepsilon\in(0,1)$ and $\mathbb{X}$-valued bounded measurable process ϕ, letting $\Phi=X-\varepsilon\phi$ in (8) and dividing both sides by ε, we get

$$
\begin{aligned}
& \mathbb{E}\left(\int_0^\tau \Lambda(s)\cdot[\phi(s),Y(s)-A(s,X(s)-\varepsilon\phi(s))]_{\mathbb{X}}\mathrm{d}s\right)\\
\leqslant\ & \varepsilon\cdot\mathbb{E}\left(\int_0^\tau \Lambda(s)\cdot\lambda_0(s)\cdot\|\phi(s)\|_{\mathbb{H}}^2\mathrm{d}s\right).
\end{aligned}
$$

By **(H4)** and the assumptions, we have

$$
1_{[0,\tau]}(\cdot)\cdot\left(\|Y(\cdot)\|_{\mathbb{X}^*}+\sup_{\varepsilon\in(0,1)}\|A(\cdot,X(\cdot)-\varepsilon\phi(\cdot))\|_{\mathbb{X}^*}\right)\in L^1(\mathfrak{A}).
$$

Hence, by **(H1)** and the dominated convergence theorem

$$
\mathbb{E}\left(\int_0^\tau \Lambda(s)\cdot[\phi(s),Y(s)-A(s,X(s))]_{\mathbb{X}}\mathrm{d}s\right)\leqslant 0.
$$

By changing ϕ to $-\phi$ and the arbitrariness of ϕ, we conclude that $Y=A(\cdot,X)$. □

The following lemma is simple and will be used in Section 4. A short proof is provided here for the reader's convenience.

Lemma 2.6. *Let* $(S,\mathcal{S})$ *be a measurable space. Let* $X:\mathbb{R}^d\times S\to\mathbb{R}^d$ *be a measurable field. Assume that for every* $s\in S$, $\mathbb{R}^d\ni x\mapsto X(x,s)\in\mathbb{R}^d$ *is a homeomorphism. Then, the inverse* $(x,s)\mapsto X^{-1}(x,s)$ *is also a measurable field, i.e.: for each* $x\in\mathbb{R}^d$, $X^{-1}(x,\cdot)$ *is* $\mathcal{S}/\mathcal{B}(\mathbb{R}^d)$*-measurable.*

Proof. Fix $x\in\mathbb{R}^d$. It suffices to prove that for any bounded open set $U\subset\mathbb{R}^d$

$$
S_1:=\{s:X^{-1}(x,s)\in\bar{U}\}\in\mathcal{S},
\tag{9}
$$

where $\bar{U}$ denotes the closure of U in $\mathbb{R}^d$.

Let Q be the set of rational points in $\mathbb{R}^d$. Then

$$
S_1=\cap_{k=1}^\infty\cup_{y\in Q\cap U}\{s:\|X(y,s)-x\|_{\mathbb{R}^d}<1/k\}=:S_2.
\tag{10}
$$

In fact, if $s\in S_1$, then there is a $y\in\bar{U}$ such that $x=X(y,s)$. Since U is open and $X(\cdot,s)$ is continuous, there exists a sequence $y_n\in U\cap Q$ such that $y_n\to y$ and $X(y_n,s)\to X(y,s)=x$. So, $s\in S_2$. On the other hand, if $s\in S_2$, then there is a sequence $y_n\in U\cap Q$ such that $\lim_{n\to\infty}\|X(y_n,s)-x\|_{\mathbb{R}^d}=0$, and so $y_n\to X^{-1}(x,s)\in\bar{U}$. (9) now follows from (10). □

3. Stochastic Evolution Equations with Random Coefficients

In this section we consider the following stochastic evolution equation:

$$\begin{cases} \mathrm{d}X(t) = A(t, X(t))\mathrm{d}t + B(t, X(t))\mathrm{d}W(t), \\ X(0) = X_0 \in \mathbb{H}, \end{cases} \tag{11}$$

where (A, B) satisfies $\mathscr{H}(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \xi, \eta_1, \eta_2, q_1, q_2)$. Here and after, one should keep in mind that $A = A_1 + A_2 \in \mathbb{Y} \subset \mathbb{X}^*$, where $A_1 \in \mathbb{X}_1^*, A_2 \in \mathbb{X}_2^*$.

Set

$$H(t,\omega) := \int_0^t \lambda_3(s,\omega)\mathrm{d}s, \tag{12}$$

and define

$$\theta_t(\omega) := \inf\{s \in [0,T] : H(s,\omega) \geqslant t\}. \tag{13}$$

Here $\inf\{\emptyset\} = T$ by convention. Then $t \mapsto \theta_t$ is continuous and θ_t is a stopping time for each t. Moreover, $\theta_t \uparrow T$ as $t \uparrow \infty$.

Set for each $m \in \mathbb{N}$

$$\mu^m(\mathrm{d}t \times \mathrm{d}\omega) := 1_{\{t \leqslant \theta_m(\omega)\}}(\mathrm{d}t \times \mathrm{d}P),$$

and define the completed measurable spaces

$$\mathfrak{M}^m := \overline{([0,T] \times \Omega, \mathcal{B}([0,T]) \times \mathcal{F})}^{\mu^m}$$

and

$$\mathfrak{M}_a^m := \overline{([0,T] \times \Omega, \mathcal{M})}^{\mu^m}.$$

We introduce the following stochastic Banach spaces for later use: for each $m \in \mathbb{N}$

$$\begin{aligned} \mathbb{K}_{1,i}^m &:= L^{\frac{q_i}{q_i-1}}(\mathfrak{M}^m, \lambda_i^{-\frac{1}{q_i-1}}(t,\omega) \cdot \mu^m(\mathrm{d}t \times \mathrm{d}\omega); \mathbb{X}_i^*), \quad i = 1, 2, \\ \mathbb{K}_{2,i}^m &:= L^{q_i}(\mathfrak{M}^m, \lambda_i(t,\omega) \cdot \mu^m(\mathrm{d}t \times \mathrm{d}\omega); \mathbb{X}_i), \quad i = 1, 2, \\ \mathbb{K}_3^m &:= L^2(\mathfrak{M}_a^m, \mu^m(\mathrm{d}t \times \mathrm{d}\omega); L_2(\mathbb{U}, \mathbb{H})), \\ \mathbb{K}_4^m &:= L^2(\mathfrak{M}^m, \mu^m(\mathrm{d}t \times \mathrm{d}\omega); \mathbb{H}), \\ \mathbb{K}_5^m &:= L^2(\mathfrak{M}^m, \lambda_3(t,\omega) \cdot \mu^m(\mathrm{d}t \times \mathrm{d}\omega); \mathbb{H}), \end{aligned}$$

where the norms are defined in a natural manner, and denoted by $\|\cdot\|_{\mathbb{K}}$, where $\mathbb{K}$ stands for the above spaces. For instance,

$$\|X\|_{\mathbb{K}_{2,i}^m} := \left[\mathbb{E}\left(\int_0^{\theta_m} \|X(t)\|_{\mathbb{X}_i}^{q_i} \cdot \lambda_i(t)\mathrm{d}t\right)\right]^{1/q_i}, \quad i = 1, 2.$$

Remark 3.1. *If λ_3 is non-random, then for some m sufficiently large, $\theta_m \equiv T$. In this case, we shall omit the superscript 'm' of $\mathbb{K}^m$.*

We need the following lemma.

Lemma 3.2. *(i) $\mathbb{K}_{i,j}^m, i, j = 1, 2$ and $\mathbb{K}_3^m, \mathbb{K}_4^m, \mathbb{K}_5^m$ are separable and reflexive Banach spaces.*

(ii) For any $Y \in \mathbb{K}_{1,i}^m$, we have $\mathbb{E}\left(\int_0^{\theta_m} \|Y(t)\|_{\mathbb{X}_i^*}\mathrm{d}t\right) \leqslant c_0 \cdot \|Y\|_{\mathbb{K}_{1,i}^m}$, where $i = 1$ or 2.

(iii) Let $\{Y_n, n \in \mathbb{N}\}$ weakly converge to Y in $\mathbb{K}_{1,i}^m$, then for any $X \in \mathbb{K}_{2,i}^m$

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^{\theta_m} [X(t), Y_n(t)]_{\mathbb{X}_i}\mathrm{d}t\right) = \mathbb{E}\left(\int_0^{\theta_m} [X(t), Y(t)]_{\mathbb{X}_i}\mathrm{d}t\right),$$

where $i = 1$ or 2.

(iv) Let $\{X_n, n \in \mathbb{N}\}$ weakly converge to X in $\mathbb{K}^m_{2,i}$, then for any $Y \in \mathbb{K}^m_{1,i}$

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^{\theta_m} [X_n(t), Y(t)]_{\mathbb{X}_i} \mathrm{d}t\right) = \mathbb{E}\left(\int_0^{\theta_m} [X(t), Y(t)]_{\mathbb{X}_i} \mathrm{d}t\right),$$

where $i = 1$ or 2. Moreover, if $\{X_n, n \in \mathbb{N}\}$ also weakly converges to $\bar{X}$ in $\mathbb{K}^m_5$, then $\bar{X}(t,\omega) = X(t,\omega)$ for μ^m-almost all (t,ω).

(v) Define a linear operator from $\mathbb{K}^m_3$ to $\mathbb{K}^m_4$ as

$$J(G) := \int_0^{\cdot\wedge\theta_m} G(s)\mathrm{d}W(s), \tag{14}$$

then J is a continuous linear operator. In particular, J is continuous with respect to the weak topologies.

Proof. (i). It follows from the separabilities and reflexivities of $\mathbb{X}_i, \mathbb{X}_i^*, i = 1, 2$, and $\mathbb{H}, L_2(\mathbb{U}, \mathbb{H})$.

(ii). By Hölder's inequality we have

$$\begin{aligned}
\mathbb{E}\left(\int_0^{\theta_m} \|Y(t)\|_{\mathbb{X}_i^*} \mathrm{d}t\right) &= \mathbb{E}\left(\int_0^{\theta_m} \|Y(t)\|_{\mathbb{X}_i^*} \lambda_2^{-1/q_i}(t) \cdot \lambda_2^{1/q_i}(t)\mathrm{d}t\right) \\
&\leqslant \|Y\|_{\mathbb{K}^m_{1,i}} \left(\int_0^T \mathbb{E}(\lambda_i(t))\mathrm{d}t\right)^{1/q_i}.
\end{aligned}$$

(iii). It follows from

$$X(\cdot) \cdot \lambda_i^{\frac{1}{q_i-1}}(\cdot) \in L^{q_i}(\mathfrak{M}^m, \lambda_i^{-\frac{1}{q_i-1}}(t,\omega) \cdot \mu^m(\mathrm{d}t \times \mathrm{d}\omega); \mathbb{X}_i) \subset (\mathbb{K}^m_{1,i})^*.$$

(iv). The first conclusion follows from

$$Y(\cdot) \cdot \lambda_i^{-1}(\cdot) \in L^{\frac{q_i}{q_i-1}}(\mathfrak{M}^m, \lambda_i(t,\omega) \cdot \mu^m(\mathrm{d}t \times \mathrm{d}\omega); \mathbb{X}_i^*) \subset (\mathbb{K}^m_{2,i})^*.$$

As for the second conclusion, by the well known Banach-Saks-Kakutani theorem, there exists a subsequence of X_n(still denoted by X_n) such that its Césaro means $\tilde{X}_n$ strongly converges to X and $\bar{X}$ in $\mathbb{K}^m_{2,i}$ and $\mathbb{K}^m_5$ respectively. Therefore, there is a subsequence $\tilde{X}_{n_k}$ such that for μ^m-almost all (t,ω), $\tilde{X}_{n_k}(t,\omega) \to X(t,\omega)$ in $\mathbb{X}$, and $\tilde{X}_{n_k}(t,\omega) \to \bar{X}(t,\omega)$ in $\mathbb{H}$. Since $\mathbb{X}$ is continuously and densely embedded in $\mathbb{H}$, we have $\bar{X}(t,\omega) = X(t,\omega)$ for μ^m-almost all (t,ω).

(v). It follows from

$$\begin{aligned}
\|J(G)\|^2_{\mathbb{K}^m_4} &= \mathbb{E}\left(\int_0^{\theta_m} \left\|\int_0^{t\wedge\theta_m} G(s)\mathrm{d}W(s)\right\|^2_{\mathbb{H}} \mathrm{d}t\right) \\
&\leqslant \int_0^T \mathbb{E}\left(\int_0^{t\wedge\theta_m} \|G(s)\|^2_{L_2(\mathbb{U},\mathbb{H})}\mathrm{d}s\right)\mathrm{d}t \\
&\leqslant T\|G\|^2_{\mathbb{K}^m_3}.
\end{aligned}$$

The proof is complete. □

Definition 3.3. *An $\mathbb{H}$-valued continuous $\mathcal{F}_t$-adapted process $X(t,\omega)$ is called a solution of Eq.(11) if for almost all $\omega \in \Omega$,*

$$X(\cdot,\omega) \in \cap_{i=1,2} L^{q_i}([0,T], \lambda_i(\cdot,\omega)\mathrm{d}t; \mathbb{X}_i)$$

and in $\mathbb{X}^*$, for all $t \in [0,T]$

$$X(t,\omega) = X_0(\omega) + \int_0^t A(s,\omega,X(s,\omega))\mathrm{d}s + \int_0^t B(s,X(s))\mathrm{d}W(s)(\omega),$$

where the first integral is understood as an $\mathbb{X}^*$-valued Bochner integral.

Remark 3.4. *Note that*

$$\int_0^t A(s,\omega,X(s,\omega))\mathrm{d}s = \int_0^t A_1(s,\omega,X(s,\omega))\mathrm{d}s + \int_0^t A_2(s,\omega,X(s,\omega))\mathrm{d}s.$$

Since X is $\mathcal{M}/\mathcal{B}(\mathbb{H})$-measurable, $1_{\mathbb{X}_i}(X)\cdot X$ is $\mathcal{M}/\mathcal{B}(\mathbb{X}_i)$-measurable by [18, Lemma 2.1] *for $i=1,2$. The above integrals are meaningful.*

We have the following estimates for the solutions of Eq.(11).

Theorem 3.5. *Assume that* **(H1)-(H4)** *hold and $X_0 \in L^2(\Omega,\mathcal{F}_0,P;\mathbb{H})$. Let $X(t)$ be any solution of Eq. (11) in the sense of Definition 3.3. Then, we have for any $m\in\mathbb{N}$*

$$\mathbb{E}\|X(\theta_m)\|_{\mathbb{H}}^2 + \sum_{i=1,2}\|X\|_{\mathbb{K}_{2,i}^m}^{q_i} + \|X\|_{\mathbb{K}_5^m}^2 \leqslant c_m\left(\mathbb{E}\|X_0\|_{\mathbb{H}}^2 + \int_0^T \mathbb{E}(\xi(s))\mathrm{d}s\right),$$

and

$$\begin{aligned}&\mathbb{E}\left(\sup_{t\in[0,\theta_m]}\|X(t)\|_{\mathbb{H}}^2\right) + \|X\|_{\mathbb{K}_4^m}^2 + \|B(\cdot,X(\cdot))\|_{\mathbb{K}_3^m}^2 + \sum_{i=1,2}\|A_i(\cdot,X(\cdot))\|_{\mathbb{K}_{1,i}^m}^{\frac{q_i}{q_i-1}}\\ \leqslant\ & c_m\left(\mathbb{E}\|X_0\|_{\mathbb{H}}^2 + \int_0^T \mathbb{E}\left(\xi(s) + \eta_1^{\frac{q_1}{q_1-1}}(s) + \eta_2^{\frac{q_2}{q_2-1}}(s)\right)\mathrm{d}s\right),\end{aligned}$$

where c_m only depends on m, T and $c_{A_i}, q_i, i=1,2$.

Proof. By Itô's formula (Theorem 2.1) and **(H3)**, we have

$$\begin{aligned}&\|X(t)\|_{\mathbb{H}}^2 - \|X_0\|_{\mathbb{H}}^2 - M(t)\\ =\ & \int_0^t \left(2[X(s),A(s,X(s))]_{\mathbb{X}} + \|B(s,X(s))\|_{L_2(\mathbb{U},\mathbb{H})}^2\right)\mathrm{d}s\\ \leqslant\ & \int_0^t\left(-\sum_{i=1,2}\left(\lambda_i(s)\cdot\|X(s)\|_{\mathbb{X}_i}^{q_i}\right) + \lambda_3(s)\cdot\|X(s)\|_{\mathbb{H}}^2 + \xi(s)\right)\mathrm{d}s,\end{aligned} \tag{15}$$

where $M(t)$ is a continuous local martingale given by

$$M(t) := 2\int_0^t \langle X(s), B(s,X(s))\mathrm{d}W(s)\rangle_{\mathbb{H}}.$$

For any $R>0$, define the stopping time

$$\tau_R := \inf\left\{t\in[0,T]: \|X(t)\|_{\mathbb{H}} \geqslant R, \int_0^t \lambda_i(s)\cdot\|X(s)\|_{\mathbb{X}_i}^{q_i}\mathrm{d}s \geqslant R, i=1,2\right\}. \tag{16}$$

Then, by Definition 3.3, $\tau_R \uparrow T$ a.s. as $R\uparrow\infty$.

By Remark 2.4 and the change of clock(cf. [28]), we know that $\{M(\theta_t\wedge\tau_R), t\geqslant 0\}$ is a continuous $\mathcal{F}_{\theta_t}$-martingale. Indeed, this follows from

$$\langle M(\theta_\cdot\wedge\tau_R)\rangle(t) \leqslant 4\int_0^{\theta_t\wedge\tau_R}\|X(s)\|_{\mathbb{H}}^2\cdot\|B(s,X(s))\|_{L_2(\mathbb{U},\mathbb{H})}^2\mathrm{d}s \leqslant c_R.$$

So, replacing t by $\theta_t\wedge\tau_R$ in (15) and taking expectations for both sides of (15), we obtain

$$\begin{aligned}
&\mathbb{E}\|X(\theta_t\wedge\tau_R)\|_{\mathbb{H}}^2-\mathbb{E}\|X_0\|_{\mathbb{H}}^2+\sum_{i=1,2}\mathbb{E}\left(\int_0^{\theta_t\wedge\tau_R}\lambda_i(s)\cdot\|X(s)\|_{\mathbb{X}_i}^{q_i}\mathrm{d}s\right)\\
\leqslant\ &\mathbb{E}\left(\int_0^{\theta_t\wedge\tau_R}\big(\lambda_3(s)\cdot\|X(s)\|_{\mathbb{H}}^2+\xi(s)\big)\,\mathrm{d}s\right)\\
=\ &\mathbb{E}\left(\int_0^{\theta_t\wedge\tau_R}\|X(s)\|_{\mathbb{H}}^2\mathrm{d}H(s)\right)+\mathbb{E}\left(\int_0^{\theta_t\wedge\tau_R}\xi(s)\mathrm{d}s\right)\\
\leqslant\ &\mathbb{E}\left(\int_0^{\theta_t}\|X(s\wedge\tau_R)\|_{\mathbb{H}}^2\mathrm{d}H(s)\right)+\int_0^T\mathbb{E}(\xi(s))\mathrm{d}s\\
=\ &\int_0^t\mathbb{E}\|X(\theta_s\wedge\tau_R)\|_{\mathbb{H}}^2\mathrm{d}s+\int_0^T\mathbb{E}(\xi(s))\mathrm{d}s,
\end{aligned}$$

where $H(s)$ is defined by (12), and in the last step we have used the variable substitution formula.

Hence, by Gronwall's inequality we have for any $t\geqslant 0$

$$\mathbb{E}\|X(\theta_t\wedge\tau_R)\|_{\mathbb{H}}^2\leqslant e^t\left(\mathbb{E}\|X_0\|_{\mathbb{H}}^2+\int_0^T\mathbb{E}(\xi(s))\mathrm{d}s\right).$$

Letting $R\to\infty$, by Fatou's lemma we obtain that for any $m\in\mathbb{N}$

$$\mathbb{E}\|X(\theta_m)\|_{\mathbb{H}}^2\leqslant e^m\left(\mathbb{E}\|X_0\|_{\mathbb{H}}^2+\int_0^T\mathbb{E}(\xi(s))\mathrm{d}s\right)$$

as well as

$$\begin{aligned}
&\sum_{i=1,2}\mathbb{E}\left(\int_0^{\theta_m}\lambda_i(s)\cdot\|X(s)\|_{\mathbb{X}_i}^{q_i}\mathrm{d}s\right)+\mathbb{E}\left(\int_0^{\theta_m}\lambda_3(s)\cdot\|X(s)\|_{\mathbb{H}}^2\mathrm{d}s\right)\\
\leqslant\ &c_m\left(\mathbb{E}\|X_0\|_{\mathbb{H}}^2+\int_0^T\mathbb{E}(\xi(s))\mathrm{d}s\right),
\end{aligned}\tag{17}$$

which gives the first estimate.

From (15), by Burkholder's inequality and Young's inequality (6) we further have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,\theta_m]}\|X(t)\|_{\mathbb{H}}^2\right)-\mathbb{E}\|X_0\|_{\mathbb{H}}^2\\
\leqslant\ &\mathbb{E}\left(\int_0^{\theta_m}\big(\lambda_3(s)\cdot\|X(s)\|_{\mathbb{H}}^2+\xi(s)\big)\,\mathrm{d}s\right)\\
&+c_0\mathbb{E}\left(\int_0^{\theta_m}\|X(s)\|_{\mathbb{H}}^2\cdot\|B(s,X(s))\|_{L_2(\mathbb{U},\mathbb{H})}^2\mathrm{d}s\right)^{1/2}\\
\leqslant\ &\int_0^T\mathbb{E}(\xi(s))\mathrm{d}s+\mathbb{E}\left(\int_0^{\theta_m}\lambda_3(s)\cdot\|X(s)\|_{\mathbb{H}}^2\mathrm{d}s\right)\\
&+\frac{1}{2}\mathbb{E}\left(\sup_{t\in[0,\theta_m]}\|X(t)\|_{\mathbb{H}}^2\right)+c_0\mathbb{E}\left(\int_0^{\theta_m}\|B(s,X(s))\|_{L_2(\mathbb{U},\mathbb{H})}^2\mathrm{d}s\right).
\end{aligned}$$

Hence

$$\mathbb{E}\left(\sup_{t\in[0,\theta_m]}\|X(t)\|_{\mathbb{H}}^2\right)\ \leqslant\ c_m\left(\mathbb{E}\|X_0\|_{\mathbb{H}}^2+\int_0^T\mathbb{E}(\xi(s))\mathrm{d}s\right)$$

$$+c_0\mathbb{E}\left(\int_0^{\theta_m}\|B(s,X(s))\|^2_{L_2(\mathbb{U},\mathbb{H})}\mathrm{d}s\right).$$

The second estimate now follows from **(H4)**, Remark 2.4 and (17). □

We now prove our main result in this section.

Theorem 3.6. *Assume that (A,B) satisfies $\mathscr{H}(\lambda_0,\lambda_1,\lambda_2,\lambda_3,\xi,\eta_1,\eta_2,q_1,q_2)$. Then for any $X_0\in L^2(\Omega,\mathcal{F}_0,P;\mathbb{H})$, there exists a unique solution to Eq.(11) in the sense of Definition 3.3.*

Proof. (Uniqueness): Let X_1 and X_2 be two solutions of Eq.(11) in the sense of Definition 3.3. For $t\geqslant 0$, define

$$\beta_t:=\inf\left\{s\in[0,T]:\int_0^s\lambda_0(r)\mathrm{d}r\geqslant t\right\},$$

and for $R>0$ and $i=1,2$, let τ_R^i be defined as in (16) corresponding to X_i. For $t_0\in(0,T)$, set $\tau_R^{t_0}:=\tau_R^1\wedge\tau_R^2\wedge t_0$. By Itô's formula(Theorem 2.1), as in the proof of Theorem 3.5 we have

$$\begin{aligned}
&\mathbb{E}\|(X_1-X_2)(\beta_t\wedge\tau_R^{t_0})\|^2_{\mathbb{H}}\\
=\;&\mathbb{E}\Bigg(\int_0^{\beta_t\wedge\tau_R^{t_0}}\Big(2[X_1(s)-X_2(s),A(s,X_1(s))-A(s,X_2(s))]_{\mathbb{X}}\\
&+\|B(s,X_1(s))-B(s,X_2(s))\|^2_{L_2(\mathbb{U},\mathbb{H})}\Big)\mathrm{d}s\Bigg)\\
\leqslant\;&\mathbb{E}\left(\int_0^{\beta_t\wedge\tau_R^{t_0}}\|X_1(s)-X_2(s)\|^2_{\mathbb{H}}\cdot\lambda_0(s)\mathrm{d}s\right)\\
\leqslant\;&\mathbb{E}\left(\int_0^t\|(X_1-X_2)(\beta_s\wedge\tau_R^{t_0})\|^2_{\mathbb{H}}\mathrm{d}s\right).
\end{aligned}$$

Using Gronwall's inequality yields that for any $t\geqslant 0$ and $R>0$

$$\mathbb{E}\|(X_1-X_2)(\beta_t\wedge\tau_R^{t_0})\|^2_{\mathbb{H}}=0.$$

Letting $R,t\to\infty$, and by Fatou's lemma we get

$$\mathbb{E}\|(X_1-X_2)(t_0)\|^2_{\mathbb{H}}=0.$$

The uniqueness is then obtained.

(Existence): We shall use Galerkin's approximation to prove the existence of solutions, and divide the proof into four steps.

(Step 1): Let $\{e_i,i\in\mathbb{N}\}\subset\mathbb{X}$ be a dense subset of $\mathbb{X}$ and a normal orthogonal basis of $\mathbb{H}$. Set

$$\Pi_n x:=\sum_{i=1}^n[e_i,x]_{\mathbb{X}}\cdot e_i,\quad x\in\mathbb{X}^*.$$

Then the mapping $\Pi_n:\mathbb{X}^*\to\mathbb{X}$ is linear and continuous, and satisfy

$$\Pi_n x=\sum_{i=1}^n\langle e_i,x\rangle_{\mathbb{H}}\cdot e_i,\quad x\in\mathbb{H}$$

and

$$[\Pi_n x,y]_{\mathbb{X}}=[\Pi_n y,x]_{\mathbb{X}},\quad x,y\in\mathbb{X}^*.$$

We also fix a normal orthogonal basis $\{f_1, f_2, \cdots\}$ of $\mathbb{U}$. Let $W_j(t) := \langle W(t), f_j\rangle_{\mathbb{U}}$ for $j \in \mathbb{N}$. Consider the following Itô type stochastic ordinary differential equation in $\mathbb{R}^n$:

$$\begin{cases} \mathrm{d}X_n^i(t) = b^i(t, X_n(t))\mathrm{d}t + \sum_{j=1}^n \sigma_j^i(t, X_n(t))\mathrm{d}W_j(t), \\ X_n^i(0) = \langle X_0, e_i\rangle_{\mathbb{H}}, \quad i = 1, \cdots, n, \end{cases} \tag{18}$$

where

$$b^i(t, x) := [e_i, A(t, x \cdot e)]_{\mathbb{X}},$$

and

$$\sigma_j^i(t, x) := \langle e_i, B(t, x \cdot e)(f_j)\rangle_{\mathbb{H}}.$$

Here $x \in \mathbb{R}^n$ and $x \cdot e := \sum_{i=1}^n x^i e_i$.

The coefficients satisfy the following conditions by **(H1)**-**(H4)**:

(i) b and σ are $\mathcal{M} \times \mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^n)$ and $\mathcal{M} \times \mathcal{B}(\mathbb{R}^n)/\mathcal{B}(L_2(\mathbb{R}^n, \mathbb{R}^n))$-measurable respectively and continuous in x.

(ii) For any $(t, \omega) \in [0, T] \times \Omega$ and $x, y \in \mathbb{R}^n$

$$\begin{aligned} &2\langle x - y, b(t, \omega, x) - b(t, \omega, y)\rangle_{\mathbb{R}^n} + \|\sigma(t, \omega, x) - \sigma(t, \omega, y)\|^2_{L_2(\mathbb{R}^n,\mathbb{R}^n)} \\ &\quad \leqslant \lambda_0(t, \omega) \cdot \|x - y\|_{\mathbb{R}^n}. \end{aligned}$$

(iii) For any $(t, \omega) \in [0, T] \times \Omega$ and $x \in \mathbb{R}^n$

$$2\langle x, b(t, \omega, x)\rangle_{\mathbb{R}^n} + \|\sigma(t, \omega, x)\|^2_{L_2(\mathbb{R}^n,\mathbb{R}^n)} \leqslant \lambda_3(t, \omega) \cdot \|x\|^2_{\mathbb{R}^n} + \xi(t, \omega).$$

(iv) For any $(t, \omega) \in [0, T] \times \Omega$ and $x \in \mathbb{R}^n$

$$\|b(t, \omega, x)\|_{\mathbb{R}^n} \leqslant c_n \sum_{i=1,2} \left(\eta_i(t, \omega) \cdot \lambda_i^{1/q_i}(t, \omega) + \lambda_i(t, \omega) \cdot \|x\|^{q_i-1}_{\mathbb{R}^n}\right).$$

By a well-known result due to Krylov [16](see also [24]), there exists a unique continuous $\mathcal{F}_t$-adapted solution denoted by $X_n^i(t)$ to Eq.(18). Moreover, if we let $X_n(t) := \sum_{i=1}^n X_n^i(t)e_i$, then we can write Eq.(18) as

$$X_n(t) = \Pi_n X_0 + \int_0^t \Pi_n A(s, X_n(s))\mathrm{d}s + \int_0^t \Pi_n B(s, X_n(s))\tilde{\Pi}_n \mathrm{d}W(s), \tag{19}$$

where $\tilde{\Pi}_n$ is the orthogonal projection from $\mathbb{U}$ to $\mathrm{span}\{f_1, \cdots, f_n\}$.

Noticing that

$$\|\Pi_n B(s, X_n(s))\tilde{\Pi}_n\|_{L_2(\mathbb{U},\mathbb{H})} \leqslant \|B(s, X_n(s))\|_{L_2(\mathbb{U},\mathbb{H})}, \tag{20}$$

and using the same method as in the proof of Theorem 3.5, by **(H4)** and Remark 2.4, we have for all $n \in \mathbb{N}$

$$\begin{aligned} &\mathbb{E}\|X_n(\theta_m)\|^2_{\mathbb{H}} + \|X_n\|^2_{\mathbb{K}_4^m} + \|X_n\|^2_{\mathbb{K}_5^m} + \|B(\cdot, X_n)\|^2_{\mathbb{K}_3^m} \\ &+ \sum_{i=1,2} \left(\|X_n\|^{q_i}_{\mathbb{K}^m_{2,i}} + \|A_i(\cdot, X_n)\|^{\frac{q_i}{q_i-1}}_{\mathbb{K}_i^m}\right) \\ \leqslant\ & c_m \left(\mathbb{E}\|X_0\|^2_{\mathbb{H}} + \int_0^T \mathbb{E}\left(\xi(s) + \eta_1^{\frac{q_1}{q_1-1}}(s) + \eta_2^{\frac{q_2}{q_2-1}}(s)\right)\mathrm{d}s\right) < +\infty, \end{aligned}$$

where $c_m > 0$ is independent of n, and $m \in \mathbb{N}$ is fixed in the next two steps.

(Step 2): By the reflexivities of Banach spaces $\mathbb{K}^m$, one may find a common subsequence n_k (denoted by k for simplicity) and $\tilde{X}^m \in \mathbb{K}^m_{2,1} \cap \mathbb{K}^m_{2,2}$, $\bar{X}^m \in \mathbb{K}_4^m \cap \mathbb{K}_5^m$, $Y_i^m \in \mathbb{K}^m_{1,i}$, $i = 1, 2$, $Z^m \in \mathbb{K}_3^m$ and $X^m_\infty \in L^2(\Omega, \mathcal{F}_{\theta_m}, P; \mathbb{H})$ such that as $k \to \infty$

$$X_k \ \rightarrow\ \tilde{X}^m \text{ weakly in } \mathbb{K}^m_{2,1} \text{ and } \mathbb{K}^m_{2,2}, \tag{21}$$

$$
\begin{aligned}
X_k &\rightarrow \bar{X}^m \text{ weakly in } \mathbb{K}_4^m \text{ and } \mathbb{K}_5^m, && (22)\\
A_i(\cdot, X_k) =: Y_{k,i} &\rightarrow Y_i^m \text{ weakly in } \mathbb{K}_{1,i}^m, \quad i = 1, 2, && (23)\\
B(\cdot, X_k)\tilde{\Pi}_k =: Z_k &\rightarrow Z^m \text{ weakly in } \mathbb{K}_3^m, && (24)\\
X_k(\theta_m) &\rightarrow X_\infty^m \text{ weakly in } L^2(\Omega, \mathcal{F}_{\theta_m}, P; \mathbb{H}). && (25)
\end{aligned}
$$

Clearly, $\tilde{X}^m$, $\bar{X}^m$, Y_1^m, Y_2^m and Z^m are $\mathcal{M}$-measurable. First of all, by (iv) of Lemma 3.2, we have

$$
\tilde{X}^m(t, \omega) = \bar{X}^m(t, \omega), \text{ for } \mu^m\text{-almost all } (t, \omega).
$$

Secondly, define

$$
X^m(t) := X_0 + \int_0^{t\wedge\theta_m} (Y_1^m(s) + Y_2^m(s))\mathrm{d}s + \int_0^{t\wedge\theta_m} Z^m(s)\mathrm{d}W(s), \tag{26}
$$

then

$$
X^m(t, \omega) = \bar{X}^m(t, \omega) \quad \text{ for } \mu^m\text{-almost all } (t, \omega). \tag{27}
$$

Indeed, let $\zeta(t)$ be any $\mathbb{H}$-valued bounded and measurable process on $(\Omega, \mathcal{F}, P)$. By (19) we have for any $k \geqslant n$

$$
\begin{aligned}
\mathbb{E}\left(\int_0^{\theta_m} \langle \Pi_n\zeta(t), X_k(t)\rangle_{\mathbb{H}}\mathrm{d}t\right) &= \mathbb{E}\left(\int_0^{\theta_m} \langle \Pi_n\zeta(t), \Pi_k X_0\rangle_{\mathbb{H}}\mathrm{d}t\right)\\
&\quad + \mathbb{E}\left(\int_0^{\theta_m}\int_0^t [\Pi_n\zeta(t), Y_{k,1}(s) + Y_{k,2}(s)]_{\mathbb{X}}\mathrm{d}s\mathrm{d}t\right)\\
&\quad + \mathbb{E}\left(\int_0^{\theta_m} \langle \Pi_n\zeta(t), J(Z_k)(t)\rangle_{\mathbb{H}}\mathrm{d}t\right),
\end{aligned}
$$

where J is defined by (14), and we have used that

$$
\langle \Pi_n\zeta(t), J(\Pi_k Z_k)(t)\rangle_{\mathbb{H}} = \langle \Pi_n\zeta(t), \Pi_k J(Z_k)(t)\rangle_{\mathbb{H}} = \langle \Pi_n\zeta(t), J(Z_k)(t)\rangle_{\mathbb{H}}.
$$

Taking limits for $k \to \infty$, and by Fubini's theorem, (22) (23) (24) and (iii), (v) of Lemma 3.2 we obtain

$$
\mathbb{E}\left(\int_0^{\theta_m} \langle \Pi_n\zeta(t), \bar{X}^m(t)\rangle_{\mathbb{H}}\mathrm{d}t\right) = \mathbb{E}\left(\int_0^{\theta_m} \langle \Pi_n\zeta(t), X^m(t)\rangle_{\mathbb{H}}\mathrm{d}t\right),
$$

which then shows (27) by the arbitrariness of ζ and n. In the following we shall not distinguish X^m, $\tilde{X}^m$ and $\bar{X}^m$. Moreover, using the same method, by (23) (24) and (25) we also have

$$
X^m(\theta_m(\omega), \omega) = X_\infty^m(\omega) \qquad \text{for } P\text{-almost all } \omega \in \Omega. \tag{28}
$$

(Step 3): Our task in this step is to show by the standard monotone argument that for μ^m-almost all $(t, \omega) \in [0, T] \times \Omega$

$$
\begin{aligned}
(A_1 + A_2)(t, \omega, X^m(t, \omega)) &= (Y_1^m + Y_2^m)(t, \omega) =: Y^m(t, \omega),\\
B(t, \omega, X^m(t, \omega)) &= Z^m(t, \omega).
\end{aligned}
$$

Set

$$
\Lambda(t, \omega) := \exp\Big\{ -\int_0^t \lambda_0(s, \omega)\mathrm{d}s\Big\}.
$$

Then $t \mapsto \Lambda(t)$ is a continuous and $\mathcal{F}_t$-adapted process, and satisfies by (7) and (13)

$$
\Lambda(t \wedge \theta_m) \leqslant \exp\big\{c_1 \cdot m\big\}, \quad t \in [0, T]. \tag{29}
$$

By (19) and Ito's formula we have

$$
\begin{aligned}
&\Lambda(\theta_m)\cdot\|X_k(\theta_m)\|_{\mathbb{H}}^2-\|\Pi_k X_0\|_{\mathbb{H}}^2-2M(\theta_m)\\
=\ &\int_0^{\theta_m}\Lambda(s)\left(2[X_k(s),A(s,X_k(s))]_{\mathbb{X}}+\|\Pi_k B(s,X_k(s))\tilde{\Pi}_k\|_{L_2(\mathbb{U},\mathbb{H})}^2\right)\mathrm{d}s\\
&-\int_0^{\theta_m}\Lambda(s)\cdot\lambda_0(s)\cdot\|X_k(s)\|_{\mathbb{H}}^2\mathrm{d}s,
\end{aligned}
$$

where $t\mapsto M(t)$ is a continuous martingale defined by

$$
M(t):=\int_0^{t\wedge\theta_m}\Lambda(s)\cdot\langle X_k(s),B(s,X_k(s))\tilde{\Pi}_k\mathrm{d}W(s)\rangle_{\mathbb{H}}.
$$

Using (20) and **(H2)**, we further have for any $\Phi\in\mathbb{K}_{2,1}^m\cap\mathbb{K}_{2,2}^m\cap\mathbb{K}_5^m$

$$
\begin{aligned}
&\Lambda(\theta_m)\cdot\|X_k(\theta_m)\|_{\mathbb{H}}^2-\|\Pi_k X_0\|_{\mathbb{H}}^2-2M(\theta_m)\\
\leqslant\ &\int_0^{\theta_m}\Lambda(s)\left(2[X_k(s),A(s,X_k(s))]_{\mathbb{X}}+\|B(s,X_k(s))\|_{L_2(\mathbb{U},\mathbb{H})}^2\right)\mathrm{d}s\\
&-\int_0^{\theta_m}\Lambda(s)\cdot\lambda_0(s)\cdot\|X_k(s)\|_{\mathbb{H}}^2\mathrm{d}s\\
\leqslant\ &\int_0^{\theta_m}\Lambda(s)\Big(2[X_k(s),A(s,\Phi(s))]_{\mathbb{X}}+2[\Phi(s),A(s,X_k(s))-A(s,\Phi(s))]_{\mathbb{X}}\\
&-\|B(s,\Phi(s))\|_{L_2(\mathbb{U},\mathbb{H})}^2+2\langle B(s,X_k(s)),B(s,\Phi(s))\rangle_{L_2(\mathbb{U},\mathbb{H})}\Big)\mathrm{d}s\\
&+\int_0^{\theta_m}\Lambda(s)\cdot\lambda_0(s)\cdot\Big(\|\Phi(s)\|_{\mathbb{H}}^2-2\langle X_k(s),\Phi(s)\rangle_{\mathbb{H}}\Big)\mathrm{d}s, \qquad (30)
\end{aligned}
$$

Since $\Phi\in\mathbb{K}_{2,1}^m\cap\mathbb{K}_{2,2}^m\cap\mathbb{K}_5^m$, we have $B(\cdot,\Phi(\cdot))\in\mathbb{K}_3^m$ by Remark 2.4. Firstly taking expectations for (30), and then taking limits for $k\to\infty$, we find by (21)-(24) and (iii) (iv) of Lemma 3.2 that

$$
\begin{aligned}
&\liminf_{k\to\infty}\mathbb{E}\Big(\Lambda(\theta_m)\cdot\|X_k(\theta_m)\|_{\mathbb{H}}^2\Big)-\mathbb{E}\|X_0\|_{\mathbb{H}}^2\\
\leqslant\ &\mathbb{E}\Bigg(\int_0^{\theta_m}\Lambda(s)\Big(2[X^m(s),A(s,\Phi(s))]_{\mathbb{X}}+2[\Phi(s),Y^m(s)-A(s,\Phi(s))]_{\mathbb{X}}\\
&-\|B(s,\Phi(s))\|_{L_2(\mathbb{U},\mathbb{H})}^2+2\langle Z^m(s),B(s,\Phi(s))\rangle_{L_2(\mathbb{U},\mathbb{H})}\Big)\mathrm{d}s\\
&+\int_0^{\theta_m}\Lambda(s)\cdot\lambda_0(s)\cdot\Big(\|\Phi(s)\|_{\mathbb{H}}^2-2\langle X^m(s),\Phi(s)\rangle_{\mathbb{H}}\Big)\mathrm{d}s\Bigg),
\end{aligned}
$$

where we have used that Λ and λ_0 are bounded on $[0,\theta_m]$.

On the other hand, noting that by (26) and Itô's formula again

$$
\begin{aligned}
&\mathbb{E}\Big(\Lambda(\theta_m)\cdot\|X^m(\theta_m)\|_{\mathbb{H}}^2\Big)-\mathbb{E}\|X_0\|_{\mathbb{H}}^2\\
=\ &\mathbb{E}\left(\int_0^{\theta_m}\Lambda(s)\left(2[X^m(s),Y^m(s)]_{\mathbb{X}}+\|Z^m(s)\|_{L_2(\mathbb{U},\mathbb{H})}^2\right)\mathrm{d}s\right)\\
&-\mathbb{E}\left(\int_0^{\theta_m}\Lambda(s)\cdot\lambda_0(s)\cdot\|X^m(s)\|_{\mathbb{H}}^2\mathrm{d}s\right),
\end{aligned}
$$

and by (25), (28) and (29)

$$\mathbb{E}\Big(\Lambda(\theta_m)\cdot\|X^m(\theta_m)\|_{\mathbb{H}}^2\Big)\leqslant\liminf_{k\to\infty}\mathbb{E}\Big(\Lambda(\theta_m)\cdot\|X_k(\theta_m)\|_{\mathbb{H}}^2\Big),$$

we finally arrive at by combining the above calculations

$$\begin{aligned}
&\mathbb{E}\left(\int_0^{\theta_m}2\Lambda(s)\cdot[X^m(s)-\Phi(s),Y^m(s)-A(s,\Phi(s))]_{\mathbb{X}}\mathrm{d}s\right)\\
&+\mathbb{E}\left(\int_0^{\theta_m}\Lambda(s)\cdot\|B(s,\Phi(s))-Z^m(s)\|_{L_2(\mathbb{U},\mathbb{H})}^2\mathrm{d}s\right)\\
\leqslant\ &\mathbb{E}\left(\int_0^{\theta_m}\Lambda(s)\cdot\lambda_0(s)\cdot\|X^m(s)-\Phi(s)\|_{\mathbb{H}}^2\mathrm{d}s\right).
\end{aligned}$$

Letting $\Phi=X^m$ in the above inequality, we obtain that $Z^m=B(\cdot,X^m)$. By Lemma 2.5 we also have $Y^m=A(\cdot,X^m)$.

(Step 4): For $m\geqslant l$, since $\theta_m(\omega)\geqslant\theta_l(\omega)$ a.s., both $X^m(\cdot,\omega)$ and $X^l(\cdot,\omega)$ solve the following equation

$$\mathrm{d}X(t)=A(t,X(t))1_{\{t\leqslant\theta_l\}}\mathrm{d}t+B(t,X(t))1_{\{t\leqslant\theta_l\}}\mathrm{d}W(s),\quad X(0)=X_0.$$

The uniqueness of solutions gives that for almost all ω

$$X^m(t,\omega)=X^l(t,\omega),\quad t\leqslant\theta_l(\omega).$$

Thus, noting that $\theta_m(\omega)\uparrow T$ a.s. as $m\uparrow\infty$, we may define a continuous $\mathcal{F}_t$-adapted $\mathbb{H}$-valued process for all $t\in(0,T)$ by

$$X(t,\omega):=X^m(t,\omega)\quad\text{ if } t<\theta_m(\omega),$$

Clearly, it is a solution of Eq.(11) in the sense of Definition 3.3. The proof is complete. □

4. Backward Stochastic Evolution Equations

In this section we consider the following type of backward stochastic evolution equation:

$$\left\{\begin{array}{l}\mathrm{d}X(t)=-A(t,X(t))\mathrm{d}t-C(t,X(t),Z(t))\mathrm{d}t+Z(t)\mathrm{d}W(t),\\ X(T)=X_T\in\mathcal{F}_T/\mathcal{B}(\mathbb{H}),\end{array}\right.\tag{31}$$

where

$$\begin{aligned}
A_i:[0,T]\times\Omega\times\mathbb{X}_i\to\mathbb{X}_i^*\ &\in\ \mathcal{M}\times\mathcal{B}(\mathbb{X}_i)/\mathcal{B}(\mathbb{X}_i^*),\quad i=1,2,\\
C:[0,T]\times\Omega\times\mathbb{H}\times L_2(\mathbb{U},\mathbb{H})\to\mathbb{H}\ &\in\ \mathcal{M}\times\mathcal{B}(\mathbb{H})\times\mathcal{B}(L_2(\mathbb{U},\mathbb{H}))/\mathcal{B}(\mathbb{H}).
\end{aligned}$$

We assume that

(HB1) $X_T\in L^2(\Omega,\mathcal{F}_T,P;\mathbb{H})$ and $(A,0)$ satisfies $\mathscr{H}(\lambda_0,\lambda_1,\lambda_2,\lambda_3,\xi,\eta_1,\eta_2,q_1,q_2)$, where λ_i, $i=0,1,2,3$ are positive constants, $q_i\geqslant 2$, $0<\xi\in L^1(\mathfrak{A})$ and

$$\mathbb{E}\left(\int_0^T|\eta_1(s)+\eta_2(s)|^2\mathrm{d}s\right)^{(q_1\vee q_2)/2}<+\infty.$$

(HB2) There exist a $c_1>0$ and an increasing concave function ρ satisfying (4) such that for all $(t,\omega)\in[0,T]\times\Omega$, $x,x'\in\mathbb{H}$ and $z,z'\in L_2(\mathbb{U},\mathbb{H})$

$$\|C(t,\omega,x,z)-C(t,\omega,x',z')\|_{\mathbb{H}}^2\leqslant c_1\left(\rho(\|x-x'\|_{\mathbb{H}}^2)+\|z-z'\|_{L_2(\mathbb{U},\mathbb{H})}^2\right).$$

(HB3) There exist a $c_2>0$ and a $0<\zeta\in L^2(\mathfrak{A})$ such that for all $(t,\omega)\in[0,T]\times\Omega$, $x\in\mathbb{H}$ and $z\in L_2(\mathbb{U},\mathbb{H})$

$$\|C(t,\omega,x,z)\|_{\mathbb{H}}\leqslant\zeta(t,\omega)+c_2\left(\|x\|_{\mathbb{H}}+\|z\|_{L_2(\mathbb{U},\mathbb{H})}\right).$$

Recalling Remark 3.1, we give the following definition.

Definition 4.1. *A pair of measurable $\mathcal{F}_t$-adapted processes (X,Z) is called a solution of Eq.(31) if*

(i) $X\in\mathbb{K}_{2,1}\cap\mathbb{K}_{2,2}\cap\mathbb{K}_4$ and $Z\in\mathbb{K}_3$, $X(0)\in L^2(\Omega,\mathcal{F}_0,P;\mathbb{H})$.

(ii) For almost all ω, $t\mapsto X(t,\omega)$ is continuous in $\mathbb{H}$ and $X(T)=X_T$ a.s..

(iii) (X,Z) satisfies that in $\mathbb{X}^*$, for almost all ω and all $t\in[0,T]$

$$X(t)=X_T+\int_t^T A(s,X(s))\mathrm{d}s+\int_t^T C(s,X(s),Z(s))\mathrm{d}s-\int_t^T Z(s)\mathrm{d}W(s).$$

Let $\gamma(t):=e^{\lambda_0 t/2}$ and make the following transformations

$$\begin{aligned}(\tilde{X}(t,\omega),\tilde{Z}(t,\omega)) &:= (\gamma(t)\cdot X(t,\omega),\gamma(t)\cdot Z(t,\omega)),\\ \tilde{A}_i(t,\omega,x) &:= \gamma(t)\cdot A_i(t,\omega,\gamma^{-1}(t)\cdot x)-\lambda_0\cdot x/2,\quad i=1,2,\\ \tilde{C}(t,\omega,x,z) &:= \gamma(t)\cdot C(t,\omega,\gamma^{-1}(t)\cdot x,\gamma^{-1}(t)\cdot z).\end{aligned}$$

Thus, we can assume $\lambda_0=0$ in **(HB1)** in the following.

We have the following uniqueness result.

Theorem 4.2. *Assume that* **(HB1)**, **(HB2)** *and* **(HB3)** *hold. Let (X,Z) and $(\tilde{X},\tilde{Z})$ be two solutions of Eq.(31) with the same terminal values X_T. Then for $(\mathrm{d}t\times\mathrm{d}P)$-almost all $(t,\omega)\in[0,T]\times\Omega$*

$$X(t,\omega)=\tilde{X}(t,\omega),\quad Z(t,\omega)=\tilde{Z}(t,\omega).$$

Proof. Set $Y(t):=X(t)-\tilde{X}(t)$. By Itô's formula(Theorem 2.1), we have

$$\begin{aligned}&\|Y(t)\|_{\mathbb{H}}^2+\int_t^T\|Z(s)-\tilde{Z}(s)\|^2_{L_2(\mathbb{U},\mathbb{H})}\mathrm{d}s\\ =\ &2\int_t^T[Y(s),A(s,X(s))-A(s,\tilde{X}(s))]_{\mathbb{X}}\mathrm{d}s\\ &+2\int_t^T\langle Y(s),C(s,X(s),Z(s))-C(s,\tilde{X}(s),\tilde{Z}(s))\rangle_{\mathbb{H}}\mathrm{d}s\\ &-2\int_t^T\langle Y(s),(Z(s)-\tilde{Z}(s))\mathrm{d}W(s)\rangle_{\mathbb{H}}.\end{aligned}$$

Taking expectations, by **(H2)**(with $\lambda_0=0$), **(HB2)** and Young's inequality (6) we have

$$\begin{aligned}&\mathbb{E}\|Y(t)\|_{\mathbb{H}}^2+\int_t^T\mathbb{E}\|Z(s)-\tilde{Z}(s)\|^2_{L_2(\mathbb{U},\mathbb{H})}\mathrm{d}s\\ \leqslant\ &c_0\mathbb{E}\left(\int_t^T\|Y(s)\|_{\mathbb{H}}\left(\rho(\|Y(s)\|^2_{\mathbb{H}})+\|Z(s)-\tilde{Z}(s)\|^2_{L_2(\mathbb{U},\mathbb{H})}\right)^{1/2}\mathrm{d}s\right)\\ \leqslant\ &c_0\int_t^T\mathbb{E}\|Y(s)\|^2_{\mathbb{H}}\mathrm{d}s+\frac{1}{2}\int_t^T\mathbb{E}\rho(\|Y(s)\|^2_{\mathbb{H}})\mathrm{d}s\\ &+\frac{1}{2}\int_t^T\mathbb{E}\|Z(s)-\tilde{Z}(s)\|^2_{L_2(\mathbb{U},\mathbb{H})}\mathrm{d}s.\end{aligned}$$

Hence, by Jensen's inequality

$$\mathbb{E}\|Y(t)\|^2_{\mathbb{H}}+\frac{1}{2}\int_t^T\mathbb{E}\|Z(s)-\tilde{Z}(s)\|^2_{L_2(\mathbb{U},\mathbb{H})}\mathrm{d}s$$

$$\leqslant \ c_0\int_t^T \mathbb{E}\|Y(s)\|_{\mathbb{H}}^2\mathrm{d}s+\frac{1}{2}\int_t^T \rho\left(\mathbb{E}\|Y(s)\|_{\mathbb{H}}^2\right)\mathrm{d}s.$$

The uniqueness follows from Lemma 2.2. □

The following finite dimensional result was proved in [1]. For completeness, we give a different proof by Yosida's approximation.

Lemma 4.3. *Assume that $\mathbb{X}=\mathbb{H}=\mathbb{X}^*=\mathbb{U}=\mathbb{R}^d$ and $C=0$, and $(A,0)$ satisfies $\mathscr{H}(0,\lambda_1,0,\lambda_3,\xi,\eta,0,q,0)$, where λ_1,λ_3 are positive constants, $q\geqslant 2$, $0<\xi\in L^1(\mathfrak{A})$ and*

$$\mathbb{E}\left(\int_0^T|\eta(s)|^2\mathrm{d}s\right)^{q/2}<+\infty.$$

Then for any $X_T\in L^q(\Omega,\mathcal{F}_T,P;\mathbb{R}^d)$, there exists a unique solution to Eq.(31) in the sense of Definition 4.1. Moreover,

$$\begin{aligned}&\mathbb{E}\left(\sup_{t\in[0,T]}\|X(t)\|_{\mathbb{R}^d}^q\right)+\mathbb{E}\left(\int_0^T\|Z(s)\|^2_{L_2(\mathbb{R}^d,\mathbb{R}^d)}\mathrm{d}s\right)^{q/2}\\ \leqslant\ & c_0\cdot\mathbb{E}\|X_T\|^q_{\mathbb{R}^d}+c_0\cdot\mathbb{E}\left(\int_0^T|\eta(s)|^2\mathrm{d}s\right)^{q/2},\end{aligned}\tag{32}$$

where c_0 only depends on q,T and λ_1.

Proof. For every $(t,\omega)\in[0,T]\times\Omega$, note that $x\mapsto A(t,\omega,x)$ is a continuous monotone function on $\mathbb{R}^d$. Let $A_\varepsilon(t,\omega,\cdot),\varepsilon>0$ be the Yosida approximation of $A(t,\omega,\cdot)$, i.e.:

$$\begin{aligned}A_\varepsilon(t,\omega,x)&:=\varepsilon^{-1}(J_\varepsilon(t,\omega,x)-x)=A(t,\omega,J_\varepsilon(t,\omega,x)),\\ J_\varepsilon(t,\omega,x)&:=(I-\varepsilon A(t,\omega,\cdot))^{-1}(x),\end{aligned}$$

then $x\mapsto J_\varepsilon(t,\omega,x)$ is a homeomorphism on $\mathbb{R}^d$ for each (t,ω) and for any $x,y\in\mathbb{R}^d$(cf. [2] [5])

(I) $\langle x-y,A_\varepsilon(t,\omega,x)-A_\varepsilon(t,\omega,y)\rangle_{\mathbb{R}^d}\leqslant 0$,
(II) $\|A_\varepsilon(t,\omega,x)-A_\varepsilon(t,\omega,y)\|_{\mathbb{R}^d}\leqslant\varepsilon^{-1}\|x-y\|_{\mathbb{R}^d}$,
(III) $\|A_\varepsilon(t,\omega,x)\|_{\mathbb{R}^d}\leqslant\|A(t,\omega,x)\|_{\mathbb{R}^d}$,
(IV) $\lim_{\varepsilon\downarrow 0}\|A_\varepsilon(t,\omega,x)-A(t,\omega,x)\|_{\mathbb{R}^d}=0$.

By Lemma 2.6, J_ε and A_ε are progressively measurable. From (I), (III) and **(H4)**, we have for any $x\in\mathbb{R}^d$

$$\langle x,A_\varepsilon(t,\omega,x)\rangle_{\mathbb{R}^d}\leqslant\|x\|_{\mathbb{R}^d}\cdot\|A(t,\omega,0)\|_{\mathbb{R}^d}\leqslant\eta(t,\omega)\cdot\lambda_1^{\frac{1}{q}}\cdot\|x\|_{\mathbb{R}^d}.\tag{33}$$

Let $(X_\varepsilon,Z_\varepsilon)$ be the unique $\mathcal{F}_t$-adapted solution of the following backward stochastic differential equation(cf. [23])

$$X_\varepsilon(t)=X_T+\int_t^T A_\varepsilon(s,X_\varepsilon(s))\mathrm{d}s-\int_t^T Z_\varepsilon(s)\mathrm{d}W(s).\tag{34}$$

By Itô's formula, we have

$$\begin{aligned}&\|X_\varepsilon(t)\|^2_{\mathbb{R}^d}+\int_t^T\|Z_\varepsilon(s)\|^2_{L_2(\mathbb{R}^d,\mathbb{R}^d)}\mathrm{d}s\\ =\ &\|X_T\|^2_{\mathbb{R}^d}+2\int_t^T\langle X_\varepsilon(s),A_\varepsilon(s,X_\varepsilon(s))\rangle_{\mathbb{R}^d}\mathrm{d}s\\ &-2\int_t^T\langle X_\varepsilon(s),Z_\varepsilon(s)\mathrm{d}W(s)\rangle_{\mathbb{R}^d},\end{aligned}\tag{35}$$

and further

$$
\begin{aligned}
& e^t\|X_\varepsilon(t)\|_{\mathbb{R}^d}^2+\int_t^T e^s\left(\|X_\varepsilon(s)\|_{\mathbb{R}^d}^2+\|Z_\varepsilon(s)\|_{L_2(\mathbb{R}^d,\mathbb{R}^d)}^2\right)\mathrm{d}s \\
= \;& e^T\|X_T\|_{\mathbb{R}^d}^2+2\int_t^T e^s\langle X_\varepsilon(s),A_\varepsilon(s,X_\varepsilon(s))\rangle_{\mathbb{R}^d}\mathrm{d}s \\
& -2\int_t^T e^s\langle X_\varepsilon(s),Z_\varepsilon(s)\mathrm{d}W(s)\rangle_{\mathbb{R}^d} \\
\leqslant\;& e^T\|X_T\|_{\mathbb{R}^d}^2+\int_t^T e^s\|X_\varepsilon(s)\|_{\mathbb{R}^d}^2\mathrm{d}s+c_0\int_t^T e^s|\eta(s)|^2\mathrm{d}s \\
& -2\int_t^T e^s\langle X_\varepsilon(s),Z_\varepsilon(s)\mathrm{d}W(s)\rangle_{\mathbb{R}^d},
\end{aligned}
\tag{36}
$$

where the second step is due to (33) and Young's inequality (6).

Taking conditional expectations for both sides of (36) with respect to $\mathcal{F}_t$, we find

$$
\begin{aligned}
e^t\|X_\varepsilon(t)\|_{\mathbb{R}^d}^2 &\leqslant e^T\mathbb{E}^{\mathcal{F}_t}\|X_T\|_{\mathbb{R}^d}^2+c_0\cdot\mathbb{E}^{\mathcal{F}_t}\int_t^T e^s|\eta(s)|^2\mathrm{d}s \\
&\leqslant e^T\mathbb{E}^{\mathcal{F}_t}\|X_T\|_{\mathbb{R}^d}^2+c_0\cdot e^T\cdot\mathbb{E}^{\mathcal{F}_t}\int_0^T |\eta(s)|^2\mathrm{d}s.
\end{aligned}
$$

Hence, by Doob's maximal inequality(cf. [28]), we have for $q>2$

$$
\mathbb{E}\left(\sup_{t\in[0,T]}\|X_\varepsilon(t)\|_{\mathbb{R}^d}^q\right)\leqslant c_0\cdot\mathbb{E}\|X_T\|_{\mathbb{R}^d}^q+c_0\cdot\mathbb{E}\left(\int_0^T|\eta(s)|^2\mathrm{d}s\right)^{q/2}. \tag{37}
$$

Hereafter c_0 only depends on q,T and λ_1.

Noting that by BDG's inequality and Young's inequality (6)

$$
\begin{aligned}
& \mathbb{E}\left|\int_0^T e^s\langle X_\varepsilon(s),Z_\varepsilon(s)\mathrm{d}W(s)\rangle_{\mathbb{R}^d}\right|^{q/2} \\
\leqslant\;& c_0\mathbb{E}\left(\int_0^T e^{2s}\|X_\varepsilon(s)\|_{\mathbb{R}^d}^2\cdot\|Z_\varepsilon(s)\|_{L_2(\mathbb{R}^d,\mathbb{R}^d)}^2\mathrm{d}s\right)^{q/4} \\
\leqslant\;& c_0\mathbb{E}\left(\sup_{t\in[0,T]}\|X_\varepsilon(t)\|_{\mathbb{R}^d}^q\right)+\frac{1}{2}\mathbb{E}\left(\int_0^T\|Z_\varepsilon(s)\|_{L_2(\mathbb{R}^d,\mathbb{R}^d)}^2\mathrm{d}s\right)^{q/2},
\end{aligned}
$$

we also have from (36)

$$
\mathbb{E}\left(\int_0^T\|Z_\varepsilon(s)\|_{L_2(\mathbb{R}^d,\mathbb{R}^d)}^2\mathrm{d}s\right)^{q/2}\leqslant c_0\cdot\mathbb{E}\|X_T\|_{\mathbb{R}^d}^q+c_0\cdot\mathbb{E}\left(\int_0^T|\eta(s)|^2\mathrm{d}s\right)^{q/2}. \tag{38}
$$

For $q=2$, from (36) and the above proof, it is easy to see that

$$
\begin{aligned}
& \mathbb{E}\left(\sup_{t\in[0,T]}\|X_\varepsilon(t)\|_{\mathbb{R}^d}^2\right)+\int_0^T\mathbb{E}\|Z_\varepsilon(s)\|_{L_2(\mathbb{R}^d,\mathbb{R}^d)}^2\mathrm{d}s \\
\leqslant\;& c_0\cdot\mathbb{E}\|X_T\|_{\mathbb{R}^d}^2+c_0\cdot\mathbb{E}\int_0^T|\eta(s)|^2\mathrm{d}s.
\end{aligned}
$$

Moreover, by (III), **(H4)** and (37)

$$
\begin{aligned}
\int_0^T \|A_\varepsilon(s, X_\varepsilon(s))\|_{\mathbb{R}^d}^{\frac{q}{q-1}} \mathrm{d}s &\leqslant \int_0^T \|A(s, X_\varepsilon(s))\|_{\mathbb{R}^d}^{\frac{q}{q-1}} \mathrm{d}s \\
&\leqslant c_0 \int_0^T \|X_\varepsilon(s)\|_{\mathbb{R}^d}^q \mathrm{d}s + c_0 \int_0^T \mathbb{E}\left(\eta^{\frac{q}{q-1}}(s)\right) \mathrm{d}s \leqslant c_0'.
\end{aligned}
$$

Therefore, there exists a subsequence $\varepsilon_n \downarrow 0$ and (X, Y, Z, X_0) such that

$$
\begin{aligned}
X_{\varepsilon_n} &\to X \text{ weakly in } \mathbb{K}_{2,1}, \\
A_{\varepsilon_n}(\cdot, X_{\varepsilon_n}(\cdot)) &\to Y \text{ weakly in } \mathbb{K}_{1,1}, \\
Z_{\varepsilon_n} &\to Z \text{ weakly in } \mathbb{K}_3, \\
X_{\varepsilon_n}(0) &\to X_0 \text{ weakly in } L^2(\Omega, \mathcal{F}_0, P; \mathbb{R}^d)
\end{aligned}
$$

as $n \to \infty$. By (37) and (38), we get (32).

Set

$$
\tilde{X}(t) := X_T + \int_t^T Y(s)\mathrm{d}s - \int_t^T Z(s)\mathrm{d}W(s).
$$

By taking weak limits for (34), we deduce that $\tilde{X}(0) = X_0$ a.s. and

$$
X(t, \omega) = \tilde{X}(t, \omega) \text{ for almost all } (t, \omega) \in [0, T] \times \Omega.
$$

It remains to show that $Y(s) = A(s, X(s))$. For any $\Phi \in \mathbb{K}_{2,1}$, by (III) (IV) and the dominated convergence theorem we have

$$
\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left(\int_0^T \langle X_{\varepsilon_n}(s) - \Phi(s), A_{\varepsilon_n}(s, \Phi(s)) - A(s, \Phi(s))\rangle_{\mathbb{R}^d} \mathrm{d}s\right) \\
\leqslant\; &\lim_{n\to\infty} \left(\|A_{\varepsilon_n}(\cdot, \Phi(\cdot)) - A(\cdot, \Phi(\cdot))\|_{\mathbb{K}_{1,1}}^{\frac{q}{q-1}} \cdot \|X_{\varepsilon_n} - \Phi\|_{L^q(\mathfrak{A})}\right) = 0.
\end{aligned} \tag{39}
$$

On the other hand, we have by (35)

$$
\begin{aligned}
&2 \liminf_{n\to\infty} \mathbb{E}\left(\int_0^T \langle X_{\varepsilon_n}(s), A_{\varepsilon_n}(s, X_{\varepsilon_n}(s))\rangle_{\mathbb{R}^d} \mathrm{d}s\right) \\
\geqslant\; &\mathbb{E}\|X_0\|_{\mathbb{R}^d}^2 - \mathbb{E}\|X_T\|_{\mathbb{R}^d}^2 + \int_0^T \mathbb{E}\|Z(s)\|_{L_2(\mathbb{R}^d, \mathbb{R}^d)}^2 \mathrm{d}s \\
=\; &2\mathbb{E}\left(\int_0^T \langle X(s), Y(s)\rangle_{\mathbb{R}^d} \mathrm{d}s\right).
\end{aligned} \tag{40}
$$

Combining (39) and (40), we have by (I)

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^T \langle X(s) - \Phi(s), Y(s) - A(s, \Phi(s)))\rangle_{\mathbb{R}^d} \mathrm{d}s\right) \\
\leqslant\; &\liminf_{n\to\infty} \mathbb{E}\left(\int_0^T \langle X_{\varepsilon_n}(s) - \Phi(s), A_{\varepsilon_n}(s, X_{\varepsilon_n}(s)) - A_{\varepsilon_n}(s, \Phi(s))\rangle_{\mathbb{R}^d} \mathrm{d}s\right) \leqslant 0,
\end{aligned}
$$

which implies that $Y = A(\cdot, X)$ by Lemma 2.5. The proof is complete. □

Remark 4.4. *When $q > 2$, it suffices to require that*

$$
\mathbb{E}\left(\int_0^T |\eta(s)|\mathrm{d}s\right)^q < +\infty.
$$

In fact, taking conditional expectations for both sides of (35) with respect to $\mathcal{F}_t$, and by (33) and Young's inequality (6) we find for any $\delta > 0$

$$\begin{aligned} \|X_\varepsilon(t)\|^2_{\mathbb{R}^d} &\leqslant \mathbb{E}^{\mathcal{F}_t}\|X_T\|^2_{\mathbb{R}^d} + q\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T \|X_\varepsilon(s)\|_{\mathbb{R}^d}\cdot\eta(s)\cdot\lambda_1^{\frac{1}{q}}\mathrm{d}s\right) \\ &\leqslant \mathbb{E}^{\mathcal{F}_t}\|X_T\|^2_{\mathbb{R}^d} + \delta\cdot\mathbb{E}^{\mathcal{F}_t}\left(\sup_{s\in[0,T]}\|X_\varepsilon(s)\|^2_{\mathbb{R}^d}\right) \\ &\quad + c_\delta\cdot\mathbb{E}^{\mathcal{F}_t}\left(\int_0^T|\eta(s)|\mathrm{d}s\right)^2. \end{aligned}$$

Hence, by Doob's maximal inequality we have for $q > 2$

$$\begin{aligned} \mathbb{E}\left(\sup_{t\in[0,T]}\|X_\varepsilon(t)\|^q_{\mathbb{R}^d}\right) &\leqslant c_0\cdot\mathbb{E}\|X_T\|^q_{\mathbb{R}^d} + \delta\cdot c_q\cdot\mathbb{E}\left(\sup_{s\in[0,T]}\|X_\varepsilon(s)\|^q_{\mathbb{R}^d}\right) \\ &\quad + c_\delta\cdot\mathbb{E}\left(\int_0^T|\eta(s)|\mathrm{d}s\right)^q. \end{aligned}$$

Letting δ be sufficiently small, we get

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|X_\varepsilon(t)\|^q_{\mathbb{R}^d}\right) \leqslant c_0\cdot\mathbb{E}\|X_T\|^q_{\mathbb{R}^d} + c_0\cdot\mathbb{E}\left(\int_0^T|\eta(s)|\mathrm{d}s\right)^q,$$

where c_0 only depends on q, λ_1 and T.

We now prove the following infinite dimensional version.

Lemma 4.5. *Assume that $C(\cdot, x, z) = C(\cdot) \in L^2(\mathfrak{A}_a; \mathbb{H})$ is independent of x and z, and* **(HB1)** *holds. Then there exists a unique solution to Eq.(31) in the sense of Definition 4.1.*

Proof. We use Galerkin's approximation to prove the existence as in the proof of Theorem 3.6. For $n \in \mathbb{N}$, let (X_n, Z_n) solve the following finite dimensional backward stochastic differential equation (Lemma 4.3)

$$X_n(t) = X_T^n + \int_t^T \Pi_n A(s, X_n(s))\mathrm{d}s + \int_t^T C_n(s)\mathrm{d}s - \int_t^T Z_n(s)\tilde{\Pi}_n\mathrm{d}W(s),$$

where Π_n and $\tilde{\Pi}_n$ are same as in Theorem 3.6, and

$$\begin{aligned} X_T^n &:= \Pi_n X_T\cdot 1_{\{\|\Pi_n X_T\|_{\mathbb{H}}\leqslant n\}} \\ C_n(s) &:= \Pi_n C(s)\cdot 1_{\{\|\Pi_n C(s)\|_{\mathbb{H}}\leqslant n\}}. \end{aligned}$$

It is easy to see that for each n and s

$$\|X_T^n\|_{\mathbb{H}} \leqslant \|X_T\|_{\mathbb{H}}, \quad \|C_n(s)\|_{\mathbb{H}} \leqslant \|C(s)\|_{\mathbb{H}}$$

and

$$\lim_{n\to\infty}\mathbb{E}\|X_T^n - X_T\|^2_{\mathbb{H}} = 0, \tag{41}$$

$$\lim_{n\to\infty}\int_0^T\mathbb{E}\|C_n(s) - C(s)\|^2_{\mathbb{H}}\mathrm{d}s = 0. \tag{42}$$

By Itô's formula and **(H3)**, we have

$$\mathbb{E}\|X_n(t)\|^2_{\mathbb{H}} + \int_t^T\mathbb{E}\|Z_n(s)\|^2_{L_2(\mathbb{U},\mathbb{H})}\mathrm{d}s \tag{43}$$

$$
\begin{aligned}
&= \mathbb{E}\|X_T^n\|_{\mathbb{H}}^2 + \int_t^T \mathbb{E}[X_n(s), A(s, X_n(s))]_{\mathbb{X}} \mathrm{d}s + 2\int_t^T \mathbb{E}\langle X_n(s), C_n(s)\rangle_{\mathbb{H}} \mathrm{d}s \\
&\leqslant \mathbb{E}\|X_T\|_{\mathbb{H}}^2 + \int_0^T \mathbb{E}(2\xi(s) + \|C(s)\|_{\mathbb{H}}^2)\mathrm{d}s \\
&\quad + \int_t^T \mathbb{E}\left(-\sum_{i=1,2} \lambda_i \cdot \|X_n(s)\|_{\mathbb{X}_i}^{q_i} + (\lambda_3+1)\cdot\|X_n(s)\|_{\mathbb{H}}^2\right)\mathrm{d}s. \qquad (44)
\end{aligned}
$$

By Gronwall's inequality we have

$$
\mathbb{E}\|X_n(t)\|_{\mathbb{H}}^2 \leqslant c_0\left(\mathbb{E}\|X_T\|_{\mathbb{H}}^2 + \int_0^T \mathbb{E}(\xi(s) + \|C(s)\|_{\mathbb{H}}^2)\mathrm{d}s\right).
$$

Hence, from (44) and **(H4)** we get

$$
\mathbb{E}\|X_n(0)\|_{\mathbb{H}}^2 + \|X_n\|_{\mathbb{K}_4}^2 + \|Z_n\|_{\mathbb{K}_3}^2 + \sum_{i=1,2}\left(\|X_n\|_{\mathbb{K}_{2,i}}^{q_i} + \|A_i(\cdot, X_n)\|_{\mathbb{K}_{1,i}}^{\frac{q_i}{q_i-1}}\right) \leqslant c_0.
$$

Hereafter, the constant c_0 is independent of n.

By the reflexivities of Banach spaces $\mathbb{K}$, one may find a subsequence n_k (denoted by k for simplicity) and $\tilde{X} \in \mathbb{K}_{2,1}\cap\mathbb{K}_{2,2}\cap\mathbb{K}_4$, $Y_i \in \mathbb{K}_{1,i}$, $i=1,2$ and $Z\in\mathbb{K}_3$ such that

$$
\begin{aligned}
X_k &\to \tilde{X} \text{ weakly in } \mathbb{K}_{2,1}, \mathbb{K}_{2,2} \text{ and } \mathbb{K}_4, \\
A_i(\cdot, X_k) =: Y_{k,i} &\to Y_i \text{ weakly in } \mathbb{K}_{1,i}, \quad i=1,2, \\
Z_k &\to Z \text{ weakly in } \mathbb{K}_3, \\
X_k(0) &\to X_0 \text{ weakly in } L^2(\Omega, \mathcal{F}_0, P; \mathbb{H}).
\end{aligned}
$$

Define $Y = Y_1 + Y_2 \in \mathbb{Y} \subset \mathbb{X}^*$ and

$$
X(t) := X_T + \int_t^T Y(s)\mathrm{d}s + \int_t^T C(s)\mathrm{d}s - \int_t^T Z(s)\mathrm{d}W(s).
$$

Then, similar to Step 2 of Theorem 3.6 one may prove that

$$
\begin{aligned}
\tilde{X}(t,\omega) &= X(t,\omega) \quad \text{for } (\mathrm{d}t\times\mathrm{d}P)\text{-almost all } (t,\omega), \\
X(0) &= X_0 \quad a.s..
\end{aligned}
$$

We now show that

$$
A(t, X(t,\omega)) = Y(t,\omega) \quad \text{for } (\mathrm{d}t\times\mathrm{d}P)\text{-almost all } (t,\omega). \qquad (45)
$$

By (43) and **(H2)**(with $\lambda_0 = 0$), we have for any $\Phi \in \mathbb{K}_2$

$$
\begin{aligned}
&\mathbb{E}\|X_k(0)\|_{\mathbb{H}}^2 + \int_0^T \mathbb{E}\|Z_k(s)\|_{L_2(\mathbb{U},\mathbb{H})}^2 \mathrm{d}s \\
\leqslant\ & \mathbb{E}\|X_T^k\|_{\mathbb{H}}^2 + 2\int_0^T \mathbb{E}[\Phi(s), A(s, X_k(s)) - A(s,\Phi(s))]_{\mathbb{X}}\mathrm{d}s \\
& +2\int_0^T \mathbb{E}[X_k(s), A(s,\Phi(s))]_{\mathbb{X}}\mathrm{d}s + 2\int_0^T \mathbb{E}\langle X_k(s), C_k(s)\rangle_{\mathbb{H}}\mathrm{d}s.
\end{aligned}
$$

Taking limits for $k\to\infty$, we find by (41) (42)

$$
\begin{aligned}
&\liminf_{k\to\infty} \mathbb{E}\|X_k(0)\|_{\mathbb{H}}^2 + \liminf_{k\to\infty}\int_0^T \mathbb{E}\|Z_k(s)\|_{L_2(\mathbb{U},\mathbb{H})}^2\mathrm{d}s \\
\leqslant\ & \mathbb{E}\|X_T\|_{\mathbb{H}}^2 + 2\int_0^T \mathbb{E}[\Phi(s), Y(s) - A(s,\Phi(s))]_{\mathbb{X}}\mathrm{d}s
\end{aligned}
$$

$$+2\int_0^T \mathbb{E}[X(s), A(s,\Phi(s))]_{\mathbb{X}}\mathrm{d}s + 2\int_0^T \mathbb{E}\langle X(s), C(s)\rangle_{\mathbb{H}}\mathrm{d}s.$$

On the other hand, noting that

$$\begin{aligned}
&\mathbb{E}\|X_0\|_{\mathbb{H}}^2 + \int_0^T \mathbb{E}\|Z(s)\|_{L_2(\mathbb{U},\mathbb{H})}^2\mathrm{d}s \\
= \ &\mathbb{E}\|X_T\|_{\mathbb{H}}^2 + 2\int_0^T \mathbb{E}[X(s), Y(s)]_{\mathbb{X}}\mathrm{d}s + 2\int_0^T \mathbb{E}\langle X(s), C(s)\rangle_{\mathbb{H}}\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\|X_0\|_{\mathbb{H}}^2 &\leqslant \liminf_{k\to\infty} \mathbb{E}\|X_k(0)\|_{\mathbb{H}}^2, \\
\int_0^T \mathbb{E}\|Z(s)\|_{L_2(\mathbb{U},\mathbb{H})}^2\mathrm{d}s &\leqslant \liminf_{k\to\infty} \int_0^T \mathbb{E}\|Z_k(s)\|_{L_2(\mathbb{U},\mathbb{H})}^2\mathrm{d}s,
\end{aligned}$$

we obtain

$$\int_0^T \mathbb{E}[X(s)-\Phi(s), Y(s) - A(s,\Phi(s))]_{\mathbb{X}}\mathrm{d}s \leqslant 0.$$

Hence $Y = A(\cdot, X)$ by Lemma 2.5. The proof is complete. □

Lemma 4.6. *Assume that $C(t,x,z) = C(t,z)$ is independent of x, and* **(HB1)**, **(HB2)** *and* **(HB3)** *hold. Then there exists a unique solution to Eq.(31) in the sense of Definition 4.1.*

Proof. Let $Z_0(t) \equiv 0$. We consider the following Picard iteration: for $n \in \mathbb{N}$, let (X_n, Z_n) solve the following equation(Lemma 4.5):

$$X_n(t) = X_T + \int_t^T A(s, X_n(s))\mathrm{d}s + \int_t^T C(s, Z_{n-1}(s))\mathrm{d}s - \int_t^T Z_n(s)\mathrm{d}W(s).$$

Set $Y_n(t) := X_{n+1}(t) - X_n(t)$. By Itô's formula, **(H2)**(with $\lambda_0 = 0$), **(HB2)** and Young's inequality, we have

$$\begin{aligned}
&\mathbb{E}\|Y_n(t)\|_{\mathbb{H}}^2 + \int_t^T \mathbb{E}\|Z_{n+1}(s) - Z_n(s)\|_{L_2(\mathbb{U},\mathbb{H})}^2\mathrm{d}s \\
= \ &\int_t^T \mathbb{E}[Y_n(s), A(s, X_{n+1}(s)) - A(s, X_n(s))]_{\mathbb{X}}\mathrm{d}s \\
&+ \int_t^T \mathbb{E}\langle Y_n(s), C(s, Z_n(s)) - C(s, Z_{n-1}(s))\rangle_{\mathbb{H}}\mathrm{d}s \\
\leqslant \ &c_0\int_t^T \mathbb{E}\|Y_n(s)\|_{\mathbb{H}}^2\mathrm{d}s + \frac{1}{2}\int_t^T \|Z_n(s) - Z_{n-1}(s)\|_{L_2(\mathbb{U},\mathbb{H})}^2\mathrm{d}s. \qquad (46)
\end{aligned}$$

Hence, for $\alpha := c_0$

$$\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\alpha t}\int_t^T \mathbb{E}\|Y_n(s)\|_{\mathbb{H}}^2\mathrm{d}s\right) + e^{\alpha t}\int_t^T \mathbb{E}\|Z_{n+1}(s) - Z_n(s)\|_{L_2(\mathbb{U},\mathbb{H})}^2\mathrm{d}s \\
\leqslant \ &\frac{e^{\alpha t}}{2}\int_t^T \|Z_n(s) - Z_{n-1}(s)\|_{L_2(\mathbb{U},\mathbb{H})}^2\mathrm{d}s =: \frac{g_n(t)}{2}.
\end{aligned}$$

Integrating both sides from 0 to T yields that

$$\int_0^T \mathbb{E}\|Y_n(s)\|_{\mathbb{H}}^2\mathrm{d}s + \int_0^T g_{n+1}(t)\mathrm{d}t \leqslant \frac{1}{2}\int_0^T g_n(t)\mathrm{d}t \leqslant \frac{1}{2^n}\int_0^T g_1(t)\mathrm{d}t =: \frac{c_0}{2^n}.$$

It then follows from (46) that

$$\int_0^T \mathbb{E}\|Z_{n+1}(s)-Z_n(s)\|^2_{L_2(\mathbb{U},\mathbb{H})}\mathrm{d}s \leqslant \frac{c_0}{2^n}+\frac{1}{2}\int_0^T \mathbb{E}\|Z_n(s)-Z_{n-1}(s)\|^2_{L_2(\mathbb{U},\mathbb{H})}\mathrm{d}s.$$

Iterating this inequality gives

$$\int_0^T \mathbb{E}\|Z_{n+1}(s)-Z_n(s)\|^2_{L_2(\mathbb{U},\mathbb{H})}\mathrm{d}s \leqslant \frac{nc_0}{2^n}.$$

Therefore, there exist an $X\in\mathbb{K}_4$ and a $Z\in\mathbb{K}_3$ such that

$$\lim_{n\to\infty}\|X_n-X\|_{\mathbb{K}_4}=0 \text{ and } \lim_{n\to\infty}\|Z_n-Z\|_{\mathbb{K}_3}=0.$$

From (46) and the above estimates, we also have

$$\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\|X_n(t)\|^2_{\mathbb{H}}<+\infty. \tag{47}$$

We now show that there exists a version $(\tilde{X},\tilde{Z})$ of (X,Z) such that $(\tilde{X},\tilde{Z})$ is a solution to Eq.(31) in the sense of Definition 4.1. In fact, let $(\tilde{X},\tilde{Z})$ solve the following equation(Lemma 4.5):

$$\tilde{X}(t)=X_T+\int_t^T A(s,\tilde{X}(s))\mathrm{d}s+\int_t^T C(s,Z(s))\mathrm{d}s-\int_t^T \tilde{Z}(s)\mathrm{d}W(s).$$

It is similar to estimate (46) that

$$\begin{aligned}
&\mathbb{E}\|X_n(t)-\tilde{X}(t)\|^2_{\mathbb{H}}+\int_t^T \mathbb{E}\|Z_n(s)-\tilde{Z}(s)\|^2_{L_2(\mathbb{U},\mathbb{H})}\mathrm{d}s\\
\leqslant\ & c_0\int_t^T \mathbb{E}\|X_n(s)-\tilde{X}(s)\|^2_{\mathbb{H}}\mathrm{d}s+\frac{1}{2}\int_t^T \|Z_{n-1}(s)-Z(s)\|^2_{L_2(\mathbb{U},\mathbb{H})}\mathrm{d}s.
\end{aligned}$$

Letting $g(t):=\limsup_{n\to\infty}\mathbb{E}\|X_n(t)-\tilde{X}(t)\|^2_{\mathbb{H}}$, by (47) and Fatou's lemma, we have

$$g(t)\leqslant c_0\int_t^T g(s)\mathrm{d}s,$$

which yields that $g(t)=0$ by Gronwall's inequality. The proof is complete. □

We now prove our main result in this section.

Theorem 4.7. *Assume that* **(HB1)**, **(HB2)** *and* **(HB3)** *hold. Then there exists a unique solution to Eq.(31) in the sense of Definition 4.1.*

Proof. Let $X_0(t)\equiv 0$. We consider the following Picard iteration: for $n\in\mathbb{N}$, let (X_n,Z_n) solve the following equation(Lemma 4.6)

$$X_n(t)=X_T+\int_t^T A(s,X_n(s))\mathrm{d}s+\int_t^T C(s,X_{n-1}(s),Z_n(s))\mathrm{d}s-\int_t^T Z_n(s)\mathrm{d}W(s).$$

First of all, by Itô's formula, **(H2)**(with $\lambda_0=0$), **(HB3)** and Young's inequality, we have

$$\begin{aligned}
&\mathbb{E}\|X_n(t)\|^2_{\mathbb{H}}+\int_t^T \mathbb{E}\|Z_n(s)\|^2_{L_2(\mathbb{U},\mathbb{H})}\mathrm{d}s\\
=\ & \mathbb{E}\|X_T\|^2_{\mathbb{H}}+2\int_t^T \mathbb{E}[X_n(s),A(s,X_n(s))]_{\mathbb{X}}\mathrm{d}s\\
&+2\int_t^T \mathbb{E}\langle X_n(s),C(s,X_{n-1}(s),Z_n(s)))\rangle_{\mathbb{H}}\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \mathbb{E}\|X_T\|_{\mathbb{H}}^2+2\int_t^T \mathbb{E}\left(\lambda_3\|X_n(s)\|_{\mathbb{H}}^2+\xi(s)\right)\mathrm{d}s\\
&\quad+2\int_t^T \mathbb{E}\left(\|X_n(s)\|_{\mathbb{H}}\Big(\zeta(s)+c_2(\|X_{n-1}(s)\|_{\mathbb{H}}+\|Z_n(s)\|_{L_2(\mathbb{U},\mathbb{H})})\Big)\right)\mathrm{d}s\\
&\leqslant \mathbb{E}\|X_T\|_{\mathbb{H}}^2+c_0\int_0^T \mathbb{E}\Big(\xi(s)+\zeta^2(s)\Big)\mathrm{d}s+c_0\int_t^T \mathbb{E}\|X_n(s)\|_{\mathbb{H}}^2\mathrm{d}s\\
&\quad+\frac{1}{4}\int_t^T \mathbb{E}\Big(\|X_{n-1}(s)\|_{\mathbb{H}}^2+\|Z_n(s)\|_{L_2(\mathbb{U},\mathbb{H})}^2\Big)\mathrm{d}s.
\end{aligned}
$$

So

$$
\begin{aligned}
&\mathbb{E}\|X_n(t)\|_{\mathbb{H}}^2+\int_t^T \mathbb{E}\|Z_n(s)\|_{L_2(\mathbb{U},\mathbb{H})}^2\mathrm{d}s\\
\leqslant\ & c_0+c_0\int_t^T \left(\mathbb{E}\|X_n(s)\|_{\mathbb{H}}^2+\mathbb{E}\|X_{n-1}(s)\|_{\mathbb{H}}^2\right)\mathrm{d}s, \qquad (48)
\end{aligned}
$$

where c_0 is independent of n.

Set

$$
g_n(t):=\max_{1\leqslant k\leqslant n}\mathbb{E}\|X_k(t)\|_{\mathbb{H}}^2.
$$

Then

$$
g_n(t)\leqslant c_0+c_0\int_t^T g_n(s)\mathrm{d}s,
$$

which gives that by Gronwall's inequality

$$
\max_{k\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\|X_k(t)\|_{\mathbb{H}}^2\leqslant\max_{n\in\mathbb{N}}\sup_{t\in[0,T]}g_n(t)<+\infty. \qquad (49)
$$

Set $Y_{n,m}(t):=X_n(t)-X_m(t)$ and $G_{n,m}(t):=Z_n(s)-Z_m(s)$. By Itô's formula, **(H2)**(with $\lambda_0=0$) and **(HB2)**, we have

$$
\begin{aligned}
&\mathbb{E}\|Y_{n,m}(t)\|_{\mathbb{H}}^2+\int_t^T \mathbb{E}\|G_{n,m}(s)\|_{L_2(\mathbb{U},\mathbb{H})}^2\mathrm{d}s\\
=\ & 2\int_t^T \mathbb{E}[Y_{n,m}(s),A(s,X_n(s))-A(s,X_m(s))]_{\mathbb{X}}\mathrm{d}s\\
&+2\int_t^T \mathbb{E}\langle Y_{n,m}(s),C(s,X_{n-1}(s),Z_n(s))-C(s,X_{m-1}(s),Z_m(s))\rangle_{\mathbb{H}}\mathrm{d}s\\
\leqslant\ & c_0\int_t^T \mathbb{E}\left(\|Y_{n,m}(s)\|_{\mathbb{H}}\Big(\rho(\|Y_{n-1,m-1}(s)\|_{\mathbb{H}}^2)+\|G_{n,m}(s)\|_{L_2(\mathbb{U},\mathbb{H})}^2\Big)^{\frac{1}{2}}\right)\mathrm{d}s.
\end{aligned}
$$

Using the same method as in estimating (48), we have

$$
\begin{aligned}
&\mathbb{E}\|Y_{n,m}(t)\|_{\mathbb{H}}^2+\int_t^T \mathbb{E}\|G_{n,m}(s)\|_{L_2(\mathbb{U},\mathbb{H})}^2\mathrm{d}s\\
\leqslant\ & c_0\int_t^T \mathbb{E}\|Y_{n,m}(s)\|_{\mathbb{H}}^2\mathrm{d}s+c_0\int_t^T \mathbb{E}\rho(\|Y_{n-1,m-1}(s)\|_{\mathbb{H}}^2)\mathrm{d}s.
\end{aligned}
$$

Set

$$
g(t):=\limsup_{n,m\to\infty}\mathbb{E}\|Y_{n,m}(t)\|_{\mathbb{H}}^2.
$$

By (49), Fatou's lemma and Jensen's inequality, we have

$$g(t) \leqslant c_0 \int_t^T (g(s) + \rho(g(s)))\mathrm{d}s.$$

So, by Lemma 2.2

$$g(t) = 0, \quad t \in [0,T].$$

Hence

$$\limsup_{n,m\to\infty} \int_0^T \left(\mathbb{E}\|Y_{n,m}(s)\|^2_{\mathbb{H}} + \mathbb{E}\|G_{n,m}(s)\|^2_{L_2(\mathbb{U},\mathbb{H})}\right) \mathrm{d}s = 0,$$

and there exist an $X \in \mathbb{K}_4$ and a $Z \in \mathbb{K}_3$ such that

$$\lim_{n\to\infty} \|X_n - X\|_{\mathbb{K}_4} = 0 \text{ and } \lim_{n\to\infty} \|Z_n - Z\|_{\mathbb{K}_3} = 0.$$

Using the same method as in the proof of Lemma 4.6, we can show that (X,Z) solves Eq.(31). The proof is thus complete. ☐

Remark 4.8. *In finite dimensional case, under rather weak assumptions on C, the authors* [1] *proved the existence and uniqueness of Eq.(31). It is interesting that the growth of C in x therein can be arbitrary (not necessary polynomial growth). We remark that in our equation, the operator A may contain a polynomial growth part in x. However, it seems difficult to extend A or C to be arbitrary growth in x when we use the cutoff technique as in* [1]*, because A is a non-linear operator and we need to take weak limits in L^p-spaces. On the other hand, if C is polynomial growth in $\mathbb{H}$ with respect to x, it will exclude the interesting case that C is a Nemytskii operator. For example, let $\varphi(r) = -|r|r$, it is not true that $L^2(0,1) \ni x \mapsto \varphi(x) \in L^2(0,1)$, but, $L^4(0,1) \ni x \mapsto \varphi(x) \in L^2(0,1)$.*

5. Stochastic Functional Integral Evolution Equations

Fix $S > 0$. For any $T \geqslant 0$, let $\mathbb{F}^T_S(\mathbb{H})$ denote the space of all continuous functions from $[-S,T]$ to $\mathbb{H}$, which is a separable Banach space under the supremum norm

$$\|f\|_{\mathbb{F}^T_S} := \sup_{s\in[-S,T]} \|f(s)\|_{\mathbb{H}}.$$

For $s \in [-S,0]$, define $\mathcal{F}_s := \mathcal{F}_0$. Suppose that $X : [-S,T] \times \Omega \to \mathbb{H}$ is a continuous $\mathcal{F}_t$-adapted process, we can associate it with another continuous $\mathbb{F}^0_S(\mathbb{H})$-valued and $\mathcal{F}_t$-adapted process as follows:

$$[0,T] \times \Omega \ni (t,\omega) \mapsto X_t(\cdot,\omega) := X(t+\cdot,\omega) \in \mathbb{F}^0_S(\mathbb{H}).$$

In the following, we shall use the following notations:

$$\|X_\cdot(0)\|_{\mathbb{F}^t_0} := \sup_{s\in[0,t]} \|X_s(0)\|_{\mathbb{H}} = \sup_{s\in[0,t]} \|X(s)\|_{\mathbb{H}} =: \|X\|_{\mathbb{F}^t_0}.$$

Consider the following stochastic functional integral evolution equation:

$$\begin{aligned} X_t(0) \;&=\; X_0(0) + \int_0^t A(s,X_s(0))\mathrm{d}s + \int_0^t C_1(s,X_s)\mathrm{d}s + \int_0^t\int_0^s C_2(s,r,X_r)\mathrm{d}r\mathrm{d}s \\ &\quad + \int_0^t D_1(s,X_s)\mathrm{d}W_s + \int_0^t\int_0^s D_2(s,r,X_r)\mathrm{d}W_r\mathrm{d}s, \end{aligned} \tag{50}$$

where X_0 is an $\mathcal{F}_0$-measurable $\mathbb{F}^0_S(\mathbb{H})$-valued random variable and $A = A_1 + A_2$,

$$\begin{aligned} A_i : [0,T]\times\Omega\times\mathbb{X}_i \;&\to\; \mathbb{X}_i^*, \quad i = 1,2, \\ C_1 : [0,T]\times\Omega\times\mathbb{F}^0_S(\mathbb{H}) \;&\to\; \mathbb{H}, \end{aligned}$$

$$\begin{array}{rcl} C_2 : [0,T]\times[0,T]\times\Omega\times\mathbb{F}_S^0(\mathbb{H}) & \to & \mathbb{H}, \\ D_1 : [0,T]\times\Omega\times\mathbb{F}_S^0(\mathbb{H}) & \to & L_2(\mathbb{U},\mathbb{H}), \\ D_2 : [0,T]\times[0,T]\times\Omega\times\mathbb{F}_S^0(\mathbb{H}) & \to & L_2(\mathbb{U},\mathbb{H}) \end{array}$$

are progressively measurable, for example, for every $0 \leqslant t \leqslant T$, the mapping $(s,\omega,x) \mapsto D_2(t,s,\omega,x)$ is $\mathcal{M}\times\mathcal{B}(\mathbb{F}_S^0(\mathbb{H}))/\mathcal{B}(L_2(\mathbb{U},\mathbb{H}))$-measurable.

We make the following assumptions:

(HF1) $X_0 \in L^2(\Omega,\mathcal{F}_0,P;\mathbb{F}_S^0(\mathbb{H}))$ and $(A,0)$ satisfies $\mathscr{H}(\lambda_0,\lambda_1,\lambda_2,\lambda_3,\xi,\eta_1,\eta_2,q_1,q_2)$, where $\lambda_i \in L^1(0,T)$, $i=0,1,2,3$ are non-random strictly positive functions, $q_i \geqslant 2$, and $0 \leqslant \eta_i \in L^{\frac{q_i}{q_i-1}}(\mathfrak{A})$, $i=1,2$, and $0\leqslant \xi \in L^1(\mathfrak{A})$.

(HF2) There exist a positive real function $\lambda_3 \in L^1(0,T)$ and an increasing concave function ρ satisfying (4) such that for all $(s,\omega)\in[0,T]\times\Omega$ and $x,y\in\mathbb{F}_S^0$

$$\begin{array}{rcl} \|C_1(s,\omega,x)-C_1(s,\omega,y)\|_{\mathbb{H}}^2 & \leqslant & \lambda_3(s)\cdot\rho(\|x-y\|_{\mathbb{F}_S^0}^2), \\ \|D_1(s,\omega,x)-D_1(s,\omega,y)\|_{L_2(\mathbb{U},\mathbb{H})}^2 & \leqslant & \lambda_3(s)\cdot\rho(\|x-y\|_{\mathbb{F}_S^0}^2). \end{array}$$

(HF3) There exists a positive real function λ_5 satisfying $t \mapsto \int_0^t \lambda_5(t,s)\mathrm{d}s \in L^1(0,T)$ such that for all $t,s\in[0,T]$, $\omega\in\Omega$ and $x,y\in\mathbb{F}_S^0$

$$\begin{array}{rcl} \|C_2(t,s,\omega,x)-C_2(t,s,\omega,y)\|_{\mathbb{H}}^2 & \leqslant & \lambda_5(t,s)\cdot\rho(\|x-y\|_{\mathbb{F}_S^0}^2), \\ \|D_2(t,s,\omega,x)-D_2(t,s,\omega,y)\|_{L_2(\mathbb{U},\mathbb{H})}^2 & \leqslant & \lambda_5(t,s)\cdot\rho(\|x-y\|_{\mathbb{F}_S^0}^2), \end{array}$$

where ρ is same as in **(HF2)**.

(HF4) There exist a positive progressively measurable process λ_6 and a positive real function λ_7 satisfying $\int_0^t [\lambda_7(t,s)+\mathbb{E}\lambda_6(t,s)]\,\mathrm{d}s \leqslant c_0\lambda_1^{2/q_1}(t)$, and $0\leqslant\zeta\in L^1(\mathfrak{A})$ such that for all $t,s\in[0,T]$, $\omega\in\Omega$ and $x\in\mathbb{F}_S^0$

$$\begin{array}{rcl} \|C_1(s,\omega,x)\|_{\mathbb{H}}^2+\|D_1(s,\omega,x)\|_{L_2(\mathbb{U},\mathbb{H})}^2 & \leqslant & c_0\lambda_1^{2/q_1}(s)\cdot(\zeta(s,\omega)+\|x\|_{\mathbb{F}_S^0}^2), \\ \|C_2(t,s,\omega,x)\|_{\mathbb{H}}^2+\|D_2(t,s,\omega,x)\|_{L_2(\mathbb{U},\mathbb{H})}^2 & \leqslant & \lambda_6(t,s,\omega)+\lambda_7(t,s)\cdot\|x\|_{\mathbb{F}_S^0}^2, \end{array}$$

where λ_1 and q_1 are same as in **(HF1)**.

Definition 5.1. *An $\mathbb{H}$-valued continuous $\mathcal{F}_t$-adapted process X on $[-S,T]$ is called a solution of Eq.(50), if*

$$\mathbb{E}\left(\|X\|_{\mathbb{F}_S^T}^2\right)+\|X_\cdot(0)\|_{\mathbb{K}_{2,1}}^{q_1}+\|X_\cdot(0)\|_{\mathbb{K}_{2,2}}^{q_2}<+\infty,$$

and (50) holds in $\mathbb{X}^*$ for all $t\in[0,T]$ a.s..

Theorem 5.2. *Under* **(HF1)-(HF4)***, there exists a unique solution to Eq.(50) in the sense of Definition 5.1.*

Proof. Let $X_t^1 \equiv X_0(\cdot)$. One constructs the following iteration sequence X_t^n for $n\in\mathbb{N}$:

$$\begin{array}{rcl} X_t^{n+1}(0) & = & X_0(0)+\displaystyle\int_0^t A(s,X_s^{n+1}(0))\mathrm{d}s \\ & & +\displaystyle\int_0^t C_1(s,X_s^n)\mathrm{d}s+\int_0^t\int_0^s C_2(s,r,X_r^n)\mathrm{d}r\mathrm{d}s \\ & & +\displaystyle\int_0^t D_1(s,X_s^n)\mathrm{d}W(s)+\int_0^t\int_0^s D_2(s,r,X_r^n)\mathrm{d}W_r\mathrm{d}s \\ & =: & X_0(0)+\displaystyle\int_0^t A^n(s,X_s^{n+1}(0))\mathrm{d}s+\int_0^t B^n(s)\mathrm{d}W(s), \end{array}$$

where

$$\begin{aligned} A^n(s,x) &:= A_1^n(s,x) + A_2(s,x) \in \mathbb{Y} \subset \mathbb{X}^*, \\ A_1^n(s,x) &:= A_1(s,x) + G^n(s) \in \mathbb{X}_1^*, \\ B^n(s) &:= D_1(s, X_s^n) \in L_2(\mathbb{U}, \mathbb{H}) \end{aligned}$$

and

$$G^n(s) := C_1(s, X_s^n) + \int_0^s C_2(s,r,X_r^n)\mathrm{d}r + \int_0^s D_2(s,r,X_r^n)\mathrm{d}W_r.$$

First of all, we clearly have for any $x, y \in \mathbb{X}$ and $s \in [0,T]$

$$2[x-y, A^n(s,x) - A^n(s,y)]_{\mathbb{X}} \leqslant \lambda_0(s) \cdot \|x-y\|_{\mathbb{H}}^2.$$

Secondly, by **(H3)** we have for any $x \in \mathbb{X}$ and $s \in [0,T]$

$$\begin{aligned} & 2[x, A^n(s,x)]_{\mathbb{X}} + \|B^n(s)\|_{L_2(\mathbb{U},\mathbb{H})}^2 \\ = \; & 2[x, A(s,x)]_{\mathbb{X}} + 2\langle x, G^n(s)\rangle_{\mathbb{H}} + \|D_1(s,X_s^n)\|_{L_2(\mathbb{U},\mathbb{H})}^2 \\ \leqslant \; & -\sum_{i=1,2} \left(\lambda_i(s) \cdot \|x\|_{\mathbb{X}_i}^{q_i}\right) + (\lambda_3(s)+1) \cdot \|x\|_{\mathbb{H}}^2 + \xi(s) \\ & + \|G^n(s)\|_{\mathbb{H}}^2 + \|D_1(s,X_s^n)\|_{L_2(\mathbb{U},\mathbb{H})}^2. \end{aligned}$$

Moreover, by the embedding $\mathbb{H} \subset \mathbb{X}_1^*$ and **(H4)** we have for any $x \in \mathbb{X}$ and $s \in [0,T]$

$$\begin{aligned} \|A_1^n(s,x)\|_{\mathbb{X}_1^*} &\leqslant \|A_1(s,x)\|_{\mathbb{X}_1^*} + \|G^n(s)\|_{\mathbb{X}_1^*} \\ &\leqslant \eta_1(s) \cdot \lambda_1^{1/q_1}(s) + c_{A_1} \cdot \lambda_1(s,\omega) \cdot \|x\|_{\mathbb{X}_1}^{q_1-1} + c_0\|G^n(s)\|_{\mathbb{H}}. \end{aligned}$$

Hence, (A^n, B^n) satisfies $\mathscr{H}(\lambda_0, \lambda_1, \lambda_2, \lambda_3+1, \xi^n, \eta_1^n, \eta_2, q_1, q_2)$, where

$$\begin{aligned} \xi^n(s) &:= \xi(s) + \|D_1(s,X_s^n)\|_{L_2(\mathbb{U},\mathbb{H})}^2 + \|G^n(s)\|_{\mathbb{H}}^2, \\ \eta_1^n(s) &:= \eta_1(s) + c_0 \cdot \lambda_1^{-1/q_1}(s) \cdot \|G^n(s)\|_{\mathbb{H}}. \end{aligned}$$

Noting that by **(HF4)**

$$\begin{aligned} \mathbb{E}\|C_1(s,X_s^n)\|_{\mathbb{H}}^2 + \mathbb{E}\|D_1(s,X_s^n)\|_{L_2(\mathbb{U},\mathbb{H})}^2 &\leqslant c_0\lambda_1^{2/q_1}(s)\left(\mathbb{E}\zeta(s) + \mathbb{E}\|X_s^n\|_{\mathbb{F}_S^0}^2\right), \\ \mathbb{E}\left\|\int_0^s C_2(s,r,X_r^n)\mathrm{d}r\right\|_{\mathbb{H}}^2 &\leqslant c_0\lambda_1^{2/q_1}(s)\left(1 + \mathbb{E}\|X^n\|_{\mathbb{F}_S^s}^2\right), \\ \mathbb{E}\left\|\int_0^s D_2(s,r,X_r^n)\mathrm{d}W_r\right\|_{\mathbb{H}}^2 &\leqslant c_0\lambda_1^{2/q_1}(s)\left(1 + \mathbb{E}\|X^n\|_{\mathbb{F}_S^s}^2\right), \end{aligned}$$

we have by $q_1 \geqslant 2$ and Young's inequality (6)

$$\begin{aligned} \mathbb{E}\left(\|G^n(s)\|_{\mathbb{H}}^{\frac{q_1}{q_1-1}}\right) &\leqslant \left(\mathbb{E}\|G^n(s)\|_{\mathbb{H}}^2\right)^{\frac{q_1}{2(q_1-1)}} \\ &\leqslant c_0 \cdot \lambda_1^{\frac{1}{q_1-1}}(s) \cdot \left(1 + \mathbb{E}\|X^n\|_{\mathbb{F}_S^s}^2\right)^{\frac{q_1}{2(q_1-1)}} \\ &\leqslant c_0 \cdot \lambda_1^{\frac{1}{q_1-1}}(s) \cdot \left(1 + \mathbb{E}\|X^n\|_{\mathbb{F}_S^s}^2\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^t \mathbb{E}\xi^n(s)\mathrm{d}s &\leqslant c_0 + c_0\int_0^t (\lambda_1(s)+1) \cdot \mathbb{E}\|X_s^n\|_{\mathbb{F}_S^0}^2 \mathrm{d}s, \\ \int_0^t \mathbb{E}\left(|\eta_1^n(s)|^{\frac{q_1}{q_1-1}}\right)\mathrm{d}s &\leqslant c_0 + c_0\int_0^t \mathbb{E}\|X^n\|_{\mathbb{F}_S^s}^2 \mathrm{d}s. \end{aligned}$$

Thus, by Theorem 3.5 we have

$$\begin{aligned}\mathbb{E}\left(\|X^{n+1}_\cdot(0)\|^2_{\mathbb{F}^t_0}\right) &\leqslant c_0\left(\mathbb{E}\|X_0(0)\|^2_{\mathbb{H}}+\int_0^t\mathbb{E}\left(\xi^n(s)+|\eta^n_1(s)|^{\frac{q_1}{q_1-1}}+|\eta_2(s)|^{\frac{q_2}{q_2-1}}\right)\mathrm{d}s\right)\\ &\leqslant c_0+c_0\int_0^t(\lambda_1(s)+1)\mathbb{E}\|X^n\|^2_{\mathbb{F}^s_S}\mathrm{d}s, \end{aligned}\tag{51}$$

where c_0 is independent of n.

By induction and Theorem 3.6, $\{X^n_t, n\in\mathbb{N}\}$ are thus well defined. Moreover, by Theorem 3.5 we have

$$\|X^n_\cdot(0)\|^{q_1}_{\mathbb{K}_{2,1}}+\|X^n_\cdot(0)\|^{q_2}_{\mathbb{K}_{2,2}}\leqslant c_n<+\infty.$$

Setting

$$g_n(t):=\sup_{k\leqslant n+1}\mathbb{E}\left(\|X^k\|^2_{\mathbb{F}^t_S}\right),$$

we then have by (51)

$$\begin{aligned}g_n(t) &\leqslant 2\sup_{k\leqslant n+1}\mathbb{E}\left(\|X^k_\cdot(0)\|^2_{\mathbb{F}^t_0}\right)+2\mathbb{E}\|X_0\|^2_{\mathbb{F}^0_S}\\ &\leqslant c_0+c_0\int_0^t(\lambda_1(s)+1)g_n(s)\mathrm{d}s.\end{aligned}$$

Applying Gronwall's inequality yields

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left(\sup_{t\in[0,T]}\|X^n_t\|^2_{\mathbb{F}^0_S}\right)=\sup_{n\in\mathbb{N}}\mathbb{E}\left(\|X^n\|^2_{\mathbb{F}^T_S}\right)<+\infty.\tag{52}$$

Next, set $Z^{n,m}_t:=X^n_t-X^m_t$. By Itô's formula, **(H2)** and **(HF2)**, we have

$$\begin{aligned}\|Z^{n+1,m+1}_t(0)\|^2_{\mathbb{H}} &= 2\int_0^t[Z^{n+1,m+1}_s(0),A^n(s,X^{n+1}_s(0))-A^m(s,X^{m+1}_s(0))]_{\mathbb{X}}\mathrm{d}s\\ &\quad+2\int_0^t\langle Z^{n+1,m+1}_s(0),(B^n(s)-B^m(s))\mathrm{d}W(s)\rangle_{\mathbb{H}}\\ &\quad+\int_0^t\|B^n(s)-B^m(s)\|^2_{L_2(\mathbb{U},\mathbb{H})}\mathrm{d}s\\ &= 2\int_0^t[Z^{n+1,m+1}_s(0),A(s,X^{n+1}_s(0))-A(s,X^{m+1}_s(0))]_{\mathbb{X}}\mathrm{d}s\\ &\quad+\int_0^t\|D_1(s,X^n_s)-D_1(s,X^m_s)\|^2_{L_2(\mathbb{U},\mathbb{H})}\mathrm{d}s+\sum_{i=1}^4 I^i_t\\ &\leqslant \int_0^t\lambda_0(s)\cdot\|Z^{n+1,m+1}_s(0)\|^2_{\mathbb{H}}\mathrm{d}s\\ &\quad+\int_0^t\lambda_3(s)\cdot\rho(\|Z^{n,m}_s\|^2_{\mathbb{F}^0_S})\mathrm{d}s+\sum_{i=1}^4 I^i_t,\end{aligned}$$

where

$$\begin{aligned}I^1_t &:= 2\int_0^t\langle Z^{n+1,m+1}_s(0),C_1(s,X^n_s)-C_1(s,X^m_s)\rangle_{\mathbb{H}}\mathrm{d}s\\ I^2_t &:= 2\int_0^t\langle Z^{n+1,m+1}_s(0),\int_0^s C_2(s,r,X^n_r)-C_2(s,r,X^m_r)\mathrm{d}r\rangle_{\mathbb{H}}\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
I_t^3 &:= 2\int_0^t \langle Z_s^{n+1,m+1}(0), (D_1(s,X_s^n)-D_1(s,X_s^m))\mathrm{d}W(s)\rangle_{\mathbb{H}} \\
I_t^4 &:= 2\int_0^t \langle Z_s^{n+1,m+1}(0), \int_0^s (D_2(s,r,X_r^n)-D_2(s,r,X_r^m))\mathrm{d}W_r\rangle_{\mathbb{H}}\mathrm{d}s.
\end{aligned}
$$

Using Burkholder's inequality and Young's inequality (6), we have by **(HF2)**

$$
\begin{aligned}
\mathbb{E}\left(\|I_\cdot^3\|_{\mathbb{F}_0^t}\right) &\leqslant c_0\mathbb{E}\left(\int_0^t \|Z_s^{n+1,m+1}(0)\|_{\mathbb{H}}^2\cdot\|D_1(s,X_s^n)-D_1(s,X_s^m)\|_{L_2(\mathbb{U},\mathbb{H})}^2\mathrm{d}s\right)^{1/2} \\
&\leqslant \frac{1}{2}\mathbb{E}\|Z_\cdot^{n+1,m+1}(0)\|_{\mathbb{F}_0^t}^2 + c_0\int_0^t \lambda_3(s)\cdot\mathbb{E}\rho(\|Z_s^{n,m}\|_{\mathbb{F}_S^0}^2)\mathrm{d}s
\end{aligned}
$$

and by **(HF3)**

$$
\begin{aligned}
\mathbb{E}\left(\|I_\cdot^4\|_{\mathbb{F}_0^t}\right) &\leqslant \int_0^t \mathbb{E}\|Z_s^{n+1,m+1}(0)\|_{\mathbb{H}}^2\mathrm{d}s + \int_0^t\int_0^s \lambda_5(s,r)\cdot\mathbb{E}\rho(\|Z_r^{n,m}\|_{\mathbb{F}_S^0}^2)\mathrm{d}r\mathrm{d}s \\
&\leqslant \int_0^t \mathbb{E}\|Z_\cdot^{n+1,m+1}(0)\|_{\mathbb{F}_0^s}^2\mathrm{d}s + \int_0^t \mathbb{E}\rho(\|Z^{n,m}\|_{\mathbb{F}_0^s}^2)\left(\int_0^s \lambda_5(s,r)\mathrm{d}r\right)\mathrm{d}s,
\end{aligned}
$$

where we have used that ρ is increasing and

$$
\sup_{r\in[0,s]}\|Z_r^{n,m}\|_{\mathbb{F}_S^0}^2 = \|Z^{n,m}\|_{\mathbb{F}_0^s}^2.
$$

Similarly,

$$
\begin{aligned}
\mathbb{E}\left(\|I_\cdot^1\|_{\mathbb{F}_0^t}\right) &\leqslant \int_0^t \mathbb{E}\|Z_\cdot^{n+1,m+1}(0)\|_{\mathbb{F}_0^s}^2\mathrm{d}s + \int_0^t \lambda_3(s)\cdot\mathbb{E}\rho(\|Z^{n,m}\|_{\mathbb{F}_0^s}^2)\mathrm{d}s, \\
\mathbb{E}\left(\|I_\cdot^2\|_{\mathbb{F}_0^t}\right) &\leqslant \int_0^t \mathbb{E}\|Z_\cdot^{n+1,m+1}(0)\|_{\mathbb{F}_0^s}^2\mathrm{d}s + \int_0^t \mathbb{E}\rho(\|Z^{n,m}\|_{\mathbb{F}_0^s}^2)\left(\int_0^s \lambda_5(s,r)\mathrm{d}r\right)\mathrm{d}s.
\end{aligned}
$$

Combining the above calculations, we obtain

$$
\begin{aligned}
\mathbb{E}\|Z_\cdot^{n+1,m+1}(0)\|_{\mathbb{F}_0^t}^2 &\leqslant 2\int_0^t (\lambda_0(s)+3)\cdot\mathbb{E}\|Z_\cdot^{n+1,m+1}(0)\|_{\mathbb{F}_0^s}^2\mathrm{d}s \\
&\quad + c_0\int_0^t \lambda_8(s)\cdot\mathbb{E}\left(\rho(\|Z^{n,m}\|_{\mathbb{F}_0^s}^2)\right)\mathrm{d}s,
\end{aligned}
$$

where $\lambda_8(s) := \lambda_3(s) + \int_0^s \lambda_5(s,r)\mathrm{d}r$.

By Gronwall's inequality and Jensen's inequality, we have

$$
\mathbb{E}\|Z_\cdot^{n+1,m+1}(0)\|_{\mathbb{F}_0^t}^2 \leqslant c_0\int_0^t \lambda_8(s)\cdot\rho(\mathbb{E}(\|Z^{n,m}\|_{\mathbb{F}_0^s}^2))\mathrm{d}s. \tag{53}
$$

Now setting

$$
g(t) := \limsup_{n,m\to\infty}\mathbb{E}\|Z_\cdot^{n+1,m+1}(0)\|_{\mathbb{F}_0^t}^2 = \limsup_{n,m\to\infty}\mathbb{E}\|Z^{n+1,m+1}\|_{\mathbb{F}_0^t}^2,
$$

we then get by (52) and Fatou's lemma

$$
g(t) \leqslant c_0\int_0^t \lambda_8(s)\cdot\rho(g(s))\mathrm{d}s.
$$

Using Lemma 2.2 yields that

$$
g(t) = 0.
$$

Therefore, there is an $\mathbb{H}$-valued continuous adapted process X such that

$$
\lim_{n\to\infty}\mathbb{E}\|X^n - X\|_{\mathbb{F}_0^T}^2 = 0.
$$

It remains to show that X_t is a solution of Eq.(50) in the sense of Definition 5.1. Let $\tilde{X}(t)$ solve the following equation(Theorem 3.6)

$$\begin{aligned}\tilde{X}(t) &= X_0 + \int_0^t A(s,\tilde{X}(s))\mathrm{d}s + \int_0^t C_1(s,X_s)\mathrm{d}s + \int_0^t\int_0^s C_2(s,r,X_r)\mathrm{d}r\mathrm{d}s \\ &\quad + \int_0^t D_1(s,s,X_s)\mathrm{d}W(s) + \int_0^t\int_0^s D_2(s,r,X_r)\mathrm{d}W_r\mathrm{d}s.\end{aligned}$$

As in estimating (53), we can prove that

$$\mathbb{E}\|X^{n+1}_\cdot(0) - \tilde{X}(\cdot)\|^2_{\mathbb{F}^t_0} \leqslant c_0 \int_0^t \lambda_8(s)\cdot\rho(\mathbb{E}\|X^n_\cdot - X_\cdot(0)\|^2_{\mathbb{F}^s_0})\mathrm{d}s.$$

Taking limits and by the dominated convergence theorem, we get

$$\lim_{n\to\infty}\mathbb{E}\|X^{n+1}_\cdot(0) - \tilde{X}(\cdot)\|^2_{\mathbb{F}^t_0} = 0.$$

So, for each $t\in[0,T]$

$$X(t) = \tilde{X}(t), \quad a.s..$$

Moreover, by Theorems 3.6 and 3.5, for almost all ω, $t\mapsto\tilde{X}(t,\omega)$ is continuous in $\mathbb{H}$, and

$$\|\tilde{X}_\cdot(0)\|^{q_1}_{\mathbb{K}_{2,1}} + \|\tilde{X}_\cdot(0)\|^{q_2}_{\mathbb{K}_{2,2}} < +\infty.$$

The uniqueness follows from the similar calculations, and the proof is thus complete. □

We now consider the following stochastic Volterra evolution equation:

$$X_t(0) = X_0(0) + \int_0^t A(s,X_s)\mathrm{d}s + \int_0^t C(t,s,X_s)\mathrm{d}s + \int_0^t D(t,s,X_s)\mathrm{d}W_s, \tag{54}$$

where X_0 is an $\mathcal{F}_0$-measurable $\mathbb{F}^0_S(\mathbb{H})$-valued random variable and $A = A_1 + A_2$,

$$\begin{aligned}A_i : [0,T]\times\Omega\times\mathbb{X}_i &\to \mathbb{X}_i^*, \quad i=1,2,\\ C : [0,T]\times[0,T]\times\Omega\times\mathbb{F}^0_S(\mathbb{H}) &\to \mathbb{H},\\ D : [0,T]\times[0,T]\times\Omega\times\mathbb{F}^0_S(\mathbb{H}) &\to L_2(\mathbb{U},\mathbb{H})\end{aligned}$$

are progressively measurable.

We make the following assumptions:

(HV1) $X_0 \in L^2(\Omega,\mathcal{F}_0,P;\mathbb{F}^0_S(\mathbb{H}))$ and $(A,0)$ satisfies $\mathscr{H}(\lambda_0,\lambda_1,\lambda_2,\lambda_3,\xi,\eta_1,\eta_2,q_1,q_2)$, where $\lambda_i \in L^1(0,T)$, $i=0,1,2,3$ are non-random strictly positive functions, $q_i\geqslant 2$, and $0\leqslant\eta_i\in L^{\frac{q_i}{q_i-1}}(\mathfrak{A})$, $i=1,2$, and $0\leqslant\xi\in L^1(\mathfrak{A})$.

(HV2) $C(\cdot,s,\omega,x)$ and $D(\cdot,s,\omega,x)$ are differentiable with respect to the first variable for all s,ω,x, and there exist a positive real function λ_5 satisfying $t\mapsto\int_0^t\lambda_5(t,s)\mathrm{d}s\in L^1(0,T)$ and an increasing concave function ρ satisfying (4) such that for all $t,s\in[0,T]$, $\omega\in\Omega$ and $x,y\in\mathbb{F}^0_S$

$$\begin{aligned}\|\partial_t C(t,s,\omega,x) - \partial_t C(t,s,\omega,y)\|^2_{\mathbb{H}} &\leqslant \lambda_5(t,s)\cdot\rho(\|x-y\|^2_{\mathbb{F}^0_S}),\\ \|\partial_t D(t,s,\omega,x) - \partial_t D(t,s,\omega,y)\|^2_{L_2(\mathbb{U},\mathbb{H})} &\leqslant \lambda_5(t,s)\cdot\rho(\|x-y\|^2_{\mathbb{F}^0_S}).\end{aligned}$$

(HV3) There exist a positive real function $\lambda_3\in L^1(0,T)$ such that for all $(s,\omega)\in[0,T]\times\Omega$ and $x,y\in\mathbb{F}^0_S$

$$\begin{aligned}\|C(s,s,\omega,x) - C(s,s,\omega,y)\|^2_{\mathbb{H}} &\leqslant \lambda_3(s)\cdot\rho(\|x-y\|^2_{\mathbb{F}^0_S}),\\ \|D(s,s,\omega,x) - D(s,s,\omega,y)\|^2_{L_2(\mathbb{U},\mathbb{H})} &\leqslant \lambda_3(s)\cdot\rho(\|x-y\|^2_{\mathbb{F}^0_S}),\end{aligned}$$

where ρ is same as in **(HV2)**.

(HV4) There exist a positive progressively measurable process λ_6 and a positive real function λ_7 satisfying $\int_0^t [\lambda_7(t,s) + \mathbb{E}\lambda_6(t,s)]\,\mathrm{d}s \leqslant c_0\lambda_1^{2/q_1}(t)$, and $0 \leqslant \zeta \in L^1(\mathfrak{A})$ such that for all $t, s \in [0,T]$, $\omega \in \Omega$ and $x \in \mathbb{F}_S^0$

$$\begin{aligned}\|C(s,s,\omega,x)\|_{\mathbb{H}}^2 + \|D(s,s,\omega,x)\|_{L_2(\mathbb{U},\mathbb{H})}^2 &\leqslant c_0\lambda_1^{2/q_1}(s)\cdot(\zeta(s,\omega) + \|x\|_{\mathbb{F}_S^0}^2),\\ \|\partial_t C(t,s,\omega,x)\|_{\mathbb{H}}^2 + \|\partial_t D(t,s,\omega,x)\|_{L_2(\mathbb{U},\mathbb{H})}^2 &\leqslant \lambda_6(t,s,\omega) + \lambda_7(t,s)\cdot\|x\|_{\mathbb{F}_S^0}^2,\end{aligned}$$

where λ_1 and q_1 are same as in **(HV1)**.

Definition 5.3. *An $\mathbb{H}$-valued continuous $\mathcal{F}_t$-adapted process X on $[-S,T]$ is called a solution of Eq.(54), if*

$$\mathbb{E}\left(\|X\|_{\mathbb{F}_S^T}^2\right) + \|X_\cdot(0)\|_{\mathbb{K}_{2,1}}^{q_1} + \|X_\cdot(0)\|_{\mathbb{K}_{2,2}}^{q_2} < +\infty,$$

and (54) holds in $\mathbb{X}^*$ for all $t \in [0,T]$ a.s..

Theorem 5.4. *Under* **(HV1)-(HV4)***, there exists a unique solution to Eq.(54) in the sense of Definition 5.3.*

Proof. Noting that by stochastic Fubini's theorem, we have

$$\begin{aligned}\int_0^t D(t,s,X_s)\mathrm{d}W_s &= \int_0^t (D(t,s,X_s) - D(s,s,X_s))\mathrm{d}W_s + \int_0^t D(s,s,X_s)\mathrm{d}W_s\\ &= \int_0^t\int_s^t \partial_r D(r,s,X_s)\mathrm{d}r\mathrm{d}W_s + \int_0^t D(s,s,X_s)\mathrm{d}W_s\\ &= \int_0^t\int_0^s \partial_s D(s,r,X_r)\mathrm{d}W_r\mathrm{d}s + \int_0^t D(s,s,X_s)\mathrm{d}W_s\end{aligned}$$

and

$$\int_0^t C(t,s,X_s)\mathrm{d}s = \int_0^t\int_0^s \partial_s C(s,r,X_r)\mathrm{d}r\mathrm{d}s + \int_0^t C(s,s,X_s)\mathrm{d}s.$$

Solving Eq.(54) is then equivalent to solving the following equation

$$\begin{aligned}X_t = X_0 &+ \int_0^t A(s,X_s(0))\mathrm{d}s + \int_0^t C(s,s,X_s)\mathrm{d}s + \int_0^t\int_0^s \partial_s C(s,r,X_r)\mathrm{d}r\mathrm{d}s\\ &+ \int_0^t D(s,s,X_s)\mathrm{d}W_s + \int_0^t\int_0^s \partial_s D(s,r,X_r)\mathrm{d}W_r\mathrm{d}s.\end{aligned}$$

The result now follows from Theorem 5.2. □

6. Applications

In this section, we discuss two applications, which in particular cover the examples given in the introduction.

6.1. **Stochastic Porous Medium Equations.** Let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}^d$. For $q \geqslant 2$, let $W_0^{1,q}(\mathcal{O})$ and $W^{-1,q^*}(\mathcal{O})$ be the usual Sobolev spaces(cf. [36][31]), where $q^* = \frac{q}{q-1}$.

By Poincaré's inequality, we consider $W_0^{1,2}(\mathcal{O})$ as a Hilbert space with the inner product

$$\langle u, v\rangle_{W_0^{1,2}} := \int_{\mathcal{O}} \nabla u(x)\cdot\nabla v(x)\mathrm{d}x.$$

For $q \geqslant 2$, since $W_0^{1,2}(\mathcal{O})$ is continuous and densely embedded in $L^2(\mathcal{O}) \subset L^{q^*}(\mathcal{O})$, we may choose

$$\mathbb{X} := L^q(\mathcal{O}) \subset W^{-1,2}(\mathcal{O}) =: \mathbb{H}.$$

The inner product in $\mathbb{H}$ is given by

$$\langle u, v\rangle_{\mathbb{H}} := \int_{\mathcal{O}} (-\Delta)^{-1/2} u(x) \cdot (-\Delta)^{-1/2} v(x) \, \mathrm{d}x, \quad u, v \in \mathbb{H}.$$

We shall identify $\mathbb{H}$ with the dual space $\mathbb{H}^* = W_0^{1,2}(\mathcal{O})$ via the corresponding Riesz isomorphism $\mathcal{R} : \mathbb{H} \to \mathbb{H}^*$ defined by $\mathcal{R}u := \langle u, \cdot\rangle_{\mathbb{H}}, u \in \mathbb{H}$. In particular, $\mathcal{R} = (-\Delta)^{-1}$ establishes an isomorphism between $W^{-1,2}(\mathcal{O})$ and $W_0^{1,2}(\mathcal{O})$(cf. [31, p.42]). Thus, we have the evolution triple(cf. [24])

$$\mathbb{X} \subset \mathbb{H} \simeq \mathbb{H}^* \subset \mathbb{X}^*,$$

where $\simeq$ is understood via $\mathcal{R} = (-\Delta)^{-1}$. Under this identification, the map

$$\Delta : W_0^{1,2}(\mathcal{O}) \to (L^q(\mathcal{O}))^*$$

can be extended to a linear isometry(cf. [24, Lemma 4.1.13])

$$\Delta : L^{q^*}(\mathcal{O}) \to (L^q(\mathcal{O}))^* = \mathbb{X}^* \tag{55}$$

such that for all $u \in L^{q^*}(\mathcal{O})$ and $v \in L^q(\mathcal{O})$

$$[v, \Delta u]_{\mathbb{X}} = -\int_{\mathcal{O}} v(x) u(x) \mathrm{d}x.$$

Indeed, for $u \in W_0^{1,2}(\mathcal{O})$ we have $\Delta u \in W^{-1,2}(\mathcal{O}) = \mathbb{H} \subset \mathbb{X}^*$. So, for any $v \in L^q(\mathcal{O}) = \mathbb{X}$

$$[v, \Delta u]_{\mathbb{X}} = \langle v, \Delta u\rangle_{\mathbb{H}} = -\int_{\mathcal{O}} v(x) u(x) \mathrm{d}x \leqslant \|v\|_{L^q} \cdot \|u\|_{L^{q^*}},$$

where the first equality is due to (3). That is,

$$\|\Delta u\|_{\mathbb{X}^*} \leqslant \|u\|_{L^{q^*}}.$$

On the other hand, for $u \not\equiv 0$ if we take $v = -\|u\|_{L^{q^*}}^{-q^*/q} \cdot |u|^{q^*-2} u$, then $\|v\|_{L^q} = 1$ and

$$\|u\|_{L^{q^*}} = [v, \Delta u]_{\mathbb{X}} \leqslant \|\Delta u\|_{\mathbb{X}^*}.$$

Thus, the desired conclusion (55) follows.

Moreover, one can prove that the map in (55) is surjective, hence,

$$(L^q(\mathcal{O}))^* = \Delta(L^{q^*}(\mathcal{O})).$$

We shall not use this below. It should be noticed that the above identification gives rise to a different dualization between $L^q(\mathcal{O})$ and $(L^q(\mathcal{O}))^*$. In particular, for all $u \in L^q(\mathcal{O})$ and $v \in \mathbb{H} \simeq \mathbb{H}^* \subset (L^q(\mathcal{O}))^*$

$$[u, v]_{\mathbb{X}} = \langle u, v\rangle_{\mathbb{H}} = \int_{\mathcal{O}} u(x)(-\Delta)^{-1} v(x) \mathrm{d}x \Big(\neq \int_{\mathcal{O}} u(x) v(x) \mathrm{d}x \Big).$$

Let φ be a real measurable function on $[0, T] \times \Omega \times \mathbb{R}$ satisfying the following assumptions:

(HP1) For each $r \in \mathbb{R}$, $(t, \omega) \mapsto \varphi(t, \omega, r)$ is a measurable adapted process.
(HP2) For each $(t, \omega) \in [0, T] \times \Omega$, $r \mapsto \varphi(t, \omega, r)$ is continuous.

(HP3) There exist $q \geqslant 2$ and positive functions $\xi, \eta, \lambda \in L^1(\mathfrak{A})$, where $\lambda(t,\omega) > 0$ for $\mathrm{d}t \times \mathrm{d}P$ almost all $(t,\omega) \in [0,T] \times \Omega$, such that for all $(t,\omega,r) \in [0,T] \times \Omega \times \mathbb{R}$

$$r \cdot \varphi(t,\omega,r) \geqslant \lambda(t,\omega) \cdot |r|^q - \xi(t,\omega),$$

and

$$|\varphi(t,\omega,r)| \leqslant \lambda(t,\omega) \cdot |r|^{q-1} + \eta^{\frac{q}{q-1}}(t,\omega) \cdot \lambda^{\frac{1}{q}}(t,\omega).$$

(HP4) For all $(t,\omega) \in [0,T] \times \Omega$ and $r, r' \in \mathbb{R}$

$$(r - r') \cdot (\varphi(t,\omega,r) - \varphi(t,\omega,r')) \geqslant 0.$$

Define the evolution operator A as follows: for $u \in \mathbb{X} = L^q(\mathcal{O})$

$$A(t,\omega,u) := \Delta\varphi(t,\omega,u).$$

Then by **(HP3)** and (55), $A(t,\omega,u) \in \mathbb{X}^*$ and for $u, v \in \mathbb{X}$

$$[v, A(t,\omega,u)]_{\mathbb{X}} = -\int_{\mathcal{O}} v(x) \cdot \varphi(t,\omega,u(x))\mathrm{d}x. \tag{56}$$

Consider the following stochastic porous medium equation with constant diffusion coefficient(cf. [31][24])

$$\begin{cases} \mathrm{d}u(t) = \Delta(\varphi(t,\omega,u(t)))\mathrm{d}t + B\mathrm{d}W(t), \\ u(t,x) = 0, \quad \forall x \in \partial\mathcal{O},\ t > 0, \\ u(0,x) = u_0(x) \in W^{-1,2}(\mathcal{O}), \end{cases} \tag{57}$$

where $B \in L_2(\mathbb{U},\mathbb{H})$. Of course, B can be some random cylindrical function or linear function in u. For simplicity, we do not discuss this case(see next subsection).

Remark 6.1. *The regularity of solution such as $\varphi(t,\omega,u(t)) \in W_0^{1,2}(\mathcal{O})$ has been studied in* [15] *and* [29] *by other tricks.*

We now check the above A satisfies **(H1)**-**(H4)**.

For **(H1)**, it is direct by (56), **(HP2)**, **(HP3)** and the dominated convergence theorem.

For **(H2)**, we have by (56) and **(HP4)**

$$\begin{aligned} & [u - v, A(t,\omega,u) - A(t,\omega,v)]_{\mathbb{X}} \\ = & -\int_{\mathcal{O}} (u(x) - v(x)) \cdot (\varphi(t,\omega,u(x)) - \varphi(t,\omega,v(x))\mathrm{d}x \leqslant 0. \end{aligned}$$

For **(H3)** and **(H4)**, we have by (56) and **(HP3)**

$$\begin{aligned} [u, A(t,\omega,u)]_{\mathbb{X}} &= -\int_{\mathcal{O}} u(x) \cdot \varphi(t,\omega,u(x))\mathrm{d}x \\ &\leqslant -\lambda(t,\omega) \cdot \|u\|_{\mathbb{X}}^q + \xi(t,\omega) \cdot \mathrm{Vol}(\mathcal{O}), \end{aligned}$$

and

$$\begin{aligned} \|A(t,\omega,u)\|_{\mathbb{X}^*} &= \left(\int_{\mathcal{O}} |\varphi(t,\omega,u(x))|^{\frac{q}{q-1}}\mathrm{d}x\right)^{\frac{q-1}{q}} \\ &\leqslant \lambda(t,\omega) \cdot \|u\|_{\mathbb{X}}^{q-1} + \eta^{\frac{q}{q-1}}(t,\omega) \cdot \lambda^{\frac{1}{q}}(t,\omega) \cdot \mathrm{Vol}(\mathcal{O}). \end{aligned}$$

Hence, A satisfies **(H1)**-**(H4)**, and Theorem 3.6 can be used to this situation. In particular, Eq.(57) contains Eq.(1) as a special case with $\varphi(t,\omega,r) = |w_t(\omega)| \cdot |r|^{p-2}r$ and $\lambda(t,\omega) = |w_t(\omega)|$.

6.2. **Stochastic Reaction Diffusion Equations.** As in the previous subsection, let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}^d$. We denote by l^2 the usual Hilbert space of square summable real number sequences.

We are given three measurable mappings

$$\begin{aligned}(a_1, \cdots, a_d) =: a: &\quad [0,T]\times\Omega\times\mathcal{O}\times\mathbb{R}\to\mathbb{R}^d,\\ b: &\quad [0,T]\times\Omega\times\mathcal{O}\times\mathbb{R}\to\mathbb{R},\\ (\sigma_1,\cdots,\sigma_j,\cdots) =: \sigma: &\quad [0,T]\times\Omega\times\mathcal{O}\times\mathbb{R}\to l^2,\end{aligned}$$

which satisfy that

(HR1) For each $(x,r)\in\mathcal{O}\times\mathbb{R}$, $(t,\omega)\mapsto a(t,\omega,x,r)$, $b(t,\omega,x,r)$ and $\sigma(t,\omega,x,r)$ are measurable adapted processes.

(HR2) For each $(t,\omega,x)\in[0,T]\times\Omega\times\mathcal{O}$, $r\mapsto a(t,\omega,x,r), b(t,\omega,x,r)$ are continuous.

(HR3) For all $(t,\omega,x)\in[0,T]\times\Omega\times\mathcal{O}$, $r,r'\in\mathbb{R}$ and $j=1,\cdots,d$

$$(r-r')\cdot(a_j(t,\omega,x,r)-a_j(t,\omega,t,r'))\geqslant 0.$$

(HR4) There exist $q_1\geqslant 2$ and positive functions $\xi_1,\eta_1,\lambda_1\in L^1(\mathfrak{A})$, where $\lambda_1(t,\omega)>0$ for $\mathrm{d}t\times\mathrm{d}P$ almost all $(t,\omega)\in[0,T]\times\Omega$, such that for all $(t,\omega,r)\in[0,T]\times\Omega\times\mathbb{R}$ and $j=1,\cdots,d$

$$r\cdot a_j(t,\omega,x,r)\geqslant\lambda_1(t,\omega)\cdot|r|^{q_1}-\xi_1(t,\omega),$$

and

$$|a_j(t,\omega,x,r)|\leqslant\lambda_1(t,\omega)\cdot|r|^{q_1-1}+\eta_1^{\frac{q_1}{q_1-1}}(t,\omega)\cdot\lambda_1^{\frac{1}{q_1}}(t,\omega).$$

(HR5) b satisfies **(HR3)** and **(HR4)** with different constant $q_2\geqslant 2$ and functions λ_2,ξ_2,η_2.

(HR6) There exist positive functions $\lambda_0,\lambda_3,\xi_3\in L^1(\mathfrak{A})$ such that for all $(t,\omega,x)\in[0,T]\times\Omega\times\mathcal{O}$ and $r,r'\in\mathbb{R}$

$$\|\sigma(t,\omega,x,r)\|_{l^2}^2\leqslant\lambda_3(t,\omega)\cdot|r|^2+\xi_3(t,\omega)$$

and

$$\|\sigma(t,\omega,x,r)-\sigma(t,\omega,x,r')\|_{l^2}^2\leqslant\lambda_0(t,\omega)\cdot|r-r'|^2,$$

where for some $c_\sigma>0$,

$$0\leqslant\lambda_0(t,\omega)\leqslant c_\sigma\cdot\lambda_3(t,\omega).$$

Consider the following stochastic reaction diffusion equation with Dirichlet boundary conditions:

$$\begin{cases}\mathrm{d}u(t,x)=\Big[\sum_{i=1}^d\partial_i a_i(t,\omega,x,\partial_i u(t,x))-b(t,\omega,x,u(t,x))\Big]\mathrm{d}t\\ \qquad\qquad\quad+\sum_{j=1}^\infty\sigma_j(t,\omega,x,u(t,x))\mathrm{d}W_j(t),\\ u(t,x)=0,\quad\forall x\in\partial\mathcal{O},\\ u(0,x)=u_0(x)\in L^2(\mathcal{O}),\end{cases}\tag{58}$$

where $W_j(t)=\langle W(t),\ell_j\rangle_{\mathbb{U}}$ and $\{\ell_j, j\in\mathbb{N}\}$ is a normal orthogonal basis of $\mathbb{U}$.

Let

$$\mathbb{X}_1:=W_0^{1,q_1}(\mathcal{O}),\quad\mathbb{X}_2:=L^{q_2}(\mathcal{O}),\quad\mathbb{H}:=L^2(\mathcal{O})$$

and

$$\mathbb{X}_1^*:=W^{-1,q_1^*}(\mathcal{O}),\quad\mathbb{X}_2^*:=L^{q_2^*}(\mathcal{O}).$$

If we identify $\mathbb{H}^*$ with $\mathbb{H}$, then

$$\mathbb{X}_1\subset\mathbb{H}\simeq\mathbb{H}^*\subset\mathbb{X}_1^*,\quad\mathbb{X}_2\subset\mathbb{H}\simeq\mathbb{H}^*\subset\mathbb{X}_2^*$$

are two evolution triples.

Now define for $u, v \in \mathbb{X}_1$

$$[v, A_1(t, \omega, u)]_{\mathbb{X}_1} := -\sum_{i=1}^{d} \int_{\mathcal{O}} a_i(t, \omega, x, \partial_i u(x)) \cdot \partial_i v(x) \mathrm{d}x \tag{59}$$

and for $u, v \in \mathbb{X}_2$

$$[v, A_2(t, \omega, u)]_{\mathbb{X}_2} := -\int_{\mathcal{O}} b(t, \omega, x, u(x)) \cdot v(x) \mathrm{d}x. \tag{60}$$

Clearly, for each $(t, \omega) \in [0, T] \times \Omega$ and $u \in \mathbb{X}_1$, $[\cdot, A_1(t, \omega, u)]_{\mathbb{X}_1} \in \mathbb{X}_1^*$ and for each $u \in \mathbb{X}_2$, $[\cdot, A_2(t, \omega, u)]_{\mathbb{X}_2} \in \mathbb{X}_2^*$. Thus,

$$A_1(t, \omega, \cdot) : \mathbb{X}_1 \to \mathbb{X}_1^*, \quad A_2(t, \omega, \cdot) : \mathbb{X}_2 \to \mathbb{X}_2^*.$$

Moreover, we also define for $u \in \mathbb{H} = L^2(\mathcal{O})$

$$B(t, \omega, u) := \sum_{j=1}^{\infty} \sigma_j(t, \omega, \cdot, u(\cdot)) \cdot \ell_j \in L_2(\mathbb{U}, \mathbb{H}).$$

We now check the above A and B satisfy **(H1)-(H4)**.

For **(H1)**, it is direct by **(HR2)**, **(HR4)**, **(HR5)** and the dominated convergence theorem.

For **(H2)**, we have by **(HR3)**, **(HR5)** and **(HR6)**

$$\begin{aligned}
& 2[u - v, A(t, \omega, u) - A(t, \omega, v)]_{\mathbb{X}} + \|B(t, \omega, u) - B(t, \omega, v)\|^2_{L_2(\mathbb{U}, \mathbb{H})} \\
= \ & -2 \sum_{i=1}^{d} \int_{\mathcal{O}} \Big(a_i(t, \omega, x, \partial_i u(x)) - a_i(t, \omega, x, \partial_i v(x)) \Big) \cdot \partial_i (u(x) - v(x)) \mathrm{d}x \\
& -2 \int_{\mathcal{O}} \Big(b(t, \omega, x, u(x)) - b(t, \omega, x, v(x)) \Big) \cdot (u(x) - v(x)) \mathrm{d}x \\
& + \int_{\mathcal{O}} \sum_{j=1}^{\infty} |\sigma_j(t, \omega, x, u(x)) - \sigma_j(t, \omega, x, v(x))|^2 \mathrm{d}x \\
\leqslant \ & 0 + 0 + \lambda_0(t, \omega) \cdot \|u - v\|^2_{\mathbb{H}}.
\end{aligned}$$

For **(H3)**, we have by **(HR4)-(HR6)**

$$\begin{aligned}
& 2[u, A(t, \omega, u)]_{\mathbb{X}} + \|B(t, \omega, u)\|^2_{L_2(\mathbb{U}, \mathbb{H})} \\
= \ & -2 \sum_{i=1}^{d} \int_{\mathcal{O}} a_i(t, \omega, x, \partial_i u(x)) \cdot \partial_i u(x) \mathrm{d}x \\
& -2 \int_{\mathcal{O}} b(t, \omega, x, u(x)) \cdot u(x) \mathrm{d}x + \int_{\mathcal{O}} \|\sigma(t, \omega, x, u(x))\|^2_{l^2} \mathrm{d}x \\
\leqslant \ & -2d\lambda_1(t, \omega) \int_{\mathcal{O}} \sum_{i=1}^{d} |\partial_i u(x)|^{q_1} \mathrm{d}x + d \cdot \xi_1(t, \omega) \cdot \mathrm{Vol}(\mathcal{O}) \\
& -2\lambda_2(t, \omega) \int_{\mathcal{O}} |u(x)|^{q_2} \mathrm{d}x + \xi_2(t, \omega) \cdot \mathrm{Vol}(\mathcal{O}) \\
& +\lambda_3(t, \omega) \int_{\mathcal{O}} |u(x)|^2 \mathrm{d}x + \xi_3(t, \omega) \cdot \mathrm{Vol}(\mathcal{O}) \\
\leqslant \ & -c_0 \sum_{i=1,2} \lambda_i(t, \omega) \cdot \|u\|^{q_i}_{\mathbb{X}_i} + \lambda_3(t, \omega) \cdot \|u\|^2_{\mathbb{H}} + \tilde{\xi}(t, \omega),
\end{aligned}$$

where $c_0 > 0$ only depends on q_1, d, and $\tilde{\xi} := \mathrm{Vol}(\mathcal{O}) \cdot (d \cdot \xi_1 + \xi_2 + \xi_3)$.

For **(H4)**, we have by (59) and **(HR4)**

$$\begin{aligned}\|A_1(t,\omega,u)\|_{\mathbb{X}_1^*} &\leqslant c_{q_1} \cdot \sum_{i=1}^d \left(\int_{\mathcal{O}} |a_i(t,\omega,x,\partial_i u(x))|^{\frac{q_1}{q_1-1}} \mathrm{d}x\right)^{\frac{q_1-1}{q_1}} \\ &\leqslant c_{q_1} \cdot \lambda_1(t,\omega) \cdot \|u\|_{\mathbb{X}_1}^{q_1-1} + \eta_1^{\frac{q_1}{q_1-1}}(t,\omega) \cdot \lambda_1^{\frac{1}{q_1}}(t,\omega) \cdot \mathrm{Vol}(\mathcal{O}) \cdot d,\end{aligned}$$

and by (60) and **(HR5)**

$$\begin{aligned}\|A_2(t,\omega,u)\|_{\mathbb{X}_2^*} &\leqslant \left(\int_{\mathcal{O}} |b(t,\omega,x,u(x))|^{\frac{q_2}{q_2-1}} \mathrm{d}x\right)^{\frac{q_2-1}{q_2}} \\ &\leqslant \lambda_2(t,\omega) \cdot \|u\|_{\mathbb{X}_2}^{q_2-1} + \eta_2^{\frac{q_2}{q_2-1}}(t,\omega) \cdot \lambda_2^{\frac{1}{q_2}}(t,\omega) \cdot \mathrm{Vol}(\mathcal{O}),\end{aligned}$$

Hence, (A, B) satisfies **(H1)-(H4)**. Thus, we may use Theorem 3.6 to Eq.(58). In particular, Eq.(2) is a special case of Eq.(58). In fact, we may take

$$\begin{aligned} a_i(t,\omega,x,r) &:= |w_t(\omega)| \cdot r, \quad i = 1, \cdots, d, \\ b(t,\omega,x,r) &:= |w_t(\omega)| \cdot |r|^{p-2} r, \\ \sigma_1(t,\omega,x,r) &:= w_t(\omega) \cdot r, \quad \sigma_i = 0, \quad i = 2, \cdots. \end{aligned}$$

Then $\lambda_1(t,\omega) = \lambda_2(t,\omega) = |w_t(\omega)|$, $\lambda_0(t,\omega) = \lambda_3(t,\omega) = |w_t(\omega)|^2$, and $\eta_i = \xi_j = 0$, $i = 1, 2, j = 1, 2, 3$.

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