Diffusion approximation for equilibrium Kawasaki dynamics in continuum

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Abstract

A Kawasaki dynamics in continuum is a dynamics of an infinite system of interacting particles in $\mathbb{R}^d$ which randomly hop over the space. In this paper, we deal with an equilibrium Kawasaki dynamics which has a Gibbs measure μ as invariant measure. We study a diffusive limit of such a dynamics, derived through a scaling of both the jump rate and time. Under weak assumptions on the potential of pair interaction, ϕ, (in particular, admitting a singularity of ϕ at zero), we prove that, on a set of smooth local functions, the generator of the scaled dynamics converges to the generator of an equilibrium diffusive dynamics of an infinite system of interacting particles. If the set on which the generators converge is a core for the diffusive generator, the latter result implies the weak convergence of finite-dimensional distributions of the corresponding equilibrium processes. In particular, if the potential ϕ is from $C_{\mathrm{b}}^3(\mathbb{R}^d)$ and sufficiently quickly converges to zero at infinity, we conclude from a result in [Choi *et al.*, J. Math. Phys. 39 (1998) 6509–6536] that the convergence of process holds when the limiting diffusion is the gradient stochastic dynamics.

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1 Introduction

A Kawasaki dynamics in continuum is a dynamics of an infinite system of interacting particles in $\mathbb{R}^d$ which randomly hop over the space. The generator of such a dynamics has the form

$$(HF)(\gamma) = -\sum_{x\in\gamma}\int_{\mathbb{R}^d} dy\, c(\gamma,x,y)(F(\gamma\setminus x\cup y)-F(\gamma)),\quad \gamma\in\Gamma. \tag{1.1}$$

Here, Γ denotes the configuration space over $\mathbb{R}^d$, i.e., the space of all locally finite subsets of $\mathbb{R}^d$, and, for simplicity of notations, we just write x instead of $\{x\}$. The coefficient $c(\gamma,x,y)$ describes the rate at which the particle x of the configuration γ jumps to y.

Let μ denote a Gibbs measure on Γ which corresponds to an activity parameter $z>0$ and a potential of pair interaction ϕ. In this paper, we will deal with a class of equilibrium Kawasaki dynamics which have μ as invariant measure. More precisely, for a fixed parameter $s\in[0,1]$, we will consider an equilibrium Kawasaki dynamics whose generator (1.1) has the coefficient $c(\gamma,x,y)$ of the form

$$c(\gamma,x,y)=c^{(s)}(\gamma,x,y)=a(x-y)\exp\big[(1-s)E(x,\gamma\setminus x)-sE(y,\gamma\setminus x)\big]. \tag{1.2}$$

Here, for any $\gamma\in\Gamma$ and $u\in\mathbb{R}^d\setminus\gamma$, $E(u,\gamma)$ denotes the relative energy of interaction between the particle at u and the configuration γ. About the function $a(\cdot)$ in (1.2) we assume that it is non-negative, bounded, has a compact support, and $a(x)$ only depends on $|x|$.

Equation (1.2) allows the following physical interpretation: particles from γ which have a high relative energy of interaction with the rest of the configuration tend to jump to places where this relative energy will be low, i.e., particles tend to jump from high energy regions to low energy regions.

Note also that the bilinear (Dirichlet) form corresponding to the generator (1.1), (1.2) admits the following representation:

$$\mathcal{E}^{(s)}(F,G)=\frac{z}{2}\int_\Gamma\mu(d\gamma)\int_{\mathbb{R}^d}dx\int_{\mathbb{R}^d}dy\,a(x-y)\exp\big[-sE(x,\gamma)-sE(y,\gamma)\big]$$
$$\times(F(\gamma\cup y)-F(\gamma\cup x))(G(\gamma\cup y)-G(\gamma\cup x)).$$

Under very mild assumptions on the Gibbs measure μ, it was proved in [11] that there indeed exists a Markov process on Γ with *cádlág* paths whose generator is given by (1.1), (1.2). We assume that the initial distribution of this dynamics is μ, and perform a diffusive scaling of this dynamics. More precisely, for each $\epsilon>0$, we consider the equilibrium Kawasaki dynamics whose jump rate is given by formula (1.2) in which $a(\cdot)$ is replaced with the function

$$a_\epsilon(\cdot):=\epsilon^{-d}a(\cdot/\epsilon), \tag{1.3}$$

and we additionally scale time, multiplying it by ϵ^{-2}. We denote the generator of the obtained dynamics by $H^{s,\epsilon}$.

So, the aim of the paper is to show that the scaled dynamics converges, as $\epsilon \to 0$, to a diffusive dynamics on the configuration space Γ. Our main result is that, under weak assumptions on the pair potential ϕ (in particular, we allow ϕ to have a singularity at zero), the generator of the scaled dynamics, $H^{s,\epsilon}$, converges, on a set of smooth local functions, to the generator of an equilibrium diffusive dynamics of an infinite system of interacting particles. The limiting diffusive generator acts as follows:

$$(H^{(s,\,\mathrm{dif})}F)(\{x_k\}_{k=1}^\infty) = c\sum_{i=1}^\infty \bigg(-\frac{1}{2}\,\Delta_{x_i}F(\{x_k\}_{k=1}^\infty)+\sum_{j\neq i}\langle\nabla_{x_i}F(\{x_k\}_{k=1}^\infty), s\nabla\phi(x_i-x_j)\rangle\bigg) \times \exp\bigg[(-2s+1)\sum_{j\neq i}\phi(x_i-x_j)\bigg], \quad (1.4)$$

where the constant c is defined by the equation (6.2) below. Using the theory of Dirichlet forms [16], we show that, for each $s \in [0,1]$, there indeed exists a diffusive Markov process on whose generator is given by (1.4).

Let us dwell upon two special cases. First, in the case $s = 1/2$, equality (1.4) becomes

$$(H^{(1/2,\,\mathrm{dif})}F)(\{x_k\}_{k=1}^\infty) = \frac{c}{2}\sum_{i=1}^\infty \bigg(-\Delta_{x_i}F(\{x_k\}_{k=1}^\infty) + \sum_{j\neq i}\langle\nabla_{x_i}F(\{x_k\}_{k=1}^\infty), \nabla\phi(x_i-x_j)\rangle\bigg),$$

which is the generator of the (infinite-dimensional) *gradient stochastic dynamics* (also called *interacting Brownian particles*), see e.g. [1, 4, 6, 13, 21, 22, 23, 28, 29] and the references therein. This stochastic process informally solves the following system of stochastic differential equations:

$$dx_i(t) = -\frac{c}{2}\sum_{j\neq i}\nabla\phi(x_i(t)-x_j(t))\,dt + \sqrt{c}\,dB_i(t), \quad i\in\mathbb{N},$$
$$\{x_i(0)\}_{i=1}^\infty = \gamma\in\Gamma,$$

where $(B_i)_{i=1}^\infty$ is a sequence of independent Brownian motions.

Second, in the case $s = 0$, equality (1.4) becomes

$$(H^{(0,\,\mathrm{dif})}F)(\{x_k\}_{k=1}^\infty) = -\frac{c}{2}\sum_{i=1}^\infty \Delta_{x_i}F(\{x_k\}_{k=1}^\infty)\exp\bigg[\sum_{j\neq i}\phi(x_i-x_j)\bigg].$$

The corresponding Markov process was studied in [10], and called there *infinite interacting diffusion particles.* This process informally solves the following system of stochastic differential equations:

$$dx_i(t) = \sqrt{c}\,\exp\bigg[\frac{1}{2}\sum_{j\neq i}\phi(x_i(t)-x_j(t))\bigg]\,dB_i(t), \quad i\in\mathbb{N},$$

$$\{x_i(0)\}_{i=1}^{\infty} = \gamma \in \Gamma.$$

If the set on which the generators converge is a core for the diffusive generator $H^{(s,\,\mathrm{dif})}$, then our main result implies the weak convergence of finite-dimensional distributions of the corresponding equilibrium processes. In particular, if the potential ϕ is from $C^3_{\mathrm{b}}(\mathbb{R}^d)$ (hence, ϕ has no singularity at zero) and sufficiently quickly converges to zero at infinity, then we conclude from a result by Choi *et al.* [2] that the weak convergence of finite-dimensional distributions holds when the limiting diffusion is the gradient stochastic dynamics.

The paper is organized as follows. In Section 2, we recall some basic facts of analysis on the configuration space Γ. In Section 3, we recall conditions which guarantee the existence of a Gibbs measure on the configuration space. In Section 4, we recall construction of the equilibrium Kawasaki dynamics in continuum. Section 5 is devoted to construction of a class of diffusion processes on Γ, which will later on appear as limiting diffusions. In Section 6, we formulate our main results. Finally, in Section 7, we present the proofs.

2 K-transform and correlation functions

The configuration space over $\mathbb{R}^d$, $d \in \mathbb{N}$, is defined as the set of all subsets of $\mathbb{R}^d$ which are locally finite:

$$\Gamma := \Gamma_{\mathbb{R}^d} := \big\{ \gamma \subset \mathbb{R}^d \mid |\gamma_\Lambda| < \infty \text{ for each } \Lambda \in \mathcal{O}_{\mathrm{c}}(\mathbb{R}^d) \big\}.$$

Here $|\cdot|$ denotes the cardinality of a set, $\gamma_\Lambda := \gamma \cap \Lambda$, and $\mathcal{O}_{\mathrm{c}}(\mathbb{R}^d)$ denotes the set of all open, relatively compact subsets of $\mathbb{R}^d$. One can identify any $\gamma \in \Gamma$ with the positive Radon measure $\sum_{x\in\gamma} \varepsilon_x \in \mathcal{M}(\mathbb{R}^d)$, where ε_x is the Dirac measure with mass at x, and $\mathcal{M}(\mathbb{R}^d)$ stands for the set of all positive Radon measures on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. The space Γ can be endowed with the relative topology as a subset of the space $\mathcal{M}(\mathbb{R}^d)$ with the vague topology, i.e., the weakest topology on Γ with respect to which all maps

$$\Gamma \ni \gamma \mapsto \langle f, \gamma\rangle := \int_{\mathbb{R}^d} f(x)\,\gamma(dx) = \sum_{x\in\gamma} f(x), \qquad f \in C_0(\mathbb{R}^d),$$

are continuous. Here, $C_0(\mathbb{R}^d)$ is the space of all continuous functions on $\mathbb{R}^d$ with compact support. We will denote by $\mathcal{B}(\Gamma)$ the Borel σ-algebra on Γ.

Next, denote by Γ_0 the space of finite configurations in $\mathbb{R}^d$:

$$\Gamma_0 := \bigsqcup_{n=0}^{\infty} \Gamma_0^{(n)}, \quad \Gamma_0^{(0)} := \{\varnothing\}, \quad \Gamma_0^{(n)} := \{\eta \subset \mathbb{R}^d \mid |\eta| = n\}, \quad n \in \mathbb{N}.$$

Evidently, $\Gamma_0 \subset \Gamma$.

Let

$$\widetilde{(\mathbb{R}^d)^n} = \{\, (x_1, \dots, x_n) \in (\mathbb{R}^d)^n \mid x_i \neq x_j \text{ for } i \neq j \,\}.$$

Let S^n be the group of all permutations of $\{1, \dots, n\}$ which acts on $\widetilde{(\mathbb{R}^d)^n}$ by permuting the coordinates. Through the natural bijection

$$\widetilde{(\mathbb{R}^d)^n}/S^n \longleftrightarrow \Gamma_0^{(n)} \tag{2.1}$$

one defines a topology on $\Gamma_0^{(n)}$. The space Γ_0 is then equipped with the topology of disjoint union. Let $\mathcal{B}(\Gamma_0)$ denote the Borel σ-algebra on Γ_0. It can be shown (see e.g. [7]) that $\mathcal{B}(\Gamma_0)$ coincides with the trace σ-algebra of $\mathcal{B}(\Gamma)$ on Γ_0. Note also that each function $k : \Gamma_0 \to \mathbb{R}$ may be identified with the sequence $(k^{(n)})_{n=0}^{\infty}$, where $k^{(0)} := k(\{\varnothing\})$ and, for each $n \in \mathbb{N}$, $k^{(n)} : \widetilde{(\mathbb{R}^d)^n} \to \mathbb{R}$ is a measurable, symmetric function.

For any $\gamma \in \Gamma$, let $\sum_{\eta \Subset \gamma}$ denote the summation over all $\eta \subset \gamma$ such that $\eta \in \Gamma_0$. For a function $G : \Gamma_0 \to \mathbb{R}$, the K-transform of G is defined by

$$(KG)(\gamma) := \sum_{\eta \Subset \gamma} G(\eta) \tag{2.2}$$

for each $\gamma \in \Gamma$ such that at least one of the series $\sum_{\eta \Subset \gamma} G^+(\eta)$, $\sum_{\eta \Subset \gamma} G^-(\eta)$ converges. Here $G^+ := \max\{0, G\}$ and $G^- := \max\{0, -G\}$.

Let us fix a probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$. The correlation measure of μ is defined by

$$\rho_\mu(A) := \int_\Gamma (K\chi_A)(\gamma)\, \mu(d\gamma), \qquad A \in \mathcal{B}(\Gamma_0),$$

where χ_A denotes the indicator of the set A. The ρ_μ is a measure on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ (see [8] for details, in particular, measurability issues). Note that $\rho_\mu(\{\varnothing\}) = 1$.

The following proposition was proved in [8], see also [14, 15]

Proposition 2.1 *Let $G \in L^1(\Gamma_0, \rho_\mu)$, then $KG \in L^1(\Gamma, \mu)$, the series in* (2.2) *is absolutely convergent for μ-a.e. $\gamma \in \Gamma$, and*

$$\|KG\|_{L^1(\mu)} \le \|K|G|\,\|_{L^1(\mu)} = \|G\|_{L^1(\rho_\mu)}.$$

Moreover, then

$$\int_{\Gamma_0} G(\eta)\, \rho_\mu(d\eta) = \int_\Gamma (KG)(\gamma)\, \mu(d\gamma). \tag{2.3}$$

The Lebesgue–Poisson measure λ on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is defined by

$$\lambda := \varepsilon_\varnothing + \sum_{n=1}^{\infty} \frac{1}{n!}\, dx^{\otimes n},$$

where $dx^{\otimes n}$ is defined via the bijection (2.1). Assume that the correlation measure ρ_μ is absolutely continuous with respect to the Lebesgue–Poisson measure λ. Denote $k_\mu := d\rho_\mu/d\lambda$. Then the corresponding functions $(k_\mu^{(n)})_{n=0}^\infty$ are called the correlation functions of the measure μ.

In what follows, we will assume that k_μ satisfies the Ruelle bound, i.e., there exists a constant $\xi > 0$ such that

$$k_\mu(\eta) \le \xi^{|\eta|} \quad \text{for all } \eta \in \Gamma_0. \tag{2.4}$$

Using (2.4), one, in particular, gets that all local moments of μ are finite:

$$\int_\Gamma |\gamma_\Lambda|^n \, \mu(d\gamma) < \infty, \quad n \in \mathbb{N},\ \Lambda \in \mathcal{O}_{\mathrm{c}}(\mathbb{R}^d). \tag{2.5}$$

We will widely use the following lemma.

Lemma 2.1 *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function which is bounded outside a set $\Lambda \in \mathcal{O}_{\mathrm{c}}(\mathbb{R}^d)$ and such that $e^f - 1 \in L^1(\mathbb{R}^d, dx)$. Let also $g,\ g_1,\ g_2 : \mathbb{R}^d \to \mathbb{R}$ be such that $e^f g,\ e^f g_1,\ e^f g_2 \in L^1(\mathbb{R}^d, dx)$. Define functions $G_1,\ G_2,\ G_3$ on Γ_0 by*

$$\begin{aligned} G_1 &= \big((e^f - 1)^{\otimes n}\big)_{n=0}^\infty, \\ G_2 &= \big(n(e^f - 1)^{\otimes(n-1)} \odot (e^f g)\big)_{n=0}^\infty, \\ G_3 &= \big(n(n-1)(e^f - 1)^{\otimes(n-2)} \odot (e^f g_1) \odot (e^f g_2)\big)_{n=0}^\infty, \end{aligned}$$

where $\odot$ denotes symmetric tensor product. Then, $G_1, G_2, G_3 \in L^1(\Gamma_0, \rho_\mu)$ and

$$\begin{aligned} (KG_1)(\gamma) &= e^{\langle f, \gamma\rangle}, \\ (KG_2)(\gamma) &= e^{\langle f, \gamma\rangle}\langle g, \gamma\rangle, \\ (KG_2)(\gamma) &= e^{\langle f, \gamma\rangle} \sum_{x_1 \in \gamma} \sum_{x_2 \in \gamma,\, x_2 \ne x_1} g_1(x_1) g_2(x_2), \end{aligned} \tag{2.6}$$

for μ-a.e. $\gamma \in \Gamma$, and so $KG_1, KG_2, KG_3 \in L^1(\Gamma, \mu)$.

Proof. Using the Ruelle bound, we clearly have that $G_1, G_2, G_3 \in L^1(\Gamma_0, \rho_\mu)$. Hence, by Proposition 2.1, we get $KG_1, KG_2, KG_3 \in L^1(\Gamma, \mu)$.

Since f is bounded on Λ^c and $e^f - 1 \in L^1(\mathbb{R}^d, dx)$, we have $f \in L^1(\Lambda^c, dx)$. Therefore, again using the Ruelle bound, we get: $\langle |f|, \gamma_{\Lambda^c}\rangle \in L^1(\Gamma, \mu)$. Hence, $\langle |f|, \gamma\rangle < \infty$ for μ-a.e. $\gamma \in \Gamma$. Furthermore, we have $g, g_1, g_2 \in L^1(\Lambda^c, dx)$, and so the functions $\langle |g|, \gamma\rangle$ and $\sum_{x_1 \in \gamma} \sum_{x_2 \in \gamma,\, x_2 \ne x_1} |g_1(x_1) g_2(x_2)|$ are finite for μ-a.e. $\gamma \in \Gamma$. Thus the functions on the right hand side of formulas (2.6) are well-defined and finite for μ-a.e. $\gamma \in \Gamma$.

Next, assume that f, g, g_1, g_2 have compact support. Then, equalities (2.6) follow by a straightforward calculation. The general case follows by approximation. $\square$

We introduce a $\star$-convolution of two functions on Γ_0, so that

$$(K(G_1 \star G_2))(\gamma) = (KG_1)(\gamma)(KG_2)(\gamma)$$

(cf. [8]). Then, we have:

$$(G_1 \star G_2)(\eta) = \sum_{(\eta_1,\eta_2,\eta_3)\in\mathcal{P}_3(\eta)} G_1(\eta_1 \cup \eta_2)G_2(\eta_2 \cup \eta_3),$$

where $\mathcal{P}_3(\eta)$ is the set of all ordered partitions of η into three parts.

For each $\Lambda \subset \mathbb{R}^d$, we denote

$$\Gamma_\Lambda := \{\gamma \in \Gamma : \gamma \subset \Lambda\}.$$

A measurable function $F : \Gamma \to \mathbb{R}$ is called local if there exists $\Lambda \in \mathcal{O}_{\mathrm{c}}(\mathbb{R}^d)$ such that

$$F(\gamma) = F(\gamma_\Lambda) \quad \text{for all } \gamma \in \Gamma.$$

For such a function F, the pre-image of F under K is given by

$$(K^{-1}F)(\eta) = \chi_{\Gamma_\Lambda}(\eta) \sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|} F(\xi), \tag{2.7}$$

see e.g. [8]. Note that, if F satisfies

$$|F(\gamma)| \le \text{const} \quad \text{for all } \gamma \in \Gamma,$$

then, by (2.7),

$$|(K^{-1}F)(\eta)| \le \chi_{\Gamma_\Lambda}(\eta) 2^{|\eta|} \,\text{const}, \quad \eta \in \Gamma_0. \tag{2.8}$$

We will also need the space $\ddot{\Gamma} := \ddot{\Gamma}_{\mathbb{R}^d}$ which consists of all multiple configurations in $\mathbb{R}^d$. So, $\ddot{\Gamma}$ is the set of all Radon $\mathbb{Z}_+ \cup \{\infty\}$-valued measures on $\mathbb{R}^d$. In particular, $\Gamma \subset \ddot{\Gamma}$. Analogously to the case of Γ, we define the vague topology on $\ddot{\Gamma}$ and the corresponding Borel σ-algebra $\mathcal{B}(\ddot{\Gamma})$. For each $\Lambda \subset \mathbb{R}^d$, we denote

$$\ddot{\Gamma}_\Lambda := \{\gamma \in \ddot{\Gamma} : \operatorname{supp}(\gamma) \subset \Lambda\}.$$

Also, by analogy, we will say that a measurable function $F : \ddot{\Gamma} \to \mathbb{R}$ is local if there exists $\Lambda \in \mathcal{O}_{\mathrm{c}}(\mathbb{R}^d)$ such that

$$F(\gamma) = F(\gamma_\Lambda) \quad \text{for all } \gamma \in \ddot{\Gamma}, \tag{2.9}$$

where $\gamma_\Lambda(dx) := \chi_\Lambda(x)\,\gamma(dx)$.

3 Gibbs measures on configuration spaces

A pair potential (without hard core) is a Borel measurable function $\phi\colon \mathbb{R}^d \to \mathbb{R}\cup\{+\infty\}$ such that $\phi(-x) = \phi(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d \setminus \{0\}$. For $\gamma \in \Gamma$ and $x \in \mathbb{R}^d \setminus \gamma$, we define the relative energy of interaction between a particle at x and the configuration γ as follows:

$$E(x,\gamma) := \begin{cases} \sum_{y\in\gamma} \phi(x-y), & \text{if } \sum_{y\in\gamma} |\phi(x-y)| < +\infty, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.1}$$

A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called a (grand canonical) Gibbs measure corresponding to the pair potential ϕ and activity $z > 0$ if it satisfies the Georgii–Nguyen–Zessin identity ([20, Theorem 2], see also [12, Theorem 2.2.4]):

$$\int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx) F(\gamma, x) = \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} z\, dx\, \exp\left[-E(x,\gamma)\right] F(\gamma \cup x, x) \tag{3.2}$$

for any measurable function $F : \Gamma \times \mathbb{R}^d \to [0, +\infty]$. We denote the set of all such measures μ by $\mathcal{G}(z, \phi)$.

Note that, by virtue of (3.1) and by applying (3.2) twice, we get, for any measurable function $U : \Gamma \times (\mathbb{R}^d)^2 \to [0, +\infty]$,

$$\int_\Gamma \mu(d\gamma) \sum_{x_1\in\gamma} \sum_{x_2\in\gamma,\, x_2\neq x_1} U(\gamma, x_1, x_2) = \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} z\, dx_1 \int_{\mathbb{R}^d} z\, dx_2$$
$$\times \exp[-E(x_1,\gamma) - E(x_2,\gamma) - \phi(x_1 - x_2)] U(\gamma \cup x_1 \cup x_2, x_1, x_2). \tag{3.3}$$

Let us now describe classes of Gibbs measures which appear in classical statistical mechanics of continuous systems [26, 27]. We will first formulate conditions on the interaction.

(S) (*Stability*) There exists $B \geq 0$ such that, for any $\eta \in \Gamma_0$,

$$\sum_{\{x,y\}\subset\eta} \phi(x-y) \geq -B|\eta|.$$

Notice that the stability condition automatically implies that the potential ϕ is semi-bounded from below.

For every $r = (r^1, \dots, r^d) \in \mathbb{Z}^d$, we define the cube

$$Q_r := \left\{ x \in \mathbb{R}^d \mid r^i - \frac{1}{2} \leq x^i < r^i + \frac{1}{2} \right\}. \tag{3.4}$$

These cubes form a partition of $\mathbb{R}^d$. For any $\gamma \in \Gamma$, we set

$$\gamma_r := \gamma_{Q_r}, \quad r \in \mathbb{Z}^d. \tag{3.5}$$

For $N \in \mathbb{N}$, let Λ_N be the cube with side length $2N-1$ centered at the origin in $\mathbb{R}^d$, Λ_N is then a union of $(2N-1)^d$ unit cubes of the form Q_r.

(SS) (*Superstability*) There exist $A > 0$ and $B \geq 0$ such that, if $\gamma \in \Gamma_{\Lambda_N}$ for some N, then

$$\sum_{\{x,y\}\subset\gamma} \phi(x-y) \geq \sum_{r\in\mathbb{Z}^d} \big(A|\gamma_r|^2 - B|\gamma_r|\big).$$

This condition is evidently stronger than (S).

(LR) (*Lower regularity*) There exists a decreasing positive function $a\colon \mathbb{N} \to \mathbb{R}_+$ such that

$$\sum_{r\in\mathbb{Z}^d} a(\|r\|) < \infty$$

and for any Λ', Λ'' which are finite unions of cubes Q_r and disjoint, with $\gamma' \in \Gamma_{\Lambda'}$, $\gamma'' \in \Gamma_{\Lambda''}$,

$$\sum_{x\in\gamma',\, y\in\gamma''} \phi(x-y) \geq -\sum_{r',r''\in\mathbb{Z}^d} a(\|r'-r''\|)|\gamma'_{r'}|\,|\gamma''_{r''}|.$$

Here, $\|\cdot\|$ denotes the maximum norm on $\mathbb{R}^d$.

(I) (*Integrability*) We have

$$\int_{\mathbb{R}^d} |e^{-\phi(x)} - 1|\, dx < +\infty.$$

We also need

(LAHT) (*Low activity–high temperature regime*) We have

$$\int_{\mathbb{R}^d} |1 - e^{-\phi(x)}|\, dx < z^{-1} \exp(-1-2B),$$

where B is as in (S).

(P) (*Positivity*) We have $\phi(x) \geq 0$ for all $x \in \mathbb{R}^d$.

A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called tempered if μ is supported by $S_\infty := \bigcup_{n=1}^\infty S_n$, where

$$S_n := \left\{ \gamma \in \Gamma \mid \forall N \in \mathbb{N} \sum_{r\in\Lambda_N\cap\mathbb{Z}^d} |\gamma_r|^2 \leq n^2 |\Lambda_N \cap \mathbb{Z}^d| \right\}.$$

By $\mathcal{G}^t(z,\phi) \subset \mathcal{G}(z,\phi)$ we denote the set of all tempered grand canonical Gibbs measures.

The following theorem is due to [19, 26, 27].

Theorem 3.1 (i) *Let* (S) *and* (LAHT) *hold. Then there exists a measure* $\mu \in \mathcal{G}(z,\phi)$ *which is constructed as a limit of finite volume Gibbs measures corresponding to empty boundary conditions* (*see* [19] *for details*).

(ii) *Let* (P) *and* (I) *hold. Then, for each* $z>0$, *there exists a Gibbs measure* $\mu \in \mathcal{G}(z,E)$ *which is constructed as a limit of finite volume Gibbs measures corresponding to empty boundary conditions.*

(iii) *Let* (SS), (I), *and* (LR) *hold. Then the set* $\mathcal{G}^t(z,\phi)$ *is non-empty for each* $z>0$.

(iv) *Let* $\mu \in \mathcal{G}(z,\phi)$ *be as either in* (i), *or in* (ii), *or in* (iii). *Then* μ *has correlation functions which satisfy the Ruelle bound* (2.4).

In what follows, we will keep a Gibbs measure $\mu \in \mathcal{G}(z,\phi)$ as in Theorem 3.1 fixed, and we will additionally assume that there exists $\Theta \in \mathcal{O}_c(\mathbb{R}^d)$ such that

$$\sup_{x\in\Theta^c} \phi(x) < \infty. \tag{3.6}$$

Since ϕ is bounded from below, (I) is now equivalent to the condition $\phi \in L^1(\Theta^c, dx)$. Furthermore, by [10, Lemma 3.1], the relative energy $E(x,\gamma)$ is finite for $dx\otimes\mu$-a.e. $(x,\gamma)\in\mathbb{R}^d\times\Gamma$, as well as $E(x,\gamma\setminus x)$ is finite for μ-a.e. $\gamma\in\Gamma$ and for all $x\in\gamma$.

4 Kawasaki dynamics

We introduce the set $\mathcal{F}C_b(C_0(\mathbb{R}^d),\Gamma)$ of all functions of the form

$$\Gamma \ni \gamma \mapsto F(\gamma) = g(\langle\varphi_1,\gamma\rangle,\dots,\langle\varphi_N,\gamma\rangle),$$

where $N\in\mathbb{N}$, $\varphi_1,\dots,\varphi_N \in C_0(\mathbb{R}^d)$, and $g\in C_b(\mathbb{R}^N)$, where $C_b(\mathbb{R}^N)$ denotes the set of all continuous bounded functions on $\mathbb{R}^N$. For each function $F:\Gamma\to\mathbb{R}$, $\gamma\in\Gamma$, and $x,y\in\mathbb{R}^d$, we denote

$$(D^{-+}_{xy}F)(\gamma) := F(\gamma\setminus x\cup y) - F(\gamma).$$

We fix any $a:\mathbb{R}^d\to[0,\infty)$ which is bounded and such that $a\in L^1(\mathbb{R}^d,dx)$. We also fix a parameter $s\in[0,1]$ and define a bilinear form

$$\begin{aligned}\mathcal{E}^{(s)}(F,G) := \frac12 \int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d}\gamma(dx)\int_{\mathbb{R}^d} dy\, a(x-y)\\ \times \exp[(1-s)E(x,\gamma\setminus x) - sE(y,\gamma\setminus x)](D^{-+}_{x,y}F)(\gamma)(D^{-+}_{x,y}G)(\gamma),\end{aligned}$$

where $F,G\in\mathcal{F}C_b(C_0(\mathbb{R}^d),\Gamma)$.

The following theorem was proved in [11].

Theorem 4.1 (i) *For each* $s\in[0,1]$, *the bilinear form* $(\mathcal{E}^{(s)},\mathcal{F}C_b(C_0(\mathbb{R}^d),\Gamma))$ *is closable on* $L^2(\Gamma,\mu)$ *and its closure will be denoted by* $(\mathcal{E}^{(s)},D(\mathcal{E}^{(s)}))$.

(ii) *For each $s \in [0,1]$, there exists a conservative Hunt process*

$$\boldsymbol{M}^{(s)} = \big(\boldsymbol{\Omega}^{(s)},\ \boldsymbol{F}^{(s)},\ (\boldsymbol{F}_t^{(s)})_{t\ge 0},\ (\boldsymbol{\Theta}_t^{(s)})_{t\ge 0},\ (\boldsymbol{X}^{(s)}(t))_{t\ge 0},\ (\boldsymbol{P}_\gamma^{(s)})_{\gamma\in\Gamma}\big)$$

on Γ (see e.g. [16, p. 92]*) which is properly associated with $(\mathcal{E}^{(s)}, D(\mathcal{E}^{(s)}))$, i.e., for all ($\mu$-versions of) $F \in L^2(\Gamma,\mu)$ and all $t>0$ the function*

$$\Gamma \ni \gamma \mapsto (p_t^{(s)}F)(\gamma) := \int_{\boldsymbol{\Omega}^{(s)}} F(\boldsymbol{X}^{(s)}(t))d\boldsymbol{P}_\gamma^{(s)}$$

is an $\mathcal{E}^{(s)}$-quasi-continuous version of $\exp\left[-tH^{(s)}\right]F$*, where $(H^{(s)}, D(H^{(s)}))$ is the generator of $(\mathcal{E}^{(s)}, D(\mathcal{E}^{(s)}))$. The $\boldsymbol{M}^{(s)}$ is up to μ-equivalence unique (cf.* [16, Chap. IV, Sect. 6]*). In particular, $\boldsymbol{M}^{(s)}$ has μ as invariant measure.*

(iii) *Assume that, if $s \in [0,1/2)$, we additionally have*:

$$\int_{\mathbb{R}^d} |e^{(1-2s)\phi(x)} - 1|dx < \infty. \tag{4.1}$$

Then, for each $s \in [0,1]$, $\mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d),\Gamma) \subset D(H^{(s)})$ and for any $F \in \mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d),\Gamma)$,

$$(H^{(s)}F)(\gamma) = -\int_{\mathbb{R}^d}\gamma(dx)\int_{\mathbb{R}^d} dy\, a(x-y) \\ \times \exp[(1-s)E(x,\gamma\setminus x) - sE(y,\gamma\setminus x)](D_{x,y}^{-+}F)(\gamma). \tag{4.2}$$

We will call the process $\boldsymbol{M}^{(s)}$ from Theorem 4.1 the Kawasaki dynamics (of continuous particles).

Remark 4.1 In Theorem 4.1 (ii), the $\boldsymbol{M}^{(s)}$ can be taken canonical, i.e., $\boldsymbol{\Omega}^{(s)}$ is the set $D([0,+\infty),\Gamma)$ of all *cádlág* functions $\omega : [0,+\infty) \to \Gamma$ (i.e., ω is right continuous on $[0,+\infty)$ and has left limits on $(0,+\infty)$), $\boldsymbol{X}^{(s)}(t)(\omega) = \omega(t)$, $t \ge 0$, $\omega \in \boldsymbol{\Omega}^{(s)}$, $(\boldsymbol{F}_t^{(s)})_{t\ge0}$ together with $\boldsymbol{F}^{(s)}$ is the correponding minimum completed admissible family (cf. [5, Section 4.1]) and $\boldsymbol{\Theta}_t^{(s)}$, $t \ge 0$, are the corresponding natural time shifts.

We denote by $\mathcal{L}_{\mathrm{b}}(\Gamma)$ the set of all local, measurable, bounded functions on Γ. Completely analogously to the proof of [9, Lemma 4.1], we conclude the following

Proposition 4.1 *Under the condition of Theorem* 4.1, (iii)*, we have*

$$\mathcal{L}_{\mathrm{b}}(\Gamma) \subset D(H^{(s)}),$$

and for each $F \in \mathcal{L}_{\mathrm{b}}(\Gamma)$, equality (4.2) *holds.*

5 Diffusions on the configuration space

We denote by $\mathcal{L}C_{\mathrm{b}}^2(\ddot\Gamma)$ the set of all functions F on $\ddot\Gamma$ which satisfy the following assumptions:

(i) F is local and continuous;

(ii) For each fixed $\gamma\in\ddot\Gamma$, the function

$$\mathbb{R}^d \ni x\mapsto F(\gamma+\varepsilon_x)$$

is twice differentiable, and the functions

$$\mathbb{R}^d\times\ddot\Gamma\ni(x,\gamma)\mapsto\nabla_x F(\gamma+\varepsilon_x), \tag{5.1}$$

$$\mathbb{R}^d\times\ddot\Gamma\ni(x,\gamma)\mapsto\nabla_x^2 F(\gamma+\varepsilon_x) \tag{5.2}$$

are continuous (here and below ∇_x denotes the gradient with respect to the x variable);

(iii) For each fixed $\gamma\in\ddot\Gamma$, the function

$$(\mathbb{R}^d)^2\ni(x,y)\mapsto F(\gamma+\varepsilon_x+\varepsilon_y)$$

is twice differentiable, and the function

$$(\mathbb{R}^d)^2\times\ddot\Gamma\ni(x,y,\gamma)\mapsto\nabla_x\nabla_y F(\gamma+\varepsilon_x+\varepsilon_y) \tag{5.3}$$

is continuous;

(iv) The functions F, (5.1), (5.2), and (5.3) are bounded.

We also introduce the set $\mathcal{F}C_{\mathrm{b}}^2(C_0^2(\mathbb{R}^d),\ddot\Gamma)$ of all functions of the form

$$\ddot\Gamma\ni\gamma\mapsto F(\gamma)=g(\langle\varphi_1,\gamma\rangle,\dots,\langle\varphi_N,\gamma\rangle),$$

where $N\in\mathbb{N}$, $\varphi_1,\dots,\varphi_N\in C_0^2(\mathbb{R}^d)$, and $g\in C_{\mathrm{b}}^2(\mathbb{R}^N)$. Here and below, $C_0^k(\mathbb{R}^d)$ and $C_{\mathrm{b}}^k(\mathbb{R}^N)$, $k\in\mathbb{N}$, denote the space of all k times continuously differentiable functions on $\mathbb{R}^d$ with compact support, respectively the space of all bounded, k times continuously differentiable functions on $\mathbb{R}^N$ with bounded derivatives. We evidently have the inclusion

$$\mathcal{F}C_{\mathrm{b}}^2(C_0^2(\mathbb{R}^d),\ddot\Gamma)\subset\mathcal{L}C_{\mathrm{b}}^2(\ddot\Gamma),$$

and therefore the set $\mathcal{L}C_{\mathrm{b}}^2(\ddot\Gamma)$ is dense in $L^2(\Gamma,\mu)$. (We have included functions from $\mathcal{L}C_{\mathrm{b}}^2(\ddot\Gamma)$ into $L^2(\Gamma,\mu)$ by taking their restriction to Γ.)

For a function $F:\Gamma\to\mathbb{R}$, a fixed $\gamma\in\Gamma$ and $x\in\gamma$, we denote

$$\nabla_x F(\gamma) := \nabla_y F(\gamma-\varepsilon_x+\varepsilon_y)\big|_{y=x}, \tag{5.4}$$

provided the gradient on the right hand side of (5.4) exists at point x.

We fix any $s\in[0,1]$, $c>0$, and define a bilinear form

$$\mathcal{E}^{(s,\,\mathrm{dif})}(F,G) := \frac{c}{2}\int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d}\gamma(dx)\langle\nabla_x F(\gamma),\nabla_x G(\gamma)\rangle \exp\big[(-2s+1)E(x,\gamma\setminus x)\big], \tag{5.5}$$

where $F,G\in\mathcal{L}C_{\mathrm{b}}^2(\ddot\Gamma)$, and we denoted by $\langle\cdot,\cdot\rangle$ the scalar product in $\mathbb{R}^d$.

It is easy to see that the integral on the right hand side of (5.5) is well defined and finite. Indeed, the function

$$\mathbb{R}^d\times\Gamma\ni(x,\gamma)\mapsto\langle\nabla_x F(\gamma+\varepsilon_x),\nabla_x G(\gamma+\varepsilon_x)\rangle_{\mathbb{R}^d}$$

is bounded, and there exists a $\Lambda\in\mathcal{O}_{\mathrm{c}}(\mathbb{R}^d)$ such that this function vanishes whenever $x\in\Lambda^c$. Therefore, the statement follows from (3.2) and Lemma 2.1, since

$$e^{-2s\phi}-1\in L^1(\mathbb{R}^d,dx)$$

(recall (3.6), which together with (I) implies the latter inclusion). Furthermore, by (3.2), we have

$$\mathcal{E}^{(s,\,\mathrm{dif})}(F,G) := \frac{c}{2}\int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d} z\,dx\,\langle\nabla_x F(\gamma+\varepsilon_x),\nabla_x G(\gamma+\varepsilon_x)\rangle\exp\big[-2sE(x,\gamma)\big] \tag{5.6}$$

for $F,G\in\mathcal{L}C_{\mathrm{b}}^2(\ddot\Gamma)$.

The following theorem extends the results of [1, 17, 21, 25, 29] (for $s=1/2$) and [10] (for $s=0$) to the case of a general $s\in[0,1]$.

Theorem 5.1 *For a fixed $s\in[0,1]$, assume that the following conditions are satisfied:*

a) *If $s\in[0,1/4)$, then*

$$e^{(-4s+1)\phi}-1\in L^1(\mathbb{R}^d,dx);$$

b) *If $s\neq 0$, then ϕ is differentiable on $\mathbb{R}^d\setminus\{0\}$, $e^{-2s\phi}$ is differentiable on $\mathbb{R}^d$, and we have*

$$\|\nabla\phi\|\in L^1(\mathbb{R}^d,e^{(-4s+1)\phi(x)}\,dx)\cap L^2(\mathbb{R}^d,e^{(-4s+1)\phi(x)}\,dx)\cap L^1(\mathbb{R}^d,e^{-2s\phi(x)}\,dx). \tag{5.7}$$

Then:

(i) *The bilinear form* $(\mathcal{E}^{(s,\,\mathrm{dif})}, \mathcal{L}C^2_{\mathrm{b}}(\ddot\Gamma))$ *is closable on* $L^2(\Gamma,\mu)$ *and its closure will be denoted by* $(\mathcal{E}^{(s,\,\mathrm{dif})}, D(\mathcal{E}^{(s,\mathrm{dif})}))$.

(ii) *Denote by* $(H^{(s,\,\mathrm{dif})}, D(H^{(s,\,\mathrm{dif})}))$ *the generator of* $(\mathcal{E}^{(s,\,\mathrm{dif})}, D(\mathcal{E}^{(s,\mathrm{dif})}))$. *Then* $\mathcal{L}C^2_{\mathrm{b}}(\ddot\Gamma) \subset D(H^{(s,\mathrm{dif})})$ *and for each* $F \in \mathcal{L}C^2_{\mathrm{b}}(\ddot\Gamma)$,

$$(H^{(s,\,\mathrm{dif})}F)(\gamma) = c\int_{\mathbb{R}^d}\gamma(dx)\Big(-\frac{1}{2}\Delta_x F(\gamma) + \sum_{u\in\gamma\setminus x}\langle\nabla_x F(\gamma), s\nabla\phi(x-u)\rangle\Big)$$
$$\times\exp\Big[(-2s+1)\sum_{v\in\gamma\setminus x}\phi(x-v)\Big]. \quad (5.8)$$

Here, $\Delta_x F(\gamma) := \Delta_u F(\gamma\setminus x\cup u)\big|_{u=x}$.

(iii) *There exists a conservative diffusion process*

$$\boldsymbol{M}^{(s,\,\mathrm{dif})} = \big(\boldsymbol{\Omega}^{(s,\,\mathrm{dif})},\ \boldsymbol{F}^{(s,\,\mathrm{dif})},\ (\boldsymbol{F}^{(s,\,\mathrm{dif})}_t)_{t\ge0},\ (\boldsymbol{\Theta}^{(s,\,\mathrm{dif})}_t)_{t\ge0},\ (\boldsymbol{X}^{(s,\,\mathrm{dif})}(t))_{t\ge0},\ (\boldsymbol{P}^{(s,\,\mathrm{dif})}_\gamma)_{\gamma\in\ddot\Gamma}\big)$$

on $\ddot\Gamma$ *(see e.g.* [16, p. 92]*) which is properly associated with* $(\mathcal{E}^{(s,\,\mathrm{dif})}, D(\mathcal{E}^{(s,\,\mathrm{dif})}))$. *The* $\boldsymbol{M}^{(s,\,\mathrm{dif})}$ *is up to* μ*-equivalence unique. In particular,* $\boldsymbol{M}^{(s,\,\mathrm{dif})}$ *has* μ *as invariant measure.*

(iv) *In the case* $d\ge 2$, *the set* $\ddot\Gamma\setminus\Gamma$ *is* $\mathcal{E}^{(s,\,\mathrm{dif})}$*-exceptional, so that* $\ddot\Gamma$ *may be replaced with* Γ *in* (iii).

Proof. Since the proof of this theorem is essentially a modification of the proof for the cases $s = 1/2$ and $s = 0$ in [1, 10, 17, 25], we will only sketch it.

Since $\|\nabla\phi\| \in L^1(\Theta^c, dx)$ (see (3.6) and (5.7)), it is easy to show, using the Ruelle bound, that

$$\sum_{u\in\gamma}\|\nabla\phi(x-u)\| < \infty \quad \text{for } dx\,\mu(d\gamma)\text{-a.a. } (x,\gamma)\in\mathbb{R}^d\times\Gamma.$$

Fix any $F, G \in \mathcal{L}C^2_{\mathrm{b}}(\ddot\Gamma)$. Let $\Lambda' \in \mathcal{O}_{\mathrm{c}}(\mathbb{R}^d)$ be such that $F(\gamma) = F(\gamma_{\Lambda'})$ and $G(\gamma) = G(\gamma_{\Lambda'})$ for all $\gamma\in\Gamma$. For $x\in\mathbb{R}^d$ and $\delta > 0$ denote by $B(x;\delta)$ the closed ball in $\mathbb{R}^d$ of radius δ centered at x. Let $\Lambda \in \mathcal{O}_{\mathrm{c}}(\mathbb{R}^d)$ be such that $B(x;1)\subset\Lambda$ for all $x\in\Lambda'$. Consider

$$\mathcal{E}^{(s,\,\mathrm{dif})}_\Lambda(F,G) := \frac{c}{2}\int_\Gamma\mu(d\gamma)\int_{\mathbb{R}^d} z\,dx\,\langle\nabla_x F(\gamma+\varepsilon_x), \nabla_x G(\gamma+\varepsilon_x)\rangle\exp\big[\langle -2s\phi(x-\cdot), \gamma_\Lambda\rangle\big].$$

Integration by parts yields

$$\mathcal{E}_\Lambda^{(s,\,\mathrm{dif})}(F,G) := c\int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d} z\,dx\left(-\frac{1}{2}\,\Delta_x F(\gamma+\varepsilon_x)+\sum_{u\in\gamma_\Lambda}\langle s\nabla\phi(x-u),\nabla_x F(\gamma+\varepsilon_x)\rangle\right)$$
$$\times\, G(\gamma+\varepsilon_x)\exp\big[\langle -2s\phi(x-\cdot),\gamma_\Lambda\rangle\big].$$

Letting $\Lambda\to\mathbb{R}^d$ and using the Ruelle bound, Lemma 2.1, and the assumptions of the theorem, we obtain:

$$\mathcal{E}^{(s,\,\mathrm{dif})}(F,G) := c\int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d} z\,dx\left(-\frac{1}{2}\,\Delta_x F(\gamma+\varepsilon_x)+\sum_{u\in\gamma}\langle s\nabla\phi(x-u),\nabla_x F(\gamma+\varepsilon_x)\rangle\right)$$
$$\times\, G(\gamma+\varepsilon_x)\exp\big[\langle -2s\phi(x-\cdot),\gamma\rangle\big].$$

Hence, by (3.2),

$$\mathcal{E}^{(s,\,\mathrm{dif})}(F,G) = \int_\Gamma (H^{(s,\,\mathrm{dif})}F)(\gamma)G(\gamma)\,\mu(d\gamma),$$

where $H^{(s,\,\mathrm{dif})}F$ is given by (5.8).

Using (3.2) and (3.3), we next have:

$$\int_\Gamma (H^{(s,\,\mathrm{dif})}F)^2(\gamma)\mu(d\gamma) = c^2\int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d} z\,dx\exp\big[\langle(-4s+1)\phi(x-\cdot),\gamma\rangle\big]$$
$$\times\Bigg\{\left(\frac{1}{2}\,\Delta_x F(\gamma\cup x)\right)^2 - \Delta_x F(\gamma\cup x)\sum_{u\in\gamma}\langle\nabla_x F(\gamma\cup x), s\nabla\phi(x-u)\rangle$$
$$+\sum_{u\in\gamma}\langle\nabla_x F(\gamma\cup x), s\nabla\phi(x-u)\rangle^2$$
$$+\sum_{u_1\in\gamma}\ \sum_{u_2\in\gamma\setminus u_1}\langle\nabla_x F(\gamma\cup x), s\nabla\phi(x-u_1)\rangle\langle\nabla_x F(\gamma\cup x), s\nabla\phi(x-u_2)\rangle\Bigg\}$$
$$+\,c^2\int_\Gamma\mu(d\gamma)\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\exp\big[\langle -2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot),\gamma\rangle$$
$$+(-4s+1)\phi(x_1-x_2)\big]\Bigg\{\frac{1}{4}\,\Delta_{x_1}F(\gamma\cup x_1\cup x_2)\Delta_{x_2}F(\gamma\cup x_1\cup x_2)$$
$$-\,\Delta_{x_1}F(\gamma\cup x_1\cup x_2)\sum_{u\in\gamma}\langle\nabla_{x_2}F(\gamma\cup x_1\cup x_2), s\nabla\phi(x_2-u)\rangle$$
$$-\,\Delta_{x_1}F(\gamma\cup x_1\cup x_2)\langle\nabla_{x_2}F(\gamma\cup x_1\cup x_2), s\nabla\phi(x_2-x_1)\rangle$$
$$+\sum_{u\in\gamma}\langle\nabla_{x_1}F(\gamma\cup x_1\cup x_2), s\phi(x_1-u)\rangle\langle\nabla_{x_2}F(\gamma\cup x_1\cup x_2), s\nabla\phi(x_2-u)\rangle$$
$$+\sum_{u_1\in\gamma}\ \sum_{u_2\in\gamma\setminus u_1}\langle\nabla_{x_1}F(\gamma\cup x_1\cup x_2), s\nabla\phi(x_1-u_1)\rangle$$
$$\times\,\langle\nabla_{x_2}F(\gamma\cup x_1\cup x_2), s\nabla\phi(x_2-u_2)\rangle$$

$$+\sum_{u\in\gamma}2\langle\nabla_{x_1}F(\gamma\cup x_1\cup x_2),s\nabla\phi(x_1-u)\rangle\langle\nabla_{x_2}F(\gamma\cup x_1\cup x_2),s\nabla\phi(x_2-x_1)\rangle$$
$$+\langle\nabla_{x_1}F(\gamma\cup x_1\cup x_2),s\nabla\phi(x_1-x_2)\rangle\langle\nabla_{x_2}F(\gamma\cup x_1\cup x_2),s\nabla\phi(x_2-x_1)\rangle\Big\}. \tag{5.9}$$

Using the Ruelle bound, Lemma 2.1, the definition of $\mathcal{L}C^2_{\mathrm{b}}(\ddot{\Gamma})$, and the assumptions of the theorem, one concludes that each integral in (5.9) is well-defined and finite.

Hence, $H^{(s,\,\mathrm{dif})}$ is the generator of the bilinear form $(\mathcal{E}^{(s,\,\mathrm{dif})},\mathcal{L}C^2_{\mathrm{b}}(\ddot{\Gamma}))$ in $L^2(\Gamma,\mu)$ This implies that $(\mathcal{E}^{(s,\,\mathrm{dif})},\mathcal{L}C^2_{\mathrm{b}}(\ddot{\Gamma}))$ is closable in $L^2(\Gamma,\mu)$, and the generator of the closed form $(\mathcal{E}^{(s,\,\mathrm{dif})},D(\mathcal{E}^{(s,\mathrm{dif})}))$ is the Friedrichs' extension of $(H^{(s,\,\mathrm{dif})},\mathcal{L}C^2_{\mathrm{b}}(\ddot{\Gamma})))$.

By using the general theory of Dirichlet forms [16], statement (iii) of the theorem can be proved analogously to [17] and [10], whereas the proof of statement (iv) is analogous to [25] and [10]. □

Remark 5.1 Note that, even when $d=1$, the finite-dimensional distributions of the process $\boldsymbol{M}^{(s,\,\mathrm{dif})}$ are concentrated on the Cartesian powers of the space Γ.

Remark 5.2 Note that, in the case of the gradient stochastic dynamics, i.e., $s=1/2$, assumption (5.7) means:

$$\nabla\phi\in L^1(\mathbb{R}^d,e^{-\phi(x)}\,dx)\cap L^2(\mathbb{R}^d,e^{-\phi(x)}\,dx).$$

and, by (5.8), we get

$$(H^{(1/2,\,\mathrm{dif})}F)(\gamma)=\frac{c}{2}\int_{\mathbb{R}^d}\gamma(dx)\bigg(-\Delta_xF(\gamma)+\sum_{u\in\gamma\setminus x}\langle\nabla_xF(\gamma),\nabla\phi(x-u)\rangle\bigg) \tag{5.10}$$

for each $F\in\mathcal{L}C^2_{\mathrm{b}}(\ddot{\Gamma})$.

6 Main results

For each $s\in[0,1]$, let us consider the Kawasaki dynamics $\boldsymbol{M}^{(s)}$ from Theorem 4.1. We will assume that $a(x)=\tilde{a}(|x|)$ for all $x\in\mathbb{R}^d$, where $\tilde{a}:[0,\infty)\to[0,\infty)$. We now perform the following scaling of this dynamics. For each $\epsilon>0$, instead of the function a, we use the function a_ϵ given by (1.3). In the obtained dynamics, we also scale time, multiplying it by ϵ^{-2}. Thus, we obtain a Kawasaki dynamics $\boldsymbol{M}^{(s,\,\epsilon)}$, which is exactly the Hunt process from Theorem 4.1 corresponding to the function $\epsilon^{-2}a_\epsilon$. We denote by $(H^{(s,\,\epsilon)},D(H^{(s,\,\epsilon)}))$ the generator of this dynamics. Under the condition of Theorem 4.1, (iii), we have, by Proposition 4.1, that $\mathcal{L}_{\mathrm{b}}(\Gamma)\subset D(H^{(s,\,\epsilon)})$ and, for each $F\in\mathcal{L}_{\mathrm{b}}(\Gamma)$,

$$(H^{(s,\epsilon)}F)(\gamma) = -\epsilon^{-d-2}\int_{\mathbb{R}^d}\gamma(dx)\int_{\mathbb{R}^d} dy\, a((x-y)/\epsilon)$$
$$\times \exp[(1-s)E(x,\gamma\setminus x) - sE(y,\gamma\setminus x)](D^{-+}_{x,y}F)(\gamma). \quad (6.1)$$

We also note the evident inclusion $\mathcal{L}C^2_{\mathrm{b}}(\ddot{\Gamma}) \subset \mathcal{L}_{\mathrm{b}}(\Gamma)$ (in the sense of functions on Γ).

Theorem 6.1 *Fix $s \in [0,1]$. Let the conditions of Theorem* 4.1, (iii) *and Theorem* 5.1 *be satisfied. Furthermore, assume that the following conditions are satisfied*:

a) *The function a has compact support.*

b) *We have $e^{-s\phi} \in C^1_{\mathrm{b}}(\mathbb{R}^d)$.*

c) *If $s \neq 0$, then there exists $\delta > 0$ such that $g_\delta \in L^1(\mathbb{R}^d, dx)$. Here,*

$$g_\delta(x) := \sup_{y\in B(x;\delta)} e^{-s\phi(y)}\|\nabla\phi(y)\|, \qquad x\in\mathbb{R}^d.$$

d) *If $s \neq 0$, then there exists $\Lambda \in \mathcal{O}_{\mathrm{c}}(\mathbb{R}^d)$ such that*

$$e^{(-3s+1)\phi}\|\nabla\phi\| \in L^1(\Lambda, dx)$$

and the function $e^{(-3s+1)\phi}\|\nabla\phi\|$ is bounded on Λ^c.

Let

$$c := \int_{\mathbb{R}^d} a(x)(x^1)^2\,dx \quad (6.2)$$

and let $(H^{(s,\,\mathrm{dif})}, D(H^{(s,\,\mathrm{dif})}))$ correspond to the above choice of the constant c (see (5.8)*). Then, for each $F \in \mathcal{L}C^2_{\mathrm{b}}(\ddot{\Gamma})$, we have*:

$$H^{(s,\epsilon)}F \to H^{(s,\,\mathrm{dif})}F \quad \text{in } L^2(\Gamma,\mu) \text{ as } \epsilon\to 0. \quad (6.3)$$

Remark 6.1 Notice that condition c) of Theorem 6.1 is slightly stronger than the condition $e^{-s\phi}\|\nabla\phi\| \in L^1(\mathbb{R}^d, dx)$.

Next, we take the canonical realizations of the processes $\boldsymbol{M}^{(s,\epsilon)}$, $\epsilon > 0$, and $\boldsymbol{M}^{(s,\,\mathrm{dif})}$ and define stochastic processes $\boldsymbol{Y}^{(s,\epsilon)} = (\boldsymbol{Y}^{(s,\epsilon)}_t)_{t\ge 0}$ and $\boldsymbol{Y}^{(s,\,\mathrm{dif})} = (\boldsymbol{Y}^{(s,\,\mathrm{dif})}_t)_{t\ge 0}$ whose law is the probability measure on $D([0,+\infty),\Gamma)$, respectively $C([0,+\infty),\Gamma)$ (replace Γ with $\ddot{\Gamma}$ if $d=1$), given by

$$\boldsymbol{Q}^{(s,\epsilon)} := \int_\Gamma \boldsymbol{P}^{(s,\epsilon)}_\gamma\,\mu(d\gamma),$$

respectively

$$\boldsymbol{Q}^{(s,\,\mathrm{dif})} := \int_\Gamma \boldsymbol{P}^{(s,\,\mathrm{dif})}_\gamma\,\mu(d\gamma).$$

Corollary 6.1 *Assume that the conditions of Theorem* 6.1 *are satisfied. Assume additionally that* $\mathcal{L}C^2_{\mathrm{b}}(\ddot\Gamma)$ *is a core for* $(H^{(s,\,\mathrm{dif})}, D(H^{(s,\,\mathrm{dif})}))$. *Then, as* $\epsilon \to 0$, *the finite-dimensional distributions of the process* $\boldsymbol{M}^{(s,\,\epsilon)}$ *weakly converge to the finite-dimensional distributions of the process* $\boldsymbol{M}^{(s,\,\mathrm{dif})}$ *with* c *given by* (6.2).

Remark 6.2 In [4], Fritz stated without proof that, in the case of a sufficiently smooth potential ϕ, the set $\mathcal{F}C^2_{\mathrm{b}}(C^2_0(\mathbb{R}^d), \ddot\Gamma)$ (and hence also $\mathcal{L}C^2_{\mathrm{b}}(\ddot\Gamma)$) is a core for the generator of the gradient stochastic dynamics, $(H^{(1/2,\,\mathrm{dif})}, D(H^{(1/2,\,\mathrm{dif})}))$.

Following [2], we will now introduce additional conditions on the potential ϕ.

Let $\alpha : [0,\infty) \to \mathbb{R}$ be any monotonic, increasing, and concave function such that:

(i) $\alpha(0) \ge 1$ and $\alpha(\lambda) \to \infty$ as $\lambda \to \infty$.

(ii) $\alpha'(\lambda) \le [1/(1+\lambda)]\alpha(\lambda)$ for $\lambda \ge 0$, and there exists a constant $c > 0$ such that $\alpha''(\lambda) \ge -c[1/(1+\lambda)]$.

For example, let $l(\lambda) := \log(1+\lambda)$, $\lambda \ge 0$. Then, for any $n \in \mathbb{N}$, the function $\alpha(\lambda) := 1 + \underbrace{l \circ \cdots \circ l}_{n \text{ times}}(\lambda)$ satisfies the above conditions.

So, in what follows we will assume:

(A) We have $\phi \in C^3_{\mathrm{b}}(\mathbb{R}^d)$, and there exist a constant c_0 and a function α that satisfies the conditions (i) and (ii) above, such that, for all $x \in \mathbb{R}^d$,

$$\|\nabla\phi(x)\| + \|\nabla^2\phi(x)\| + \|\nabla^3\phi(x)\| \le \exp[-c_0 \log(1+|x|^2)\alpha(1+|x|^2)].$$

Corollary 6.2 *Let* $\mu \in \mathcal{G}^t(z,\phi)$ *be as in Theorem* 3.1, (iii). *Assume that the assumptions of Theorem* 6.1 *for* $s = 1/2$, *as well as assumption* (A) *are satisfied. Then, as* $\epsilon \to 0$, *the finite-dimensional distributions of the process* $\boldsymbol{M}^{(1/2,\,\epsilon)}$ *weakly converge to the finite-dimensional distributions of the process* $\boldsymbol{M}^{(1/2,\,\mathrm{dif})}$ *with* c *given by* (6.2).

Remark 6.3 Assume that the assumptions of Corollary 6.2 are satisfied. To prove this corollary, we will easily extend the convergence (6.3) to a wider set of local functions, and then use the result of [2] stating that this set of functions is a core for $(H^{(1/2,\,\mathrm{dif})}, D(H^{(1/2,\,\mathrm{dif})}))$. However, we believe that, using the technique of the K-transform, it should be possible to deduce from [2] that the set $\mathcal{F}C^2_{\mathrm{b}}(C^2_0(\mathbb{R}^d), \ddot\Gamma)$ is a core for $(H^{(1/2,\,\mathrm{dif})}, D(H^{(1/2,\,\mathrm{dif})}))$ (the latter statement has also been conjectured in [2]). Then, Corrolary 6.2 would become a special case of Corllary 6.1.

7 Proofs

Proof of Theorem 6.1. Denote the support of the function a by Δ. By a), the set Δ is bounded and hence $r := \sup_{h\in\Delta}|h| < \infty$. Recall δ from condition c) of the theorem. In what follows, we will assume that $\epsilon \in (0, \delta/r)$. Then

$$|\epsilon h| < \delta \quad \text{for all } h \in \Delta. \tag{7.1}$$

Fix any $F \in \mathcal{L}C^2_{\mathrm{b}}(\ddot{\Gamma})$. By (6.1),

$$(H^{(s,\epsilon)}F)(\gamma) = -\epsilon^{-2}\int_{\mathbb{R}^d}\gamma(dx)\int_\Delta dh\, a(h)$$
$$\times \exp[(1-s)E(x,\gamma\setminus x) - sE(x+\varepsilon h, \gamma\setminus x)](F(\gamma\setminus x\cup(x+\varepsilon h)) - F(\gamma)).$$

Using (3.2) and (3.3), we have:

$$\int_\Gamma (H^{(s,\epsilon)}F)^2(\gamma)\,\mu(d\gamma)$$
$$= \epsilon^{-4}\int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d} z\,dx\int_\Delta dh_1\int_\Delta dh_2\, a(h_1)a(h_2)(F(\gamma\cup(x+\epsilon h_1)) - F(\gamma\cup x))$$
$$\times (F(\gamma\cup(x+\epsilon h_2)) - F(\gamma\cup x))$$
$$\times \exp\left[\sum_{u\in\gamma}((-2s+1)\phi(x-u) - s\phi(x+\epsilon h_1 - u) - s\phi(x+\epsilon h_2 - u))\right]$$
$$+ \epsilon^{-4}\int_\Gamma \mu(d\gamma)\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_\Delta dh_1\int_\Delta dh_2\, a(h_1)a(h_2)$$
$$\times (F(\gamma\cup(x_1+\epsilon h_1)\cup x_2) - F(\gamma\cup x_1\cup x_2))$$
$$\times (F(\gamma\cup x_1\cup(x_2+\epsilon h_2)) - F(\gamma\cup x_1\cup x_2))$$
$$\times \exp\Big[(1-2s)\phi(x_1-x_2) - s\phi(x_1+\epsilon h_1 - x_2) - s\phi(x_2+\epsilon h_2 - x_1)$$
$$+ \sum_{u\in\gamma}(-s\phi(x_1-u) - s\phi(x_2-u) - s\phi(x_1+\epsilon h_1-u) - s\phi(x_2+\epsilon h_2 - u))\Big].$$

(Here and below, our calculations are justified by the assumptions of the theorem and Lemma 2.1). Hence, by Lemma 2.1, we get:

$$\int_\Gamma (H^{(s,\epsilon)}F)^2(\gamma)\,\mu(d\gamma)$$
$$= \epsilon^{-4}\int_{\mathbb{R}^d} z\,dx\int_\Delta dh_1\int_\Delta dh_2\, a(h_1)a(h_2)\int_\Gamma \mu(d\gamma)(F(\gamma\cup(x+\epsilon h_1)) - F(\gamma\cup x))$$
$$\times (F(\gamma\cup(x+\epsilon h_2)) - F(\gamma\cup x))$$

$$\times K\left(\left(\left(e^{(-2s+1)\phi(x-\cdot)-s\phi(x+\epsilon h_1-\cdot)-s\phi(x+\epsilon h_2-\cdot)}-1\right)^{\otimes n}\right)_{n=0}^{\infty}\right)(\gamma)$$
$$+\epsilon^{-4}\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_\Delta dh_1\int_\Delta dh_2\,a(h_1)a(h_2)$$
$$\times\exp\big[(1-2s)\phi(x_1-x_2)-s\phi(x_1+\epsilon h_1-x_2)-s\phi(x_2+\epsilon h_2-x_1)\big]$$
$$\times\int_\Gamma \mu(d\gamma)(F(\gamma\cup(x_1+\epsilon h_1)\cup x_2)-F(\gamma\cup x_1\cup x_2))$$
$$\times(F(\gamma\cup x_1\cup(x_2+\epsilon h_2))-F(\gamma\cup x_1\cup x_2))$$
$$\times K\left(\left(\left(e^{-s\phi(x_1-\cdot)-s\phi(x_2-\cdot)-s\phi(x_1+\epsilon h_1-\cdot)-s\phi(x_2+\epsilon h_2-\cdot)}-1\right)^{\otimes n}\right)_{n=0}^{\infty}\right)(\gamma). \quad (7.2)$$

For each $\gamma\in\Gamma$ and $x,h\in\mathbb{R}^d$, denote by $y_1(\gamma,x,h)$ a point in the segment $[x,x+h]$ such that

$$F(\gamma\cup(x+h))-F(\gamma\cup x)=\langle\nabla_x F(\gamma\cup x),h\rangle+\frac{1}{2}\langle\nabla_y^2F(\gamma\cup y),h^{\otimes 2}\rangle\big|_{y=y_1(\gamma,x,h)}. \quad (7.3)$$

Also, for each $x,h\in\mathbb{R}^d$, we denote by $y_2(x,h)$ a point in the segment $[x,x+h]$ such that

$$e^{-s\phi(x+h)}=e^{-s\phi(x)}+e^{-s\phi(y_2(x,h))}\langle -s\nabla\phi(y_2(x,h)),h\rangle. \quad (7.4)$$

Note that the existence of $y_1(\gamma,x,h)$ and $y_2(x,h)$ follows from the definition of the $\mathcal{L}C_{\mathrm{b}}^2(\ddot\Gamma)$ and assumption b) of the theorem, respectively. Note also that, by (7.1),

$$e^{-s\phi(y_2(x,\epsilon h))}\|\nabla\phi(y_2(x,\epsilon h))\|\le g_\delta(x),\quad x\in\mathbb{R}^d,\ h\in\Delta.$$

Now, by (7.2), (7.3), and (7.4), we have:

$$\int_\Gamma (H^{(s,\epsilon)}F)^2(\gamma)\,\mu(d\gamma)$$
$$=\int_{\mathbb{R}^d} z\,dx\int_\Delta dh_1\int_\Delta dh_2\,a(h_1)a(h_2)\int_\Gamma\mu(d\gamma)\big(\epsilon^{-2}F_{-2}^{(1)}(\gamma,x,h_1,h_2)$$
$$+\epsilon^{-1}F_{-1}^{(1)}(\gamma,x,h_1,h_2,\epsilon)+F_0^{(1)}(\gamma,x,h_1,h_2,\epsilon)\big)(KG^{(1)}(\cdot,x,h_1,h_2,\epsilon))(\gamma)$$
$$+\int_{\mathbb{R}^d} z\,dx_1\int_{\mathbb{R}^d} z\,dx_2\int_\Delta dh_1\int_\Delta dh_2\,a(h_1)a(h_2)$$
$$\times\big(e^{(1-4s)\phi(x_1-x_2)}+\epsilon u_1(x_1,x_2,h_1,h_2,\epsilon)+\epsilon^2u_2(x_1,x_2,h_1,h_2,\epsilon)\big)$$
$$\times\int_\Gamma\mu(d\gamma)\big(\epsilon^{-2}F_{-2}^{(2)}(\gamma,x_1,x_2,h_1,h_2)+\epsilon^{-1}F_{-1}^{(2)}(\gamma,x_1,x_2,h_1,h_2,\epsilon)$$
$$+F_0^{(2)}(\gamma,x_1,x_2,h_1,h_2,\epsilon)\big)(KG^{(2)}(\cdot,x_1,x_2,h_1,h_2,\epsilon))(\gamma). \quad (7.5)$$

Here,

$$u_1(x_1,x_2,h_1,h_2,\epsilon):=e^{(1-3s)\phi(x_1-x_2)}\Big(e^{-s\phi(y_2(x_1-x_2,\epsilon h_1))}\langle -s\nabla\phi(y_2(x_1-x_2,\epsilon h_1)),h_1\rangle$$

$$
\begin{aligned}
&\quad + e^{-s\phi(y_2(x_2-x_1,\epsilon h_2))}\langle -s\nabla\phi(y_2(x_2-x_1,\epsilon h_2)),h_2\rangle\big),\\
&u_2(x_1,x_2,h_1,h_2,\epsilon) := e^{(1-2s)\phi(x_1-x_2)}e^{-s\phi(y_2(x_1-x_2,\epsilon h_1))}e^{-s\phi(y_2(x_2-x_1,\epsilon h_2))}\\
&\quad \times \langle -s\nabla\phi(y_2(x_1-x_2,\epsilon h_1)),h_1\rangle\langle -s\nabla\phi(y_2(x_2-x_1,\epsilon h_2)),h_2\rangle,
\end{aligned}
\tag{7.6}
$$

and

$$
\begin{aligned}
&F^{(1)}_{-2}(\gamma,x,h_1,h_2) := \langle \nabla_x F(\gamma\cup x),h_1\rangle\langle \nabla_x F(\gamma\cup x),h_2\rangle,\\
&F^{(1)}_{-1}(\gamma,x,h_1,h_2,\epsilon) := \langle \nabla_x F(\gamma\cup x),h_1\rangle(1/2)\langle \nabla^2_y F(\gamma\cup y),h_2^{\otimes 2}\rangle\big|_{y=y_1(\gamma,x,\epsilon h_2)}\\
&\quad + \langle \nabla_x F(\gamma\cup x),h_2\rangle(1/2)\langle \nabla^2_y F(\gamma\cup y),h_1^{\otimes 2}\rangle\big|_{y=y_1(\gamma,x,\epsilon h_1)},\\
&F^{(1)}_{0}(\gamma,x,h_1,h_2,\epsilon) = (1/4)\langle \nabla^2_y F(\gamma\cup y),h_1^{\otimes 2}\rangle\big|_{y=y_1(\gamma,x,\epsilon h_1)}\\
&\quad \times \langle \nabla^2_y F(\gamma\cup y),h_2^{\otimes 2}\rangle\big|_{y=y_1(\gamma,x,\epsilon h_2)},
\end{aligned}
\tag{7.7}
$$

and

$$
\begin{aligned}
&F^{(2)}_{-2}(\gamma,x_1,x_2,h_1,h_2) := \langle \nabla_{x_1} F(\gamma\cup x_1\cup x_2),h_1\rangle\langle \nabla_{x_2} F(\gamma\cup x_1\cup x_2),h_2\rangle,\\
&F^{(2)}_{-1}(\gamma,x_1,x_2,h_1,h_2,\epsilon)\\
&\quad := \langle \nabla_{x_1} F(\gamma\cup x_1\cup x_2),h_1\rangle(1/2)\langle \nabla^2_y F(\gamma\cup x_1\cup y),h_2^{\otimes 2}\rangle\big|_{y=y_1(\gamma\cup x_1,x_2,\epsilon h_2)}\\
&\quad\quad + \langle \nabla_{x_2} F(\gamma\cup x_1\cup x_2),h_2\rangle(1/2)\langle \nabla^2_y F(\gamma\cup y\cup x_2),h_1^{\otimes 2}\rangle\big|_{y=y_1(\gamma\cup x_2,x_1,\epsilon h_1)},\\
&F^{(1)}_{0}(\gamma,x,h_1,h_2,\epsilon) := (1/4)\langle \nabla^2_y F(\gamma\cup y\cup x_2),h_1^{\otimes 2}\rangle\big|_{y=y_1(\gamma\cup x_2,x_1,\epsilon h_1)}\\
&\quad\quad \times \langle \nabla^2_y F(\gamma\cup x_1\cup y),h_2^{\otimes 2}\rangle\big|_{y=y_1(\gamma\cup x_1,x_2,\epsilon h_2)},
\end{aligned}
\tag{7.8}
$$

and

$$
\begin{aligned}
&G^{(1)}(\cdot,x,h_1,h_2,\epsilon)\\
&\quad := \big((e^{(-4s+1)\phi(x-\cdot)}-1+\epsilon g^{(1)}_1(\cdot,x,h_1,h_2,\epsilon)+\epsilon^2 g^{(1)}_2(\cdot,x,h_1,h_2,\epsilon))^{\otimes n}\big)_{n=0}^{\infty},\\
&G^{(2)}(\cdot,x_1,x_2,h_1,h_2,\epsilon) := \big((e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}-1\\
&\quad\quad +\epsilon g^{(2)}_1(\cdot,x_1,x_2,h_1,h_2,\epsilon)+\epsilon^2 g^{(2)}_2(\cdot,x_1,x_2,h_1,h_2,\epsilon))^{\otimes n}\big)_{n=0}^{\infty},
\end{aligned}
$$

where

$$
\begin{aligned}
&g^{(1)}_1(\cdot,x,h_1,h_2,\epsilon) := e^{(-3s+1)\phi(x-\cdot)}\big(e^{-s\phi(y_2(x-\cdot,\epsilon h_1))}\langle -s\nabla\phi(y_2(x-\cdot,\epsilon h_1)),h_1\rangle\\
&\quad + e^{-s\phi(y_2(x-\cdot,\epsilon h_2))}\langle -s\nabla\phi(y_2(x-\cdot,\epsilon h_2)),h_2\rangle\big),\\
&g^{(1)}_2(\cdot,x,h_1,h_2,\epsilon) := e^{(-2s+1)\phi(x-\cdot)}e^{-s\phi(y_2(x-\cdot,\epsilon h_1))}\langle -s\nabla\phi(y_2(x-\cdot,\epsilon h_1)),h_1\rangle\\
&\quad \times e^{-s\phi(y_2(x-\cdot,\epsilon h_2))}\langle -s\nabla\phi(y_2(x-\cdot,\epsilon h_2)),h_2\rangle,\\
&g^{(2)}_1(\cdot,x_1,x_2,h_1,h_2,\epsilon) := e^{-s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}e^{-s\phi(y_2(x_1-\cdot,\epsilon h_1))}\langle -s\nabla\phi(y_2(x_1-\cdot,\epsilon h_1)),h_1\rangle
\end{aligned}
$$

$$+ e^{-2s\phi(x_1-\cdot)-s\phi(x_2-\cdot)} e^{-s\phi(y_2(x_2-\cdot,\epsilon h_2))} \langle -s\nabla\phi(y_2(x_2-\cdot,\epsilon h_2)), h_2\rangle,$$

$$g_1^{(2)}(\cdot, x_1, x_2, h_1, h_2, \epsilon) := e^{-s\phi(x_1-\cdot)-s\phi(x_2-\cdot)} e^{-s\phi(y_2(x_1-\cdot,\epsilon h_1))} e^{-s\phi(y_2(x_2-\cdot,\epsilon h_2))}$$
$$\times \langle -s\nabla\phi(y_2(x_1-\cdot,\epsilon h_1)), h_1\rangle \langle -s\nabla\phi(y_2(x_2-\cdot,\epsilon h_2)), h_2\rangle. \tag{7.9}$$

Since F is a local function, so are $F_j^{(i)}$, $i = 1, 2$, $j = -2, -1, 0$, as functions of $\gamma \in \Gamma$. Then, by (7.5), we get

$$\int_\Gamma (H^{(s,\epsilon)}F)^2(\gamma)\,\mu(d\gamma)$$
$$= \int_{\mathbb{R}^d} z\,dx \int_\Delta dh_1 \int_\Delta dh_2\, a(h_1)a(h_2)$$
$$\times \int_{\Gamma_0} \rho_\mu(d\eta)\big(K^{-1}\big(\epsilon^{-2}F_{-2}^{(1)}(\cdot, x, h_1, h_2) + \epsilon^{-1}F_{-1}^{(1)}(\cdot, x, h_1, h_2, \epsilon)$$
$$+ F_0^{(1)}(\cdot, x, h_1, h_2, \epsilon)\big) \star G^{(1)}(\cdot, x, h_1, h_2, \epsilon)\big)(\eta)$$
$$+ \int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2 \int_\Delta dh_1 \int_\Delta dh_2\, a(h_1)a(h_2)$$
$$\times \big(e^{(1-4s)\phi(x_1-x_2)} + \epsilon u_1(x_1, x_2, h_1, h_2, \epsilon) + \epsilon^2 u_2(x_1, x_2, h_1, h_2, \epsilon)\big)$$
$$\times \int_{\Gamma_0} \rho_\mu(d\eta)\big(K^{-1}\big(\epsilon^{-2}F_{-2}^{(2)}(\cdot, x_1, x_2, h_1, h_2) + \epsilon^{-1}F_{-1}^{(2)}(\cdot, x_1, x_2, h_1, h_2, \epsilon)$$
$$+ F_0^{(2)}(\cdot, x_1, x_2, h_1, h_2, \epsilon)\big) \star G^{(2)}(\cdot, x_1, x_2, h_1, h_2, \epsilon)\big)(\eta). \tag{7.10}$$

Collecting the coefficients by powers of ϵ, we get:

$$\int_\Gamma (H^{(s,\epsilon)}F)^2(\gamma)\mu(d\gamma) = c_{-2}(\epsilon)\epsilon^{-2} + c_{-1}(\epsilon)\epsilon^{-1} + c_0(\epsilon) + c_1(\epsilon)\epsilon, \tag{7.11}$$

where

$$c_{-2}(\epsilon) = \int_{\mathbb{R}^d} z\,dx \int_\Delta dh_1 \int_\Delta dh_2\, a(h_1)a(h_2) \int_{\Gamma_0} \rho_\mu(d\eta)$$
$$\times \big(K^{-1}F_{-2}^{(1)}(\cdot, x, h_1, h_2) \star \big(\big(e^{(-4s+1)\phi(x-\cdot)} - 1\big)^{\otimes n}\big)_{n=0}^\infty\big)(\eta)$$
$$+ \int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2 \int_\Delta dh_1 \int_\Delta dh_2\, e^{(1-4s)\phi(x_1-x_2)}$$
$$\times \int_{\Gamma_0} \rho_\mu(d\eta)\big(K^{-1}F_{-2}^{(2)}(\cdot, x_1, x_2, h_1, h_2) \star \big(\big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)} - 1\big)^{\otimes n}\big)_{n=0}^\infty\big)(\eta),$$

$$c_{-1}(\epsilon) = \int_{\mathbb{R}^d} z\,dx \int_\Delta dh_1 \int_\Delta dh_2\, a(h_1)a(h_2) \int_{\Gamma_0} \rho_\mu(d\eta)\Big[\big(K^{-1}F_{-2}^{(1)}(\cdot, x, h_1, h_2)$$
$$\star \big(n\big(e^{(-4s+1)\phi(x-\cdot)} - 1\big)^{\otimes(n-1)} \odot g_1^{(1)}(\cdot, x, h_1, h_2, \epsilon)\big)_{n=0}^\infty\big)(\eta)$$
$$+ \big(K^{-1}F_{-1}^{(1)}(\cdot, x, h_1, h_2, \epsilon) \star \big(\big(e^{(-4s+1)\phi(x-\cdot)} - 1\big)^{\otimes n}\big)_{n=0}^\infty\big)(\eta)\Big]$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2 \int_\Delta dh_1 \int_\Delta dh_2\, a(h_1)a(h_2)\Big[e^{(1-4s)\phi(x_1-x_2)} \\
&\times \int_{\Gamma_0} \rho_\mu(d\eta)\big(K^{-1}F^{(2)}_{-1}(\cdot,x_1,x_2,h_1,h_2,\epsilon)\star\big(\big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}-1\big)^{\otimes n}\big)_{n=0}^{\infty}\big)(\eta) \\
&+u_1(x_1,x_2,h_1,h_2,\epsilon)\int_{\Gamma_0}\rho_\mu(d\eta)\big(K^{-1}F^{(2)}_{-2}(\cdot,x_1,x_2,h_1,h_2) \\
&\star\big(\big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}-1\big)^{\otimes n}\big)_{n=0}^{\infty}\big)(\eta) \\
&+e^{(1-4s\phi(x_1-x_2))}\int_{\Gamma_0}\big(K^{-1}F^{(2)}_{-2}(\cdot,x_1,x_2,h_1,h_2) \\
&\star\big(n\big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}-1\big)^{\otimes(n-1)}\odot g^{(2)}_1(\cdot,x_1,x_2,h_1,h_2,\epsilon)\big)_{n=0}^{\infty}\big)(\eta)\Big],
\end{aligned}
$$

$$
\begin{aligned}
c_0(\epsilon)=&\int_{\mathbb{R}^d} z\,dx \int_\Delta dh_1\int_\Delta dh_2\, a(h_1)a(h_2)\int_{\Gamma_0}\rho_\mu(d\eta) \\
&\times\Big[\big(K^{-1}F^{(1)}_{-2}(\cdot,x,h_1,h_2) \\
&\star\big(n(n-1)(1/2)\big(e^{(1-4s)\phi(x-\cdot)}-1\big)^{\otimes(n-2)}\odot\big(g^{(1)}_1(\cdot,x,h_1,h_2,\epsilon)\big)^{\otimes 2} \\
&+n\big(e^{(1-4s)\phi(x-\cdot)}-1\big)^{\otimes(n-1)}\odot g^{(1)}_2(\cdot,x,h_1,h_2,\epsilon)\big)_{n=0}^{\infty}\big)(\eta) \\
&+\big(K^{-1}F^{(1)}_{-1}(\cdot,x,h_1,h_2,\epsilon) \\
&\star\big(n\big(e^{(1-4s)\phi(x-\cdot)}-1\big)^{\otimes(n-1)}\odot g^{(1)}_1(\cdot,x,h_1,h_2,\epsilon)\big)_{n=0}^{\infty}\big)(\eta) \\
&+\big(K^{-1}F^{(1)}_{0}(\cdot,x,h_1,h_2,\epsilon)\star\big(\big(e^{(1-4s)\phi(x-\cdot)}-1\big)^{\otimes n}\big)_{n=0}^{\infty}\big)(\eta)\Big] \\
&+\int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2 \int_\Delta dh_1 \int_\Delta dh_2\, a(h_1)a(h_2)\Big[e^{(1-4s)\phi(x_1-x_2)} \\
&\times\int_{\Gamma_0}\rho_\mu(d\eta)\big(K^{-1}F^{(2)}_{0}(\cdot,x_1,x_2,h_1,h_2,\epsilon) \\
&\star\big(\big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}-1\big)^{\otimes n}\big)_{n=0}^{\infty}\big)(\eta) \\
&+e^{(1-4s)\phi(x_1-x_2)}\int_{\Gamma_0}\rho_\mu(d\eta)\big(K^{-1}F^{(2)}_{-1}(\cdot,x_1,x_2,h_1,h_2,\epsilon) \\
&\star\big(n\big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}-1\big)^{\otimes(n-1)}\odot g^{(2)}_1(\cdot,x_1,x_2,h_1,h_2,\epsilon)\big)_{n=0}^{\infty}\big)(\eta) \\
&+e^{(1-4s)\phi(x_1-x_2)}\int_{\Gamma_0}\rho_\mu(d\eta)\big(K^{-1}F^{(2)}_{-2}(\cdot,x_1,x_2,h_1,h_2) \\
&\star\big(n(n-1)(1/2)\big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}-1\big)^{\otimes(n-2)}\odot\big(g^{(2)}_1(\cdot,x_1,x_2,h_1,h_2,\epsilon)\big)^{\otimes 2} \\
&+n\big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}-1\big)^{\otimes(n-1)}\odot g^{(2)}_1(\cdot,x_1,x_2,h_1,h_2,\epsilon)\big)_{n=0}^{\infty}\big)\big)(\eta) \\
&+u_1(x_1,x_2,h_1,h_2,\epsilon)\int_{\Gamma_0}\rho_\mu(d\eta)\big(K^{-1}F^{(2)}_{-1}(\cdot,x_1,x_2,h_1,h_2,\epsilon)
\end{aligned}
$$

$$\star\big(\big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}-1\big)^{\otimes n}\big)_{n=0}^{\infty}\big)(\eta)$$
$$+\,u_1(x_1,x_2,h_1,h_2,\epsilon)\int_{\Gamma_0}\rho_\mu(d\eta)\big(K^{-1}F^{(2)}_{-2}(\cdot,x_1,x_2,h_1,h_2)$$
$$\star\big(n\big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}-1\big)^{\otimes(n-1)}\odot g_1^{(2)}(\cdot,x_1,x_2,h_1,h_2,\epsilon)\big)_{n=0}^{\infty}\big)(\eta)$$
$$+\,u_2(x_1,x_2,h_1,h_2,\epsilon)\int_{\Gamma_0}\rho_\mu(d\eta)\big(K^{-1}F^{(2)}_{-2}(\cdot,x_1,x_2,h_1,h_2)$$
$$\star\big(\big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}-1\big)^{\otimes n}\big)_{n=0}^{\infty}\big)(\eta)\Big], \tag{7.12}$$

and $c_1(\epsilon)$ is defined so that equality (7.11) holds, i.e., by subtracting from the right hand side of (7.10) the expression $c_{-2}(\epsilon)\epsilon^{-2}+c_{-1}(\epsilon)\epsilon^{-1}+c_0(\epsilon)$, given through (7.12), and dividing by ϵ.

We evidently have:

$$\int_{\mathbb{R}^d} a(h)h^i\,dh=0,\quad i\in\{1,\dots,d\}, \tag{7.13}$$

and therefore

$$c_{-2}(\epsilon)=c_{-1}(\epsilon)=0.$$

Furthermore, as easily seen $a_1(\epsilon)=O(\epsilon)$ as $\epsilon\to 0$.

Below, we denote $\phi_i'(x):=(\partial/\partial x^i)\phi(x)$ and $\phi_i''(x):=(\partial^2/\partial(x^i)^2)\phi(x)$. So, using (7.13), the equalities

$$\int_{\mathbb{R}^d} a(h)h^ih^j\,dh=0,\quad i,j\in\{1,\dots,d\},\ i\neq j,$$
$$\int_{\mathbb{R}^d} a(h)(h^i)^2\,dh=c,\quad i\in\{1,\dots,d\},$$

and the dominated convergence theorem, we get

$$\lim_{\epsilon\to 0}\int_\Gamma (H^{(s,\epsilon)}F)^2(\gamma)\,\mu(d\gamma)=\lim_{\epsilon\to 0}c_0(\epsilon)$$
$$=c^2\sum_{i,j=1,\dots,d}\Big[\int_{\mathbb{R}^d} z\,dx\int_{\Gamma_0}\rho_\mu(d\eta)\Big[\big(K^{-1}(\partial/\partial x^i)F(\cdot\cup x)(\partial/\partial x^j)F(\cdot\cup x)$$
$$\star\big(n(n-1)\big(e^{(1-4s)\phi(x-\cdot)}-1\big)^{\otimes(n-2)}$$
$$\odot\big(e^{(1-4s)\phi(x-\cdot)}(-s)\phi_i'(x-\cdot)\big)\odot\big(e^{(1-4s)\phi(x-\cdot)}(-s)\phi_j'(x-\cdot)\big)$$
$$+\,n\big(e^{(1-4s)\phi(x-\cdot)}-1\big)^{\otimes(n-1)}\odot\big(e^{(-4s+1)\phi(x-\cdot)}(-s)\phi_i'(x-\cdot)(-s)\phi_j'(x-\cdot)\big)\big)_{n=0}^{\infty}\big)(\eta)$$
$$+\big(K^{-1}(\partial/\partial x^i)F(\cdot\cup x)(\partial^2/(\partial x^j)^2)F(\cdot\cup x)$$
$$\star\big(n\big(e^{(1-4s)\phi(x-\cdot)}-1\big)^{\otimes(n-1)}\odot\big(e^{(1-4s)\phi(x-\cdot)}(-s)\phi_i'(x-\cdot)\big)\big)_{n=0}^{\infty}\big)(\eta)$$

$$
\begin{aligned}
&+ \big(K^{-1}(1/4)(\partial^2/(\partial x^i)^2)F(\cdot\cup x)(\partial^2/(\partial x^j)^2)F(\cdot\cup x) \star \big(\big(e^{(1-4s)\phi(x-\cdot)}-1\big)^{\otimes n}\big)_{n=0}^{\infty}\big)(\eta)\Big]\\
&+ \int_{\mathbb{R}^d} z\,dx_1 \int_{\mathbb{R}^d} z\,dx_2 \bigg(e^{(1-4s)\phi(x_1-x_2)} \int_{\Gamma_0} \rho_\mu(d\eta)\\
&\times \Big[\big(K^{-1}(1/4)(\partial^2/(\partial x_1^i)^2)F(\cdot\cup x_1\cup x_2)(\partial^2/(\partial x_2^j)^2)F(\cdot\cup x_1\cup x_2)\\
&\star \big(\big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}-1\big)^{\otimes n}\big)_{n=0}^{\infty}\big)(\eta)\\
&+ \big(K^{-1}(\partial/\partial x_1^i)F(\cdot\cup x_1\cup x_2)(\partial^2/(\partial x_2^j)^2)F(\cdot\cup x_1\cup x_2)\\
&\star \big(n\big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}-1\big)^{\otimes(n-1)} \odot \big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}(-s)\phi_i'(x_1-\cdot)\big)\big)_{n=0}^{\infty}\big)(\eta)\\
&+ \big(K^{-1}(\partial/\partial x_1^i)F(\cdot\cup x_1\cup x_2)(\partial/\partial x_2^j)F(\cdot\cup x_1\cup x_2)\\
&\star \big(n(n-1)\big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}-1\big)^{\otimes(n-2)}\\
&\odot \big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}(-s)\phi_i'(x_1-\cdot)\big) \odot \big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}(-s)\phi_j'(x_2-\cdot)\big)\\
&+ n\big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}-1\big)^{\otimes(n-1)}\\
&\odot \big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}(-s)\phi_i'(x_1-\cdot)\phi_j'(x_2-\cdot)\big)\big)_{n=0}^{\infty}\big)(\eta)\Big]\\
&- s\phi_i'(x_1-x_2)\int_{\Gamma_0}\rho_\mu(d\eta)\big(K^{-1}(\partial/\partial x_1^i)F(\cdot\cup x_1\cup x_2)(\partial^2/(\partial x_2^j)^2)F(\cdot\cup x_1\cup x_2)\\
&\star \big(\big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}-1\big)^{\otimes n}\big)_{n=0}^{\infty}\big)(\eta)\\
&- 2s\phi_i'(x_1-x_2)\int_{\Gamma_0}\rho_\mu(d\eta)\big((\partial/\partial x_1^i)F(\gamma\cup x_1\cup x_2)(\partial/\partial x_2^j)F(\gamma\cup x_1\cup x_2)\\
&\star \big(n\big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}-1\big)^{\otimes(n-1)} \odot \big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}(-s)\phi_j'(x_2-\cdot)\big)\big)_{n=0}^{\infty}\big)(\eta)\\
&- s\phi_i'(x_1-x_2)(-s)\phi_j'(x_2-x_1)\\
&\times \int_{\Gamma_0}\rho_\mu(d\eta)\big(K^{-1}(\partial/\partial x_1^i)F(\gamma\cup x_1\cup x_2)(\partial/\partial x_2^j)F(\gamma\cup x_1\cup x_2)\\
&\star \big(\big(e^{-2s\phi(x_1-\cdot)-2s\phi(x_2-\cdot)}-1\big)^{\otimes n}\big)_{n=0}^{\infty}\big)(\eta)\bigg). \qquad (7.14)
\end{aligned}
$$

By Lemma 2.1, the expression in (7.14) is equal to the right hand side of equality (5.9). Hence,

$$
\lim_{\epsilon\to 0}\int_\Gamma (H^{(s,\,\epsilon)}F)^2(\gamma)\,\mu(d\gamma) = \int_\Gamma (H^{(s,\,\mathrm{dif})}F)^2(\gamma)\mu(d\gamma). \qquad (7.15)
$$

Analogously, one may also prove that

$$
\lim_{\epsilon\to 0}\int_\Gamma (H^{(s,\,\epsilon)}F)(\gamma)(H^{(s,\,\mathrm{dif})}F)(\gamma)\,\mu(d\gamma) = \int_\Gamma (H^{(s,\,\mathrm{dif})}F)^2(\gamma)\mu(d\gamma). \qquad (7.16)
$$

Now, (6.3) follows from (7.15) and (7.16). □

Proof of Corollary 6.1. By Theorem 6.1, [3, Chapter 3, Theorem 3.17], and the assumption of the corollary, we see that, for each $t \geq 0$, $e^{-tH^{(s,\epsilon)}} \to e^{-tH^{(s,\,\mathrm{dif})}}$ strongly in $L^2(\Gamma, \mu)$ as $\epsilon \to 0$. To conclude from here the weak convergence of finite-dimensional distributions, we proceed as follows.

We fix any $0 \leq t_1 < t_2 < \cdots < t_n$, $n \in \mathbb{N}$. For $\varepsilon \geq 0$, denote by $\mu^{\varepsilon}_{t_1,\dots,t_n}$ the finite-dimensional distribution of the process $\boldsymbol{Y}^{(s,\epsilon)}$ at times $t_1, \dots, t_n$, which is a probability measure on Γ^n. Since Γ is a Polish space (see e.g. [18]), by [24, Chapter II, Theorem 3.2], the measure μ is tight on Γ. Since all the marginal distributions of the measure $\mu^{\varepsilon}_{t_1,\dots,t_n}$ are μ, we therefore conclude that the set $\{\mu^{\varepsilon}_{t_1,\dots,t_n} \mid \varepsilon > 0\}$ is pre-compact in the space $\mathcal{M}(\Gamma^n)$ of the probability measures on Γ^n with respect to the weak topology, see e.g. [24, Chapter II, Section 6]. Hence, the weak convergence of finite-dimensional distributions follows from the strong convergence of the semigroups. □

Proof of Corollary 6.2. Let us first recall the following estimate of the correlation functions of the measure μ, which is stronger than the usual Ruelle bound (2.4).

Lemma 7.1 ([27]) *Let $\mu \in \mathcal{G}^t(z, \phi)$ be as in Theorem* 3.1, (iii). *Then, there exist $\xi, \psi > 0$ such that*

$$k_\mu(\eta) \leq \xi^{|\eta|} \exp\left[-\psi \sum_{r \in \mathbb{Z}^d} |\eta_r|^2\right], \quad \text{for all } \eta \in \Gamma_0 \tag{7.17}$$

(*recall the notations* (3.4) *and* (3.5)).

We denote by $\mathcal{D}$ the set of all functions F on $\ddot{\Gamma}$ which satisfy the same assumptions as functions from $\mathcal{L}C^2_{\mathrm{b}}(\ddot{\Gamma})$, accept for condition (iv), which now reads as follows:

(iv) Let Λ be the minimal subset of $\mathbb{R}^d$ which is a finite union of Q_r cubes, and such that (2.9) holds for this set Λ. Then there exist $\zeta > 0$, $\tau > 0$ and $0 < p < 1$ (depending on function F) such that

$$|F(\gamma)| \vee \|\nabla_x F(\gamma + \varepsilon_x)\| \vee \|\nabla^2_x F(\gamma + \varepsilon_x)\| \\ \leq \zeta^{|\gamma|} \exp\left[\tau\left(1 + \sum_{u \in \gamma} \sum_{v \in \gamma,\, v \neq u} \phi^+(u - v)\right)^p\right] \quad \text{for all } \gamma \in \ddot{\Gamma}_\Lambda,\ x \in \Lambda. \tag{7.18}$$

Here, $\phi^+(u) := \phi(u) \vee 0$, $u \in \mathbb{R}^d$.

Analogously to (2.8), we conclude from (2.7) that, for F as in the condition (iv) above, we have

$$|(K^{-1}F)(\eta)| \leq \chi_{\Gamma_\Lambda}(\eta)(2\zeta)^{|\eta|} \exp\left[\tau\left(1 + \sum_{u \in \eta} \sum_{v \in \eta,\, v \neq u} \phi^+(u - v)\right)^p\right], \quad \eta \in \Gamma_0. \tag{7.19}$$

Next, for each $\eta \in \Gamma_\Lambda$, we have:

$$\sum_{u\in\eta}\sum_{v\in\eta,\, v\neq u}\phi^+(u-v) \le \sup_{u\in\mathbb{R}^d}\phi^+(u)\Big(\sum_{r\in\mathbb{Z}^d}|\eta_r|\Big)^2$$
$$\le \sup_{u\in\mathbb{R}^d}\phi^+(u)\,\mathrm{vol}(\Lambda)\sum_{r\in\mathbb{Z}^d}|\eta_r|^2. \tag{7.20}$$

By (7.19) and (7.20),

$$|(K^{-1}F)(\eta)| \le \chi_{\Gamma_\Lambda}(\eta)(2\zeta)^{|\eta|}\exp\left[\tau\Big(1+\sigma\sum_{r\in\mathbb{Z}^d}|\eta_r|^2\Big)^p\right], \tag{7.21}$$

where

$$\sigma := \sup_{u\in\mathbb{R}^d}\phi^+(u)\,\mathrm{vol}(\Lambda).$$

Hence, by Lemma 7.1 and (7.21), we easily have: $F \in L^2(\Gamma,\mu)$. Furthermore, it is not hard to show that the statement of Proposition 4.1 holds with $\mathcal{D}$ replacing $\mathcal{L}_\mathrm{b}(\Gamma)$.

Next, it follows from [2] that there exists a subset $\mathcal{D}_1$ of $\mathcal{D}$ such that

$$\mathcal{D}_1 \subset D(H^{1/2,\,\mathrm{dif}}),$$

for each $F \in \mathcal{D}_1$ equality (5.10) holds, and $\mathcal{D}_1$ is a core for $(H^{1/2,\,\mathrm{dif}}, D(H^{1/2,\,\mathrm{dif}}))$. Therefore, analogously to the proof of Corollary 6.1, we see that, in order to prove Corollary 6.2, it suffices to show the convergence (6.3) for each $F \in \mathcal{D}_1$. But this can be done completely analogously to the proof of Theorem 6.1. One only needs to use Lemma 7.1 instead of the Ruelle bound (2.4), the definition of $\mathcal{D}$, estimate (7.21), as well as analogous estimates for the functions

$$K^{-1}(\partial/\partial x^i)F(\cdot+\varepsilon_x),\ K^{-1}(\partial/\partial x^i)F(\cdot+\varepsilon_x+\varepsilon_y),\ K^{-1}(\partial/\partial x^i)(\partial/\partial x^j)F(\cdot+\varepsilon_x),$$
$$K^{-1}(\partial/\partial x^i)(\partial/\partial x^j)F(\cdot+\varepsilon_x+\varepsilon_y),\quad x,y\in\Lambda,\ i,j=1,\dots,d,\quad \square.$$

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